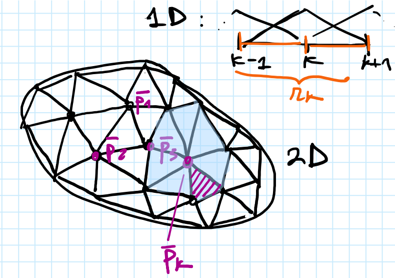
FEM - Poisson equation 2D

Monday, December 9, 2024 11:14 AM



PIECEWISE Linear interpolation on TRIANGLES

HP: $f(x,y)$ where $f(x,y)$ is known at <u>NODES</u>

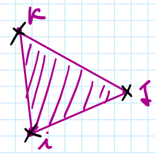
$\bar{P}_1 = (x_1, y_1) \Rightarrow f_1 = f(x_1, y_1)$

$\vdots$

$\bar{P}_k = (x_k, y_k) \Rightarrow f_k = f(x_k, y_k)$

Ω_k = union of all elements that have $\bar{P}_k$ as one of their VERTICES

Focus on $W_{ijk} \in \Omega_k$



VERTICES :

$\bar{P}_i = (x_i, y_i)$

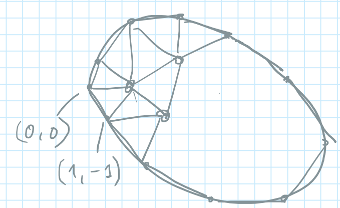
$\bar{P}_j = (x_j, y_j)$

$\bar{P}_k = (x_k, y_k)$

GOAL: define a set of LINEAR SHAPE FUNCTIONS [interpol.] on the ELEMENT

AREA of W_{ijk} :

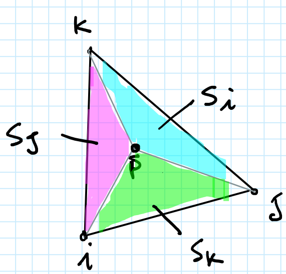
$$S = \frac{1}{2} \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$



Introduce a point inside W_{ijk}

$\bar{P} = (x, y) \in W_{ijk}$

Defines 3 SUB TRIANGLES inside W_{ijk}



if $\bar{P}$ moves closer to NODE i

$S_i \uparrow$, $S_j \downarrow$, $S_k \downarrow$

$$S_i = \frac{1}{2} \det \begin{bmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$

$$S_j = \frac{1}{2} \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x & y \\ 1 & x_k & y_k \end{bmatrix}$$

$$S_k = \frac{1}{2} \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x & y \end{bmatrix}$$

SHAPE FUNCTIONS :

$$\begin{cases} L_i(x,y) = \dfrac{S_i(x,y)}{S} \\ L_j(x,y) = \dfrac{S_j(x,y)}{S} \\ L_k(x,y) = \dfrac{S_k(x,y)}{S} \end{cases}$$

AREA COORDINATES

[BARYCENTRIC coordinates]

NOT INDEPENDENT!

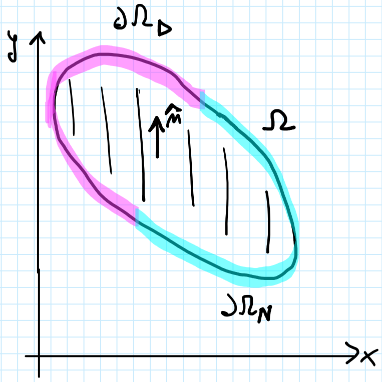
$$\boxed{L_i(x,y) + L_j(x,y) + L_k(x,y) = 1} \quad \forall x, y \in W_{ijk}$$

Ex: if $P = (x_j, y_j) \rightarrow \begin{cases} S_i = \frac{0}{S} = 0 \\ S_j = \frac{S}{S} = 1 \\ S_k = 0/S = 0 \end{cases}$

