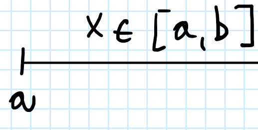
FEM Poisson's equation 1D

Monday, December 2, 2024 1:42 PM

1 Finite Element Method



Hp: 1D

$$\nabla \cdot (p \nabla \varphi) = t \xrightarrow{\partial/\partial y = 0 \quad \partial/\partial z = 0} \boxed{\frac{d}{dx}\left[p(x)\frac{d\varphi}{dx}\right] = t(x)}$$

$p(x) = \varepsilon_r$

$t(x) = -\rho/\varepsilon_0$

ELECTROSTATIC POTENTIAL

$$\boxed{\frac{d}{dx}\left[\varepsilon_r \frac{d\varphi}{dx}\right] = -\rho/\varepsilon_0}$$

Formulation [STRONG FORMULATION]

"Find $\varphi(x)$ such that this ⇒ is true for every $x \in [a,b]$"

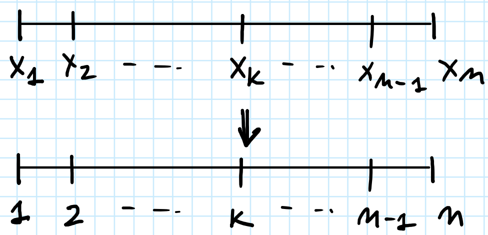
$$\begin{cases} \frac{d}{dx}\left[p(x)\frac{d\varphi}{dx}\right] = t(x) & x \in]a,b[\\ \varphi(a) = \varphi_a & \text{Dirichlet BC}, \; x = a \\ \left.\frac{d\varphi}{dx}\right|_b = \varphi'_b & \text{Neumann BC}, \; x = \end{cases}$$

Dirichlet BC: prescribing value of φ in a

Neumann BC: prescribing value of $d\varphi/dx$ in b

DISCRETIZATION

- introduce n-points between a and b
  ↓
  n-NODES ⇒ n-1 ~~intervals~~ ELEMENTS



shorthand notation

if GRID is uniform : $\Delta = \frac{L}{n-1} = \frac{b-a}{n-1}$

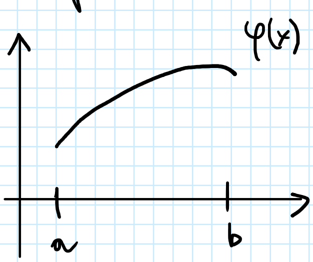
~ tanh — ~ 10^{-7} m — ~ 10^{-1} m

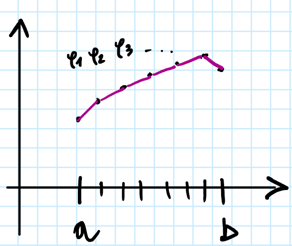
INTERPOLATION

$$\varphi(x) \longrightarrow \tilde{\varphi}(x)$$

original unknown function

PIECEWISE INTERPOLATION of the unknown function



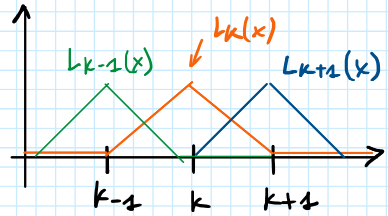


"Find $\varphi_1, \varphi_2, \ldots \varphi_n$"

now $\tilde{\varphi}(x)$ is an APPROXIMATION of the true solution

$$\tilde{\varphi}(x) = \varphi_1 L_1(x) + \varphi_2 L_2(x) + \dots + \varphi_k L_k(x) + \dots, \varphi_{n-1} L_{n-1}(x) + \varphi_n L_n(x)$$

HAT FUNCTIONS as interpolant functions [SHAPE FUNCTIONS]



