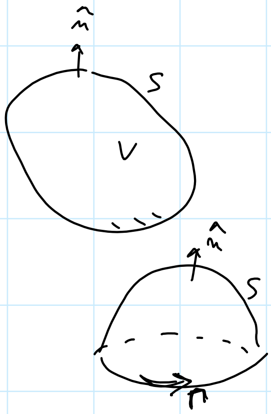
Electromagnetic in matter

MAXWELL'S EQNS



$$\nabla\cdot\bar{D} = \rho \longrightarrow \oint_S \bar{D}\cdot d\bar{S} = Q$$

$$\nabla\cdot\bar{B} = 0 \longrightarrow \oint_S \bar{B}\cdot d\bar{S} = 0$$

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} \longrightarrow \oint_\Gamma \bar{E}\cdot d\bar{l} = -\frac{d}{dt}\int_S \bar{B}\cdot d\bar{S} = -\frac{d\Phi_{B,\Gamma}}{dt}$$

$$\nabla\times\bar{H} = \bar{J} + \frac{\partial\bar{D}}{\partial t} \longrightarrow \oint_\Gamma \bar{H}\cdot d\bar{l} = i + i_d$$

CONTINUITY EQ

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\bar{J} = 0 \longrightarrow \frac{dQ}{dt} + i = 0 \Rightarrow \boxed{i = -\frac{dQ}{dt}}$$

MATERIAL CONSTITUTIVE LAWS

Hp. linear & isotropic materials

$$\begin{cases} \bar{D} = \varepsilon_0\varepsilon_r\bar{E} \\ \bar{B} = \mu_0\mu_r\bar{H} \\ \bar{J} = \sigma(\bar{E} + \bar{E}_i) \end{cases}$$

SIGMA (electric conductivity S/m)

LOCAL OHM'S LAW ⇒ $\bar{J} = \sigma(\bar{E} + \bar{E}_i)$

$\bar{E}$: ELECTROSTATIC (conservative) el. fields

$\bar{E}_i$: ELECTROMOTIVE (non-conservative) el fields [IMPRESSED]

EXAMPLE PROBLEM infinitely long cylinder carrying a current i

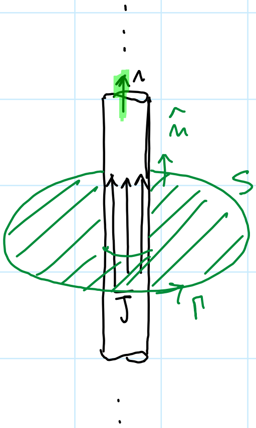
GOAL: compute $\bar{H}$ using a SCALAR POTENTIAL (ψ)

$\Downarrow$

if $\nabla\times\bar{H} = 0$ $\forall P$ in a SCD

↓

if $\underbrace{\nabla \times \bar{H} = 0}_{\bar{H} = -\nabla\psi}$ ∀ p in a SCD



if $D = \mathbb{R}^3 \Rightarrow$ inside cylinder $\bar{J} \neq 0$

$\nabla \times \bar{H} = \bar{J} \Rightarrow$ NON CONSERVATIVE

IDEA: restrict domain : $D' = \mathbb{R}^3 \setminus cyl$

$\nabla \times \bar{H} = 0 \quad \forall \bar{p} \in D' \Rightarrow D'$ not a SCD

$\cancel{\bar{H} = -\nabla\psi}$

⇓

S bounded by $\Gamma \notin D'$

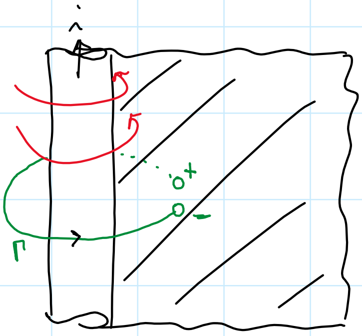
check: $\oint_\Gamma \bar{H} \cdot d\bar{l} = i \Rightarrow \bar{H}$ not CONSERVATIVE in D'

↓

IDEA: use a SEMI-INFINITE DIAPHRAM (infinitely thin)

$D'' = D \setminus cyl \setminus diaphram \qquad \nabla \times \bar{H} = 0$ on a SCD

