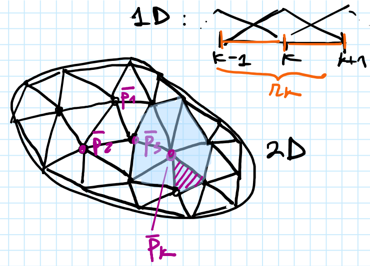
FEM - Poisson equation 2D

Monday, December 9, 2024 11:14 AM

PIECEWISE Linear interpolation on TRIANGLES

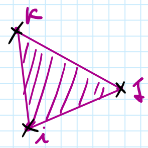


HP: $f(x,y)$ where $f(x,y)$ is known at NODES

$$\bar{P}_1 = (x_1, y_1) \Rightarrow f_1 = f(x_1, y_1)$$
$$\vdots$$
$$\bar{P}_k = (x_k, y_k) \Rightarrow f_k = f(x_k, y_k)$$

Ω_k = union of all elements that have $\bar{P}_k$ as one of their VERTICES

Focus on $W_{ijk} \in \Omega_k$



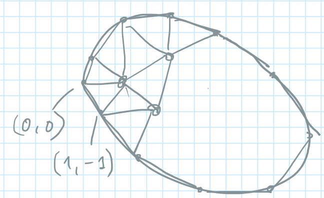
VERTICES :

$$\bar{P}_i = (x_i, y_i)$$
$$\bar{P}_j = (x_j, y_j)$$
$$\bar{P}_k = (x_k, y_k)$$

GOAL: define a set of LINEAR SHAPE FUNCTIONS [interpol.] on the ELEMENT

AREA of W_{ijk} :

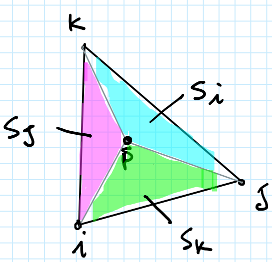
$$S = \frac{1}{2} \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$



Introduce a point inside W_{ijk}

$$\bar{P} = (x, y) \in W_{ijk}$$

Defines 3 SUB TRIANGLES inside W_{ijk}



if $\bar{P}$ moves closer to NODE i

$S_i \uparrow$, $S_j \downarrow$, $S_k \downarrow$

$$S_i = \frac{1}{2} \det \begin{bmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$

$$S_j = \frac{1}{2} \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x & y \\ 1 & x_k & y_k \end{bmatrix}$$

$$S_k = \frac{1}{2} \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x & y \end{bmatrix}$$

SHAPE FUNCTIONS :

$$\begin{cases} L_i(x,y) = \dfrac{S_i(x,y)}{S} \\ L_j(x,y) = \dfrac{S_j(x,y)}{S} \\ L_k(x,y) = \dfrac{S_k(x,y)}{S} \end{cases}$$

AREA COORDINATES

[BARYCENTRIC coordinates]

$\Rightarrow$ NOT INDEPENDENT!

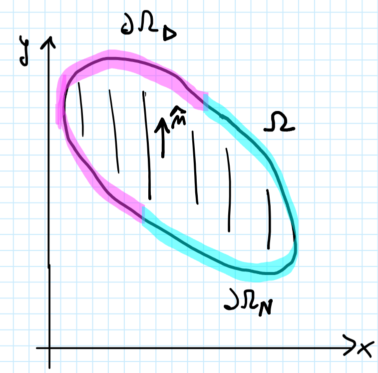
$$\boxed{L_i(x,y) + L_j(x,y) + L_k(x,y) = 1} \quad \forall x,y \in W_{ijk}$$

Ex: if $P = (x_j, y_j) \rightarrow \begin{cases} S_i = \dfrac{0}{S} = 0 \\ S_j = \dfrac{S}{S} = 1 \\ S_k = 0/S = 0 \end{cases}$

$\forall (x,y) \in W_{ijk}$

$$\boxed{f(x,y) = f_i L_i(x,y) + f_j L_j(x,y) + f_k L_k(x,y)}$$

FEM – Poisson Equation in 2D



Hp: $\partial/\partial z = 0$ $\qquad \varphi(x,y) = \varphi$

$p(x,y) = p$

$t(x,y) = t$

FORMULATION (strong form)

$$\begin{cases} \nabla \cdot (p \nabla \varphi) = t & \Omega \\ \varphi = \varphi_0 & \partial\Omega_D \\ \dfrac{\partial \varphi}{\partial n} = \varphi_0' & \partial\Omega_N \end{cases}$$

