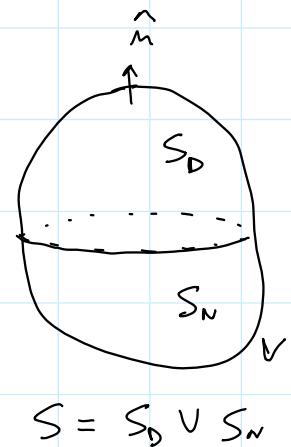


MIXED BOUNDARY CONDITIONSDIRICHLET on S_D NEUMANN on S_N

$$\begin{aligned} & \nabla^2 \varphi = t \\ \rightarrow S_D: & \varphi = \varphi_0 \\ \rightarrow S_N: & \frac{\partial \varphi}{\partial \hat{n}} = \varphi'_0 \end{aligned} \quad (1)$$



Suppose: φ - ELECTRIC POTENTIAL

$$\begin{aligned} \bar{E} &= -\nabla \varphi \\ \bar{E} \cdot \hat{n} &= -\nabla \varphi \cdot \hat{n} \\ E_n &= -\frac{\partial \varphi}{\partial n} \end{aligned}$$

Proof by contradiction. 2 solutions φ_1, φ_2 that satisfy (1)

$$\begin{aligned} & \nabla^2 \varphi_1 = t \\ S_D: & \varphi_1 = \varphi_0 \\ S_N: & \frac{\partial \varphi_1}{\partial n} = \varphi'_0 \end{aligned} \quad \left\{ \begin{array}{l} \nabla^2 \varphi_2 = t \\ \varphi_2 = \varphi_0 \\ \frac{\partial \varphi_2}{\partial n} = \varphi'_0 \end{array} \right.$$

φ_3 [DIFFERENCE FIELD] GOAL: prove that $\varphi_3 = 0$

$$\begin{aligned} & \nabla^2 \varphi_1 - \nabla^2 \varphi_2 = t - t = 0 \Rightarrow \nabla^2 (\varphi_3 - \varphi_2) = 0 \Rightarrow \boxed{\nabla^2 \varphi_3 = 0} \quad \varphi_3 \text{ is HARMONIC} \\ S_D: & \varphi_1 - \varphi_2 = \varphi_0 - \varphi_0 = 0 \Rightarrow \boxed{\varphi_3 = 0} \\ S_N: & \frac{\partial \varphi_1}{\partial n} - \frac{\partial \varphi_2}{\partial n} = \varphi'_0 - \varphi'_0 = 0 \Rightarrow \boxed{\frac{\partial \varphi_3}{\partial n} = 0} \end{aligned}$$

CANNOT USE φ_3 of mean theorem

Recall. GREEN'S FIRST IDENTITY

$$\varphi, \psi \in C_2 \quad \oint_S \varphi \nabla \psi \cdot d\hat{s} = \int_V (\nabla \varphi \cdot \nabla \psi + \varphi \nabla^2 \psi) dV$$

ASSUME. $\varphi = \varphi_3, \psi = \varphi_3$

$\hat{n} dS$

ASSUME . $\Psi = \Psi_3$, $\Psi = \Psi_3$ $\hat{n} dS$

$$\oint_S \Psi_3 \nabla \Psi_3 \cdot \hat{n} dS = \int_V (\nabla \Psi_3^2 + \Psi_3 \nabla^2 \Psi_3) dV$$

$\Psi_3 = 0$ on S_D \downarrow

$$\int_{S_D} \Psi_3 \frac{\partial \Psi_3}{\partial n} dS + \int_{S_N} \Psi_3 \frac{\partial \Psi_3}{\partial n} dS = \int_V \nabla \Psi_3^2 dV + \int_V \Psi_3 \nabla^2 \Psi_3 dV$$

Ψ_3 is HARMONIC on V

$$\int_V \nabla \Psi_3^2 dV = 0 \rightarrow \Psi_3 \text{ MUST fulfill this identity}$$

$$\geq 0$$

$$\geq 0$$

$$\geq 0$$

$\nabla \Psi_3$ must be $= 0 \forall \bar{p} \in V$ (Ψ_3 must be UNIFORM)

