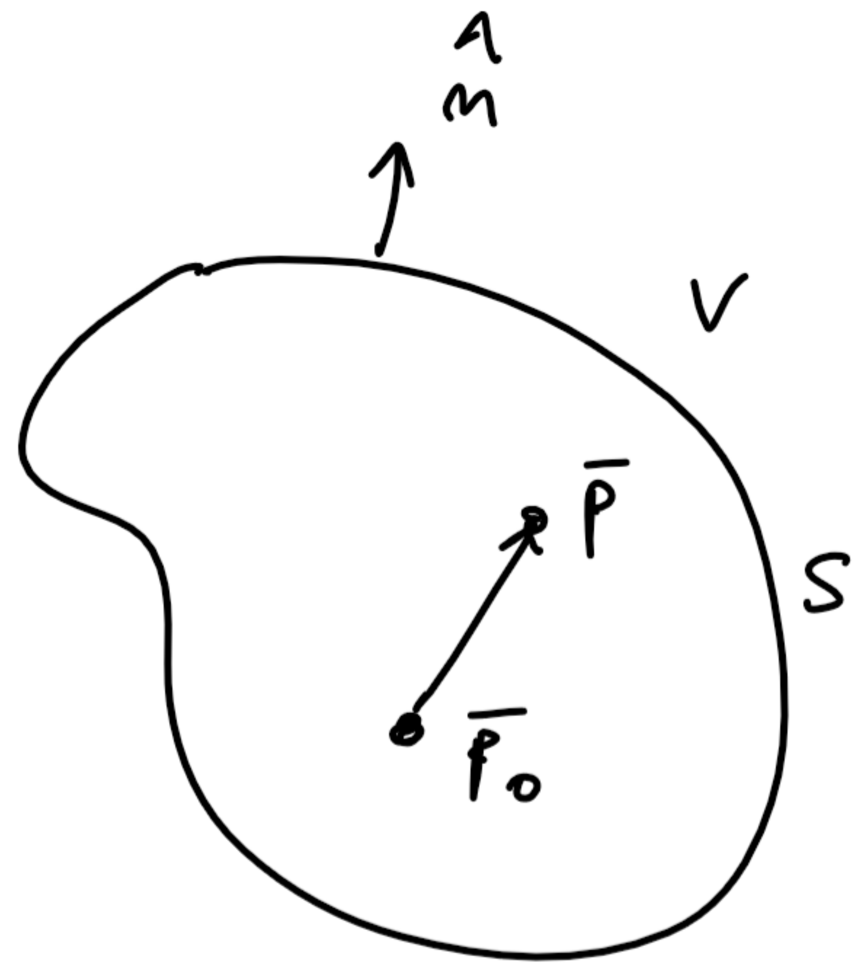
HARMONIC FUNCTIONS

consider SCD V,



DEF: φ is HARMONIC if $\nabla^2 \varphi = 0 \quad \forall \bar{P} \in V$

HARM. $\begin{cases} \varphi = K \in \mathbb{R} \\ \varphi = ax + by + c \\ \varphi = 1/r \end{cases}$, $\bar{r}$: vector distance between $\bar{P}$ and $\bar{P}_0$