PIECEWISE POLYNOMIAL interpolation $\qquad \varphi \rightarrow \tilde{\varphi} \in C_0$

$$\tilde{\varphi} = \sum_{k=1}^{n} \varphi_k \underbrace{L_k(x,y)}_{L_k} = \varphi_1 L_1(x,y) + \dots + \varphi_n L_n(x,y)$$

WEIGHTED RESIDUALS APPROACH

$$r(x,y) = \underbrace{\nabla \cdot (p \nabla \tilde{\varphi}) - t}_{=0 \text{ for } \varphi} \neq 0$$

WEIGHTING FUNCTION w

"find nodal values of $\tilde{\varphi}$ such that ..."

$$\int_\Omega w(x,y)\, r(x,y)\, dS = 0$$

RESIDUAL

$$\boxed{\int_\Omega \overset{g}{w}\, \nabla \cdot (\overset{k}{p} \nabla \overset{f}{\tilde{\varphi}})\, dS} = \int_\Omega w t\, dS \qquad \text{problem } \tilde{\varphi} \in C_0$$

VECTOR IDENTITY :

f, g, k generic scalar functions

$$\boxed{\nabla \cdot (g k \nabla f) = g \nabla \cdot (k \nabla f) + k \nabla f \cdot \nabla g}$$

$$\boxed{\int_\Omega \nabla \cdot (w p \nabla \tilde{\varphi})\, dS} - \boxed{\int_\Omega p\, \nabla w \cdot \nabla \tilde{\varphi}} = \int_\Omega w t\, dS$$

$\Downarrow$

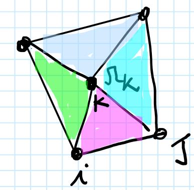
$$\oint_{\partial\Omega} w\, p \nabla\tilde{\varphi} \cdot d\bar{l}$$

$$\Rightarrow \boxed{\int_{\Omega} p \nabla w \cdot \nabla\tilde{\varphi}\, dS = \int_{\partial\Omega} w\, p \nabla\tilde{\varphi} \cdot d\bar{l} - \int_{\Omega} w t\, dS}$$ WEAK FORMULATION

GALERKIN'S CHOICE $\quad w(x,y) \to L_k(x,y)$, for $k = 1, 2, \dots, m$

$\Downarrow$

$L_k(x,y) \neq 0$ only on Ω_k, for $x, y \in \Omega_k$



$\Rightarrow$ RESTRICT domain of integration $\Omega \to \Omega_k$

$$\Rightarrow \underbrace{\int_{\Omega_k} p \nabla L_k \cdot \nabla\tilde{\varphi}\, dS}_{\mathrm{I}} = \underbrace{\int_{\partial\Omega} L_k\, p \nabla\tilde{\varphi} \cdot d\bar{l}}_{\mathrm{II}} - \underbrace{\int_{\Omega_k} L_k t\, dS}_{\mathrm{III}}, \quad \text{for } k = 1, 2, \dots, m$$

if need to derive algebraic expression for node k:

- SPLIT integrals on SUPPORT DOMAIN into Σ integrals on elements $\in \Omega_k$

$$\sum_{w_i \in \Omega_k} \int_{w_i} p \nabla L_k \cdot \nabla\tilde{\varphi}\, dS = \int_{\partial\Omega} L_k\, p \nabla\tilde{\varphi} \cdot d\bar{l} - \sum_{w_i \in \Omega_k} \int_{w_i} L_k t\, dS \qquad \text{Eq. } K$$

