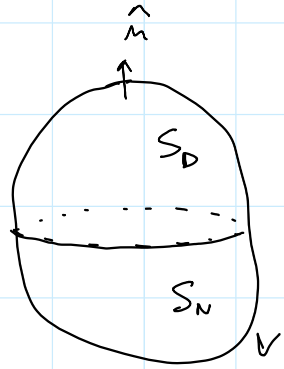
MIXED BOUNDARY CONDITIONS. DIRICHLET on S_D

NEUMANN on S_N



$S = S_D \cup S_N$

$$\begin{array}{ll} V & \nabla^2 \varphi = t \\ S_D & \varphi = \varphi_0 \\ S_N & \dfrac{\partial \varphi}{\partial n} = \varphi_0' \end{array} \quad (1)$$

Suppose: φ - ELECTRIC POTENTIAL

$$E = -\nabla \varphi$$

$$E \cdot \hat{n} = -\nabla \varphi \cdot \hat{n}$$

$$E_n = -\frac{\partial \varphi}{\partial n}$$

$\Updownarrow$

φ_0'

Proof by contradiction. 2 solutions φ_1, φ_2 that satisfy (1)

$$\begin{array}{ll} V & \nabla^2 \varphi_1 = t \\ S_D & \varphi_1 = \varphi_0 \\ S_N & \dfrac{\partial \varphi_1}{\partial n} = \varphi_0' \end{array} \qquad \begin{array}{l} \nabla^2 \varphi_2 = t \\ \varphi_2 = \varphi_0 \\ \dfrac{\partial \varphi_2}{\partial n} = \varphi_0' \end{array}$$

GOAL: prove that $\varphi_3 = 0$

$$\begin{array}{ll} V & \nabla^2 \varphi_1 - \nabla^2 \varphi_2 = t - t = 0 \Rightarrow \nabla^2 \overbrace{(\varphi_1 - \varphi_2)}^{\varphi_3 \text{ [DIFFERENCE FIELD]}} = 0 \Rightarrow \boxed{\nabla^2 \varphi_3 = 0} \ \varphi_3 \text{ is HARMONIC} \\ S_D & \varphi_1 - \varphi_2 = \varphi_0 - \varphi_0 = 0 \Rightarrow \boxed{\varphi_3 = 0} \\ S_N & \partial \varphi_1 / \partial n - \partial \varphi_2 / \partial n = \varphi_0' - \varphi_0' = 0 \Rightarrow \boxed{\dfrac{\partial \varphi_3}{\partial n} = 0} \end{array}$$

CANNOT USE K3 of mean theorem

Recall. GREEN'S FIRST IDENTITY

$$\varphi, \psi \in C_2 \qquad \oint_S \varphi \nabla \psi \cdot d\mathbf{S} = \int_V \left(\nabla \varphi \cdot \nabla \psi + \varphi \nabla^2 \psi \right) dV$$

ASSUME. $\varphi = \varphi_3 \ , \ \psi = \varphi_3$

$\hat{n}\, dS$

ASSUME . $\varphi = \varphi_3$, $\psi = \varphi_3$

$$\oint_S \varphi_3 \underbrace{\nabla\varphi_3 \cdot d\bar{S}}_{\frac{\partial \varphi_3}{\partial n} dS} = \int_V \left(\nabla\varphi_3^2 + \varphi_3 \nabla^2 \varphi_3 \right) dV$$

$\hat{n}\, dS$

$\varphi_3 = 0$ on S_D ; ? ; $\partial\varphi_3/\partial n = 0$; φ_3 is HARMONIC on V

$$\cancel{\int_{S_D} \varphi_3 \frac{\partial\varphi_3}{\partial n} dS} + \cancel{\int_{S_N} \varphi_3 \frac{\partial\varphi_3}{\partial n} dS} = \int_V \nabla\varphi_3^2\, dV + \cancel{\int_V \varphi_3 \nabla^2 \varphi_3\, dV}$$

$$\underbrace{\int_V \overbrace{\nabla\varphi_3^2}^{\geq 0} dV}_{\geq 0} = 0$$ $\Rightarrow$ φ_3 MUST fulfill this identity

