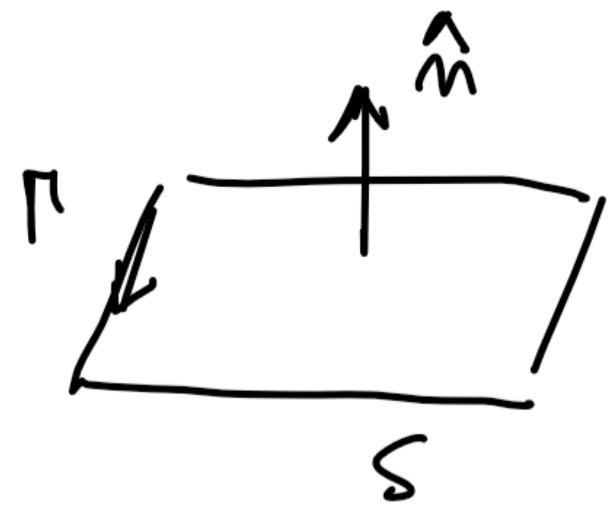
Curl: $\nabla \times \bar{U} \cdot \hat{n} = \lim_{\Delta S \to 0} \dfrac{\oint_\Gamma \bar{U} \cdot d\bar{l}}{\Delta S}$



↓

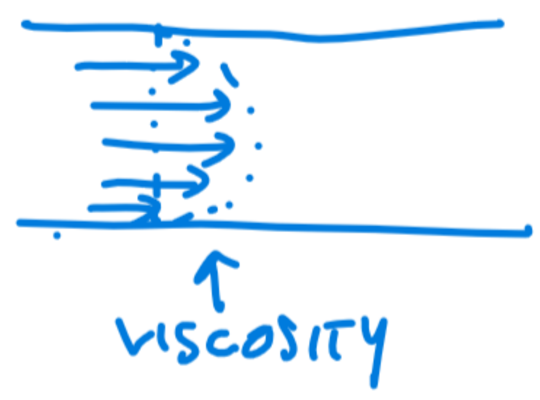
CARTESIAN COORDS

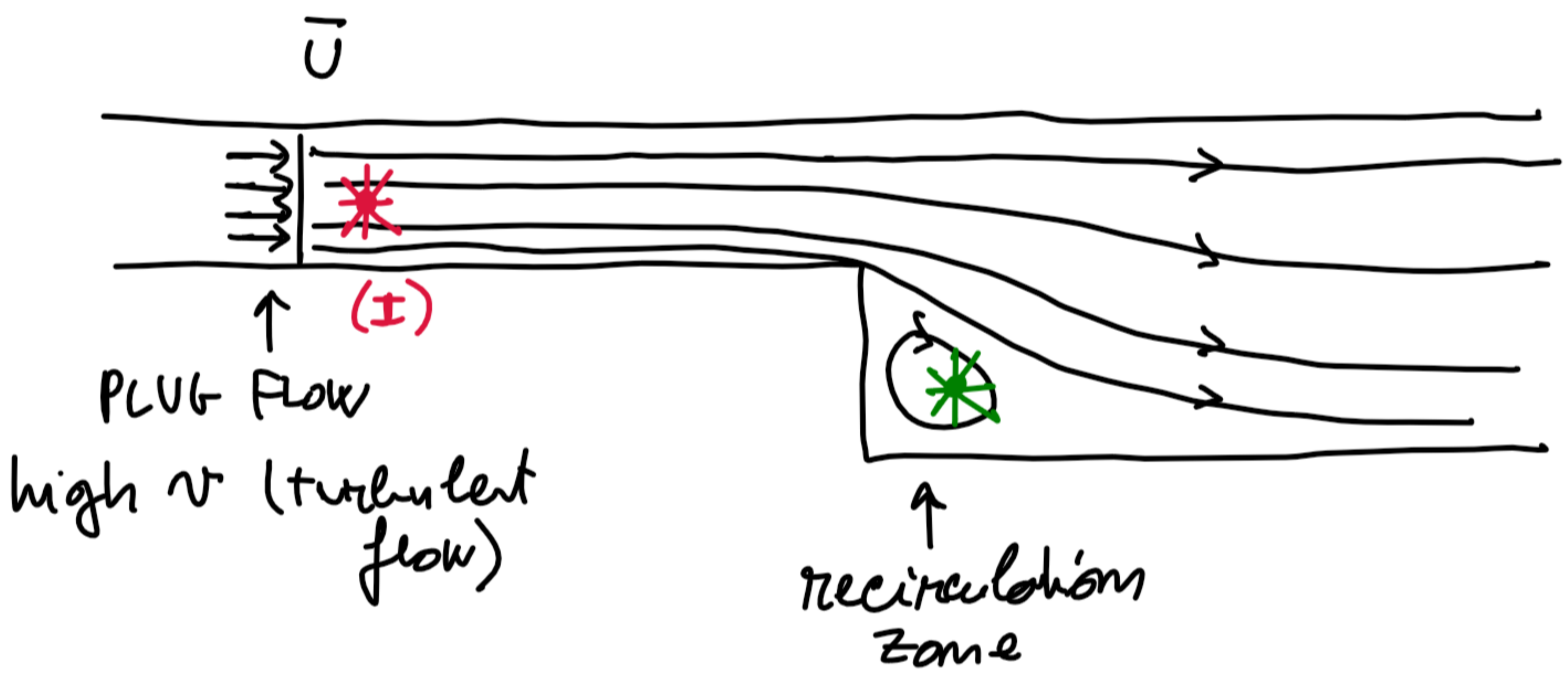
$$\nabla \times \bar{U} = \mathrm{DET} \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ U_x & U_y & U_z \end{bmatrix} =$$