WEIGHTED RESIDUALS APPROACH

if exact solution $\varphi(x)$ $\longrightarrow$ $\boxed{\frac{d}{dx}\left[p(x)\frac{d\varphi}{dx}\right] - t(x) = 0}$

we have switched from $\varphi(x) \Rightarrow \tilde{\varphi}(x)$

$$\underbrace{\frac{d}{dx}\left[p(x)\frac{d\tilde{\varphi}}{dx}\right] - t(x)}_{r(x) \text{ RESIDUAL}} \neq 0$$

$\Rightarrow$ CANNOT BE 0 $\forall x$

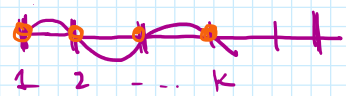
W.R. approach : REQUIRE that the WEIGHTED RESIDUAL is ZERO over the domain

$$\int_a^b w(x)\, r(x)\, dx = 0$$

WEIGHTING FUNCTION

(ALESSANDRO'S CHOICE)

$w(x) \Rightarrow \delta(x - x_k)$



$$\int_a^b \underbrace{w(x)}_{f(x)} \frac{d}{dx}\underbrace{\left[\overbrace{p(x)\frac{d\tilde{\varphi}}{dx}}^{g(x)}\right]}_{g'(x)} dx = \int_a^b w(x)\, t(x)\, dx$$

$\Rightarrow$ $\tilde{\varphi} \in C_0$ PIECEWISE linear function

$$\int_a^b f(x) g'(x)\, dx = \Big[f(x) g(x)\Big]_a^b - \int_a^b f'(x) g(x)\, dx$$

integration by parts

$$\left[w(x)\, p(x)\frac{d\tilde{\varphi}}{dx}\right]_a^b - \int_a^b \frac{dw}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx = \int_a^b w(x)\, t(x)\, dx$$

NO SECOND DERIVATIVES of $\tilde{\varphi} \in C_0$

$$\left[w(x)\, p(x) \frac{d\tilde{\varphi}}{dx} \right]_a^b - \int_a^b \frac{dw}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx = \int_a^b w(x)\, t(x)\, dx$$

NO SECOND DERIVATIVES of $\tilde{\varphi} \in C_0$

$$\boxed{\int_a^b \frac{dw}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx = \left[w(x)\, p(x) \frac{d\tilde{\varphi}}{dx} \right]_a^b - \int_a^b w(x)\, t(x)\, dx}$$

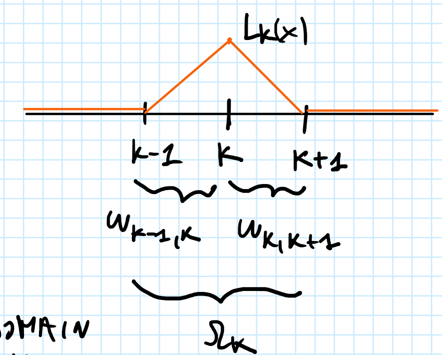
WEAK FORMULATION

GALERKIN'S CHOICE $\rightarrow$ $\boxed{W(x) \Rightarrow L_k(x)}$

residual weighted by hat functions !!

WEAK Formulation + GALERKIN'S CHOICE $\tilde{\varphi} \in C_0$

$$\boxed{\int_{\Omega_k} \frac{dL_k}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx = \left[L_k(x)\, p(x) \frac{d\tilde{\varphi}}{dx} \right]_a^b - \int_{\Omega_k} L_k(x)\, t(x)\, dx \qquad \boxed{k = 1, 2, \dots, n-1, n}}$$



