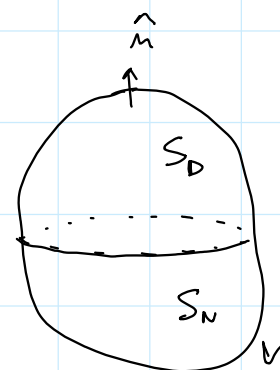


MIXED BOUNDARY CONDITIONS.

DIRECTLET on S_D

NEUMANN on S_N

$$\begin{aligned} & V \left\{ \begin{array}{l} \nabla^2 \varphi = t \\ \varphi = \varphi_0 \\ \frac{\partial \varphi}{\partial n} = \varphi'_0 \end{array} \right. \quad (1) \\ & \rightarrow S_D \\ & \rightarrow S_N \end{aligned}$$



$$S = S_D \cup S_N$$

Suppose: φ - ELECTRIC POTENTIAL

$$\begin{aligned} \vec{E} &= -\nabla \varphi \\ \vec{E} \cdot \hat{n} &= -\nabla \varphi \cdot \hat{n} \\ \vec{E}_n &= -\frac{\partial \varphi}{\partial n} \\ \varphi'_0 &\end{aligned}$$

Proof by contradiction. 2 solutions φ_1, φ_2 that satisfy (1)

$$\begin{aligned} V \left\{ \begin{array}{l} \nabla^2 \varphi_1 = t \\ \varphi_1 = \varphi_0 \\ \frac{\partial \varphi_1}{\partial n} = \varphi'_0 \end{array} \right. & \quad \left\{ \begin{array}{l} \nabla^2 \varphi_2 = t \\ \varphi_2 = \varphi_0 \\ \frac{\partial \varphi_2}{\partial n} = \varphi'_0 \end{array} \right. \end{aligned}$$

φ_3 [DIFFERENCE FIELD] GOAL: prove that $\varphi_3 = 0$

$$\begin{aligned} V \left\{ \begin{array}{l} \nabla^2 \varphi_1 - \nabla^2 \varphi_2 = t - t = 0 \\ \varphi_1 - \varphi_2 = \varphi_0 - \varphi_0 = 0 \\ \frac{\partial \varphi_1}{\partial n} - \frac{\partial \varphi_2}{\partial n} = \varphi'_0 - \varphi'_0 = 0 \end{array} \right. & \Rightarrow \nabla^2 (\varphi_1 - \varphi_2) = 0 \Rightarrow \boxed{\nabla^2 \varphi_3 = 0} \quad \varphi_3 \text{ is HARMONIC} \\ S_D \left\{ \begin{array}{l} \varphi_1 - \varphi_2 = \varphi_0 - \varphi_0 = 0 \\ \frac{\partial \varphi_1}{\partial n} - \frac{\partial \varphi_2}{\partial n} = \varphi'_0 - \varphi'_0 = 0 \end{array} \right. & \Rightarrow \boxed{\varphi_3 = 0} \\ S_N \left\{ \begin{array}{l} \frac{\partial \varphi_1}{\partial n} - \frac{\partial \varphi_2}{\partial n} = \varphi'_0 - \varphi'_0 = 0 \end{array} \right. & \Rightarrow \boxed{\frac{\partial \varphi_3}{\partial n} = 0} \end{aligned}$$

CANNOT USE K_3 of mean theorem

Recall. GREEN'S FIRST IDENTITY

$$\varphi, \psi \in C_2 \quad \oint_S \varphi \nabla \psi \cdot \vec{dS} = \int_V (\nabla \varphi \cdot \nabla \psi + \varphi \nabla^2 \psi) dV$$

ASSUME. $\varphi = \varphi_3$, $\psi = \varphi_3$

$\vec{n} dS$

Assume $\psi = \psi_2$, $\gamma = \psi_3$

$$\oint_S \psi_3 \nabla \psi_3 \cdot d\vec{S} = \int_V (\nabla \psi_3^2 + \psi_3 \nabla^2 \psi_3) dV$$