$\uparrow$ for all elements $\in \Omega_k$

L_{k+1} if equation for node L_{k+1}

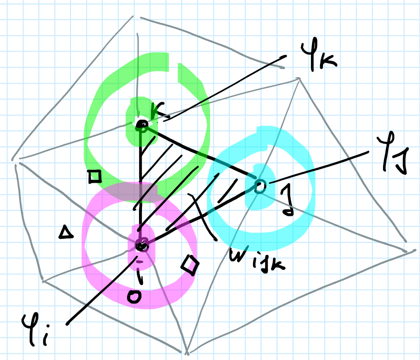
Element-centered approach

WRITE $[K]_{el}$ $\quad [rhs]_{el}$

Element w_{ijk} contributes to 3 nodal equations

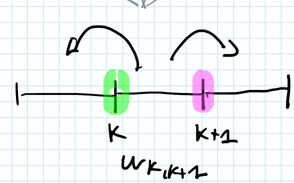
$$\begin{cases} \text{eq for node } i \\ \text{" " " } j \\ \text{" " " } k \end{cases}$$

$\rightarrow$ Equation for node i
3 coeffs from w_{ijk}
" " □
" " △
" " ○
" " ◇



$\boxed{\text{Term I}}$

$w_{ijk} \rightarrow$ eq i



$$\int_{w_{ijk}} p \nabla L_i \cdot \nabla\tilde{\varphi}\, dS =$$

on w_{ijk} $\quad \tilde{\varphi} = \varphi_i L_i(x,y) + \varphi_j L_j(x,y) + \varphi_k L_k(x,y)$

$$= \int_{W_{ijk}} p \, \nabla L_i \cdot \nabla \left[\varphi_i L_i + \varphi_j L_k + \varphi_k L_k \right] dS =$$

$$= \left[\int_{W_{ijk}} p \, \nabla L_i \cdot \nabla L_i \, dS \right] \varphi_i + \left[\int_{W_{ijk}} p \, \nabla L_i \cdot \nabla L_j \, dS \right] \varphi_j + \left[\int_{W_{ijk}} p \, \nabla L_i \cdot \nabla L_k \, dS \right] \varphi_k$$

$W_{ijk} \rightarrow$ eq j

$\tilde{\varphi} = \varphi_i L_i + \varphi_j L_j + \varphi_k L_k$

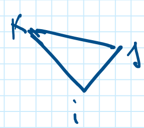
$$\int_{W_{ijk}} p \, \nabla L_j \cdot \nabla \tilde{\varphi} \, dS = \left[\int_{W_{ijk}} p \nabla L_j \cdot \nabla L_i \, dS \right] \varphi_i + \left[\int_{W_{ijk}} p \nabla L_j \cdot \nabla L_j \, dS \right] \varphi_j + \left[\int_{W_{ijk}} p \nabla L_j \cdot \nabla L_k \, dS \right] \varphi_k$$

$W_{ijk} \rightarrow$ eq k

$$\int_{W_{ijk}} p \nabla L_k \cdot \nabla \tilde{\varphi} \, dS = \left[\int_{W_{ijk}} p \nabla L_k \cdot \nabla L_i \, dS \right] \varphi_i + \left[\int_{W_{ijk}} p \nabla L_k \cdot \nabla L_j \, dS \right] \varphi_j + \left[\int_{W_{ijk}} p \nabla L_k \cdot \nabla L_k \, dS \right] \varphi_k$$

in 1D $\rightarrow$ 

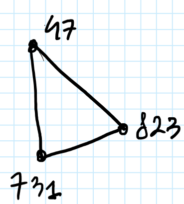
$\rightarrow$ 4 contributions [coefficients] $\rightarrow$ 2 eq k, 2 eq $k+1$ $\rightarrow [K]_{el} = 2 \times 2$

in 2D $\rightarrow$ 

$\rightarrow$ 9 coeffs. $\rightarrow$ 3 eq i, 3 eq j, 3 eq k $\rightarrow [K]_{el} = 3 \times 3$

$\Rightarrow$ i, j, k are not necessarily adjacent

Ex:



$$\nabla L_i \cdot \nabla L_k = \left(\frac{\partial L_i}{\partial x} \hat{i} + \frac{\partial L_i}{\partial y} \hat{j} \right) \cdot \left(\frac{\partial L_k}{\partial x} \hat{i} + \frac{\partial L_k}{\partial y} \hat{j} \right)$$

$$= \frac{\partial L_i}{\partial x} \frac{\partial L_k}{\partial x} + \frac{\partial L_i}{\partial y} \frac{\partial L_k}{\partial y}$$

$$[K]_{el,ijk} = \int_{W_{ijk}} \begin{bmatrix} p \nabla L_i \cdot \nabla L_i & p \nabla L_i \cdot \nabla L_j & p \nabla L_i \cdot \nabla L_k \\ p \nabla L_j \cdot \nabla L_i & p \nabla L_j \cdot \nabla L_j & p \nabla L_j \cdot \nabla L_k \\ p \nabla L_k \cdot \nabla L_i & p \nabla L_k \cdot \nabla L_j & p \nabla L_k \cdot \nabla L_k \end{bmatrix} dS =$$

