Asymptotic Alpha: The Paradox of Never Reaching Your Target and Still Making Bank

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Abstract

This paper presents a sophisticated investment strategy that integrates the continuous-time Kelly Criterion, geometric Brownian motion, Bayesian updating of market parameters, and a logistic scaling function inspired by Zeno's paradox. The strategy dynamically adjusts investment positions in response to new market information while managing risk through adaptive mechanisms. We expand on the implementation of Bayesian updates, detailing how to update beliefs about market drift and volatility in real-time. The incorporation of Kullback-Leibler (KL) divergence allows for model uncertainty to be addressed, ensuring a cautious approach to investment decisions, while the Zeno-inspired logistic scaling function modulates position sizes as asset prices approach predefined targets.

1 Introduction

Investing in financial markets requires balancing the pursuit of returns with the management of risk. Traditional strategies often rely on static assumptions about market behavior, which may not hold in dynamic environments. To address this challenge, we propose an investment strategy that adapts to new information using Bayesian inference, adjusts position sizes based on the continuous-time Kelly Criterion, and incorporates a logistic scaling function inspired by Zeno's paradox to manage exposure as target prices are approached.

Our strategy seeks to improve upon traditional approaches by dynamically adjusting positions as market conditions evolve, using probabilistic updating methods to incorporate new information in a coherent manner. By combining geometric Brownian motion with Bayesian updating and the Kelly Criterion, the strategy can respond adaptively to changes in drift and volatility while maintaining a balance between risk and reward. The Zeno-inspired logistic scaling function ensures that positions are reduced as the target price is approached, preventing over-commitment to any one outcome.

2 Continuous-Time Kelly Criterion

At the core of the strategy is the continuous-time Kelly Criterion, which determines the optimal fraction of wealth to invest in a risky asset. The Kelly Criterion balances expected returns against the asset's volatility and the investor's risk aversion. In its continuous form, the optimal investment fraction f^* is given by:

$$f^* = \frac{\mu - r}{\gamma \sigma^2},$$

where:

- μ is the expected return (drift) of the asset,
- r is the risk-free rate,
- σ is the volatility of the asset, and
- γ is the investor's risk-aversion coefficient.

This formulation accounts for continuous compounding and aligns investment decisions with both return expectations and risk considerations. The result is a dynamic position-sizing mechanism that adjusts based on real-time estimates of the asset's drift and volatility, ensuring that capital is allocated in proportion to the investor's confidence in future returns and tolerance for risk.

3 Geometric Brownian Motion for Stock Prices

We model the asset price P_t using geometric Brownian motion (GBM), capturing the stochastic nature of financial markets. The dynamics of P_t are described by the stochastic differential equation:

$$dP_t = \mu P_t \, dt + \sigma P_t \, dW_t,$$

where:

- dP_t is the change in the asset price,
- $\mu P_t \, dt$ represents the deterministic trend component,
- $\sigma P_t \, dW_t$ represents the stochastic component, and
- dW_t is the increment of a Wiener process (standard Brownian motion).

This model is widely used in financial mathematics to simulate asset price movements over time. The continuous-time framework allows us to incorporate the randomness inherent in financial markets while still modeling an underlying drift that can be dynamically updated as new information is obtained.

4 Bayesian Updating of Market Parameters

To adapt to changing market conditions, we employ Bayesian inference to update our beliefs about the drift μ and volatility σ based on new data. This approach allows the strategy to incorporate uncertainty and adjust investment positions dynamically, in line with current market conditions.

4.1 Prior Distribution

We begin with prior distributions for μ and σ^2. For μ, we assume a normal prior distribution:

$$\mu \sim N(\mu_0, \tau_0^2),$$

where μ_0 is the initial estimate of the drift and τ_0^2 is the variance. For σ^2, we assume an inverse gamma distribution:

$$\sigma^2 \sim \text{Inverse Gamma}(\alpha_0, \beta_0),$$

where α_0 and β_0 are shape and scale parameters, respectively.

4.2 Likelihood Function

Assuming that asset returns are normally distributed, the likelihood function based on observed returns $\{r_1, r_2, \ldots, r_n\}$ is given by:

$$L(\{r_i\}|\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \exp\left(-\frac{(r_i - \mu\Delta t)^2}{2\sigma^2\Delta t}\right),$$

where Δt is the time increment between observations. This likelihood function serves as the basis for updating our beliefs about μ and σ^2 as new data becomes available.