$\Downarrow$

$\nabla\varphi_3$ must be $= 0 \ \forall \bar{p} \in V$ (φ_3 must be UNIFORM)

$\varphi_3 = 0 \ \forall \bar{p} \in S_D$

$\downarrow$

φ_3 has to be $= 0$ everywhere (including S_N)

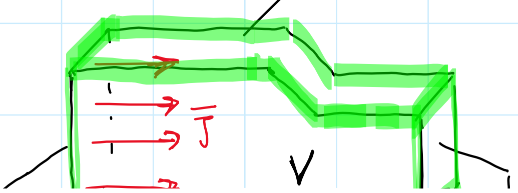
$\Rightarrow$ if $\varphi_3 = 0 \Rightarrow \varphi_1 - \varphi_2 = \cancel{\varphi_3} \Rightarrow \boxed{\varphi_1 = \varphi_2}$

$\uparrow$

SOLUTION IS UNIQUE

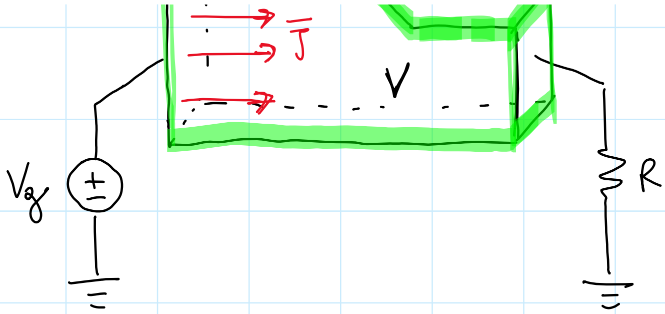
Problem: STEADY-STATE CURRENT conduction problem with an external electric circuit

UNIFORM σ (S/m)



GOAL :

WRITE a FORMULATION to find $\bar{J}$ and $\bar{E}$ inside the conductor



to find $\bar{J}$ and $\bar{E}$ inside the conductor

V is a SCD

$\bar{E}$ is CONSERVATIVE

internal points of V

$$\begin{cases} \nabla \times \bar{E} = -\cancel{\frac{\partial \bar{B}}{\partial t}} = 0 \Rightarrow \bar{E} = -\nabla\varphi \\ \nabla \cdot \left(\bar{J} + \cancel{\frac{\partial \bar{D}}{\partial t}}\right) = 0 \Rightarrow \nabla \cdot \bar{J} = 0 \\ \bar{J} = \sigma \bar{E} \end{cases}$$

$\bar{J} = \sigma\bar{E}$ $\quad$ $\bar{E} = -\nabla\varphi$

$$\rightarrow \nabla \cdot \bar{J} = 0 \Rightarrow \nabla \cdot (\sigma \bar{E}) = 0 \Rightarrow \nabla \cdot (-\sigma \nabla \varphi) = 0$$

Since σ UNIFORM $\quad -\sigma \nabla \cdot (\nabla\varphi) = 0$

$$-\cancel{\sigma}\nabla^2\varphi = 0$$

If σ is UNIFORM no role in the SPACE DISTRIBUTION of φ

$$\boxed{\nabla^2\varphi = 0}$$

LAPLACE EQUATION (φ is HARMONIC in V)

$\Downarrow$

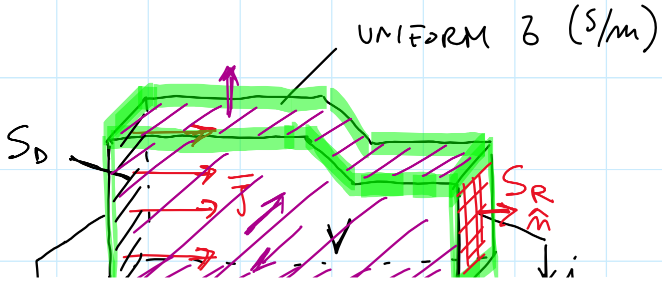
Behavior of φ influenced ONLY BY BC

For internal points $\forall \bar{P} \in V$ $\rightarrow$ Find φ such that $\nabla^2\varphi = 0$

