L1

15 September 2025 11:02

Topology

METRIC SPACE . SET equipped with notion of DISTANCE

Ex. EUCLIDEAN SPACE $\mathbb{R}^3$ $(n=3)$

ENSEMBLE of N-TUPLES

Ex. $n=2$ $\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}$ $\begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix}$ ....

P_1 $\quad$ P_2

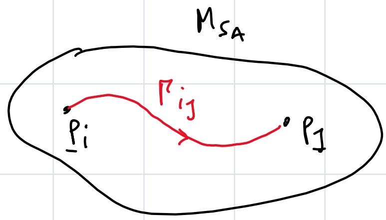
measured with METRIC

for $n=2$:

$$D_{12} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

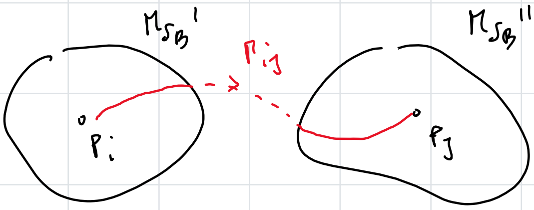
CONNECTED METRIC SPACE (M_S) . for any two points $P_i, P_j \in M_S$, M_S is connected if $\exists\, \Gamma_{ij}$ connecting P_i and P_j, where $\Gamma_{ij} \in M_S$

(A)



M_{SA} IS CONNECTED

(B)



M_{SB} IS NOT CONNECTED

$$M_{SB} = M_{SB}' \cup M_{SB}''$$

DOMAIN . CONNECTED SUBSET of a METRIC SPACE

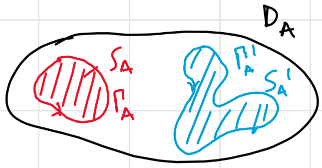
- M_{SA}, M_{SB}', M_{SB}'' ARE DOMAINS
- M_{SB} IS NOT DOMAIN

2D

SIMPLY CONNECTED DOMAIN (SCD)

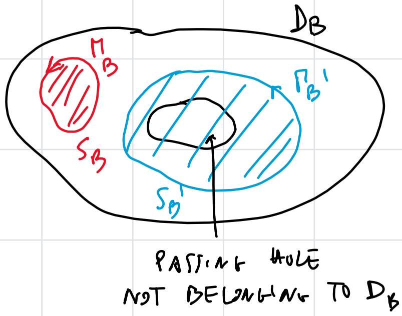
D is a SCD if $\forall$ (for all) Γ (CLOSED PATH) $\in$ D there exist a surface S $\in$ D

(A)



$S_A, S_{A'} \in D_A$, D_A IS SCD

(B)



$S_B \in D_B$

$S_B' \notin D_B \Rightarrow D_B$ is NOT a SCD

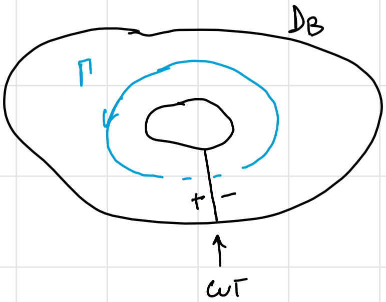
$\Downarrow$

D_B is MULTIPLY-CONNECTED DOMAIN

$\Downarrow$

CAN BE REDUCED to SCD

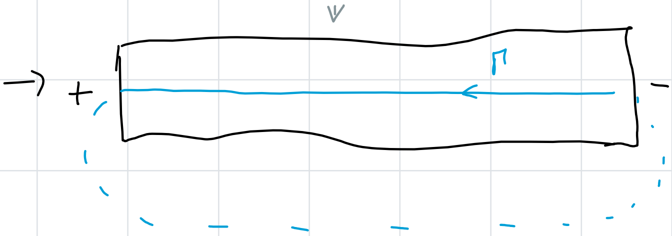
$\rightarrow$ INFINITESIMAL CUT



$\rightarrow$ NOT possible to define $S \notin D_B$

$\Downarrow$

REDUCED D_B to a SCD



MULTIPLICITY of a DOMAIN

(number) of infinitesimal cuts to reduce a multiply connected domain to a SCD

MULTIPLICITY = 2

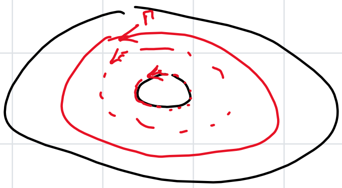
Simply connected domain (ALTERNAT. DEF.)