$\forall (x,y) \in W_{ijk}$

$$\boxed{f(x,y) = f_i L_i(x,y) + f_J L_J(x,y) + f_k L_k(x,y)}$$

FEM – Poisson Equation in 2D



Hp: $\partial/\partial z = 0$

$\varphi(x,y) = \varphi$

$p(x,y) = p$

$t(x,y) = t$

FORMULATION (strong form)

$$\begin{cases} \nabla\cdot(p\nabla\varphi) = t & \Omega \\ \varphi = \varphi_0 & \partial\Omega_D \\ \dfrac{\partial\varphi}{\partial n} = \varphi'_0 & \partial\Omega_N \end{cases}$$

PIECEWISE POLYNOMIAL interpolation $\varphi \rightarrow \tilde{\varphi} \in C_0$

$$\tilde{\varphi} = \sum_{k=1}^{n} \varphi_k \underset{\substack{\uparrow \\ L_K}}{L_k(x,y)} = \varphi_1 L_1(x,y) + \dots + \varphi_n L_n(x,y)$$

WEIGHTED RESIDUALS APPROACH

$$r(x,y) = \underbrace{\nabla\cdot(p\nabla\tilde{\varphi}) - t}_{=0 \text{ for } \varphi} \neq 0$$

"find nodal values of $\tilde{\varphi}$ such that ..."

WEIGHTING FUNCTION w

$$\int_\Omega w(x,y)\, r(x,y)\, dS = 0$$

RESIDUAL

g k f

$$\boxed{\int_\Omega w\, \nabla\cdot(p\nabla\tilde{\varphi})\, dS} = \int_\Omega w t \, dS \qquad \text{problem } \tilde{\varphi} \in C_0$$

VECTOR IDENTITY:

$$\boxed{\boxed{\nabla\cdot(g k \nabla f)} = g\,\nabla\cdot(k\nabla f) + \boxed{k \nabla f \cdot \nabla g}}$$

f, g, k generic scalar functions

$$\boxed{\int \nabla\cdot(w p \nabla\tilde{\varphi})\, dS} - \boxed{\int p \nabla w \cdot \nabla\tilde{\varphi}} = \int w t \, dS$$

$$\boxed{\int_\Omega \nabla\cdot(w\, p\, \nabla\tilde{\varphi})\, dS} - \boxed{\int_\Omega p\, \nabla w \cdot \nabla\tilde{\varphi}} = \int_\Omega w\, t\, dS$$

$$\Downarrow$$

$$\oint_{\partial\Omega} w\, p\, \nabla\tilde{\varphi}\cdot d\bar{l}$$

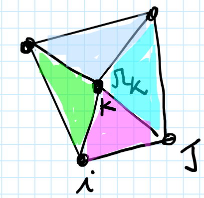
$$\Rightarrow \boxed{\int_\Omega p\, \nabla w \cdot \nabla\tilde{\varphi}\, dS = \int_{\partial\Omega} w\, p\, \nabla\tilde{\varphi}\cdot d\bar{l} - \int_\Omega w\, t\, dS}$$

WEAK FORMULATION

GALERKIN'S CHOICE $\quad w(x,y) \rightarrow L_k(x,y)$, for $k = 1, 2, \dots, m$

$\Downarrow$

$L_k(x,y) \neq 0$ only on Ω_k, for $x, y \in \Omega_k$



$\Rightarrow$ RESTRICT domain of integration $\Omega \rightarrow \Omega_k$

$$\Rightarrow \underbrace{\int_{\Omega_k} p\, \nabla L_k \cdot \nabla\tilde{\varphi}\, dS}_{\mathrm{I}} = \underbrace{\int_{\partial\Omega} L_k\, p\, \nabla\tilde{\varphi}\cdot d\bar{l}}_{\mathrm{II}} - \underbrace{\int_{\Omega_k} L_k\, t\, dS}_{\mathrm{III}}, \quad \text{for } k = 1, 2, \dots, m$$

if need to derive algebraic expression for node k :

