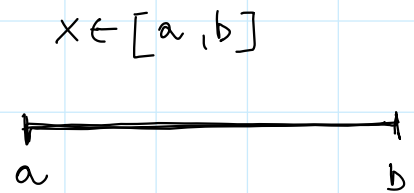# Finite Element method FEM

1D

$X \in [a, b]$

$$\nabla \cdot (P(x) \nabla \varphi) = t(x) \qquad Hp: 1D$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$$

$$\boxed{\frac{d}{dx}\left(P(x)\frac{d\varphi}{dx}\right) = t(x)}$$

(STRONG) FORMULATION of problem

REQUIREMENT: φ TWICE DIFFERENTIABLE

"Find the function $\varphi(x)$ that SATISFIES this ⟹ " ⟷
$$\begin{cases} \frac{d}{dx}\left(P(x)\frac{d\varphi}{dx}\right) = t(x) & , \forall x \in]a, b[\\[2mm] \varphi(a) = \varphi_a & X \in a \quad \text{DIRICHLET} \\[2mm] \frac{d\varphi}{dx}\Big|_b = \varphi'_b & X \in b \quad \text{NEUMANN} \end{cases}$$
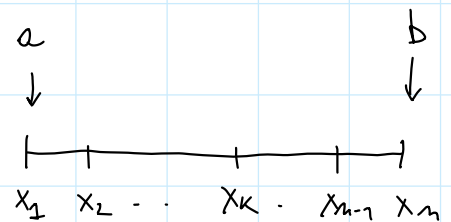
DOMAIN DISCRETIZATION (MESHING)

a   b

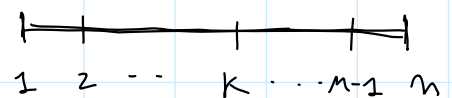$$[a, b] \Rightarrow \{x_1, x_2, \ldots x_k, \ldots x_n\}$$

⟹

introduce n-nodes ⟹ n-1 ~~intervals~~

ELEMENTS

$x_1 \quad x_2 \cdots \quad x_k \cdot \quad x_{n-1} \, x_n$

if MESH is UNIFORM

SPACING $\Delta = \dfrac{L}{n-1}$ ⟵ b-a length of domain

$1 \quad 2 \cdots \quad k \cdots n-1 \quad n$

INTERPOLATION

interpolated solution

$$\varphi(x) \implies \tilde{\varphi}(x) \qquad : \text{need to choose interpolating function}$$

⟹

PIECEWISE LINEAR POLYNOMIAL (HAT FUNCTIONS)

~ .
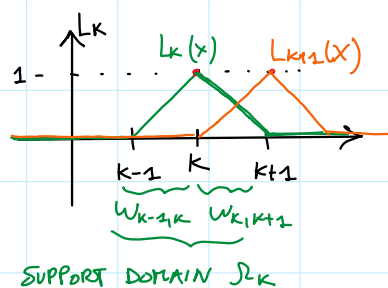
$$\tilde{\varphi}(x) = \sum_{k=1}^{m} \varphi_k L_k(x) = \varphi_1 L_1(x) + \varphi_2 L_2(x) + \ldots + \varphi_k L_k(k) + \ldots \varphi_m L_m(x)$$

aim to find $\varphi_1, \varphi_2, \ldots \varphi_m$

consequence of interpolation:

EXACT solution. $\varphi(x)$ $\Rightarrow$ $\dfrac{d}{dx}\left(p(x)\dfrac{d\varphi}{dx}\right) - t(x) = 0$

INTERPOLATED solution $\tilde{\varphi}(x) \Rightarrow$ $\underbrace{\dfrac{d}{dx}\left(p(x)\dfrac{d\tilde{\varphi}}{dx}\right) - t(x)}_{} \neq 0$.

