order 0:

$$f(x) = 1 \qquad \int_{-1}^{1} f(x)\,dx = 2 \Rightarrow w_1 f(x_1) + w_2 f(x_2) = 2$$

$$f(x) = 1$$

$$\Rightarrow \boxed{w_1 \cdot 1 + w_2 \cdot 1 = 2}$$

order 1

$$f(x) = x \qquad \int_{-1}^{1} f(x)\,dx = 0 \Rightarrow \boxed{w_1 x_1 + w_2 x_2 = 0}$$

order 2

$$f(x) = x^2 \qquad \int_{-1}^{1} f(x)\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = 2/3$$

$$\boxed{w_1 x_1^2 + w_2 x_2^2 = 2/3}$$

order 3

$$f(x) = x^3 \qquad \int_{-1}^{1} f(x)\,dx = \left[x^4/4\right]_{-1}^{1} = 0$$

$$\boxed{w_1 x_1^3 + w_2 x_2^3 = 0}$$

NONLINEAR SYSTEM

$$\begin{cases} w_1 + w_2 = 2 & \text{Eq1} \\ w_1 x_1 + w_2 x_2 = 0 & \text{Eq2} \\ w_1 x_1^2 + w_2 x_2^2 = 2/3 & \text{Eq3} \\ w_1 x_1^3 + w_2 x_2^3 = 0 & \text{Eq4} \end{cases}$$

Hp: EQUAL WEIGHTS $w_1 = w_2$

$$\boxed{w_1 = w_2 = 1}$$

$\downarrow$ Eq2

$$1 \cdot x_1 + 1\, x_2 = 0$$

$$\rightarrow \boxed{x_2 = -x_1} \quad \text{SYMMETRY}$$

Eq4: SATISFIED by Eq. weights and symmetry

$$1\, x_1^3 + 1\, x_2^3 = 0 \rightarrow x_2 = -x_1 \rightarrow -x_2^3 + x_2^3 = 0 \quad \text{ALWAYS TRUE}$$

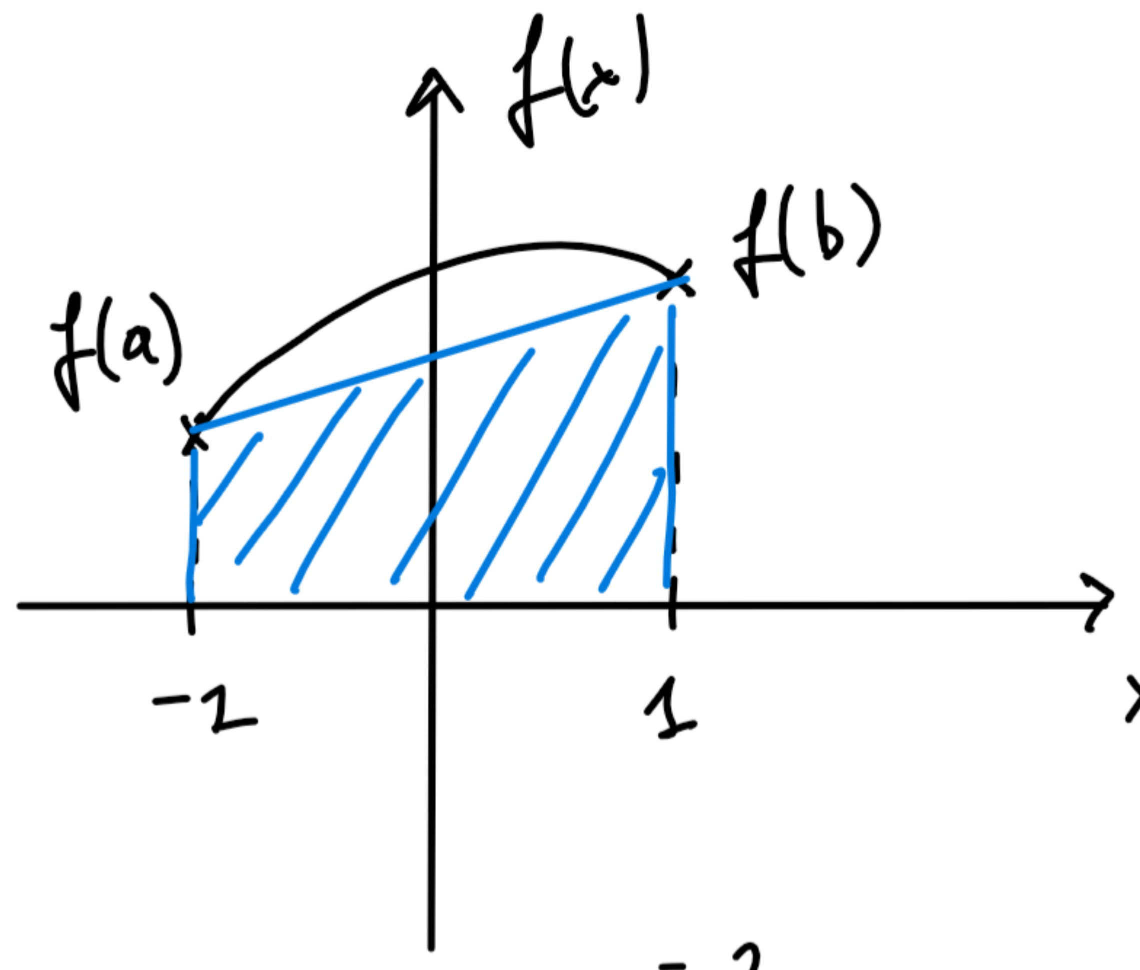
Eq3:

$$x_1^2 + x_2^2 = 2/3 \rightarrow \cancel{2} x_1^2 = \cancel{2}/3 \rightarrow x_1 = \pm 1/\sqrt{3}$$

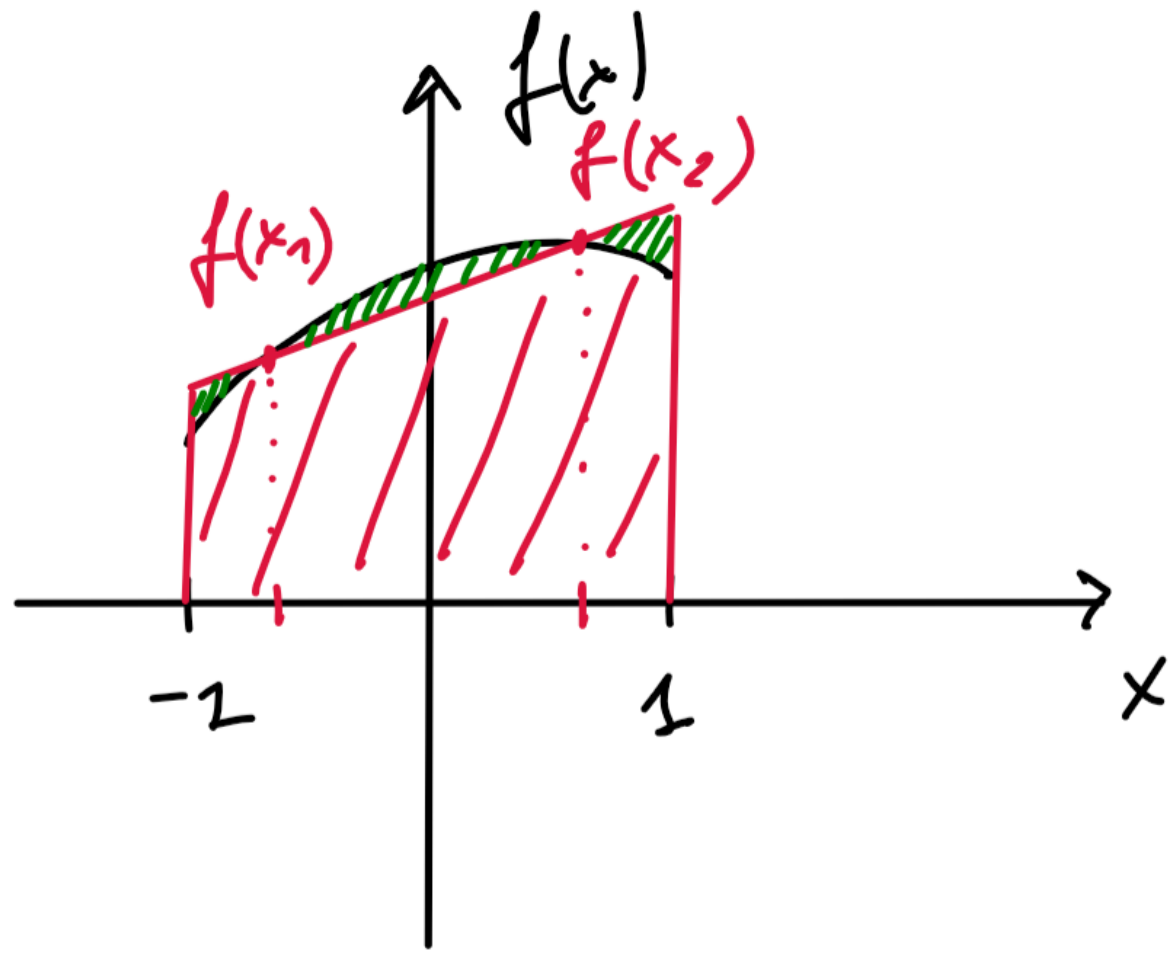
$$x_2 = \mp 1/\sqrt{3}$$

$$x_2 > x_1 \Rightarrow \boxed{\begin{aligned} x_1 &= -1/\sqrt{3} \\ x_2 &= 1/\sqrt{3} \end{aligned}}$$

TRAPEZOIDAL RULE



GAUSS $n = 2$



$$\int_{-1}^{1} f(x)\,dx = \frac{\overset{=2}{\Delta}}{2}\left[f(a)+f(b)\right] = f(a)+f(b)$$

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) = 1 \cdot f\left(-1/\sqrt{3}\right) + 1\, f\left(1/\sqrt{3}\right)$$

ADVANTAGE : better accuracy with same # of function evaluations