$\Downarrow$

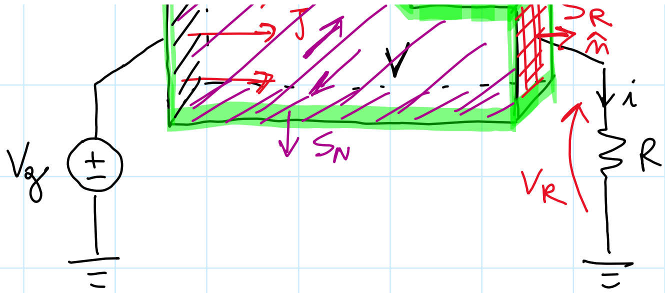
$$\varphi \Rightarrow \bar{E} = -\nabla\varphi \Rightarrow \bar{J} = \sigma\bar{E}$$

BOUNDARY CONDITIONS



$V \{ \nabla^2\varphi = 0$

$$
\begin{cases}
V & \nabla^2\varphi = 0 \\
S_D & \varphi = V_g \\
S_N & \partial\varphi/\partial n = 0 \\
S_R & \varphi = V_R
\end{cases}
$$



$\forall \bar{p} \in S_N . \quad \underset{\Uparrow}{\bar{J}\cdot\hat{n} = 0} \Rightarrow \sigma \bar{E}\cdot\hat{n} = 0 \Rightarrow -\cancel{\sigma}\underbrace{\nabla\varphi\cdot\hat{n}}_{\partial\varphi/\partial n} = 0 \Rightarrow \boxed{\partial\varphi/\partial n = 0}$

no current through S_N $\qquad$ $\bar{E} = -\nabla\varphi$

Need express V_R through φ $\qquad$ $\bar{E} = -\nabla\varphi$ $\qquad$ UNIFORM

$$\varphi = V_R = R\, i = R\int_{S_R} \sigma \bar{E}\cdot d\bar{S} = -R\int_{S_R} \sigma\nabla\varphi\cdot d\bar{S}$$

$i = \int_{S_R} \bar{J}\cdot d\bar{S}$ $\qquad$ GRADIENT

$$= -R\sigma\int_{S_R} \partial\varphi/\partial n \, dS$$

NO SKIN EFFECT
$\bar{J}$, $\bar{E}$, $\partial\varphi/\partial n$ UNIFORM on S_R

$$= -R\sigma\frac{\partial\varphi}{\partial n}\underbrace{\int_{S_R} dS}_{S_R}$$

$$\varphi = -R\sigma\frac{\partial\varphi}{\partial n}S_R$$

$$\underset{\alpha}{1}\varphi + \overbrace{R\sigma S_R}^{\beta}\frac{\partial\varphi}{\partial n} = 0$$

$\Rightarrow$ GENERAL FORM : $\boxed{\alpha\varphi + \beta\, \partial\varphi/\partial n + \gamma = 0}$ ROBIN BOUNDARY CONDITION

linear combination between Dirichlet an Neumann BCs

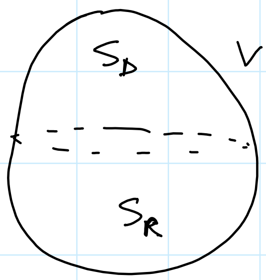
FORMULATION:

$$\begin{cases} V & \nabla^2\varphi = 0 \\ S_D & \varphi = V_g \\ S_N & \partial\varphi/\partial n = 0 \\ S_R & \varphi + R\,\sigma\,S_R\,\partial\varphi/\partial n = 0 \end{cases}$$

