Finite Difference Method (FDM) - 1D Poisson's Equation

FDM:

GOAL: solve a PdE numerically

$$D[f(x)] = 0$$

$$\Big\downarrow \text{F.d.M}$$

$$f_1, f_2, \cdots, f_k, \cdots f_n$$

$$\Big\downarrow$$

$$[K]\{f\} = \{Rhs\}$$

RIGHT-HAND-SIDE ARRAY

1. Define GRID of NODES to discretize the physical domain
2. Express derivatives in the mathematical formulation through Finite Difference Approximations
3. Assemble on solve a LINEAR SYSTEM to find NODAL VALUES of the unknown function

1D Poisson's Equation [ELLIPTIC EQUATION]

$$\nabla \cdot (p \nabla \varphi) = t \;\xrightarrow{\text{if } p \text{ UNIFORM}}\; \nabla \cdot \nabla \varphi = \underbrace{t/p}_{t} \;\rightarrow\; \boxed{\nabla^2 \varphi = t}$$

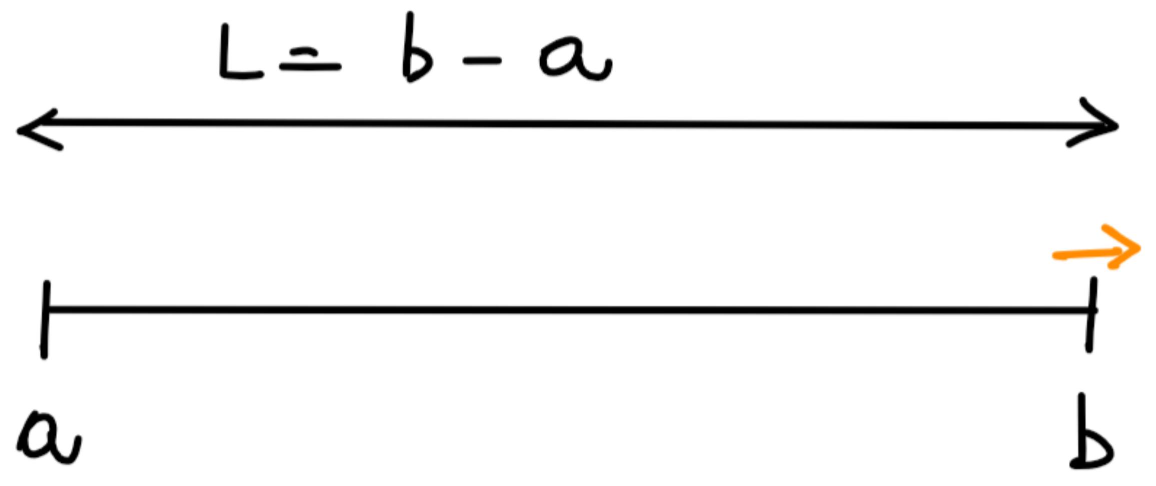
EXAMPLE Eq for ELECTROSTATIC

$$\nabla \cdot (\varepsilon_r \nabla \varphi) = -\rho/\varepsilon_0$$

Hp: $\underbrace{\partial/\partial y = 0 \quad \partial/\partial z = 0}_{\varphi = \varphi(x)} \;\xrightarrow[\text{CART. COORDS.}]{}\; \boxed{\dfrac{d^2\varphi}{dx^2} = t}\,, \quad x \in [a, b]$

$$\begin{cases} \dfrac{d^2\varphi}{dx^2} = t\,, \quad x \in \,]a, b[& \text{INTERNAL POINTS} \\ \varphi(a) = \varphi_a & \text{DIRICHLET} \\ \left.\dfrac{d\varphi}{dx}\right|_b = \varphi'_b & \text{NEUMANN BC} \end{cases}$$

