Lagrange polynomials $\mathcal{L}(x)$

$$\tilde{f}(x) = f_1 \mathcal{L}_1(x) + f_2 \mathcal{L}_2(x) + \dots + f_k \mathcal{L}_k(x) + \dots + f_n \mathcal{L}_n(x)$$

KNOWN VALUES of $f(x)$

$$\mathcal{L}_k(x) = \frac{\prod_{i=1, i\neq k}^{n} (x - x_i)}{\prod_{i=1, i\neq k}^{n} (x_k - x_i)}$$

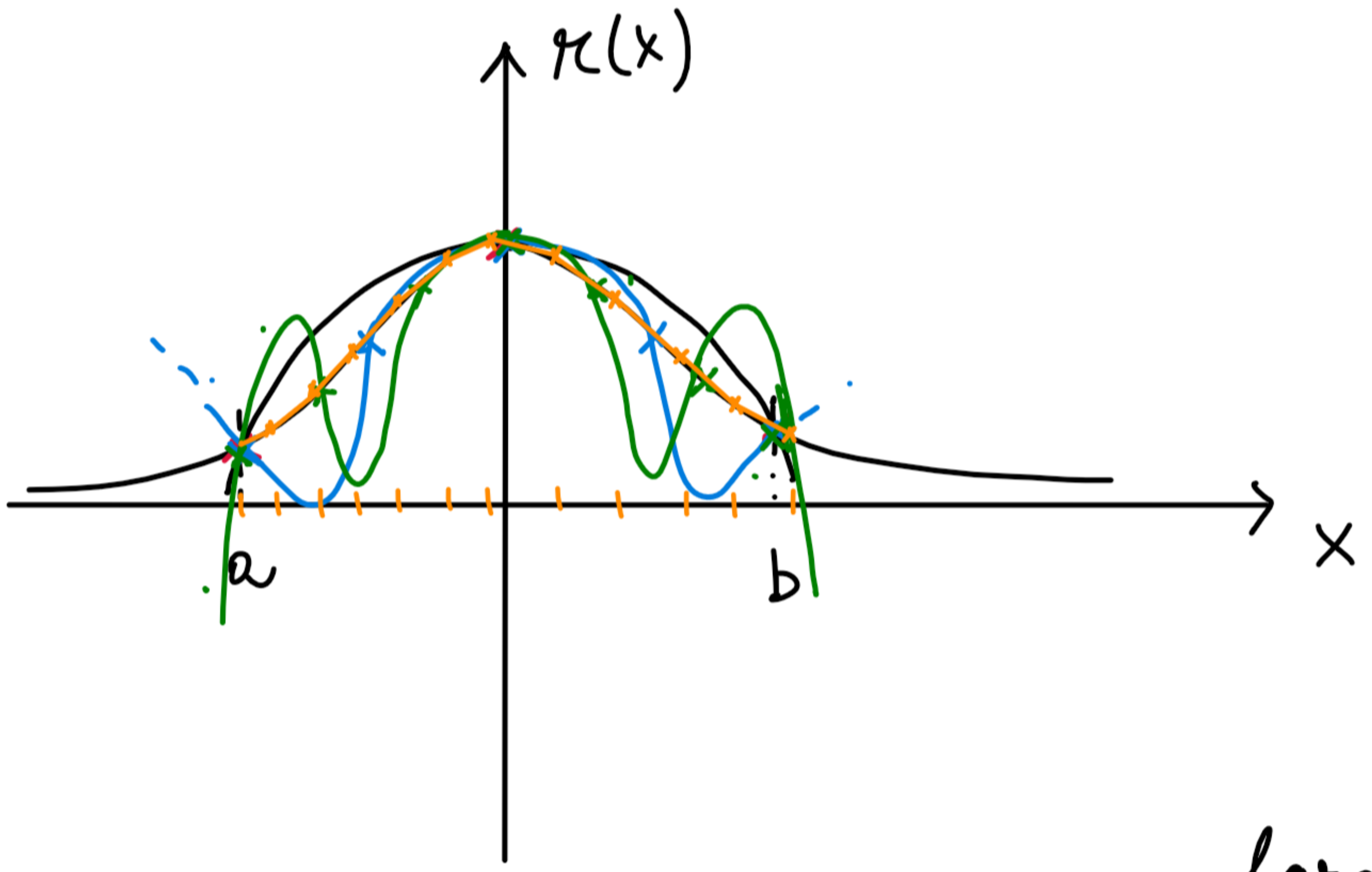
$n-1$ degree polynomials

$\mathcal{L}_k(x_k) = 1$

$\mathcal{L}_k(x_{i\neq k}) = 0$

Fulfill the conditions of the VANDERMOND SYSTEM

Example: RUNGE function $r(x) = \dfrac{1}{1 + \alpha x^2}$ $\qquad \alpha = 25$



# points		DEGREE
$np = 3$	$\rightarrow$	$n = 2$
$np = 5$	$\rightarrow$	$n = 4$
$np = 7$	$\rightarrow$	$n = 6$

Higher polynomial degrees $\Downarrow$ larger oscillations near the interval boundaries