$$= \left(\partial U_z/\partial y - \partial U_y/\partial z \right) \hat{\imath} - \left(\partial U_z/\partial x - \partial U_x/\partial z \right) \hat{\jmath} + \left(\partial U_y/\partial x - \partial U_x/\partial y \right) \hat{k}$$

Example: Fluid in a duct

LAMINAR



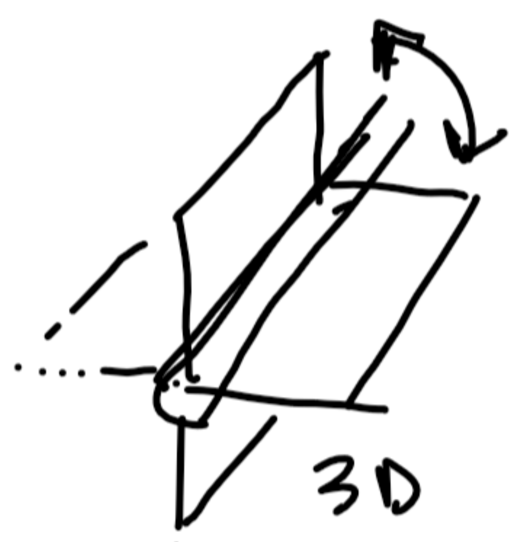


I: no spinning
⇒ $\nabla \times \bar{U} = 0$

II: clockwise spinning
$\nabla \times \bar{U} \neq 0$
⇓
curl "into the page"

- CURL - METER (bladed wheel)