SUPPORT DOMAIN of node K

$\Rightarrow L_k(x) \neq 0$ only on Ω_k

$\Rightarrow$ restrict domain of integration from $[a, b] \Rightarrow \Omega_k$

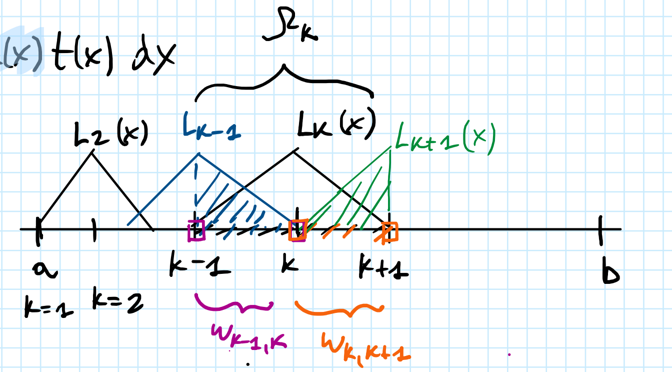
find $\varphi_1, \varphi_2, \dots \varphi_n$ such that the n-expressions hold true

$$\tilde{\varphi} = \sum_{i=1}^{n} \varphi_i L_i(x)$$

internal nodes $\quad k = 2, 3, \dots, n-1$

$$\int_{\Omega_k} \frac{dL_k}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx = \cancel{\left[L_k(x)\, p(x) \frac{d\tilde{\varphi}}{dx} \right]_a^b} - \int_{\Omega_k} L_k(x)\, t(x)\, dx$$

$$\boxed{\Omega_k = \underline{w_{k-1,k}} \cup \underline{w_{k,k+1}}}$$



$$\underbrace{\left[\int \frac{dL_k}{dx} p(x) \frac{dL_{k-1}}{dx}\, dx \right]}_{w_{k-1,k}} \varphi_{k-1} + \underbrace{\left[\int \frac{dL_k}{dx} p(x) \frac{dL_k}{dx} \right]}_{w_{k-1,k}} \varphi_k + \dots$$

$\uparrow K_L \qquad \uparrow K_{c1}$

$$\left[\int \frac{dL_k}{dx} p(x) \frac{dL_k}{dx}\, dx \right] \varphi_k + \left[\int \frac{dL_k}{dx} p(x) \frac{dL_{k+1}}{dx}\, dx \right] \varphi_{k+1} =$$

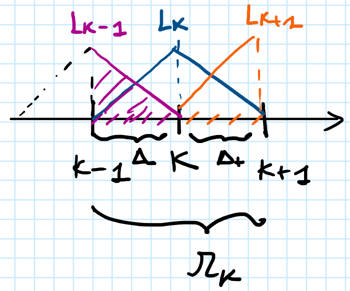
$$\tilde{\varphi} = \begin{cases} \boxed{\varphi_{k-1} L_{k-1}(x) + \varphi_k L_k(x)} & \text{in } w_{k-1,k} \\ \boxed{\varphi_k L_k(x) + \varphi_{k+1} L_{k+1}(x)} & \text{in } w_{k,k+1} \end{cases}$$

$$\underbrace{\left[\int_{w_{k,k+1}} \frac{dL_k}{dx}\, p(x)\, \frac{dL_k}{dx}\, dx\right]}_{K_{C''}}\varphi_k + \underbrace{\left[\int_{w_{k,k+1}} \frac{dL_k}{dx}\, p(x)\, \frac{dL_{k+1}}{dx}\, dx\right]}_{K_R}\varphi_{k+1} =$$

$$\underbrace{\left[-\int_{w_{k-1,k}} L_k(x)\, t(x)\, dx - \int_{w_{k,k+1}} L_k(x)\, t(x)\, dx\right]}_{K_{RHS}}$$

$$\Rightarrow \boxed{K_L \varphi_{k-1} + (K_{C'} + K_{C''})\varphi_k + K_R \varphi_{k+1} = K_{RHS}}$$