$\Downarrow$

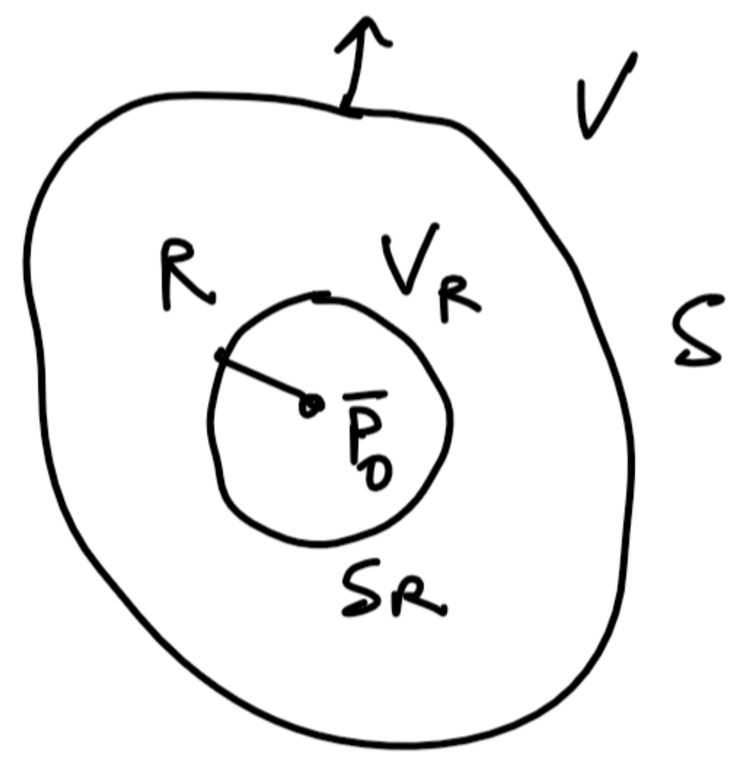
SCALAR potential produced by a point charge

$$\varphi_0(\bar{P}) = \frac{1}{4\pi\varepsilon_0} \, q/r \rightsquigarrow \text{ALSO an HARMONIC FUNCTION}$$

MEAN VALUE theorem for HARMONIC FUNCTIONS

" For any spherical surface S_R with radius R, centered in $\bar{P}_0$ ":

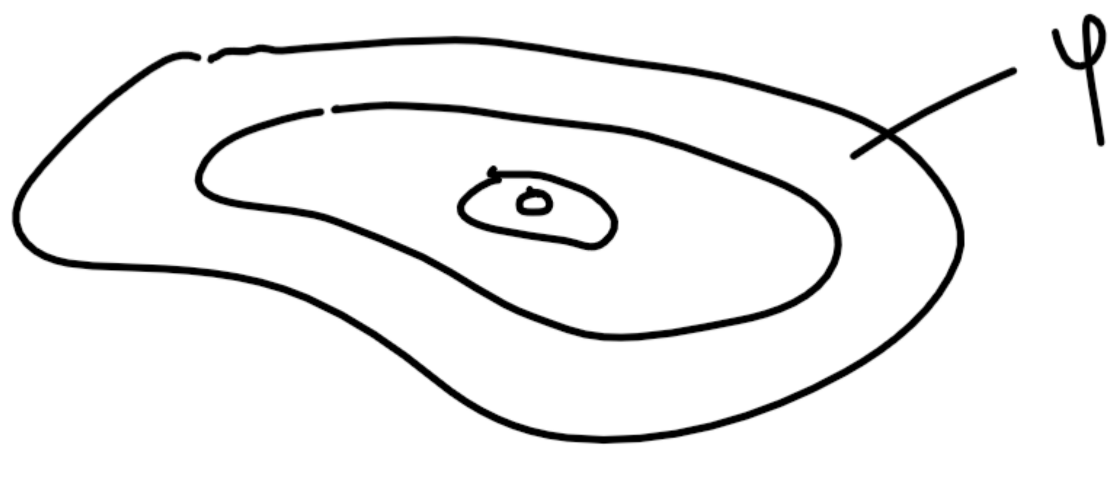
$$\underset{\substack{\uparrow \\ \text{generic HARMONIC FUNCTION}}}{\varphi(\bar{P}_0)} = \underbrace{\frac{1}{4\pi R^2} \int_{S_R} \varphi(\bar{P}) \, dS}_{\text{AVERAGE of } \varphi \text{ on surface}} = \underbrace{\frac{1}{4/3\pi R^3} \int_{V_R} \varphi(\bar{P}) \, dV}_{\text{AVERAGE of } \varphi \text{ on Volume}}$$



