L6

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GREEN'S IDENTITIES

- Consider $\varphi, \psi \in C_2$ in SCD

$\bar{F} = \varphi \nabla \psi$

$$\boxed{\nabla \cdot \bar{F} = \nabla \cdot (\varphi \nabla \psi) = \nabla \varphi \cdot \nabla \psi + \varphi \nabla^2 \psi}$$

VECTOR ID. : $\nabla \cdot (f \nabla g) = \nabla f \cdot \nabla g + f \nabla^2 g$

$$\int_V \nabla \cdot \bar{F} \, dV = \int_V (\nabla \varphi \cdot \nabla \psi + \varphi \nabla^2 \psi) \, dV$$

$\downarrow$ DIV TH

$$\oint_S \varphi \nabla \psi \cdot d\bar{S} = \; - \; - \; - \; -$$

$$\Downarrow$$

$$\nabla \psi \cdot \hat{n} \, dS = \frac{\partial \psi}{\partial n} \, dS$$

$$\oint_S \varphi \frac{\partial \psi}{\partial n} \, dS = \int_V \nabla \varphi \cdot \nabla \psi \, dV + \int_V \varphi \nabla^2 \psi \, dV$$

GREEN'S FIRST IDENTITY

$$\int_V \underset{f'}{\underset{\uparrow}{\nabla \varphi}} \cdot \underset{g}{\underset{\uparrow}{\nabla \psi}} \, dV = \oint_S \varphi \frac{\partial \psi}{\partial n} \, dS - \int_V \varphi \nabla^2 \psi \, dV \quad (1)$$

MULTI-DIMENSIONAL VERSION of integration by parts!

$$\int_a^b f' g \, dx = [f g]_a^b - \int_a^b f g' \, dx$$

if $\bar{F} = \psi \nabla \varphi$ .. repeat passages.

$$\int_V \nabla \psi \cdot \nabla \varphi \, dV = \oint_S \psi \frac{\partial \varphi}{\partial n} dS - \int_V \psi \nabla^2 \varphi \, dV \quad (2)$$

Subtract (1) and (2) side by side

$$\int_V \cancel{\nabla \varphi \cdot \nabla \psi - \nabla \psi \cdot \nabla \varphi} \, dV = \oint_S \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \right) dS - \int_V \left(\varphi \nabla^2 \psi - \psi \nabla^2 \varphi \right) dV$$

$$\Downarrow$$

$$\boxed{\oint_S \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \right) dS = \int_V \left(\varphi \nabla^2 \psi - \psi \nabla^2 \varphi \right) dV}$$

GREEN'S SECOND IDENTITY

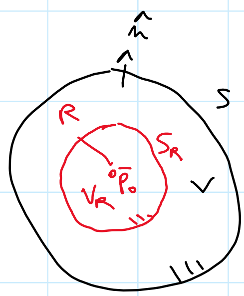
Conversion of volume integrals of laplacians to surface integrals

MEAN VALUE theorem for HARMONIC FUNCTIONS

Consider a SCD V : $\boxed{\varphi \text{ is harmonic if } \nabla^2 \varphi = 0 \ \forall \bar{p} \in V}$

Examples :
$$\begin{cases} \varphi = K, \quad K \in \mathbb{R} & \text{constant functions} \\ \varphi = ax + by + c & \text{first-order polynomial} \\ \varphi = 1/r \end{cases}$$

vector distance between $\bar{P}_0$ and $\bar{P}$



Theorem : For any spherical surface S_R with radius R centered around a point $\bar{p}_0$

$$\varphi(\bar{P}_0) = \underbrace{\frac{1}{4\pi R^2}\int_{S_R} \varphi(\bar{P})\, dS}_{\text{AVERAGE of } \varphi \text{ on the surface } S_R} = \underbrace{\frac{1}{4/3\,\pi R^3}\int_{V_R} \varphi(\bar{P})\, dV}_{\text{AVERAGE of } \varphi \text{ on volume } V_R}$$

$\uparrow$

φ generic harmonic function

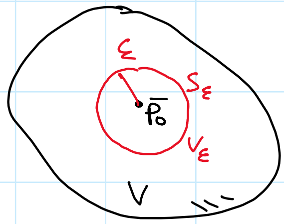
COROLLARIES.

