Astronomy from 4 perspectives: the Dark Universe

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prepared by: Florence participants and BMS

exercise: Supernova-cosmology and dark energy
Solutions

1. ***light-propagation in FLRW-spacetimes***
   Photons travel along null geodesics, $\mathrm{d}s^2 = 0$, in any spacetime.

   (a)

   (b)

   (c)

   (d)

   (e)

2. ***light-propagation in perturbed metrics***

$$ds^2 = \left(1 + 2\frac{\Phi}{c^2}\right)c^2dt^2 - \left(1 - 2\frac{\Phi}{c^2}\right)dx^2 \tag{I}$$

With $ds^2 = 0$:

$$\left(1 + \frac{2\Phi}{c^2}\right)c^2dt^2 = \left(1 - \frac{2\Phi}{c^2}\right)dx^2 \tag{II}$$

$$\frac{dx}{dt} = \pm c\sqrt{\frac{1 + \frac{2\Phi}{c^2}}{1 - \frac{2\Phi}{c^2}}} \tag{III}$$

With $\frac{1}{1-\epsilon} \approx 1 + \epsilon$ for small ϵ:

$$\frac{dx}{dt} \approx \pm c\left(1 + \frac{2\Phi}{c^2}\right) \tag{IV}$$

For a non-zero Φ this is not equal to c!
We assign an effective index of refraction by:

$$n(\Phi) = \frac{dx/dt}{c} \approx \left(1 + \frac{2\Phi}{c^2}\right) \tag{V}$$

3. ***classical potentials including a cosmological constant***
   The field equation of classical gravity including a cosmological constant λ is given by

$$\Delta\Phi = C(n)G\rho + \lambda \tag{VI}$$

(a) field calculation
Using the n-dimensional Laplacian

$$\Delta = \frac{1}{r^{n-1}}\left(\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) + \Delta_{S^{n-1}}\right)$$

Assuming spherical symmetry one may neglect the angular Laplace-Beltrami operator $\Delta_{S^{n-1}}$. Respecting the total mass

$$M = C(n)\int_0^r \mathrm{d}r'\,(r')^{n-1}\,\rho(r')$$

Now it is possible to simply integrate the field equation starting with:

$$\Delta\Phi = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial\Phi}{\partial r}\right) \tag{VII}$$

$$= C(n)G\rho(r) + \lambda \tag{VIII}$$

$$r^{n-1}\frac{\partial\Phi}{\partial r} = \int_0^r \mathrm{d}r'\left(C(n)G\,(r')^{n-1}\,\rho\,(r') + (r')^{n-1}\,\lambda\right) \tag{IX}$$

$$= GM + \frac{\lambda}{n}r^n \tag{X}$$

$$\frac{\partial\Phi}{\partial r} = \frac{GM}{r^{n-1}} + \frac{\lambda r}{n} \tag{XI}$$

$$\Phi = -\frac{GM}{(n-2)\,r^{n-2}} + \frac{\lambda r^2}{2n} \tag{XII}$$

(b) power-law solutions

Following the calculation one may see that each source term corresponds to an individual power-law:

$$C(n)G\rho(r) \quad\Rightarrow\quad -\frac{GM}{(n-2)\,r^{n-2}}$$

$$\lambda \quad\Rightarrow\quad \frac{\lambda r^2}{2n}$$

Whereas the λ-term corresponds to a repulsive potential.

(c) equilibrium

To find an equilibrium distance one must set $\Phi\left(r_{\mathrm{eq}}\right) = 0$

$$\frac{GM}{(n-2)\,r_{\mathrm{eq}}^{n-2}} = \frac{\lambda r_{\mathrm{eq}}^2}{2n} \tag{XIII}$$

$$\frac{\lambda r_{\mathrm{eq}}^n}{2n} = \frac{GM}{n-2} \tag{XIV}$$

from which follows immediatly:

$$r_{\mathrm{eq}} = \sqrt[n]{\frac{GM}{\lambda}\frac{2n}{n-2}} \tag{XV}$$

4. ***physics close to the horizon***

   Why is it necessary to observe supernovae at the Hubble distance c/H_0 to see the dimming in accelerated cosmologies? Please start at considering the curvature scale of the Universe: A convenient quantisation of curvature might be the Ricci-scalar $R = 6H^2(1-q)$ for flat FLRW-models.