Piecewise Linear Interpolation (1D)

IDEA: construct interpolating function $\tilde{f}(x)$ in $[a, b]$ from a set PIECEWISE LINEAR POLYNOMIALS

- each polynomial is defined on a $\underbrace{\text{SUBSET of } [a,b]}$

introduce n-points [NODES]

$n-1$ sub-intervals

$$\tilde{f}(x) = f_1 L_1(x) + f_2 L_2(x) + \dots + f_K L_K(x) + \dots + f_m L_m(x)$$

$\downarrow$

BASIS FUNCTIONS

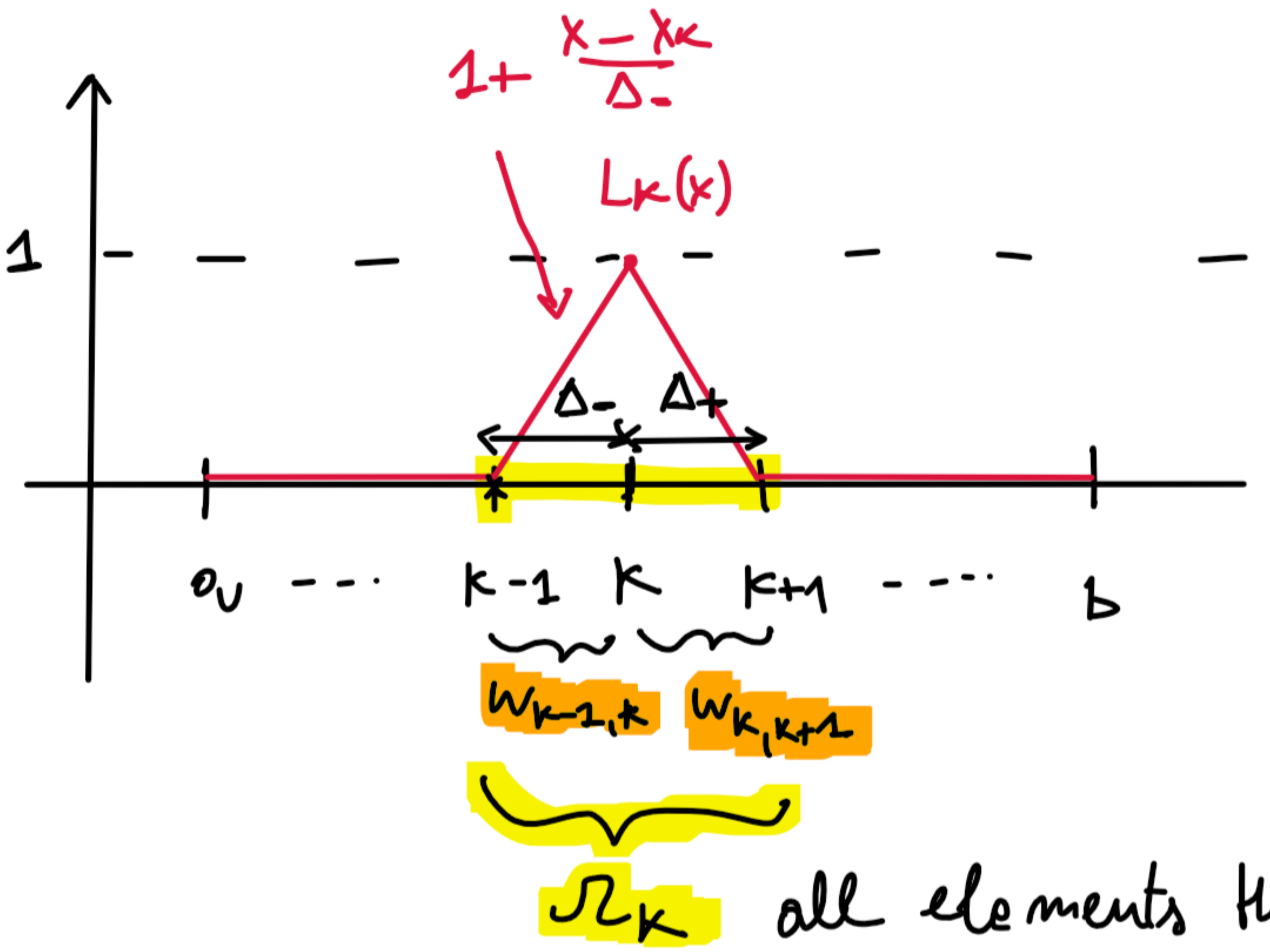
HAT FUNCTIONS

REQUIREMENTS

$$L_K(x) = \begin{cases} 1, & x = x_K \\ 0, & x \notin \Omega_K \end{cases} \quad + \text{ LINEARITY}$$

