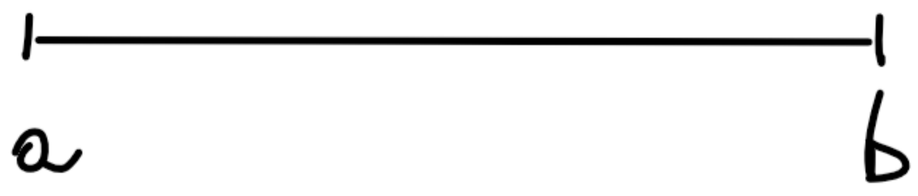
$$\nabla^2 \varphi = t \longrightarrow \frac{d^2\varphi}{dx^2} = t(x)$$

generic SCALAR

$\partial/\partial y = 0$

$\partial/\partial z = 0$



Poisson eq. for electric potential $\boxed{\nabla^2 \varphi = -\rho/\varepsilon_0}$

[V]

Electric Field

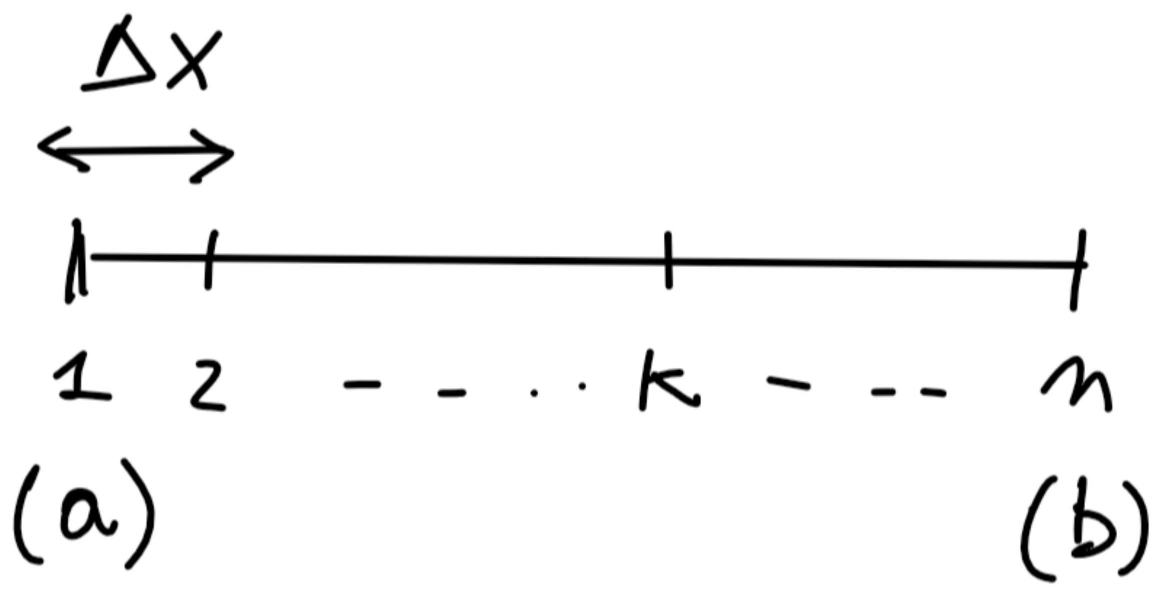
$$\bar{E} = -\nabla\varphi$$

$\downarrow$ 1D

$$\boxed{\bar{E} = -d\varphi/dx}$$

Known from 1D_FDM

On a discretized domain
n nodes



internal nodes 2, 3 ..., k, ... n-1

$$\left.\frac{d\varphi}{dx}\right|_k = \frac{\varphi_{k+1} - \varphi_{k-1}}{2\Delta x} + O(\Delta x^2)$$ centered difference

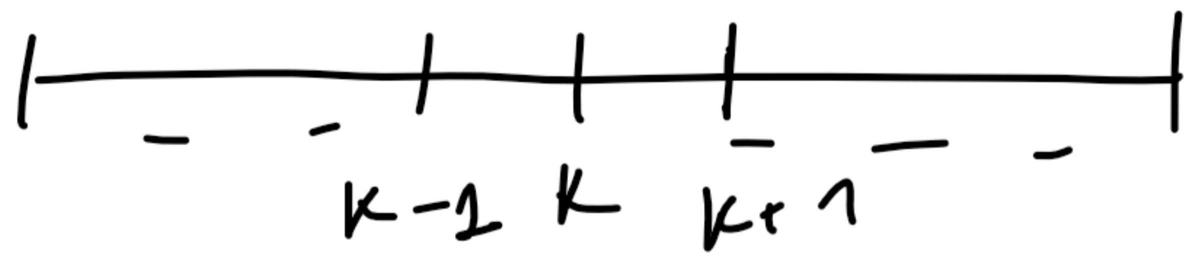
node 1: $$\left.\frac{d\varphi}{dx}\right|_1 = \frac{\varphi_2 - \varphi_1}{\Delta x} + O(\Delta x)$$ FORWARD difference

node n: $$\left.\frac{d\varphi}{dx}\right|_n = \frac{\varphi_n - \varphi_{n-1}}{\Delta x} + O(\Delta x)$$ BACKWARD difference

