Kepler-problem: direct numerical integration of the equation of motion

$\mathcal{L} = T - V$ classical Lagrange function.

$$\mathcal{L} = \frac{m}{2} \cdot (\dot{r}^2 + r^2 \dot{\varphi}^2) + \frac{GM}{r^\alpha}$$

mechanical similarity: $\frac{\mathcal{L}}{m} \to \mathcal{L}$

$$\mathcal{L} = \frac{1}{2}(\dot{r}^2 + r^2 \dot{\varphi}^2) + \frac{GM}{r^\alpha}$$

Euler-Lagrange equations.

$$\frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = r^2 \dot{\varphi} \to \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 2 r \dot{r} \dot{\varphi} + r^2 \ddot{\varphi}$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{-\alpha G M}{r^{\alpha+1}} + r \dot{\varphi}^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \dot{r} \to \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} = \ddot{r}$$

auxiliary variables:

$$s = \dot{r}$$
$$\psi = \dot{\varphi}$$

equations of motion:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\partial \mathcal{L}}{\partial \varphi} \to 2 r \dot{r} \dot{\varphi} + r^2 \ddot{\varphi} = 0 \quad | : r^2$$

$$2 \frac{\dot{r}}{r} \dot{\varphi} + \ddot{\varphi} = 0$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \to \ddot{r} = -\alpha \cdot \frac{GM}{r^{\alpha+1}} + r \dot{\varphi}^2$$

Kepler-problem: direct numerical integration of the eqn. of motion

$$\dot{\psi} = -2\frac{s}{r}\psi$$
$$\dot{\varphi} = \psi$$
$$\dot{s} = -\alpha\frac{GM}{r^{\alpha+1}}$$
$$\dot{r} = s.$$

$$\longrightarrow \quad \frac{d}{dt}\begin{pmatrix} r \\ s \\ \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} s \\ -\alpha\frac{GM}{r^{\alpha+1}} + r\psi^2 \\ \psi \\ -2\frac{s}{r}\psi \end{pmatrix}$$

initial conditions for a circular orbit

$$\frac{v^2}{r} = \alpha\frac{GM}{r^{\alpha+1}}$$

$$v = r\dot{\varphi} \longrightarrow \quad r\dot{\varphi}^2 = \alpha\frac{GM}{r^{\alpha+1}}, \quad \dot{\varphi} = \sqrt{\frac{\alpha GM}{r^{\alpha+2}}} = \psi$$

energy and angular momentum.

$$E_{kin} = T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) = \frac{1}{2}(s^2 + r^2\psi^2)$$

$$E_{pot} = \frac{GM}{r^\alpha}.$$

(in the code $GM = 1$).

$$L = r^2\dot{\varphi} = r^2\psi$$

ellipticity $\epsilon \sim$ add a little energy.

$E_{kin}(t)$, $E_{pot}(t)$ or φ

$L(t) \simeq$ conserved.

$E_{kin} + E_{pot} \simeq$ conserved

$\langle E_{kin}\rangle$, $\langle E_{pot}\rangle \longrightarrow$ agrees with virial theorem?

Kepler-problem in $\phi \sim \frac{1}{r^n}$ potentials.

$$\epsilon = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{GmM}{r^n}$$ for the generalised Kepler problem.

conservation of energy and angular momentum

orbit in a plane, due to fixed angular momentum direction

$$L = mrv = mr^2\dot{\varphi}, \text{ with } v = r\dot{\varphi}$$

$$\dot{r} = \frac{dr}{dt} = \frac{d\varphi}{dt}\frac{dr}{d\varphi}$$

$$\frac{d\varphi}{dt} = \frac{L}{mr^2} \longrightarrow \frac{dr}{dt} = \frac{L}{mr^2}\frac{dr}{d\varphi}$$

$$\underline{\frac{\epsilon}{m}} = \frac{1}{2}\left(\left(\frac{L}{mr^2}\frac{dr}{d\varphi}\right)^2 + r^2\left(\frac{L}{mr^2}\right)^2\right) - \frac{GM}{r^n}$$

specific energy (use mechanical similarity)

→ possible because of inertial force

$\frac{L}{m} \rightarrow L$ specific angular momentum.

