Finite Difference Method FDM 1D

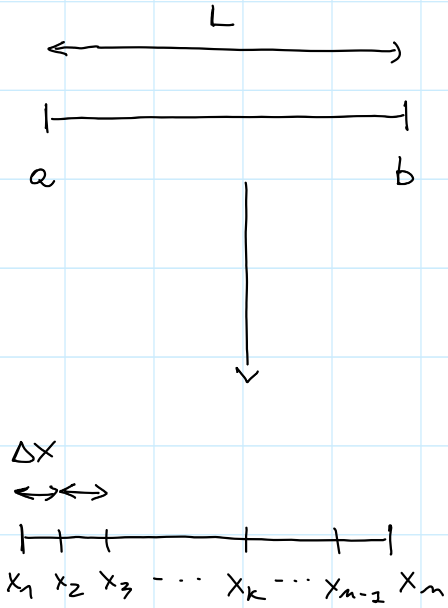
GOAL: solve a 1D Poisson problem using finite differences

$$\nabla^2 \varphi = t \xrightarrow[\text{Hp: } \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial z} = 0]{} \boxed{\frac{d^2\varphi}{dx^2} = t}, \quad x \in [a, b]$$

$$\Rightarrow \varphi = \varphi(x)$$

ASSUME CARTESIAN COORDINATES

$$(1)\begin{cases} \dfrac{d^2\varphi}{dx^2} = t & \forall x \in]a,b[\\ \varphi(a) = \varphi_a & x = a \quad \text{DIRICHLET BC} \\ \left.\dfrac{d\varphi}{dn}\right|_b = \varphi'_b & x = b \quad \text{NEUMANN BC} \end{cases}$$



GOAL: write a DISCRETE formulation

① GRID of nodes to represent domain

n - nodes with UNIFORM SPACING

$x_1 \equiv a$

$x_n \equiv b$

$$\Delta x = \frac{L}{n-1}$$

number of intervals

LINEAR MAPPING between coordinates and indices

$$x_k = a + \Delta x(k-1)$$

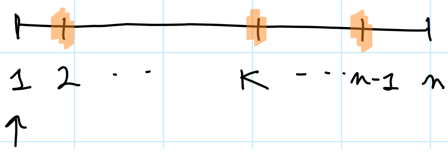
EXAMPLE:

$$x_2 = a + \Delta x(2-1) = a + \Delta x$$

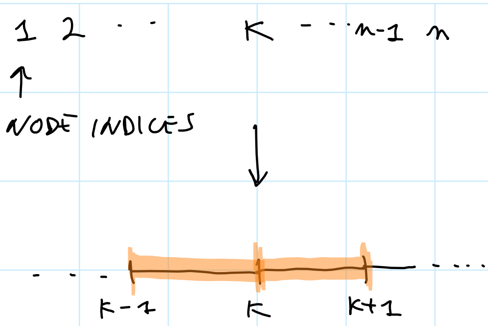
② Use finite difference formulas to express derivatives

INTERNAL NODES

$$\left.\frac{d^2\varphi}{dx^2}\right|_k = \frac{\varphi_{k-1} - 2\varphi_k + \varphi_{k+1}}{\Delta x^2} + O(\Delta x^2)$$



$$\left.\frac{d^2\varphi}{dx^2}\right|_k = \frac{\cdots}{\Delta X^2} + O(\Delta X^2)$$



$$\boxed{\left.\frac{d^2\varphi}{dx^2}\right|_K = t(K)}$$ ⇐ known term

$$\boxed{\frac{\tilde{\varphi}_{K-1} - 2\tilde{\varphi}_K + \tilde{\varphi}_{K+1}}{\Delta X^2} = t_K}$$

⇒ yields an approximation (with second-order accuracy) of the unknown function φ

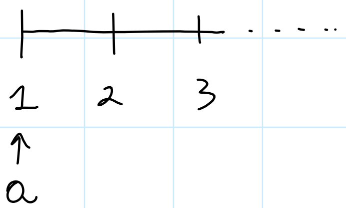
NOTATION we will use φ instead of $\tilde{\varphi}$ for conciseness

$$\begin{cases} \varphi_1 - 2\varphi_2 + \varphi_3 = t_2 \Delta X^2 & (2) \\ \vdots \\ \boxed{\varphi_{K-1} - 2\varphi_K + \varphi_{K+1} = t_K \Delta X^2} & \text{(for node } K) \\ \vdots \\ \varphi_{n-2} - 2\varphi_{n-1} + \varphi_n = t_{n-1} \Delta X^2 & (n-1) \end{cases}$$

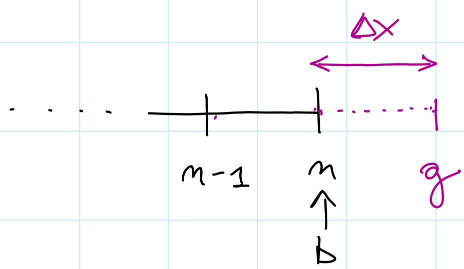
algebraic expression for nodes $2, 3, \dots, K, \dots n-1$

BOUNDARIES

a $\quad \boxed{\varphi(a) = \varphi_1 = \varphi_a}$



b . $\left.\dfrac{d\varphi}{dn}\right|_b = \varphi'_b$



IDEA: express derivative using FD

$$\left.\frac{d\varphi}{dn}\right|_b = \left.\frac{d\varphi}{dx}\right|_b = \frac{\varphi_n - \varphi_{n-1}}{\Delta X} + O(\Delta X)$$ First-order accurate

