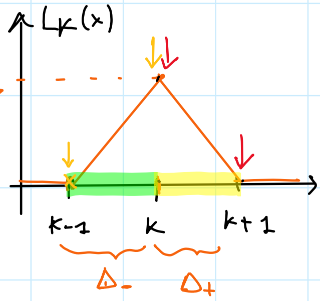
L16

17 November 2025 11:44

L 16 on an element $\tilde{f}(x)$ depends on only TWO DATA POINTS and
" " hat functions

if $x \in W_{k,k+1} \Rightarrow \tilde{f}(x) = f_k L_k(x) + f_{k+1} L_{k+1}(x)$



Definition of $L_k(x)$

$$L_k(x) \begin{cases} 1 + \dfrac{x - x_k}{\Delta_-}, & x \in W_{k-1,k} \\ 1 - \dfrac{x - x_k}{\Delta_+}, & x \in W_{k,k+1} \\ 0, & x \notin \underbrace{W_{k-1,k} \cup W_{k,k+1}}_{\Omega_k} \end{cases}$$

if $x = x_{k-1}$ $\quad L_k(\underset{x = x_{k-1}}{\underset{\uparrow}{x}}) = 1 + \dfrac{\overbrace{(x_{k-1} - x_k)}^{-\Delta_-}}{\underset{x_k - x_{k-1}}{\underset{\uparrow}{\Delta_-}}} = 1 - 1 = 0$

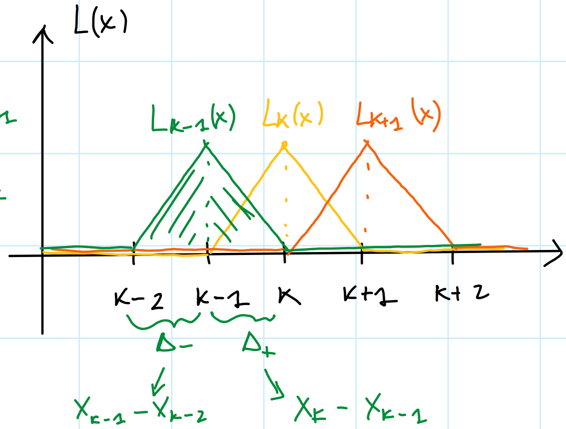
if $x = x_k$ $\quad L_k(\underset{x = x_k}{\underset{\uparrow}{x}}) = 1 + \dfrac{\cancel{x_k - x_k}^{\,0}}{\Delta_-} = 1$

if $x = x_k$ $\quad L_k(\underset{x = x_k}{\underset{\uparrow}{x}}) = 1 - \dfrac{\overbrace{\cancel{x_k - x_k}}^{=0}}{\Delta_+} = 1$

if $x = x_{k+1}$ $\quad L_k(\underset{x = x_{k+1}}{\underset{\uparrow}{x}}) = 1 - \dfrac{\overbrace{x_{k+1} - x_k}^{\Delta_+}}{\Delta_+} = 0$

Example:

$$L_{k-1}(x) = \begin{cases} 1 + \dfrac{x - x_{k-1}}{\Delta_-}, & x \in W_{k-2,k-1} \\ 1 - \dfrac{x - x_{k-1}}{\Delta_+}, & x \in W_{k-1,k} \\ 0, & x \notin \Omega_{k-1} \end{cases}$$



Boundary nodes

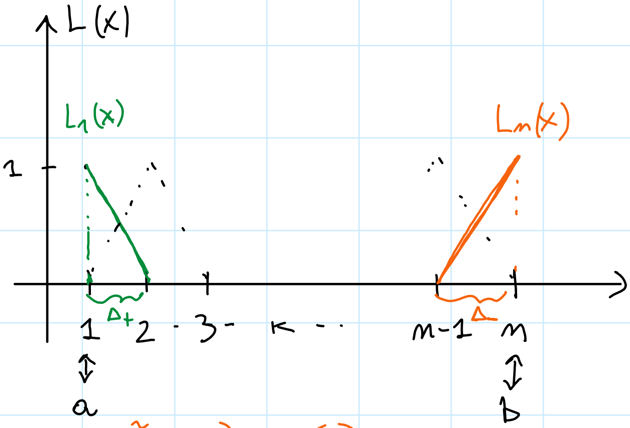
$\{1 - \dfrac{x - x_1}{\ldots}, x \in W_{1,2}$