$\uparrow$ SUPPORT DOMAIN of node K

$$\Omega_K = \underset{\uparrow \text{ ELEMENT}}{W_{K-1,K}} \cup W_{K,K+1}$$



all elements that share node K

$$L_K(x) = \begin{cases} 1 + \frac{x - x_K}{\Delta_-}, & x \in W_{K-1,K} \\ 1 - \frac{x - x_K}{\Delta_+}, & x \in W_{K,K+1} \\ 0, & x \notin \Omega_K \end{cases}$$

$$W_{K-1,K} = [x_{K-1}, x_K]$$

$$W_{K,K+1} = [x_K, x_{K+1}]$$

$$\Delta_- = x_K - x_{K-1}$$

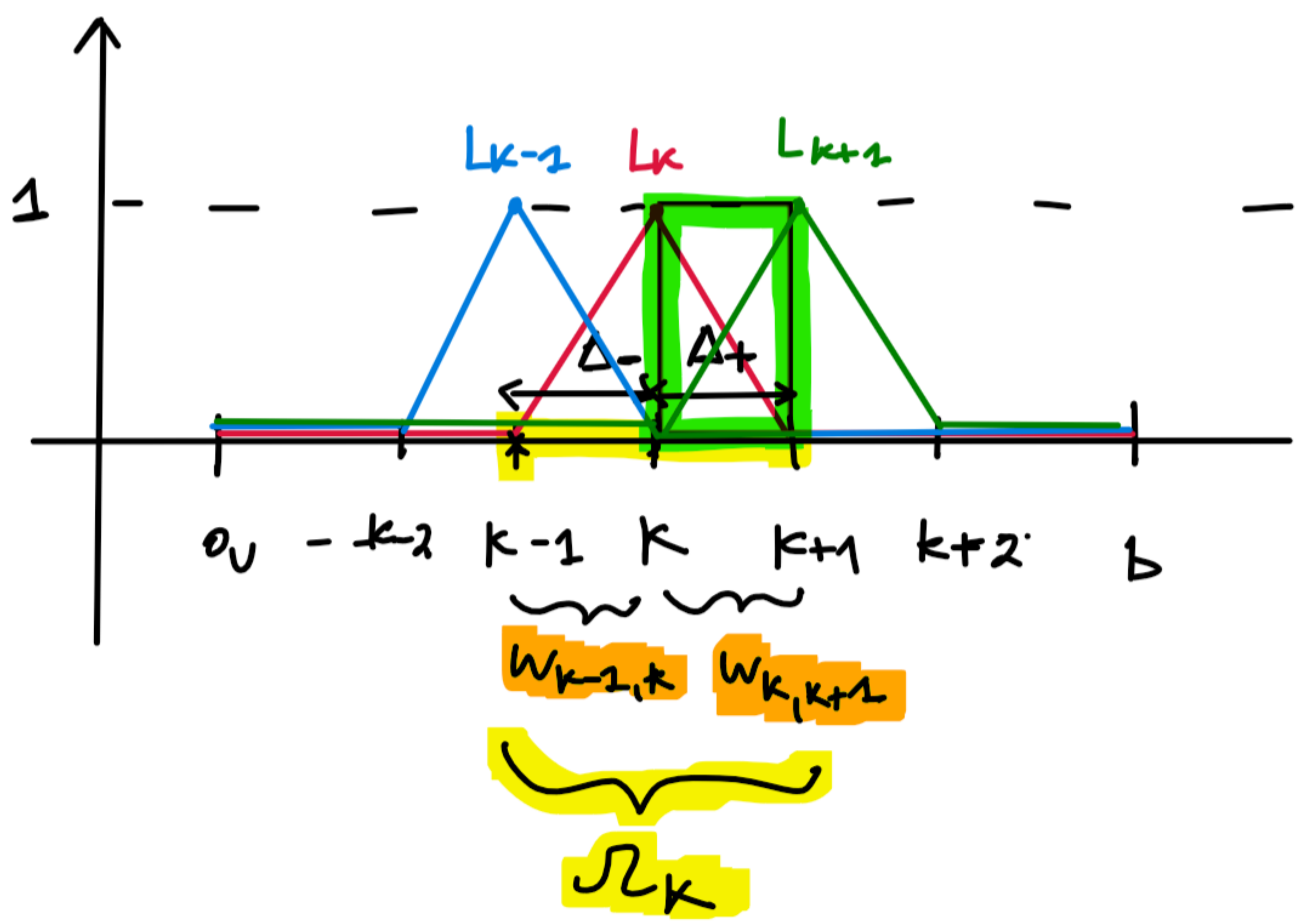
$$\Delta_+ = x_{K+1} - x_K$$

CHECK! :

in $x = x_{K-1} \rightarrow 1 + \frac{\overbrace{x_{K-1} - x_K}^{= -\Delta_-}}{\Delta_-} = 1 - 1 = 0$ ✓

in $x = x_K \rightarrow 1 + \cancel{\frac{x_K - x_K}{\Delta_-}}^{0} = 1$ ✓

DRAW $L_{K-1}(x)$, $L_{K+1}(x)$:



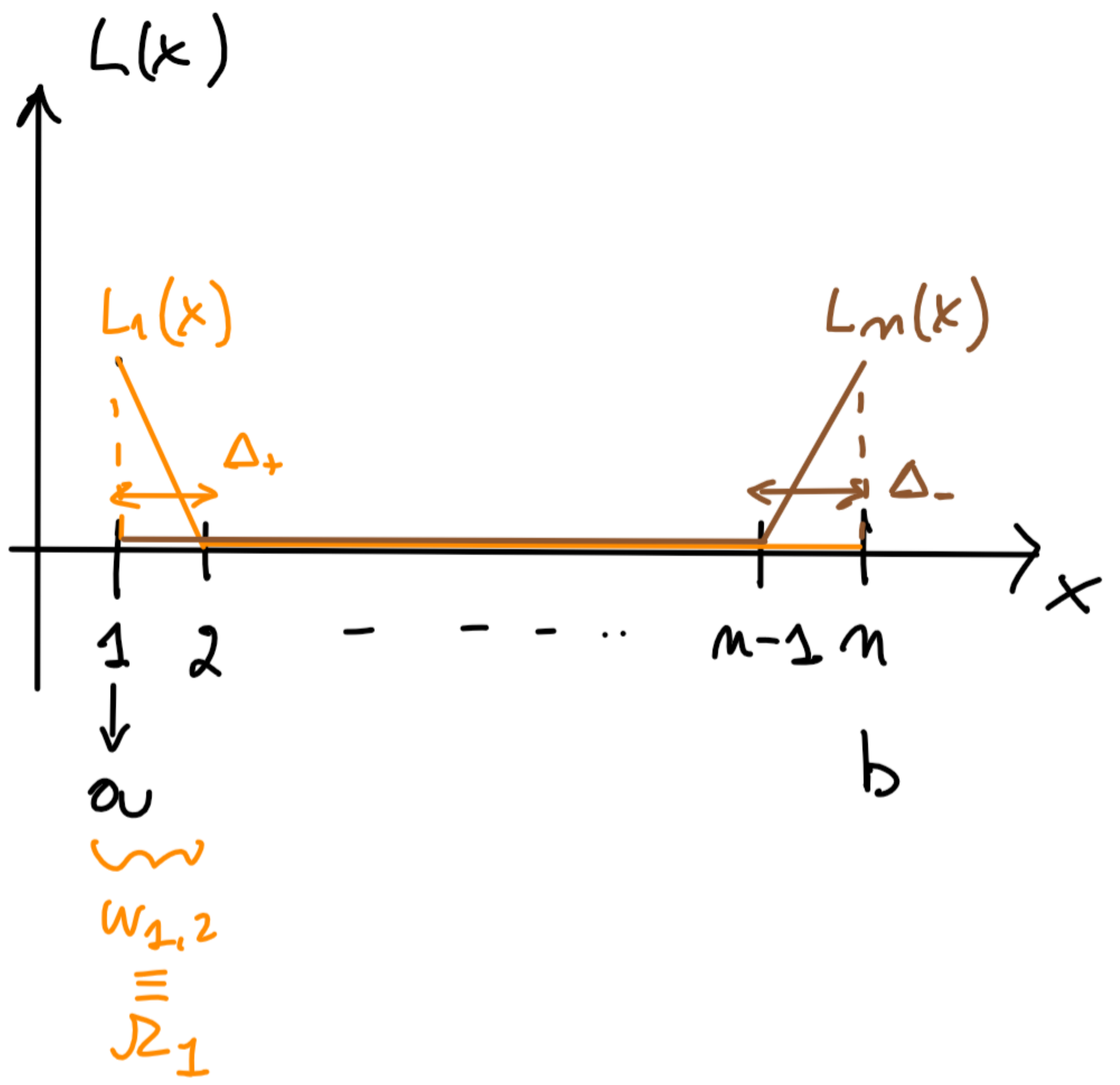
$\rightarrow$ FOR each element w, only TWO HAT FUNCTIONS are $\neq 0$

