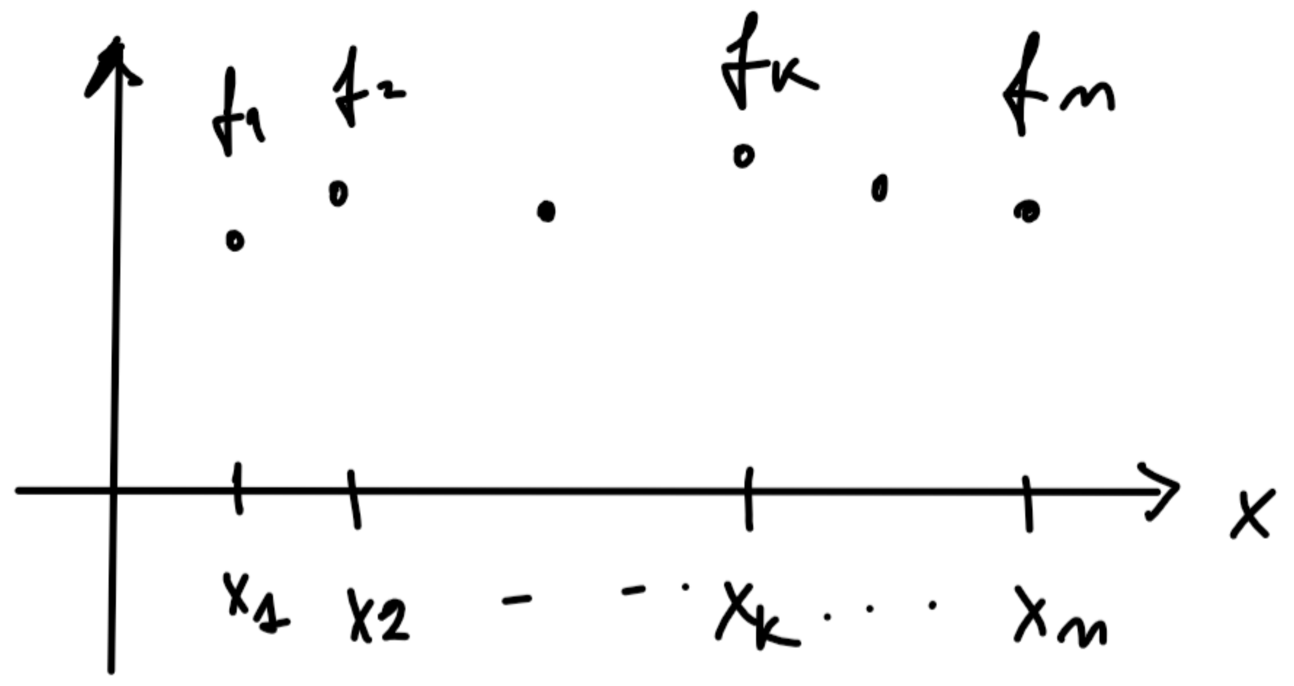
# Polynomial interpolation




$$\tilde{f}(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k + \dots a_{n-1} x^{n-1}$$

$n-1$ degree polynomial

## CONDITIONS for interpolating function

$$\begin{array}{l} x_1: \\ x_2: \\ \vdots \\ x_k: \\ \vdots \\ x_n: \end{array} \left\{ \begin{array}{l} \tilde{f}(x_1) = f(x_1) = f_1 \\ \tilde{f}(x_2) = f(x_2) = f_2 \\ \\ \tilde{f}(x_k) = \dots = f_k \\ \\ \tilde{f}(x_n) = \dots = f_n \end{array} \right.$$

using polynomial form for $\tilde{f}(x)$

$$\begin{array}{l} x_1: \\ x_2: \\ \vdots \\ x_n: \end{array} \left\{ \begin{array}{l} a_0 + a_1 x_1^1 + a_2 x_1^2 + \dots + a_{n-1} x_1^{n-1} = f_1 \\ a_0 + a_1 x_2^1 + a_2 x_2^2 + \dots + a_{n-1} x_2^{n-1} = f_2 \\ \vdots \\ a_0 + a_1 x_n^1 + a_2 x_n^2 + \dots + a_{n-1} x_n^{n-1} = f_n \end{array} \right.$$

