Topology $\rightsquigarrow$ DOMAIN

METRIC SPACE : SET with the notion of DISTANCE (e.g. $\mathbb{R}^3$ EUCLIDEAN SPACE)

SET ↙ ensemble of n-tuples

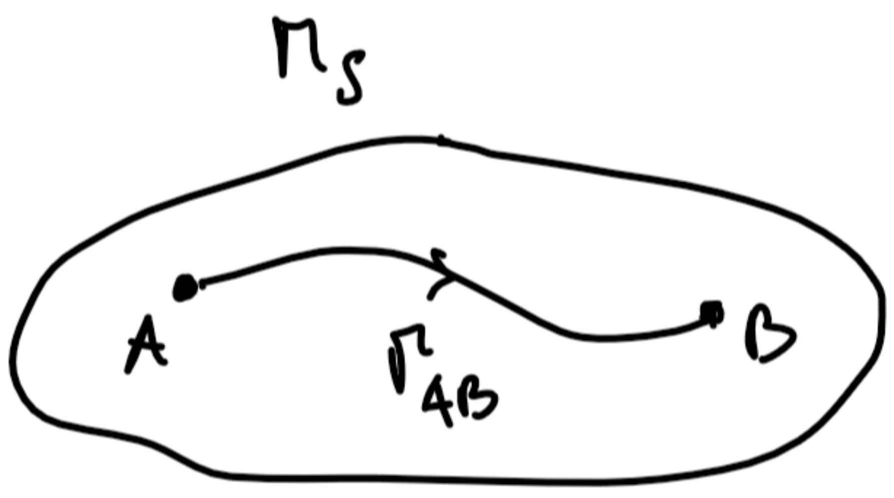
$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} \dots$$

p_1 p_2

DISTANCE ↑ measured by a distance function ⇓ METRIC

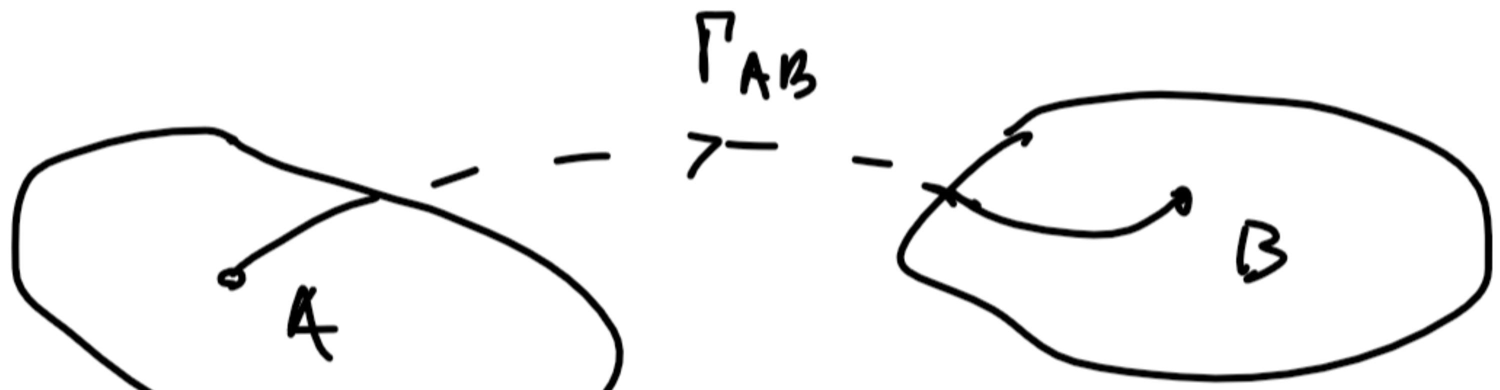
CONNECTED $\underbrace{\text{METRIC SPACE}}_{M_S}$: for any two points A e $B \in M_S$, M_S is connected if $\exists\, \Gamma_{AB}$ connecting A and B, $\Gamma_{AB} \in M_S$ (fully contained in M_S)

(A)



M_S is CONNECTED

(B)



