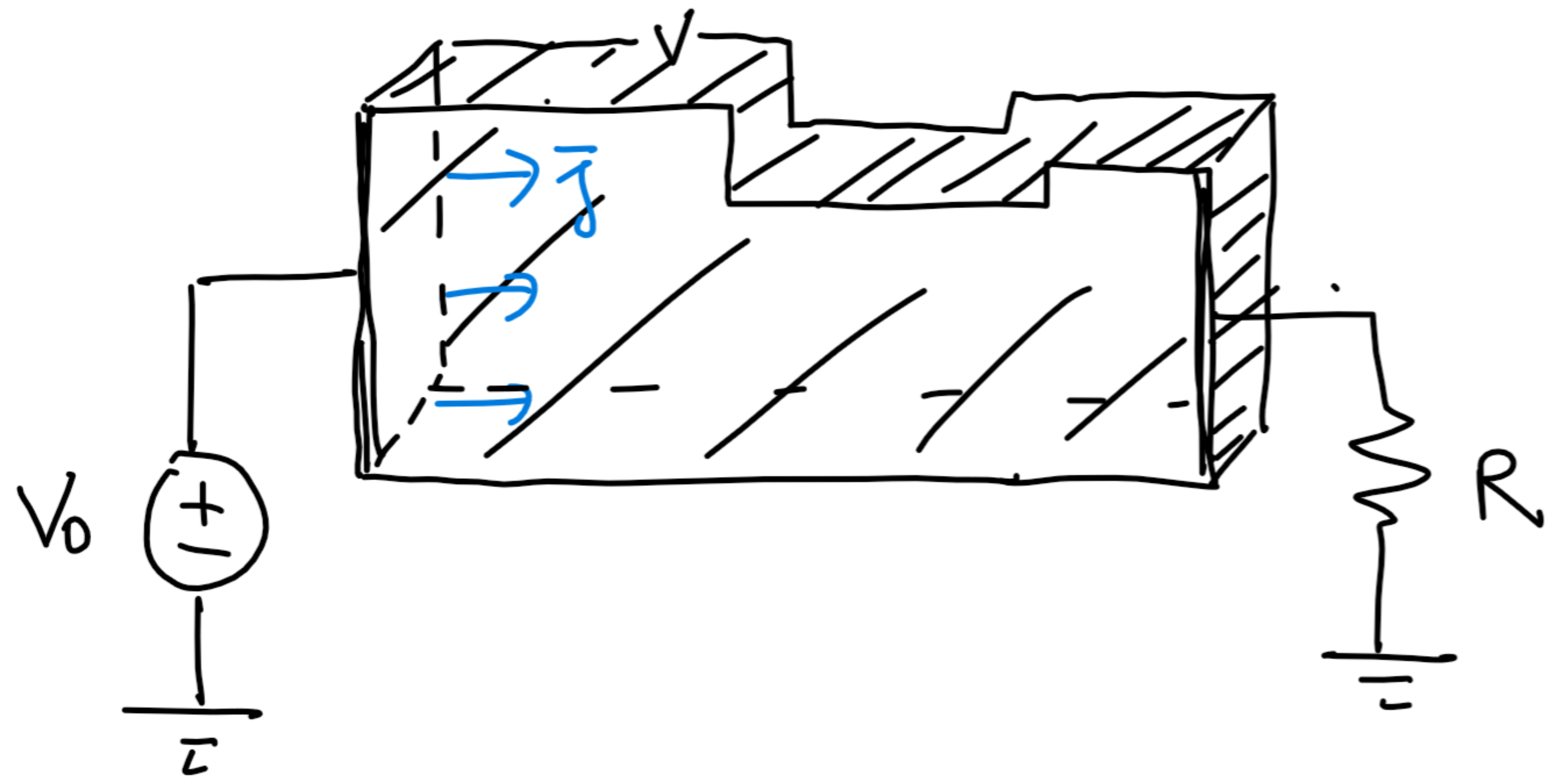
PROBLEM UNIFORMLY CONDUCTIVE ($\sigma = \sigma_0$) PLATE (STEADY-STATE)

