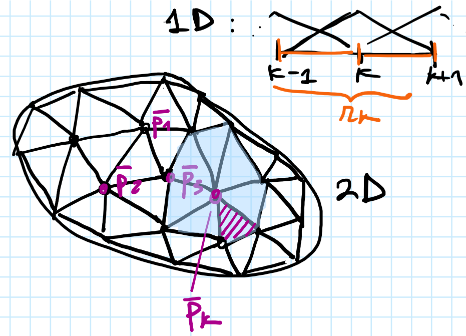
# PIECEWISE linear interpolation on TRIANGLES




HP: $f(x,y)$ where $f(x,y)$ is known at NODES

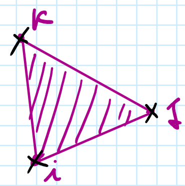
$\bar{P}_1 = (x_1, y_1) \Rightarrow f_1 = f(x_1, y_1)$

$\vdots$

$\bar{P}_k = (x_k, y_k) \Rightarrow f_k = f(x_k, y_k)$

$\Omega_k$ = union of all elements that have $\bar{P}_k$ as one of their VERTICES

Focus on $W_{ijk} \in \Omega_k$




VERTICES :

$\bar{P}_i = (x_i, y_i)$

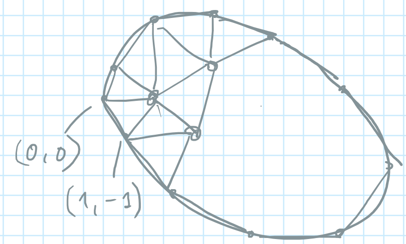
$\bar{P}_j = (x_j, y_j)$

$\bar{P}_k = (x_k, y_k)$

GOAL: define a set of LINEAR SHAPE FUNCTIONS [interpol.] on the ELEMENT

AREA of $W_{ijk}$ :

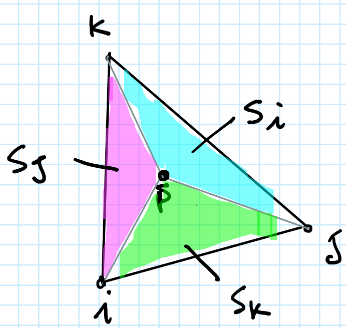
$$S = \frac{1}{2} \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$




Introduce a point inside $W_{ijk}$

$\bar{P} = (x,y) \in W_{ijk}$

Defines 3 SUB TRIANGLES inside $W_{ijk}$




if $\bar{P}$ moves closer to NODE $i$

$S_i \uparrow$, $S_j \downarrow$, $S_k \downarrow$

$$S_i = \frac{1}{2} \det \begin{bmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$

$$S_j = \frac{1}{2} \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x & y \\ 1 & x_k & y_k \end{bmatrix}$$

$$S_k = \frac{1}{2} \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x & y \end{bmatrix}$$

SHAPE FUNCTIONS :

$$\begin{cases} L_i(x,y) = \dfrac{S_i(x,y)}{S} \\ L_j(x,y) = \dfrac{S_j(x,y)}{S} \end{cases}$$

AREA COORDINATES

[BARYCENTRIC coordinates]

NOT INDEPENDENT!

$$\begin{cases} L_j(x,y) = \dfrac{S_j(x,y)}{S} \\ L_k(x,y) = \dfrac{S_k(x,y)}{S} \end{cases}$$

NOT INDEPENDENT!

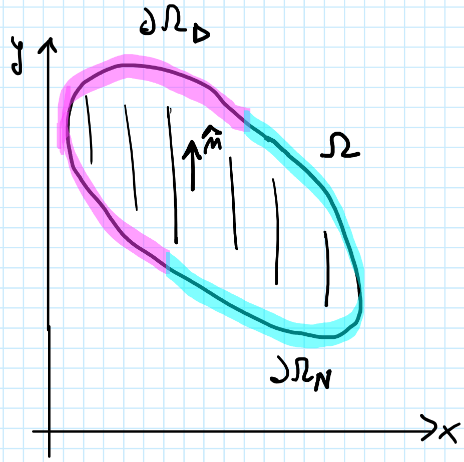
$$\boxed{L_i(x,y) + L_j(x,y) + L_k(x,y) = 1} \quad \forall x,y \in w_{ijk}$$

Ex: if $P = (x_j, y_j) \rightarrow \begin{cases} S_i = \dfrac{0}{S} = 0 \\ S_j = \dfrac{S}{S} = 1 \\ S_k = 0/S = 0 \end{cases}$

$\forall (x,y) \in w_{ijk}$

$$\boxed{f(x,y) = f_i L_i(x,y) + f_j L_j(x,y) + f_k L_k(x,y)}$$

## FEM – Poisson Equation in 2D




Hp: $\partial/\partial z = 0$ $\quad \varphi(x,y) = \varphi$

$p(x,y) = p$

$t(x,y) = t$

### FORMULATION (strong form)

$$\begin{cases} \nabla \cdot (p \nabla \varphi) = t & \Omega \\ \varphi = \varphi_0 & \partial\Omega_D \\ \dfrac{\partial \varphi}{\partial n} = \varphi_0' & \partial\Omega_N \end{cases}$$

### PIECEWISE POLYNOMIAL interpolation $\quad \varphi \rightarrow \tilde{\varphi} \in C_0$

$$\tilde{\varphi} = \sum_{k=1}^{n} \varphi_k \underset{\uparrow\, L_k}{L_k(x,y)} = \varphi_1 L_1(x,y) + \dots + \varphi_n L_n(x,y)$$

### WEIGHTED RESIDUALS APPROACH

$$r(x,y) = \underbrace{\nabla\cdot(p\nabla\tilde{\varphi}) - t}_{=0 \text{ for } \varphi} \neq 0$$

WEIGHTING FUNCTION $w$

"find nodal values of $\tilde{\varphi}$ such that ..."

$$\int_\Omega w(x,y)\, r(x,y)\, dS = 0$$

RESIDUAL

$$\boxed{\int_\Omega w\, \nabla\cdot(p\nabla\tilde{\varphi})\, dS} = \int_\Omega w t\, dS$$

$g$, $k$, $f$

problem $\tilde{\varphi} \in C_0$

VECTOR IDENTITY :

$$\boxed{\nabla\cdot(gk\nabla f) = g\nabla\cdot(k\nabla f) + k\nabla f\cdot\nabla g}$$

$f, g, k$ generic SCALAR functions

$$\int_\Omega \nabla\cdot(w p\nabla\tilde{\varphi})\, dS - \int_\Omega p\, \nabla w\cdot\nabla\tilde{\varphi} = \int_\Omega w t\, dS$$

$$\Downarrow$$

$$\oint_{\partial\Omega} w p\nabla\tilde{\varphi}\cdot d\bar{l}$$

$$\Rightarrow \boxed{\int_\Omega p\nabla w\cdot\nabla\tilde{\varphi}\, dS = \oint_{\partial\Omega} w p\nabla\tilde{\varphi}\cdot d\bar{l} - \int_\Omega w t\, dS}$$

WEAK FORMULATION