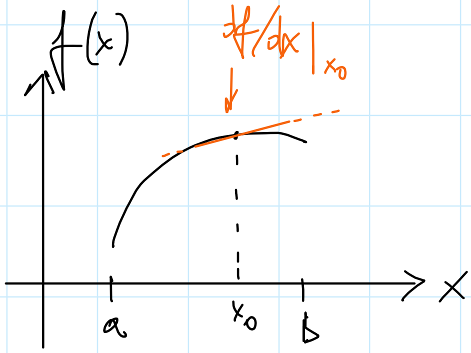
L9

20 October 2025 13:29

Numerical differentiation



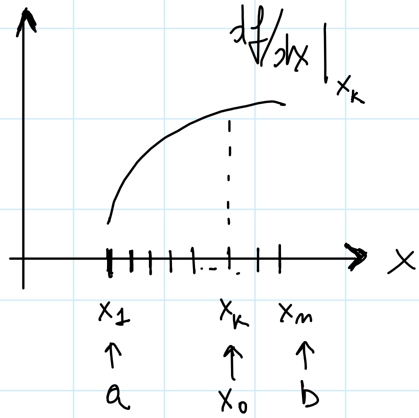
GOAL. $f(x) \in [a,b]$

$\Downarrow$

compute $\left.\dfrac{df}{dx}\right|_{x_0}$

ANALYTICALLY: $\left.\dfrac{df}{dx}\right|_{x_0} = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$

NUMERICALLY:



→ SUBDIVISION $[a,b]$ n-nodes $\begin{pmatrix} n-1 \\ \text{INTERVALS} \end{pmatrix}$

$\updownarrow$ points

$\downarrow$

$[x_1, x_2, \dots, x_k, \dots, x_n]$

$f(x_1) = f(a)$; $f(x_k) = f(x_0)$ ...

$\downarrow$ $\qquad\qquad\qquad\qquad \downarrow$

f_1 -- f_k - - f_n

IDEA: ESTIMATE df/dx from $\underbrace{\text{NODAL VALUES of } f(x)}_{f_1,\, f_2,\, \dots\, f_k,\, \dots\, f_n}$

$\uparrow$

$\sim$

→ TAYLOR EXPANSION of $f(x)$ around $f(x_0)$

$\updownarrow$

f_k

$$f(x) = \left[f(x_0) + \left.\frac{df}{dx}\right|_{x_0}(x-x_0) + \frac{1}{2}\left.\frac{d^2 f}{dx^2}\right|_{x_0}(x-x_0)^2 \right] + \underset{\Uparrow}{O}(x-x_0)^3$$

"BIG-O NOTATION"

$\exists\, M \in \mathbb{R}^+ : |E(x)| \leq M(x-x_0)^3$ for $(x-x_0) \to 0$

$\uparrow$ $\qquad\qquad \uparrow$

"there exist a

ERROR introduced by truncating TAYLOR

"there exist a positive real number ..."

ERROR introduced by truncating Taylor series to third-order

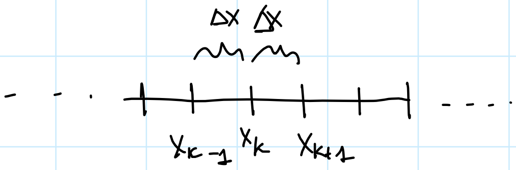
$$|E(x)| = \left| f(x) - \left[f(x_0) + \frac{df}{dx}\Big|_{x_0}(x-x_0) + \frac{1}{2}\frac{d^2f}{dx^2}\Big|_{x_0}(x-x_0)^2 \right] \right|$$

→ If taylor series is truncated to order N, the error goes to zero as $(x-x_0)^N$ when $(x-x_0) \to 0$

if $N = 2$ if $(x-x_0) \Rightarrow E(x)$

$$(x'-x_0) = \frac{1}{2}(x-x_0) \Rightarrow E'(x) = \left(\frac{1}{2}\right)^N E(x) = \frac{1}{4}E(x)$$

HP: UNIFORM SPACING between $x_1, x_2, \dots, x_n$



$$\rightarrow \begin{aligned} x_{k+1} - x_k &= \Delta x \\ x_k - x_{k-1} &= \Delta x \end{aligned}$$

TAYLOR EXP.: around x_0

$$f(x) = \underset{\substack{\uparrow \\ f(x_0)}}{f_k} + \frac{df}{dx}\Big|_k \underset{\substack{\uparrow \\ (x-x_0)}}{\Delta x} + \frac{1}{2}\frac{d^2f}{dx^2}\Delta x^2 + \theta(\Delta x^3) \quad [1]$$

TAYLOR EXP from k to k+1

TRUNCATED to third-order

$$f_{k+1} \overset{\downarrow}{=} f_k + \frac{df}{dx}\Big|_k \Delta x + \frac{1}{2}\frac{d^2f}{dx^2}\Big|_k \Delta x^2 + \theta(\Delta x^3)$$

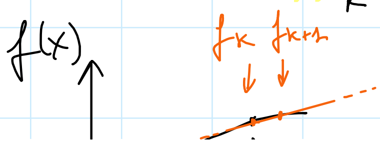
TRUNCATE TO SECOND-ORDER.