DISADVANTAGE : $f(x)$ must be known analytically / numerically

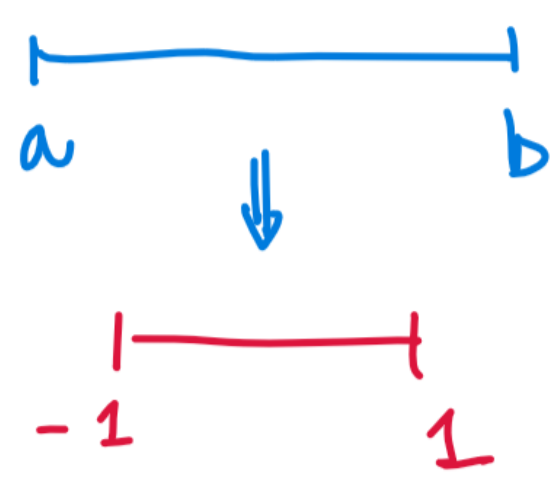
$n = 3 \qquad m = 2\overset{3}{n} - 1 = 5$

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

$$\begin{cases} w_1 = 5/9 \\ w_2 = 8/9 \\ w_3 = 5/9 \end{cases} \begin{matrix} \longleftrightarrow \\ \longleftrightarrow \\ \longleftrightarrow \end{matrix} \begin{cases} x_1 = -\sqrt{3/5} \\ x_2 = 0 \\ x_3 = +\sqrt{3/5} \end{cases}$$

GAUSS integration for ARBITRARY intervals

$$\int_a^b f(x)\,dx \Longrightarrow \int_{-1}^{1} f(\tau)\,d\tau = \sum_{i=1}^{m} w_i f(\tau_i)$$