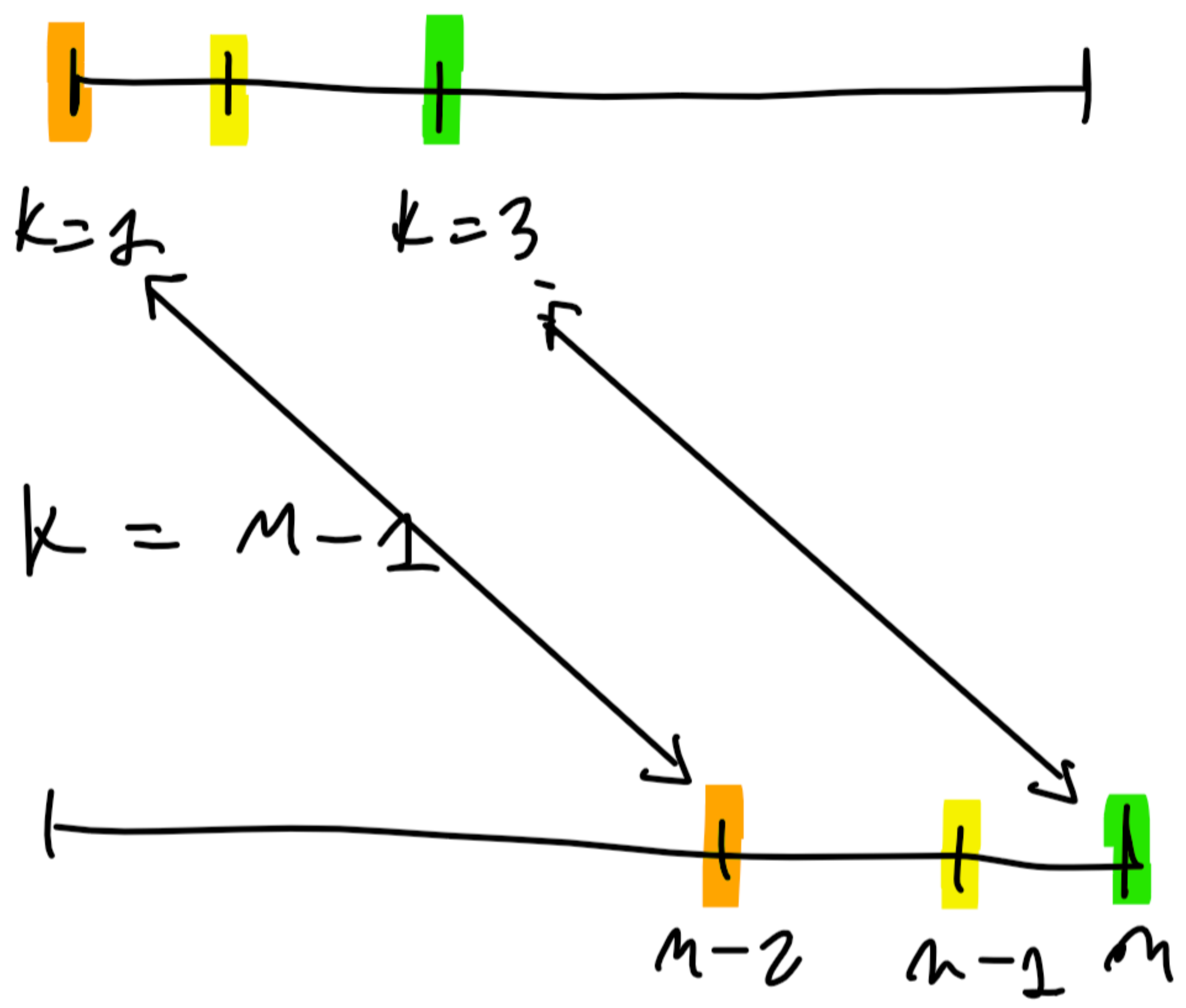
If second-order needed:

$$\left.\frac{d\varphi}{dx}\right|_1 = \frac{-3\varphi_1 + 4\varphi_2 - \varphi_3}{2\Delta x} + O(\Delta x^2)$$

$$\left.\frac{d\varphi}{dx}\right|_n = \frac{+3\varphi_n - 4\varphi_{n-1} + \varphi_{n-2}}{2\Delta y} + O(\Delta x^2)$$