K_1 A harmonic function has NO LOCAL EXTREMA on the internal points of its domain of definition

Proof AB ABSURDUM: assume there <u>IS</u> a LOCAL MAXIMUM

Def of LOCAL MAXIMUM : there $\exists$ an infinitesimal sphere V_ε with radius ε in which.

$$\begin{cases} \varphi(\bar{P}_0) - \varphi(\bar{P}) > 0 & \forall \bar{P} \in V_\varepsilon,\ \bar{P} \neq \bar{P}_0 \\ \varphi(\bar{P}_0) - \varphi(\bar{P}) = 0, & \bar{P} = \bar{P}_0 \end{cases}$$



Linear comb. of HARMONIC FUNCTIONS

$\Downarrow$

$\varphi' = \varphi(\bar{P}_0) - \varphi(\bar{P}) \Rightarrow \varphi'(\bar{P})$ <u>is</u> HARMONIC

check mean value theorem for $\varphi'(\bar{P})$

$$\underbrace{\varphi'(\bar{P}_0)}_{\substack{\Downarrow \\ =0 \text{ by def.}}} = \frac{1}{4/3\,\pi R^3}\int_{V_\varepsilon} \underbrace{\varphi'(\bar{P})}_{\substack{\Downarrow \\ >0}}\, dV$$

$\Rightarrow$ the <u>only</u> solution that satisfies the mean value theorem is $\varphi'(\bar{P}) = 0 \ \forall \bar{P} \in V_\varepsilon$

$\Downarrow$

Proof that there can be no maximum within V_ε

K_2. "Any extrema of a harmonic function φ on a SCD <u>must</u> be located on the domain BOUNDARY"

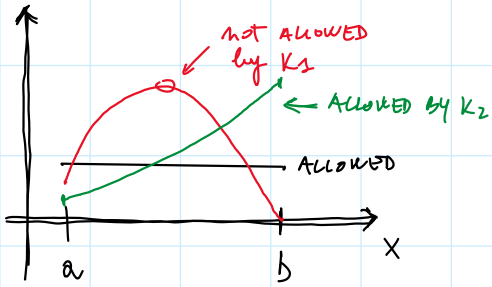
<u>Examples 1D</u>

$\uparrow$

not ALLOWED by K_1

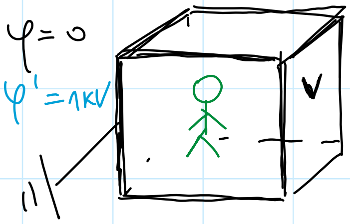
Examples 1D

Hp: φ IS HARMONIC
$(\nabla^2\varphi = 0 \ \forall x \in [a,b])$



K_3: "If a harmonic function φ is UNIFORM on the entire boundary S of a SCD V, φ must also be uniform within the domain V"

Example: Faraday cage, suppose $\varphi \Rightarrow$ ELECTRIC POTENTIAL



$\Rightarrow \rho = 0$ INSIDE CAGE

$\Downarrow$

$\forall \bar{P} \in V \quad \nabla^2\varphi = -\cancel{\rho/\varepsilon_0} = 0$

$\Downarrow$

φ is HARMONIC within the cage

$\varphi = 0 \ \forall \bar{P} \in S \Rightarrow$ For K_3 $\Rightarrow \varphi =$ UNIFORM

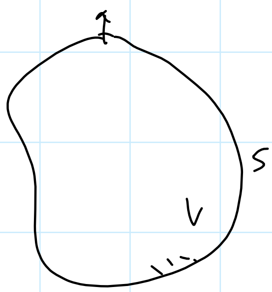
$\Downarrow$

$\bar{E} = -\nabla\varphi = 0 \Rightarrow$ NO ELECTRIC FIELD!