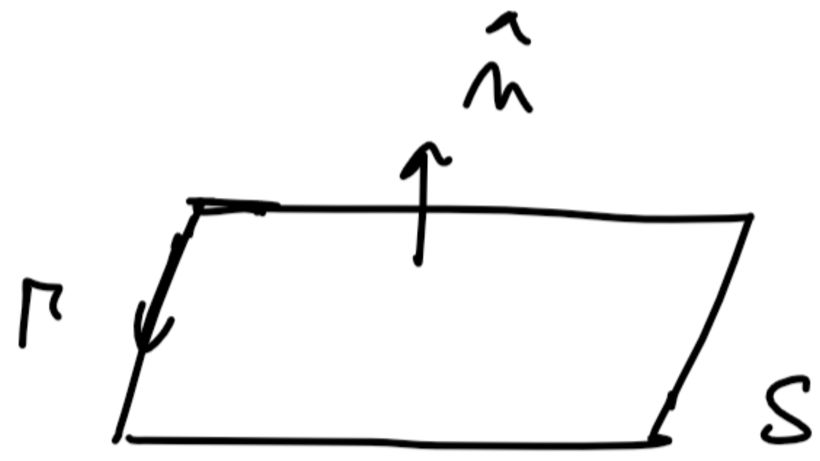
2D



3D

STOKES THEOREM

- $\bar{U} \in C_1$ on an open surface S (SCD)



$$\underbrace{\int_S \nabla \times \bar{U} \cdot d\bar{S}}_{\Sigma \text{ of local circulations}} = \underbrace{\oint_\Gamma \bar{U} \cdot d\bar{l}}_{\text{Circulation of } \bar{U} \text{ along } \Gamma}$$

↓

LOCAL CIRCULATION

LINKED FLUX:

the flux of the curl of $\bar{U}$ does not depend on S, only on the boundary Γ of S

$\bar{B} \Rightarrow \nabla \times \bar{A}$ $\quad \int \bar{B} \cdot d\bar{S} \Rightarrow$ LINKED FLUX