FIND: $\bar{J}$ on the plate

$\partial/\partial t = 0$



- STEADY-STATE CURRENT (DC conduction problem)

SCALAR ELECTRIC POTENTIAL

$$\begin{cases} \nabla \times \bar{E} = -\underbrace{\cancel{\partial \bar{B}/\partial t}}_{\text{steady state}} = 0 \Rightarrow \bar{E} \text{ is curl-free} \rightarrow \bar{E} = -\nabla \varphi \\ \nabla \cdot \left(\bar{J} + \cancel{\partial \bar{D}/\partial t}\right) = \nabla \cdot \bar{J} = 0 \Rightarrow \bar{J} \text{ is div-free} \rightarrow \nabla \cdot (\sigma \bar{E}) = 0 \end{cases}$$

CONSTITUTIVE REL. $\bar{J} = \sigma \bar{E}$

$\downarrow \bar{E} = -\nabla \varphi$

$$\nabla \cdot (-\sigma \nabla \varphi) = 0$$

σ UNIFORM

$$-\sigma \nabla \cdot (\nabla \varphi) = 0$$

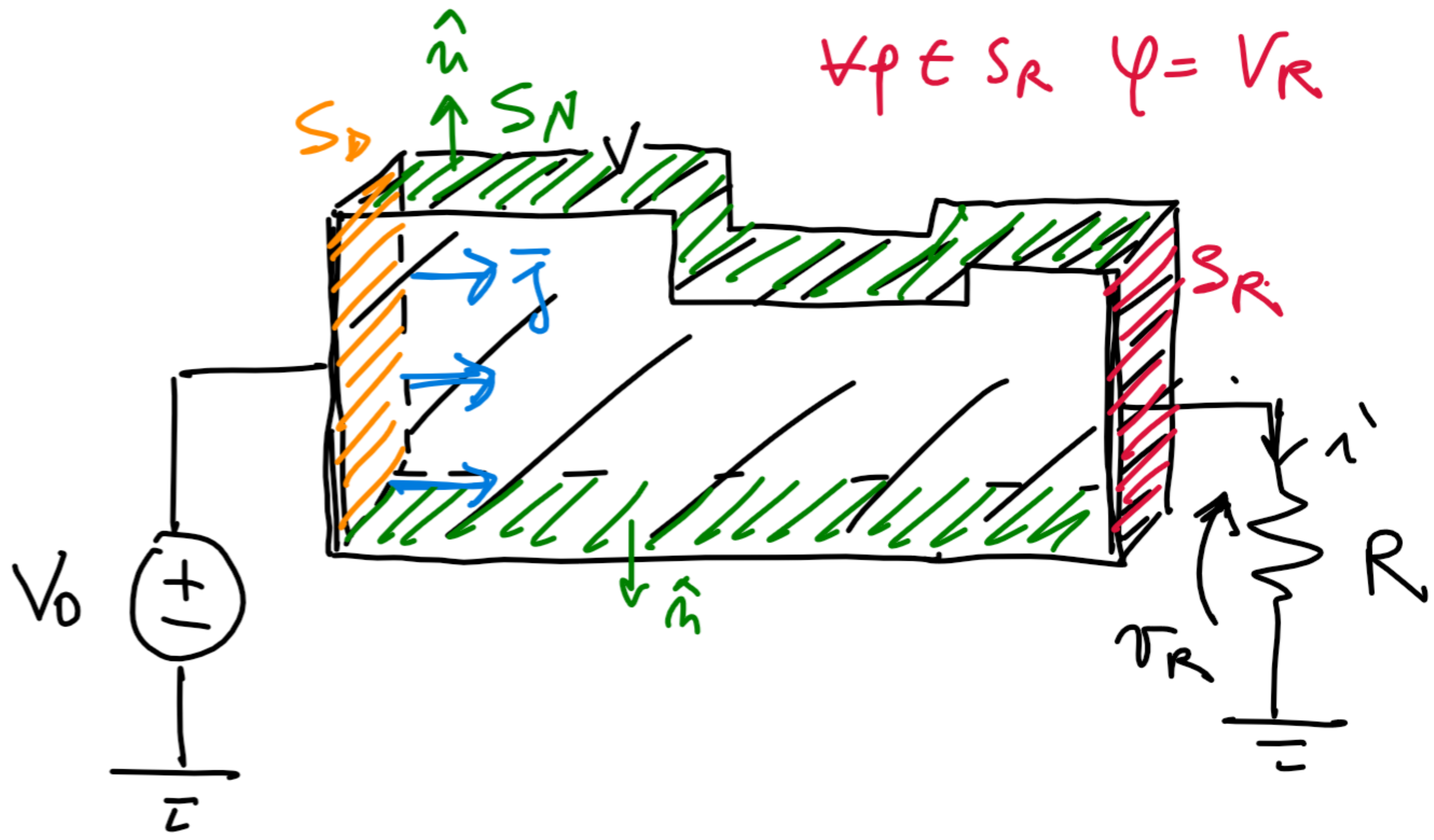
$$\boxed{-\cancel{\sigma} \nabla^2 \varphi = 0}$$