- SPLIT integrals on SUPPORT DOMAIN into Σ integrals on elements $\in \Omega_k$

$$\sum_{W_i \in \Omega_k} \int_{W_i} p\, \nabla L_k \cdot \nabla\tilde{\varphi}\, dS = \int_{\partial\Omega} L_k\, p\, \nabla\tilde{\varphi}\cdot d\bar{l} - \sum_{W_i \in \Omega_k} \int_{W_i} L_k\, t\, dS \qquad \text{Eq. } K$$

$\uparrow$ for all elements $\in \Omega_k$

L_{k+1} if equation for node L_{k+1}

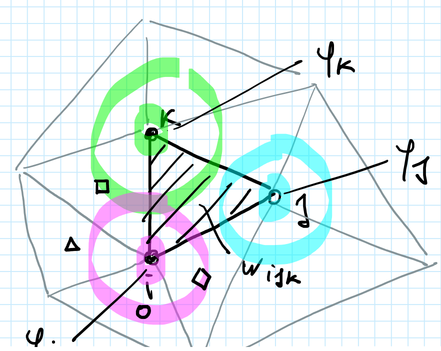
Element-centered approach

WRITE $[K]_{el}$ $\quad [rhs]_{el}$

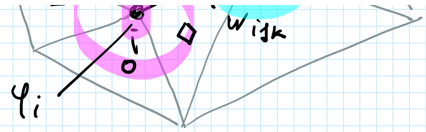
Element W_{ijk} contributes to 3 nodal equations

$$\begin{cases} \text{eq for node } i \\ \text{'' \quad '' \quad '' \quad } J \\ \text{'' \quad '' \quad '' \quad } K \end{cases}$$

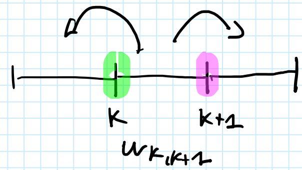
$\Rightarrow$ Equation for node i		
3 coeffs from		W_{ijk}
''	''	□
''	''	△
''	''	○
''	''	◇



Term I



Term I



$w_{ijk} \rightarrow$ eq i

$$\int_{w_{ijk}} p\, \nabla L_i \cdot \nabla \tilde{\varphi}\, dS =$$

on w_{ijk} $\quad \tilde{\varphi} = \varphi_i L_i(x,y) + \varphi_j L_j(x,y) + \varphi_k L_k(x,y)$

$$= \int_{w_{ijk}} p\, \nabla L_i \cdot \nabla \left[\varphi_i L_i + \varphi_j L_j + \varphi_k L_k \right] dS =$$

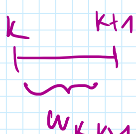
$$= \left[\int_{w_{ijk}} p\, \nabla L_i \cdot \nabla L_i\, dS \right] \varphi_i + \left[\int_{w_{ijk}} p\, \nabla L_i \cdot \nabla L_j\, dS \right] \varphi_j + \left[\int_{w_{ijk}} p\, \nabla L_i \cdot \nabla L_k\, dS \right] \varphi_k$$

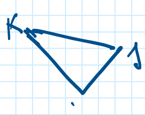
$w_{ijk} \rightarrow$ eq j $\qquad \tilde{\varphi} = \varphi_i L_i + \varphi_j L_j + \varphi_k L_k$

$$\int_{w_{ijk}} p\, \nabla L_j \cdot \nabla \tilde{\varphi}\, dS = \left[\int_{w_{ijk}} p\, \nabla L_j \cdot \nabla L_i\, dS \right] \varphi_i + \left[\int_{w_{ijk}} p\, \nabla L_j \cdot \nabla L_j\, dS \right] \varphi_j + \left[\int_{w_{ijk}} p\, \nabla L_j \cdot \nabla L_k\, dS \right] \varphi_k$$