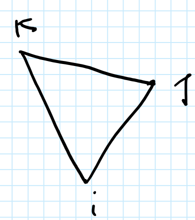
$$
= \int_{w_{ijk}} P \begin{bmatrix}
\frac{\partial L_i}{\partial x}\frac{\partial L_i}{\partial x} + \frac{\partial L_i}{\partial y}\frac{\partial L_i}{\partial y} & \frac{\partial L_i}{\partial x}\frac{\partial L_j}{\partial x} + \frac{\partial L_i}{\partial y}\frac{\partial L_j}{\partial y} & \frac{\partial L_i}{\partial x}\frac{\partial L_k}{\partial x} + \frac{\partial L_i}{\partial y}\frac{\partial L_k}{\partial y} \\
\frac{\partial L_j}{\partial x}\frac{\partial L_i}{\partial x} + \frac{\partial L_j}{\partial y}\frac{\partial L_i}{\partial y} & \frac{\partial L_j}{\partial x}\frac{\partial L_j}{\partial x} + \frac{\partial L_j}{\partial y}\frac{\partial L_j}{\partial y} & \frac{\partial L_j}{\partial x}\frac{\partial L_k}{\partial x} + \frac{\partial L_j}{\partial y}\frac{\partial L_k}{\partial y} \\
\frac{\partial L_k}{\partial x}\frac{\partial L_i}{\partial x} + \frac{\partial L_k}{\partial y}\frac{\partial L_i}{\partial y} & \frac{\partial L_k}{\partial x}\frac{\partial L_j}{\partial x} + \frac{\partial L_k}{\partial y}\frac{\partial L_j}{\partial y} & \frac{\partial L_k}{\partial x}\frac{\partial L_k}{\partial x} + \frac{\partial L_k}{\partial y}\frac{\partial L_k}{\partial y}
\end{bmatrix} dS =
$$

$$
= \int_{w_{ijk}} P \underbrace{\begin{bmatrix}
\frac{\partial L_i}{\partial x} & \frac{\partial L_i}{\partial y} \\
\frac{\partial L_j}{\partial x} & \frac{\partial L_j}{\partial y} \\
\frac{\partial L_k}{\partial x} & \frac{\partial L_k}{\partial y}
\end{bmatrix}}_{[\nabla L]^T_{el,ijk}}
\underbrace{\begin{bmatrix}
\frac{\partial L_i}{\partial x} & \frac{\partial L_j}{\partial x} & \frac{\partial L_k}{\partial x} \\
\frac{\partial L_i}{\partial y} & \frac{\partial L_j}{\partial y} & \frac{\partial L_k}{\partial y}
\end{bmatrix}}_{[\nabla L]_{el,ijk}} dS =
$$

GRADIENT ELEMENT MATRIX

$$
\boxed{[K]_{el,ijk} = \int_{w_{ijk}} P \, [\nabla L]^T_{el,ijk} \cdot [\nabla L]_{el,ijk} \, dS}
$$

Term III : $-\int_{\Omega_k} L_k t \, dS$



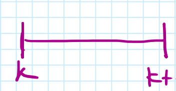
$$\Downarrow$$

$$
-\sum_{w_i \in \Omega_k} \int_{w_i} L_k t \, dS
$$

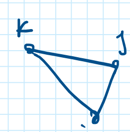
Contributions of element w_{ijk} to eqns → node i, → node j, → node k