STOKES theorem and Solenoidal fields

→ Flux of $\bar{F} = \nabla \times \bar{U}$ over OPEN SURFACE ⟶ Boundary

STOKES

$$\int_{S'} \nabla \times \bar{U} \cdot d\bar{S} = \oint_{\Gamma} \bar{U} \cdot d\bar{\ell}$$

$$\int_{S''} \nabla \times \bar{U} \cdot d\bar{S} = \oint_{\Gamma} \bar{U} \cdot d\bar{\ell}$$

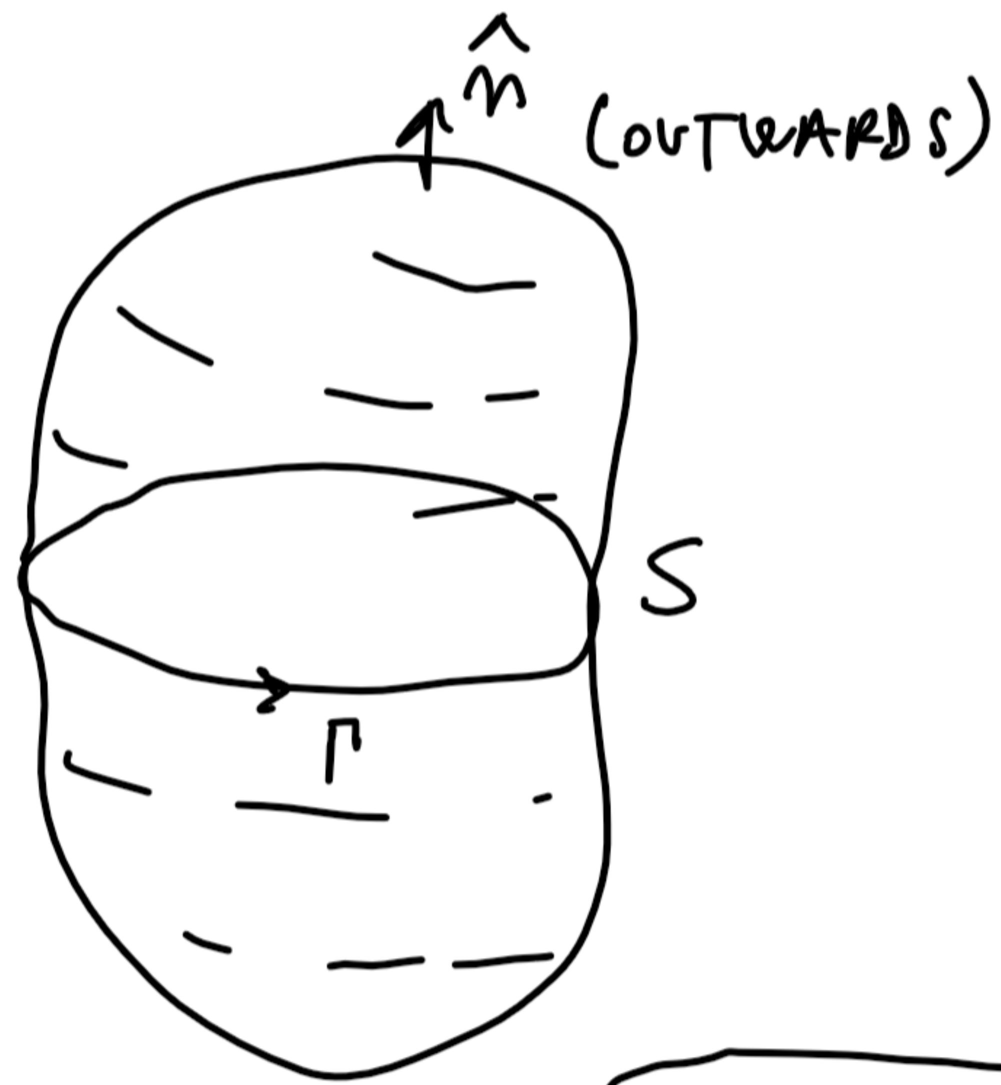
$$\Rightarrow \int_{S'} \nabla \times \bar{U} \cdot d\bar{S} \overset{(1)}{=} \int_{S''} \nabla \times \bar{U} \cdot d\bar{S} \Rightarrow \text{EXPECTED}$$

