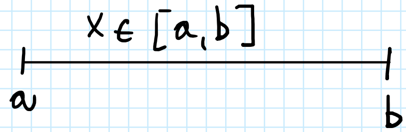
FEM Poisson's equation 1D

Monday, December 2, 2024 1:42 PM

Finite Element Method



Hp: 1D

$$\nabla \cdot (p \nabla \varphi) = t \xrightarrow{\partial/\partial y = 0 \quad \partial/\partial z = 0} \boxed{\frac{d}{dx}\left[p(x)\frac{d\varphi}{dx}\right] = t(x)}$$

$p(x) = \varepsilon_r$

$t(x) = -\rho/\varepsilon_0$

ELECTROSTATIC POTENTIAL

$$\boxed{\frac{d}{dx}\left[\varepsilon_r \frac{d\varphi}{dx}\right] = -\rho/\varepsilon_0}$$

Formulation [STRONG FORMULATION]

"Find $\varphi(x)$ such that this $\Rightarrow$ is true for every $x \in [a,b]$"

$$\begin{cases} \frac{d}{dx}\left[p(x)\frac{d\varphi}{dx}\right] = t(x) & x \in]a,b[\\ \varphi(a) = \varphi_a & \text{Dirichlet BC}, \ x = a \\ \left.\frac{d\varphi}{dx}\right|_b = \varphi'_b & \text{Neumann BC}, \ x = \end{cases}$$

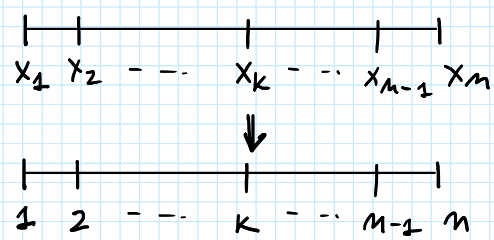
Dirichlet BC: prescribing value of φ in a

Neumann BC: prescribing value of $d\varphi/dx$ in b

DISCRETIZATION

- introduce n-points between a and b

n-NODES $\Rightarrow$ $n-1$ ~~intervals~~ ELEMENTS



if GRID is uniform : $\Delta = \frac{L}{n-1} = \frac{b-a}{n-1}$

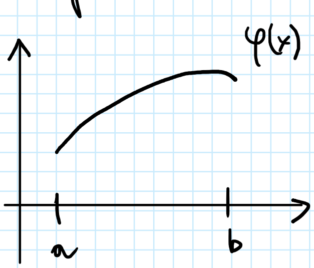
$\sim \tanh$ — $\sim 10^{-7}$ m, $\sim 10^{-1}$ m

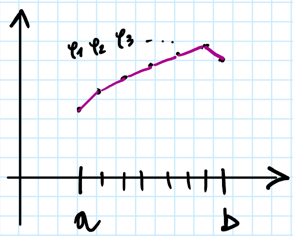
INTERPOLATION

$$\varphi(x) \longrightarrow \tilde{\varphi}(x)$$

original unknown function

PIECEWISE INTERPOLATION of the unknown function



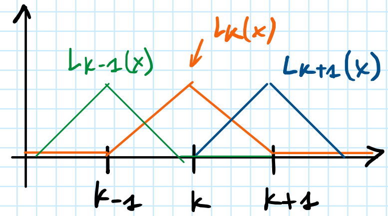


"Find $\varphi_1, \varphi_2, \dots \varphi_n$"

now $\tilde{\varphi}(x)$ is an APPROXIMATION of the true solution

$$\tilde{\varphi}(x) = \varphi_1 L_1(x) + \varphi_2 L_2(x) + \dots + \varphi_k L_k(x) + \dots, \varphi_{n-1} L_{n-1}(x) + \varphi_n L_n(x)$$

HAT FUNCTIONS as interpolant functions
[SHAPE FUNCTIONS]



WEIGHTED RESIDUALS APPROACH

if exact solution $\varphi(x)$ $\longrightarrow$ $\boxed{\frac{d}{dx}\left[p(x)\frac{d\varphi}{dx}\right] - t(x) = 0}$

we have switched from $\varphi(x) \Rightarrow \tilde{\varphi}(x)$

$$\underbrace{\frac{d}{dx}\left[p(x)\frac{d\tilde{\varphi}}{dx}\right] - t(x)}_{r(x)\ \text{RESIDUAL}} \neq 0$$

$r(x)$ RESIDUAL $\Rightarrow$ CANNOT BE 0 $\forall x$

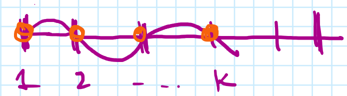
W.R. approach : REQUIRE that the WEIGHTED RESIDUAL is ZERO over the domain

$$\int_a^b w(x)\, r(x)\, dx = 0$$

WEIGHTING FUNCTION

(ALESSANDRO'S CHOICE)

$w(x) \Rightarrow \delta(x - x_k)$



$$\int_a^b \underbrace{w(x)}_{f(x)} \underbrace{\frac{d}{dx}\Big[\overbrace{p(x)\frac{d\tilde{\varphi}}{dx}}^{g(x)}\Big]}_{g'(x)} dx = \int_a^b w(x)\, t(x)\, dx$$

$\Rightarrow \tilde{\varphi} \in C_0$ PIECEWISE linear function

$$\int_a^b f(x) g'(x)\, dx = \Big[f(x) g(x)\Big]_a^b - \int_a^b f'(x) g(x)\, dx$$

integration by parts

$$\left[w(x)\, p(x) \frac{d\tilde{\varphi}}{dx}\right]_a^b - \int_a^b \frac{dw}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx = \int_a^b w(x)\, t(x)\, dx$$