→ Solve to get $a_0, a_1, a_2, \dots a_{n-1}$

$[V]$

$$\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{array} \begin{bmatrix} 1 & x_1^1 & x_1^2 & \dots & x_1^k & \dots & x_1^{n-1} \\ 1 & x_2^1 & x_2^2 & \dots & x_2^k & \dots & x_2^{n-1} \\ & \vdots & & & \vdots & & \vdots \\ 1 & x_k^1 & x_k^2 & \dots & x_k^k & \dots & x_k^{n-1} \\ & \vdots & & & \vdots & & \vdots \\ 1 & x_n^1 & x_n^2 & \dots & x_n^k & \dots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_k \\ \vdots \\ f_n \end{bmatrix}$$

VANDERMONDE matrix

$$\downarrow$$

$$\rightarrow [V][a] = [f]$$

there exist $[V^{-1}]$ such that : $[V^{-1}]$ : inverse of $[V]$

$$[V][a] = [f]$$

$$\underbrace{[V^{-1}][V]}[a] = [V^{-1}][f]$$

$$[I_d]\cdot[a] = [V^{-1}][f]$$

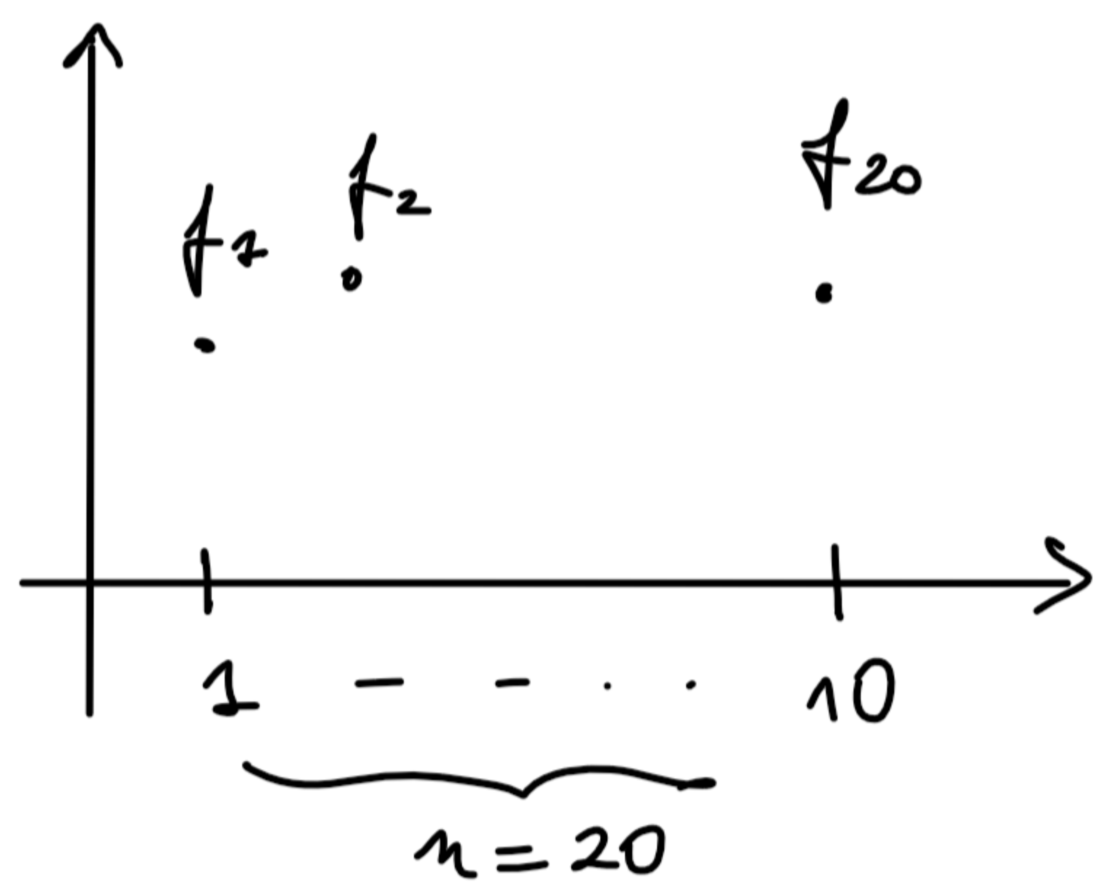
$$[a] = [V^{-1}][f]$$

$$[a] = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

- $[V]$ is linearly independent it <u>CAN</u> be inverted in THEORY!
  RANK $n$

- in PRACTICE $[V]$ is often $\boxed{\text{ILL-CONDITIONED matrix}}$

EXAMPLE : $[x_1, x_2, \ldots x_n]$ : $x_1 = 1$, $x_n = 10$, $n = 20$




$$[V] = \begin{array}{c} \begin{array}{cccccc} & x_1 = 1 & & & & n-1 \end{array} \\ \begin{bmatrix} 1 & 1^1 & 1^2 & \ldots & 1^{19} \\ - & - & - & - & - \\ 1 & 10^1 & 10^2 & \ldots & 10^{19} \end{bmatrix} \\ \begin{array}{cccccc} & x_n = 10 & & & & \end{array} \end{array}$$

