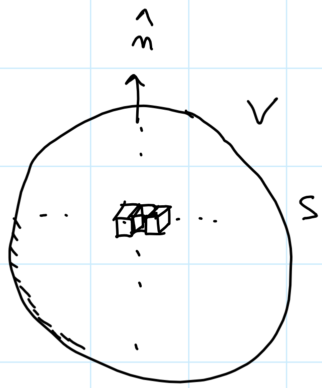
DIVERGENCE THEOREM

$\bar{U} \in C_1$ $\rightarrow$ $\bar{U}$ is continuous
$\rightarrow$ partial deriv of $\bar{U}$ are continuous

$$\underbrace{\int_V \underbrace{\nabla \bar{U}}_{\text{LOCAL FLUX}} dV}_{\Sigma \text{ of LOCAL FLUXES over } V} = \underbrace{\oint_S \bar{U} \cdot d\bar{S}}_{\text{Flux of } \bar{U} \text{ through } S}$$



implication for SOLENOIDAL FIELDS, hp. $\bar{U}$ is SOLENOIDAL

def: $\bar{U}$ is SOLENOIDAL if $\oint_S \bar{U} \, d\bar{S} = 0 \ \forall S \in D$

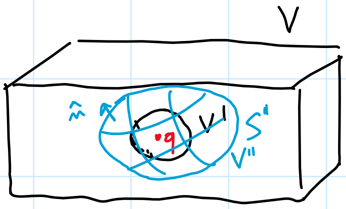
$$\underbrace{\oint_S \bar{U} \cdot d\bar{S}}_{=0} \overset{\text{GAUSS}}{=} \underbrace{\int_V \nabla \bar{v} \, dV}_{=0} \Rightarrow \underbrace{\nabla \cdot \bar{U} = 0}_{\bar{U} \text{ is DIV-FREE}} \ \forall p \in D$$

① $\bar{U}$ is SOLENOIDAL $\xrightarrow{\text{(ALWAYS)}}$ $\bar{U}$ is DIVERGENCE-FREE

② $\bar{U}$ is DIV-FREE $\xrightarrow{\text{(if D SCD)}}$ $\bar{U}$ is SOLENOIDAL ?

YES only if D is a SCD

Example - SOLID DIELECTRIC with a VOID



$D = V - V' \Rightarrow$ in D : $\nabla \cdot \bar{E} = \cancel{\rho/\varepsilon_0} = 0$

$\Downarrow$

~~$\bar{E}$ is SOLENOIDAL in D~~

$\Downarrow$

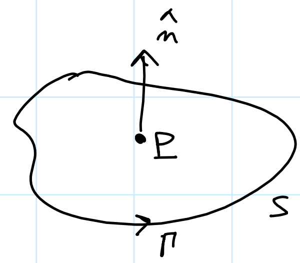
NO because D is NOT SCD

CURL : $\mathbb{R}^3 \to \mathbb{R}^3$

- the $\nabla \times (\cdot\,\cdot)$ describes the TENDENCY of a V.F. to CIRCULATE AROUND A POINT

Hp: $\bar{U} \in C_1$

def: $$\nabla \times \bar{U}(\underline{P}) \cdot \hat{n} = \lim_{\substack{\Delta S \to 0 \\ \uparrow \\ \text{AREA of } S}} \underbrace{\frac{\oint_{\Gamma} \bar{U} \cdot d\bar{l}}{\Delta S}}_{\frac{\text{circulation}}{\text{area}}}$$



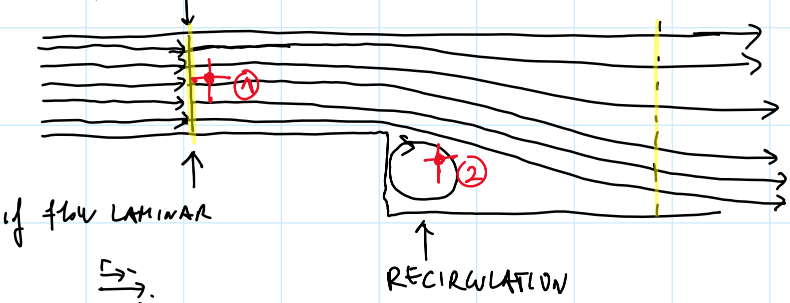
CART. COORD.:

$$\nabla \times \bar{U}(\underline{P}) = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ U_x & U_y & U_z \end{bmatrix} =$$

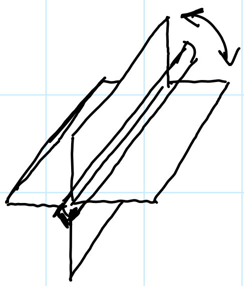
$$= \underbrace{\left(\frac{\partial}{\partial y} U_z - \frac{\partial}{\partial z} U_y\right)}_{\text{component of } \nabla \times \bar{U} \text{ in } \hat{i} \text{ direction}} \hat{i} - \left(\frac{\partial}{\partial x} U_z - \frac{\partial}{\partial z} U_x\right) \hat{j} + \left(\frac{\partial}{\partial x} U_y - \frac{\partial}{\partial y} U_x\right) \hat{k}$$