$$\partial\varphi/\partial n \rightarrow \frac{d\varphi}{dx}$$



1. GRID of nodes

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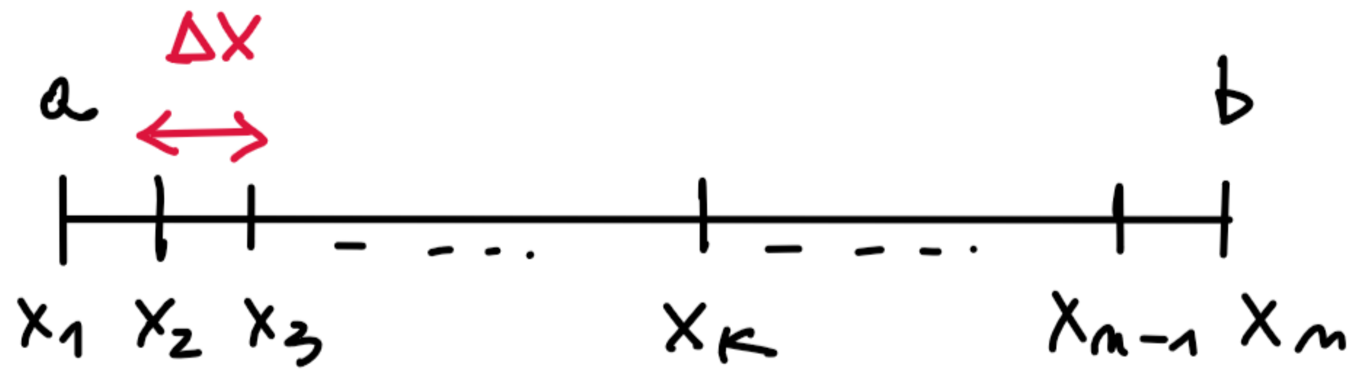
n nodes with uniform spacing Δx



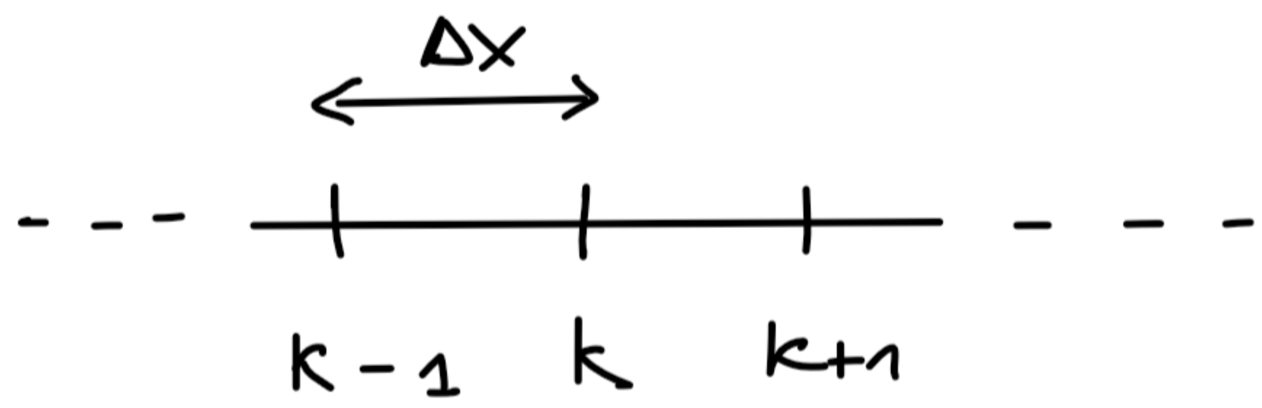
$x_1 \equiv a$

$x_n \equiv b$

$$\Delta x = \frac{L}{\underbrace{n-1}_{\#\text{ of intervals}}} = \frac{b-a}{n-1}$$

2. Finite difference formulas to express derivatives

$$\left.\frac{d^2\varphi}{dx^2}\right|_k = \frac{\varphi_{k+1} - 2\varphi_k + \varphi_{k-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$



"SOURCE TERM" KNOWN

$$\frac{\varphi_{k+1} - 2\varphi_k + \varphi_{k-1}}{\Delta x^2} = t_k$$

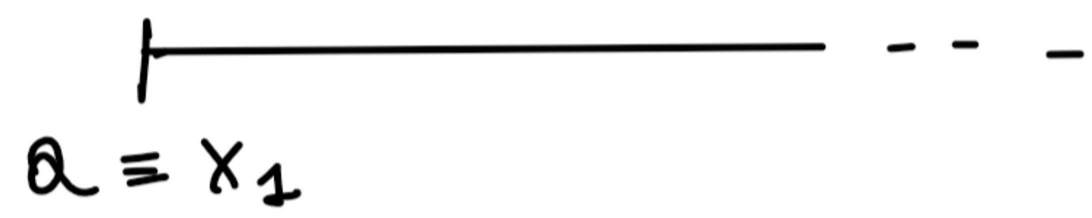
$$\boxed{\varphi_{k+1} - 2\varphi_k + \varphi_{k-1} = \Delta x^2 t_k}$$

