Interpolation

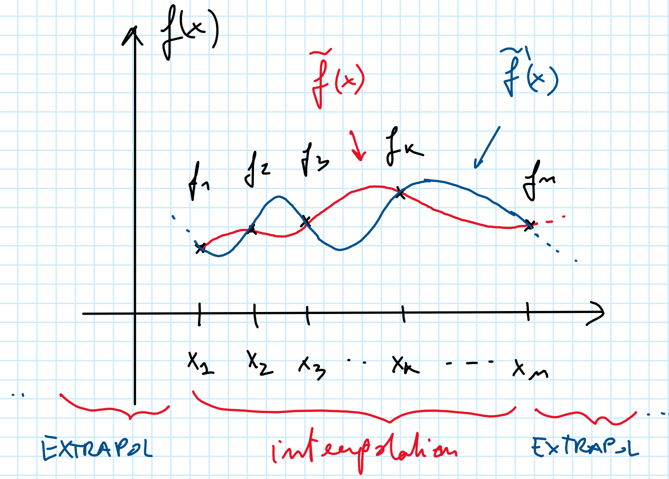
DATA POINTS

Def given $f(x)$, known only at certain points $x_1, x_2, \dots, x_n$

$f(x_1) = f_1,\ f(x_2) = f_2, \dots\ f(x_k) = f_k \dots$

interpolation is the process of constructing $\tilde{f}(x)$ such that.

$$\underbrace{\tilde{f}(x_k)}_{\text{DATA POINT}} = \underbrace{f_k}_{f(x_k)}, \quad \forall\ k = 1, \dots, n \qquad (1)$$



$\Rightarrow \tilde{f}(x)$ can be used to ESTIMATE $f(x)\ \forall \in [x_1, \dots x_n]$

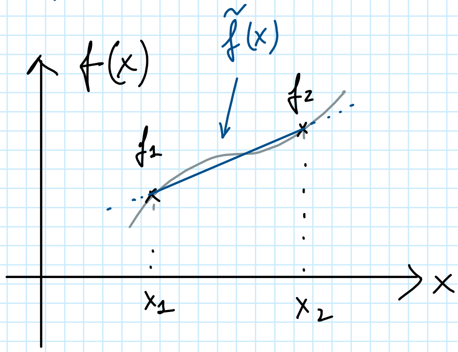
NOTE: $\tilde{f}(x)$ is not UNIQUE $\rightarrow \infty$ of $\tilde{f}(x)$ that satisfy the CONDITION (1)

for UNIQUENESS $\Rightarrow$ RESTRICTION of $\tilde{f}(x)$

Polynomial interpolation

- restrict $\tilde{f}(x)$ polynomial of degree $n-1$ (n - DATA POINTS)

2 DATA POINTS



coefficients

$$\tilde{f}(x) = a_0 + a_1 x$$

$\leftarrow$ Polynomial of order 1

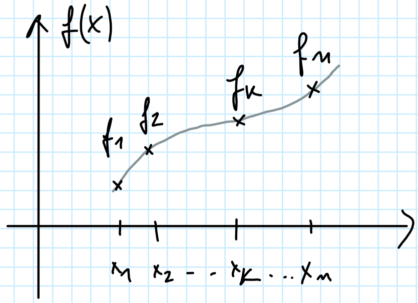
$\Downarrow$

"find a_0 and a_1 such that".

$$\begin{cases} \tilde{f}(x_1) = a_0 + a_1 x_1 = f_1 \\ \tilde{f}(x_2) = a_0 + a_1 x_2 = f_2 \end{cases}$$

$$\begin{cases} \tilde{f}(x_1) = a_0 + a_1 x_1 = f_1 \\ \tilde{f}(x_2) = a_0 + a_1 x_2 = f_2 \end{cases}$$

n DATA POINTS



$$\tilde{f}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1}$$

"find $a_0, a_1, \ldots, a_{n-1}$ such that:"

$$\begin{cases} \tilde{f}(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_{n-1} x_1^{n-1} = f_1 \\ \tilde{f}(x_2) = a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_{n-1} x_2^{n-1} = f_2 \\ \vdots \\ \tilde{f}(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_{n-1} x_n^{n-1} = f_n \end{cases}$$

matrix form:

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ & & \vdots & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}}_{[V]} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