$w_{ijk} \rightarrow$ eq k

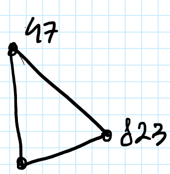
$$\int_{w_{ijk}} p\, \nabla L_k \cdot \nabla \tilde{\varphi}\, dS = \left[\int_{w_{ijk}} p\, \nabla L_k \cdot \nabla L_i\, dS \right] \varphi_i + \left[\int_{w_{ijk}} p\, \nabla L_k \cdot \nabla L_j\, dS \right] \varphi_j + \left[\int_{w_{ijk}} p\, \nabla L_k \cdot \nabla L_k\, dS \right] \varphi_k$$

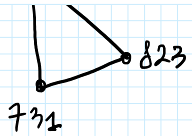
in 1D $\rightarrow$  $\rightarrow$ 4 contributions [coefficients] $\rightarrow$ 2 eq k, 2 eq $k+1$ $\rightarrow [K]_{el} = 2 \times 2$

in 2D $\rightarrow$  $\rightarrow$ 9 coeffs. $\rightarrow$ 3 eq i, 3 eq j, 3 eq k $\rightarrow [K]_{el} = 3 \times 3$

$\Rightarrow$ i, j, k are not necessarily adjacent

Ex:





$$\nabla L_i \cdot \nabla L_k = \left(\frac{\partial L_i}{\partial x}\hat{i} + \frac{\partial L_i}{\partial y}\hat{j}\right)\cdot\left(\frac{\partial L_k}{\partial x}\hat{i} + \frac{\partial L_k}{\partial y}\hat{j}\right) = \partial L_i/\partial x \, \partial L_k/\partial x + \partial L_i/\partial y \, \partial L_k/\partial y$$

$$[K]_{el,ijk} = \int_{\omega_{ijk}} \begin{bmatrix} p\nabla L_i\cdot\nabla L_i & p\nabla L_i\cdot\nabla L_j & p\nabla L_i\cdot\nabla L_k \\ p\nabla L_j\cdot\nabla L_i & p\nabla L_j\cdot\nabla L_j & p\nabla L_j\cdot\nabla L_k \\ p\nabla L_k\cdot\nabla L_i & p\nabla L_k\cdot\nabla L_j & p\nabla L_k\cdot\nabla L_k \end{bmatrix} dS =$$

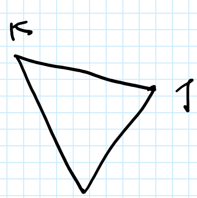
$$= \int_{\omega_{ijk}} p \begin{bmatrix} \partial L_i/\partial x \, \partial L_i/\partial x + \partial L_i/\partial y \, \partial L_i/\partial y & \partial L_i/\partial x \, \partial L_j/\partial x + \partial L_i/\partial y \, \partial L_j/\partial y & \partial L_i/\partial x \, \partial L_k/\partial x + \partial L_i/\partial y \, \partial L_k/\partial y \\ \partial L_j/\partial x \, \partial L_i/\partial x + \partial L_j/\partial y \, \partial L_i/\partial y & \partial L_j/\partial x \, \partial L_j/\partial x + \partial L_j/\partial y \, \partial L_j/\partial y & \partial L_j/\partial x \, \partial L_k/\partial x + \partial L_j/\partial y \, \partial L_k/\partial y \\ \frac{\partial L_k}{\partial x}\frac{\partial L_i}{\partial x} + \frac{\partial L_k}{\partial y}\frac{\partial L_i}{\partial y} & \frac{\partial L_k}{\partial x}\frac{\partial L_j}{\partial x} + \frac{\partial L_k}{\partial y}\frac{\partial L_j}{\partial y} & \frac{\partial L_k}{\partial x}\frac{\partial L_k}{\partial x} + \frac{\partial L_k}{\partial y}\frac{\partial L_k}{\partial y} \end{bmatrix} dS =$$

$$= \int_{\omega_{ijk}} p \underbrace{\begin{bmatrix} \frac{\partial L_i}{\partial x} & \frac{\partial L_i}{\partial y} \\ \frac{\partial L_j}{\partial x} & \frac{\partial L_j}{\partial y} \\ \frac{\partial L_k}{\partial x} & \frac{\partial L_k}{\partial y} \end{bmatrix}}_{[\nabla L]^T_{el,ijk}} \underbrace{\begin{bmatrix} \frac{\partial L_i}{\partial x} & \frac{\partial L_j}{\partial x} & \frac{\partial L_k}{\partial x} \\ \frac{\partial L_i}{\partial y} & \frac{\partial L_j}{\partial y} & \frac{\partial L_k}{\partial y} \end{bmatrix}}_{[\nabla L]_{el,ijk}} dS =$$