ALGEBRAIC EXPRESSION for internal nodes

$k = 2, 3, \dots, k, \dots n-1$

BOUNDARY CONDITIONS

node a: $\varphi(a) = \varphi_a$



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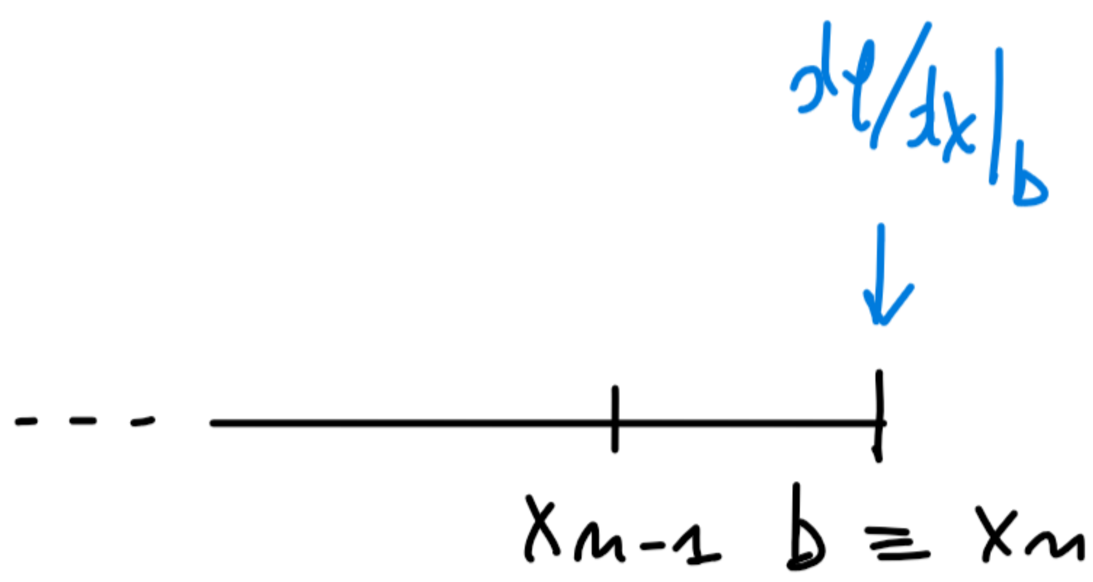
$$\boxed{\varphi_1 = \varphi_a} \quad k = 1$$