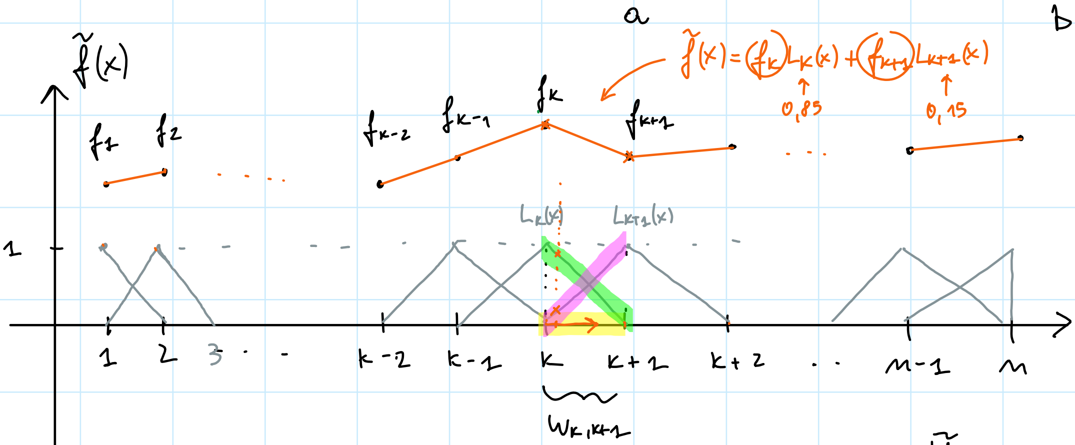
$L(x)$

Boundary nodes

$$L_1(x)=\begin{cases}1-\dfrac{x-x_1}{\Delta_+}, & x\in W_{1,2}\\[4pt] 0, & x\notin \Omega_1\equiv W_{1,2}\end{cases}$$

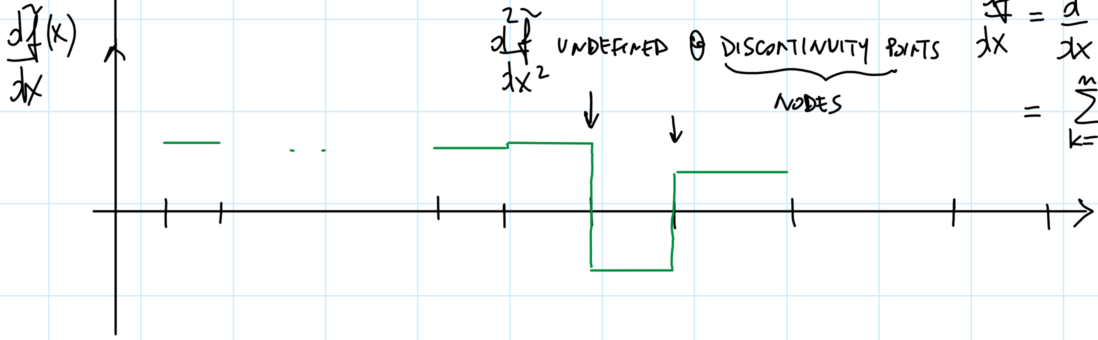
$$L_m(x)=\begin{cases}1+\dfrac{x-x_m}{\Delta_-}, & x\in W_{m-1,m}\\[4pt] 0, & x\notin \Omega_m\equiv W_{m-1,m}\end{cases}$$