$k = 2$

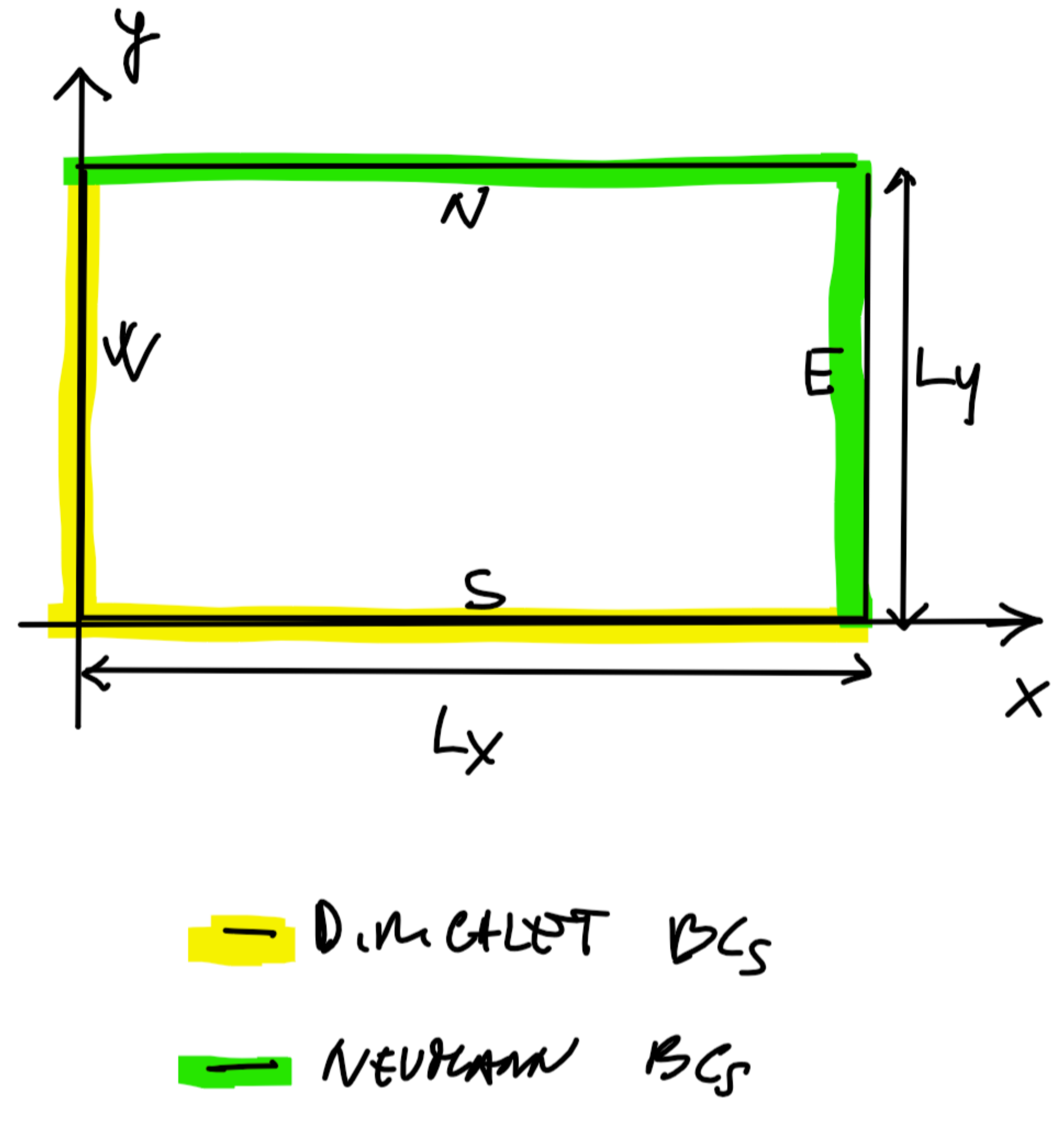


Finite difference method – 2D Poisson Equation

GOAL: $\nabla^2 \varphi = t$

HP: $\partial/\partial z = 0$, RECTANGULAR Domain

$$\begin{cases} \partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 = t(x,y) \\ \varphi(x,0) = \varphi_S \\ \varphi(0,y) = \varphi_W \\ \left.\dfrac{\partial\varphi}{\partial x}\right|_{x=L_x} = \varphi'_E \\ \left.\dfrac{\partial\varphi}{\partial y}\right|_{y=L_y} = \varphi'_N \end{cases}$$