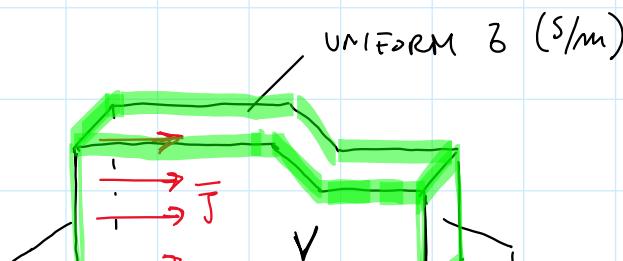
$$\left. \begin{array}{l} \Psi_3 = 0 \\ \forall \bar{p} \in S_D \end{array} \right\}$$

Ψ_3 has to be $= 0$ everywhere (including S_N)

$$\Rightarrow \text{if } \Psi_3 = 0 \Rightarrow \Psi_1 - \Psi_2 = \cancel{\Psi_3} \Rightarrow \boxed{\Psi_1 = \Psi_2}$$

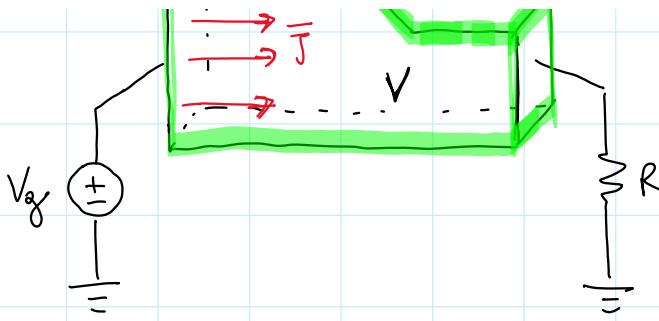
SOLUTION IS UNIQUE

Problem : STATIONARY STATE current conduction problem with an external electric circuit



GOAL :

WRITE a FORMULATION
to find \bar{J} and \bar{E} inside
the conductor



to find \bar{J} and \bar{E} inside the conductor

internal points
of V

$$\nabla \times \bar{E} = - \frac{\partial \bar{B}}{\partial t} = 0 \Rightarrow \bar{E} = -\nabla \varphi$$

$$\nabla \cdot (\bar{J} + \frac{\partial \bar{B}}{\partial t}) = 0 \Rightarrow \nabla \cdot \bar{J} = 0$$

$$\bar{J} = \sigma \bar{E}$$

$$\bar{E} = -\nabla \varphi$$

$$\rightarrow \nabla \cdot \bar{J} = 0 \Rightarrow \nabla \cdot (\sigma \bar{E}) = 0 \Rightarrow \nabla \cdot (-\sigma \nabla \varphi) = 0$$

Since σ UNIFORM

$$-\sigma \nabla \cdot (\nabla \varphi) = 0$$

$$-\sigma \nabla^2 \varphi = 0$$

$$\boxed{\nabla^2 \varphi = 0}$$

LAPLACE EQUATION (φ is HARMONIC in V)

If σ is UNIFORM no role in the SPACE DISTRIBUTION of φ

Behavior of φ influenced ONLY BY BC

For internal points

$$\forall \bar{P} \in V$$

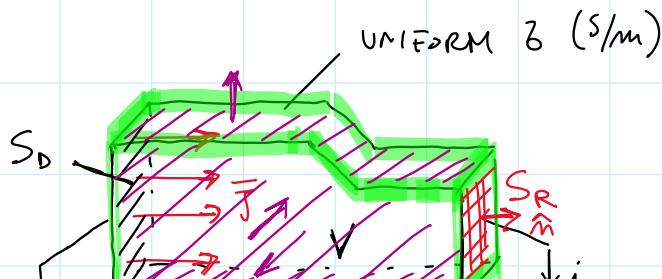
\rightarrow Find φ such that $\nabla^2 \varphi = 0$



$$\varphi \Rightarrow \bar{E} = -\nabla \varphi \Rightarrow \bar{J} = \sigma \bar{E}$$