$\dfrac{d^2\tilde f}{dx^2}$ UNDEFINED @ $\underbrace{\text{DISCONTINUITY POINTS}}_{\text{NODES}}$

$$\frac{d\tilde f}{dx}=\frac{d}{dx}\left(\sum_{k=1}^{m} L_k(x)\, f_k\right)=\sum_{k=1}^{m} f_k\,\frac{dL_k(x)}{dx}$$



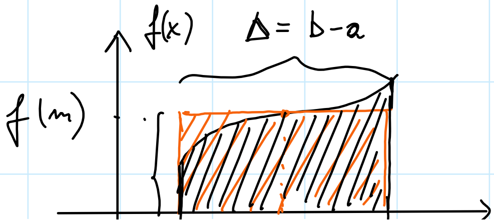
EXAMPLE: $L_k(x)\begin{cases}1+\dfrac{x-x_k}{\Delta_-}\\[4pt] 1-\dfrac{x-x_k}{\Delta_+}\\[4pt] 0\end{cases} \Rightarrow \dfrac{dL_k}{dx}=\begin{cases}1/\Delta_-, & x\in W_{k-1,k}\\[4pt] -1/\Delta_+, & x\in W_{k,k+1}\\[4pt] 0\end{cases}$

Numerical integration (1D)

goal: $\displaystyle\int_a^b f(x)\,dx$ numerically

RECTANGLES METHOD (midpoint)

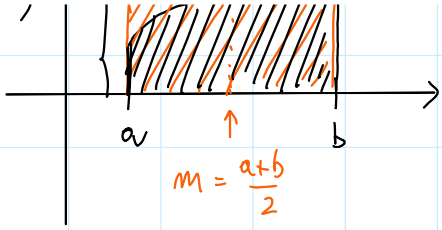
$$\int_a^b f(x)\,dx = f(m)\,\Delta + O(\Delta^3)$$



$$\int_a^b f(x)\,dx = f(m)\,\Delta + \mathcal{O}(\Delta^3)$$

$$\simeq f(m)\,\Delta$$

third-order accuracy



⚠ For FINITE differences formulas

$\mathcal{O}(\Delta x^h)$ → when Δx is halved, error is divided by 2^h ⇒ TRUE for INTEGRATION

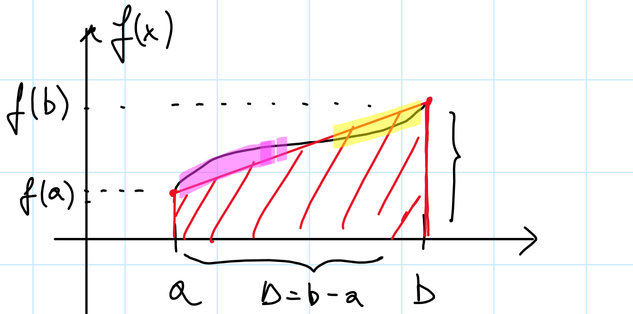
→ able to solve without error a problem whose solution is a polynomial of order h ⇒ NOT TRUE for integration

no Taylor series

TRAPEZOIDAL RULE

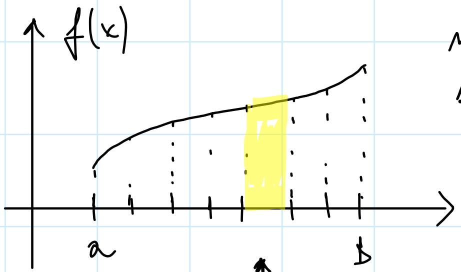
$$\int_a^b f(x)\,dx = \frac{1}{2}\left[f(a)+f(b)\right]\Delta + \mathcal{O}(\Delta^3)$$