$\Gamma_{AB} \notin M_S$

M_S NOT CONNECTED

$$M_S = M_S' \cup M_S''$$

<u>DOMAIN</u> : connected subset of a metric space

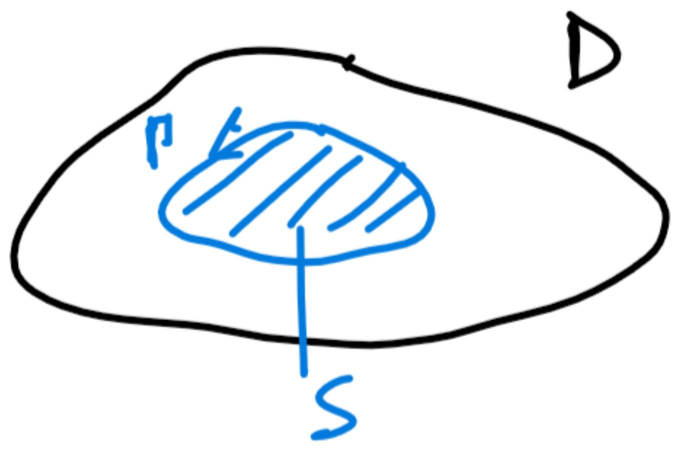
(A) is a DOMAIN

(B) is <u>not</u> a DOMAIN

... kinds of possible connections

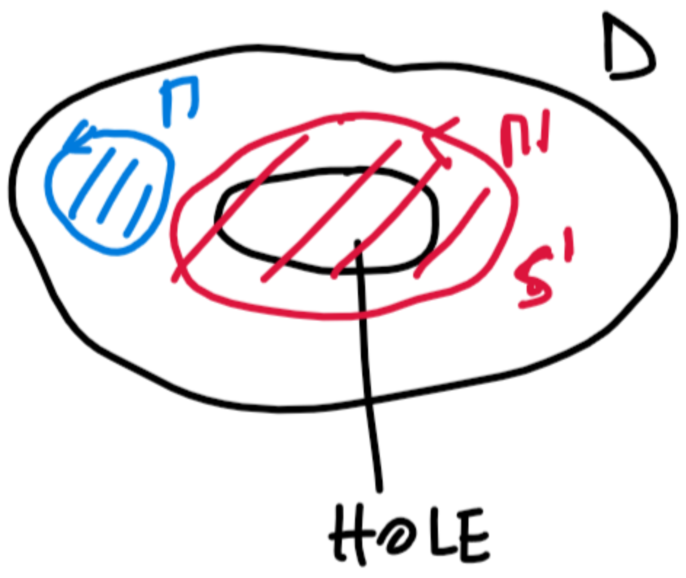
SIMPLY - CONNECTED DOMAIN (SCD) (2D) : D is a SCD if $\forall\, \Gamma \in D$ there exist a surface S entirely $\in D$

(A)



$S \in D \;\; \forall\, \Gamma \;\Rightarrow\;$ SCD

(B)



$\Gamma \to S \in D$

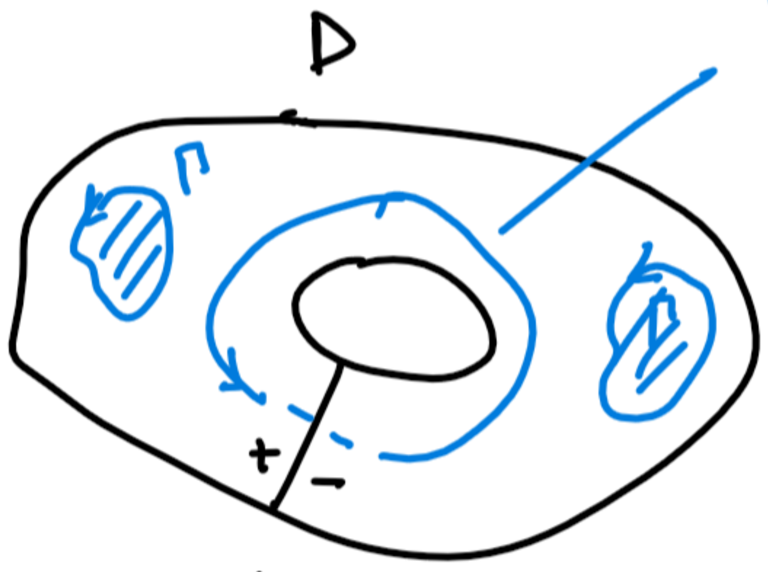
$S' \notin D \Rightarrow$ not SCD

$\Downarrow$

MULTIPLY - CONNECTED DOMAIN

• every D with a passing hole is MULTIPLY - CONNECTED

Γ not a closed PATH

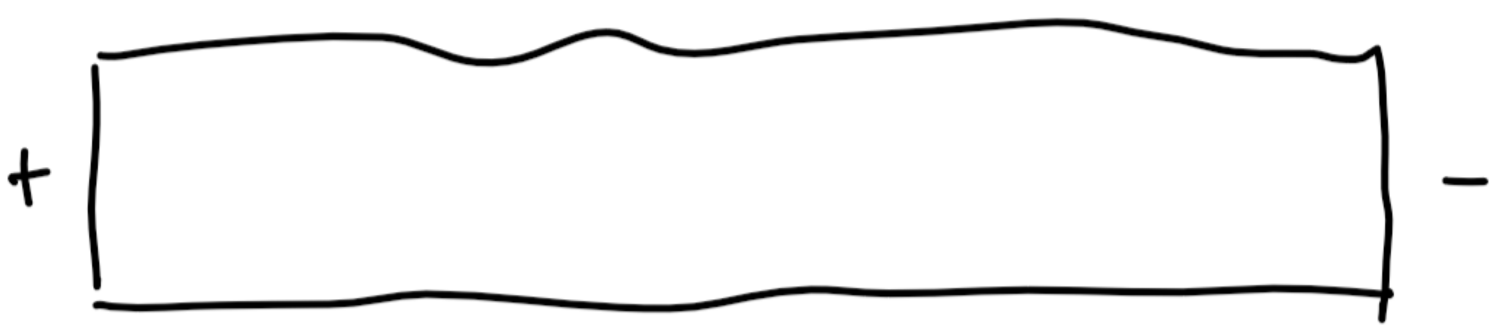


$\Rightarrow$ we can no longer define $S \notin D$

↳ D was <u>REDUCED</u> to a SCD (by cutting)