$x \in [a,b] \dashrightarrow \tau \in [-1,1]$

find a MAP between x and τ $\Longrightarrow$ linear transformation

$$\begin{cases} x = m\tau + q & \\ a = m(-1) + q & \text{in } \tau = -1 \to x = a \quad (1) \\ b = m(1) + q & \text{in } \tau = 1 \to x = b \quad (2) \end{cases}$$

(2) − (1) $b - a = m + \cancel{q} - (-m + \cancel{q})$

$$\Rightarrow \boxed{m = \frac{b-a}{2}}$$

(1) : $a = -m + q = -\frac{b-a}{2} + q$

$$a + \frac{b-a}{2} = q \quad \to \quad \frac{2a + b - a}{2} = q \quad \to \quad \boxed{q = \frac{a+b}{2}}$$

Linear transf. :

$$\boxed{x = \underset{\text{SCALING}}{\underset{\Uparrow}{\frac{b-a}{2}}}\tau + \underset{\text{DISPLACEMENT}}{\underset{\uparrow}{\frac{a+b}{2}}}}$$

change of differential

$$\boxed{dx = \frac{b-a}{2}\, d\tau}$$

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{b-a}{2}\tau + \frac{a+b}{2}\right)\frac{b-a}{2}\,d\tau$$

$$= \underset{m}{\underset{\uparrow}{\frac{b-a}{2}}} \int_{-1}^{1} f\left(\underset{m}{\underset{\uparrow}{\frac{b-a}{2}}}\tau + \underset{q}{\underset{\uparrow}{\frac{a+b}{2}}}\right) d\tau = \sum_{i=1}^{n} w_i\, f\left(\frac{b-a}{2}\tau_i + \frac{a+b}{2}\right)$$

(1) ROUTINE for GAUSS integration $n = 1, 2, 3$ on $[a, b]$

(2) ROUTINE for integration on subdomains → $\boxed{\text{ADAPTIVE}}$

PSEUDO - CODE

$$[int] = \text{INT\_ADT}(a, b, f, n, tol)$$

\# GAUSS points → n

TOLERANCE → tol

$$I_1 = \int_a^b f(x)\,dx \longrightarrow \text{int\_GAUSS}(a, b, f, n)$$

$$\boxed{mp = \frac{a+b}{2}}$$

$$I_2 = \int_a^{mp} f(x)\,dx + \int_{mp}^{b} f(x)\,dx$$

$$err = \left| \frac{I_1 - I_2}{I_2} \right|$$

if $err < tol$ ← 10^{-6}

  $int = I_2$

else

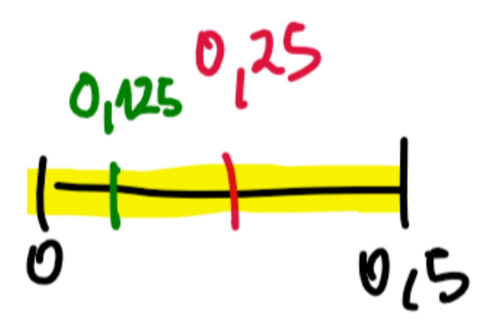
  RECURSIVITY

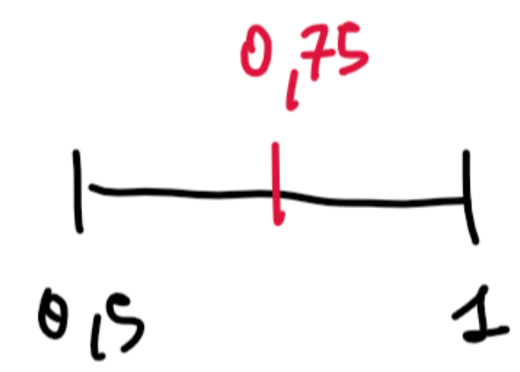
  $I_L = \text{int\_ADT}(a, mp, f, n, tol)$

  $I_R = \text{int\_ADT}(mp, b, f, n, tol)$

  $int = I_L + I_R$

end





end