$\Rightarrow$ SAME SHAPE AS FDM



$$L_{k-1}(x) = \begin{cases} 1 - \dfrac{(x - x_{k-1})}{\Delta_-} \end{cases} \Rightarrow \frac{dL_{k-1}}{dx} = \begin{cases} -1/\Delta_- \end{cases}$$

$$L_k(x) = \begin{cases} 1 + \left(\dfrac{x - x_k}{\Delta_-}\right) \leftarrow w_{k-1,k} \\ 1 - \left(\dfrac{x - x_k}{\Delta_+}\right) \leftarrow w_{k,k+1} \end{cases} \Rightarrow \frac{dL_k}{dx} = \begin{cases} 1/\Delta_- , & \boxed{x \in w_{k-1,k}} \\ -1/\Delta_+ , & \boxed{x \in w_{k,k+1}} \end{cases}$$

$$L_{k+1}(x) = \begin{cases} 1 + \left(\dfrac{x - x_{k+1}}{\Delta_+}\right) \end{cases} \Rightarrow \frac{dL_{k+1}}{dx} = \begin{cases} 1/\Delta_+ \end{cases}$$

GENERAL

$$\underbrace{\left[\int_{w_{k-1,k}} \overset{1/\Delta_-}{\frac{dL_k}{dx}}\, p(x)\, \overset{-(1/\Delta_-)}{\frac{dL_{k-1}}{dx}}\, dx\right]}_{K_L}\varphi_{k-1} + \underbrace{\left[\int_{w_{k-1,k}} \overset{1/\Delta_-}{\frac{dL_k}{dx}}\, p(x)\, \overset{1/\Delta_-}{\frac{dL_k}{dx}}\right]}_{K_{C'}}\varphi_k + \dots$$

$$\underbrace{\left[\int_{w_{k,k+1}} \overset{-1/\Delta_+}{\frac{dL_k}{dx}}\, p(x)\, \overset{-1/\Delta_+}{\frac{dL_k}{dx}}\, dx\right]}_{K_{C''}}\varphi_k + \underbrace{\left[\int_{w_{k,k+1}} \overset{-1/\Delta_+}{\frac{dL_k}{dx}}\, p(x)\, \overset{1/\Delta_+}{\frac{dL_{k+1}}{dx}}\, dx\right]}_{K_R}\varphi_{k+1} =$$

$$\underbrace{\left[-\int_{w_{k-1,k}} L_k(x)\, t(x)\, dx - \int_{w_{k,k+1}} L_k(x)\, t(x)\, dx\right]}_{K_{RHS}}$$

$$-\frac{1}{\Delta_-^2}\overbrace{\left[\int_{w_{k-1,k}} p(x)\, dx\right]}^{P_{k-1,k}}\varphi_{k-1} + \frac{1}{\Delta_-^2}\overbrace{\left[\int_{w_{k-1,k}} p(x)\, dx\right]}^{P_{k-1,k}}\varphi_k + \frac{1}{\Delta_+^2}\overbrace{\left[\int_{w_{k,k+1}} p(x)\, dx\right]}^{P_{k,k+1}}\varphi_k - \frac{1}{\Delta_+^2}\overbrace{\left[\int_{w_{k,k+1}} p(x)\, dx\right]}^{P_{k,k+1}}\varphi_{k+1} =$$

$$= -\int \left[1 + \left(\frac{x - x_k}{\Delta_-}\right)\right] \cdot t(x)\, dx - \int \left[1 - \left(\frac{x - x_k}{\Delta_+}\right)\right] \cdot t(x)\, dx$$

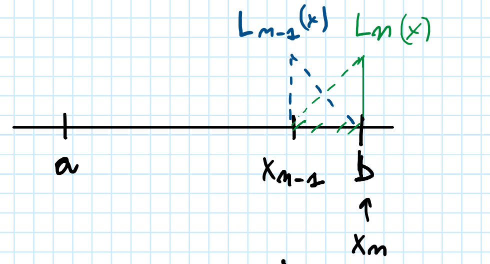
$$= -\underbrace{\int_{w_{k-1,k}} \left[1+\left(\frac{x-x_k}{\Delta_-}\right)\right]\cdot t(x)\,dx}_{S_{k-1,k}} - \underbrace{\int_{w_{k,k+1}} \left[1-\left(\frac{x-x_k}{\Delta_+}\right)\right]\cdot t(x)\,dx}_{S_{k,k+1}}$$