BOUNDARY CONDITIONS

$$V \int \nabla^2 \varphi = 0$$



$$\begin{aligned} V & \left\{ \nabla^2 \varphi = 0 \right. \\ S_D & \left. \varphi = V_g \right. \\ S_N & \frac{\partial \varphi}{\partial n} = 0 \\ S_R & \varphi = V_R \end{aligned}$$



$$\forall \vec{q} \in S_N : \vec{J} \cdot \hat{n} = 0 \Rightarrow b \vec{E} \cdot \hat{n} = 0 \Rightarrow -b \nabla \varphi \cdot \hat{n} = 0 \Rightarrow \boxed{\frac{\partial \varphi}{\partial n} = 0}$$

\uparrow \uparrow
 no current through $\vec{E} = -\nabla \varphi$
 S_N $\frac{\partial \varphi}{\partial n}$

Need express V_R through φ

$$\varphi = V_R = R i = R \int_{S_R} \vec{J} \cdot d\vec{S}$$

$$= R \int_{S_R} b \vec{E} \cdot d\vec{S} = -R \int_{S_R} b \nabla \varphi \cdot d\vec{S}$$

UNIFORM

$$= -R b \int_{S_R} \frac{\partial \varphi}{\partial n} dS$$

GRADIENT

\downarrow
 NO SKIN EFFECT
 $\vec{J}, \vec{E}, \frac{\partial \varphi}{\partial n}$ UNIFORM
 on S_R

$$\varphi = -R b \frac{\partial \varphi}{\partial n} S_R$$

$$\cancel{\varphi} + R b S_R \frac{\partial \varphi}{\partial n} = 0$$

\Rightarrow GENERAL FORM:

$$\boxed{\alpha \varphi + \beta \frac{\partial \varphi}{\partial n} + \gamma = 0}$$

ROBIN
BOUNDARY
CONDITION

linear combination between
Dirichlet and Neumann BCs

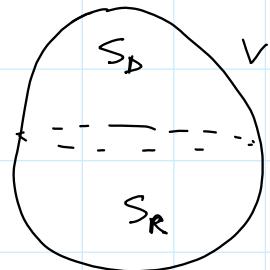
FORMULATION:

$$\begin{cases} \nabla^2 \varphi = 0 \\ S_D \quad \varphi = V_g \\ S_N \quad \frac{\partial \varphi}{\partial n} = 0 \\ S_R \quad \varphi + R_B S_R \frac{\partial \varphi}{\partial n} = 0 \end{cases}$$

↑↑↑
Known electrical/geometric (also V_g)

UNIQUENESS theorem for Poisson problems with Robin BC

$$\begin{cases} \nabla^2 \varphi = t \\ S_D \quad \varphi = \varphi_0 \\ S_R \quad \alpha \varphi + \beta \frac{\partial \varphi}{\partial n} + \gamma = 0 \end{cases}$$



$$S = S_D \cup S_R$$

Proof ABSURDUM \rightarrow assume solutions φ_1, φ_2

GOAL: prove that $\varphi_3 = \varphi_1 - \varphi_2 = 0$

$$\begin{cases} \nabla^2 \varphi_1 - \nabla^2 \varphi_2 = \boxed{\nabla^2 \varphi_3 = 0} \\ S_D \quad \varphi_1 - \varphi_2 = \boxed{\varphi_3 = 0} \\ S_R \quad \alpha \varphi_1 + \beta \frac{\partial \varphi_1}{\partial n} + \gamma - (\alpha \varphi_2 + \beta \frac{\partial \varphi_2}{\partial n} + \gamma) = 0 \\ \alpha(\varphi_1 - \varphi_2) + \beta \left(\frac{\partial \varphi_1}{\partial n} - \frac{\partial \varphi_2}{\partial n} \right) = 0 \\ \boxed{\alpha \varphi_3 + \beta \frac{\partial \varphi_3}{\partial n} = 0} \quad \text{on } S_R \end{cases}$$