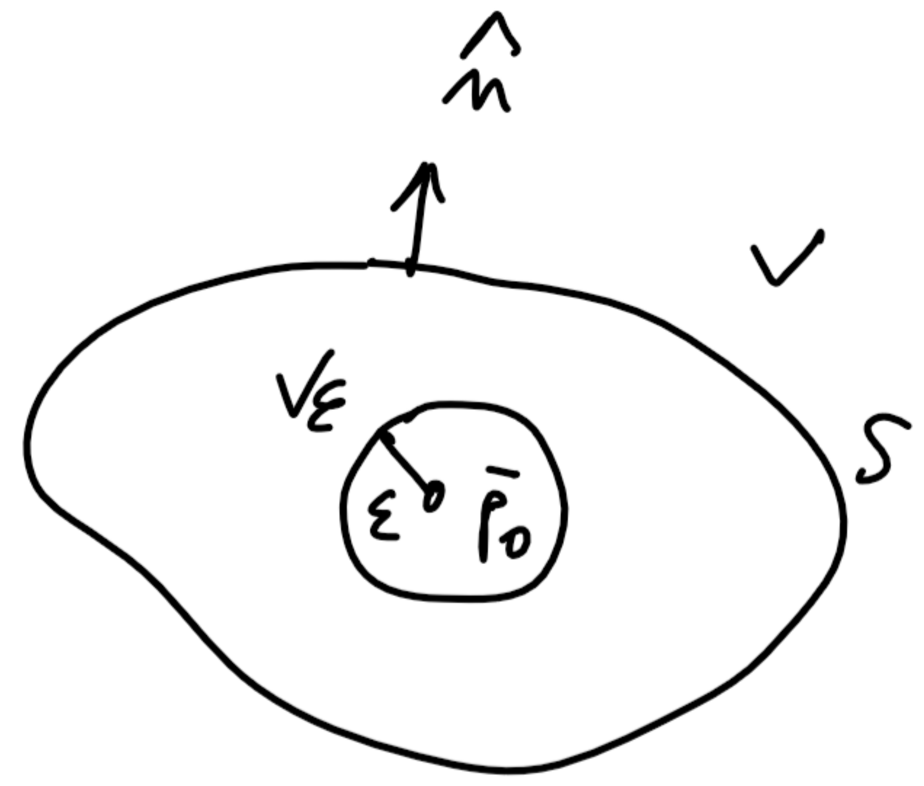
COROLLARIES of mean value theorem

no LOCAL MAX / MIN

K_1 : " A harmonic function has NO LOCAL EXTREMA on the internal points of its domain of definition "



Proof by contradiction:



$\Rightarrow$ ASSUME there is a local maximum in $\bar{P}_0$

Def of LOCAL MAX : there $\exists$ an infinitesimal sphere V_ε, with radius ε in which

$$\Rightarrow \quad \varphi(\bar{P}_0) - \varphi(\bar{P}) > 0 \;, \forall \bar{P} \in V_\varepsilon \;, \bar{P} \neq \bar{P}_0$$