↑ SCD



$$\int_\Gamma \bar{H} \cdot dl = i$$

↓

$$\int_+^- -\nabla\psi \cdot d\bar{l} = i$$

↓

$\partial\psi / \partial l \; dl$

$$-\int_+^- \partial\psi / \partial l \; dl = i$$

How to set BC for ψ when solving $\boxed{\nabla^2\psi = 0}$

⇓

$$-\left[\psi_- - \psi_+\right] = i \Rightarrow \boxed{\psi_+ - \psi_- = i}$$

Poynting theorem

Hp. linear and isotropic materials

$$\left| \; \nabla \times \bar{E} = -\partial\bar{B}/\partial t \Rightarrow \bar{H} \cdot \nabla \times E = \bar{H} \cdot (-\partial\bar{B}/\partial t) \qquad (1) \right.$$

$$\left\{\begin{array}{ll} \nabla\times\bar E = -\partial\bar B/\partial t \Rightarrow \bar H\cdot\nabla\times\bar E = \bar H\cdot(-\partial B/\partial t) & (1) \\ \nabla\times\bar H = \bar J+\partial\bar D/\partial t \Rightarrow \bar E\cdot\nabla\times\bar H = \bar E\cdot(\bar J+\partial\bar D/\partial t) & (2) \\ \bar D=\varepsilon\bar E,\ \varepsilon=\varepsilon_0\varepsilon_r & \bar J=\sigma(\bar E+\bar E_i) \\ \bar B=\mu\bar H,\ \mu=\mu_0\mu_r & \end{array}\right.$$

① − ②

$$\underbrace{\bar H\cdot\nabla\times\bar E - \bar E\cdot\nabla\times\bar H}_{I} = \bar H(-\partial\bar B/\partial t) - \bar E\cdot(\bar J+\partial\bar D/\partial t)$$

$$= \underbrace{-\bar E\cdot\bar J}_{II} - \underbrace{\bar E\cdot\partial\bar D/\partial t}_{III} - \underbrace{\bar H\cdot\partial\bar B/\partial t}_{IV}$$

(I) $$\nabla\cdot(\overset{\bar E}{\overset{\Updownarrow}{\bar A}}\times\overset{\bar H}{\overset{\Updownarrow}{\bar B}}) = \bar B\cdot(\nabla\times\bar A) - \bar A\cdot(\nabla\times\bar B)$$

$$\Downarrow$$

$$\nabla\cdot(\bar E\times\bar H)$$

Joule   Power density due to ELECTROMOTIVE FIELDS

(II) $$\bar E\cdot\bar J = (\bar J/\sigma - \bar E_i)\cdot\bar J = J^2/\sigma - \bar E_i\cdot\bar J$$

$\Downarrow$

$\bar J=\sigma(\bar E+\bar E_i)$

$\bar E = \bar J/\sigma - \bar E_i$

$\Uparrow$ W/m^3 … $\Uparrow$ Power density

(III) $$\bar E\cdot\partial\bar D/\partial t \underset{\uparrow}{=} \varepsilon\bar E\cdot\frac{\partial\bar E}{\partial t} = \frac12\varepsilon\frac{\partial E^2}{\partial t}$$

$\bar D=\varepsilon\bar E$

(Hp: ε constant over time)

(IV) $$\bar H\cdot\partial\bar B/\partial t \underset{|}{=} \frac1\mu\bar B\cdot\partial\bar B/\partial t = \frac{1}{2\mu}\frac{\partial B^2}{\partial t}$$

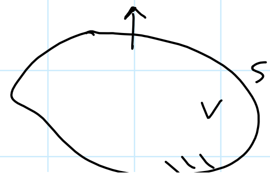
$\bar H = 1/\mu\,\bar B$

$$\nabla\cdot(\bar E\times\bar H) = -J^2/\sigma + \bar E_i\cdot\bar J - \frac12\varepsilon\frac{\partial E^2}{\partial t} - \frac{1}{2\mu}\frac{\partial B^2}{\partial t}$$

$$+\bar E_i\cdot\bar J = +J^2/\sigma + \frac12\varepsilon\frac{\partial E^2}{\partial t} + \frac{1}{2\mu}\frac{\partial B^2}{\partial t} + \nabla\cdot(\underbrace{\bar E\times\bar H}_{})$$

$\bar S$: Poynting VECTOR

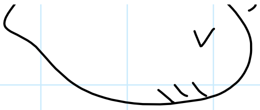
Consider V :



Hp. V CONSTANT over time

$$\downarrow$$

$$\int_V \nabla\cdot(\bar E\times\bar H)\,d\Omega$$



over time

$$\int_V \nabla\cdot(\vec E\times\vec H)\,dv$$

POWER DUE TO...