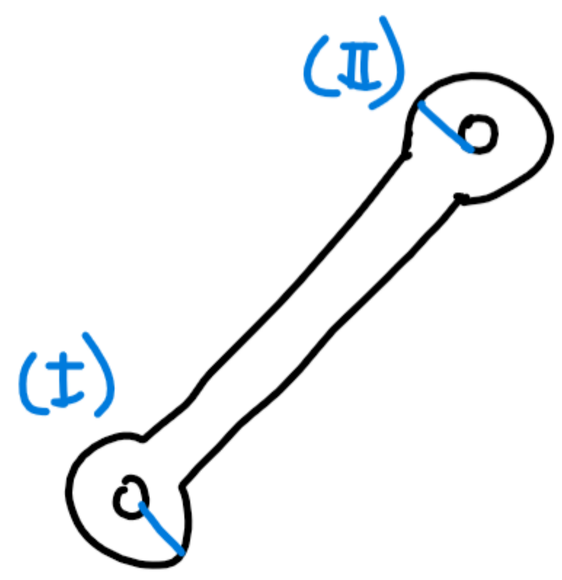
infinitesimal CUT

$\Downarrow$



"number"

MULTIPLICITY of a DOMAIN : # of cuts to REDUCE domain to a SCD



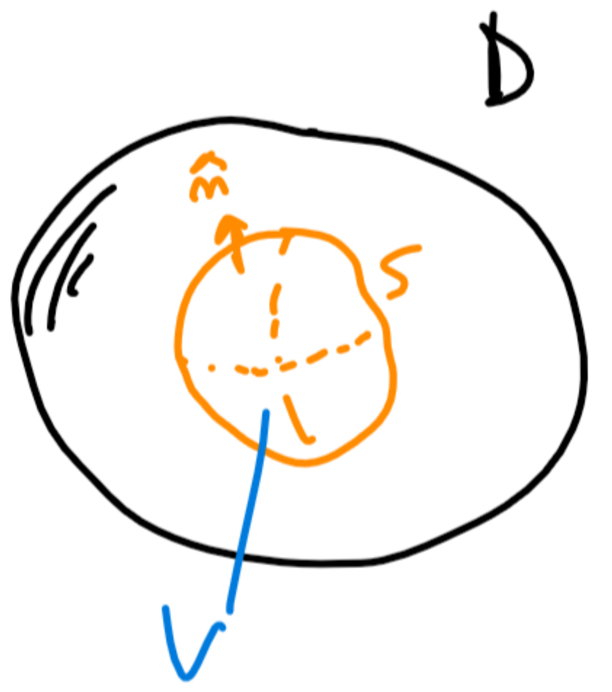
Connecting Rod
multiplicity 2

SCD (alternantive def) : a domain D is a SCD if ANY closed path Γ can be shrunk to a single point by a CONTINUOUS DEFORMATION

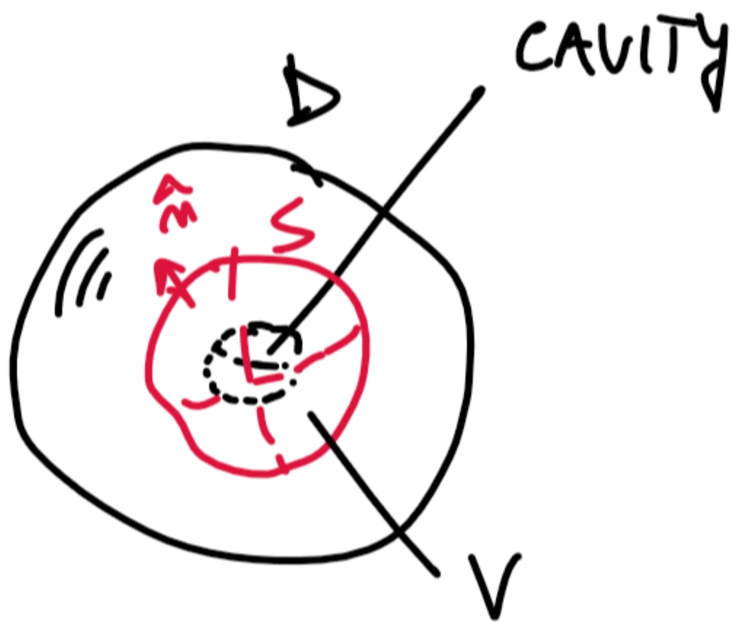
3D :

2D : closed curve $\rightarrow$ surface $\notin D$
3D : closed surfs. $\rightarrow$ volumes $\notin D$

SCD (3D) : D is a SCD if $\forall$ closed surfaces $S \in D$ there exist a Volume V $\underbrace{\text{defined by}}_{\text{ENCLOSED BY } S}$ S within D

(A) 

$\Rightarrow V \in D \;\; \forall S \in D \Rightarrow D$ is SCD

(B) 

