Homework 05 - 1D FEM

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Complete the missing steps in the `FEM_1D_main.mlx` main file

- Complete "Step 1.1: check with analytic solution" from the file `FEM_1D_main.mlx`
- Complete "Step 4: support for Neumann BCs" from the `FEM_1D_main.mlx` file

Convergence

The goal of this step is to check the order of accuracy of our FEM discretization, to see if it achieves the second order as expected. To do that, we can try to monitor the error when the mesh is progressively refined. This procedure is commonly used to verify that a numerical implementation achieves its expected theoretical accuracy.

Read the next paragraph, where an expression for the order of convergence is derived. Try to apply it to Step 1, using the analytical solution $\varphi(x) = \frac{t_0}{2p_0}x^2 + \left(\frac{\varphi_L - \varphi_0 - \frac{t_0}{2p_0}L^2}{L}\right)x + \varphi_0$ to compute the error. Use the following expression to evaluate the global error for progressively finer meshes

```
np = [10,20,40,80]; % number of nodes

% compute phi corresponding to the meshes given by np
% ...

err(i) = sqrt(mean((phi(:)-exact_sol(:)).^2))
```

Find p and check if it matches the expected second-order convergence

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Expression for the order of convergence

For a numerical discretization with **order of convergence** p, the global error err is expected to decrease with the characteristic mesh h, as:

$$\mathrm{err}(h) = Mh^p,$$

where:

- in 1D, $h = \Delta x$
- M is a constant independent of the mesh size
- p is the order of convergence of the method
- $\mathrm{err}(h)$ is a global error norm

Consider two simulations performed on meshes with different sizes h_1 and h_2. Let the corresponding global errors be:

$$\mathrm{err}_1 = Mh_1^p, \qquad \mathrm{err}_2 = Mh_2^p.$$

Taking the ratio of the two errors eliminates the unknown constant M:

$$\frac{\mathrm{err}_1}{\mathrm{err}_2} = \left(\frac{h_1}{h_2}\right)^p.$$

By taking the natural logarithm of both sides, we obtain:

$$\ln\left(\frac{\mathrm{err}_1}{\mathrm{err}_2}\right) = \ln\left(\frac{h_1}{h_2}\right).$$

Solving for the order of convergence p gives:

$$\boxed{p = \frac{\ln(\mathrm{err}_1/\mathrm{err}_2)}{\ln(h_1/h_2)}}$$

This formula provides the **observed order of convergence** of the numerical method.

- If $p \approx 2$, the FEM implementation exhibits **second-order accuracy**, as expected for piecewise linear shape functions.
- If $p \approx 1$, the method is only first-order accurate.