node b: $\left.\frac{d\varphi}{dx}\right|_b = \varphi'_b$

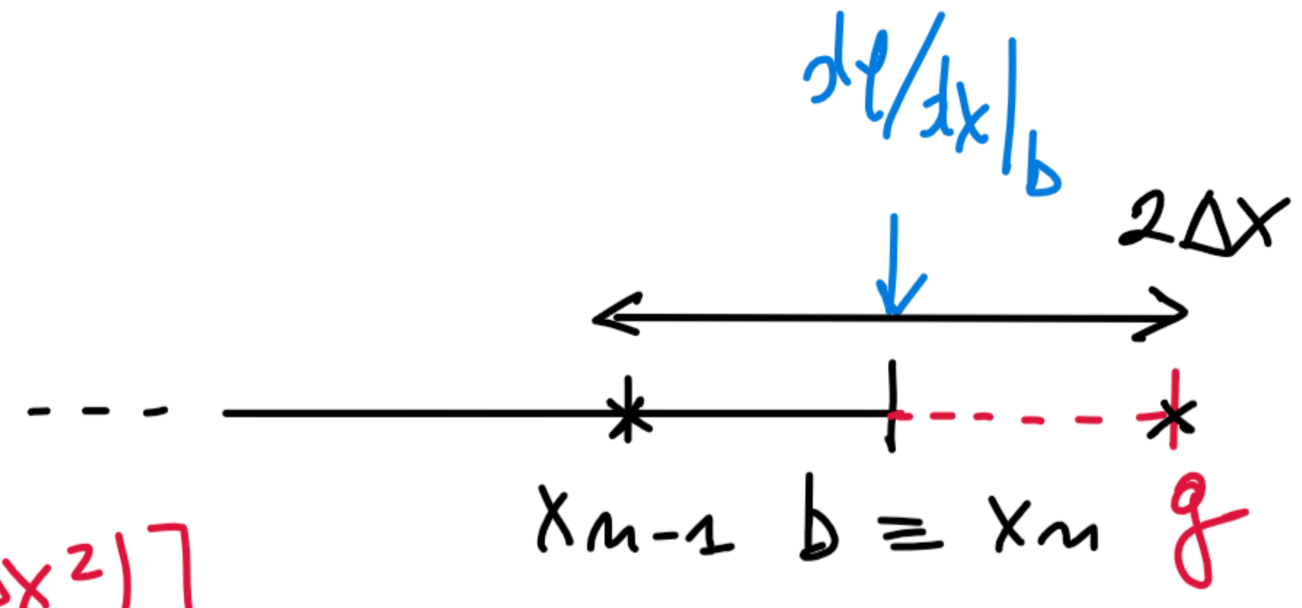


Idea 1: use BWD $\left.\frac{d\varphi}{dx}\right|_b = \frac{\varphi_n - \varphi_{n-1}}{\Delta x} + \mathcal{O}(\Delta x)$

$$\boxed{\varphi_n - \varphi_{n-1} = \Delta x\, \varphi'_b} \quad k = n$$

$\Rightarrow$ Only First-order accurate

GOAL: 2^{nd}-order-accurate expression for $d\varphi/dx|_b$



→ GHOST-NODE TECHNIQUE

CENTERED $[\mathcal{O}(\Delta x^2)]$

$$(1)\quad \begin{cases} \left.\dfrac{d\varphi}{dx}\right|_b = \dfrac{\varphi_g - \varphi_{m-1}}{2\Delta x} = \varphi'_b \\[2ex] \left.\dfrac{d^2\varphi}{dx^2}\right|_b = t_b \Rightarrow \dfrac{\varphi_g - 2\varphi_m + \varphi_{m-1}}{\cancel{\Delta x^2}} = t_b \Delta x^2 \end{cases}$$

→ ADD ADDITIONAL EQUATION

Hp: t KNOWN in b

$$(1)\quad \varphi_g = 2\Delta x\, \varphi'_b + \varphi_{m-1}$$

$$(2)\quad 2\Delta x\, \varphi'_b + \varphi_{m-1} - 2\varphi_m + \varphi_{m-1} = \Delta x^2 t_b$$

$$2\varphi_{m-1} - 2\varphi_m = \Delta x^2 t_b - 2\Delta x\, \varphi'_b$$

$$\boxed{\varphi_{m-1} - \varphi_m = \Delta x \left(\frac{\Delta x}{2} t_b - \varphi'_b \right)}$$

2^{nd}-order accurate expression for node $m \equiv b$

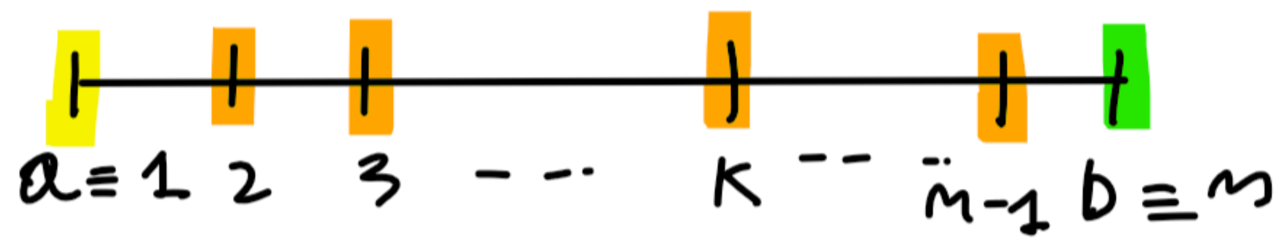
- Assemble Linear System $[K]\{\varphi\} = \{Rhs\}$

BOUNDARY CONDITIONS $t_1, t_2, \ldots, t_k, \ldots t_m$

$$\varphi_1 = \varphi_a$$

$$\varphi_{k+1} - 2\varphi_k + \varphi_{k-1} = \Delta x^2 t_k$$

$$\varphi_{m-1} - \varphi_m = \Delta x \left(\frac{\Delta x}{2} t_b - \varphi'_b \right)$$



$$\begin{array}{l} k=1 \\ k=2 \\ k=3 \\ \\ \\ \\ k=m \end{array} \begin{bmatrix} 1 & & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & -2 & 1 & \\ & & & \ddots & \ddots & \ddots \\ & & & & 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_k \\ \vdots \\ \varphi_m \end{bmatrix}}_{\{\varphi\}} = \underbrace{\begin{bmatrix} \varphi_a \\ \Delta x^2 t_2 \\ \vdots \\ \vdots \\ \Delta x^2 t_k \\ \vdots \\ \Delta x \left(\frac{\Delta x}{2} t_b - \varphi'_b \right) \end{bmatrix}}_{\{Rhs\}}$$