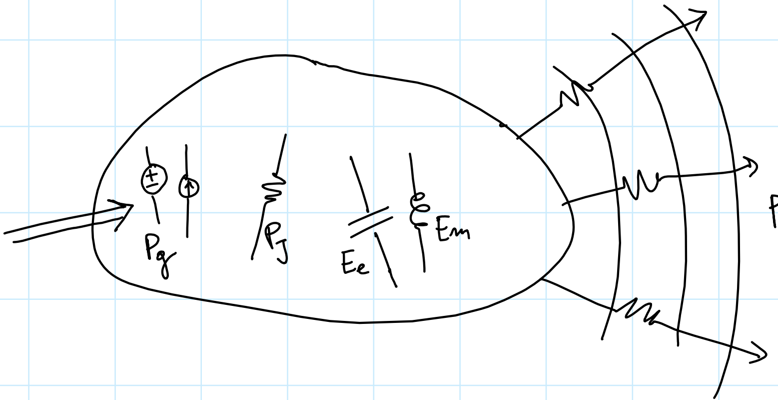
$$\underbrace{\int_V \vec E_i\cdot\vec J\,dV}_{\text{ELECTROMOTIVE FIELDS}} = \underbrace{\int_V J^2/\sigma\,dv}_{\text{Joule losses}} + \frac{d}{dt}\left[\underbrace{\int_V \frac{1}{2}\varepsilon E^2\,dV}_{E_e} + \underbrace{\int_V \frac{1}{2\mu}B^2\,dV}_{E_m}\right] + \underbrace{\oint_S \vec E\times\vec H\cdot d\vec S}_{\text{Poynting vector's FLUX}}$$

E_e ↓ ELECTRIC ENERGY (ELECTROSTATIC...)

E_m ↓ MAGNETIC ENERGY

Poynting vector's FLUX ⇓ RADIATED ENERGY per unit of time

$$+\,d/dt\,(E_e+E_m)$$

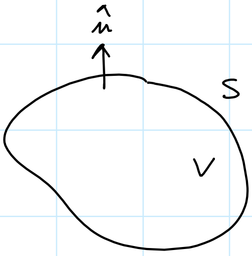


$$P_g = P_J + d/dt\,(E_e+E_m) + P_p$$

UNIQUENESS theorem for ELECTROMAGNETIC PROBLEMS

EM system evolving in time $t\geq t_0$

Hp: linear and ISOTROPIC materials

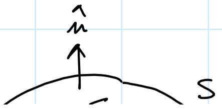


THEOREM: the following quantities must be PRESCRIBED to grant a unique solution

SOURCE TERMS $\vec E_i$

$\forall p\in V,\ \forall t\geq t_0$

ONE of the following must hold



BOUNDARY CONDITIONS

$\forall t \geq t_0$

ONE of the following must hold

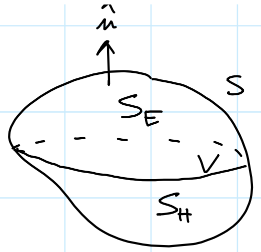
① $\bar{E}_t \quad \forall p \in S$

↑ tangential component of $\bar{E}$, $\bar{E}_t = \hat{n} \times \bar{E}$

② $\bar{H}_t \quad \forall p \in S$

③ $\bar{E}_t \quad \forall p \in S_E$

$\bar{H}_t \quad \forall p \in S_H$



$S = S_E \cup S_H$

INITIAL CONDITIONS

$\forall p \in V, \; t = t_0$

$$\bar{E}, \; \bar{H}$$

$$\Downarrow$$

$$\bar{J} = \sigma(\underset{\uparrow \; \forall p \in V, \, t \geq t_0}{\bar{E} + \bar{E}_i}) \; ; \; \bar{D} = \varepsilon \bar{E} \; ; \; \bar{B} = \mu \bar{H}$$

Hp. TWO SOLUTIONS

Sol. 1 $\begin{bmatrix} \bar{E}_1 \\ \bar{H}_1 \end{bmatrix}$ Sol. 2 $\begin{bmatrix} \bar{E}_2 \\ \bar{H}_2 \end{bmatrix}$

Proof of theorem: AB ABSURDUM proof (by contradiction)

consider Sol 3 Sol 1 − Sol 2

$$\begin{bmatrix} \bar{E}_3 \\ \bar{H}_3 \end{bmatrix} = \begin{bmatrix} \bar{E}_1 - \bar{E}_2 \\ \bar{H}_1 - \bar{H}_2 \end{bmatrix} \Rightarrow$$

IF CAN PROVE that

$\bar{E}_3 = 0$, $\bar{H}_3 = 0$ $\forall t \geq t_0$

⇒ Solution is unique

Local ohm's LAW.