$$\varphi(\bar{P}_0) - \varphi(\bar{P}) = 0 \;, \bar{P} = \bar{P}_0$$

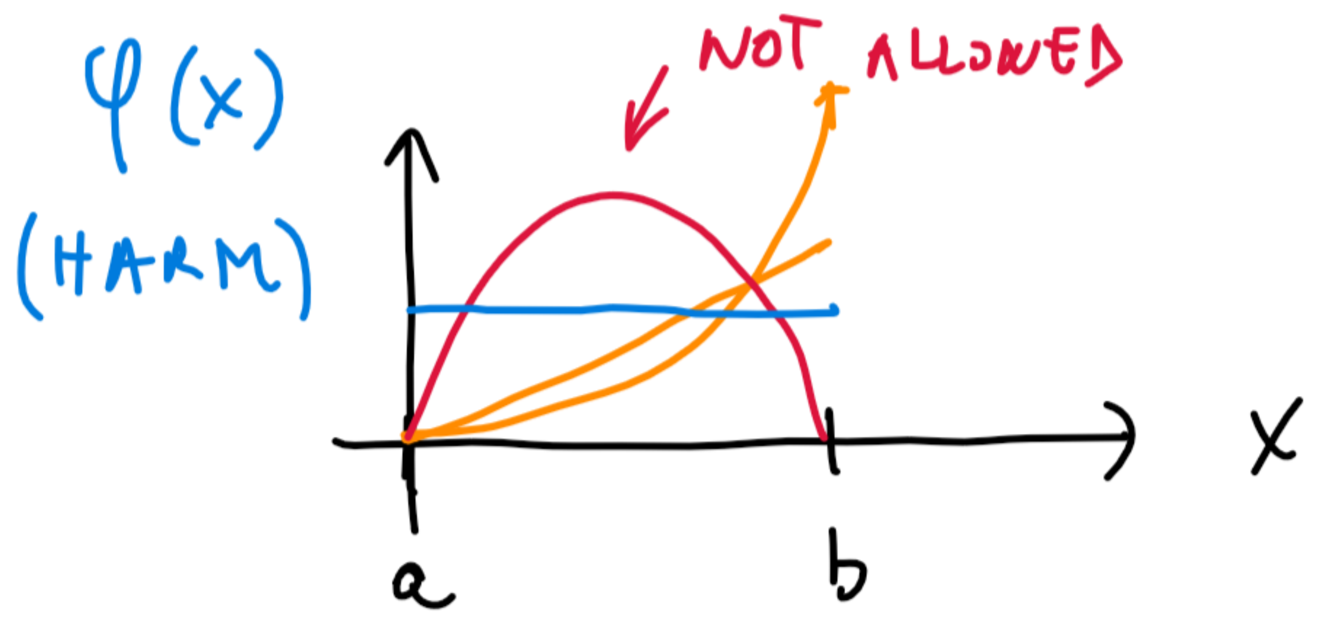
Define a SCALAR FUNCTION φ' :

$$\varphi'(\bar{P}) = \varphi(\bar{P}_0) - \varphi(\bar{P}) \quad \Rightarrow \underbrace{\varphi' \text{ is HARMONIC!}}_{\text{MUST RESPECT MEAN VALUE theorem for HARM. FUNCTIONS}}$$

$$\underbrace{\varphi'(\bar{P}_0)}_{= 0 \text{ by def}} = \frac{1}{4/3 \pi \varepsilon^3} \int_{V_\varepsilon} \underbrace{\varphi'(\bar{P})}_{\geq 0 \; \forall \bar{P} \in V_\varepsilon} dV$$

$\Rightarrow$ the only solution that satisfies the equation is $\varphi' = 0 \;\forall \bar{P} \in V_\varepsilon$

$\Rightarrow$ there CANNOT BE any MAXIMUM



φ SCALAR EL_POT

$\nabla^2 \varphi = \cancel{-\rho/\varepsilon_0} = 0$

$\varphi = \varphi_0$

$E = -\nabla \varphi$

K2 : "Any extrema of an harmonic function φ on a simply connected domain V MUST BE located on the domain BOUNDARY

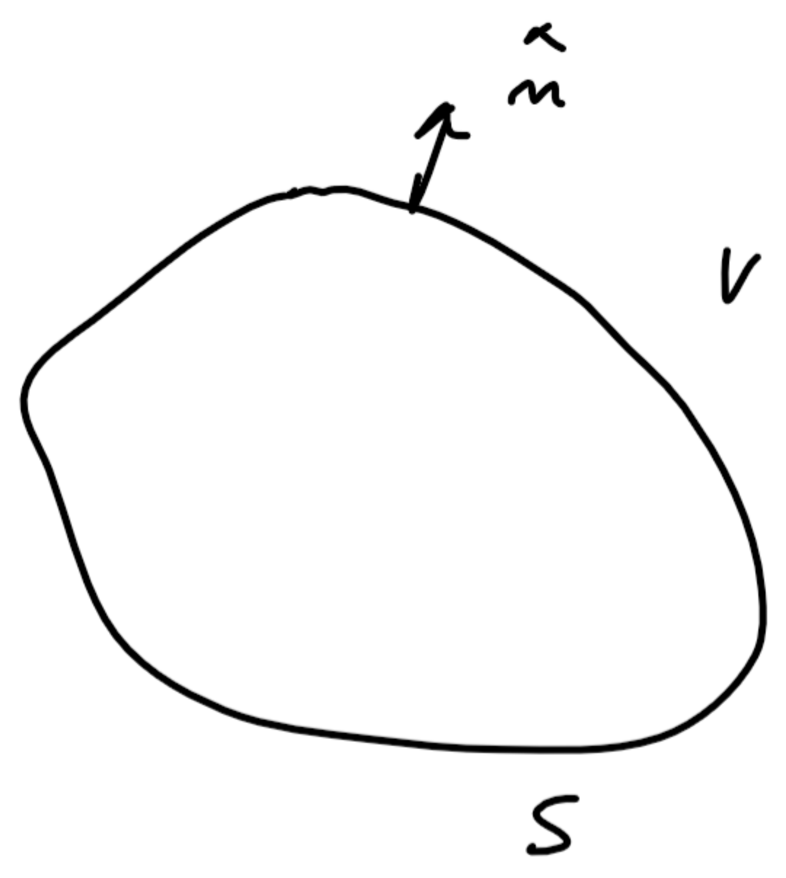
K3 : "If a harmonic function φ is uniform on the entire boundary S of a SCD V, φ must also be uniform within domain V