$\uparrow\ \uparrow\ \uparrow$ Known electrical/geometric (also V_g)

UNIQUENESS theorem for POISSON PROBLEMS with ROBIN BC

$$\begin{cases} V & \nabla^2\varphi = t \\ S_D & \varphi = \varphi_0 \\ S_R & \alpha\varphi + \beta\,\partial\varphi/\partial n + \gamma = 0 \end{cases}$$



$S = S_D \cup S_R$

Proof AB ABSURDUM $\rightarrow$ assume solutions φ_1, φ_2

GOAL: prove that $\varphi_3 = \varphi_1 - \varphi_2 = 0$

$$\begin{cases} V & \nabla^2\varphi_1 - \nabla^2\varphi_2 = \boxed{\nabla^2\varphi_3 = 0} \\ S_D & \varphi_1 - \varphi_2 = \boxed{\varphi_3 = 0} \\ S_R & \alpha\varphi_1 + \beta\,\partial\varphi_1/\partial n + \cancel{\gamma} - \left(\alpha\varphi_2 + \beta\,\partial\varphi_2/\partial n + \cancel{\gamma}\right) = 0 \end{cases}$$

$$\alpha(\varphi_1 - \alpha_2) + \beta\left(\partial\varphi_1/\partial n - \partial\varphi_2/\partial n\right) = 0$$

$$\boxed{\alpha\varphi_3 + \beta\,\partial\varphi_3/\partial n = 0}$$ on S_R

GREEN'S FIRST IDENTITY - application to φ_3

$\varphi = \varphi_3\ ,\ \psi = \varphi_3$ $\qquad$ $\nabla^2\varphi_3 = 0$

$\varphi = \varphi_3$, $\psi = \varphi_3$

$\nabla^2 \varphi_3 = 0$

$$\oint_S \varphi_3 \nabla \varphi_3 \cdot d\bar{S} = \int_V \left(\nabla \varphi_3^2 + \cancel{\varphi_3 \nabla^2 \varphi_3} \right) dV$$

$= 0$ on S_D

$$\int_{S_D} \cancel{\varphi_3} \nabla \varphi_3 \cdot d\bar{S} + \boxed{\int_{S_R} \varphi_3 \, \partial \varphi_3 / \partial n \, dS = \int_V \nabla \varphi_3^2 \, dV}$$

$$\partial \varphi_3 / \partial n = -\alpha/\beta \, \varphi_3$$

$$\underbrace{\int_{S_R} -\alpha/\beta \, \underbrace{\varphi_3^2}_{>0} \, dS}_{LHS(\varphi_3)} = \underbrace{\int_V \nabla \varphi_3^2 \, dV}_{RHS(\varphi_3)}$$

IF $\alpha/\beta > 0$ $\quad \underbrace{LHS(\varphi_3)}_{\leq 0} = \underbrace{RHS(\varphi_3)}_{\geq 0}$ $\Rightarrow$ $LHS \neq RHS \ \forall \ \varphi_3 \neq 0$

$\Downarrow$

Solution is unique

$\varphi_3 = 0 \Rightarrow \varphi_1 = \varphi_2$

IF $\alpha/\beta < 0$ $\quad \underbrace{LHS(\varphi_3)}_{\geq 0} = \underbrace{RHS(\varphi_3)}_{\geq 0}$ $\Rightarrow$ there $\exists$ $\varphi_3 \neq 0$ such that $LHS(\varphi_3) = RHS(\varphi_3)$

if $\varphi_3 \neq 0$

$\Rightarrow \varphi_1 \neq \varphi_2$

Solution is NOT UNIQUE

For DIRICHLET + ROBIN poisson problems solution is unique only for $\boxed{\alpha/\beta > 0}$

