2. (a) We need to maximize $\sum_{t=0}^{T-1}\left[(x_{t+1}-x_t)^2+(y_{t+1}-y_t)^2\right]^{1/2}$

or $\max c^T v$ where $c=[\underbrace{1,\dots 1}_{T \text{ ones}}]^T$

and $v=[v_0, v_1, \dots, v_{T-1}]^T$

subject to $x_T = x_f$, $y_T = y_f$, $\theta_{T-1} = \theta_f$ and $\theta_0 = 90^\circ$

$x_f - x_0 = \sum_{t=0}^{T-1} v_t \cos\theta_t$, $y_f - y_0 = \sum_{t=0}^{T-1} v_t \sin\theta_t$

Writing this as
$$\begin{bmatrix}\cos\theta_0 & \cos\theta_1 & \dots & \cos\theta_{T-1}\\ \sin\theta_0 & \sin\theta_1 & \dots & \sin\theta_{T-1}\end{bmatrix}\begin{bmatrix}v_0\\ v_1\\ \vdots\\ v_{T-1}\end{bmatrix}=\begin{bmatrix}x_f-x_0\\ y_f-y_0\end{bmatrix}$$

$\therefore$ $\max c^T(v-\vec{v}_{min}) + c^T\vec{v}_{min}$

Subject to
$$\begin{bmatrix}\cos\theta_0 & \dots & \cos\theta_{T-1}\\ \sin\theta_0 & \dots & \sin\theta_{T-1}\end{bmatrix}\begin{bmatrix}v_0-v_{min}\\ v_1-v_{min}\\ \vdots\\ v_{T-1}-v_{min}\end{bmatrix}=\begin{bmatrix}x_f-x_0-\sum\cos\theta_t \, v_{min}\\ y_f-y_0-\sum_{t=0}^{T-1}\cos\theta_t \, v_{min}\end{bmatrix}$$

$(v-\vec{v}_{min}) \geq 0$

where $\vec{v}_{min} = [\underbrace{v_{min} \dots v_{min}}_{T \text{ times}}]^T$

or $\min -c^T(v-v_{min})$

Subject to
$$\begin{bmatrix}\cos\theta_0 & \dots & \cos\theta_{T-1}\\ \sin\theta_0 & \dots & \sin\theta_{T-1}\end{bmatrix}\begin{bmatrix}v_0-v_{min}\\ \vdots\\ v_{T-1}-v_{min}\end{bmatrix}=\begin{bmatrix}x_f-x_0-\sum_{t=0}^{T-1}\cos\theta_t \times v_{min}\\ y_f-y_0-\sum_{t=0}^{T-1}\sin\theta_t \times v_{min}\end{bmatrix}$$

$v-\vec{v}_{min} \geq 0$

(b) I have made all plots and described the results in the attached pdf

2

d) $\min -c^T(v-\vec{v}_{min})$

Subject to $\begin{bmatrix}\cos\theta_0 \dots \cos\theta_{T-1}\\ \sin\theta_0 \dots \sin\theta_{T-1}\end{bmatrix}\begin{bmatrix}v_0-v_{min}\\ \vdots \\ v_{T-1}-v_{min}\end{bmatrix} = \begin{bmatrix}x_f - x_0 - \sum_0^{T-1} v_{min}\cos\theta_t\\ y_f - y_0 - \sum_0^{T-1} v_{min}\sin\theta_t\end{bmatrix}$.

$v-\vec{v}_{min} \geq 0$

let $v' = v - \vec{v}_{min}$, $c' = -c = [-1, -1, \dots -1]^T$

$A = \begin{bmatrix}\cos\theta_0 \dots \cos\theta_{T-1}\\ \sin\theta_0 \dots \sin\theta_{T-1}\end{bmatrix}$ $b = \begin{bmatrix}x_f - x_0 - \sum_0^{T-1} v_{min}\cos\theta_t\\ y_f - y_0 - \sum_0^{T-1} v_{min}\sin\theta_t\end{bmatrix}$

let Basis $= \{1, T\}$. Reshuffle columns of A to express it as $[B\ N]$ and also reshuffle v_t's

$B = \begin{bmatrix}\cos\theta_{T-1} & \cos\theta_0\\ \sin\theta_{T-1} & \sin\theta_0\end{bmatrix}$, $N = \begin{bmatrix}\cos\theta_1 \dots \cos\theta_{T-2}\\ \sin\theta_1 \dots \sin\theta_{T-2}\end{bmatrix}$