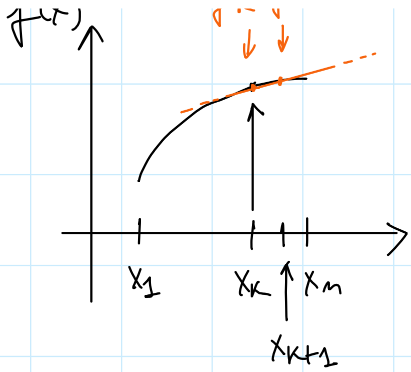
$$\Rightarrow f_{k+1} = f_k + \frac{df}{dx}\Big|_k \Delta x + \theta(\Delta x^2)$$

FORWARD finite difference

$$\frac{df}{dx}\Big|_k = \frac{f_{k+1} - f_k}{\Delta x} + \underbrace{\frac{\theta(\Delta x^2)}{\Delta x}}_{\theta(\Delta x)}$$

FIRST-ORDER DERIVATIVE FINITE difference formula





$\mathcal{O}(\Delta x)$

$\Uparrow$

FIRST-ORDER accurate

$$+\frac{df}{dx}\overbrace{(x_{k-1}-x_k)}^{<0} = -\frac{df}{dx}\Delta x$$

TAYLOR EXPANSION from x_k to x_{k-1}:

$$f_{k-1} = f_k - \frac{df}{dx}\Big|_k \Delta x + \frac{1}{2}\frac{d^2f}{dx^2}\Delta x^2 + \mathcal{O}(\Delta x^3) \quad [2]$$

TRUNCATION to second-order

$$f_{k-1} = f_k - \frac{df}{dx}\Big|_k \Delta x + \mathcal{O}(\Delta x^2)$$

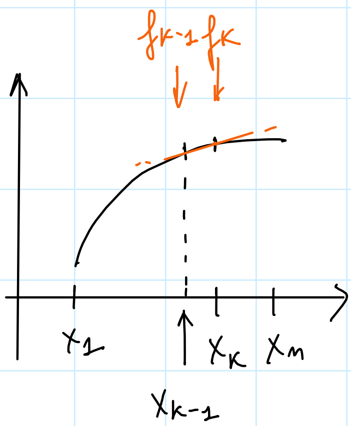
BACKWARD Finite difference formula

$$\frac{df}{dx}\Big|_k = \frac{f_k - f_{k-1}}{\Delta x} + \mathcal{O}(\Delta x)$$

First-order accuracy

$$\Updownarrow$$

$$\frac{df}{dx}\Big|_k \approx \frac{f_k - f_{k-1}}{\Delta x}$$



THIRD-ORDER

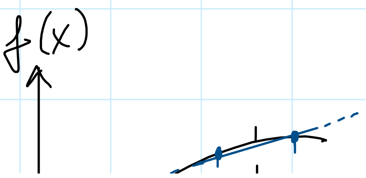
Combining [1] - [2]

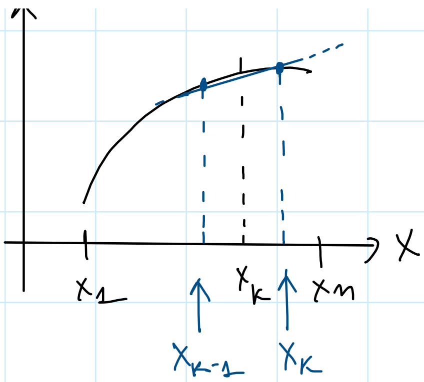
$$f_{k+1} - f_{k-1} = 0\cancel{f_k} + 2\frac{df}{dx}\Big|_k \Delta x + 0\cancel{\frac{d^2f}{dx^2}}\Delta x^2 + \mathcal{O}(\Delta x^3)$$

$$\Rightarrow \frac{df}{dx}\Big|_k = \frac{f_{k+1} - f_{k-1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

CENTERED finite difference formula for first-order derivative

ACCURACY: SECOND-ORDER





ROUNDoff errors
(finite amount of digits)

$\neq$

ERRORS. FW / BW $\longrightarrow$ $\Delta x \rightarrow E(x)$

$\frac{\Delta x}{2} \rightarrow \frac{E(x)}{2}$

C (centered) $\longrightarrow$ $\Delta x \rightarrow E(x)$

$\frac{\Delta x}{2} \rightarrow \frac{E(x)}{4}$

TRUNCATION errors
(finite amount of terms in TAYLOR SERIES to express derivatives in terms of algebraic expressions)