$\psi_3 = 0$ on S_D \downarrow $\frac{\partial \psi_3}{\partial n} = 0$ \downarrow ψ_3 is HARMONIC on V

$$\int_{S_D} \psi_3 \frac{\partial \psi_3}{\partial n} dS + \int_{S_N} \psi_3 \frac{\partial \psi_3}{\partial n} dS = \int_V \nabla \psi_3^2 dV + \int_V \psi_3 \nabla^2 \psi_3 dV$$

$$\underbrace{\int_V \nabla \psi_3^2 dV}_{\geq 0} = 0 \rightarrow \psi_3 \text{ must fulfill this identity}$$

$$\Downarrow$$

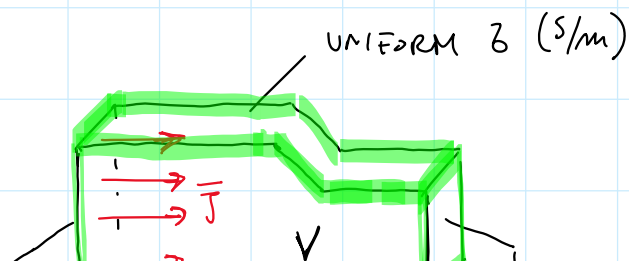
$$\left\{ \begin{array}{l} \nabla \psi_3 \text{ must be } = 0 \quad \forall \vec{p} \in V \\ \psi_3 = 0 \quad \forall \vec{p} \in S_D \end{array} \right. \quad (\psi_3 \text{ must be UNIFORM})$$

ψ_3 has to be $= 0$ everywhere (including S_N)

$$\Rightarrow \text{if } \psi_3 = 0 \Rightarrow \psi_1 - \psi_2 = \cancel{\psi_3} \Rightarrow \boxed{\psi_1 = \psi_2}$$

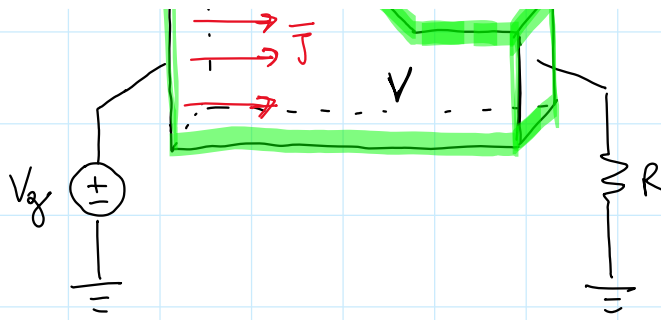
\uparrow
SOLUTION IS UNIQUE

Problem: STEADY-STATE CURRENT conduction problem with an external electric circuit



GOAL:

WRITE a FORMULATION to find \vec{J} and \vec{E} inside the conductor



to find \bar{J} and \bar{E} inside the conductor

internal points of V

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = 0 \Rightarrow \bar{E} = -\nabla \varphi$$

$$\nabla \cdot (\bar{J} + \frac{\partial \bar{D}}{\partial t}) = 0 \Rightarrow \nabla \cdot \bar{J} = 0$$

$$\bar{J} = \beta \bar{E}$$

$$\bar{J} = \beta \bar{E}$$

$$\bar{E} = -\nabla \varphi$$

$$\rightarrow \nabla \cdot \bar{J} = 0 \Rightarrow \nabla \cdot (\beta \bar{E}) = 0 \Rightarrow \nabla \cdot (-\beta \nabla \varphi) = 0$$

Since β UNIFORM

$$-\beta \nabla \cdot (\nabla \varphi) = 0$$

$$-\beta \nabla^2 \varphi = 0$$

If β is UNIFORM no hole in the SPACE DISTRIBUTION of φ

$$\boxed{\nabla^2 \varphi = 0}$$

LAPLACE EQUATION (φ is HARMONIC in V)

Behavior of φ influenced ONLY BY BC

For internal points $\forall p \in V$

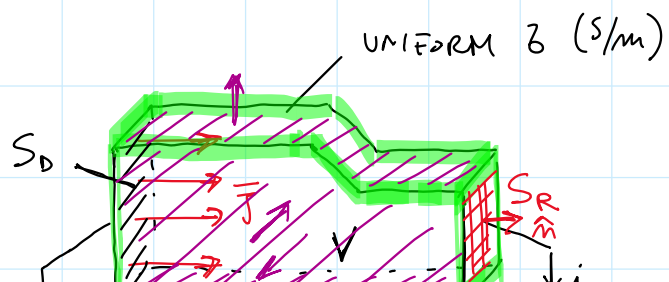
\rightarrow Find φ such that $\nabla^2 \varphi = 0$