$\mathcal{R}(x)$ RESIDUAL

NOTE. IMPOSSIBLE to have
$\mathcal{R}(x) = 0 \; \forall x$

WEIGHTED RESIDUALS APPROACH

Require that
the WEIGHTED integral
of $\mathcal{R}(x)$ is ZERO

$$\int_a^b W(x)\, \mathcal{R}(x)\, dx = 0 \qquad 1 \text{ eqn}$$

↑ WEIGHTING FUNCTION (STILL TO BE defined...)

$$\int_a^b W(x)\left[\dfrac{d}{dx}\left(p(x)\dfrac{d\tilde{\varphi}}{dx}\right) - t(x)\right] dx = 0$$

$\tilde{\varphi} \in C_0$ $\tilde{\varphi}$ can be differentiated but $\dfrac{d\tilde{\varphi}}{dx}$ is not CONTINUOUS
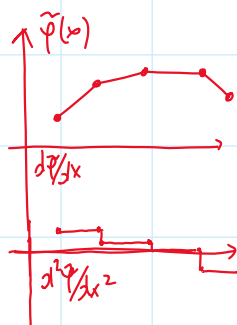
Second derivative is not defined at GRID NODES

EXAMPLE.

$\tilde{\varphi}(x)$

$\dfrac{d\tilde{\varphi}}{dx}$

$\dfrac{d^2\tilde{\varphi}}{dx^2}$

integration by parts.

$$\int_a^b w(x) \left[\frac{d}{dx}\left(P(x)\frac{d\tilde{\varphi}}{dx} \right) - t(x) \right] dx = 0$$

g' over $\frac{d}{dx}\left(P(x)\frac{d\tilde{\varphi}}{dx}\right)$, f is $w(x)$, g is $P(x)\frac{d\tilde{\varphi}}{dx}$

$$\int_a^b w(x) \frac{d}{dx}\left(P(x)\frac{d\tilde{\varphi}}{dx} \right) dx = \int_a^b w(x)\, t(x)\, dx$$
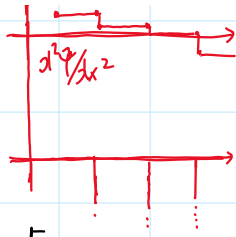
$$\int_a^b f\, g'\, dx = \left[f(x)\,g(x) \right]_a^b - \int_a^b f'\, g\, dx$$

$w(x)$ $P(x)\frac{d\tilde{\varphi}}{dx}$

$$\left[w(x)\, P(x)\, \frac{d\tilde{\varphi}}{dx} \right]_a^b - \int_a^b \frac{dw}{dx}\, P(x)\, \frac{d\tilde{\varphi}}{dx}\, dx = \int_a^b w(x)\, t(x)\, dx$$

WEAK FORMULATION of Poisson's Eq

$$\boxed{\int_a^b \frac{dw}{dx}\, P(x)\, \frac{d\tilde{\varphi}}{dx}\, dx = \left[w(x)\, P(x)\, \frac{d\tilde{\varphi}}{dx} \right]_a^b - \int_a^b w(x)\, t(x)\, dx}$$

↑ BOUNDARY TERM

✓ no 2nd derivatives of C_0 functions

✓ some type of information of STRONG FORMULATION
  ⇒ expressed using NODAL VALUES

"Find $\varphi_1, \varphi_2, \ldots$ that SATISFY the WEAK FORMULATION"

✗ still 1 eqn ☹

HAT FUNCTIONS || SHAPE FUNCTIONS

GALERKIN'S CHOICE   ⇒ $W(x) ⇒ L_k(x)$

$$\int_a^b \frac{dL_k}{dx}\, P(x)\, \frac{d\tilde{\varphi}}{dx}\, dx = \left[L_k(x)\, P(x)\, \frac{d\tilde{\varphi}}{dx} \right]_a^b - \int_a^b L_k(x)\, t(x)\, dx \qquad k = 1, 2, \ldots K, \ldots m$$

IDEA. find $\varphi_1, \varphi_2, \ldots \varphi_k \ldots$ such that all the expressions below are satisfied simultaneously

$$\left(\int_a^b \frac{dL_k}{dx} P(x) \frac{d\tilde{\varphi}}{dx} \ldots \left[\ldots \frac{d\tilde{\varphi}}{dx} \right]_a^b \int_a^b \ldots t(x) \ldots \right.$$

$$\begin{cases} \int_a^b \frac{dL_1}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx = \left[L_1(x) p(x) \frac{d\tilde{\varphi}}{dx} \right]_a^b - \int_a^b L_1(x)\, t(x)\, dx \\[2em] \int_a^b \frac{dL_2}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx = \left[L_2(x) p(x) \frac{d\tilde{\varphi}}{dx} \right]_a^b - \int_a^b L_2(x)\, t(x)\, dx \\[1em] \qquad\qquad\qquad . \qquad\qquad\qquad\qquad\qquad . \qquad\qquad\qquad . \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ \int_a^b \frac{dL_m}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx = \left[L_m(x) p(x) \frac{d\tilde{\varphi}}{dx} \right]_a^b - \int_a^b L_m(x)\, t(x)\, dx \end{cases}$$

