L19

02 December 2025 18:03

$$[K]\{\varphi\} = \{Rhs\}$$

$$\{\varphi\} = \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \end{Bmatrix}$$

Shorthand notation

$$\int_{w_{k-1,k}} p(x)\,dx \Rightarrow P_{k-1,k}$$

$$\begin{array}{c|c}
 & 1 \quad\quad 2 \quad\quad 3 \quad\quad \cdots \quad\quad m-1 \quad\quad m \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} &
\begin{bmatrix}
1 & & & & & \\
-\frac{P_{12}}{\Delta_-^2} & \left(\frac{P_{12}}{\Delta_-^2}+\frac{P_{23}}{\Delta_+^2}\right) & -\frac{P_{23}}{\Delta_+^2} & & & \\
 & -\frac{P_{23}}{\Delta_-^2} & \left(\frac{P_{23}}{\Delta_-^2}+\frac{P_{34}}{\Delta_+^2}\right) & -\frac{P_{34}}{\Delta_+^2} & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & & -\frac{P_{m-1,m}}{\Delta^2} & \frac{P_{m-1,m}}{\Delta^2}
\end{bmatrix}
\end{array}
\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_k \\ \vdots \\ \varphi_m \end{bmatrix}
=
\begin{bmatrix} \varphi_a \\ S_{1,2}+S_{2,3} \\ S_{2,3}+S_{3,4} \\ \vdots \\ \vdots \\ p(b)\varphi'_b + S_{m-1,m} \end{bmatrix}
\quad (\varphi_1 = \varphi_a)$$

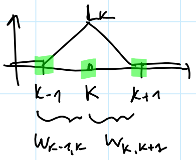
Rhs

if uniform mesh: $\Delta_- = \Delta_+ = \Delta x$, if $p(x) = $ CONSTANT $\Rightarrow [K]$ SAME AS FDM !

$[K]$: SPARSE, TRI-DIAGONAL MATRIX

$\Downarrow$

because Hat functions are "LOCAL" $L_k(x) \neq 0$ only over 2 elements



3 nodes $\Rightarrow$ 3 non-zero entries for each row of K

in FEM: the user can change the degree of weighting polynomial functions

US: 1^{st}-order $L_k(x)$

$\downarrow$

if 1^{st}-order piecewise $L_k(x) \Rightarrow$ ACCURACY $\mathcal{O}(\Delta^2)$

So for MATRIX ASSEMBLY in a node-wise order

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for i = 1 : n_nodes
    FILL row i of matrix [K]
end
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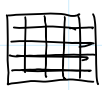
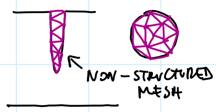
$\rightarrow$ Fine in 1D

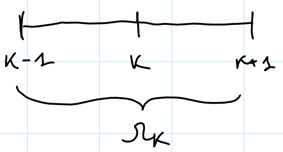
ELEMENT-WISE MATRIX ASSEMBLY

STRUCTURED MESH $\rightarrow$





1D :

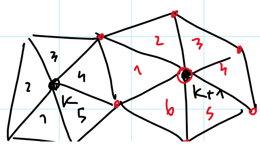


each support domain has 2 elements.

2D / 3D : TRIANGULAR MESHES

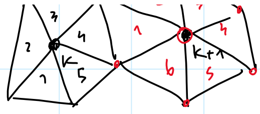
$\downarrow$

TRIANGLES (NON-STRUCTURED MESH)



each support domain has 2 elements (involves 3 nodes)

SAME # of NONZEROS for each ROW



Ω_K : 5 triangles

Ω_{K+1} 6 triangles

# of nonzeros for each row of [K] depends on the mesh

coeffs. in [K]

What is the "CONTRIBUTION" to [K] from a SINGLE ELEMENT?

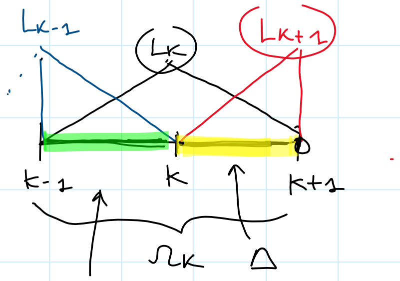
for $i = 1 . M_{EL}$
  Fill entries of [K] based on i_{EL} → each element involves SAME NUMBER of nodes
end

1D: 2 nodes

2D: 3 nodes

EXAMPLE in 1D for $W_{K,K+1}$ involves 2 nodes → K, → K+1

node K:



$\tilde{\varphi} = \varphi_{K-1} L_{K-1}(x) + \varphi_K L_K(x)$

$(x_{K+1} - x_K)$

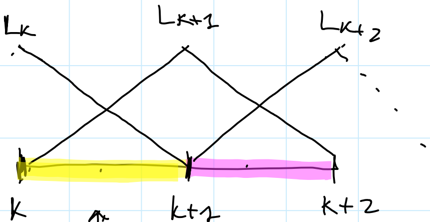
$$\left(\int_{W_{K-1,K}} \frac{dL_K}{dx} p(x) \frac{dL_{K-1}}{dx} dx\right)\varphi_{K-1} + \left(\int_{W_{K-1,K}} \frac{dL_K}{dx} p(x) \frac{dL_K}{dx} dx\right)\varphi_K +$$

$$\left(\int_{W_{K,K+1}} \frac{dL_K}{dx} p(x) \frac{dL_K}{dx} dx\right)\varphi_K + \left(\int_{W_{K,K+1}} \frac{dL_K}{dx} p(x) \frac{dL_{K+1}}{dx} dx\right)\varphi_{K+1} =$$