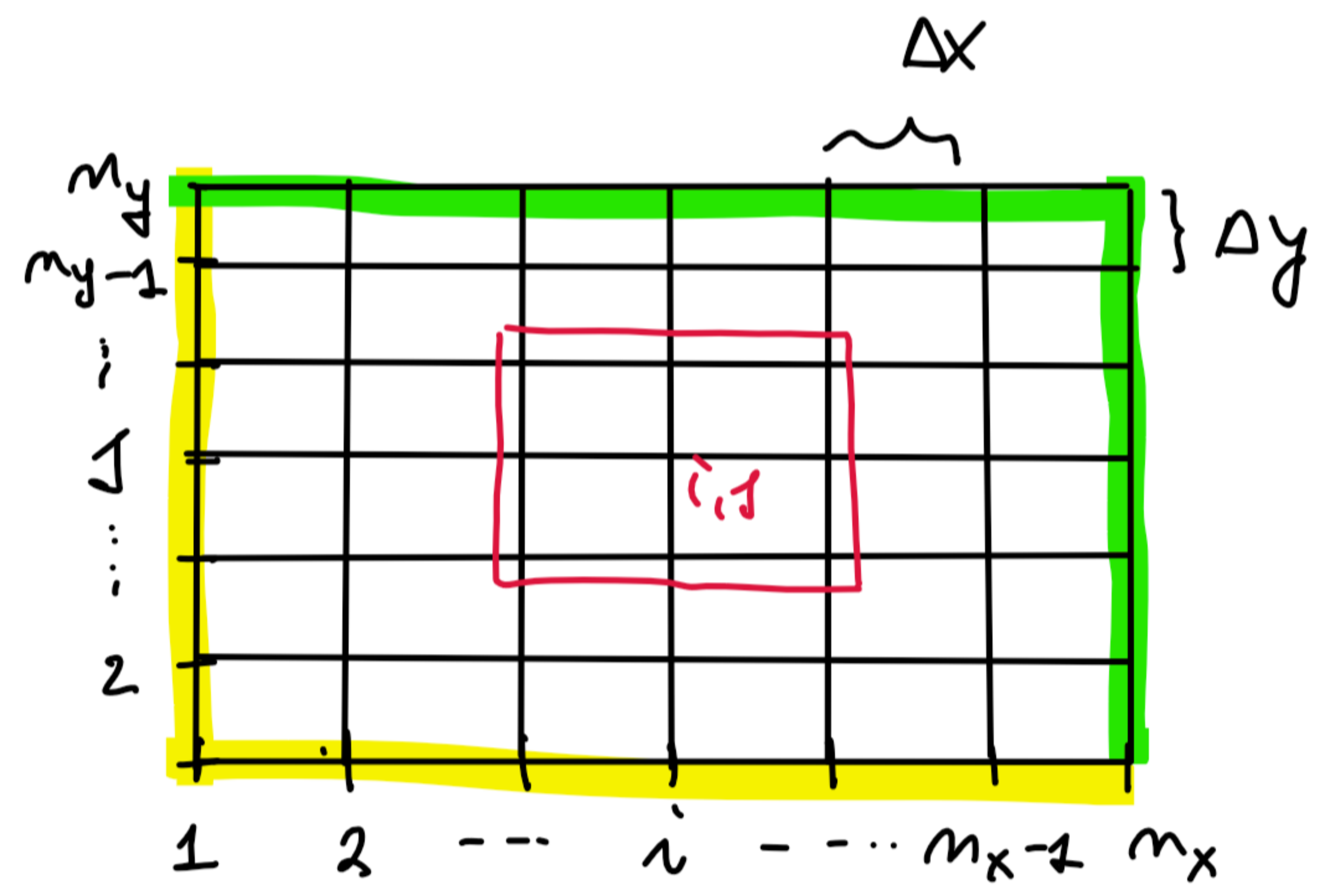
GRID DISCRETIZATION

2D uniform GRID

n_x nodes along x

n_y nodes along y

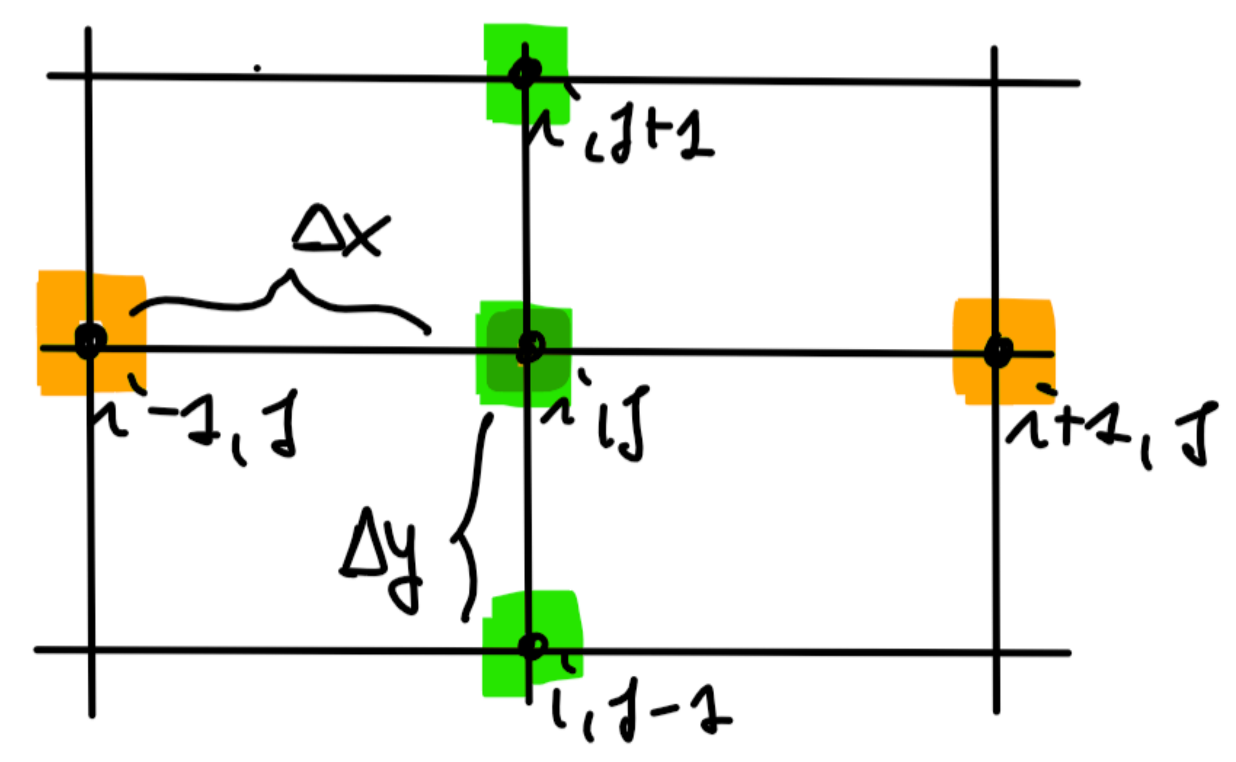
$$\Delta x = \frac{L_x}{n_x - 1} \qquad \Delta y = \frac{L_y}{n_y - 1}$$



Internal nodes (i, j)

$$\left.\frac{\partial^2\varphi}{\partial x^2}\right|_{i,j} = \frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

$$\left.\frac{\partial^2\varphi}{\partial y^2}\right|_{i,j} = \frac{\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}}{\Delta y^2} + \mathcal{O}(\Delta y^2)$$