- φ is HARMONIC
- If σ UNIFORM the SPACE DISTRIBUTION of φ is independent of σ

... FORMULATION

$$\begin{array}{ll} V & \nabla^2 \varphi = 0 \quad \forall \bar{P} \in V \\ S_D & \varphi = \varphi_0 \quad \forall \bar{P} \in S_D \\ S_N & \partial \varphi / \partial n = 0 \quad \forall \bar{P} \in S_N \\ S_R & \varphi = v_R = R\, i \end{array}$$

$\forall \bar{P} \in S_R \quad \varphi = V_R$



$\forall \bar{P} \in S_N: \bar{J} \cdot \hat{n} = J_n = 0$

$\bar{J} \cdot \hat{n} = \sigma \bar{E} \cdot \hat{n}$

$J_n = -\sigma \nabla \varphi \cdot \hat{n}$

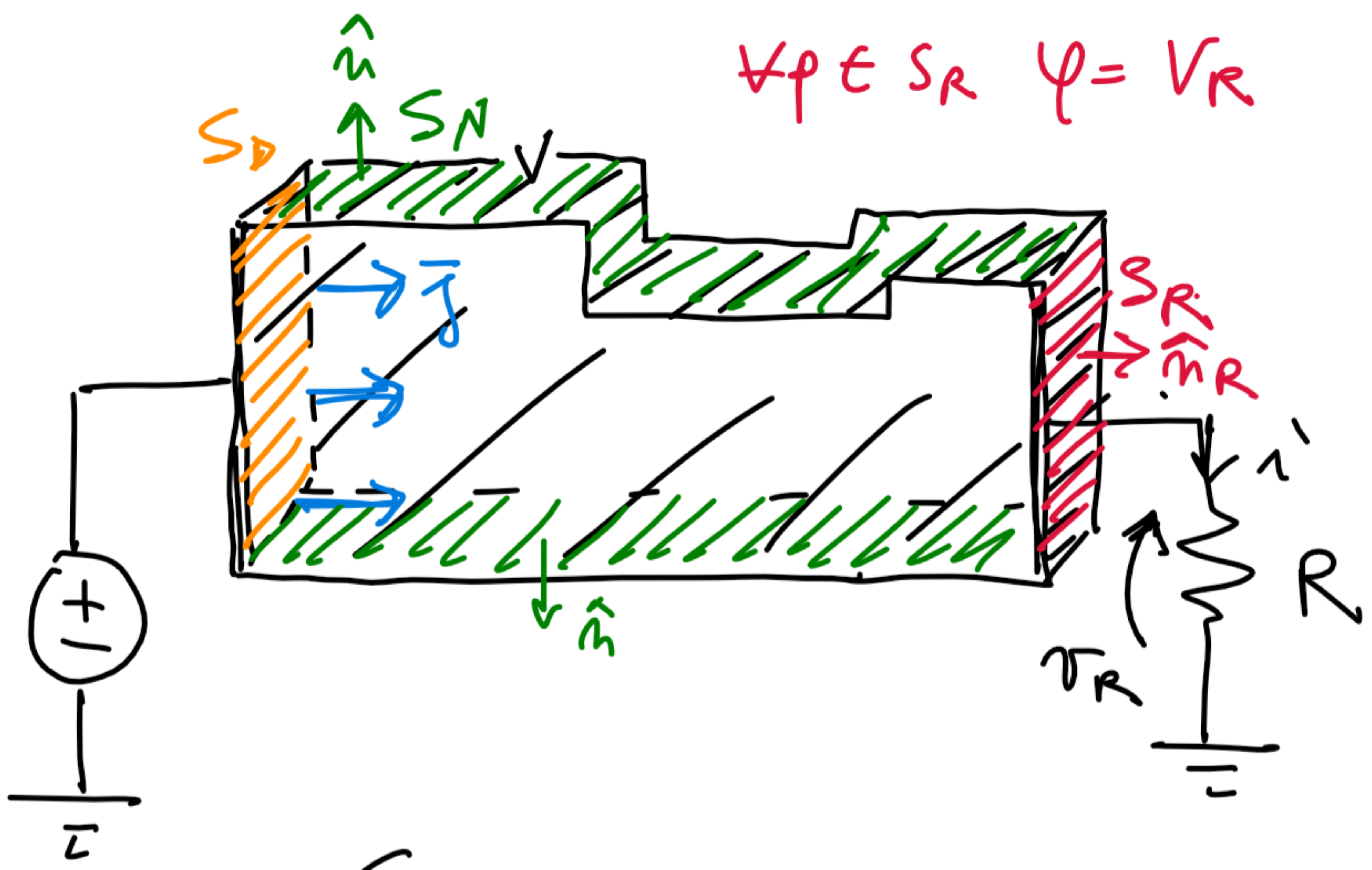
$$\boxed{J_n = -\sigma \, \partial \varphi / \partial n}$$