$c_B = [-1, -1]^T$, $c_N = [-1, \dots, -1]^T$ $v_B = \begin{bmatrix}v'_{T-1}\\ v'_0\end{bmatrix}$; $v_N = \begin{bmatrix}v_1'\\ \vdots\\ v'_{T-2}\end{bmatrix}$

Also $\theta_0 = \pi/2$, $\theta_{T-1} = 0$

$\therefore B = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $B^{-1} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Since B is invertible therefore basic feasible solution is $v_B = B^{-1}b$, $v_N = 0$

or $v'_{T-1} = x_f - x_0 - \sum_0^{T-1} v_{min}\cos\theta_t$, $v_0' = y_f - y_0 - \sum_0^{T-1} v_{min}\sin\theta_t$

and rest $v_t' = 0$

The optimality test is compute $c_j - c_B^T B^{-1} A_j$ (for j such that $A_j \notin B$) and $\exists l$ such that $c_l - c_B^T B^{-1} A_l < 0$ then solution is not optimal

let $l \neq \{0, T-1\}$. Then $c_l = -1$, $c_B^T = [-1, -1]$

$B^{-1} = I$, ~~$A_l = \begin{bmatrix}\cos\theta_{l-1}\\ \sin\theta_{l-1}\end{bmatrix}$~~ $A_l = \begin{bmatrix}\cos\theta_{l-1}\\ \sin\theta_{l-1}\end{bmatrix}$

$$c_l - c_B^T B^{-1} A_l = -1 - [-1, -1]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta_{l-1}\\ \sin\theta_{l-1}\end{bmatrix}$$

$$= \cos\theta_{l-1} + \sin\theta_{l-1} - 1 \qquad l \in \{2, \dots T-1\}$$

$$= \sqrt{2}\sin(\theta_{l-1} + \pi/4) - 1$$

Since $0 \le \theta_{l-1} \le \pi/2$, $\pi/4 \le \theta_{l-1} + \pi/4 \le 3\pi/4$

$\therefore \sqrt{2}\sin(\theta_{l-1} + \pi/4) - 1 > 0$ for all l

$\therefore$ Solution is optimal.

$v_t = v_{min}$ for $t \neq \{0, T-1\}$

and $v_{T-1} = x_f - x_0 + v_{min} - \sum_{t=0}^{T-1} v_{min}\cos\theta_t$

and $v_0 = y_f - y_0 + v_{min} + \sum_{t=0}^{T-1} v_{min}\sin\theta_t$

(derived before as $v_B' = B^{-1}b$ and $v_N' = 0$)

2c) $x_f - x_0 = \sum_{i=0}^{T-1} v_i\cos\theta_i$, $y_f - y_0 = \sum_{i=0}^{T-1} v_i\sin\theta_i$

$x_f + y_f = \sum_{i=0}^{T-1} v_i(\cos\theta_i + \sin\theta_i)$ $[x_0 = y_0 = 0]$

$x_f + y_f = \sqrt{2}\sum_{i=0}^{T-1} v_i \sin(\theta_i + \pi/4)$

$\sqrt{2}\sum_{i=0}^{T-1} v_i\sin(\theta_i + \pi/4) \ge \sqrt{2} \times T \times \frac{v_{min}}{\sqrt{2}}$ (as $\sin(\theta_i + \pi/4) \ge \frac{1}{\sqrt{2}}$ for $\theta_i \in [0, \pi/2]$)

$\therefore x_f + y_f \ge T v_{min}$

or $v_{min} \le \frac{x_f + y_f}{T}$

4. $\arg\min x^T A_0 x + 2b_0^T x + c_0$

s.t $x^T A_i x + 2b_i^T x + c_i \le 0 \quad i = 1, 2, \dots, m$

$$\mathcal{L}(x, \lambda_1, \dots, \lambda_m) = x^T A_0 x + 2b_0^T x + c_0 + \sum_{i=1}^{m} \lambda_i (x^T A_i x + 2b_i^T x + c_i)$$

where $\lambda_1, \dots, \lambda_m$ are lagrange multipliers

$$\mathcal{L}(x, \lambda_1, \dots, \lambda_m) = x^T \left(A_0 + \sum_{i=1}^{m} \lambda_i A_i\right) x + 2\left(b_0 + \sum_{i=1}^{m} \lambda_i b_i\right)^T x + \left(c_0 + \sum_{i=1}^{m} \lambda_i c_i\right)$$