a domain D is a SCD if ANY closed path Γ can be SHRUNK to a single point by means of a CONTINUOUS TRANSFORMATION

EXAMPLE

SCD

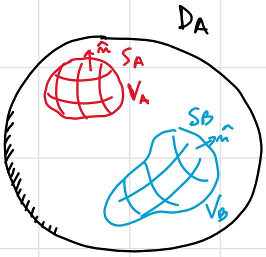
MULTIPLY connected DOMAIN



3D

SCD : D is a SCD if $\forall$ closed SURFACES $S \in D$, there exist a volume V, enclosed by S, $\in D$

Ⓐ



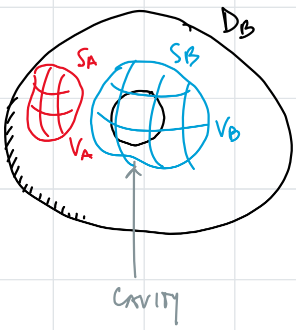
$V_A \in D_A$

$V_B \in D_A$

. . .

$\rightarrow$ D_A IS a SCD

Ⓑ



$V_A \in D_B$

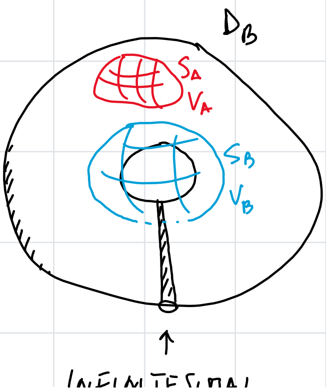
$V_B \notin D_B$

. . .

$\rightarrow$ D_B IS NOT a SCD

$\downarrow$

MULTIPLY CONNECTED DOMAIN (MULTIPLICITY = 1)



S_B CANNOT be defined thanks to the puncture

$\Rightarrow$ D_B IS a SCD

↑
INFINITESIMAL
PUNCTURE

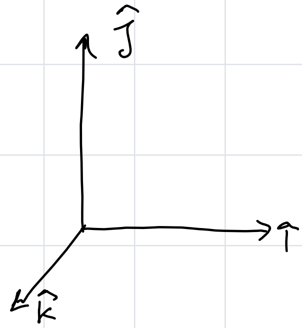
SCD (3D) . a 3D domain is SCD if any closed surface S can be shrunk to a single point by means of a CONTINUOUS TRANSFORMATION
(ALT.)

Differential Operators

GRADIENT $\mathbb{R} \to \mathbb{R}_3$

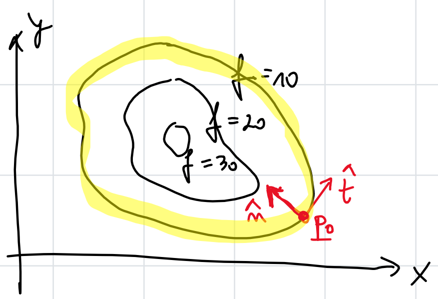
- the $\nabla(\dots)$ describes the PARTIAL DERIVATIVES of a SCALAR function

Def : $\underbrace{\nabla f \cdot \hat{n}}_{\text{projection of grad along direction } \hat{n}} = \partial f / \partial n$



in CART. COORDINATES : $\nabla f = \dfrac{\partial f}{\partial x}\hat{\imath} + \dfrac{\partial f}{\partial y}\hat{j} + \dfrac{\partial f}{\partial z}\hat{k}$

ISO-LINES of f Hp $f = f(x,y)$



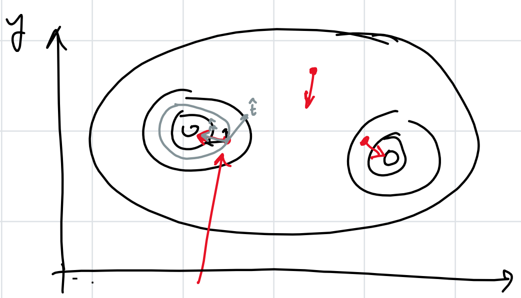
$\left.\dfrac{\partial f}{\partial t}\right|_{P_0} = 0 \Rightarrow \nabla f \cdot \hat{t} = 0$