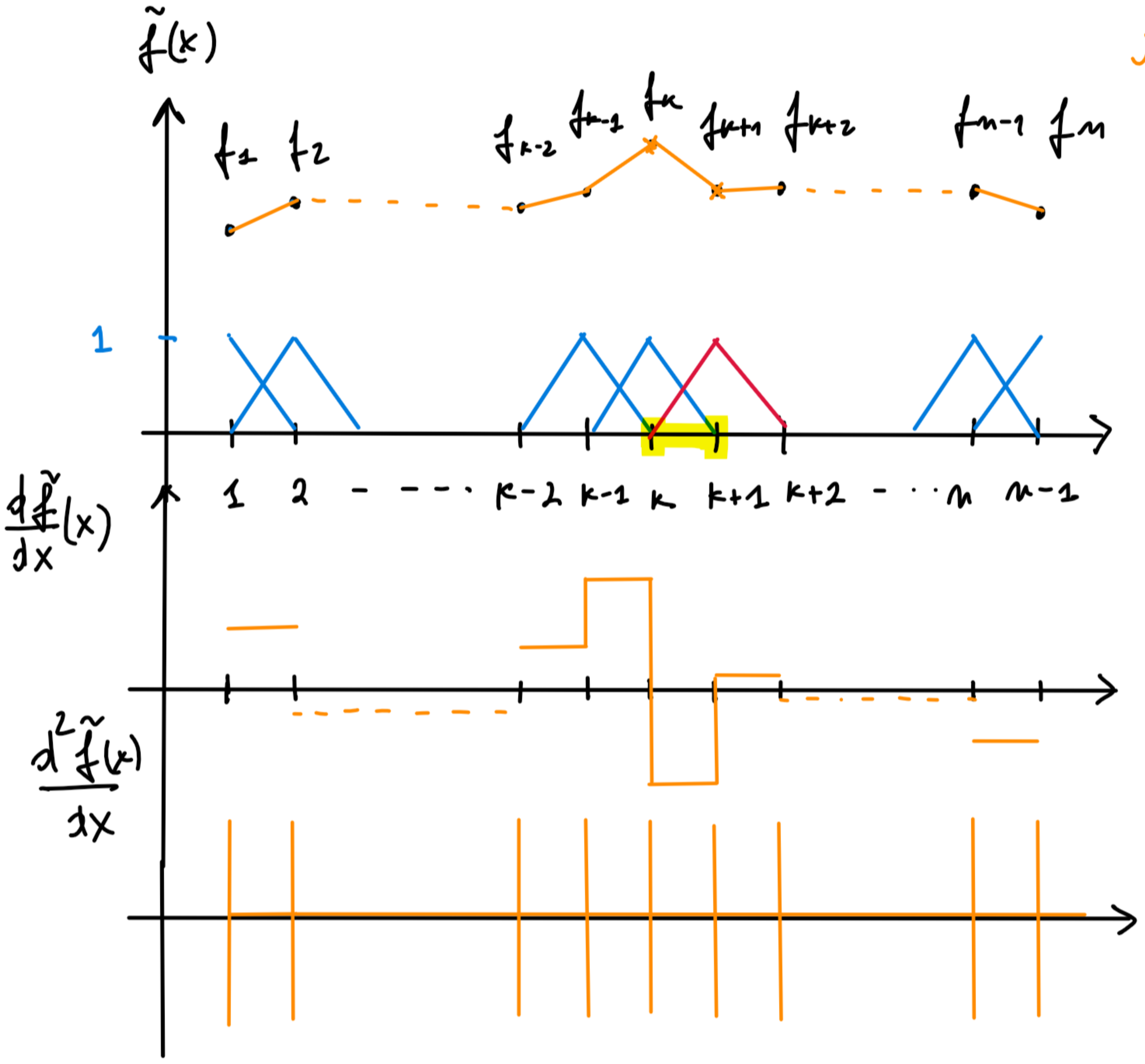
$$W_{K,K+1} \rightarrow \begin{cases} L_K(x) \neq 0 \\ L_{K+1}(x) \neq 0 \end{cases}$$

BOUNDARY NODES

$$L_1(x) = \begin{cases} 1 - \dfrac{x - x_1}{\Delta_+} & \nearrow \dfrac{dL_1}{dx} = -1/\Delta_+ \\ 0, & x \notin \Omega_1 \equiv w_{1,2} \end{cases}$$

$$L_n(x) = \begin{cases} 1 + \dfrac{x - x_n}{\Delta_-} & \nearrow \dfrac{dL_n}{dx} = 1/\Delta_- \\ 0, & x \notin \Omega_n \equiv w_{n-1,n} \end{cases}$$