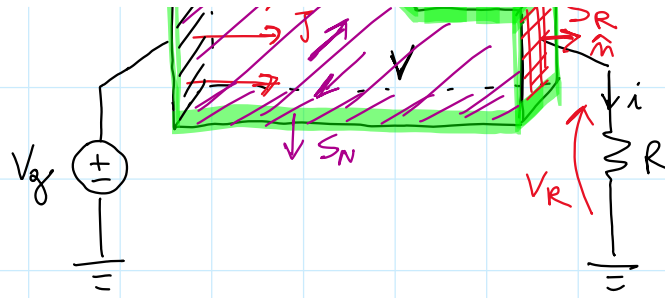
$$\varphi \Rightarrow \bar{E} = -\nabla \varphi \Rightarrow \bar{J} = \beta \bar{E}$$

BOUNDARY CONDITIONS

$$V \mid \nabla^2 \varphi = 0$$



$$\begin{aligned} V & \left\{ \begin{aligned} \nabla^2 \psi &= 0 \\ S_D & \left\{ \begin{aligned} \psi &= V_g \\ S_N & \left\{ \begin{aligned} \frac{\partial \psi}{\partial n} &= 0 \\ S_R & \left\{ \begin{aligned} \psi &= V_R \end{aligned} \end{aligned} \end{aligned} \end{aligned} \end{aligned} \end{aligned}$$



$$\forall \vec{r} \in S_N. \quad \vec{J} \cdot \hat{n} = 0 \Rightarrow \oint \vec{E} \cdot \hat{n} = 0 \Rightarrow -\oint \nabla \psi \cdot \hat{n} = 0 \Rightarrow \boxed{\frac{\partial \psi}{\partial n} = 0}$$

\uparrow no current through S_N \uparrow $\vec{E} = -\nabla \psi$ \uparrow $\frac{\partial \psi}{\partial n}$

Need express V_R through ψ

$$\begin{aligned} \psi &= V_R = R \underset{\substack{\uparrow \\ \int_{S_R} \vec{J} \cdot d\vec{S}}}{i} = R \int_{S_R} \oint \vec{E} \cdot d\vec{S} = -R \int_{S_R} \oint \nabla \psi \cdot d\vec{S} \\ &= -R \oint_{S_R} \frac{\partial \psi}{\partial n} dS \\ &= -R \oint_{S_R} \frac{\partial \psi}{\partial n} dS \end{aligned}$$

UNIFORM \downarrow
 GRADIENT \downarrow
 NO SKIN EFFECT \uparrow
 $\vec{J}, \vec{E}, \frac{\partial \psi}{\partial n}$ UNIFORM on S_R

$$\psi = -R \oint_{S_R} \frac{\partial \psi}{\partial n} dS$$

$$\alpha \psi + \beta \frac{\partial \psi}{\partial n} = 0$$

\Rightarrow GENERAL FORM :

$$\boxed{\alpha \psi + \beta \frac{\partial \psi}{\partial n} + \gamma = 0}$$

ROBIN
BOUNDARY
CONDITION

linear combination between
Dirichlet and Neumann BCs

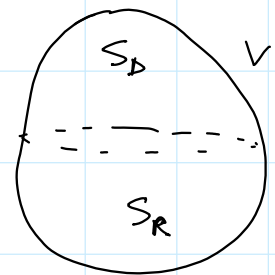
FORMULATION:

$$\begin{cases} V & \nabla^2 \varphi = 0 \\ S_D & \varphi = V_g \\ S_N & \partial \varphi / \partial n = 0 \\ S_R & \varphi + R \partial \varphi / \partial n = 0 \end{cases}$$

$\uparrow \uparrow \uparrow$
 known electrical/geometric (also V_g)

UNIQUENESS theorem for Poisson problems with Robin BC

$$\begin{cases} V & \nabla^2 \varphi = t \\ S_D & \varphi = p_0 \\ S_R & \alpha \varphi + \beta \partial \varphi / \partial n + \gamma = 0 \end{cases}$$



$$S = S_D \cup S_R$$

Proof BY ABSURDUM \rightarrow assume solutions φ_1, φ_2

GOAL: prove that $\varphi_3 = \varphi_1 - \varphi_2 = 0$

$$\begin{cases} V & \nabla^2 \varphi_1 - \nabla^2 \varphi_2 = \boxed{\nabla^2 \varphi_3 = 0} \\ S_D & \varphi_1 - \varphi_2 = \boxed{\varphi_3 = 0} \\ S_R & \alpha \varphi_1 + \beta \partial \varphi_1 / \partial n + \cancel{\gamma} - (\alpha \varphi_2 + \beta \partial \varphi_2 / \partial n + \cancel{\gamma}) = 0 \\ & \alpha (\varphi_1 - \varphi_2) + \beta (\partial \varphi_1 / \partial n - \partial \varphi_2 / \partial n) = 0 \\ & \boxed{\alpha \varphi_3 + \beta \partial \varphi_3 / \partial n = 0} \text{ on } S_R \end{cases}$$