$[V]$ VANDERMONDE MATRIX

$\Rightarrow$ Solve $[V]\{a\} = \{f\}$ $\Rightarrow$ find $[V^{-1}]$ inverse of $[V]$

$$\underbrace{[V^{-1}][V]}_{[I_d]}\{a\} = [V^{-1}]\{f\}$$

$$\underbrace{[I_d]\{a\}}_{\{a\}}$$

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \\ & & \cdot 1 \end{bmatrix} \rightarrow$$

$$\begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix}$$

$[V^{-1}]$ $\{a\}$ $\Downarrow$ $\begin{pmatrix} \vdots \\ f_n \end{pmatrix}$

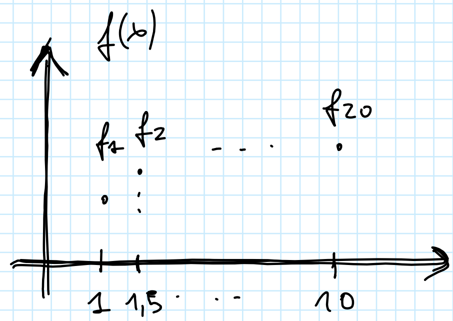
$$\{a\} = [V^{-1}]\{f\}$$

$[V]$ is invertible if $\mathrm{Det}[V] \neq 0 \Rightarrow [V]$ is linearly independent. ✓

· numerically ·

$[V]$ often ILL-CONDITIONED matrix

EXAMPLE: $f(x) \to \tilde{f}(x)$ $\quad n = 20$, $x_1 = 1$, $x_{20} = 10$



$$[V] = \begin{bmatrix} 1 & 1 & 1^2 & \cdots & 1^{19} \\ 1 & 1{,}5 & 1{,}5^2 & \cdots & 1{,}5^{19} \\ \vdots & & & & \vdots \\ 1 & 10 & 10^2 & \cdots & 10^{19} \end{bmatrix}$$

V_k · k-th entry of $[V]$

$$\frac{\min(V_k)}{\max(V_k)} = \frac{1}{10^{19}} = 10^{-19} \qquad \varepsilon \sim 2{,}2 \; 10^{-16}$$

$\Rightarrow$ first rows are "AS IF" they were ZEROS

$\Downarrow$

MATRIX is "numerically" SINGULAR

κ

CONDITION number (of a matrix): unified measure of the sensitivity of the solution of $[A]\{x\} = \{b\}$ to SMALL PERTURBATIONS in $[A]$ or $\{b\}$

internal perturbations (roundoff errors)

external perturbations

ILL-conditioned linear system: HUGE VARIATIONS in $\{x\}$ for small variations in $\{b\}$

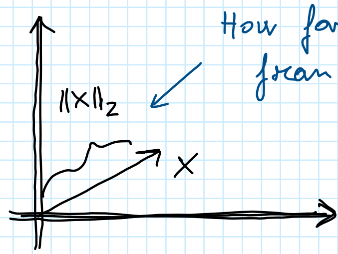
Def: $\kappa([A]) = \|A\|_2 \, \|A^{-1}\|_2$

How for $\{x\}$ extends

Def: $\kappa([A]) = \|A\|_2 \, \|A^{-1}\|_2$

2-NORM (EUCLIDEAN NORM) of a VECTOR

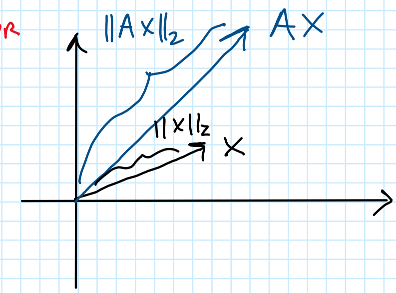
$$\|x\|_2 = \sqrt{\sum_{k=1}^{m} x_k^2}$$



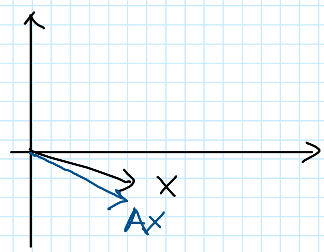
2-norm of a matrix

$$\|A\|_2 = \max \left(\frac{\overbrace{\|Ax\|_2}^{Ax = \text{VECTOR}}}{\|x\|_2} \right) = M$$

$\underbrace{\qquad}$ STRETCHING of x induced by A



$$\frac{1}{\|A^{-1}\|_2} = \min \left(\frac{\|Ax\|_2}{\|x\|_2} \right) = m$$



$$\kappa = \underbrace{\|A\|_2}_{M} \cdot \underbrace{\|A^{-1}\|_2}_{1/m} = \frac{M}{m}$$

M ← maximum stretching of x induced by [A]

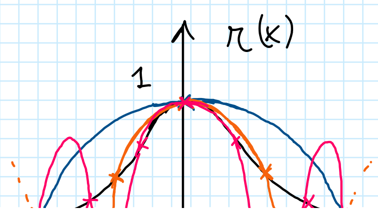
m ← MINIMUM " " " " "

if $\kappa \sim 1 \Rightarrow$ System is STABLE with respect to perturbations $\Rightarrow$ WELL-COND.