GREEN'S FIRST IDENTITY - application to φ_3

$$\Psi = \varphi_3, \quad \Upsilon = \varphi_3$$

$$\nabla^2 \varphi_3 = 0$$

$$\varphi = \varphi_3, \quad \nabla \cdot \nabla \varphi = \nabla^2 \varphi = 0$$

$$\oint_S \varphi_3 \nabla \varphi_3 \cdot d\bar{S} = \int_V (\nabla \varphi_3)^2 + \varphi_3 \cancel{\nabla^2 \varphi_3} dV$$

$$= 0 \text{ on } S_D$$

$$\int_S \cancel{\varphi_3} \nabla \varphi_3 \cdot d\bar{S} + \left[\int_{S_R} \varphi_3 \frac{\partial \varphi_3}{\partial n} dS \right] = \int_V \nabla \varphi_3^2 dV$$

$$\frac{\partial \varphi_3}{\partial n} = -\alpha/\beta \varphi_3$$

$$\underbrace{\int_{S_R} -\frac{\alpha}{\beta} \varphi_3^2 dS}_{LHS(\varphi_3)} = \underbrace{\int_V \nabla \varphi_3^2 dV}_{RHS(\varphi_3)}$$

$$\text{IF } \frac{\alpha}{\beta} > 0 \quad \underbrace{LHS(\varphi_3)}_{\leq 0} = \underbrace{RHS(\varphi_3)}_{\geq 0} \Rightarrow \begin{aligned} LHS &\neq RHS &+ \\ \varphi_3 &\neq 0 & \downarrow \\ \text{Solution is unique} & & \\ \varphi_3 = 0 &\Rightarrow \varphi_1 = \varphi_2 & \end{aligned}$$

$$\text{IF } \frac{\alpha}{\beta} < 0 \quad \underbrace{LHS(\varphi_3)}_{\geq 0} = \underbrace{RHS(\varphi_3)}_{\geq 0} \Rightarrow \begin{aligned} \text{there } &\exists \varphi_3 \neq 0 \\ \text{such that } & \\ LHS(\varphi_3) &= RHS(\varphi_3) \end{aligned}$$

$$\begin{aligned} \text{if } \varphi_3 \neq 0 \\ \Rightarrow \varphi_1 \neq \varphi_2 \end{aligned}$$

For DIRICHLET + RUMIN poisson problems

solution is unique only for $\frac{\alpha}{\beta} > 0$

Solution is not unique

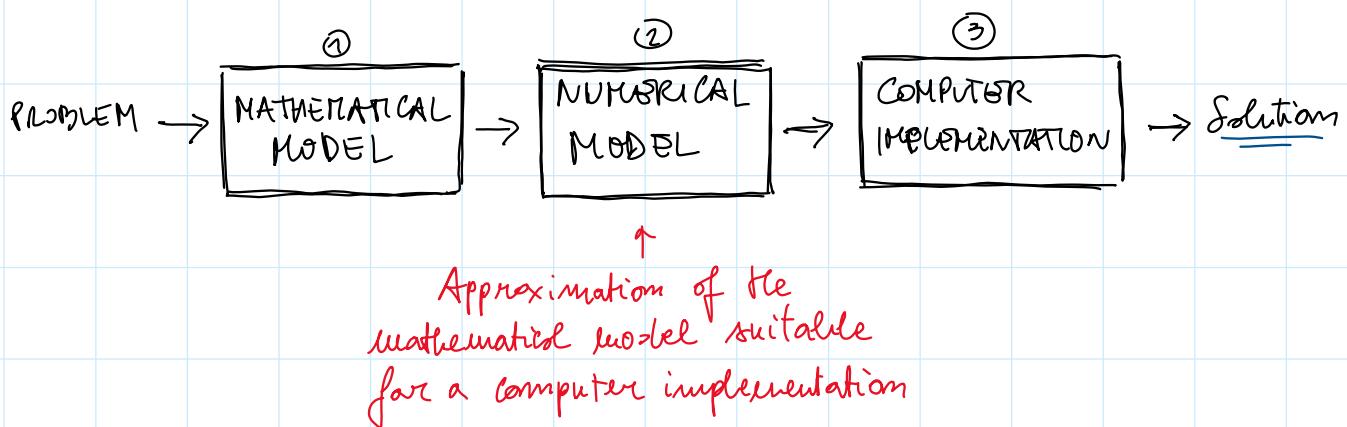
Steady-state conduction

Steady-state conduction

$$\alpha \varphi + \underbrace{\beta R S_R \frac{\partial \varphi}{\partial m}}_{\beta} = 0 \quad \alpha = 1 \quad \beta = \beta R S_R \quad \Rightarrow \alpha/\beta > 0$$