To get second-order accuracy, write a centered FD formula for $\left.\frac{d\varphi}{dx}\right|_b$

GHOST NODE TECHNIQUE:

CENTERED FINITE difference formula for $d\varphi/dx$

$$\begin{cases} \left.\dfrac{d\varphi}{dx}\right|_b = \dfrac{\varphi_g - \varphi_{n-1}}{2\Delta x} + \mathcal{O}(\Delta x^2) \Rightarrow \dfrac{\varphi_g - \varphi_{n-1}}{2\Delta x} = \varphi'_b \\ \dfrac{\varphi_{n-1} - 2\varphi_n + \varphi_g}{\cancel{\Delta x^2}} = t_n \Delta x^2 \end{cases}$$

$\left.\dfrac{d^2\varphi}{dx^2}\right|_n$

Derive φ_g: $\varphi_g - \varphi_{n-1} = 2\Delta x \varphi'_b$

$$\boxed{\varphi_g = 2\Delta x \varphi'_b + \varphi_{n-1}}$$

$$\varphi_{n-1} - 2\varphi_n + 2\Delta x \varphi'_b + \varphi_{n-1} = t_n \Delta x^2$$

$$\cancel{2}\varphi_{n-1} - \cancel{2}\varphi_n = \frac{t_n \Delta x^2}{2} - \cancel{2}\Delta x \varphi'_b$$

$$\boxed{\varphi_{n-1} - \varphi_n = \Delta x \left(\frac{\Delta x}{2} t_n - \varphi'_b \right)}$$

Algebraic equation for node n (second-order)

Assemble a linear system with the algebraic expressions:

DISCRETIZED VERSION (1) $\Rightarrow [K]\{\varphi\} = \{Rhs\}$

1 2 ... k ... n-1 n

1: $\varphi_1 = \varphi_a$

2, 3, ... n-1: $\varphi_{k-1} - 2\varphi_k + \varphi_{k+1} = t_k \Delta x^2$

n: $\varphi_{n-1} - \varphi_n = \Delta x \left(\frac{\Delta x}{2} t_n - \varphi'_b \right)$

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \end{array} \overbrace{\begin{bmatrix} 1 & & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \end{bmatrix}}^{[K]} \overbrace{\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \end{Bmatrix}}^{\{\varphi\}} = \overbrace{\begin{Bmatrix} \varphi_a \\ t_2 \Delta x^2 \\ \vdots \end{Bmatrix}}^{\{Rhs\}}$$

$$
\begin{array}{c}
\vdots \\ k \\ \vdots \\ n
\end{array}
\begin{bmatrix}
\ddots & \ddots & \ddots & & \\
& 1 & -2 & 1 & \\
& & \ddots & \ddots & \ddots \\
& & & 1 & -1
\end{bmatrix}
\begin{bmatrix}
\vdots \\ \varphi_k \\ \vdots \\ \varphi_{n-1} \\ \varphi_n
\end{bmatrix}
=
\begin{bmatrix}
\vdots \\ t_k \Delta x^2 \\ \vdots \\ t_{n-1} \Delta x^2 \\ \Delta x\left(\frac{\Delta x}{2} t_n - \varphi'_b\right)
\end{bmatrix}
$$

$[K]$ is a TRI-DIAGONAL matrix and a SPARSE MATRIX

$\Downarrow$

$[K]$: $n \times n$ matrix

if $n = 1000$, $[K]$ has 10^6 elements but only three elements are non-zeros for each row corresponding to an internal node

Property of FDM

$\Downarrow$

produces SPARSE MATRICES

Diagonally-dominant matrices

$[A]$ square matrix with n-rows

- $[A]$ is STRICTLY DIAGONALLY-DOMINANT if

$$|a_{i,i}| > \sum_{j=1,\, j\neq i}^{n} |a_{i,j}| \quad \forall\, i = 1, \dots, n$$

$\Uparrow$

generic element of matrix $[A]$ on the main diagonal

Example

$$
[A] = \begin{array}{cccc|c}
j=1 & j=2 & \dots & j=n & \\
\hline
1 & 0 & 0 & 0 & i=1 \\
1 & -2 & 1 & 0 & i=2 \\
0 & 1 & -2 & 1 & \vdots \\
0 & 0 & 0 & 1 & i=n
\end{array}
$$

$\Downarrow$

$[A]$ is NOT a STRICTLY diagonally-dominant matrix

i_1: $1 > 0$

i_2: $2 = 1 + 1 = 2$

i_3: $\dots$

i_4: $1 > 0$

however, $[A]$ is WEAKLY diagonally-dominant

$$|a_{i,i}| \geq \sum_{j=1,\, j\neq i}^{n} |a_{i,j}| \quad \forall\, i = 1, \dots, n$$

Solution IS UNIQUE $\leftarrow$ ① if $[A]$ is STRICTLY DIAGONALLY DOMINANT

in general → Solution IS UNIQUE → ① if [A] is STRICTLY DIAGONALLY DOMINANT
→ ② if [A] is WEAKLY DIAGONALLY DOMINANT
+
[A] has at least a row with STRONG DIAGONAL DOMINANCE

in general → solution is NOT UNIQUE
if [A] is weakly diagonally dominant