UNIQUENESS – POISSON PROBLEMS W/ DIRICHLET BOUNDARY CONDITIONS

SCALAR POTENTIAL

$$(1)\ \begin{cases} \nabla^2 \varphi = t & \forall\, \bar{P} \in V \\ \varphi = \varphi_0 & \forall\, \bar{P} \in S \end{cases}$$

DIRICHLET BC



- We want to prove that solution is unique

AB ABSURDUM (by contradiction)

ASSUME : SOLUTIONS $\varphi_1(\bar{P})$ $\varphi_2(\bar{P})$ → SATISFY (1)

$$\begin{cases} \nabla^2 \varphi_1 = t & \forall\, \bar{P} \in V \\ \varphi_1 = \varphi_0 & \forall\, \bar{P} \in S \end{cases} \qquad \begin{cases} \nabla^2 \varphi_2 = t \\ \varphi_2 = \varphi_0 \end{cases}$$

⇒ Define "Difference solution" $\varphi_3 = \varphi_1 - \varphi_2$ $\forall\, \bar{P} \in V$

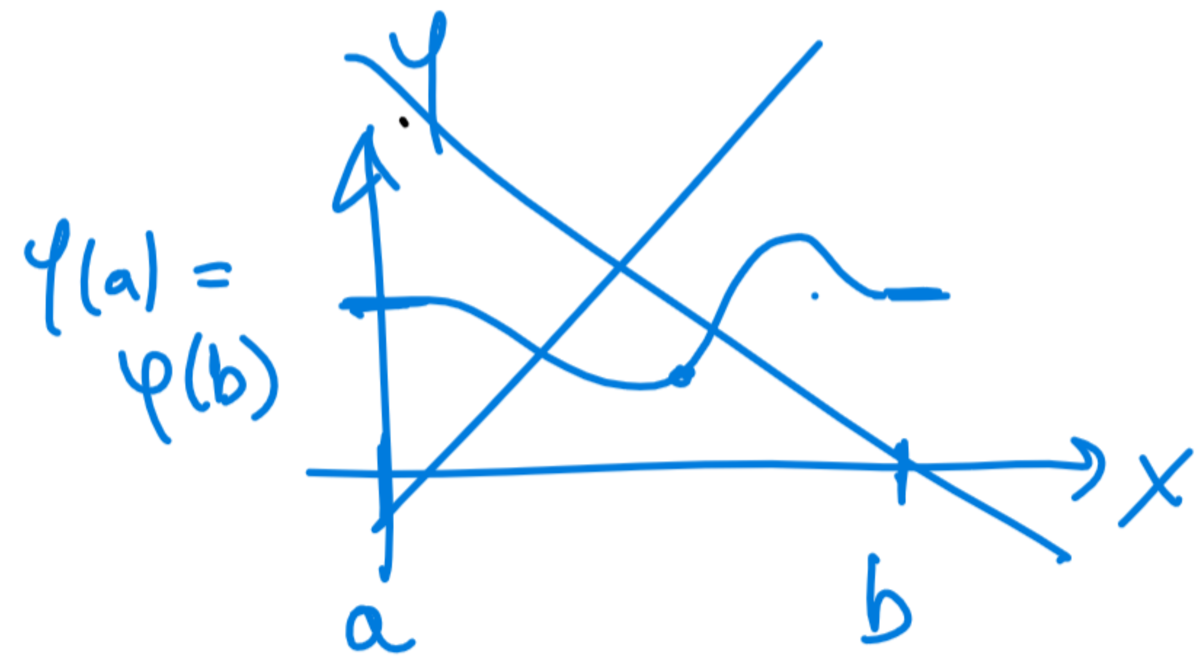
internal points / BCs