-- if the surface is CLOSED

$$\oint_{S} \nabla \times \bar{U} \cdot d\bar{S} = \quad S = S' \cup S''$$

$\hat{n} \cdot \hat{n}' = 1$ $\qquad$ $\hat{n} \cdot \hat{n}'' = -1$

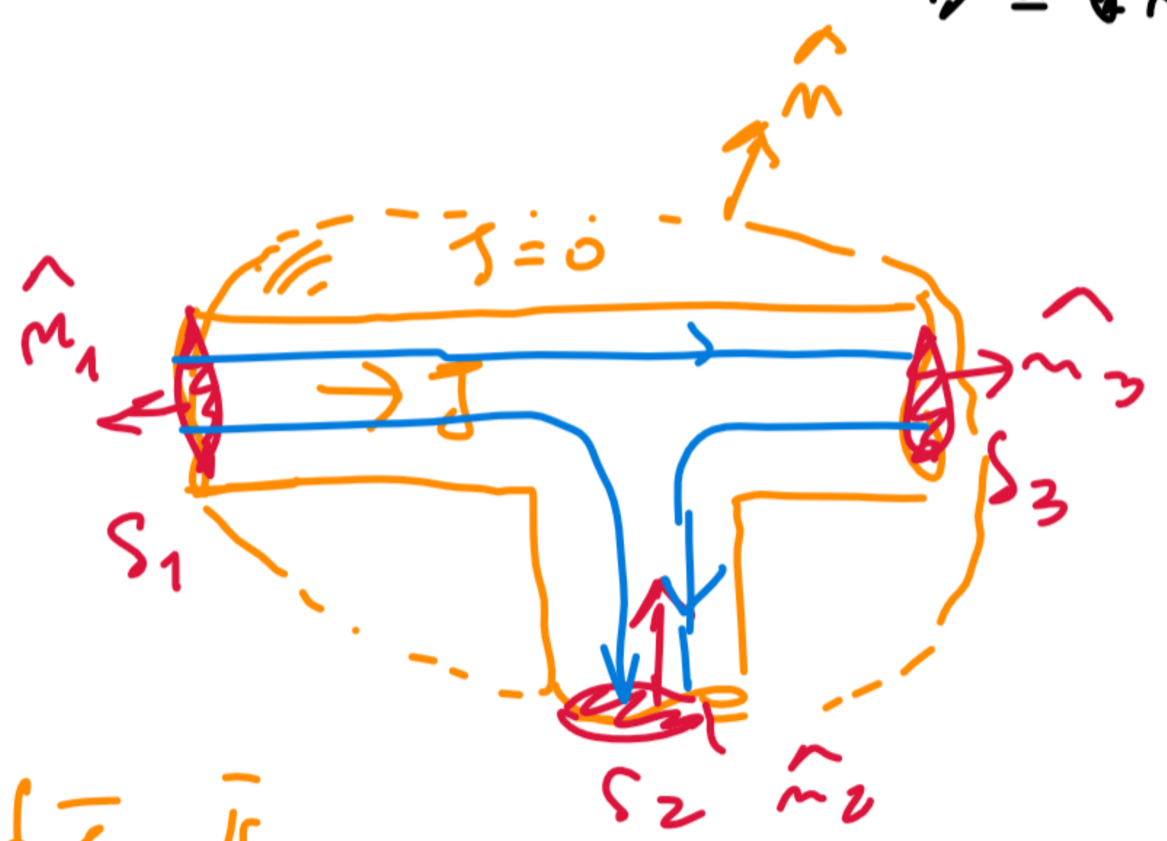
$$= \int_{S'} \nabla \times \bar{U} \cdot d\bar{S} - \int_{S''} \nabla \times \bar{U} \cdot d\bar{S} \overset{(1)}{=} 0$$