w_{ijk} → node i $\quad \int_{w_{ijk}} L_i t \, dS$

→ node j $\quad \int_{w_{ijk}} L_j t \, dS$

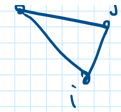
in 1D 

⇒ 2 contrib ↙ k ↘ k+1

in 2D 

⇒ 3 contributions

$\int_{w_{ijk}}$ in 2D



$\Rightarrow$ 3 contributions $i \swarrow \quad \downarrow j \quad \searrow k$

$\rightarrow$ node k $\quad \int_{w_{ijk}} L_k t \, dS$

Source term of Poisson's eq on element w_{ijk}

$$[rhs]_{el,ijk} = -\int_{w_{ijk}} \begin{bmatrix} L_i t \\ L_j t \\ L_k t \end{bmatrix} dS = -\int_{w_{ijk}} [L]_{el,ijk}\, t(x,y)\, dS$$

$$\begin{bmatrix} L_i(x,y) \\ L_j(x,y) \\ L_k(x,y) \end{bmatrix} = [L]_{el,ijk}$$

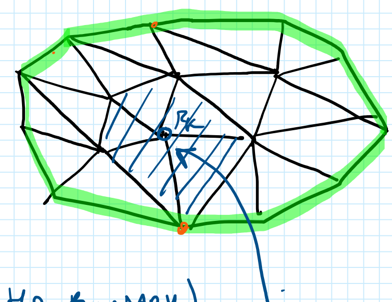
ELEMENT shape function array

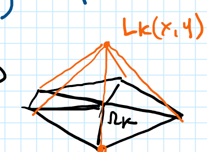
Term II : $\quad \underbrace{\int_{\partial\Omega} L_k\, p\, \nabla\tilde{\varphi} \cdot d\bar{l}}_{II} \Rightarrow$ on the boundary

$\Downarrow$ to be evaluated only for nodes $\in \partial\Omega_N, \partial\Omega_D$

$\Downarrow$ For all internal nodes $L_k = 0$

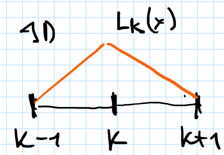
(if k is internal node, $L_k = 0$ on the BOUNDARY)





DIRICHLET nodes $\quad$ if node $\in \partial\Omega_D$

$$\boxed{\varphi(x_k, y_k) = \varphi_0} \Rightarrow \text{no weighted residuals!}$$

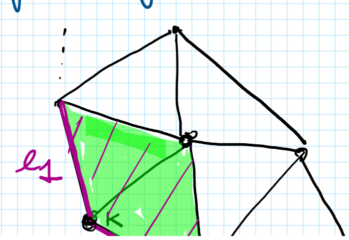


Neumann nodes ... to do!

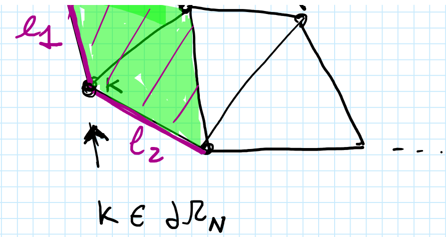
For next lesson : DOWNLOAD $\boxed{\text{GMSH}}$ put it in GITHUB folder for 2D Fem

$$\int_{\partial\Omega} L_k\, p\, \nabla\tilde{\varphi} \cdot d\bar{l} = \int_{e_1} L_k\, p\, \nabla\tilde{\varphi} \cdot d\bar{l} + \int_{e_2} L_k \nabla p \nabla\tilde{\varphi} \cdot d\bar{l} =$$

$\updownarrow$ φ' KNOWN



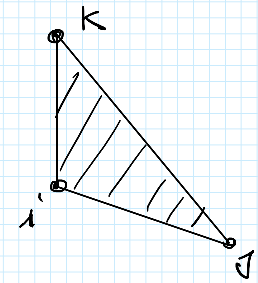
$\partial\Omega$ $\downarrow$ φ' KNOWN $\qquad \ell_1 \qquad \ell_2$