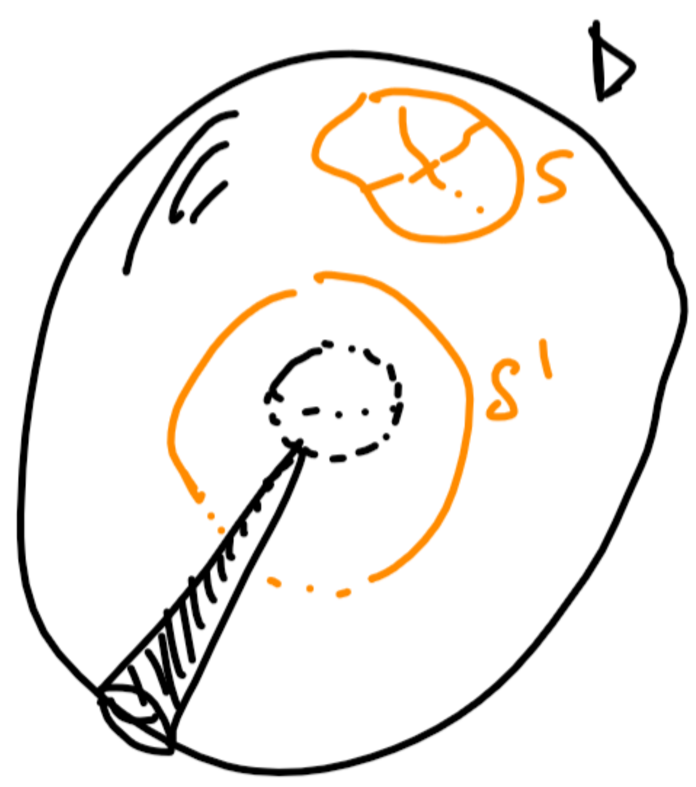
$\Rightarrow V \notin D \;\; \forall S \in D \Rightarrow D$ is not SCD

$\Downarrow$

MULTIPLY - connected domain
(each 3D domain with a CAVITY is a MCD)

D = sphere - CAVITY

--- PUNCTURES



$\uparrow$
infinitesimal puncture

$\Rightarrow$ no longer define problematic surface S' $\Rightarrow$ SCD

Multiplicity of a 3D domain
$\Rightarrow$ # of punctures to make to reduce the Domain to a SCD

SCD (3D) : a 3D domain is SCD if any closed S can be reduced to a single point by a continuous deformation

Differential Operators

GRADIENT

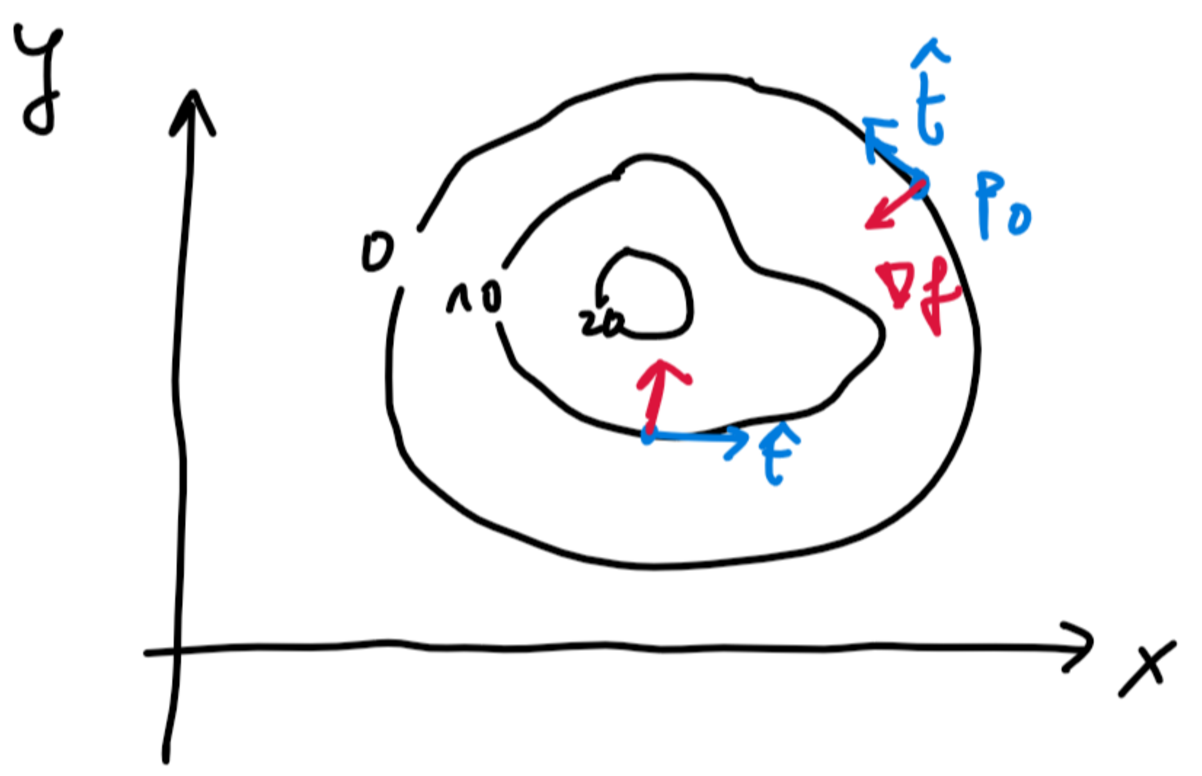
- the $\nabla(\cdot)$ describes the partial derivatives of a function