⇒ Because of Stokes theorem, FLUX of $\bar{F} = \nabla \times \bar{U}$ over a closed surface is 0

⟹ $\bar{F}$ a SOLENOIDAL FIELD

$\bar{B} = \nabla \times \bar{A}$



$$\oint_S \bar{J} \cdot d\bar{S} = 0$$

$\hookrightarrow \int_{S_1} \cdots - \int_{S_2} \cdots + \int_{S_3} \cdots$

↓

$\bar{J} = \nabla \times \bar{T}$ $\quad$ $\bar{J}$ allows for a vector potential when $\partial \bar{D}/\partial t \sim 0$

Stokes theorem on CONSERVATIVE FIELDS

$\bar{U}$ such that $\nabla \times \bar{U} = 0 \; \forall \bar{P} \in D$, D a SCD

CURL-FREE $\bar{U}$

$$\int_S \cancel{\nabla \times \bar{U}}_{\,0} \cdot d\bar{S} = \oint_{\Gamma} \bar{U} \cdot d\bar{\ell} = 0$$

On a SCD, $\bar{U}$ CURL-FREE ⇒ circulation of $\bar{U}$ is ZERO for all possible Γ

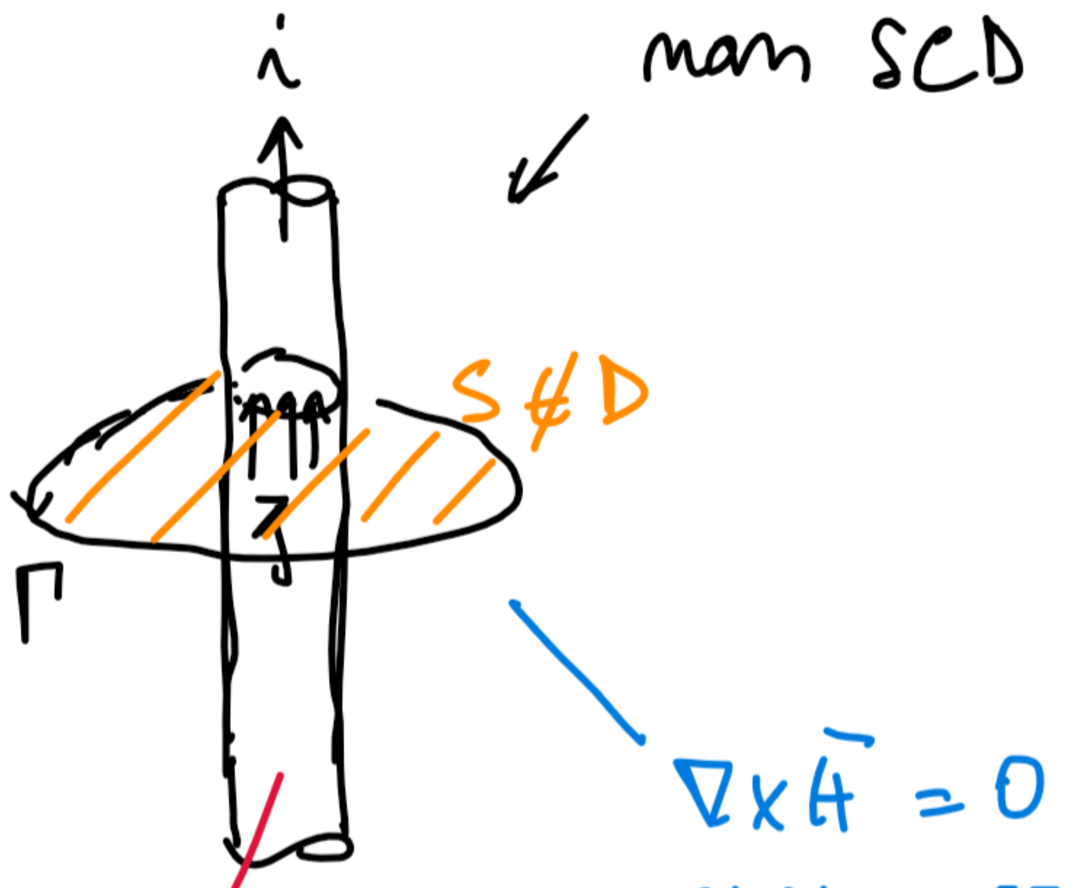
⇒ $\bar{U}$ is CONSERVATIVE ⇒ $\bar{U} = \nabla f$

if $\nabla \times \bar{H} = 0$ on every point of a SCD

$\Rightarrow \bar{H} = -\nabla\psi \Rightarrow \psi$: magnetic SCALAR POTENTIAL

EXAMPLE : $D = \mathbb{R}^3 -$ infinite cylinder. We want $\bar{H}$ in this domain