nodes $2, 3, \dots m-1$

BOUNDARIES

$x = a$ DIRICHLET BC $\varphi(a) = \varphi_a$

$$\boxed{\varphi_1 = \varphi_a}$$ NODE 1



$x = b$ NEUMANN BC $\left.\frac{d\varphi}{dx}\right|_b = \varphi'_b$

$\Omega_m = w_{m-1,m}$

$$\int_{\Omega_k} \frac{dL_k}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\,dx = \left[L_k(x)\, p(x)\frac{d\tilde{\varphi}}{dx}\right]_a^b - \int_{\Omega_k} L_k(x)\, t(x)\,dx$$

↓ FOR NODE m

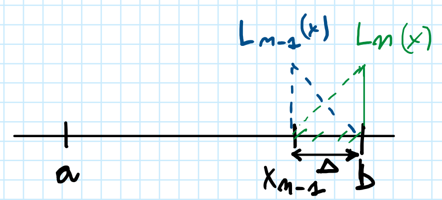
$$\tilde{\varphi} = \varphi_{m-1} L_{m-1}(x) + \varphi_m L_m(x)$$

$$\left[\int_{w_{m-1,m}} \underbrace{\frac{dL_m}{dx}}_{1/\Delta} p(x) \underbrace{\frac{dL_{m-1}}{dx}}_{-1/\Delta}\,dx\right]\varphi_{m-1} + \left[\int_{w_{m-1,m}} \underbrace{\frac{dL_m}{dx}}_{1/\Delta} p(x) \underbrace{\frac{dL_m}{dx}}_{1/\Delta}\,dx\right]\varphi_m =$$

$$= \left[\underbrace{L_m(b)}_{=1}\, p(b) \underbrace{\left.\frac{d\tilde{\varphi}}{dx}\right|_b}_{\substack{\text{Prescribed} \\ = \varphi'_b}} - \cancel{\underbrace{L_m(a)}_{=0}\, p(a) \left.\frac{d\tilde{\varphi}}{dx}\right|_a}\right] - \int_{w_{m-1,m}} L_m(x)\, t(x)\,dx$$

$$L_m(x) = 1 + \left(\frac{x-x_m}{\Delta}\right) \rightarrow \frac{dL_m}{dx} = 1/\Delta$$

$$L_{m-1}(x) = 1 - \left(\frac{x-x_{m-1}}{\Delta}\right) \rightarrow \frac{dL_{m-1}}{dx} = -1/\Delta$$



$\Delta = x_m - x_{m-1}$

$$\boxed{-\frac{1}{\Delta^2}\left[\int_{w_{m-1,m}} p(x)\,dx\right]\varphi_{m-1} + \frac{1}{\Delta^2}\left[\int_{w_{m-1,m}} p(x)\,dx\right]\varphi_m = p(b)\varphi'_b - \int_{w_{m-1,m}}\left[1+\frac{x-x_m}{\Delta}\right]t(x)\,dx}$$ NODE m

