L13

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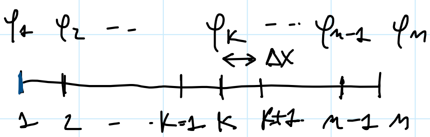
$$\nabla^2 \varphi = t \overset{1D}{\Rightarrow} \frac{d^2\varphi}{dx^2} = t \Rightarrow \varphi$$



GOAL. $\nabla \varphi \overset{1D}{=} \frac{d\varphi}{dx}$

$$\left.\frac{d\varphi}{dx}\right|_K = \frac{\varphi_{K+1} - \varphi_{K-1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$ ok for internal nodes

ONE-SIDED difference formulas for boundary nodes.

NODE 1: $\left.\frac{d\varphi}{dx}\right|_1 = \frac{\varphi_2 - \varphi_1}{\Delta x} + \mathcal{O}(\Delta x^1)$

NODE N: $\left.\frac{d\varphi}{dx}\right|_n = \frac{\varphi_n - \varphi_{n-1}}{\Delta x} + \mathcal{O}(\Delta x^1)$

} BOUNDARY NODES

Homework. try to use second-order one-sided expressions to find $\left.d\varphi/dx\right|_1$ and $\left.d\varphi/dx\right|_n$

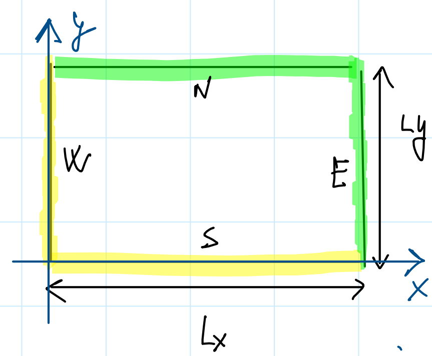
FDM 2D

$$\nabla^2 \varphi = t, \quad \partial\varphi/\partial z = 0 \Rightarrow \varphi = \varphi(x,y)$$

if CARTESIAN COORDINATES:

$$\nabla^2 \varphi = t \Rightarrow \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = t(x,y)$$

Hp: RECTANGULAR DOMAIN



Formulation

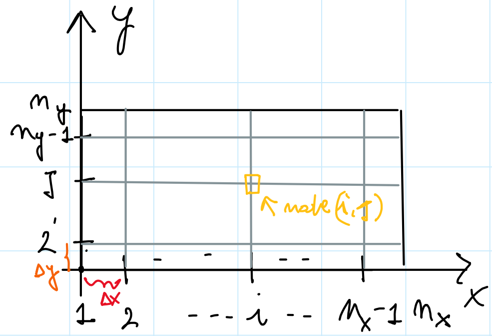
$$\left\{ \partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 = t(x,y) \right.$$

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2} = t(x,y) \\ \left.\begin{array}{l} \varphi(x, y=0) = \varphi_S \\ \varphi(x=0, y) = \varphi_W \end{array}\right\} \text{DIRICHLET BCs} \\ \left.\begin{array}{l} \left.\dfrac{\partial\varphi}{\partial x}\right|_E = \varphi'_E \\ \left.\dfrac{\partial\varphi}{\partial y}\right|_N = \varphi'_N \end{array}\right\} \text{NEUMANN BCs} \end{cases}$$

DISCRETIZATION :

n_x nodes along x – direction $\Leftrightarrow L_x$

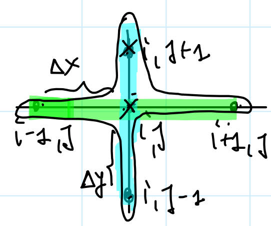
n_y nodes along y – direction $\Leftrightarrow L_y$



Total number of nodes; $N = n_x \cdot n_y$

$\Delta x = \dfrac{L_x}{n_x - 1}$; $\Delta y = \dfrac{L_y}{n_y - 1}$

zoom (i,j)



INTERNAL NODES (i,j)