$(\partial/\partial t = 0)$



$\nabla \times \bar{H} = 0$

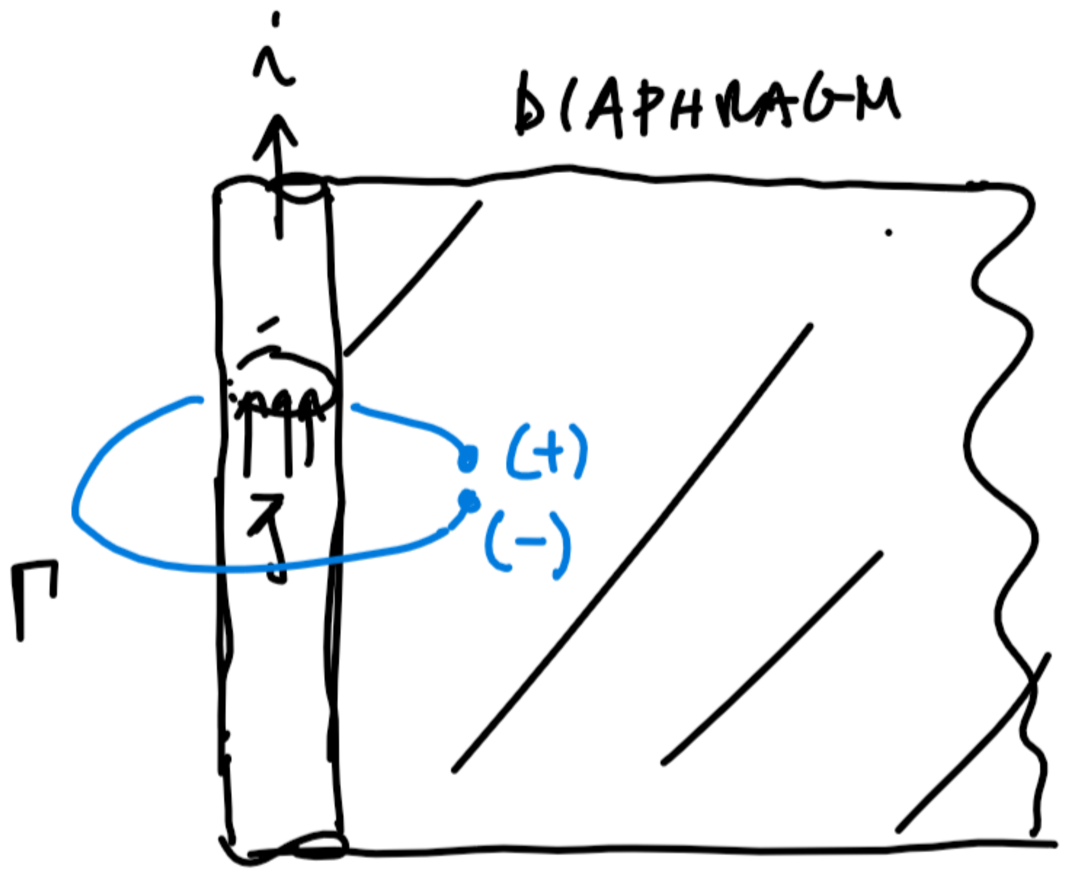
CURL-FREE

$\Rightarrow \cancel{\bar{H} = -\nabla\psi}$

$\nabla \times \bar{H} = \bar{J}$

$\Downarrow$

$\oint_\Gamma \bar{H}\cdot d\bar{l} = i \longrightarrow$



$\Downarrow$

$\int_{(+)}^{(-)} \bar{H}\cdot d\bar{l} = i$

$\{\mathbb{R}^3 - cyl - DIAPHRAGM\}$

→ in the REDUCED SCD, $\nabla \times \bar{H} = 0$ everywhere

$\Rightarrow \bar{H} = -\nabla\psi$

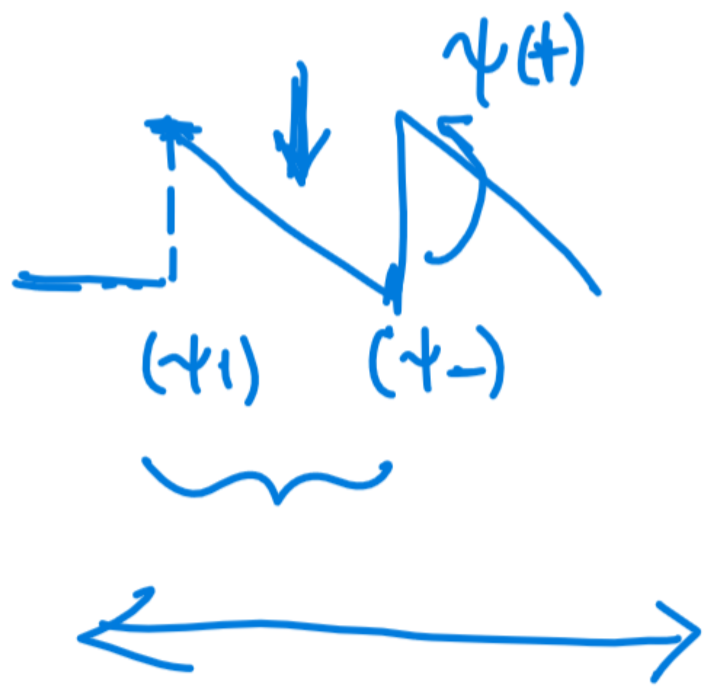
$\rightarrow \int_{(+)}^{(-)} -\nabla\psi \cdot d\bar{l} = \boxed{\psi_+ - \psi_- = i}$