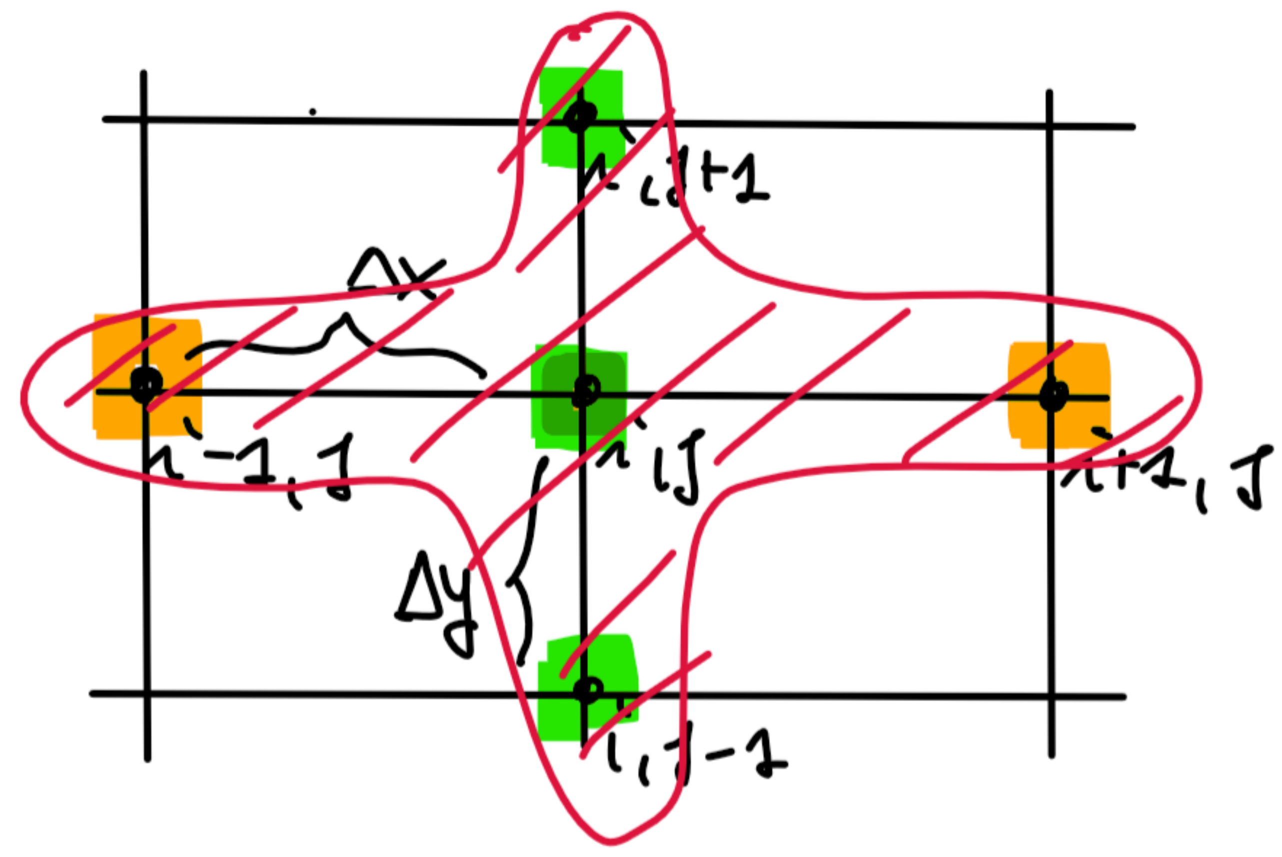


for $\begin{cases} x = i \\ y = j \end{cases}$ $\quad \left.\dfrac{\partial^2\varphi}{\partial x^2}\right|_{i,j} + \left.\dfrac{\partial^2\varphi}{\partial y^2}\right|_{i,j} = t_{i,j}$

internal nodes

$$\boxed{\frac{1}{\Delta x^2}\varphi_{i+1,j} + \frac{1}{\Delta y^2}\varphi_{i,j+1} - 2\left(1/\Delta x^2 + 1/\Delta y^2\right)\varphi_{i,j} + \frac{1}{\Delta x^2}\varphi_{i-1,j} + \frac{1}{\Delta y^2}\varphi_{i,j-1} = t_{i,j}}$$

- in 1D : 3-points STENCIL
- 2D : 5-points STENCIL



if $\Delta x = \Delta y = \Delta$, and $\underbrace{t = 0}_{\varphi \text{ is HARMONIC}}$

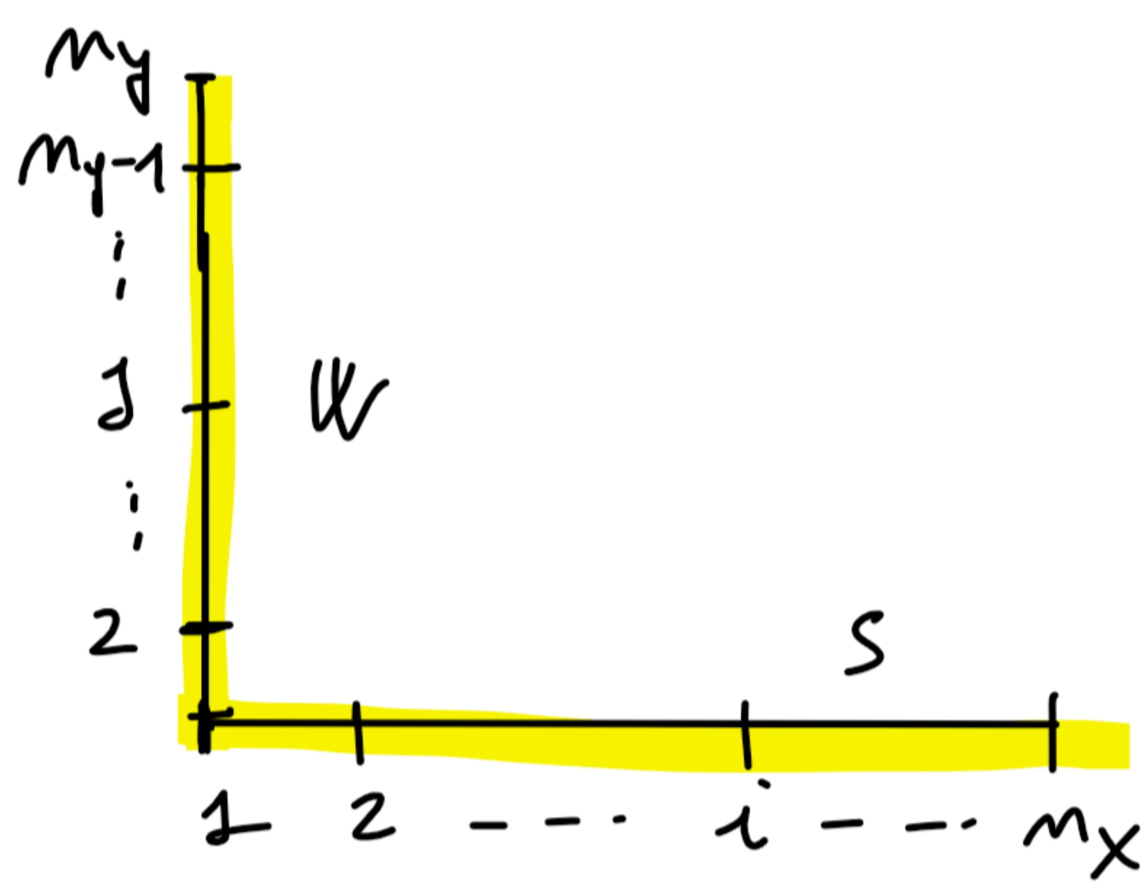
$\Downarrow$

$$1/\cancel{\Delta^2}\,\varphi_{i+1,j} + \frac{1}{\cancel{\Delta^2}}\varphi_{i,j+1} - 2\left(\frac{2}{\cancel{\Delta^2}}\right)\varphi_{i,j} + 1/\cancel{\Delta^2}\,\varphi_{i-1,j} + 1/\cancel{\Delta^2}\,\varphi_{i,j-1} = 0$$

$$\varphi_{i,j} = \frac{1}{4}\left[\varphi_{i+1,j} + \varphi_{i,j+1} + \varphi_{i-1,j} + \varphi_{i,j-1}\right]$$

$\downarrow$

$\varphi_{i,j}$ is the AVERAGE of neighbour nodes (MEAN VALUE theorem for HARMONIC FUNCTIONS)