NO SECOND DERIVATIVES of $\tilde{\varphi} \in C_0$

$$\left[w(x)\,p(x)\,\frac{d\tilde{\varphi}}{dx}\right]_a^b - \int_a^b \frac{dw}{dx}\,p(x)\,\frac{d\tilde{\varphi}}{dx}\,dx = \int_a^b w(x)\,t(x)\,dx$$

NO SECOND DERIVATIVES of $\tilde{\varphi} \in C_0$

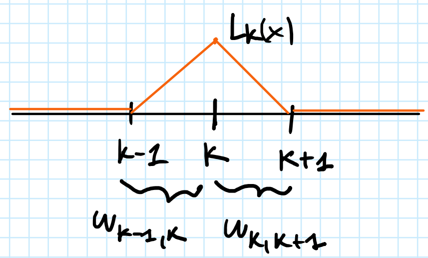
$$\boxed{\int_a^b \frac{dw}{dx}\,p(x)\,\frac{d\tilde{\varphi}}{dx}\,dx = \left[w(x)\,p(x)\,\frac{d\tilde{\varphi}}{dx}\right]_a^b - \int_a^b w(x)\,t(x)\,dx}$$

WEAK FORMULATION

GALERKIN'S CHOICE $\rightarrow$ $\boxed{W(x) \Rightarrow L_K(x)}$

residual weighted by hat functions !!

$$\int_{\Omega_K} \frac{dL_K}{dx}\,p(x)\,\frac{d\tilde{\varphi}}{dx}\,dx = \left[L_K(x)\,p(x)\,\frac{d\tilde{\varphi}}{dx}\right]_a^b - \int_{\Omega_K} L_K(x)\,t(x)\,dx \qquad \boxed{K = 1, 2, \dots, n-1, n}$$



SUPPORT DOMAIN of node K

$\Rightarrow$ $L_K(x) \neq 0$ only on Ω_K

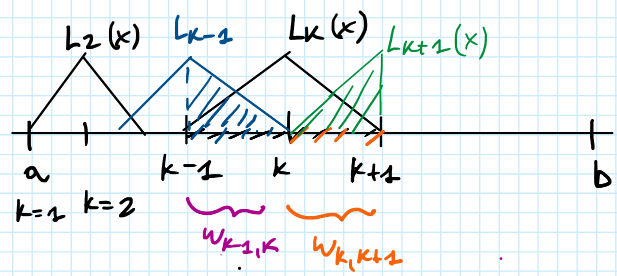
$\Rightarrow$ restrict domain of integration from $[a,b] \Rightarrow \Omega_K$

find $\varphi_1, \varphi_2, \dots \varphi_n$ such that the n-expressions hold true

internal nodes $\quad K = 2, 3, \dots, n-1$

$$\int_{\Omega_K} \frac{dL_K}{dx}\,p(x)\,\frac{d\tilde{\varphi}}{dx}\,dx = \cancel{\left[L_K(x)\,p(x)\,\frac{d\tilde{\varphi}}{dx}\right]_a^b} - \int_{\Omega_K} L_K(x)\,t(x)\,dx$$

$$\Omega_K = \underline{w_{K-1,K}} \cup \underline{w_{K,K+1}}$$



$$\left[\int_{w_{K-1,K}} \frac{dL_K}{dx}\,p(x)\,\frac{dL_{K-1}}{dx}\,dx\right]\varphi_{K-1} + \left[\int_{w_{K-1,K}} \frac{dL_K}{dx}\,p(x)\,\frac{dL_K}{dx}\right]\varphi_K + \dots$$

$$\tilde{\varphi} = \begin{cases} \boxed{\varphi_{K-1}\,L_{K-1}(x) + \varphi_K\,L_K(x)} & \text{in } w_{K-1,K} \\ \boxed{\varphi_K\,L_K(x) + \varphi_{K+1}\,L_{K+1}(x)} & \text{in } w_{K,K+1} \end{cases}$$

$$\left[\int \frac{dL_K}{dx}\,p(x)\,\frac{dL_K}{dx}\,dx\right]\varphi_K + \left[\int \frac{dL_K}{dx}\,p(x)\,\frac{dL_{K+1}}{dx}\,dx\right]\varphi_{K+1} =$$

$$\Bigg[\underset{w_{k,k+1}}{\int} \frac{dL_k}{dx}\, p(x)\, \frac{dL_k}{dx}\, dx\Bigg] \varphi_k + \Bigg[\underset{w_{k,k+1}}{\int} \frac{dL_k}{dx}\, p(x)\, \frac{dL_{k+1}}{dx}\, dx\Bigg] \varphi_{k+1} =$$

$$- \underset{w_{k-1,k}}{\int} L_k(x)\, t(x)\, dx - \underset{w_{k,k+1}}{\int} L_k(x)\, t(x)\, dx$$

$$L_{k-1}(x) = \begin{cases} \ldots \\ \\ \end{cases} \Rightarrow \quad \frac{dL_{k-1}}{dx} = \begin{cases} \\ \\ \end{cases}$$

$$L_k(x) = \begin{cases} \\ \\ \end{cases} \Rightarrow \quad \frac{dL_k}{dx} = \begin{cases} \\ \\ \end{cases}$$

$$L_{k+1}(x) = \begin{cases} \\ \\ \end{cases} \Rightarrow \quad \frac{dL_{k+1}}{dx} = \begin{cases} \\ \\ \end{cases}$$