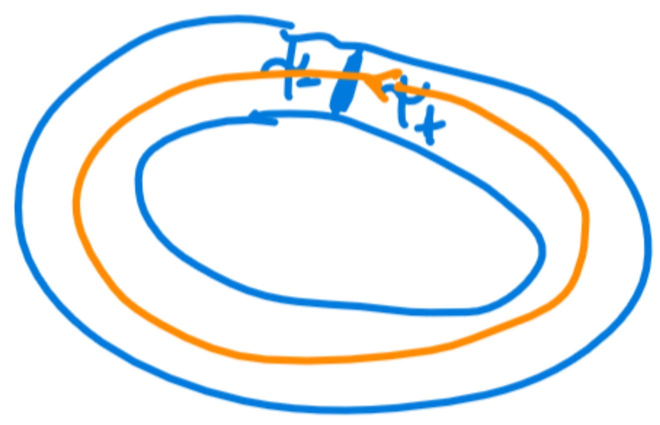
$\Downarrow$

$\psi \Rightarrow -\nabla\psi = \bar{H}$

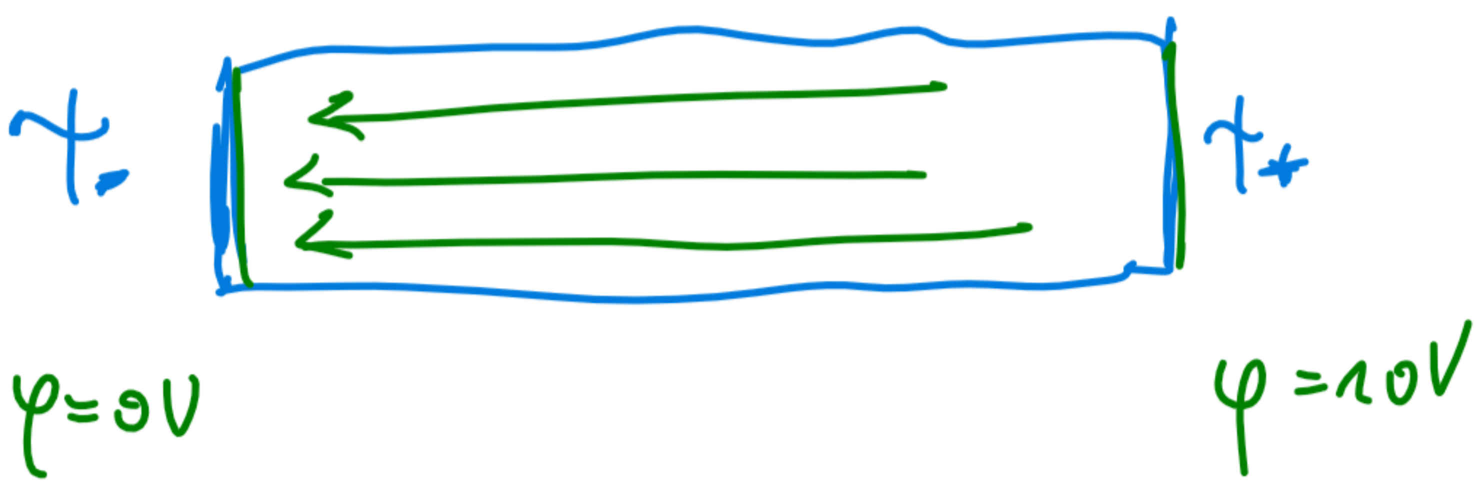
⇒ there is a DISCONTINUITY in ψ

⇒ DUE TO the CURRENT in the cylinder





$\Downarrow$



$\bar{E} = -\nabla\varphi$