$$\simeq \tfrac{1}{2}\left[f(a)+f(b)\right]\Delta$$



area trapezoid. $\left[f(a)+f(b)\right]\Delta/2$

On multiple sub-domains



$n-1$ elements

$\Delta = \frac{b-a}{n-1}$ ⇒ Δ is smaller

∀ sub-interval: apply → rectangles / → trapezoidal

[sum on n-elements of LOCAL error $\mathcal{O}(\Delta^3)$]
GLOBAL ERROR

RECT: $$\int_a^b f(x)\,dx = \sum_{k=1}^{n-1} f(m_k)\,\Delta + \mathcal{O}(\Delta^2)$$

midpoint of element k

GLOBAL error has increased but SMALLER Δ

if $n_{ints} = 1$ $\quad \Delta = b-a$

$n_{ints} = n$ $\quad \Delta = \frac{b-a}{n-1}$

TRAPZ: $$\int_a^b f(x)\,dx = \sum_{k=1}^{n-1} \frac{1}{2}\left[f(x_k) + f(x_{k+1})\right]\Delta + \mathcal{O}(\Delta^2)$$

GAUSS integration (GAUSS QUADRATURE)

IDEA. compute a numerical integral as a WEIGHTED sum of $f(x)$ evaluated at a given number of points

$$\int_{-1}^{1} f(x) \simeq \sum_{k=1}^{n} \underbrace{w_k}_{\text{WEIGHTS}} \, f(\underbrace{x_k}_{\text{GAUSS POINTS}})$$

<u>order of accuracy</u> <u>m</u> : $2n-1$

example: if $n=2$ $\Rightarrow$ $m = 2\overset{2}{n}-1 = 3$ $\Rightarrow$ if $n=2 \Rightarrow$ integrate a polynomial of order 3 without ERROR

Derivation of <u>QUADRATURE RULES</u> (values of W_k and X_k to obtain a certain degree of accuracy)

$n = 1$
$\downarrow$
$m = 2n-1 = 1$

Gauss formula integrate with no error $\rightarrow$ polynomial order 0
" " " 1

Order 0 . $f_1(x) \overset{\text{example}}{=} 1$ $\quad \int_{-1}^{1} \overbrace{f_1(x)}^{=1} dx = [x]_{-1}^{1} = 1-(-1) = 2$

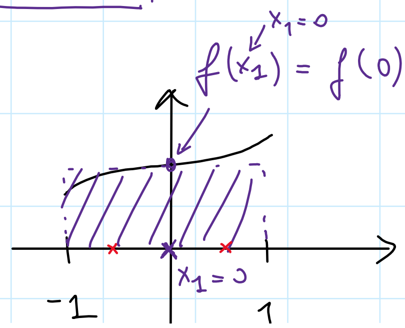
$\Rightarrow w_1 \cdot \underbrace{f_1(x_1)}_{x_1} = \boxed{w_1 \; 1 = 2}$ $\rightarrow$ FULFILLED to integrate exactly polynomial functions order 0

order 1 . $f_2(x) = x$ $\quad \int_{-1}^{1} \overbrace{f_2(x)}^{x} dx = \left[x^2/2 \right]_{-1}^{1} = 0$

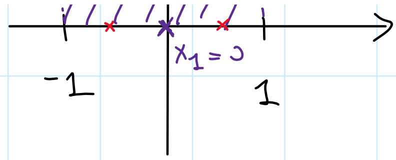
$\Rightarrow w_1 \cdot f_2(x_1) = \boxed{w_1 \cdot x_1 = 0}$

order 0 polyn $\quad w_1 \cdot 1 = 2 \Rightarrow w_1 = 2$

order 1 polyn $\quad w_1 \cdot x_1 = 0 \Rightarrow x_1 = 0$



order 1 polyn $\quad \left| \underset{\uparrow\, =2}{w_1} \cdot x_1 = 0 \right. \Rightarrow \boxed{x_1 = 0}$



$$\int_{-1}^{1} f(x)\,dx = \overset{\Delta = 1-(-1) = 2}{2}\, f(0)$$

weight $= 2$ $\qquad$ gauss point

QUADRATURE RULE for $n=1$ $\equiv$ rectangles method

$n = 2 \quad . \quad m = n - 1 = 3 \rightarrow$ REQUIRE no error for polynomials order $\rightarrow 0, 1, 2, 3$