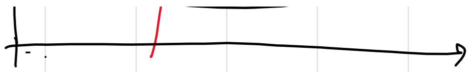
↓

∇f IS DIRECTED along $\hat{n}$

↓

DIRECTION of MAXIMUM INCREASE of the FUNCTION





$$\nabla f = -10\,\hat{\imath} + 1\,\hat{\jmath}$$

$$\nabla f = -11\,\hat{n} + 0\,\hat{t}$$

Hp: $\bar{U} = \nabla f$

$\uparrow$

VECTOR FIELD that can be expressed as the gradient of a SCALAR fun

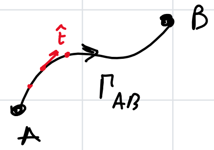
LINE INTEGRAL of $\bar{U} = \nabla f$

$$\int_A^B \bar{U} \cdot d\bar{\ell} = \int_A^B \nabla f \cdot d\bar{\ell} = \int_A^B df = f(B) - f(A)$$

$d\bar{\ell} = \hat{e}\, d\ell$

$$\frac{\partial f}{\partial \ell}\, d\ell = df$$

$\Downarrow$ RESULT depends only on BOUNDARIES of Γ



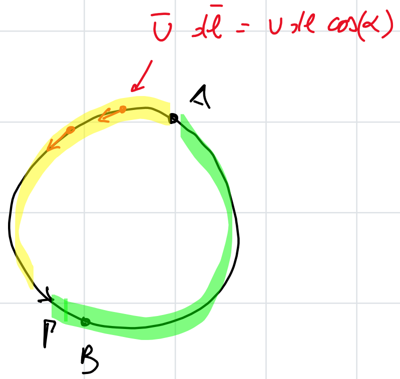
f is a SCALAR POTENTIAL for $\bar{U}$, $\bar{U}$ is a CONSERVATIVE field

CIRCULATION of $\bar{U} = \nabla f$

$$\oint_\Gamma \bar{U} \cdot d\bar{\ell} = \oint_\Gamma \nabla f \cdot d\bar{\ell} =$$

$$\int_A^B \nabla f \, d\bar{\ell} + \int_B^A \nabla f \cdot d\bar{\ell} = \cancel{f(B)} - \cancel{f(A)} + \cancel{f(A)} - \cancel{f(B)} = 0$$

since $\nabla f \cdot \hat{n} = \partial f / \partial n$



$\Rightarrow$ GRADIENT of ANY function f is CIRCULATION - FREE

CONSERVATIVE VECTOR FIELD

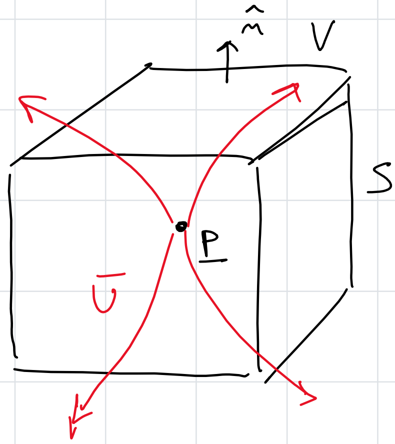
if a vector field (e.g. $\bar{U}$) is circulation free $\forall$ possible Γ

$\Rightarrow$ $\bar{U}$ is a CONSERVATIVE field if a vector field (e.g. $\bar U$) is circulation free $\forall$ point $\underline{P}$

$\Rightarrow$ $\bar U$ is a CONSERVATIVE field

DIVERGENCE OPERATOR $\mathbb{R}^3 \to \mathbb{R}$

- the $\nabla \cdot (\,..\,)$ describes the SPACE-DISTRIBUTION of SOURCES/SINKS of a vector field



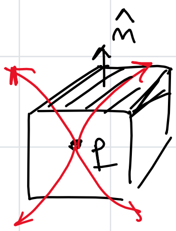
$\underline{P} \in V$, bounded by S

FLUX of $\bar U$ through CLOSED surface S bounding V

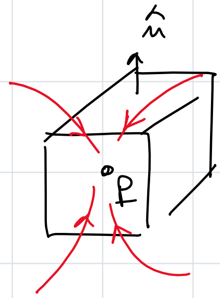
$$\nabla \cdot \bar U(\underline{P}) = \lim_{\Delta V \to 0} \frac{\oint_S \bar U \cdot d\bar S}{\Delta V} = \underbrace{\left[\frac{\text{flux}}{\text{volume}}\right]}_{\text{LOCAL FLUX}}$$

volume of V

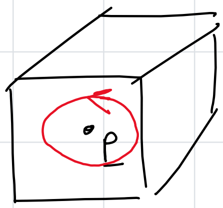
Examples



$\nabla \cdot \bar U(\underline{P}) > 0$



$\nabla \cdot \bar U(\underline{P}) < 0$



$\nabla \cdot \bar U(\underline{P}) = 0$