GREEN'S FIRST IDENTITY - application to φ_3

$$\varphi = \varphi_3, \quad \gamma = \varphi_3$$

$$\nabla^2 \varphi_3 = 0$$

$$\psi = \psi_3, \quad \gamma = \psi_3$$

$$\oint_S \psi_3 \nabla \psi_3 \cdot d\vec{S} = \int_V \left(\nabla \psi_3^2 + \cancel{\psi_3 \nabla^2 \psi_3} \right) dV$$

$$\downarrow$$

$$= 0 \text{ on } S_D$$

$$\int_{S_D} \cancel{\psi_3} \nabla \psi_3 \cdot d\vec{S} + \boxed{\int_{S_R} \psi_3 \frac{\partial \psi_3}{\partial n} dS = \int_V \nabla \psi_3^2 dV}$$

$$\frac{\partial \psi_3}{\partial n} = -\frac{\alpha}{\beta} \psi_3$$

$$\underbrace{\int_{S_R} -\frac{\alpha}{\beta} \underbrace{\psi_3^2}_{>0} dS}_{\text{LHS}(\psi_3)} = \underbrace{\int_V \nabla \psi_3^2 dV}_{\text{RHS}(\psi_3)}$$

$$\text{IF } \frac{\alpha}{\beta} > 0 \quad \underbrace{\text{LHS}(\psi_3)}_{\leq 0} = \underbrace{\text{RHS}(\psi_3)}_{\geq 0} \Rightarrow \text{LHS} \neq \text{RHS} \quad \psi_3 \neq 0$$

↓
Solution is unique
 $\psi_3 = 0 \Rightarrow \psi_1 = \psi_2$

$$\text{IF } \frac{\alpha}{\beta} < 0 \quad \underbrace{\text{LHS}(\psi_3)}_{\geq 0} = \underbrace{\text{RHS}(\psi_3)}_{\geq 0} \Rightarrow \text{there } \exists \psi_3 \neq 0 \text{ such that } \text{LHS}(\psi_3) = \text{RHS}(\psi_3)$$

$$\text{if } \psi_3 \neq 0 \Rightarrow \psi_1 \neq \psi_2$$

For DIRICHLET + ROBIN problem problems
solution is unique only for $\boxed{\frac{\alpha}{\beta} > 0}$

Solution is not
UNIQUE

Steady-state conduction

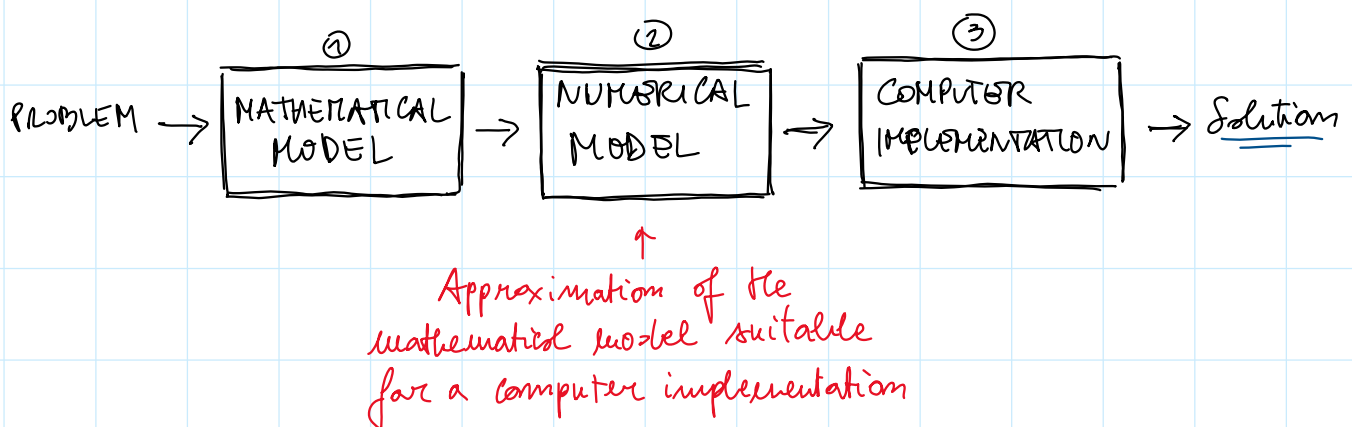
Steady-state conduction

$$\alpha \psi + \underbrace{\beta R S_R}_{\beta} \frac{\partial \psi}{\partial n} = 0$$

$$\alpha = 1 \quad \beta = \beta R S_R \\ \Rightarrow \alpha/\beta > 0$$

NUMERICAL ANALYSIS

GOAL: solve mathematical or physical problems with a computer



Sources of error

MATH. MODEL \longrightarrow physical approximation

EX: $\nabla \times \vec{H} = \vec{J} + \cancel{\frac{\partial \vec{D}}{\partial t}}$