Kel (1,1) Kel (1,2)

$$= -\int_{W_{K-1,K}} L_K(x) t(x)\, dx - \int_{W_{K,K+1}} L_K(x) t(x)\, dx$$

Rhsel (1,1)



node K+1:

Kel (2,1) Kel (2,2)

$$\left(\int_{W_{K,K+1}} \frac{dL_{K+1}}{dx} p(x) \frac{dL_K}{dx} dx\right)\varphi_K + \left(\int_{W_{K,K+1}} \frac{dL_{K+1}}{dx} p(x) \frac{dL_{K+1}}{dx} dx\right)\varphi_{K+1} +$$

$\tilde{\varphi} = \varphi_K L_K(x) + \varphi_{K+1} L_{K+2}(x)$

$$+\left(\int_{W_{K+1,K+2}} \frac{dL_{K+1}}{dx} p(x) \frac{dL_{K+1}}{dx} dx\right)\varphi_{K+1} + \left(\int_{W_{K+1,K+2}} \frac{dL_{K+1}}{dx} p(x) \frac{dL_{K+2}}{dx} dx\right)\varphi_{K+2} = \ldots$$

Rhs el (2,1)

$w_{k+1,k+2}$ $\qquad$ $w_{k+1,k+2}$

$Rhs_{el}(2,1)$

$$= -\int_{w_{k,k+1}} L_{k+1}(x)\, t(x)\, dx \;-\; \int_{w_{k+1,k+2}} L_{k+1}(x)\, t(x)\, dx$$

IDEA: collect all coefficients due to $w_{k,k+1}$ into an "ELEMENT MATRIX"

$$\left[K_{el,k,k+1}\right] = \begin{bmatrix} \left(\int_{w_{k,k+1}} \frac{dL_k}{dx} p(x) \frac{dL_k}{dx}\, dx\right) & \left(\int_{w_{k,k+1}} \frac{dL_k}{dx} p(x) \frac{dL_{k+1}}{dx}\, dx\right) \\ \left(\int_{w_{k,k+1}} \frac{dL_{k+1}}{dx} p(x) \frac{dL_k}{dx}\, dx\right) & \left(\int_{w_{k,k+1}} \frac{dL_{k+1}}{dx} p(x) \frac{dL_{k+1}}{dx}\, dx\right) \end{bmatrix}$$

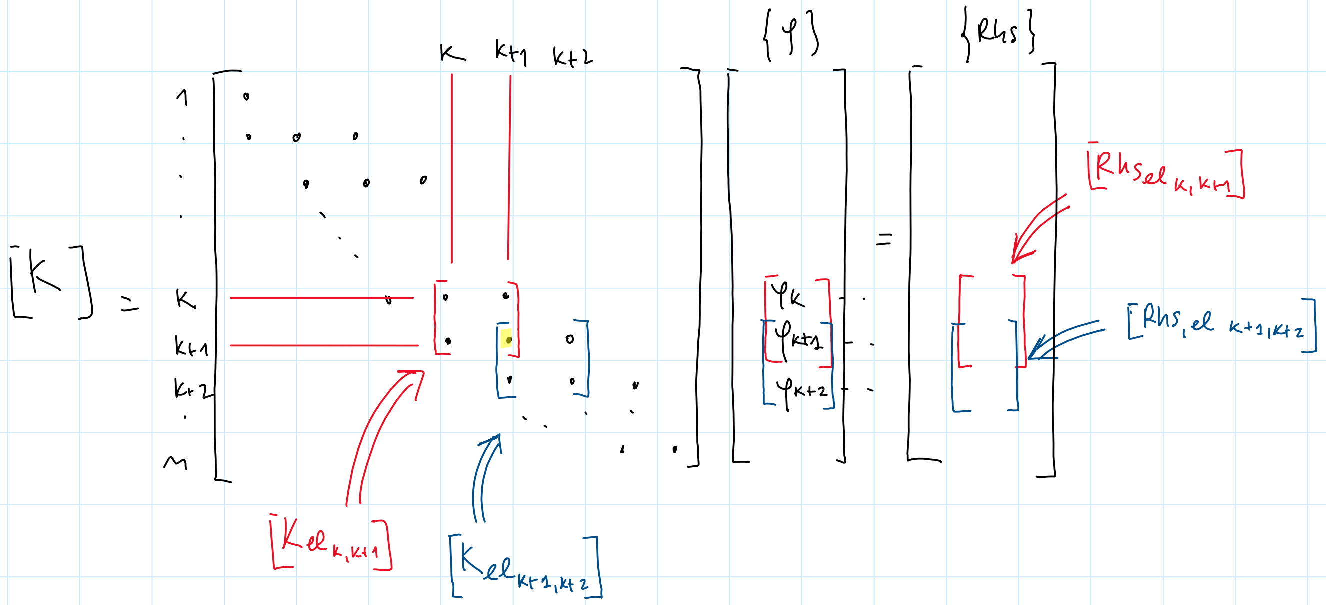
$\uparrow$

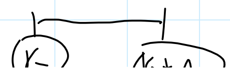
2×2

$$\left[Rhs_{el,k,k+1}\right] = \begin{bmatrix} -\int_{w_{k,k+1}} L_k(x)\, t(x)\, dx \\ -\int_{w_{k,k+1}} L_{k+1}(x)\, t(x)\, dx \end{bmatrix}$$

$\uparrow$

2×1





k $\quad$ $k+1$