$\bar{E}' = -\nabla\varphi' = 0$

UNIQUENESS for POISSON PROBLEMS on SCD V

$$\begin{cases} \nabla^2\varphi = t & \forall \bar{P} \in V \\ \varphi = \varphi_0 & \forall \bar{P} \in S \quad \text{(DIRICHLET BC)} \end{cases} \quad (1)$$



Suppose 2 solutions $\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix}$ satisfying (1)

$\forall \bar{P} \in V$ $\quad \begin{cases} \nabla^2\varphi_1 = t \\ \varphi_1 = \varphi_0 \end{cases} \ (2) \quad \begin{cases} \nabla^2\varphi_2 = t \\ \varphi_2 = \varphi_0 \end{cases} \ (3)$

$\forall \bar{P} \in S$

$\forall \bar{p} \in S \quad \left\{ \varphi_1 = \varphi_0 \right. \quad (2) \qquad \left\{ \varphi_2 = \varphi_0 \right. \quad (3)$

Consider $\varphi_3 = \varphi_1 - \varphi_2$ GOAL: show that $\varphi_3 = 0$

difference field is HARMONIC!

(2) − (3).

$$\forall \bar{p} \in V \quad \nabla^2 \varphi_1 - \nabla^2 \varphi_2 = t - t = 0 \implies \boxed{\nabla^2 \varphi_3 = 0}$$

$$\forall \bar{p} \in S \quad \underbrace{\varphi_1 - \varphi_2 = \varphi_0 - \varphi_0 = 0}_{= 0 \ \forall \bar{p} \in S} \implies \boxed{\varphi_3 = 0}$$

φ_3 UNIFORMLY $= 0$ on the boundary

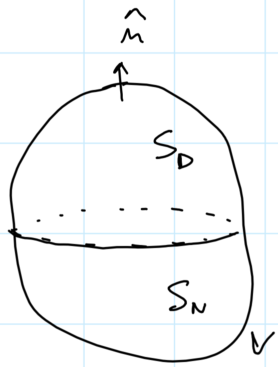
$$\underbrace{\text{For } K_3 . \ \varphi_3 \text{ is UNIFORM on boundary} \implies \varphi_3 = 0 \ \forall \bar{p} \in V}$$

$$\varphi_3 = 0 \ \forall \bar{p} \in V \text{ and } \forall \bar{p} \in S$$

$$\implies \cancel{\varphi_3} \overset{=0}{=} \varphi_1 - \varphi_2 \implies \boxed{\varphi_1 = \varphi_2} \implies \boxed{\text{Solution is UNIQUE}}$$

MIXED BOUNDARY CONDITIONS. DIRICHLET on S_D

NEUMANN on S_N



$S = S_D \cup S_N$

$$\begin{cases} V & \nabla^2 \varphi = t \\ S_D & \varphi = \varphi_0 \\ S_N & \dfrac{\partial \varphi}{\partial n} = \varphi_0' \end{cases} \qquad (1)$$

Suppose: φ - ELECTRIC POTENTIAL

$$E = -\nabla \varphi$$

$$E \cdot \hat{n} = -\nabla \varphi \cdot \hat{n}$$

$$E_n = -\frac{\partial \varphi}{\partial n}$$

$\Updownarrow$

φ_0'

Proof by contradiction. 2 solutions φ_1, φ_2 that satisfy (1)

$$\begin{cases} V & \nabla^2 \varphi_1 = t \\ S_D & \varphi_1 = \varphi_0 \\ S_N & \dfrac{\partial \varphi_1}{\partial n} = \varphi_0' \end{cases} \qquad \begin{cases} \nabla^2 \varphi_2 = t \\ \varphi_2 = \varphi_0 \\ \dfrac{\partial \varphi_2}{\partial n} = \varphi_0' \end{cases}$$