WE NEED NEUMANN BC

GOAL, express $\varphi = Ri$ as a function of φ

$$i = \int_{S_R} \bar{J} \cdot d\bar{S}$$

$\forall p \in S_R \quad \varphi = V_R$



$$\varphi = R \int_{S_R} \bar{J} \cdot d\bar{S} \qquad \sigma\bar{E} = -\sigma\nabla\varphi$$

$$= R \int -\sigma \nabla\varphi \cdot d\bar{S} \qquad d\bar{S} = \hat{n}_R \, dS$$

$$= -R\sigma \int_{S_R} \partial\varphi/\partial n \, dS = -R\sigma\, \partial\varphi/\partial n \underbrace{\int_{S_R} dS}_{S_R} = -R\sigma\, \partial\varphi/\partial n \, S_R$$

DC conduction

$\bar{J}$ is uniform $\Rightarrow$ $\partial\varphi/\partial t$ uniform

$\rightarrow$ RECAP: $\forall \bar{p} \in S_R$, $$\boxed{\varphi + R\sigma S_R \, \partial\varphi/\partial n = 0}$$

ROBIN BOUNDARY CONDITION

{ MIXED BCs, "IMPEDANCE BCs" }

FORMULATION

$$\begin{cases} V & \nabla^2 \varphi = 0 \\ S_D & \varphi = \varphi_0 \\ S_N & \partial\varphi/\partial n = 0 \\ S_R & 1\cdot\varphi + \overbrace{R\sigma S_R}^{\beta} \, \partial\varphi/\partial n = 0 \end{cases}$$

GENERAL FORM: $$\boxed{\alpha\varphi + \beta \, \partial\varphi/\partial n = \gamma}$$

if $\alpha = 0 \Rightarrow$ NEUMANN BC

if $\beta = 0 \Rightarrow$ DIRICHLET BC

UNIQUENESS THEOREM – POISSON PROBLEMS W/ ROBIN BCs

$$\begin{cases} \nabla^2 \varphi = t, & \forall \bar{P} \in V \\ \varphi = \varphi_0, & \forall \bar{P} \in S_D \\ \alpha\varphi + \beta \, \partial\varphi/\partial n = \gamma, & \forall \bar{P} \in S_R \end{cases}$$