DEFINITION $\rightarrow$ $\nabla f \cdot \hat{n} = \partial f / \partial n$

vector field which - projected along $\hat{n}$ - gives directional derivative of f in the $\hat{n}$ direction

CARTESIAN COORDINATES : $\nabla f = \partial f/\partial x \, \hat{i} + \partial f/\partial y \, \hat{j} + \partial f/\partial z \, \hat{k}$

ISO-LINES of f



- $\partial f/\partial t = \nabla f \cdot \hat{t} = 0$

$\downarrow$

- ∇f ALWAYS $\perp$ to the ISO-LINES

$\Downarrow$

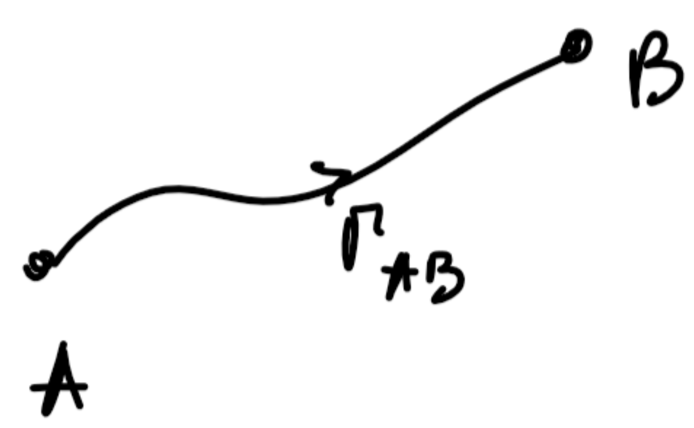
GRADIENT of a function POINTS to where f INCREASES most rapidly

CONSERVATIVE FUNCTIONS

- vector field $\bar{U} = \nabla f$

$\downarrow$

LINE INTEGRAL



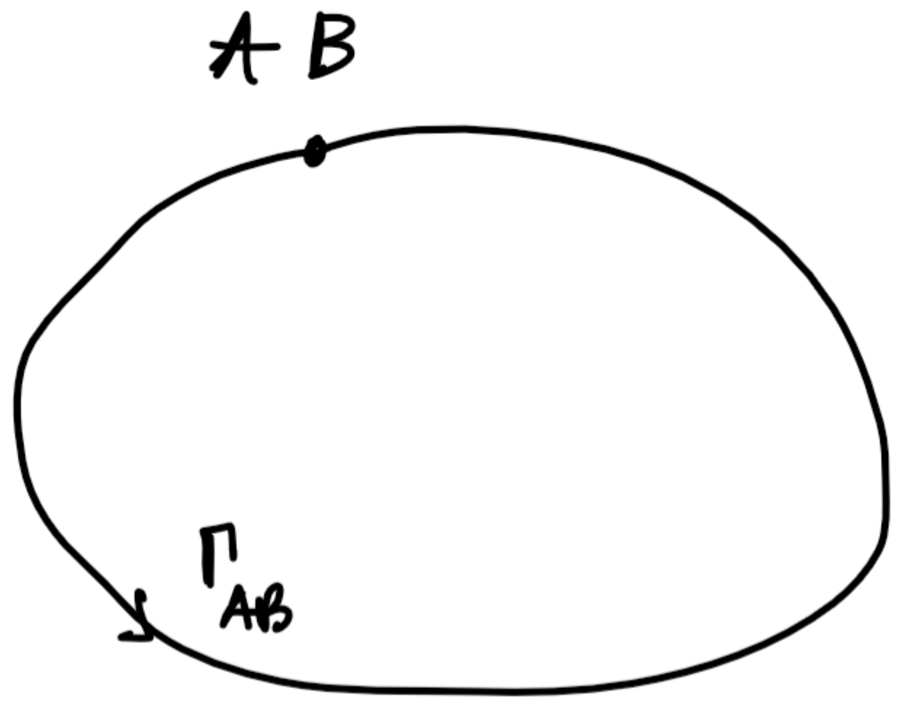
$$\int_A^B \bar{U} \cdot \underset{\hat{l}\,dl}{\underset{\uparrow}{d\bar{l}}} = \int_A^B \nabla f \cdot \hat{l} \, dl =$$

def. grad

$$= \int_A^B \partial f/\partial l \, dl = f(B) - f(A)$$

CIRCULATION of $\bar{U} = \nabla f$

$$\oint_{\Gamma} \bar{U} \cdot d\bar{l} = f(B) - f(A) = 0$$
$$\downarrow$$
$$f(B) = f(A)$$



$\Rightarrow$ GRADIENT of ANY function f is a CIRCULATION-FREE vector field

CIRCULATION FREE (for all Γ) $\Rightarrow$ CONSERVATIVE

DIVERGENCE

- the $\nabla \cdot (\cdot)$ describes the space-distribution of SOURCES or SINKS of a vector field