$$\begin{cases} \nabla^2 \varphi_3 = \nabla^2(\varphi_1 - \varphi_2) = \underbrace{\nabla^2 \varphi_1}_{t} - \underbrace{\nabla^2 \varphi_2}_{t} = 0 \\ \varphi_3 = \underbrace{\varphi_1}_{\varphi_0} - \underbrace{\varphi_2}_{\varphi_0} = 0 \quad \forall\, \bar{P} \in S \end{cases}$$

φ_3 is HARMONIC

⇒ φ_3 HARMONIC on internal points

φ_3 is ZERO on BOUNDARY



K_3 : if $\underbrace{\varphi_3 \text{ is UNIFORM on BOUNDARY}}_{=0}$ ⇒ φ_3 is uniform in internal points

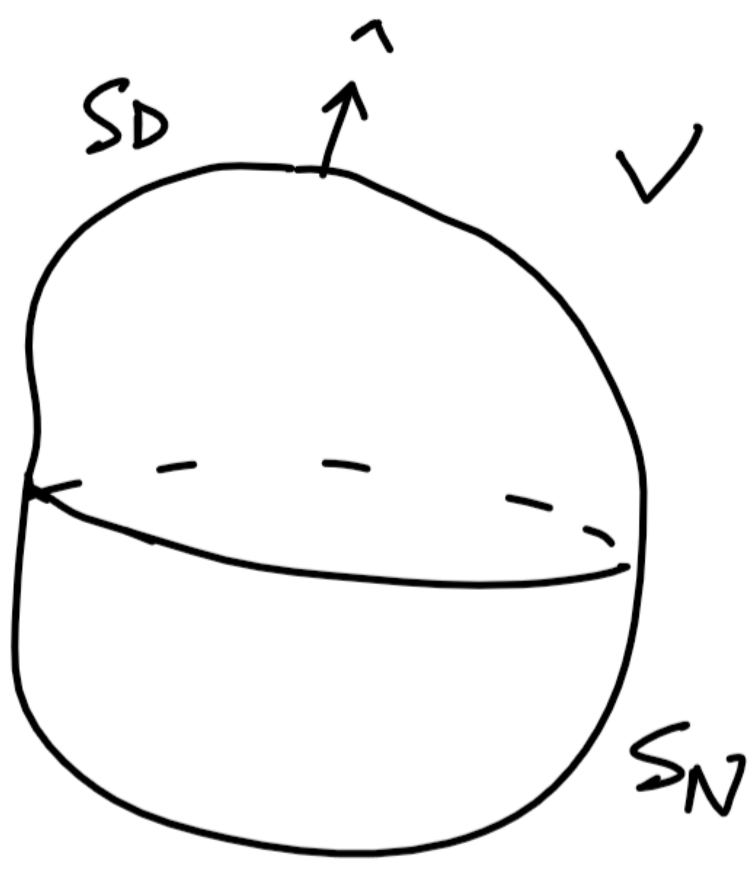
$= 0$ ⟶ $= 0$ on internal points

$\varphi_3 = 0$ $\forall\, \bar{P} \in V$ and S ⇒ $\varphi_1 = \varphi_2$ ⇒ Problem has UNIQUE SOLUTION

UNIQUENESS – POISSON PROBLEMS W/ DIRICHLET AND NEUMANN Conditions

$$
\begin{cases}
\nabla^2 \varphi = t \leftarrow t(\bar{P}) & \forall \bar{P} \in V \\
\varphi = \varphi_0 & \forall \bar{P} \in S_D \\
\underbrace{\partial \varphi / \partial n = \varphi_0'}_{\text{NEUMANN BC}} & \forall \bar{P} \in S_N
\end{cases}
$$