$\min[V] = 1$

$\max[V] = 10^{19}$

$\varepsilon \simeq 2{,}2\ 10^{-16}$

<u>CONDITION NUMBER</u> $K$ : unified measure of the sensitivity of a linear system $[A][x]=[b]$ to ERRORS

CONDITION NUMBER $K$ : unified measure of the sensitivity of a linear system $[A][x]=[b]$ to ERRORS

- EXTERNAL ERRORS: how sensitive is SOLUTION $[x]$ to small changes in $[b]$
- INTERNAL ERRORS: how sensitive is SOLUTION to roundoff errors in $[A]$

$K \sim 1 \rightarrow$ SYSTEM is STABLE errors not significantly magnified

$K \gg 1 \rightarrow$ SYSTEM is UNSTABLE errors are significantly amplified

Def: $$\underbrace{K([A])}_{\text{cond. numb. for matrix } [A]} = \underbrace{\|A\|_2}_{M} \cdot \underbrace{\|A^{-1}\|_2}_{1/m} = M/m$$

p-norm of vector $[x]$

$$\|X\|_p = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \qquad x_i : \text{ i-th entry of } [x]$$

2-norm [Euclidean norm]

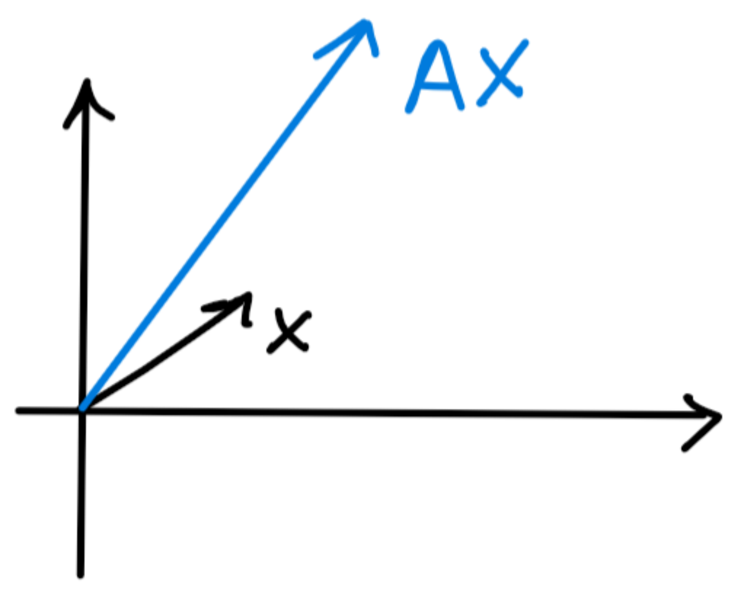
$$\|X\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

MAGNITUDE of $[x]$

how far $[x]$ extends from the origin

For a matrix:

- $$\|A\|_2 = \max\left(\frac{\|Ax\|_2}{\|x\|_2}\right) = M$$

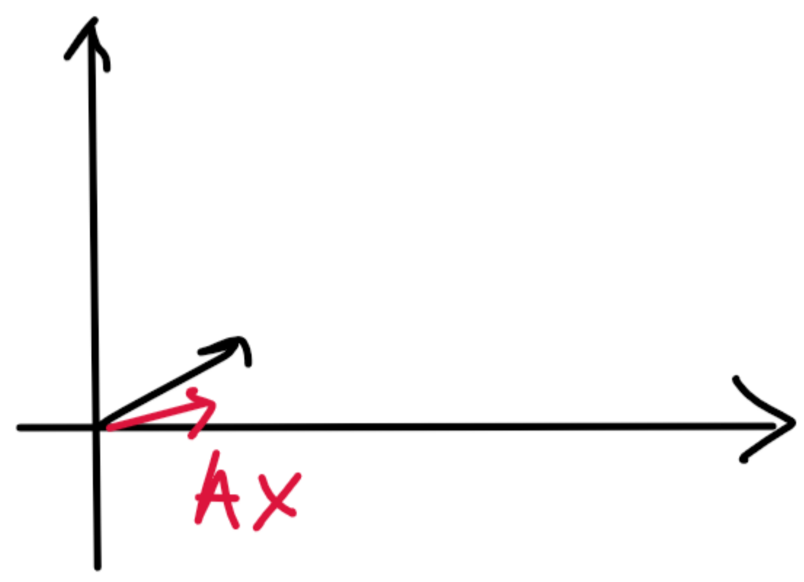



recall: $[A]$ matrix
$[x]$ vector

$[A][x]$ is a vector

MAXIMUM STRETCHING of $X$ due to $A$

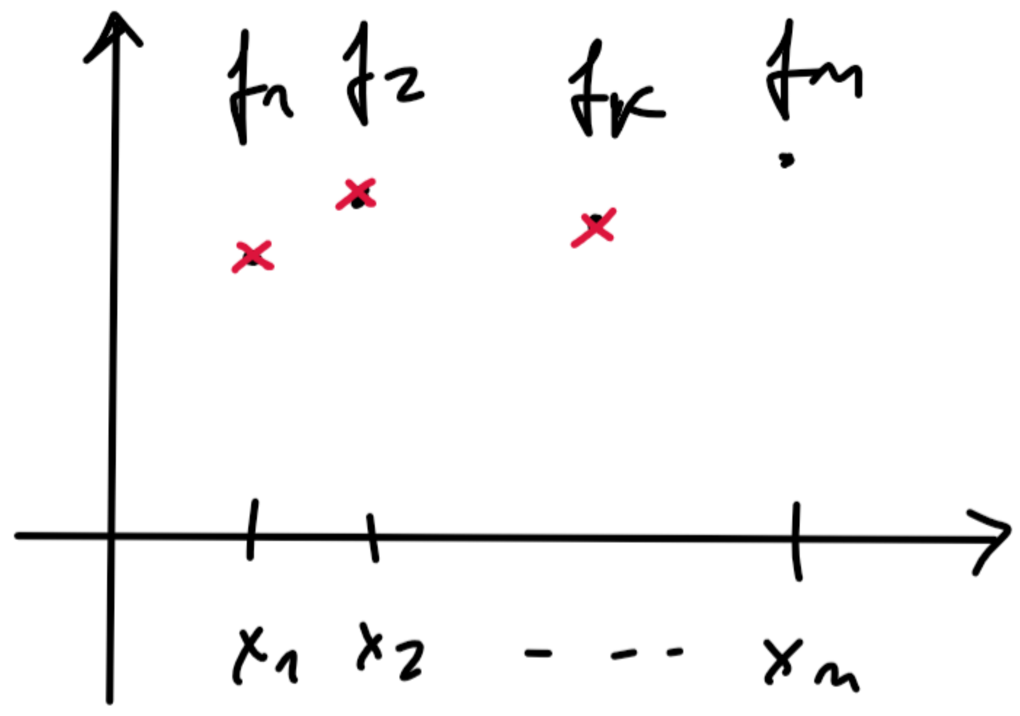
- $$\frac{1}{\|A^{-1}\|_2} = \min\left(\frac{\|Ax\|_2}{\|x\|_2}\right) = m$$




MINIMUM STRETCHING of $X$ due to $A$

we need different strategy to perform polynomial interpolation

## LAGRANGE POLYNOMIALS




$$\tilde{f}(x) = f_1 \mathcal{L}_1(x) + f_2 \mathcal{L}_2(x) + \dots + f_k \mathcal{L}_k(x) + \dots f_n \mathcal{L}_n(x)$$

$n$ - polynomials of degree $n-1$

$$\mathcal{L}_1(x) = \frac{(x - x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} \leftarrow n-1 \text{ products}$$

$$\mathcal{L}_1(x_1) = \frac{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} = 1 \quad ; \quad \mathcal{L}_1(x_2) = 0 \quad \mathcal{L}_1(x_k) = 0 \quad \dots$$

$x_2 - x_2 = 0$

$$\mathcal{L}_2(x) = \frac{(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} \rightarrow \begin{array}{l} \mathcal{L}_2(x_1) = 0 \\ \mathcal{L}_2(x_2) = 1 \\ \vdots \\ \mathcal{L}_2(x_n) = 0 \end{array}$$

$$\mathcal{L}_k(x) = \frac{\prod_{i=1, i\neq k}^{n} (x - x_i)}{\prod_{i=1, i\neq k}^{n} (x_k - x_i)}$$

$$\mathcal{L}_2(x) = \frac{(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} \leftarrow \alpha$$

$$\mathcal{L}_2(x_n) = \frac{(x_n-x_1)(x_n-x_3)\cancel{(x_n-x_n)}\cdots(x_n-x_n)}{(x_2-x_1)(x_2-x_3)\cdots} = 0$$