NUMERICAL ANALYSIS

GOAL: Solve mathematical or physical problems with a computer



Sources of error

MATH. MODEL \longrightarrow physical approximation EX: $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{B}}{\partial t}$

\rightarrow NUM. MODEL \longrightarrow TRUNCATION ERRORS

TAYLOR series to express mathematical operators

\rightarrow COMPUT. IMPL \longrightarrow ROUND OFF ERRORS

REPRESENT a NUMBER with a FINITE AMOUNT of DIGITS

NUMBER REPRESENTATION

POSITIONAL REPRESENTATION: position of digits indicates the power of the BASE which multiplies the given digit

the given digit

$$\text{Ex: } (3012)_{10} = 3 \cdot 10^3 + 0 \cdot 10^2 + 1 \cdot 10^1 + 2 \cdot 10^0$$

INTEGERS. $q_B = a_m B^m + a_{m-1} B^{m-1} + \dots + a_1 B^1 + a_0 B^0$

↑
DIGITS.
 $a_k \in \mathbb{N}$
 $0 \leq a_k \leq B-1$
 $a_m \neq 0$

↑
BASE
 $B \in \mathbb{N}$
 $B \geq 2$

otherwise. 032, 32, 0032

REALS: $x = \lfloor x \rfloor + \text{frac}(x)$

↑
integer part

$\text{pos}_3 \quad \text{pos}_0 \quad \text{pos}_{-1}$
↓ ↓ ↓

Ex. $3012.401 = 3 \cdot 10^3 + 0 \cdot 10^2 + 1 \cdot 10^1 + 2 \cdot 10^0 + 4 \cdot 10^{-1} + 0 \cdot 10^{-2} + 1 \cdot 10^{-3}$

↑
RADIX
POINT

$$(x)_B = a_m B^m + a_{m-1} B^{m-1} + \dots + a_1 B^1 + a_0 B^0 + \dots + b_1 B^{-1} + b_2 B^{-2} + \dots + b_m B^{-m}$$

↑
DIGITS of fractional part

$$b_h \in \mathbb{N}; 0 \leq b_h \leq B-1, b_m \neq 0$$

↓
to avoid: $3.12 = 3.120 \dots$

FIXED POINT REPRESENTATION

Given: the number of digits for the number $\rightarrow t$
 the " " " " the fractional part $\rightarrow q$
 the base $\rightarrow \beta$

FIXED Point SET

$$X(\beta, t, q) = \left\{ x \in \mathbb{R} = \text{sign}(x) \left[\sum_{k=0}^{t-(q+1)} a_k \beta^k + \sum_{k=0}^q b_k \beta^{-k} \right] \right\}$$

CAPITAL X

↑ integer part ↑ fractional part

Example: $X(\beta=10, t=4, q=1)$

$$\text{MAX}(x) = \underbrace{9 \cdot 10^2 + 9 \cdot 10^1 + 9 \cdot 10^0}_{3 \text{ DIGITS}} + \underbrace{9 \cdot 10^{-1}}_{1 \text{ DIGIT}} = 999.9$$

↑ RANGE $\sim 10^4$

$$\text{MIN}(x) = 0 \cdot 10^2 + 0 \cdot 10^1 + 0 \cdot 10^0 + 1 \cdot 10^{-1} = 0.1$$

<u>SPACUNG</u>	$L \times 1$	$\text{frac}(x)$	
	0 0 0 . 1		
	0 0 0 . 2	}	$\Delta = 0.1$
	0 0 0 . 3		
	0 0 0 . 9		
	0 0 0 . 9	}	$\Delta = \beta^{-q}$
	0 1 2 . 3		
	0 1 2 . 4	}	$\Delta = 0.1$
	.		

↑ UNIFORM SPACUNG