for $x \in w_{k,k+1}$

$\tilde{f}(x) = f_k L_k(x) + f_{k+1} L_{k+1}(x)$

$\downarrow$

PIECEWISE LINEAR

$\frac{d\tilde{f}}{dx}$: PIECEWISE CONSTANT but DISCONTINUOUS

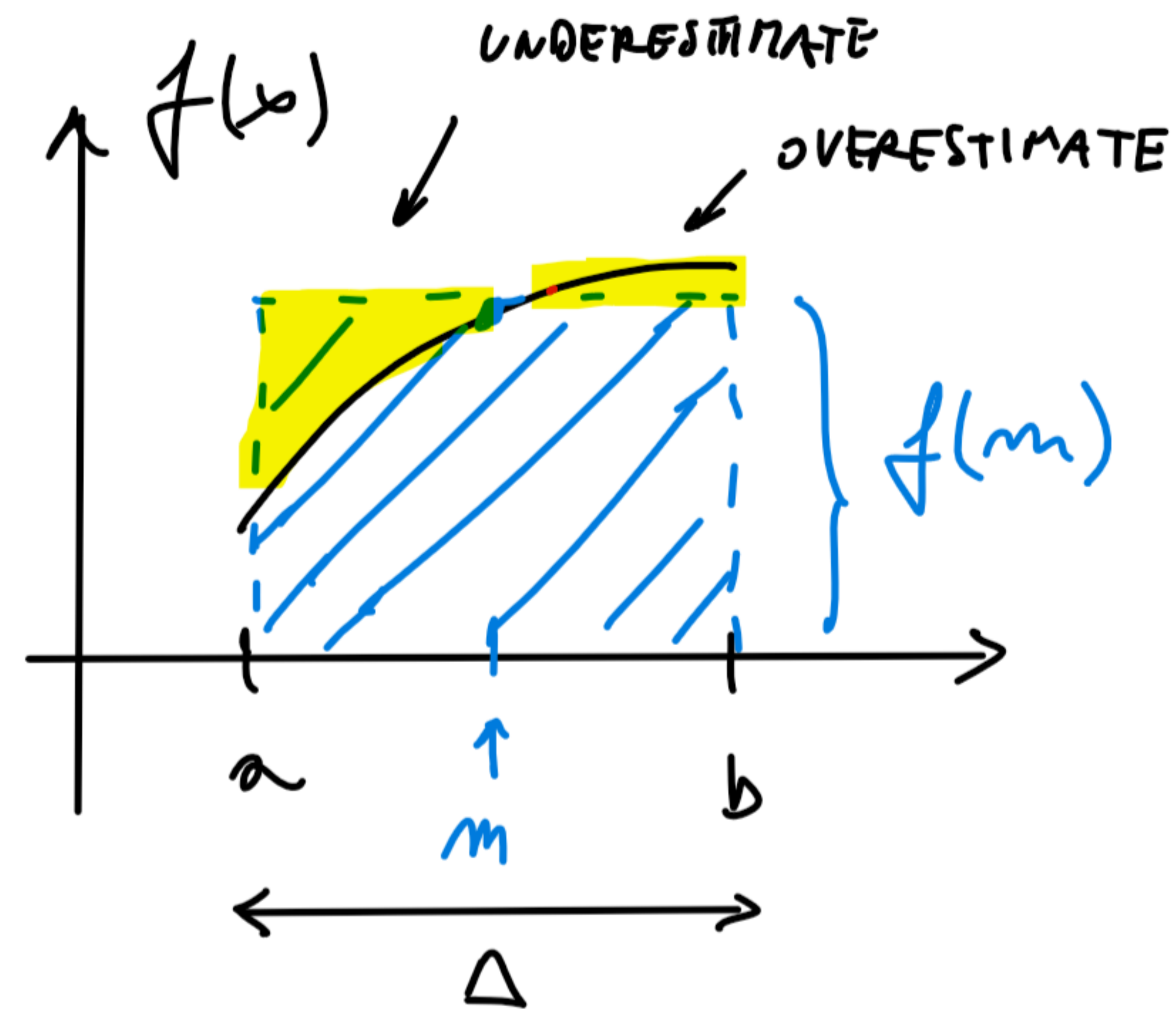
$\frac{d^2\tilde{f}(x)}{dx^2}$: not defined at discontinuity points of $\frac{d\tilde{f}}{dx}$

