

Astronomy from 4 perspectives: the Dark Universe

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play with data: virial theorem and periodic motion

In these exercises we will explore the virial theorem by solving equations of motions numerically and measuring averaged energies from the solutions.

1. virial theorem and the harmonic oscillator

The script `harmonic.py` generates a solution to the harmonic oscillator equation $\ddot{x} = -x$ by transforming it with the definition $y = \dot{x}$ into a coupled first order equation,

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (\text{I})$$

where the angular frequency is set to $\omega = 1$ for the numerics.

- (a) Is the total energy conserved?
- (b) Please run the script and measure the average kinetic and potential energies: Do you find $\langle T \rangle = \langle V \rangle$ for the harmonic oscillator?
- (c) Is $\langle T \rangle = \langle V \rangle$ true for any initial condition?

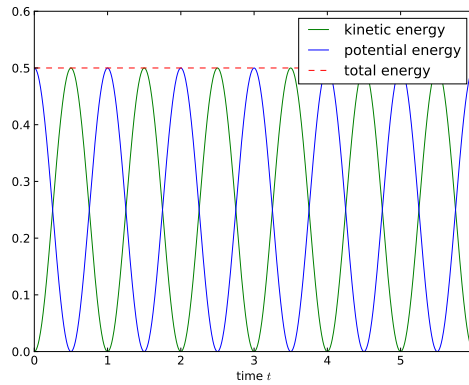


Figure 1: numerical solutions to the harmonic oscillator

2. virial relationship in the anharmonic oscillator

The virial theorem makes a prediction for the average kinetic and potential energies in any system with a scale-free potential, for instance the anharmonic oscillator with the potential $\Phi \propto x^{2n}$. The script `anharmonic.py` solves the equation of motion. The constant of proportionality is set to 1.

- (a) What's the equation of motion for the potential $\Phi = x^{2n}/(2n)$, and what's the corresponding coupled first order system?
- (b) Why are there with increasing n phases of constant velocity and a sawtooth pattern in position?
- (c) Please measure the average kinetic and potential energies over many oscillations: What's their ratio r as a function of n ? Why does $r = \langle T \rangle / \langle V \rangle$ increase with n ?

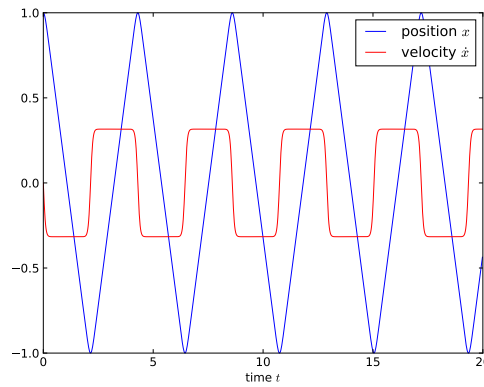


Figure 2: numerical solutions to the anharmonic oscillator for $n = 10$

3. virial theorem in the (generalised) Kepler-problem

Please simulate Kepler orbits with the script `kepler.py`: The equation of motion of a particle in the potential $\Phi \propto 1/r^\alpha$ can be derived from the Lagrange density

$$\mathcal{L} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{GM}{r^\alpha} \quad (\text{II})$$

Please derive the equation of motion with the Euler-Lagrange-equations and convert the resulting second-order equations in a set of first coupled order equations.

- Change the total energy of the planet by setting δ to a value unequal to 0 and observe the change in the orbit. The product constants GM is set to $GM = 1$ for the numerics.
- Change the value of α to a number different from 1: Are the orbits still closed? NB: The problem becomes unstable if α is too large, try to experiment in the range $\alpha = 0.8 \dots 1.2$.
- Can you generate precession motion and orbits lagging behind by choosing a suitable α ?
- Is it possible to have bound systems for $\alpha \geq 2$? What's the physical reason?
- What's the virial relationship and how does it depend on α ?

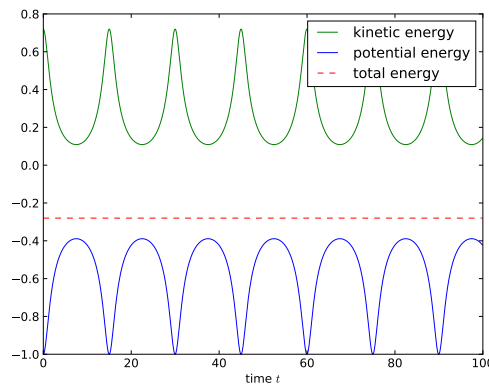


Figure 3: kinetic and potential energy in Kepler-orbits