Boundary conditions

SOUTH EDGE :

$$\boxed{\varphi_{i,1} = \varphi_S \quad , \quad i = 1, 2, \dots m_x}$$

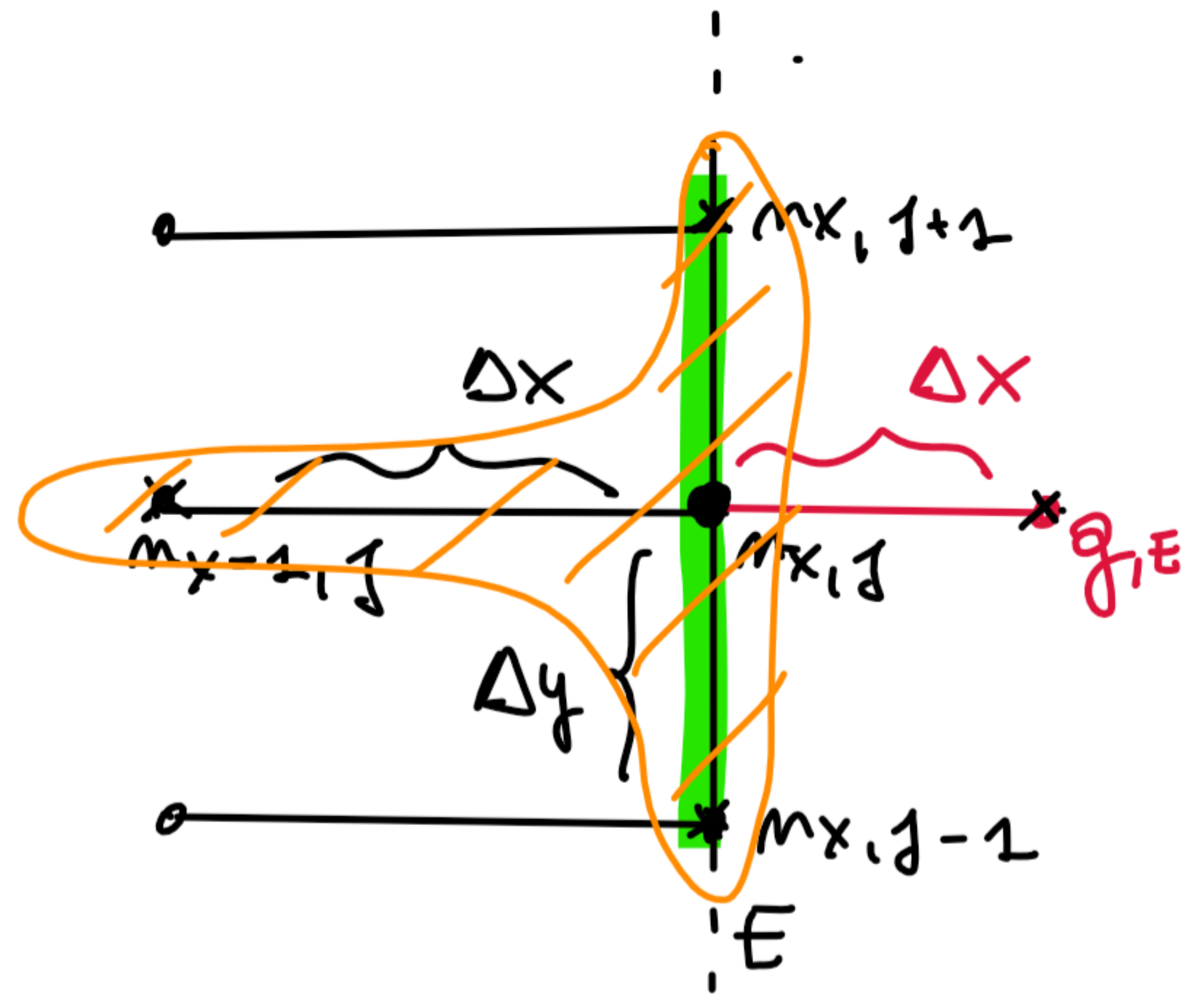
WEST EDGE

$$\boxed{\varphi_{1,j} = \varphi_W \quad , \quad j = 1, 2, \dots m_y}$$

S-W CORNER : $\boxed{\varphi_{1,1} = \frac{\varphi_S + \varphi_W}{2}}$

EAST EDGE

$$\begin{cases} \left.\dfrac{\partial\varphi}{\partial x}\right|_{mx,J} = \varphi'_E \\[2ex] \left.\dfrac{\partial^2\varphi}{\partial x^2}\right|_{mx,J} + \left.\dfrac{\partial^2\varphi}{\partial y^2}\right|_{mx,J} = t_{mx,J} \end{cases}$$



$$\begin{cases} \dfrac{\varphi_{g,E} - \varphi_{mx-1,J}}{2\Delta x} = \varphi'_E \implies \varphi_{g,E} = \varphi_{mx-1,J} + 2\Delta x\,\varphi'_E \\[2ex] \dfrac{1}{\Delta x^2}\varphi_{g,E} + \dfrac{1}{\Delta y^2}\varphi_{mx,J+1} - 2\left(\dfrac{1}{\Delta x^2}+\dfrac{1}{\Delta y^2}\right)\varphi_{mx,J} + \dfrac{1}{\Delta x^2}\varphi_{mx-1,J} + --- \\ \qquad + \cdots \dfrac{1}{\Delta y^2}\varphi_{mx,J-1} = t_{mx,J} \end{cases}$$