4.3 Posterior Distribution

Using Bayes' theorem, the posterior distribution is proportional to the product of the likelihood and the prior:

$$p(\mu, \sigma^2|\{r_i\}) \propto L(\{r_i\}|\mu, \sigma^2) \times p(\mu, \sigma^2).$$

From this posterior, we can derive updated estimates for both the drift and the volatility of the asset, ensuring that the strategy adapts as new market data are observed.

4.4 Updating the Parameters

For the posterior of μ given σ^2, the update is:

$$\tau_n^2 = \left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)^{-1},$$

$$\mu_n = \tau_n^2 \left(\sum_{i=1}^{n} \frac{r_i}{\sigma^2} + \frac{\mu_0}{\tau_0^2} \right).$$

For the posterior of σ^2, we have:

$$\alpha_n = \alpha_0 + \frac{n}{2},$$

$$\beta_n = \beta_0 + \frac{1}{2} \sum_{i=1}^{n} (r_i - \bar{r})^2,$$

where $\bar{r}$ is the sample mean of the returns. These updates allow the strategy to adjust both drift and volatility estimates dynamically.

5 Incorporating KL Divergence for Adaptive Adjustment

To account for model uncertainty and prevent overconfidence in our estimates, we incorporate the Kullback-Leibler (KL) Divergence between the prior and posterior distributions as a penalizing term. For normal distributions, the KL Divergence between two distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_0, \sigma_0^2)$ is given by:

$$D_{KL}(N_1 \| N_0) = \frac{1}{2} \left(\frac{(\mu_0 - \mu_1)^2}{\sigma_0^2} + \frac{\sigma_1^2}{\sigma_0^2} - 1 + \ln \left(\frac{\sigma_0^2}{\sigma_1^2} \right) \right).$$

The KL Divergence serves as a measure of how much our beliefs about the market have shifted. A large divergence indicates high uncertainty, and we adjust the investment amount downward accordingly.

6 Adjusted Investment Amount

The investment amount a_{t+1} is modified by incorporating the KL Divergence, penalizing overconfidence in market estimates:

$$a_{t+1} = f^* \cdot W_t \cdot S(P_t) \cdot e^{-\beta D_{KL}},$$

where:

- W_t is the total wealth at time t,
- $S(P_t)$ is the logistic scaling function,
- β controls the sensitivity to the KL Divergence.

The exponential term $e^{-\beta D_{KL}}$ reduces the investment size when there is significant divergence between prior and posterior beliefs, reflecting increased uncertainty.

7 Logistic Scaling Function Inspired by Zeno's Paradox

To emulate the diminishing steps of Zeno's paradox, we employ a logistic scaling function that decreases the investment size as the asset price approaches a target T:

$$S(P_t) = \frac{1}{1 + e^{k(P_t - T)}},$$

where k determines the steepness of the scaling effect. This function ensures that investment exposure diminishes smoothly as P_t approaches T, preventing abrupt changes in position sizes.

8 Practical Implementation

Data Handling and Initialization

Implementing the strategy requires access to high-quality price data for the asset and appropriate time increments Δt (e.g., daily returns). Initial priors for μ_0, τ_0^2, α_0, and β_0 should be based on historical data or expert judgment. The risk-aversion coefficient γ should be set according to the investor's preferences.

Computational Methods and Risk Management

Analytical solutions can be used where available, but for more complex scenarios, numerical methods such as Markov Chain Monte Carlo (MCMC) or variational inference may be required. Position limits should be implemented to prevent overexposure, and stop-loss rules should be in place to exit positions under adverse conditions. Diversification across multiple assets can also mitigate idiosyncratic risk.

9 Conclusion

This paper has presented an investment strategy that dynamically adjusts position sizes using the continuous-time Kelly Criterion, Bayesian updates of market parameters, and a logistic scaling function inspired by Zeno's paradox. The strategy adapts to new information while accounting for uncertainty through the use of KL Divergence. Though SPY offers a suitable case study, the framework is generalizable to other asset classes, including single-name equities, sector-specific ETFs, and commodities futures. Further research and fine-tuning of parameters, particularly in different market environments, will help refine the strategy's performance across various asset classes. I don't know why anyone would use this. It's a joke.