$\kappa \gg 1 \Rightarrow$ " " UNSTABLE " " " " $\Rightarrow$ ILL-COND

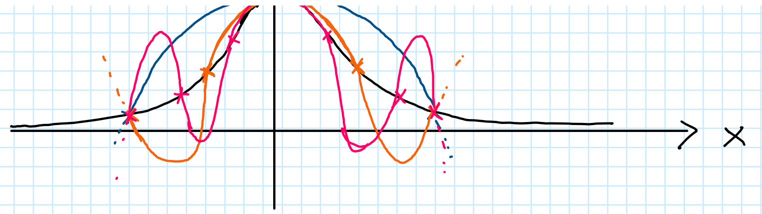
Polynomial interpolation is OSCILLATORY

Ex. RUNGE function $r(x) = \dfrac{1}{1+\alpha x^2}$, $\alpha = 25$



Polynomial interp

# points	Degree
m = 3	2



$n = 3$	2
$n = 5$	4
$n = 7$	6

if we increase the polynomial order $\Rightarrow$ larger oscillations near the boundaries of the sampling interval

PIECEWISE LINEAR INTERPOLATION (1D)

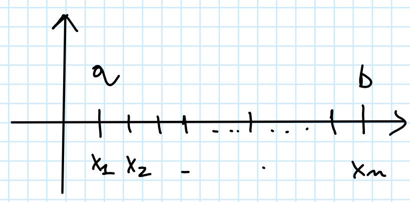
IDEA: construct an interpolating function $\tilde{f}(x)$ in $[a,b]$ from a SET of PIECEWISE-LINEAR POLYNOMIALS

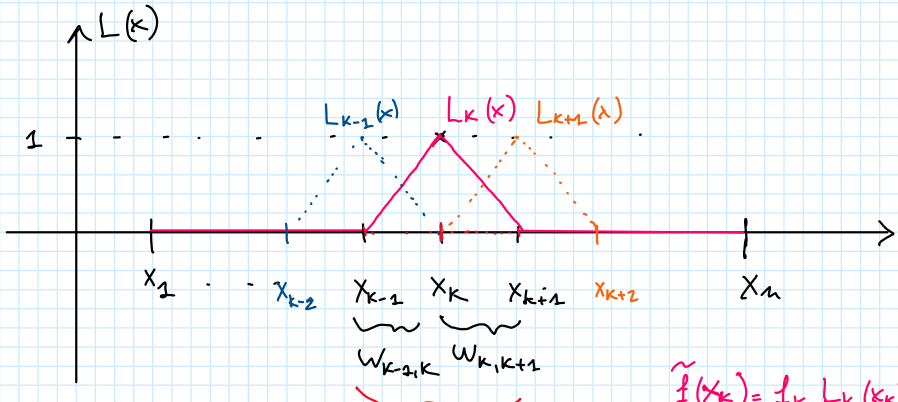
"HAT" FUNCTIONS (BASIS FUNCTIONS)

$$\tilde{f}(x) = L_1(x) f_1 + L_2(x) f_2 + \dots + L_k(x) f_k + \dots L_n(x) f_n$$

each polynomial is $\neq 0$ only on a SUBSET of $[a,b]$

STRATEGY: introduce set of n-points (nodes) to subdivide $[a,b]$ $n-1$ ELEMENTS (INTERVALS)





$\underbrace{\quad}_{W_{k-1,k}}$ $\underbrace{\quad}_{W_{k,k+1}}$

SUPPORT DOMAIN of node k Ω_k

$$\tilde{f}(x_k) = f_k \underbrace{L_k(x_k)}_{=1} + \cancel{f_{k+1} L_{k+1}(x_k)}$$

must be $= 0$ to have $\tilde{f}(x_k) = f_k$

$W_{k,k+1} = [x_k, x_{k+1}]$

$W_{k-1,k} = [x_{k-1}, x_k]$

$\Omega_k = W_{k-1,k} \cup W_{k,k+1}$

REQUIREMENTS:

$$\begin{cases} L_k(x_k) = 1 \\ L_k(x) = 0, \ \forall x \notin \Omega_k \\ L_k(x) \text{ linear} \end{cases}$$