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = t(x,y) \quad \text{for} \quad \begin{array}{l} x \to i \\ y \to j \end{array}$$

$$\left.\frac{\partial^2\varphi}{\partial x^2}\right|_{i,j} = \frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{\Delta x^2} + O(\Delta x^2)$$

$$\left.\frac{\partial^2\varphi}{\partial y^2}\right|_{i,j} = \frac{\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}}{\Delta y^2} + O(\Delta y^2)$$

REMINDER of the error that we make when we approximate $\left.\dfrac{\partial^2\varphi}{\partial x^2}\right|_{i,j} = \dfrac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{\Delta x^2}$

DISCRETIZED EXPRESSION FOR NODE i,j

$$\frac{\varphi_{i+1,j} - \boxed{2\varphi_{i,j}} + \varphi_{i-1,j}}{\Delta x^2} + \frac{\varphi_{i,j+1} - \boxed{2\varphi_{i,j}} + \varphi_{i,j-1}}{\Delta y^2} = t_{i,j}$$

$$\Rightarrow \frac{1}{\Delta x^2}\varphi_{i+1,j} + \frac{1}{\Delta y^2}\varphi_{i,j+1} - 2\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\varphi_{i,j} + \frac{1}{\Delta x^2}\varphi_{i-1,j} + \frac{1}{\Delta y^2}\varphi_{i,j-1} = t_{i,j}$$

"FIVE – POINTS STENCIL" –

"$\boxed{\text{FIVE - POINTS STENCIL}}$" -

ensemble of nodal values involved in the computation of the unknown function for a given node

1D - THREE-POINTS STENCIL

node K

$$\boxed{\frac{\varphi_{K+1} - 2\varphi_K + \varphi_{K-1}}{\Delta x^2} = t_K}$$

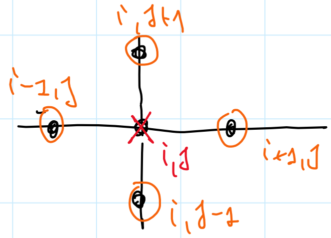
Hp: $\underbrace{\Delta x = \Delta y}_{\Delta}$, $t = 0 \Rightarrow \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \iff$ HARMONIC PROBLEM

$$\frac{1}{\cancel{\Delta^2}}\varphi_{i+1,j} + \frac{1}{\cancel{\Delta^2}}\varphi_{i,j+1} - 2\overbrace{\left(\frac{1}{\cancel{\Delta^2}} + \frac{1}{\cancel{\Delta^2}}\right)}^{2}\varphi_{i,j} + \frac{1}{\cancel{\Delta^2}}\varphi_{i-1,j} + \frac{1}{\cancel{\Delta^2}}\varphi_{i,j-1} = \cancel{t_{i,j}\Delta^2}$$

$$1\cdot\varphi_{i+1,j} + 1\,\varphi_{i,j+1} - 4\varphi_{i,j} + \varphi_{i-1,j} + \varphi_{i,j-1} = \cancel{t_{i,j}\Delta^2} = 0$$

$$\varphi_{i,j} = \underbrace{\frac{1}{4}\left[\varphi_{i+1,j} + \varphi_{i,j+1} + \varphi_{i-1,j} + \varphi_{i,j-1}\right]}_{\text{MEAN of neighbours of node } i,j}$$

MEAN VALUE theorem for harmonic functions



BOUNDARY CONDITIONS

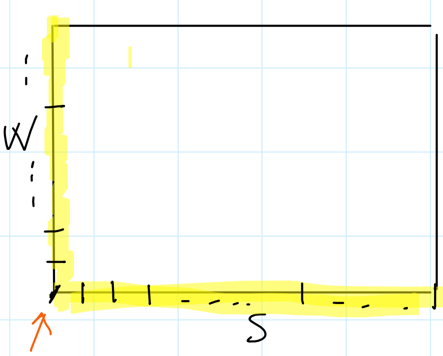
DIRICHLET EDGES:

$\varphi(x, y=0) = \varphi_S$

$\varphi_{i,j}$, for $i = 2, n_x$, $j = 1$ $= \varphi_S$

$\varphi(x=0, y) = \varphi_W$

$\varphi_{i,j}$, for $i = 1$, $j = 2, n_y$ $= \varphi_W$



SW CORNER, $i = 1$, $j = 1$ $\Rightarrow$ AVERAGE

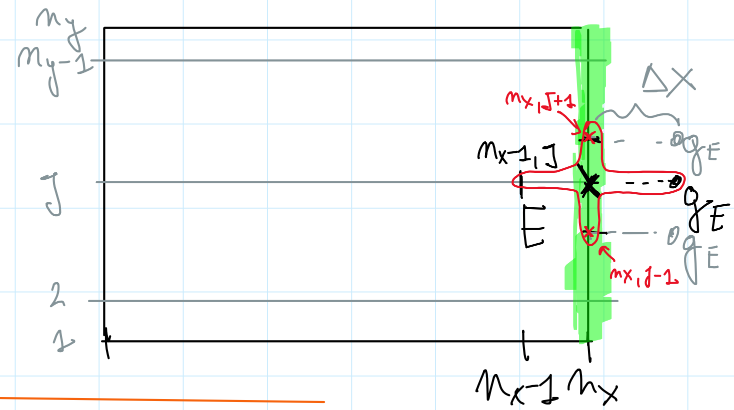
SW CORNER, $i=1,\ j=1 \Rightarrow$ AVERAGE

$$\varphi_{1,1} = \frac{\varphi_S + \varphi_W}{2}$$

$$\frac{\varphi_{mx,j} - \varphi_{mx-1,j}}{\Delta x} = \varphi'_E \quad \text{[FIRST-ORDER F.D.]}$$

EAST EDGE

$$\begin{cases} \left.\dfrac{\partial \varphi}{\partial y}\right|_E = \varphi'_E \\[2ex] \left.\dfrac{\partial^2 \varphi}{\partial x^2}\right|_{mx,j} + \left.\dfrac{\partial^2 \varphi}{\partial y^2}\right|_{mx,j} = t_{mx,j} \end{cases}$$



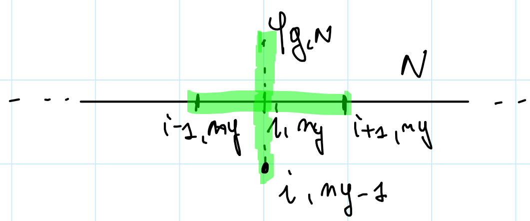
$$\rightarrow \begin{cases} \dfrac{\varphi_{g,E} - \varphi_{mx-1,j}}{2\Delta x} = \varphi'_E \Rightarrow \boxed{\varphi_{g,E} = \varphi'_E 2\Delta x + \varphi_{mx-1,j}} \\[2ex] \underbrace{\dfrac{1}{\Delta x^2}\varphi_{g,E}}_{\boxed{\frac{\varphi'_E 2\cancel{\Delta x}}{\Delta x^{\cancel{2}}} + \frac{\varphi_{mx-1,j}}{\Delta x^2}}} + \dfrac{1}{\Delta y^2}\varphi_{mx,j+1} - 2\left(\dfrac{1}{\Delta x^2} + \dfrac{1}{\Delta y^2}\right)\varphi_{mx,j} + \dfrac{1}{\Delta x^2}\varphi_{mx-1,j} + \dfrac{1}{\Delta y^2}\varphi_{mx,j-1} = t_{mx,j} \end{cases}$$

↑ goes RHS

$\Downarrow$ 4-POINTS STENCIL

$$\boxed{\frac{1}{\Delta y^2}\varphi_{mx,j+1} - 2\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\varphi_{mx,j} + \frac{2}{\Delta x^2}\varphi_{mx-1,j} + \frac{1}{\Delta y^2}\varphi_{mx,j-1} = t_{mx,j} - \frac{2\varphi'_E}{\Delta x}}$$

NORTH EDGE



TRY TO DERIVE EXPRESSIO