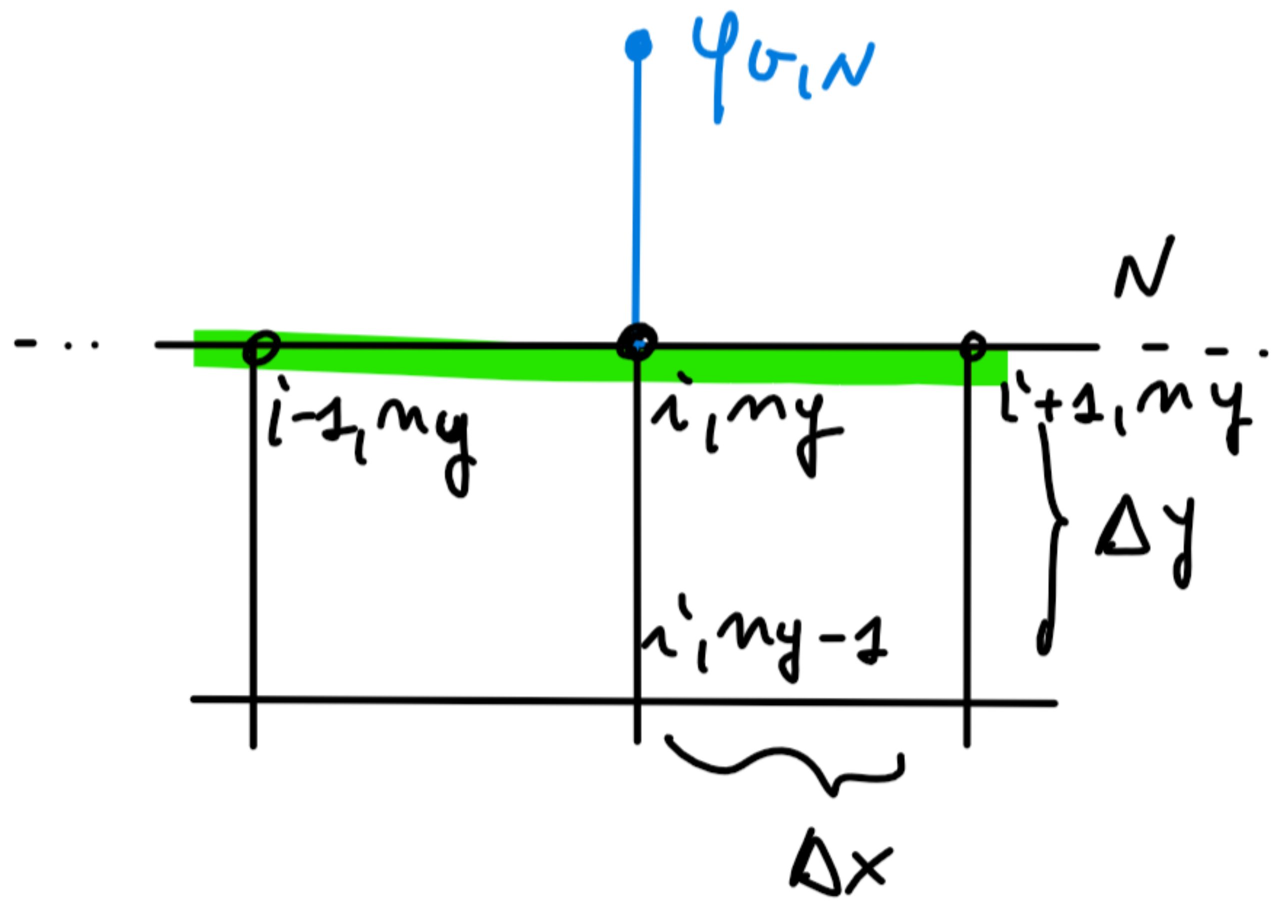
EAST SIDE

$$\boxed{\begin{aligned} &\frac{1}{\Delta y^2}\varphi_{mx,J+1} - 2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\varphi_{mx,J} + \frac{2}{\Delta x^2}\varphi_{mx-1,J} + \frac{1}{\Delta y^2}\varphi_{mx,J-1} = --- \\ &\qquad = --\, t_{mx,J} - \frac{2\cancel{\Delta x}\,\varphi'_E}{\Delta x^{\cancel{2}}} \end{aligned}}$$

4 Points stencil

NORTH SIDE

$$\begin{cases} \left.\partial\varphi/\partial y\right|_{i,my} = \varphi'_N \\[2ex] \left.\dfrac{\partial^2\varphi}{\partial x^2}\right|_{i,my} + \left.\dfrac{\partial^2\varphi}{\partial y^2}\right|_{i,my} = t_{i,my} \end{cases}$$