$\frac{\epsilon}{m} \rightarrow \epsilon$ specific total energy

$$\epsilon = \frac{1}{2}\left(\left(\frac{L}{r^2}\frac{dr}{d\varphi}\right)^2 + r^2\left(\frac{L}{r^2}\right)^2\right) - \frac{GM}{r^n}$$

$$\epsilon = \frac{1}{2}\left(\frac{L^2}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{L^2}{r^2}\right) - \frac{GM}{r^n}$$

solve for $\frac{dr}{d\varphi}$

$$\sqrt{\left(2\left(\epsilon + \frac{GM}{r^n}\right) - \frac{L^2}{r^2}\right)\frac{r^4}{L^2}} = \frac{dr}{d\varphi}$$

$$\rightarrow \frac{dr}{d\varphi} = \frac{r^2}{L}\cdot\sqrt{2\left(\epsilon + \frac{GM}{r^n}\right) - \frac{L^2}{r^2}}.$$ (for numerical solution with φ as an ev. parameter)

Kepler problem in $\phi \sim \frac{1}{r^n}$ potentials.

solution for the orbit by separation of variables.

$$\int d\varphi = \varphi = \int dr \frac{\frac{L}{r^2}}{\sqrt{2\left(\epsilon + \frac{GM}{r^n}\right) - \frac{L^2}{r^2}}}.$$

effective potential → minimum defines circular orbit with a constant r.

$$\phi = \underbrace{-\frac{GM}{r^n}}_{\text{attractive gravitational term}} + \underbrace{\frac{L^2}{2r^2}}_{\text{repulsive (equally gravitational) centrifugal term.}}$$

effective potential

$$\frac{d\phi}{dr} = 0 = n\frac{GM}{r^{n+1}} - \frac{L^2}{r^3}$$

$$nGM = L^2 \cdot r^{n+1-3} = L^2 \cdot r^{n-2}$$

$$L = \sqrt{\frac{nGM}{r^{n-2}}}$$ fixes specific angular momentum for a given r. to make sure that $\frac{d\phi}{dr} = 0$ and the orbit is circular.

energy: from r_{ini} (the initial condition)

$$\epsilon = \frac{v^2}{2} - \frac{GM}{r^n} = \frac{1}{2}\frac{L^2}{r^2} - \frac{GM}{r^n}.$$

$$v = r \cdot \dot{\varphi} \rightarrow v^2 = r^2 \dot{\varphi}^2 = \frac{L^2}{r^2}.$$

$$L = r^2 \dot{\varphi} \rightarrow \dot{\varphi} = \frac{L}{r^2}$$

Kepler-problem in $\phi \sim \frac{1}{r^n}$ potentials.

introduce inverse radius $u = \frac{1}{r}$, $r = \frac{1}{u}$

$$\frac{dr}{d\varphi} = \frac{r^2}{L} \sqrt{2\left(\epsilon + \frac{GM}{r^n}\right) - \frac{L^2}{r^2}}$$

$$\frac{dr}{d\varphi} = \frac{du}{d\varphi} \frac{dr}{du}, \quad \frac{dr}{du} = -\frac{1}{u^2}, \quad \frac{du}{dr} = -u^2$$

$\longrightarrow$

$$\frac{du}{d\varphi} = \underbrace{\frac{du}{dr}} \cdot \frac{1}{Lu^2} \cdot \sqrt{2(\epsilon + GMu^n) - L^2u^2}$$

$$\frac{du}{d\varphi} = \frac{-1}{L} \cdot \sqrt{2(\epsilon + GMu^n) - L^2u^2}$$

$$\boxed{\frac{du}{d\varphi} = \sqrt{\frac{2(\epsilon + GMu^n)}{L^2} - u^2}.}$$