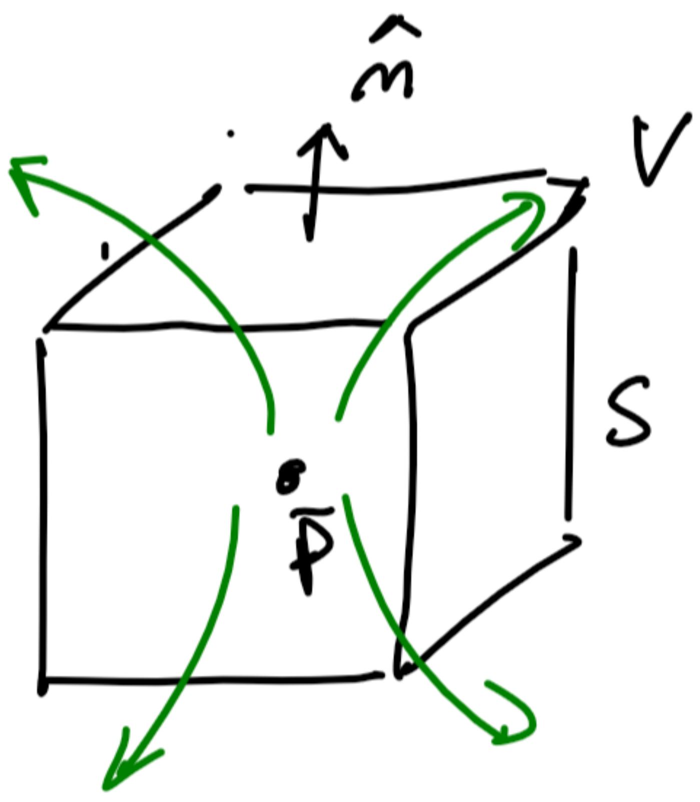
$P \in V$, bounded by S

Flux of $\bar{U}$ through S

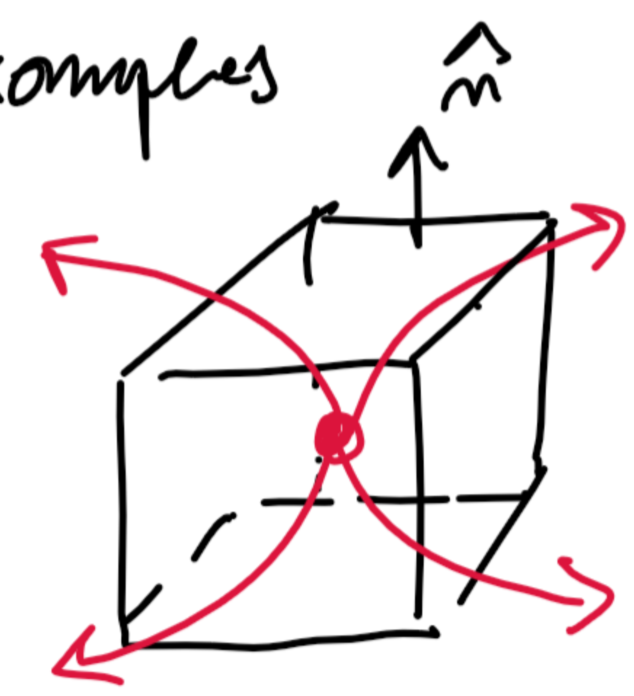
$$\nabla \cdot \bar{U} = \lim_{\Delta V \to 0} \frac{\Phi_{\bar{U},S}}{\Delta V} = \frac{\int_S \bar{U} \cdot d\bar{S}}{\Delta V} =$$

Volume of the domain V

$$\left[\frac{\text{flux of } \bar{U} \text{ through } S}{\Delta V} \right] = \left[\frac{\text{flux}}{\text{volume}} \right]$$