S_D



$S = S_D \cup S_N$

if φ : electric potential $\bar{E} = -\nabla\varphi$

$\bar{E} \cdot \hat{m} = -\nabla\varphi \cdot \hat{n}$

$$\boxed{E_n = -\partial\varphi / \partial n}$$

GOAL : check uniqueness $\rightarrow$ ASSUME $\boxed{\text{Solutions: } \varphi_1 , \varphi_2}$

$\varphi_3 = \varphi_1 - \varphi_2$

$$
\begin{array}{l}
V: \\ S_D \\ S_N
\end{array}
\begin{cases}
\nabla^2 \varphi_1 = t \\
\varphi_1 = \varphi_0 \\
\partial\varphi_1 / \partial n = \varphi_0'
\end{cases}
\qquad
\begin{cases}
\nabla^2 \varphi_2 = t \\
\varphi_2 = \varphi_0 \\
\partial\varphi_2 / \partial n = \varphi_0'
\end{cases}
$$

For φ_3 :

$$
\begin{array}{l}
V \\ S_D \\ S_N
\end{array}
\begin{cases}
\nabla^2 \varphi_3 = \nabla^2(\varphi_1 - \varphi_2) = 0 \\
\varphi_3 = \varphi_1 - \varphi_2 = \varphi_0 - \varphi_0 = 0 \\
\partial\varphi_3 / \partial n = \underbrace{\partial\varphi_1 / \partial n}_{\varphi_0'} - \underbrace{\partial\varphi_2 / \partial n}_{\varphi_0'} = 0
\end{cases}
$$

$\Rightarrow$ CANNOT USE K_3 of Mean value theorem

$\Rightarrow$ WE DON'T KNOW φ_3 over S_N , just it's DERIVATIVE!

GREEN'S IDENTITY 1 $\quad \varphi, \psi$

$$\int_V \left(\nabla\varphi\cdot\nabla\psi + \varphi\nabla^2\psi\right)dV = \oint_S \varphi\nabla\psi\cdot d\vec{S}$$

→ ASSUME $\varphi, \psi = \varphi_3$

$$\int_V \left(\nabla\varphi_3\cdot\nabla\varphi_3 + \underbrace{\cancel{\varphi_3\nabla^2\varphi_3}}_{=0}\right)dV = \oint_S \varphi_3 \overbrace{\nabla\varphi_3\cdot d\vec{S}}^{\nabla\varphi_3\cdot\hat{n}\,dS}$$

$$\int_V (\nabla\varphi_3)^2\,dV = \oint_S \varphi_3 \frac{\partial\varphi_3}{\partial n}\,dS$$

$$\int_V (\nabla\varphi_3)^2\,dV = \underbrace{\cancel{\int_{S_D} \varphi_3 \frac{\partial\varphi_3}{\partial n}\,dS}}_{=0,\ \varphi_3=0\ S_D} + \underbrace{\int_{S_N} \varphi_3 \frac{\partial\varphi_3}{\partial n}\,dS}_{=0,\ \partial\varphi_3/\partial n=0\ S_N}$$

$$\int_V \underbrace{(\nabla\varphi_3)^2}_{\geq 0}\,dV = 0$$

⇒ integrand function $(\nabla\varphi_3)^2$ must be $=0$

⇓

→ $\varphi_3 =$ UNIFORM over whole VOLUME

since $\boxed{\varphi_3 = 0 \text{ on } S_D}$

⇒ $\varphi_3 = 0$ EVERYWHERE !

$\cancel{\varphi_3} = \varphi_1 - \varphi_2 \Rightarrow \varphi_1 = \varphi_2$

Solution is UNIQUE !

if WE HAD NEUMANN BCs on whole S
⇒ solution DEFINED up to A CONSTANT

↑

⇒ For a UNIQUE sol, the must be at least one point of the domain with a DIRICHLET BC

PROBLEM UNIFORMLY CONDUCTIVE ($\sigma = \sigma_0$) PLATE (STEADY-STATE)

Find: $\bar{J}$ on the plate