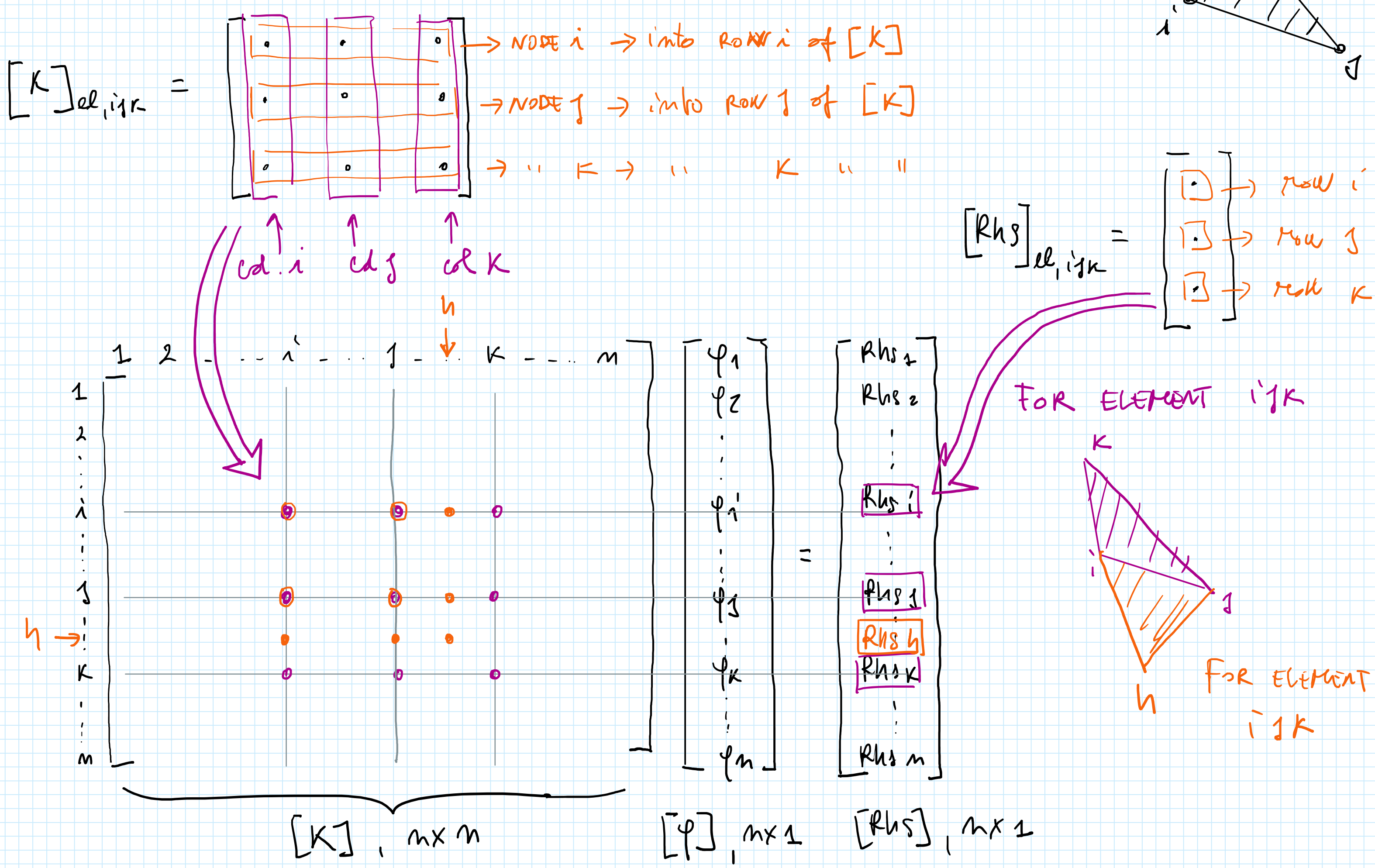
$$\nabla\tilde{\varphi}\cdot\hat{n} = \partial\tilde{\varphi}/\partial n$$

$$\Rightarrow \;=\; \int_{\ell_1} L_k\, p\, \underbrace{\partial\tilde{\varphi}/\partial n}_{\frac{\partial\tilde{\varphi}}{\partial n}=\varphi'}\, d\ell \;+\; \int_{\ell_2} L_k\, p\, \underbrace{\partial\tilde{\varphi}/\partial n}_{\partial\tilde{\varphi}/\partial n=\varphi'}\, d\ell$$