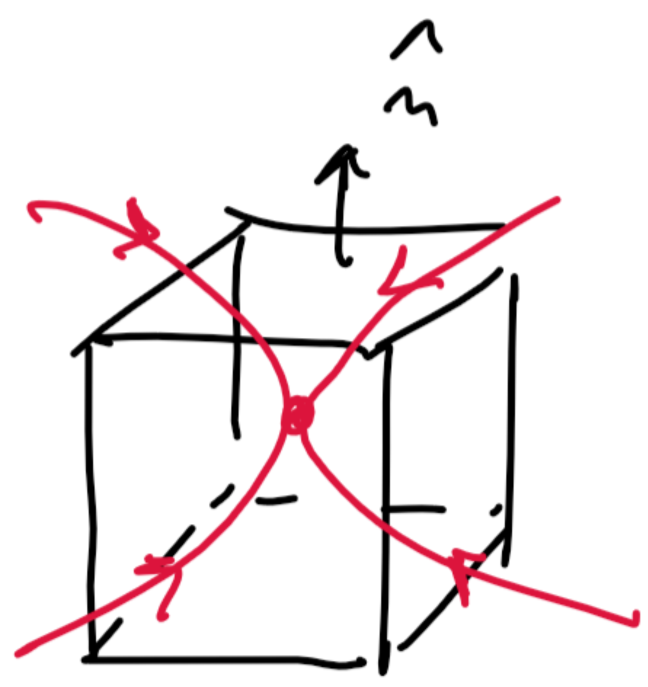


Examples

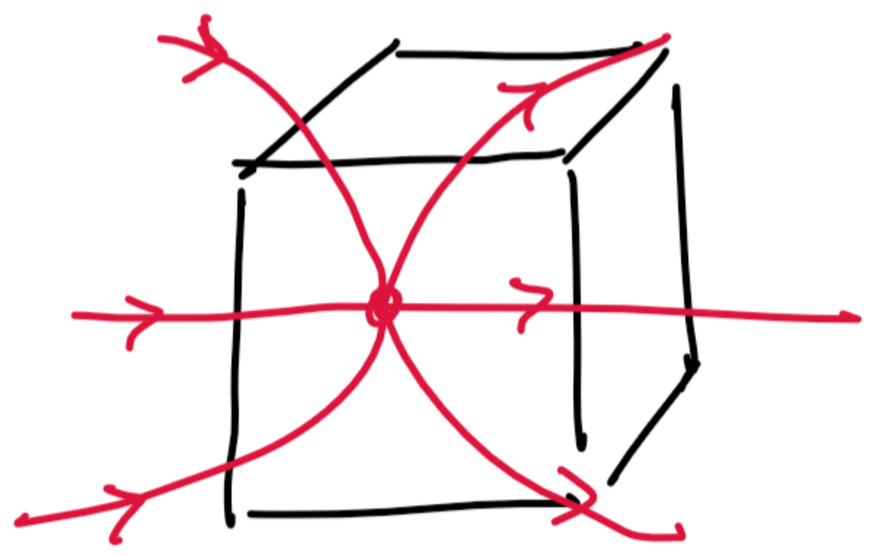


$\nabla \cdot \bar{U} > 0$

{field lines same direction as the normal}



$\nabla \cdot \bar{U} < 0$



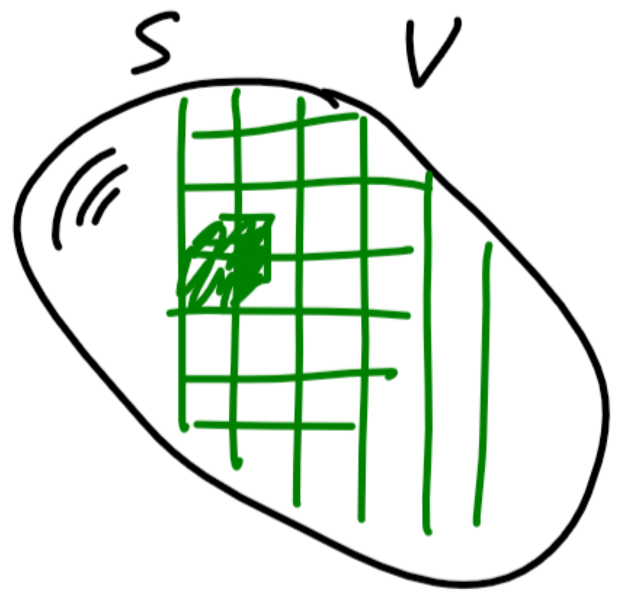
$\nabla \cdot \bar{U} = 0$

DIVERGENCE theorem

$\bar{U} \in C_1$ → $\bar{U}$ CONTINUOUS → PARTIAL derivatives of $\bar{U}$ are continuous — on a SCD

$$\underbrace{\int_V \nabla \cdot \bar{U}\, dV}_{\sum \text{"local" fluxes} \\ \text{total flux leaving } V} = \underbrace{\oint_S \bar{U} \cdot d\bar{S}}_{\text{Flux of } \bar{U} \\ \text{through } S}$$

$\nabla \cdot \bar{U}$: Flux out/in of a inf. volume



⇒ conservation law for vector fields

DIVERGENCE theorem – SOLENOIDAL FIELDS

Def: $\bar{U}$ is solenoidal if $\oint_S \bar{U} \cdot d\bar{S} = 0 \quad \forall S \in D$ — Domain where $\bar{U}$ is defined