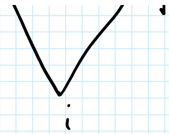
GRADIENT ELEMENT MATRIX

$$\boxed{[K]_{el,ijk} = \int_{\omega_{ijk}} p \, [\nabla L]^T_{el,ijk} \cdot [\nabla L]_{el,ijk} \, dS}$$

Term III : $-\int_{\partial\Omega_k} L_k t \, dS$



Term III : $-\int_{\Omega_K} L_K t \, dS$



$$\Downarrow$$

$$-\sum_{w_i \in \Omega_K} \int_{w_i} L_K t \, dS$$

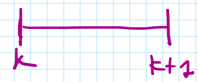
Contributions of element w_{ijk} to eqns → node i, → node j, → node k

w_{ijk} → node i $\int_{w_{ijk}} L_i t \, dS$

→ node j $\int_{w_{ijk}} L_j t \, dS$

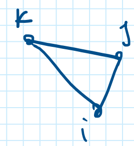
→ node k $\int_{w_{ijk}} L_k t \, dS$

in 1D



⇒ 2 contrib → k, → k+1

in 2D



⇒ 3 contributions → i, → j, → k

SOURCE TERM of Poisson's eq on element w_{ijk}

$$\Downarrow$$

$$[rhs]_{el,ijk} = -\int_{w_{ijk}} \begin{bmatrix} L_i t \\ L_j t \\ L_k t \end{bmatrix} dS = -\int_{w_{ijk}} [L]_{el,ijk} \, t(x,y) \, dS$$

$$\begin{bmatrix} L_i(x,y) \\ L_j(x,y) \\ L_k(x,y) \end{bmatrix} = [L]_{el,ijk}$$

ELEMENT shape function array

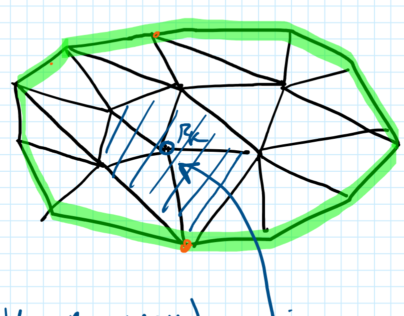
Term II : $\underbrace{\int_{\partial\Omega} L_K \, p \, \nabla\tilde{\varphi} \cdot d\bar{l}}_{\text{II}}$ ⇒ on the boundary

$\Downarrow$

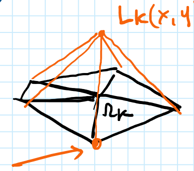
For all internal nodes $L_K = 0$

(if K is internal node, $L_K = 0$ on the BOUNDARY)

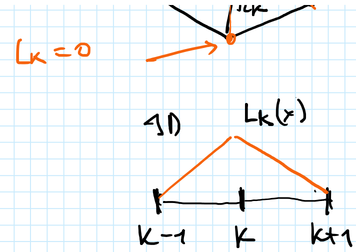
↓ to be evaluated only for nodes $\in \partial\Omega_N, \partial\Omega_D$



2D



DIRICHLET nodes   if node $\in \partial\Omega_D$



DIRICHLET nodes if node $\in \partial\Omega_D$

$$\boxed{\varphi(x_K, y_K) = \varphi_0} \Rightarrow \text{no weighted residuals!}$$

Neumann nodes ... to do!

For next lesson: DOWNLOAD $\boxed{\text{GMSH}}$ put it in GITHUB folder for 2D Fem

https://gmsh.info/#Download