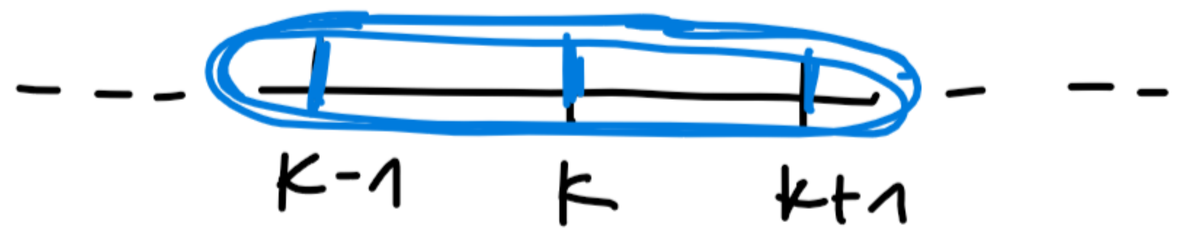
$t_b = t_m$

[K] is

- TRI-DIAGONAL
- SPARSE MATRIX (most elements are zeros)

$\Downarrow$

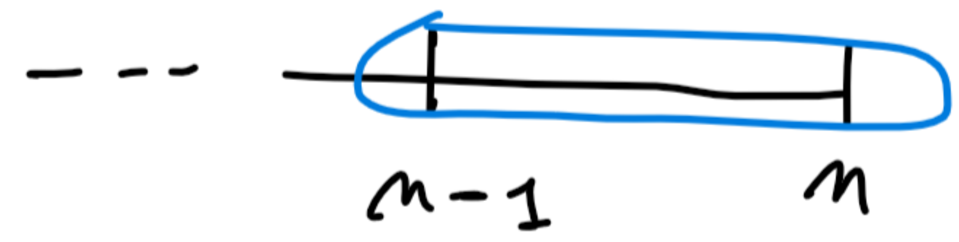
$\boxed{\text{FdM yields SPARSE MATRICES}}$



$$\left.\frac{d^2\varphi}{dx^2}\right|_k = f\big(\underbrace{\varphi_{k+1}, \varphi_k, \varphi_{k-1}}_{\substack{\text{THREE-NODES STENCIL} \\ \text{(for internal nodes)}}}\big)$$

EXAMPLE: for Neumann BC in b

TWO-NODES STENCIL



- [K] is $n \times n$ matrix

Diagonally-Dominant matrices

[A] square matrix with n-rows

- [A] is STRICTLY DIAGONALLY-DOMINANT if:

$$|a_{i,i}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}| \quad \forall\ i = 1, \dots, n$$

generic entry of [A] on the diagonal

EXAMPLE: 4 nodes Dirichlet a Neumann b

$$[K] : \begin{array}{c} K_{2,2} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \quad \begin{array}{l} 1 > 0 \\ 2 = (1+1) \\ 2 = (1+1) \\ 1 = (1) \end{array}$$

[K] is $\underline{\underline{\text{NOT}}}$ strictly DIAG. DOM.

- [A] is WEAKLY DIAGONALLY DOMINANT if $\Rightarrow$ [K]

$$|a_{i,i}| \geq \sum_{j=1, j\neq i}^{n} |a_{i,j}| \quad \forall\ i = 1, \dots, n$$

$\rightarrow$ DOES not have UNIQUE SOLUTION

- WEAKLY-DIAGONALLY DOMINANT matrices that have AT LEAST ONE ROW where $|a_{i,i}| > \sum_{j=1, j\neq i}^{n} |a_{i,j}|$

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IRREDUCIBLY diagonally dominant : $\boxed{\text{they ARE INVERTIBLE}}$

$K_{2,2}$

$$[K] : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{array}{l} 1 > 0 \\ 2 = (1+1) \\ 2 = (1+1) \\ 1 = (1) \end{array}$$

→ this makes [K] IRREDUCIBLE

↓

DIRICHLET BC

uniqueness of solution

Homework:

Build [K] with $n=4$ as if Neumann BCs on a, b

Study RANK of [K]

→ try also in MATLAB $\underbrace{K \backslash t}_{\text{solve linear system}}$, $t = \text{rand}(4,1)$