↑ System of m-eqns to find $\varphi_1, \varphi_2, \ldots \varphi_m$

IDEA: strong form. → def. a polynomial → multiply by → derive WEAK FORM
approx for UNKNOWN FUN $W(x)$ integrate
$\varphi \to \tilde{\varphi}$ over domain
↓ (GALERKIN'S CHOICE)

$\tilde{\varphi}$ ← Solve ← System of m-equations (ALGEBRAIC)

· Consider weak form for node K (Hp. INTERNAL NODE) $K = 2, 3, \ldots K, \ldots m-1$

$$\int_a^b \frac{dL_K}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx = \left[L_K(x) p(x) \frac{d\tilde{\varphi}}{dx} \right]_a^b - \int_a^b L_K(x)\, t(x)\, dx$$
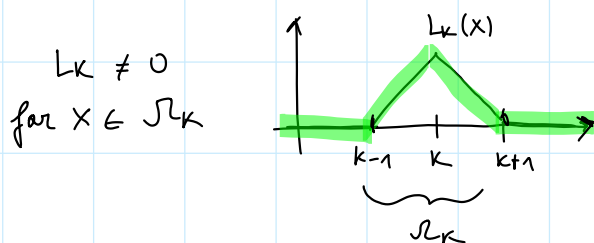
$L_K \neq 0$
for $x \in \Omega_K$

$L_K(x)$
$k-1 \quad k \quad k+1$
Ω_K

RESTRICT integration to support domain of given node

$$\int_{\Omega_K} \frac{dL_K}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx = \cancel{\left[L_K(x) p(x) \frac{d\tilde{\varphi}}{dx} \right]_a^b} - \int_{\Omega_K} L_K(x)\, t(x)\, dx$$

↑ $L_K(x) = 0$ for $x = a$, $x = b$

$\Omega_K = W_{K-1,K} \cup W_{K,K+1}$

$L_{K-1}(x)$ $L_K(x)$ $L_{K+1}(x)$
$k-1 \quad k \quad k+1$
Ω_K
$W_{K-1,K}$ $W_{K,K+1}$
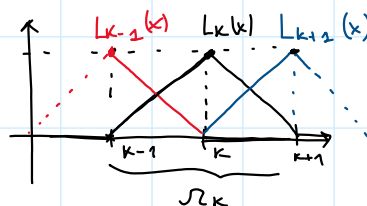
$\int \frac{dL_K}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx$ + $\int \frac{dL_K}{dx} p(x) \frac{d\tilde{\varphi}}{dx}\, dx$ =

$$\int_{w_{k-1,k}} \frac{dL_k}{dx} p(x) \frac{d\tilde{\psi}}{dx} dx + \int_{w_{k,k+1}} \frac{dL_k}{dx} p(x) \frac{d\tilde{\psi}}{dx} dx =$$

$$\overbrace{}^{w_{k-1,k}} \underbrace{}_{w_{k,k+1}}$$

$$-\int L_k(x) t(x)\, dx - \int_{w_{k,k+1}} L_k(x) t(x)\, dx$$

$$\tilde{\psi} = \begin{cases} \boxed{\psi_{k-1} L_{k-1}(x) + \psi_k L_k(x)} & \forall x \in w_{k-1,k} \\[2mm] \boxed{\psi_k L_k(x) + \psi_{k+1} L_{k+1}(x)} & \forall x \in w_{k,k+1} \end{cases}$$

$$\underbrace{\left(\int_{w_{k-1,k}} \frac{dL_k}{dx} p(x) \frac{dL_{k-1}}{dx} dx \right)}_{k_L} \psi_{k-1} + \underbrace{\left(\int_{w_{k-1,k}} \frac{dL_k}{dx} p(x) \frac{dL_k}{dx} dx \right)}_{k_c{}'} \psi_k +$$

