Homework 05 - 1D FEM

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Complete the missing steps in the `FEM_1D_main.mlx` main file

- Complete "Step 1.1: check with analytic solution" from the file `FEM_1D_main.mlx`

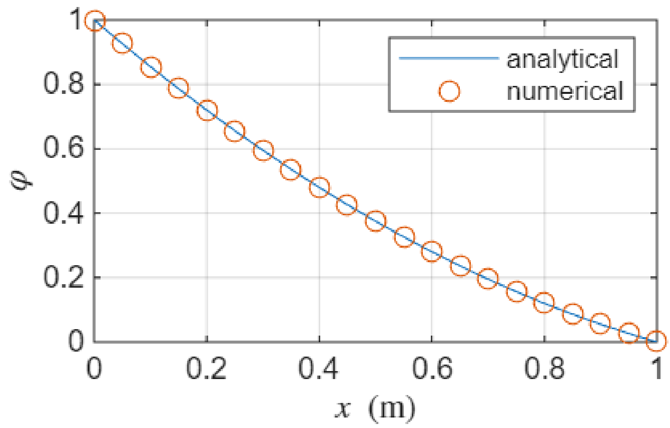

Figure 1: Proposed solution

- Complete "Step 4: support for Neumann BCs" from the `FEM_1D_main.mlx` file You should get something like the following:

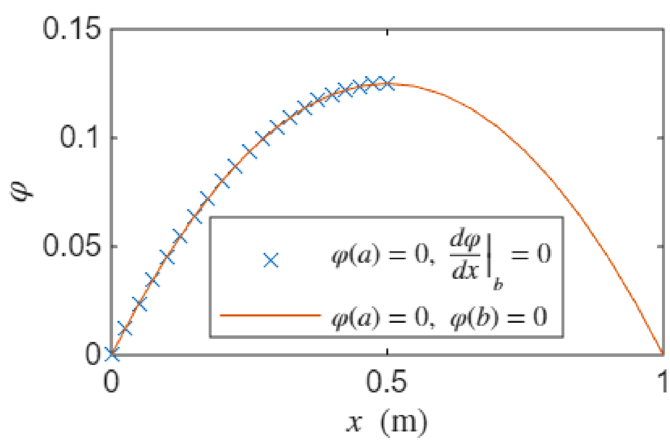

Figure 2: Proposed solution

Convergence

The goal of this exercise is to verify the order of convergence of the 1D FEM code developed in the previous steps.

For a Poisson problem discretized with piecewise linear finite elements and a sufficiently smooth solution, the theory predicts **second–order convergence in the L2 norm**. In this exercise, we will verify this property numerically.

To do that, we can try to monitor the error when the mesh is progressively refined.

To measure the convergence rate, we need: - a numerical solution $\varphi(x)$ computed by the FEM code; - an analytical (exact) solution $\varphi_{ex}(x)$ to compare against.

For nontrivial convergence tests, the right–hand side must be non-constant. Otherwise the exact solution would be a polynomial reproduced exactly by the finite element space, and the numerical solution would coincide with the exact one (up to machine precision), making the convergence study meaningless.

Therefore, we use the **Method of Manufactured Solutions (MMS)**:

1. Choose (or, better, `manufacture`) an arbitrary behavior φ (smooth and non-polynomial)
2. Insert it into the PDE to derive the corresponding forcing term $t(x)$ that makes the manufactured solution satisfy the PDE
3. Solve the resulting PDE numerically
4. Compare the numerical and analytical solutions

The resulting error will then be evaluated for a sequence of refined meshes and used to estimate the **convergence order**.

Manufactured solution derivation

We consider the 1D Poisson problem

$$-\frac{d}{dx}\left(p(x)\frac{d\varphi}{dx}\right) = t(x), \quad x \in (0, L),$$

We choose: - Domain length: $L = 1$ - Constant diffusion coefficient: $p(x) = p_0 = 1$ - Dirichlet boundary conditions $\varphi_a = \varphi(0) = 0,\ \varphi_b = \varphi(L) = 5$

We choose the following manufactured solution:

$$\varphi_{\text{exact}}(x) = \varphi_a + (\varphi_b - \varphi_a)\frac{x}{L} + \sin\left(\frac{3\pi x}{L}\right)$$

This function: - satisfies the boundary conditions; - is smooth; - is **not a polynomial**, ensuring a non-zero FEM error.