Steady-state conduction

Steady-state conduction

$$\alpha\varphi + \underbrace{bRS_R}_{\beta}\,\partial\varphi/\partial m = 0 \qquad \alpha = 1 \qquad \beta = bRS_R$$

$$\Rightarrow \alpha/\beta > 0$$

NUMERICAL ANALYSIS

GOAL. solve mathematical or physical problems with a computer

PROBLEM → ① MATHEMATICAL MODEL → ② NUMERICAL MODEL → ③ COMPUTER IMPLEMENTATION → Solution

↑ Approximation of the mathematical model suitable for a computer implementation

Sources of error

MATH. MODEL ⟶ physical approximation EX: $\nabla \times \bar{H} = \bar{J} + \cancel{\partial \bar{D}/\partial t}$

→ NUM. MODEL ⟶ TRUNCATION ERRORS

TAYLOR SERIES to express mathematical operators

→ COMPUT. IMPL ⟶ ROUNDOFF ERRORS

REPRESENT a NUMBER with a FINITE AMOUNT of DIGITS

NUMBER REPRESENTATION

POSITIONAL REPRESENTATION. position of digits indicates the power of the BASE which multiplies the given digit the given digit

Ex. $(3012)_{10} = 3 \cdot 10^{3} + 0 \cdot 10^{2} + 1 \cdot 10^{1} + 2 \cdot 10^{0}$

INTEGERS. $q_\beta = a_n \beta^n + a_{n-1} \beta^{n-1} + \dots + a_1 \beta^1 + a_0 \beta^0$

DIGITS.

$a_k \in \mathbb{N}$

$0 \leq a_k \leq \beta - 1$

$a_n \neq 0$

otherwise. 032, 32, 0032

BASE

$\beta \in \mathbb{N}$

$\beta \geq 2$

REALS: $X = \lfloor X \rfloor + frac(x)$

integer part

Pos 3 Pos 0 Pos −1

Ex. $3012.401 = 3 \cdot 10^3 + 0 \cdot 10^2 + 1 \cdot 10^1 + 2 \cdot 10^0 + 4 \cdot 10^{-1} + 0 \cdot 10^{-2} + 1 \cdot 10^{-3}$

RADIX POINT

$$(X)_\beta = a_n \beta^n + a_{n-1} \beta^{n-1} + \dots + a_1 \beta^1 + a_0 \beta^0 + \dots$$
$$+ b_1 \beta^{-1} + b_2 \beta^{-2} + \dots + b_m \beta^{-m}$$

Digits of fractional part

$b_h \in \mathbb{N}$; $0 \leq b_h \leq \beta - 1$, $b_m \neq 0$

to avoid: $3.12 = 3.120 \dots$

FIXED POINT REPRESENTATION

Given: the number of digits for the number $\rightarrow t$

the " " " " the fractional part $\rightarrow q$

the base $\rightarrow \beta$

FIXED Point SET

REAL NUMBER to be represented

$$X(\beta, t, q) = \left\{ x \in \mathbb{R} = \mathrm{sign}(x) \left[\underbrace{\sum_{k=0}^{t-(q+1)} a_k \beta^k}_{\text{integer part}} + \underbrace{\sum_{k=0}^{q} b_k \beta^{-k}}_{\text{fractional part}} \right] \right\}$$

Example: $X(\beta = 10, t = 4, q = 1)$

$$\max = \underbrace{9 \cdot 10^2 + 9 \cdot 10^1 + 9 \cdot 10^0}_{3 \text{ DIGITS}} + \underbrace{9 \cdot 10^{-1}}_{1 \text{ DIGIT}} = 999.9$$

$$\min = 0 \cdot 10^2 + 0 \cdot 10^1 + 0 \cdot 10^0 + 1 \cdot 10^{-1} = 0.1$$

SPACING

$\lfloor x \rfloor$				$\mathrm{frac}(x)$	
0	0	0	.	1	$\Delta = 0.1$
0	0	0	.	2	
0	0	0	.	3	
0	⋮	0	.	9	UNIFORM SPACING $\Delta = \beta^{-q}$
	⋮			⋮	
0	1	2	.	3	$\Delta = 0.1$
0	1	2	.	4	
	⋮				