$$\frac{d}{dx}(\psi_{k-1} L_{k-1}(x)) = \\ = \frac{dL_{k-1}}{dx} \psi_{k-1}$$

$$+ \underbrace{\left(\int_{w_{k,k+1}} \frac{dL_k}{dx} p(x) \frac{dL_k}{dx} dx \right)}_{k_c{}''} \psi_k + \underbrace{\left(\int_{w_{k,k+1}} \frac{dL_k}{dx} p(x) \frac{dL_{k+1}}{dx} dx \right)}_{k_R} \psi_{k+1} = \dots$$

$$= \underbrace{-\int_{w_{k-1,k}} L_k(x) t(x)\, dx}_{S_{k-1,k}} - \underbrace{\int_{w_{k,k+1}} L_k(x) t(x)\, dx}_{S_{k,k+1}}$$
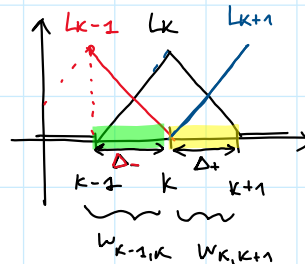
$$\Rightarrow \boxed{ k_L \psi_{k-1} + (k_c{}' + k_c{}'') \psi_k + k_R \psi_{k+1} = S_{k-1,k} + S_{k,k+1} }$$

Same shape as finite difference method!

HAT FUNCTIONS DERIVATIVES

$$L_{k-1}(x) = 1 - \frac{x - x_{k-1}}{\Delta_-} \qquad \frac{dL_{k-1}}{dx} = \boxed{- \frac{1}{\Delta_-}}$$

$$L_k(x) = \begin{cases} 1 + \dfrac{x - x_k}{\Delta_-} & x \in w_{k-1,k} \\[2mm] 1 - \dfrac{x - x_k}{\Delta_+} & x \in w_{k,k+1} \end{cases} \qquad \frac{dL_k}{dx} = \begin{cases} \boxed{\frac{1}{\Delta_-}} \\[2mm] \boxed{- \frac{1}{\Delta_+}} \end{cases}$$

$$L_{k+1}(x) = 1 + \frac{x - x_{k+1}}{\Delta_+} \qquad \frac{dL_{k+1}}{dx} = \frac{1}{\Delta_+}$$

Substitute.

$$\left(\int_{w_{k-1,k}} \frac{1}{\Delta_-} p(x) \left(-\frac{1}{\Delta_-}\right) dx \right) \psi_{k-1} + \left(\int_{w_{k-1,k}} \frac{1}{\Delta_-} p(x) \frac{1}{\Delta_-} dx \right) \psi_k +$$

$$\left(\int \left(-\frac{1}{\Delta_+}\right) p(x) \left(-\frac{1}{\Delta_+}\right) dx \right) \psi_k + \left(\int -\frac{1}{\Delta_+} p(x) \frac{1}{\Delta_+} dx \right) \psi_{k+1} =$$

$$\left(\int_{w_{k,k+2}}\left(-\tfrac{1}{\Delta_+}\right)P(x)\left(-\tfrac{1}{\Delta_+}\right)dx\right)\varphi_k + \left(\int_{w_{k,k+1}}-\tfrac{1}{\Delta_+}P(x)\tfrac{1}{\Delta_+}\,dx\right)\varphi_{k+2} =$$

$$-\int_{w_{k-1,k}}\left(1+\tfrac{x-x_k}{\Delta_-}\right)t(x)\,dx \;-\; \int_{w_{k,k+2}}\left(1-\tfrac{x-y_k}{\Delta_+}\right)t(x)\,dx$$

$$\color{red}{-\tfrac{1}{\Delta_-^2}\left(\int_{w_{k-1,k}}P(x)\,dx\right)\varphi_{k-1}} + \color{red}{\tfrac{1}{\Delta_-^2}\left(\int_{w_{k-1,k}}P(x)\,dx\right)\varphi_k} + \color{blue}{\tfrac{1}{\Delta_+^2}\left(\int_{w_{k,k+1}}P(x)\,dx\right)\varphi_k} - \color{blue}{\tfrac{1}{\Delta_+^2}\left(\int_{w_{k,k+2}}P(x)\,dx\right)\varphi_{k+1}} =$$