…now we just have to derive a right hand side that satisfies the formulation with the manufactured solution

Since `p(x) = 1`, the PDE can be rewritten as:

$$-\varphi''(x) = t(x).$$

The second derivative of the sine term is:

$$\frac{d^2}{dx^2}\sin\left(\frac{3\pi x}{L}\right) = -\left(\frac{3\pi}{L}\right)^2 \sin\left(\frac{3\pi x}{L}\right).$$

The linear part has zero second derivative, therefore:

$$t(x) = \left(\frac{3\pi}{L}\right)^2 \sin\left(\frac{3\pi x}{L}\right).$$

Now we have to define an error function, which in this case is known as the **L2 norm of the error** is defined as

$$\text{err} = \left(\int_0^L (\varphi_{\text{exact}}(x) - \varphi(x))^2\, dx\right)^{1/2}.$$

We evaluate the integral element by element using Gaussian quadrature (use your `int_gauss.m` function). The global L2 error is then obtained as

$$\text{err} = \left(\sum_{el} \int_{el} (\varphi_{\text{exact}} - \varphi)^2 \, dx \right)^{1/2}.$$

This global error measure will be evaluated for different mesh sizes h and used to study how the error decays as the mesh is refined.

Expression for the order of convergence

To connect the computed L2 error to the convergence analysis introduced above, we assume that for a numerical discretization with **order of convergence** p, the global error err decreases with the characteristic mesh size h as:

$$\text{err}(h) = M h^p,$$

where:

- in 1D, $h = \Delta x$
- M is a constant independent of the mesh size
- p is the order of convergence of the method
- $\text{err}(h)$ is a global error norm

Consider two simulations performed on meshes with different sizes h_1 and h_2. Let the corresponding global errors be:

$$\text{err}_1 = M h_1^p, \qquad \text{err}_2 = M h_2^p.$$

Taking the ratio of the two errors eliminates the unknown constant M:

$$\frac{\text{err}_1}{\text{err}_2} = \left(\frac{h_1}{h_2} \right)^p.$$

By taking the natural logarithm of both sides, we obtain:

$$\ln \left(\frac{\text{err}_1}{\text{err}_2} \right) = p \ln \left(\frac{h_1}{h_2} \right).$$

Solving for the order of convergence p gives:

$$\boxed{p = \frac{\ln(\text{err}_1/\text{err}_2)}{\ln(h_1/h_2)}}$$

This formula provides the **observed order of convergence** of the numerical method.

- If $p \approx 2$, the FEM implementation exhibits **second-order accuracy**, as expected for piecewise linear shape functions.
- If $p \approx 1$, the method is only first-order accurate.

Expected solution

```
np = 10, h=1.111e-01, err_L2=6.244e-02
np = 20, h=5.263e-02, err_L2=1.439e-02
np = 40, h=2.564e-02, err_L2=3.435e-03
np = 80, h=1.266e-02, err_L2=8.383e-04

p ~ [1.96 1.99 2]
```

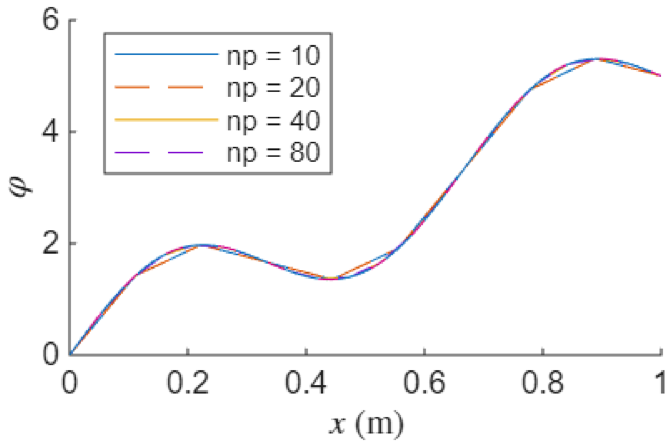

Figure 3: Proposed solution