The above lagragian is quadratic in x.
Therefore it ~~is~~ ~~f~~ is convex if and only if hessian is p.s.d or $2(A_0 + \sum_{i=1}^{m} \lambda_i A_i)$ is p.s.d
Since A_0 is p.d and A_i's are p.s.d and $\lambda_i \ge 0$ therefore sum of positive definite matrix with positive semi definite matrix gives a positive definite matrix

$\therefore$ The above lagrangian has a global minima w.r.t x

$$\nabla_x \mathcal{L}(x^*, \lambda_1, \dots, \lambda_m) = 0 \Rightarrow 2\left(A_0 + \sum_{i=1}^{m} \lambda_i A_i\right) x^* + 2\left(b_0 + \sum_{i=1}^{m} \lambda_i b_i\right) = 0$$

$$\text{or } x^* = -\left(A_0 + \sum_{i=1}^{m} \lambda_i A_i\right)^{-1} \left(b_0 + \sum_{i=1}^{m} \lambda_i b_i\right)$$

(since $A_0 + \sum_{i=1}^{m} \lambda_i A_i$ is p.d therefore it is invertible)

$$g(\lambda_1, \dots, \lambda_m) = -\left(b_0 + \sum_{i=1}^{m} \lambda_i b_i\right)^T \left(A_0 + \sum_{i=1}^{m} \lambda_i A_i\right)^{-1} \left(b_0 + \sum_{i=1}^{m} \lambda_i b_i\right) + \left(c_0 + \sum_{i=1}^{m} \lambda_i c_i\right)$$

in $\lambda_1, \dots, \lambda_m \ge 0$

Dual problem $\rightarrow$ $\max -(b_0+\sum_{i=1}^{m}\lambda_i b_i)^T(A_0+\sum_{i=1}^{m}\lambda_i A_i)^{-1}(b_0+\sum_{i=1}^{m}\lambda_i b_i)$
$+(c_0+\sum_{i=1}^{m}\lambda_i c_i)$

s.t $\lambda_1,\lambda_2,\dots,\lambda_m \geq 0$

3. $\arg\min \frac{1}{2}\|Aw-b\|^2$
s.t $\|w-w^0\|^2 \leq r^2$

The lagrangian of the above ~~p~~ function is

$$\mathcal{L}(w,\lambda) = \frac{1}{2}\|Aw-b\|^2+\lambda(\|w-w^0\|^2-r^2)$$

$$=\frac{1}{2}w^TA^TAw - \frac{2b^TAw}{2}+\frac{b^Tb}{2}+\lambda w^Tw - 2\lambda w^{0T}w + \lambda\|w^0\|^2 - \lambda r^2$$

$$\mathcal{L}(w,\lambda) = \frac{1}{2}w^T(A^TA+\lambda I)w-(b^TA+\lambda w^{0T})w+\lambda\|w^0\|^2+\frac{\|b\|^2}{2}-\lambda r^2$$

KKT conditions

1. $\nabla_w \mathcal{L}(w^*,\lambda^*)=0$

$$(A^TA+\lambda^* I)w^*-(A^Tb+\lambda w^0)=0$$
$$(A^TA+\lambda^* I)w^* = A^Tb+\lambda w^0$$

2. $\lambda^* \geq 0$

3. $\lambda^*(\|w-w^0\|^2-r^2)=0$

4. $\|w-w^0\|^2 \leq r^2$

If $\|w-w^0\|^2<r^2$ (inactive) then ~~$\lambda^* \to 0$~~ $\lambda^*=0$ (from 3)

from (1) $A^TAw^* = A^Tb$

or $w^* = (A^TA)^{-1}A^Tb$ (inverse exists if A^TA is full column rank)

w^* has to satisfy (4)

$$\|(A^TA)^{-1}A^Tb-w^0\|^2<r^2$$ (Since constraint is inactive)

5) a $\arg\min_{x \in S} \frac{1}{2}\|x-y\|^2$, $S = \{x \in \mathbb{R}^n : Ax \le b\}$