\longrightarrow NUM. MODEL \longrightarrow TRUNCATION ERRORS
Taylor series to express mathematical operators

\longrightarrow COMPUT. IMPL \longrightarrow ROUND OFF ERRORS
REPRESENT a NUMBER with a FINITE AMOUNT of DIGITS

NUMBER REPRESENTATION

POSITIONAL REPRESENTATION: position of digits indicates the power of the BASE which multiplies the given digit

the given digit

$$\text{Ex. } (3012)_{10} = 3 \cdot 10^3 + 0 \cdot 10^2 + 1 \cdot 10^1 + 2 \cdot 10^0$$

$$\text{INTegers. } q_{\beta} = a_m \beta^m + a_{m-1} \beta^{m-1} + \dots + a_1 \beta^1 + a_0 \beta^0$$

DIGITS.

$$a_k \in \mathbb{N}$$

$$0 \leq a_k \leq \beta - 1$$

$$a_m \neq 0$$

BASE

$$\beta \in \mathbb{N}$$

$$\beta \geq 2$$

otherwise . 032 , 32 , 0032

$$\text{REALS: } X = \lfloor X \rfloor + \text{frac}(x)$$

integer part

pos 3 pos 0 pos -1

$$\text{Ex. } 3012.401 = 3 \cdot 10^3 + 0 \cdot 10^2 + 1 \cdot 10^1 + 2 \cdot 10^0 + 4 \cdot 10^{-1} + 0 \cdot 10^{-2} + 1 \cdot 10^{-3}$$

RADIX
POINT

$$(X)_{\beta} = a_m \beta^m + a_{m-1} \beta^{m-1} + \dots + a_1 \beta^1 + a_0 \beta^0 + \dots + b_1 \beta^{-1} + b_2 \beta^{-2} + \dots + b_m \beta^{-m}$$

DIGITS of fractional part

$$b_h \in \mathbb{N} ; 0 \leq b_h \leq \beta - 1 , b_m \neq 0$$

↓

to avoid: $3.12 = 3.120 \dots$

FIXED POINT REPRESENTATION

Given: the number of digits for the number $\rightarrow t$
 the " " " " the fractional part $\rightarrow q$
 the base $\rightarrow \beta$

FIXED POINT SET

REAL NUMBER to be represented

$$\underset{\substack{\uparrow \\ \text{CAPITAL X}}}{X}(\beta, t, q) = \left\{ x \in \mathbb{R} = \text{sign}(x) \left[\underbrace{\sum_{k=0}^{t-(q+1)} a_k \beta^k}_{\substack{\uparrow \\ \text{integer part}}} + \underbrace{\sum_{k=0}^q b_k \beta^{-k}}_{\substack{\uparrow \\ \text{fractional part}}} \right] \right\}$$

Example: $X(\beta=10, t=4, q=1)$

$$\text{MAX}(x) = \underbrace{9 \cdot 10^2 + 9 \cdot 10^1 + 9 \cdot 10^0}_{\substack{\uparrow \\ 3 \text{ DIGITS}}} + \underbrace{9 \cdot 10^{-1}}_{\substack{\uparrow \\ 1 \text{ DIGIT}}} = 999.9$$

$$\text{MIN}(x) = 0 \cdot 10^2 + 0 \cdot 10^1 + 0 \cdot 10^0 + 1 \cdot 10^{-1} = 0.1$$

\updownarrow RANGE $\sim 10^4$

	$L \times J$	$\text{frac}(x)$	
SPACING	0 0	0 . 1	$\Delta = 0.1$
	0 0	0 . 2	
	0 0	0 . 3	
	0 \vdots	0 . \vdots	UNIFORM SPACING $\Delta = \beta^{-q}$
	0 0	0 . 9	
	0 \vdots	\vdots	
	0 1	2 . 3	
0 1	2 . 4	$\Delta = 0.1$	