$$\bar{J}_3 = \underset{\sigma(\bar{E}_1 + \bar{E}_i)}{\underset{\uparrow}{\bar{J}_1}} - \underset{\sigma(\bar{E}_2 + \bar{E}_i)}{\underset{\uparrow}{\bar{J}_2}} = \sigma(\bar{E}_1 - \bar{E}_2) \Rightarrow \bar{E}_{i_3} = 0$$

⇓

DIFFERENCE Field has no source terms

Sol 1 & 2 must have $\underline{\underline{\text{some}}}$ impressed fields

Apply Poynting theorem to solution 3 (difference field)

$=0$ $=0$

$$\int_V \overset{=0}{\cancel{\bar{E}_{i_3}\cdot\bar{J}_3}}\, dV = \int_V \frac{J_3^2}{\sigma}\, dV + \frac{\partial}{\partial t}\left[\int_V \frac{1}{2}\varepsilon E_3^2\, dV + \int_V \frac{1}{2\mu} B_3^2\, dV\right] + \cancel{\oint_S \bar{E}_3\times\bar{H}_3\cdot d\bar{S}} = 0$$

$\Downarrow$

$\bar{E}_{i_3} = 0$

$$\bar{E}_3 = \bar{E}_{3,t} + E_{3,n}\hat{n}$$

$$\bar{H}_3 = \bar{H}_{3,t} + H_{3,n}\hat{n}$$

PARALLEL

$$\bar{E}_3\times\bar{H}_3 = \underbrace{\cancel{\bar{E}_{3,t}\times\bar{H}_{3,t}}}_{I} + \underbrace{\bar{E}_{3,t}\times H_{3,n}\hat{n}}_{II} + \underbrace{E_{3,n}\hat{n}\times\bar{H}_{3,t}}_{III} + \underbrace{\cancel{E_{3,n}\hat{n}\times H_{3,n}\hat{n}}}_{IV}$$

$\Downarrow$ for the same $\bar{P}$,

EITHER $\left\{\begin{array}{l}\bar{H}_{3,t}\\ \bar{E}_{3,t}\end{array}\right. = 0$

$\Uparrow$ TANGENT to S

$\Uparrow$ TANGENT TO S

$= 0$

FLUX through $\hat{n}$ of II and III is ZERO

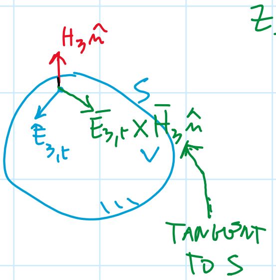
Example if we chose to set $\bar{E}_t$ as BOUNDARY CONDITION

$\bar{E}_{1,t} = \bar{E}_{t_0}$

$\bar{E}_{2,t} = \bar{E}_{t_0}$

"the" given value of $\bar{E}_t$ chosen as BC

$\Rightarrow \bar{E}_{3,t} = \bar{E}_{1,t} - \bar{E}_{2,t} = 0$



$\varepsilon, \mu \geq 0$

$$\underbrace{\int_V \frac{J^2}{\sigma}\, dV}_{\geq 0} = \underbrace{-\frac{\partial}{\partial t}\underbrace{\left[\int_V \frac{1}{2}\varepsilon E^2\, dV + \int_V \frac{1}{2\mu} B^2\, dV\right]}_{\geq 0}}_{\leq 0}$$

$\uparrow \geq 0$

TERM 1 TERM 2

$t = t_0 \Rightarrow$ INITIAL CONDITIONS ARE PRESCRIBED

$\bar{J}_1 = \bar{J}_2 = \bar{J}$

$\bar{E}_1 = \bar{E}_2 = \bar{E}$

$\bar{B}_1 = \bar{B}_2 = \bar{B}$

$$\Rightarrow \boxed{\boxed{\bar{J}_3, \bar{E}_3, \bar{B}_3 = 0}}$$

$t > t_0$

$$\left.\begin{array}{l}\bar{J}_1 \neq \bar{J}_2 \\ \bar{E}_1 \neq \bar{E}_2 \\ \bar{B}_1 \neq \bar{B}_2\end{array}\right. \longrightarrow \boxed{\bar{J}_3,\ \bar{E}_3,\ \bar{B}_3 \neq 0}$$

$$\Downarrow$$

ABSURD: TERM 1 ≠ TERM 2

only possible solution $\boxed{\bar{J}_3,\ \bar{E}_3,\ \bar{B}_3 = 0 \quad \forall t \geq t_0}$

$$\Downarrow$$

$$\bar{H}_3,\ \bar{D}_3 = 0$$

$\Rightarrow$ Sol 1 = Sol 2

$\Rightarrow$ Solution is UNIQUE