GOAL : WRITE LINEAR SYSTEM $[K][\varphi] = [rhs]$

NOTATION: $\displaystyle\int_{w_{k,k+1}} p(x)\,dx = p_{k,k+1}$

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} 1 & 2 & 3 & \cdots & k & \cdots & n \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \boxed{3} \\ \vdots \\ k \\ \vdots \\ n \end{array} &
\begin{bmatrix}
1 & & & & & & \\
-\frac{p_{1,2}}{\Delta_-^2} & \frac{p_{1,2}}{\Delta_-^2}+\frac{p_{2,3}}{\Delta_+^2} & -\frac{p_{2,3}}{\Delta_+^2} & & & & \\
 & -\frac{p_{2,3}}{\Delta_-^2} & \frac{p_{2,3}}{\Delta_-^2}+\frac{p_{3,4}}{\Delta_+^2} & -\frac{p_{3,4}}{\Delta_+^2} & & & \\
 & & \ddots & \ddots & \ddots & & \\
 & & & -\frac{p_{k-1,k}}{\Delta_-^2} & \frac{p_{k-1,k}}{\Delta_-^2}+\frac{p_{k,k+1}}{\Delta_+^2} & -\frac{p_{k,k+1}}{\Delta_+^2} & \\
 & & & & \ddots & \ddots & \ddots \\
 & & & & & -\frac{p_{n-1,n}}{\Delta^2} & \frac{p_{n-1,n}}{\Delta^2}
\end{bmatrix}
\end{array}
\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_k \\ \vdots \\ \varphi_{n-1} \\ \varphi_n \end{bmatrix}
=
\begin{bmatrix} \varphi_a \\ S_{1,2}+S_{2,3} \\ \vdots \\ S_{k-1,k}+S_{k,k+1} \\ \vdots \\ p(b)\varphi'_b + S_{n-1,n} \end{bmatrix}
$$

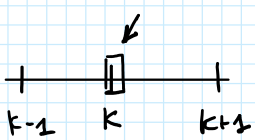
"the" Δ_+ of node 2

"the" Δ_+ of node 3

$\underbrace{\qquad\qquad\qquad}_{[K]}$ TRI-DIAGONAL SPARSE

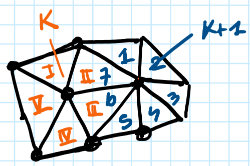
FEM. (Piecewise Linear Shape functions) $\Rightarrow O(\Delta^2)$

So far .... ASSEMBLY matrix $[K]$ from the point of view of NODES
GOOD for 1D, less practical for 2D, 3D
$\Downarrow$
TRIANGULAR GRIDS



$\downarrow$

$\Omega_k = w_{k-1,k} \cup w_{k,k+1}$



How many ELEMENTS in

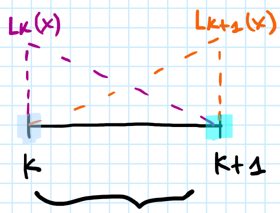
Ω_k : 5 elements

Ω_{k+1} : 7 elements

$\rightarrow$ # of non-zero entries in $[K]$ in rows k, $k+1$ is different

ELEMENT POINT of VIEW

isolate contributions of each element to $[K]$ and assemble by cycling over elements



$w_{k,k+1}$

Expression NODE k.

$$\left[\int_{w_{k-1,k}} \frac{dL_k}{dx} p(x) \frac{dL_{k-1}}{dx}\, dx\right]\varphi_{k-1} + \left[\int_{w_{k-1,k}} \frac{dL_k}{dx} p(x) \frac{dL_k}{dx}\, dx\right]\varphi_k + \dots$$

$$+ \boxed{\left[\int_{w_{k,k+1}} \frac{dL_k}{dx} p(x) \frac{dL_k}{dx}\, dx\right]\varphi_k} + \boxed{\left[\int_{w_{k,k+1}} \frac{dL_k}{dx} p(x) \frac{dL_{k+1}}{dx}\, dx\right]\varphi_{k+1}} = \dots$$

$K_{ee}[1,1]$ $\qquad$ $K_{ee}[1,2]$

$$- \int_{w_{k-1,k}} L_k(x)\, t(x)\, dx - \boxed{\int_{w_{k,k+1}} L_k(x)\, t(x)\, dx}$$

Expression node $k+1$

$K_{ee}[2,1]$ $\qquad$ $K_{ee}[2,2]$

$$\boxed{\left[\int_{w_{k,k+1}} \frac{dL_{k+1}}{dx} p(x) \frac{dL_k}{dx}\, dx\right]\varphi_k} + \boxed{\left[\int_{w_{k,k+1}} \frac{dL_{k+1}}{dx} p(x) \frac{dL_{k+1}}{dx}\, dx\right]\varphi_{k+1}} + \dots$$

$$+ \dots \left[\int_{w_{k+1,k+2}} \frac{dL_{k+1}}{dx} p(x) \frac{dL_{k+1}}{dx}\, dx\right]\varphi_{k+1} + \left[\int_{w_{k+1,k+2}} \frac{dL_{k+1}}{dx} p(x) \frac{dL_{k+2}}{dx}\, dx\right]\varphi_{k+2} =$$