NUMERICAL INTEGRATION

GOAL: $\int_a^b f(x)\,dx$ numerically

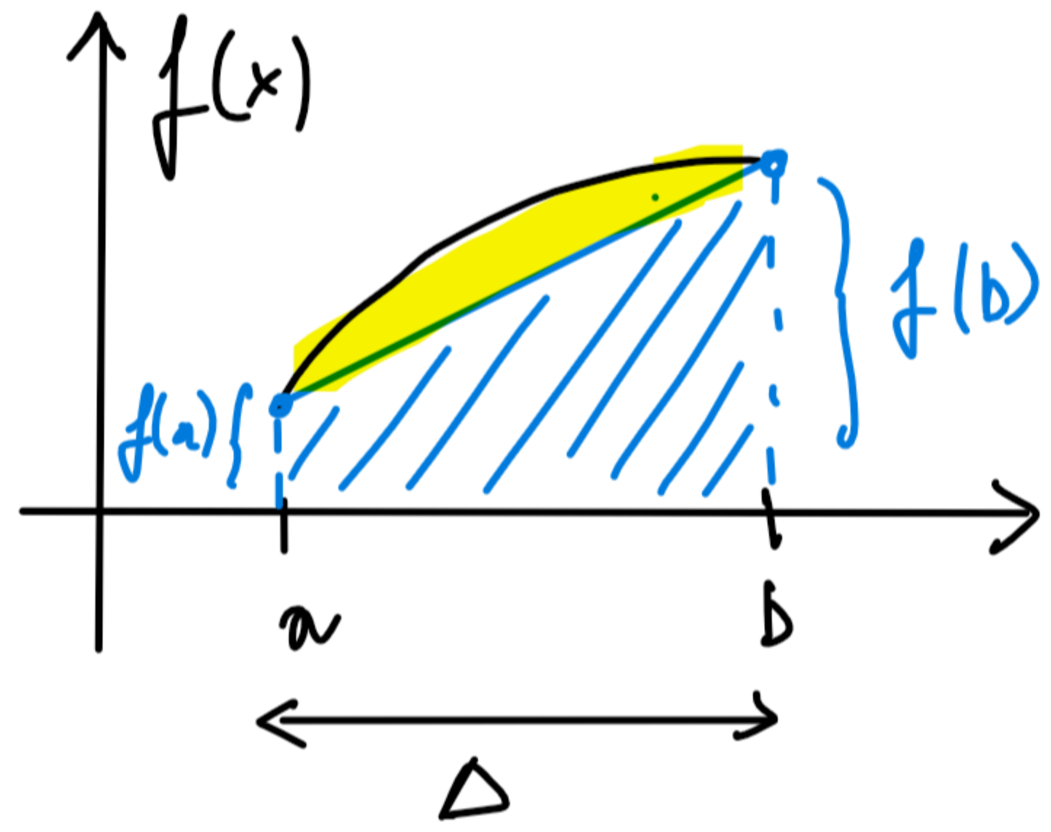
RECTANGLES METHOD (midpoint method)

$$\int_a^b f(x)\,dx \simeq \underset{\substack{\uparrow \\ = \frac{a+b}{2}}}{f(m)} \cdot \underset{\substack{\nwarrow \\ b-a}}{\Delta} + \mathcal{O}(\Delta^3)$$