$S = S_D \cup S_R$

ASSUME: 2 valid solutions φ_1, φ_2

$\Rightarrow$ DIFFERENCE FIELD $\quad \varphi_3 = \varphi_1 - \varphi_2$

φ_3 must obey formulation

$$\begin{array}{l} V \\ S_D \\ S_R \end{array}\begin{cases} \nabla^2 \varphi_3 = \nabla^2(\varphi_1 - \varphi_2) = t - t = 0 & \varphi_3 \text{ is HARM.} \\ \varphi_3 = \varphi_1 - \varphi_2 = \varphi_0 - \varphi_0 = 0 \\ \alpha\varphi_3 + \beta \cdot \partial\varphi_3/\partial n = 0 \end{cases}$$

$$\longrightarrow \begin{cases} \alpha\varphi_1 + \beta \, \partial\varphi_1/\partial n = \gamma & (1) \\ \alpha\varphi_2 + \beta \, \partial\varphi_2/\partial n = \gamma & (2) \end{cases}$$

$$(1)-(2) \quad \alpha\underbrace{(\varphi_1 - \varphi_2)}_{\varphi_3} + \beta\underbrace{\left(\frac{\partial\varphi_1}{\partial n} - \frac{\partial\varphi_2}{\partial n}\right)}_{\partial\varphi_3/\partial n} = 0$$

Ist GREEN IDENTITY

$$\oint_S \psi\nabla\varphi \cdot d\bar{S} = \int_V (\nabla\psi \cdot \nabla\varphi + \psi\nabla^2\varphi)\, dV$$

take $\psi = \varphi = \varphi_3$

$$\oint_S \varphi_3 \nabla\varphi_3 \cdot d\bar{S} = \int_V \Big[(\nabla\varphi_3)^2 + \cancel{\varphi_3 \nabla^2 \varphi_3}^{\,=0 \ \varphi_3 \text{ HARMONIC}}\Big] dV$$

$\downarrow$

$$\cancel{\int_{S_D} \varphi_3 \nabla\varphi_3 \cdot d\bar{S}}^{\,0} + \int_{S_R} \varphi_3 \nabla\varphi_3 \cdot d\bar{S} = \int_V (\nabla\varphi_3)^2 \, dV$$

$\uparrow \varphi_3 = 0 \quad \forall \bar{P} \; S_D$

$$\alpha\varphi_3 + \beta\,\partial\varphi_3/\partial n = 0$$

$$\boxed{\partial\varphi_3/\partial n = -\frac{\alpha}{\beta}\varphi_3}$$

$$\int_{S_R} \varphi_3 \boxed{\partial\varphi_3/\partial n}\, dS = \int_V (\nabla\varphi_3)^2 \, dV$$

$$\Rightarrow \int_{S_R} -\alpha/\beta \, \varphi_3^2 \, dS = \int_V (\nabla\varphi_3)^2 \, dV$$

$$\underbrace{\int_{S_R} -\alpha/\beta \underbrace{\varphi_3^2}_{\geq 0} \, dS}_{\leq 0} = \underbrace{\int_V (\nabla \varphi_3)^2 \, dV}_{\geq 0, \ \forall \varphi_3}$$

if $\alpha/\beta > 0 \Rightarrow \leq 0$

if $\alpha/\beta > 0$ the only value that satisfies the expression is $\boxed{\varphi_3 = 0}$

$\Downarrow$

$\varphi_3 = \varphi_1 - \varphi_2 \Rightarrow \varphi_1$ and φ_2 ARE THE SAME SOLUTION

$\Downarrow$

UNIQUE!

$\downarrow$

DC conduction problem

$\alpha = 1$

$\beta = R \, \delta \, S_R$ $\rightarrow \alpha/\beta > 0$

if $\alpha/\beta < 0 \Rightarrow$ CAN satisfy expression with φ_3

$\Rightarrow$ SOLUTION is NOT UNIQUE