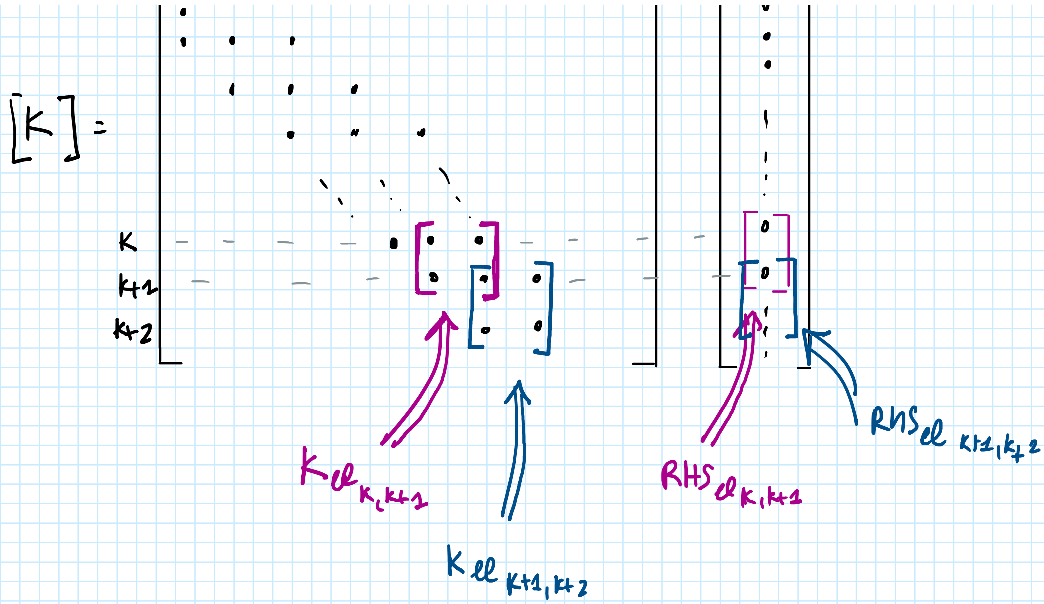
$$- \boxed{\int_{w_{k,k+1}} L_{k+1}(x)\, t(x)\, dx} - \int_{w_{k+2,k+3}} L_{k+1}(x)\, t(x)\, dx$$

Define: ELEMENT MATRIX $[2 \times 2]$ $\qquad$ RHS ELEMENT ARRAY $[2 \times 1]$

$$K_{ee_{k,k+1}} = \begin{bmatrix} \int_{w_{k,k+1}} \frac{dL_k}{dx} p(x) \frac{dL_k}{dx}\, dx & \int_{w_{k,k+1}} \frac{dL_k}{dx} p(x) \frac{dL_{k+1}}{dx}\, dx \\ \int_{w_{k,k+1}} \frac{dL_{k+1}}{dx} p(x) \frac{dL_k}{dx}\, dx & \int_{w_{k,k+1}} \frac{dL_{k+1}}{dx} p(x) \frac{dL_{k+1}}{dx}\, dx \end{bmatrix}$$

$$RHS_{ee_{k,k+1}} = \begin{bmatrix} -\int_{w_{k,k+1}} L_k(x)\, t(x)\, dx \\ -\int_{w_{k,k+1}} L_{k+1}(x)\, t(x)\, dx \end{bmatrix}$$

$$\begin{array}{c} \quad\quad\quad k \quad k+1 \\ \begin{bmatrix} \vdots & & \\ \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \vdots \end{bmatrix} \end{array}$$


[K] =
K
K+1
K+2
Kel K,K+1
Kel K+1,K+2
RHSel K,K+1
RHSel K+1,K+2