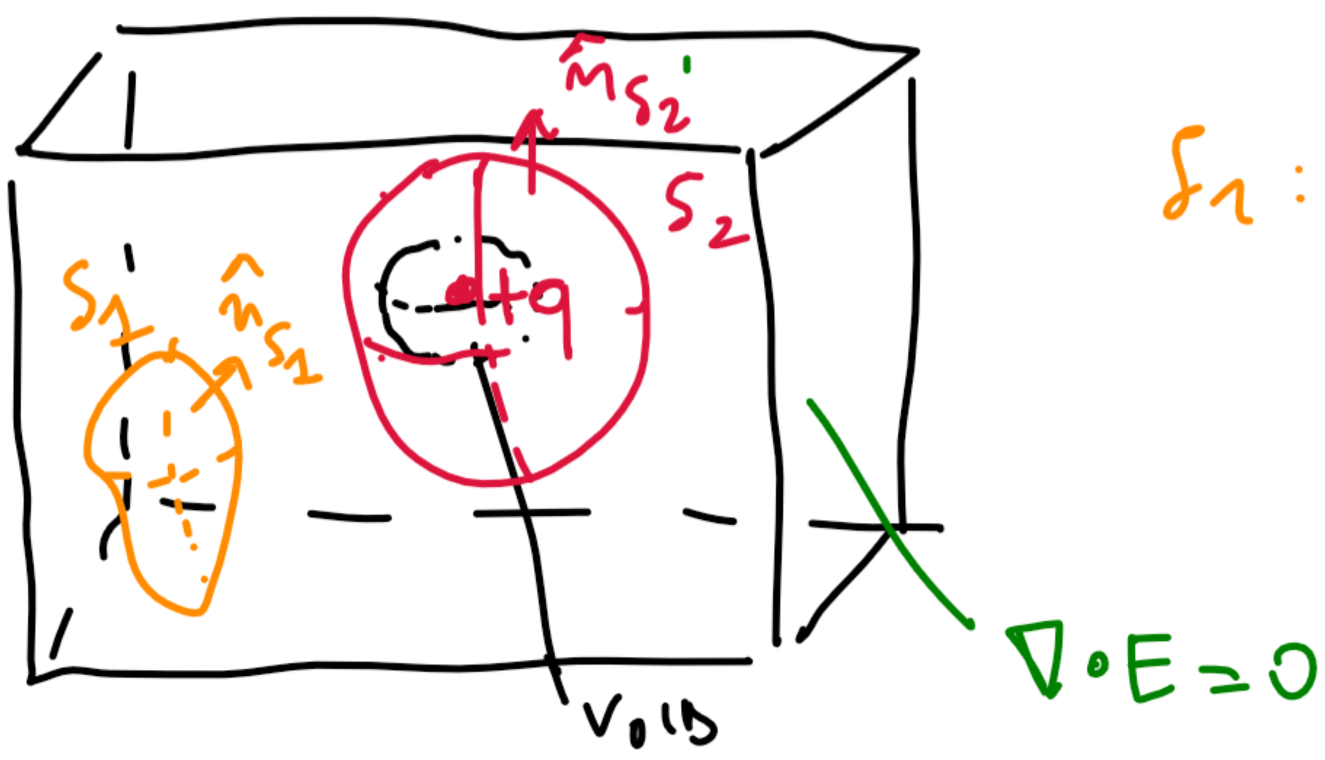
$$\underbrace{\oint_S \bar{U} \cdot d\bar{S} = 0 \;\xRightarrow{\text{DIVERG. th.}}\; \nabla \cdot \bar{U} = 0 \quad \forall \bar{P} \in D}$$

for all $S \in D$

(1) if a field is SOLENOIDAL ⟹ field is DIVERGENCE-FREE

(2) field is div-free ⟹ field is solenoidal ⇒ TRUE ONLY if D is SCD

EXAMPLE: solid dielectric with a void ⇒ non SCD



S_1: $\underbrace{\oint_{S_1} E \cdot dS = 0}_{\bar{E} \text{ is SOLENOIDAL}}$ ⇒ because $\nabla \cdot E = 0$ outside the cavity → FALSE D is not SCD

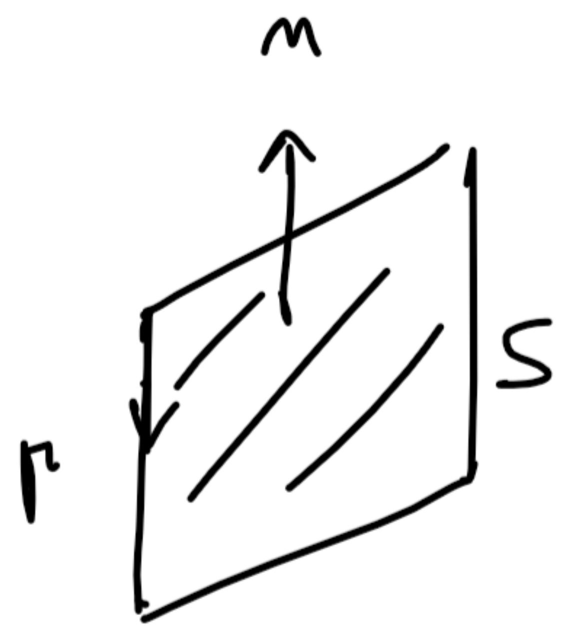
S_2: $\oint_{S_2} \bar{E} \cdot d\bar{S} = +q/\varepsilon_0$ ⇒ E not SOLENOIDAL

CURL : $\mathbb{R}^3 \to \mathbb{R}^3$

- the $\nabla \times (\cdot)$ operator describes the tendency of field lines to CIRCULATE around a point

$\overline{U} \in C_1$ on a SCD D

$$\underbrace{\nabla \times \overline{U} \cdot \hat{n}}_{\text{component of CURL along } \hat{n}} = \lim_{\Delta S \to 0} \frac{C_{\overline{U},\Gamma}}{\Delta S} \doteq \frac{\oint_\Gamma \overline{U} \cdot d\overline{l}}{\Delta S} =$$



surface of domain S

$$\left[\frac{\text{CIRCULATION}}{\text{SURFACE}} \right]$$