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2)$$

4 degrees of freedom

order 0 $\quad f_1(x) = 1 \rightarrow \int_{-1}^{1} f_1(x)\,dx = 2 \quad \Rightarrow w_1 \underbrace{f_1(x_1)}_{1} + w_2 \underbrace{f_1(x_2)}_{1} = 2$

$\Rightarrow \boxed{w_1 + w_2 = 2}$

order 1 . $\quad f_2(x) = x \rightarrow \int_{-1}^{1} f_2(x)\,dx = 0 \quad \Rightarrow w_1 \underbrace{f_2(x_1)}_{=x_1} + w_2 \underbrace{f_2(x_2)}_{=x_2} = 0$

$\Rightarrow \boxed{w_1 x_1 + w_2 x_2 = 0}$

order 2 . $\quad f_3(x) = x^2 \rightarrow \int_{-1}^{1} f_3(x)\,dx = 2/3 \quad \Rightarrow \boxed{w_1 x_1^2 + w_2 x_2^2 = 2/3}$

$\left[x^3/3 \right]_{-1}^{1} = 1/3 - (-1/3)$

$= 2/3$

order 3 : $\quad f_4(x) = x^3 \quad \int_{-1}^{1} f_4(x)\,dx = 0 \quad \Rightarrow \boxed{w_1 x_1^3 + w_2 x_2^3 = 0}$

Find w_1, x_1, w_2, x_2

$\left| w_1 + w_2 = 2 \right.$ $\quad$ $\Rightarrow$ Hp. $w_1 = w_2$ $\quad \Rightarrow \boxed{w_1 = w_2 = 1}$

nonlinear $\left| w_1 x_1 + w_2 x_2 = 0 \right. \Rightarrow 1 x_1 + 1 x_2 = 0 \Rightarrow \boxed{x_1 = -x_2}$

nonlinear system

$$\begin{cases} w_1 + w_2 = 2 \\ w_1 x_1 + w_2 x_2 = 0 \\ w_1 x_1^2 + w_2 x_2^2 = 2/3 \\ w_1 x_1^3 + w_2 x_2^3 = 0 \end{cases}$$

$\Leftarrow$ Hp. $w_1 = w_2$ $\Rightarrow$ $\boxed{w_1 = w_2 = 1}$

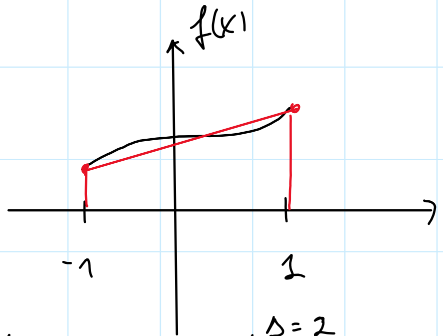
$\Rightarrow 1 x_1 + 1 x_2 = 0 \Rightarrow \boxed{x_1 = -x_2}$

$1 \cdot x_1^3 + 1(-x_1)^3 = 0 \Rightarrow$ SATISFIED by every value of x_1

$x_1^2 + \underset{\substack{\uparrow \\ x_2 = -x_1}}{x_2^2} = 2/3 \Rightarrow \cancel{2} x_1^2 = \cancel{2}/3 \Rightarrow \boxed{\begin{aligned} x_1 &= \pm 1/\sqrt{3} \\ x_2 &= \mp 1/\sqrt{3} \\ w_1 &= w_2 = 1 \end{aligned}}$