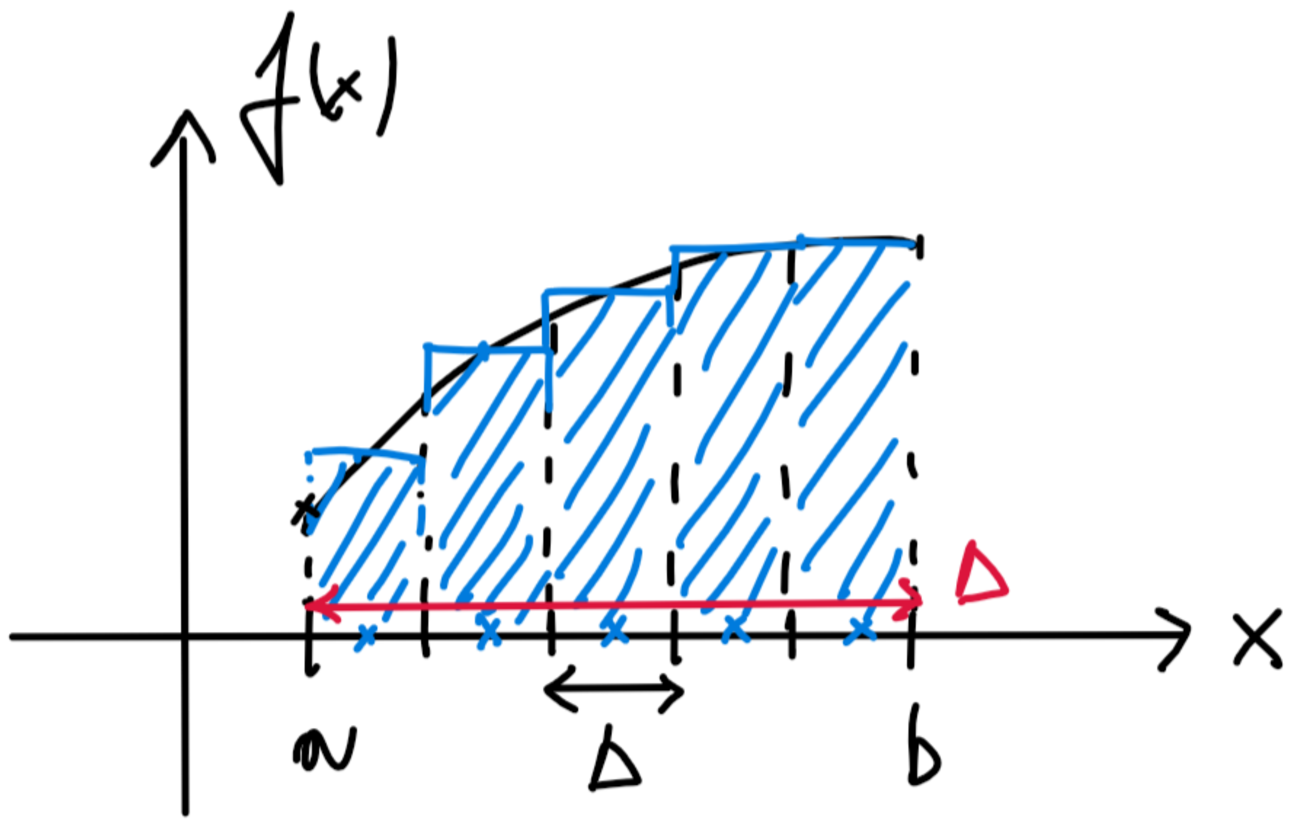
TRAPEZOIDAL RULE

$$\int_a^b f(x)\,dx \simeq \frac{\Delta}{2}\left[f(a)+f(b)\right] + \mathcal{O}(\Delta^3)$$



for more accuracy:

$[a,b] \rightarrow$ n nodes ; $n-1$ sub-intervals [ELEMENTS] ; $\Delta = \dfrac{b-a}{n-1}$



TRUNCATION ERROR

GLOBAL ERROR $\neq$ LOCAL ERROR

$\mathcal{O}(\Delta^3)$ on a single element

Rectangles method: $\displaystyle\int_a^b f(x)\,dx \simeq \sum_{i=1}^{n-1} f\left(\frac{x_{i+1}+x_i}{2}\right)\Delta + \mathcal{O}(\Delta^2)$

TRAPEZOIDAL RULE: $\displaystyle\int_a^b f(x)\,dx \simeq \sum_{i=1}^{n-1} \frac{\Delta}{2}\left[f(x_i)+f(x_{i+1})\right] + \mathcal{O}(\Delta^2)$

$n \propto \dfrac{1}{\Delta}$

$\Delta \propto \dfrac{1}{n} \rightarrow$ sum n-contributions $\rightarrow \dfrac{\mathcal{O}(\Delta^3)}{\Delta} = \mathcal{O}(\Delta^2)$

GAUSS INTEGRATION (GAUSS QUADRATURE) [1D]

IDEA: compute numerical integral as a WEIGHTED SUM of $f(x)$ evaluated at DIFFERENT POINTS

$$\int_{-1}^{1} f(x)\,dx \simeq \sum_{i=1}^{n} w_i\, f(x_i)$$

n ← n GAUSS POINTS

w_i ← i-th WEIGHT

x_i ← i-th GAUSS POINT

ORDER of ACCURACY: $\boxed{m = 2n - 1}$

For $n = 2$ → $m = 2\cdot 2 - 1 = 3$ → INTEGRATE EXACTLY polynomial function up to degree 3

Derivation of QUADRATURE RULES

$n = 1$ $\quad \int_{-1}^{1} f(x)\,dx = w_1 f(x_1)$ $\quad m = 2\cdot 1 - 1 = 1$

REQUIRE: exact integration of polynomials order 0, 1

deg 0: $f(x) = 1 \longrightarrow \int_{-1}^{1} 1\,dx = [x]_{-1}^{1} = 2 \Rightarrow w_1 \underbrace{f(x_1)}_{1} = 2$ $\Rightarrow w_1 \cdot 1 = 2$

deg 1: $f(x) = x \longrightarrow \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0 \Rightarrow w_1 \underbrace{f(x_1)}_{x} = 0$ $\Rightarrow w_1 x = 0$

0: $\boxed{w_1 = 2}$

1: $\underset{\uparrow\, = 2}{w_1} x = 0 \Rightarrow \boxed{x = 0}$

$$\Rightarrow \int_{-1}^{1} f(x)\,dx = 2\, f(0)$$

2 ← $\Delta = 1-(-1)$

RECTANGLES METHOD

$n = 2$

$$\int_{-1}^{1} f(x)\,dx = \boxed{w_1 f(x_1) + w_2 f(x_2)}$$

$m = 2 \cdot 2 - 1 = 3$

$\Rightarrow$ 2 weights w_1, w_2 / 2 Gauss nodes x_1, x_2 $\Rightarrow$ 4 degrees of freedom

REQUIRE no error for polynomials order $0 \div 3$

order 0:

$f(x) = 1$

$$\int_{-1}^{1} f(x)\,dx = 2 \Rightarrow w_1 f(x_1) + w_2 f(x_2) = 2$$

$f(x) = 1$

$$\Rightarrow \boxed{w_1 \cdot 1 + w_2 \cdot 1 = 2}$$

order 1

$f(x) = x$

$$\int_{-1}^{1} f(x)\,dx = 0 \Rightarrow \boxed{w_1 x_1 + w_2 x_2 = 0}$$

order 2

$f(x) = x^2$

$$\int_{-1}^{1} f(x)\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = 2/3$$

$$\boxed{w_1 x_1^2 + w_2 x_2^2 = 2/3}$$