Example flow in a DUCT

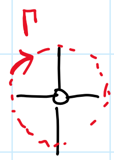
Hp. very TURBULENT FLOW (High Re)



CURL-MOTOR :

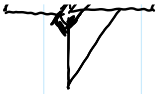


$\rightarrow$



① NO ROTATION $\Rightarrow$ NO CURL of FLOW field

② CLOCK-WISE ROTATION $\Rightarrow$ CURL IS INTO the PAGE



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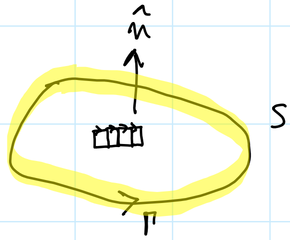
STOKES THEOREM

$\bar{U} \in C_1$ on a surface S



$$\underbrace{\int_S \underbrace{\nabla \times \bar{U}}_{\text{LOCAL CIRC.}} \cdot d\bar{S}}_{\Sigma \text{ of LOCAL CIRC.}} = \underbrace{\oint_\Gamma \bar{U} \cdot d\bar{\ell}}_{\text{circulation of } \bar{U} \text{ along } \Gamma}$$

LINKED FLUX — $\overbrace{\text{flux of the CURL}}^{\text{Flux of } (\nabla \times \bar{U})}$ of a V.F through S depends only on the BOUNDARY of S

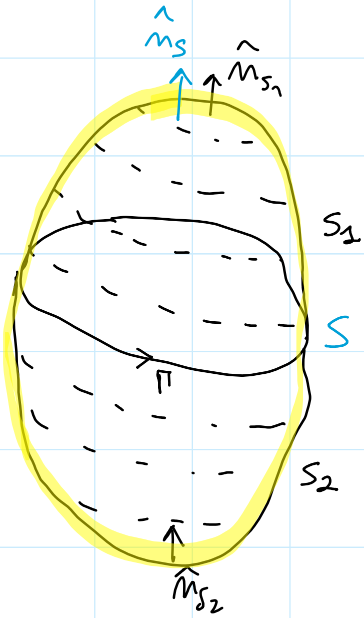
Ex: $\vec{B} = \nabla \times \bar{A} \Rightarrow \int_S \bar{B} \cdot d\bar{S}$ is a LINKED FLUX

Flux of $\nabla \times \bar{U}$ through a CLOSED SURFACE ?

(1) $\int_{S_1} \nabla \times \bar{U} \cdot d\bar{S} \overset{\text{STOKES}}{=} \oint_\Gamma \bar{U} \cdot d\bar{\ell}$, S_1 open SURFACE bounded by Γ

$\qquad | = \qquad\qquad | =$

(2) $\int_{S_2} \nabla \times \bar{U} \, d\bar{S} = \oint_\Gamma \bar{U} \cdot d\bar{\ell}$



$\rightarrow \int_{S_1} \nabla \times \bar{U} \cdot d\bar{S} = \int_{S_2} \nabla \times \bar{U} \cdot d\bar{S}$ EXPECTED!

$S = S_1 \cup S_2$

$$\oint_S \nabla \times \bar{U} \cdot d\bar{S} = \underbrace{\int_{S_1} \nabla \times \bar{U} \cdot d\bar{S}}_{\hat{n}_{S_1} \cdot \hat{n}_S = 1} - \underbrace{\int_{S_2} \nabla \times \bar{U} \cdot d\bar{S}}_{\hat{n}_{S_2} \cdot \hat{n}_S = -1} = 0$$

$$\underbrace{\quad}_{\hat{n}_{S_1} \cdot \hat{n}_S = 1} \qquad \underbrace{\quad}_{\hat{n}_{S_2} \, \hat{n}_S = -1}$$

$\Rightarrow$ since S is ARBITRARY, $\underbrace{\oint_S \nabla \times \bar{U} \cdot d\bar{S} = 0 \quad \forall \text{ CLOSED } S}$

$\nabla \times \bar{U}$ <u>is</u> A SOLENOIDAL FIELD

<u>implication of STOKES theorem for CONSERVATIVE fields</u>

Def: $\bar{U}$ is conservative if $\oint_\Gamma \bar{U} \, d\bar{\ell} = 0 \quad \forall \, \Gamma \in D$

Hp: $\bar{U}$ <u>is</u> conservative

$$\underbrace{\oint_\Gamma \bar{U} \cdot d\bar{\ell}}_{= 0 \, \forall \Gamma \in D} \underset{\text{ST. Th}}{\overset{\uparrow}{=}} \underbrace{\int_S \nabla \times \bar{U} \cdot d\bar{S}}_{= 0 \, \forall S \in D} \Longrightarrow \underbrace{\nabla \times \bar{U} = 0}_{\text{CURL-FREE}} \quad \forall P \in D$$

