Astronomy from 4 perspectives: the Dark Universe

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exercise: Dark matter and galaxy rotation curves

1. ***harmonic oscillator and energy types***
   The harmonic oscillator is described the the differential equation $\ddot{x} = -g/l\,x$, and performs harmonic oscillations $x(t) \propto \exp(\pm \mathrm{i}\omega t)$ with $\omega^2 = g/l$.

   (a) Please show that $\langle T \rangle = \langle V \rangle$ with the kinetic energy T and the potential energy V. The brackets $\langle \ldots \rangle$ are time averages over one oscillation period τ,

   $$\langle T \rangle = \frac{1}{\tau}\int_0^\tau \mathrm{d}t\, T(t) \quad \text{and} \quad \langle V \rangle = \frac{1}{\tau}\int_0^\tau \mathrm{d}t\, V(t) \tag{I}$$

   which is defined as $\tau = 2\pi/\omega$, and the specific energies $T(t) = \dot{x}^2/2$ and $V(t) = gx^2/(2\mathrm{l})$.

   (b) Could you predict the proportionality between $\langle T \rangle$ and $\langle V \rangle$ from the isochrony of the harmonic oscillator?

   The probability of finding the oscillator at a certain amplitude x is inversely proportional to the velocity: $\mathrm{d}x/\mathrm{d}t = \upsilon$, such that $\Delta t = \Delta x/\upsilon$. If the range of motion is divided into equidistant intervals Δx, the probability p of seeing the oscillator in one of those is proportional to the time it spends there, i.e. proportional to $1/|\upsilon|$.

   (c) Please normalise p and draw the function $p(\upsilon)$: If you look randomly at a harmonic oscillator, at what stage in its oscillation are you most likely to see it?

   (d) Please define averages

   $$\langle T \rangle = \int \mathrm{d}\upsilon\, p(\upsilon) T(\upsilon) \quad \text{and} \quad \langle V \rangle = \int \mathrm{d}\upsilon\, p(\upsilon) V(\upsilon) \tag{II}$$

   and compute both integrals. You can use energy conservation for the second integral to express V in terms of the velocity υ. Are the results identical to the previous computation? Be careful to take the positive sign of p into account, by using the symmetry of the integrand.

   (e) Why is there no issue with convergence when the probability density $p \to \infty$ at $\upsilon \to 0$?

   (f) Is the virial relation $\langle T \rangle = \langle V \rangle$ as well valid for a circular orbit in a spherically symmetric harmonic potential?

   (g) Is it valid as well for any other Lissajous-figure?

2. ***flat rotation curves***
   Let's consider the motion of stars inside a galaxy with the density profile of a *singular isothermal sphere*, which is $\rho \propto r^{-2}$. The singular isothermal sphere describes the density of dark matter well on scales of the galactic disc.

   (a) Please show by solving the Poisson equation $\Delta\Phi = 4\pi G\rho$,

   $$\Delta\Phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho, \tag{III}$$

   for a spherically symmetric density profile $\rho \propto r^{-2}$ that rotation curves are flat.

(b) Please compute the mean kinetic $\langle T\rangle$ and mean potential energy $\langle V\rangle$ for the circular motion in an isothermal sphere as a function of r.

(c) Is it possible in this case to decompose the circular orbiting motion into two uncoupled orthogonal harmonic oscillations?

(d) What would the density profile need to be such that stars would perform harmonic oscillations through the centre of the galaxy, i.e. for the potential to be quadratic, $\Phi \propto r^2$?

3. ***MoND, the Solar system and the Milky Way***

Let's assume that we can change the acceleration due to gradients in the gravitational potential $\nabla\Phi$ in an empirical way,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} \rightarrow \frac{\mathrm{d}\Phi}{\mathrm{d}r} + a_0, \tag{IV}$$

as it would be relevant for a circular motion around the Milky Way centre in a spherically symmetric potential.

(a) What would be the effect on a rotation curve from the density profile $\rho \propto r^{-\alpha}$?

(b) The parameter a_0 would need to be chosen small: Please estimate an upper bound on the value of a_0 from the orbital acceleration of the Solar system on its passage around the Milky Way center. You can find all necessary data on Wikipedia.

(c) Please think of a way to visualise the numerical value of a_0.

(d) At what distance from the Earth's surface would the gravitational acceleration be a_0?