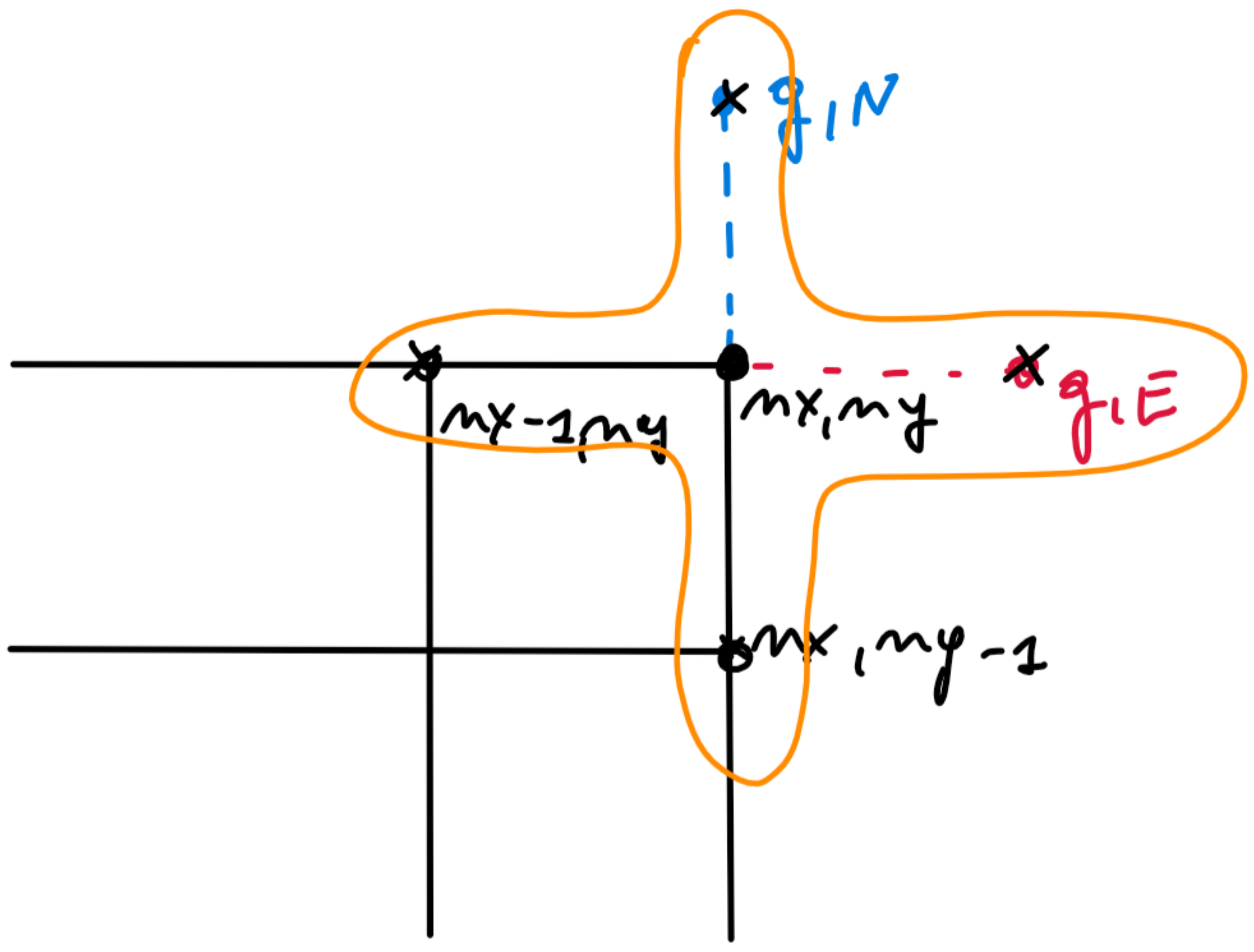


Show that :

NORTH EDGE

$$\boxed{\begin{aligned} &\frac{1}{\Delta x^2}\varphi_{i+1,my} - 2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\varphi_{i,my} + \frac{1}{\Delta x^2}\varphi_{i-1,my} + --- \\ &\qquad --- \frac{2}{\Delta y^2}\varphi_{i,my-1} = t_{i,my} - \frac{2\Delta y\,\varphi'_N}{\Delta y^2} \end{aligned}}$$

NORTH – EAST CORNER



$$
\begin{cases}
\dfrac{\varphi_{g,E} - \varphi_{mx-1,my}}{2\Delta x} = \varphi'_E \\[2ex]
\dfrac{\varphi_{g,N} - \varphi_{mx,my-1}}{2\Delta y} = \varphi'_N \\[2ex]
\frac{1}{\Delta x^2}\varphi_{g,E} + \frac{1}{\Delta y^2}\varphi_{g,N} - 2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\varphi_{mx,my} + \frac{1}{\Delta x^2}\varphi_{mx-1,my} + \frac{1}{\Delta y^2}\varphi_{mx,my-1} = t_{mx,my}
\end{cases}
$$

→ Substituting .... CORNER

$$\varphi_{g,E} = \varphi_{mx-1,my} + 2\Delta x\, \varphi'_E \qquad \varphi_{g,N} = \varphi_{mx,my-1} + 2\Delta y\, \varphi'_N$$

$$-2\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\varphi_{mx,my} + \frac{2}{\Delta x}\varphi_{mx-1,my} + \frac{2}{\Delta y}\varphi_{mx,my-1} = \dots$$

N-E corner

$$= t_{mx,my} - \frac{2\varphi'_E}{\Delta x} - \frac{2\varphi'_N}{\Delta y}$$

$$\boxed{\frac{1}{\Delta x^2}\varphi_{i+1,j} + \frac{1}{\Delta y^2}\varphi_{i,j+1} - 2\left(1/\Delta x^2 + 1/\Delta y^2\right)\varphi_{i,j} + \frac{1}{\Delta x^2}\varphi_{i-1,j} + \frac{1}{\Delta y^2}\varphi_{i,j-1} = t_{i,j}}$$

INTERNAL NODES

SOUTH EDGE :

$$\boxed{\varphi_{i,1} = \varphi_s \quad , \quad i = 1, 2, \dots n_x}$$

WEST EDGE

$$\boxed{\varphi_{1,j} = \varphi_w \quad , \quad j = 1, 2, \dots n_y}$$

S-W CORNER : $\boxed{\varphi_{1,1} = \frac{\varphi_s + \varphi_w}{2}}$

EAST SIDE

$$1/\Delta y^2\, \varphi_{nx,j+1} - 2\left(1/\Delta x^2 + 1/\Delta y^2\right)\varphi_{nx,j} + \frac{2}{\Delta x^2}\varphi_{nx-1,j} + \frac{1}{\Delta y^2}\varphi_{nx,j-1} = \dots$$

$$= \dots t_{nx,j} - \frac{2\cancel{\Delta x}\,\varphi'_E}{\Delta x^{\cancel{2}}}$$

NORTH EDGE

$$1/\Delta x^2\, \varphi_{i+1,ny} - 2\left(1/\Delta x^2 + 1/\Delta y^2\right)\varphi_{i,ny} + 1/\Delta x^2\,\varphi_{i-1,ny} + \dots$$

$$\dots 2/\Delta y^2\,\varphi_{i,ny-1} = t_{i,ny} - \frac{2\Delta y\,\varphi'_N}{\Delta y^2}$$

$$-2\left(1/\Delta x^2 + 1/\Delta y^2\right)\varphi_{nx,ny} + 2/\Delta x\,\varphi_{nx-1,ny} + 2/\Delta y\,\varphi_{nx,ny-1} = \dots$$

N-E corner

$$= t_{nx,ny} - \frac{2\varphi'_E}{\Delta x} - \frac{2\varphi'_N}{\Delta y}$$

Need to ASSEMBLE a linear system : $[K]\{\varphi\} = \{RHS\}$

1D : $[K] : n \times n$

2D: $[K] : (n_x \times n_y) \times (n_x \times n_y)$ if $n_x = 4$, $n_y = 4$ $\rightarrow [K] : 16 \times 16$