   (a)

   (b)

5. ***measure cosmic acceleration***

The luminosity distance $d_{\mathrm{lum}}(z)$ in a spatially flat FLRW-universe is given by

$$d_{\mathrm{lum}}(z) = (1+z)\int_0^z \mathrm{d}z' \, \frac{1}{H(z')} \tag{XVI}$$

with the Hubble function $H(z)$. Let's assume that the Universe is filled with a cosmological fluid up to the critical density with a fluid with equation of state w, such that the Hubble function is

$$H(z) = H_0(1+z)^{\frac{3(1+w)}{2}}. \tag{XVII}$$

(a) By definition:

$$H = \frac{\dot{a}}{a} \text{ and } q = -\frac{\ddot{a}a}{\dot{a}^2}$$

It follows

$$\dot{H} = \frac{\ddot{a}a - \dot{a}^2}{a^2} = \frac{\ddot{a}a}{a^2} - H^2$$

So we get

$$\frac{\dot{H}}{H^2} = \frac{\ddot{a}a}{\dot{a}^2} - 1 = -q - 1$$
$$q = -(\frac{\dot{H}}{H^2} + 1)$$

We also have

$$H = H_0 \cdot (1+z)^{\frac{3(1+w)}{2}} = H_0 \cdot a^{\frac{-3(1+w)}{2}}$$

and

$$\begin{aligned}\dot{H} &= H_0\left(\frac{-3(1+w)}{2}\right) \cdot a^{\frac{-3(1+w)}{2}} \cdot \dot{a} \\ &= H_0 \cdot a^{\frac{-3(1+w)}{2}} \cdot \frac{\dot{a}}{a} \cdot \left(\frac{-3(1+w)}{2}\right) \\ &= H^2 \cdot \left(\frac{-3(1+w)}{2}\right)\end{aligned}$$

so

$$q = -\left(\frac{-3(1+w)}{2} + 1\right) = \frac{1}{2}(3w+1)$$

and obviously

$$q < 0 \text{ for } w < -\frac{1}{3}$$
$$q > 0 \text{ for } w > -\frac{1}{3}$$

(b) First, we consider the case $w = -\frac{1}{3}$ (non-accelerating universe):

$$H = H_0(1+z)^{\frac{3(1+w)}{2}} = H_0(1+z)$$

$$
\begin{aligned}
d_{lum,1} &= (1+z)\int_0^z \frac{1}{H(z')}\,\mathrm{d}z' \\
&= (1+z)\int_0^z \frac{1}{H_0(1+z')}\,\mathrm{d}z' \\
&= \frac{1+z}{H_0} ln(1+z)
\end{aligned}
$$

Now, we consider the case $w < -\frac{1}{3}$ (accelerating universe):

$$
\begin{aligned}
d_{lum,2} &= (1+z)\int_0^z \frac{1}{H(z')}\,\mathrm{d}z' \\
&= \frac{1+z}{H_0}\int_0^z (1+z')^{\frac{-3(1+w)}{2}}\,\mathrm{d}z' \\
&= \frac{1+z}{H_0}\left[(1+z')^{\frac{-3(1+w)+2}{2}} \cdot \frac{2}{-3(1+w)+2}\right]_0^z \\
&= \frac{1+z}{H_0}\left(\frac{2}{-3(1+w)+2}\right)\left[(1+z')^{\frac{-3(1+w)+2}{2}} - 1\right]
\end{aligned}
$$

It follows: $d_{lum_2}(z) > d_{lum_1}(z)$, because the exponent $\frac{-3(1+w)+2}{2}$ is positive ($w < -\frac{1}{3}$), so $d_{lum_2}(z)$ is growing faster, than the logarithmic function $d_{lum_1}(z)$.

(c)

(d)

(e)