Δ_-,Δ_+ .. only depend on MESH

$$= -\int_{w_{k-1,k}}\left(1+\tfrac{x-x_k}{\Delta_-}\right)t(x)\,dx \;-\; \int_{w_{k,k+2}}\left(1-\tfrac{x-y_k}{\Delta_+}\right)t(x)\,dx$$

BOUNDARY NODES

$$x_1 = a \quad\Rightarrow\quad \varphi(a) = \varphi_a \quad \text{DIRICHLET BC}$$

$$\text{equation 1} \;\Rightarrow\; \boxed{\varphi_1 = \varphi_a}$$

$$x_m = b \;\Rightarrow\; \frac{d\varphi}{dx}\Big|_b = \varphi'_b \quad \text{NEUMANN BC}$$
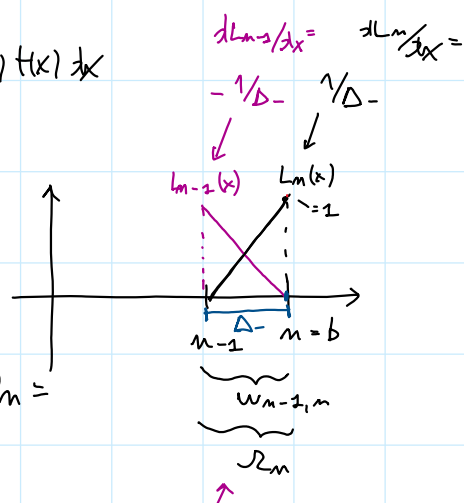
consider weak form for node k

$$\int_{\Omega_k}\frac{dL_k}{dx}P(x)\frac{d\varphi}{dx}\,dx = \left[L_k(x)P(x)\frac{d\varphi}{dx}\right]_a^b - \int_a^b L_k(x)\,t(x)\,dx$$

$$\Rightarrow\; x_m = b \quad \color{magenta}{\text{support domain has only 1 element } (w_{m-1,m})}$$

$dL_{m-1}/dx = -\tfrac{1}{\Delta_-}$ $dL_m/dx = \tfrac{1}{\Delta_-}$

$$m: \left(\int_{w_{m-1,m}}\frac{dL_m}{dx}\,P(x)\,\frac{dL_{m-1}}{dx}\right)\varphi_{m-1} + \left(\int_{w_{m-1,m}}\frac{dL_m}{dx}P(x)\frac{dL_m}{dx}\right)\varphi_m =$$

(over terms, arrows: $\tfrac{1}{\Delta_-}$, $-\tfrac{1}{\Delta_-}$, $\tfrac{1}{\Delta_-}$)

$L_{m-1}(x)$ $L_m(x) = 1$

Δ_- $w_{m-1,m}$ Ω_m

$$= \left[L_m(b)P(b)\frac{d\widetilde{\varphi}}{dx}\Big|_b \;\color{red}{-\; L_m(a)P(a)\frac{d\widetilde{\varphi}}{dx}\Big|_a}\right] + \cdots$$

$=1$ φ'_b

$\widetilde{\varphi} = \varphi_{m-1}L_{m-1}(x) + \underline{\underline{\varphi_m L_m(x)}}$

$$\cdots - \int L_m(x)\,t(x)\,dx$$

$$\quad x - x_m$$

$$\cdots \quad - \int_{w_{m-1,m}} L_m(x)\, t(x)\, dx$$

$$\searrow \quad 1 + \frac{x - x_m}{\Delta_-}$$

NEUHANN BC

$$m: \quad -\frac{1}{\Delta_-^2}\left(\int_{w_{m-1,m}} p(x)\, dx\right)\varphi_{m-1} + \frac{1}{\Delta_-^2}\left(\int_{w_{m-1,m}} p(x)\, dx\right)\varphi_m = p(b)\,\varphi_D - \int_{w_{m-1,m}}\left(1 + \frac{x - x_m}{\Delta_-}\right)t(x)\, dx$$

$$\Rightarrow \quad \text{Assemble matrix } [K]$$