① $\bar{U}$ is CONSERVATIVE $\xRightarrow{\text{always}}$ $\bar{U}$ is CURL-FREE

② $\bar{U}$ is CURL-FREE $\xRightarrow{\text{if D SCD}}$ $\bar{U}$ is CONSERVATIVE
only if D is SCD

EICHUNG

<u>GAUGE FREEDOM</u> , Hp $\bar{U}$ on a SCD

$\bar{U}$ CONSERVATIVE

$$\oint_\Gamma \bar{U}\cdot d\bar{l} = 0 \quad \forall \Gamma \in D$$
$$\Updownarrow \text{SCD}$$
$$\rightarrow \nabla\times\bar{U} = 0 \quad \forall P \in D$$

$$\Rightarrow \quad \bar{U} = \nabla f \leftarrow \text{SCALAR POTENTIAL}$$

f is defined "UP TO A CONSTANT"

f^* another potential for $\bar{U}$

$$f^* = f + K, \quad K \in \mathbb{R}$$

$$\nabla f^* = \nabla(f+K) =$$
$$\underbrace{\nabla f}_{\bar{U}} + \underbrace{\cancel{\nabla K}}_{=0} \Rightarrow \nabla f^* = \nabla f$$

$\bar{U}$ is SOLENOIDAL

$$\oint_S \bar{U}\cdot d\bar{S} = 0 \quad \forall S \in D$$
$$\Updownarrow \text{SCD}$$
$$\nabla\cdot\bar{U} = 0 \quad \forall P \in D$$

$$\Rightarrow \quad \bar{U} = \nabla\times\bar{F} \swarrow \text{VECTOR POTENTIAL}$$

$\bar{F}$ is defined "UP TO a CONSERVATIVE FUNCTION"

$\bar{F}^*$: another v. potential for $\bar{U}$

$$\bar{F}^* = \bar{F} + \nabla f$$

$$\nabla\times\bar{F}^* = \nabla\times(\bar{F} + \nabla f) =$$
$$\underbrace{\nabla\times\bar{F}}_{\bar{U}} + \underbrace{\cancel{\nabla\times(\nabla f)}}_{=0 \ \text{CONS. FIELD}} = \nabla\times\bar{F}$$

COMPUTATION of a CONSERVATIVE FIELD

$\bar{U}$ conservative on a SCD D

$$\begin{cases} \nabla\times\bar{U} = 0 \Rightarrow \bar{U} = \nabla f \\ \nabla\cdot\bar{U} = t \Rightarrow \nabla\cdot(\nabla f) = t \Rightarrow \boxed{\nabla^2 f = t} \end{cases}$$

SCALAR POISSON PROBLEM

t: KNOWN value; UNKNOWN: f

in CART COORDS: $\nabla^2 f = \boxed{\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = t}$

PDE
ELLIPTIC PDE

COMPUTATION of A SOLENOIDAL FIELD

COMPUTATION of A SOLENOIDAL FIELD

$\bar{U}$ solenoidal, SCD GOAL: Find $\bar{U}$ $\Rightarrow$ Find $\bar{F}$ $\rightarrow$ take curl of $\bar{F}$ to find $\bar{U}$

$$\rightarrow \begin{cases} \nabla \cdot \bar{U} = 0 \Rightarrow \bar{U} = \nabla \times \bar{F} \\ \nabla \times \bar{U} = \underset{\text{known}}{\bar{T}} \Rightarrow \underbrace{\nabla \times (\underbrace{\nabla \times \bar{F}}_{\bar{U}}) = \bar{T}}_{\text{VECTOR ANALYSIS}} \end{cases}$$

$$\cancel{\nabla(\nabla \cdot \bar{F})} - \nabla^2 \bar{F} = \bar{T}$$

SET ARBITRARILY

$\nabla \cdot \bar{F} = 0$

COULOMB GAUGE

$$\Downarrow$$

$$\boxed{\nabla^2 \bar{F} = -\bar{T}}$$

VECTOR POISSON PROBLEM

CART. COORDS.

$$\begin{cases} \nabla^2 F_x = -T_x \\ \nabla^2 F_y = -T_y \\ \nabla^2 F_z = -T_z \end{cases} \rightarrow \text{set of 3 SCALAR POISSON PROBLEMS}$$

HELMHOLTZ THEOREM : any vector field $\bar{U}$ can be decomposed into a CONSERVATIVE and a SOLENOIDAL component

$$\bar{U} = \underset{\text{CONSERV.}}{\bar{U}_c} + \underset{\text{SOL.}}{\bar{U}_s}$$

$\bar{U}_c = \nabla f$ $\bar{U}_s = \nabla \times \bar{F}$

$\nabla^2 f = t$ $\nabla^2 \bar{F} = -\bar{T}$