$\Rightarrow$ To be ADDED to RHS for boundary nodes



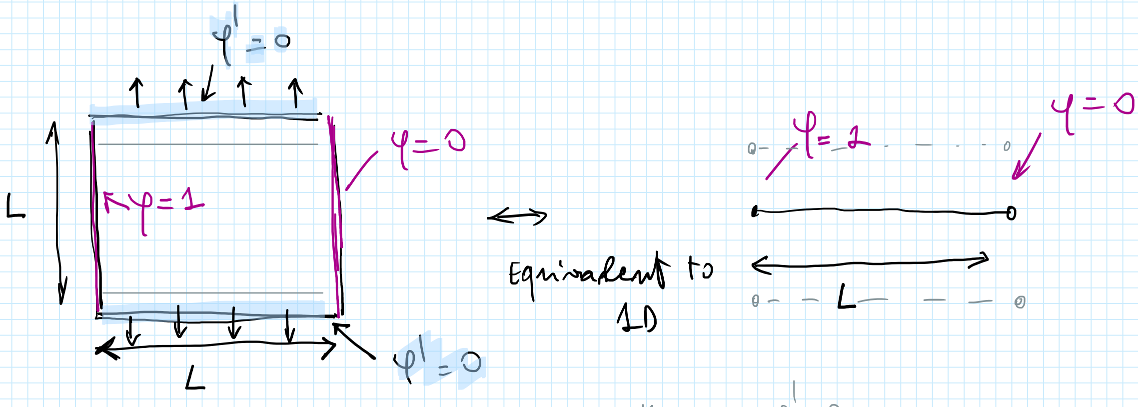
Linear System assembly $\qquad [K][\varphi] = [Rhs]$

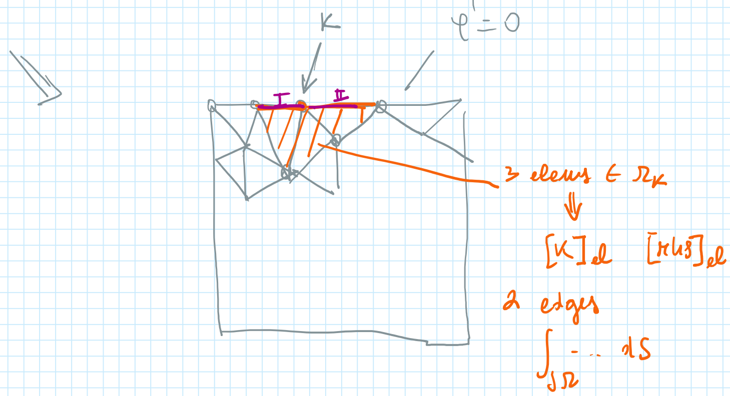


- $[K]$ is SPARSE, symmetric

$\Rightarrow$ IRREDUCIBLY DIAGONALLY DOMINANT (granted by the DIRICHLET BC)

$\varphi' = 0$

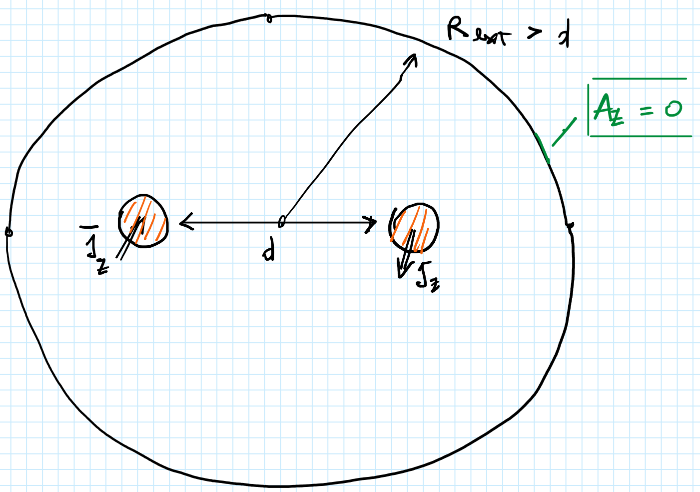


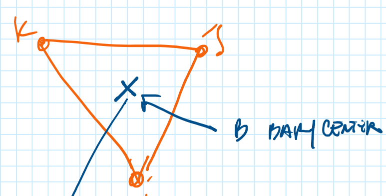


Transmission line

$$\nabla \cdot \left(\frac{1}{\mu_r} \nabla A_z \right) = \mu_0 J_z$$

unknown





$L_i(x,y) = 1/3$

$L_j(x,y) = 1/3$

$L_k(x,y) = 1/3$