Consider the lagrangian

$$\mathcal{L}(x,\lambda) = \frac{1}{2}\|x-y\|^2 + \lambda^T(Ax-b) = \frac{1}{2}x^Tx + (\lambda^TA - y^T)x + \frac{\|y\|^2}{2} - \lambda^Tb$$

$$g(\lambda) = \arg\min_{x \in \mathbb{R}^n} \mathcal{L}(x,\lambda)$$

$\mathcal{L}(x,\lambda)$ is convex in x since its hessian is identity. $\nabla_x \mathcal{L}(x^*,\lambda) = 0$ is sufficient

$$\nabla_x \mathcal{L}(x^*,\lambda) = 0 \Rightarrow x^* + (A^T\lambda - y) = 0 \Rightarrow x^* = y - A^T\lambda$$

$$\mathcal{L}(x^*,\lambda) = -\frac{\|y - A^T\lambda\|^2}{2} + \frac{\|y\|^2}{2} - \lambda^Tb$$

$$g(\lambda) = -\frac{\lambda^T(AA^T)\lambda}{2} + (Ay-b)^T\lambda$$

$$g(\lambda) = -\frac{1}{2}\lambda^T(AA^T)\lambda + (Ay-b)^T\lambda$$

dual problem $\rightarrow \max_{\lambda \ge 0} g(\lambda)$ ~~min~~

b) ~~Since~~ $\max_{\lambda \ge 0} g(\lambda)$ is equivalent to $\min_{\lambda \ge 0} -g(\lambda)$

$\min_{\lambda \ge 0} \frac{1}{2}\lambda^T(AA^T)\lambda - (Ay-b)^T\lambda$. Hessian of this which is AA^T is p.s.d. Then $L = \lambda_{max}(AA^T)$ [maximum eigen value of A]

$i = 1, \dots m$

$$\lambda_{k+1} = P_c(\lambda_k - \frac{1}{L}\nabla(-g(\lambda_k)))$$

c is the convex set $\{\lambda \in \mathbb{R}^m : \lambda \ge 0\}$

let $f(\lambda) = -g(\lambda) = \frac{1}{2}\lambda^T(AA^T)\lambda - (Ay-b)^T\lambda$

$$\nabla f(\lambda_k) = (AA^T)\lambda_k - (Ay-b)$$

$$\lambda_{k+1} = P_c\left(\left(I - \frac{1}{L}AA^T\right)\lambda_k + (Ay-b)\right)$$

c) Algorithm: Find projection x of point z onto $Ax=b$

1. Initialize $x = z$ $(=(0,0,0))$, iterations $=0$
2. Compute $cons(z) = \{i : a_i^T x - b[i] \,!= 0\}$
3. While($cons(z) \,!= \phi$) { $\left(A = \begin{bmatrix} \leftarrow a_1^T \rightarrow \\ \leftarrow a_2^T \rightarrow \end{bmatrix} \right)$

    $y =$ projection$(x, A[0,:], b[0])$

    $x =$ projection$(y, A[1,:], b[1])$ ($\because$ projection onto 1st and 2nd rows of A)

    compute $cons(x) = \{i : a_i^T x - b[i] \,!= 0\}$

    iterations $+= 1$

}

The above algorithm computes projection onto first row and then onto second row.
Projection on a row is given by

$$x = z - \frac{(a_i^T z - b)}{\|a_i\|^2} a$$

Above algorithm stops when the $cons(x)$ set is ϕ or when x ~~is~~ ~~in~~ is in the set ~~Ax~~ $\{x : Ax = b\}$. Solution obtained $= [0.78571, 0.07143, -1.14286]^T$

Analytical solution:

$$z = [0,0,0]^T$$

$$\min_{x \in \mathbb{R}^3} \frac{1}{2}\|x - z\|^2 \quad Ax = b, \qquad A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Writing lagrangian for this problem

$$\mathcal{L}(x,\mu) = \frac{1}{2}\|x - z\|^2 + \mu^T(Ax - b)$$

KKT conditions

1. $\nabla_x \mathcal{L}(x^*, \mu) = 0 \Rightarrow x - z + A^T\mu = 0$

    or $x = z - A^T\mu$

    $Ax = Az - AA^T\mu$

$AA^T\mu = Az - b \Rightarrow \mu = (AA^T)^{-1}(Az - b)$

$\therefore x = z - A^T(AA^T)^{-1}(Az - b)$

Putting values of z, A and ~~be~~ b we get

$$x = \left(I - A^T(AA^T)^{-1}A\right)z + A^T(AA^T)^{-1}b$$

$$x = [0.78571, 0.07143, -1.14286]^T$$

This is the same solution obtained ~~after~~ from the algorithm.