QUADRATURE RULE for $n = 2$

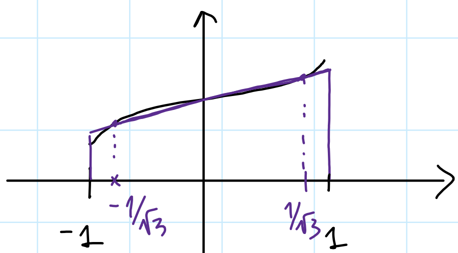
TRAP. RULE



$$\int_{-1}^{1} f(x)\,dx \simeq \frac{\overset{\Delta = 2}{\Delta}}{2}\left[f(-1) + f(1)\right]$$

$$= \underset{\substack{\uparrow \\ w_1 = 1}}{1} f(\underset{\substack{\uparrow \\ x_1 = -1}}{-1}) + \underset{\substack{\uparrow \\ w_2 = 1}}{1} f(\underset{\substack{\uparrow \\ x_2 = 1}}{1})$$

GAUSS ($n = 2$)



$$\int_{-1}^{1} f(x)\,dx \simeq \underbrace{w_1 f(x_1) + w_2 f(x_2)}_{1 \cdot f(-1/\sqrt{3}) + 1 \cdot f(1/\sqrt{3})}$$

$n = 3 \qquad m = 2n - 1 = 5$

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

$$\begin{cases} w_1 = 5/9 \\ w_2 = 8/9 \end{cases} \qquad \begin{cases} x_1 = -\sqrt{3/5} \\ x_2 = 0 \end{cases}$$

$$\begin{cases} w_2 = 8/9 \\ w_3 = 5/9 \end{cases} \qquad \begin{cases} x_2 = 0 \\ x_3 = \sqrt{3/5} \end{cases}$$

GAUSS QUADRATURE for arbitrary intervals

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(\tau)\,d\tau \simeq \sum_{k=1}^{n} w_k f(\tau_k)$$

$$x \in [a,b] \qquad \tau \in [-1,1]$$

find a MAP between $[a,b] \Leftrightarrow [-1,1]$

$\Rightarrow$ linear transformation $x \Leftrightarrow \tau$

requirements (find m and q)

$$\begin{cases} x = m\tau + q & \text{(linearity)} \\ x=a: \quad a = m(-1) + q \\ x=b: \quad b = m(1) + q \end{cases} \rightarrow \quad b - a = m + \cancel{q} - (-m + \cancel{q})$$

$$\Rightarrow b - a = 2m$$

$$\Rightarrow \boxed{m = \frac{b-a}{2}}$$

to find q:

$$a = -\frac{b-a}{2} + q \Rightarrow q = a + \frac{b-a}{2} = \frac{2a+b-a}{2} = \boxed{\frac{a+b}{2} = q}$$

Linear transformation:

$$\underset{[a,b]}{x} = m\tau + q = \frac{b-a}{2}\tau + \frac{a+b}{2}$$

differential: $\frac{dx}{d\tau} = m \Rightarrow \boxed{dx = m\,d\tau = \frac{b-a}{2}\,d\tau}$

<u>differential</u> . $\frac{dx}{d\tau} = m \Rightarrow \boxed{dx = m\, d\tau = \frac{b-a}{2}\, d\tau}$

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(m\tau + q)\,\overbrace{\frac{b-a}{2}}^{m}\,d\tau = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}\tau + \frac{a+b}{2}\right) d\tau$$

$$\Downarrow$$

$$\simeq \sum_{k=1}^{m} w_k \, \frac{b-a}{2} \, f\left(\frac{b-a}{2}\tau_k + \frac{a+b}{2}\right)$$

if $m = 1$ $\quad w_1 = 2\,,\ x_k \to \tau_k = 0$

$$\int_a^b f(x)\,dx \simeq 2\,\frac{b-a}{2}\, f\left(\cancel{\frac{b-a}{2}}\,0 + \frac{a+b}{2}\right)$$