Origin of the Moon-forming impactor

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1 Introduction

One of the Moon formation scenarios suggested is the so-called giant-impact theory. In this hypothesis, a small planetary body called Theia, with mass comparable to that of Mars impacted the Earth and material ejected from this collision kept orbiting the Earth. This material then have been accreted and formed the Moon. The alternative scenario, of the Moon forming from material that initially orbited the Earth doesn't explain the different iron abundances of the two bodies. The giant-impact theory also supports that the impactor has been initially formed in an orbital distance similar to that of Earth's due to the similar oxygen abundances of the two bodies.

In order for such a big body to form in an Earth-like orbit around the Sun and not collide with it when it was smaller, it would have to form in the Earth-Sun Lagrange points L_4 or L_5. Each of these points form an equilateral triangle with the Earth and the Sun. As the body grows it will remain on them and if perturbed by other bodies that encounter it or collide with it, it will orbit around these stable points.

In Belbruno and Gott 2005, it is shown that the body's orbit will deviate from a stable one around L_4 (or L_5) independently of its mass, when the magnitude of the perturbation in its velocity reaches some value. They even find a value in this perturbation that would lead to direct collision of the impactor with the Earth.

This project's goal is to demonstrate that a breakout in the orbit of a Mars-sized body orbiting the Earth-Sun L_4 can happen for some velocity perturbation and that its resulting trajectory can threaten the Earth, either by colliding with it, or by being captured in an orbit around it.

2 Theory and Methods

The code used for this work is included in Appendix A.

2.1 Equations of motion for the three-body problem

In order to study how the impactor's orbit changes in time, we need to numerically solve the three-body problem. In a system containing the Sun (object 2), the Earth (object 1) and the small body (impactor, object 3), which are considered point masses, the equations of motion for their positions $\boldsymbol{r}_i = (x_i, y_i, z_i)$, where $i = 1, 2, 3$ are:

$$\begin{aligned}
\ddot{\boldsymbol{r}}_1 &= -Gm_2\frac{\boldsymbol{r_1}-\boldsymbol{r_2}}{|\boldsymbol{r_1}-\boldsymbol{r_2}|^3} - Gm_3\frac{\boldsymbol{r_1}-\boldsymbol{r_3}}{|\boldsymbol{r_1}-\boldsymbol{r_3}|^3}\\
\ddot{\boldsymbol{r}}_2 &= -Gm_3\frac{\boldsymbol{r_2}-\boldsymbol{r_3}}{|\boldsymbol{r_2}-\boldsymbol{r_3}|^3} - Gm_1\frac{\boldsymbol{r_2}-\boldsymbol{r_1}}{|\boldsymbol{r_2}-\boldsymbol{r_1}|^3}\\
\ddot{\boldsymbol{r}}_3 &= -Gm_1\frac{\boldsymbol{r_3}-\boldsymbol{r_1}}{|\boldsymbol{r_3}-\boldsymbol{r_1}|^3} - Gm_2\frac{\boldsymbol{r_3}-\boldsymbol{r_2}}{|\boldsymbol{r_3}-\boldsymbol{r_2}|^3}
\end{aligned} \tag{1}$$

with G being the gravitational constant, m_1, m_2, m_3 the masses of the three bodies and $\ddot{\boldsymbol{r}}_{\boldsymbol{i}}$ denoting the second time derivative of the position vector.

Even though in our case the center of mass will lie on the Sun (or very close to it), after computing $\boldsymbol{r_i}$, which are relative to the origin, we change our coordinates to the center of mass frame of reference. The position of the center of mass relative to the origin will be:

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r_1} + m_2\boldsymbol{r_2} + m_3\boldsymbol{r_3}}{m_1 + m_2 + m_3} \tag{2}$$

and the bodies' positions relative to it are:

$$\boldsymbol{r_{i,CM}} = \boldsymbol{R} - \boldsymbol{r_i} \tag{3}$$

2.2 Leapfrog integration

Since the system of equations (1) cannot be solved analytically, we implement a time reversible, second-order integration method called leapfrog. We chose this method, instead of others (like the Euler method), as it does not become unstable as time increases. In the leapfrog method, position is calculated in each full time step $(t_j, t_{j+1}, ...)$, whereas velocity $\boldsymbol{v}$ in each half time step $(t_{j-1/2}, t_{j+1/2}, ...)$, with the difference between these two time increments being $t_{j+1/2} - t_j = dt/2$, where dt is the constant time interval (The Art of Computational Science 2004/01/25). Omitting the index i (which denotes the object):

$$\begin{aligned} \boldsymbol{r_j} &= \boldsymbol{r_{j-1}} + \boldsymbol{v_{j-1/2}}dt \\ \boldsymbol{v_{j+1/2}} &= \boldsymbol{v_{j-1/2}} + \boldsymbol{a_j}dt \end{aligned} \tag{4}$$

with $\boldsymbol{a_j}$ being calculated by Eq. (1), as $\boldsymbol{a_j} = \ddot{\boldsymbol{r}}_{\boldsymbol{i}}\boldsymbol{(t_j)}$. The above equations can be written for integer times as:

$$\begin{aligned} \boldsymbol{r_{j+1}} &= \boldsymbol{r_j} + \boldsymbol{v_j}dt + \boldsymbol{a_i}\frac{(dt)^2}{2} \\ \boldsymbol{v_{j+1}} &= \boldsymbol{v_j} + (\boldsymbol{a_j} + \boldsymbol{a_{j+1}})\frac{dt}{2} \end{aligned} \tag{5}$$

which means that before calculating $\boldsymbol{v_{j+1}}$ we should find $\boldsymbol{a_{j+1}} = \ddot{\boldsymbol{r}}_{\boldsymbol{i}}\boldsymbol{(t_{j+1})}$, again from Eq. (1).

2.3 Initial conditions and velocity perturbation

The system will be studied only in two dimensions (x, y). Before using the leapfrog method as presented in Eq. (5) to solve the three-body problem we compared it with the analytical Keplerian motion of the two-body problem, of just the Sun and the Earth, to assure it works.

In order to integrate the system of equations (1), we also need the initial position and velocity vectors for all three bodies, as well as their masses. Earth's mass is the unit used, so $m_1 = 1$, the mass of the Sun is set to $m_2 = 333 \times 10^3$ and the impactor's mass m_3 will vary.

The Sun is placed in the origin, with no initial velocity, the Earth 1 AU away from it with a velocity equal to its current orbital speed (29.78 km/s) and the impactor on the L_4 Lagrange point, which orbits ahead of Earth and was found by rotating Earth's initial position by 60°. Its initial velocity is equal in magnitude to that of Earth.

To achieve the breakout of the small object's orbit from around L_4 to orbits that can lead to close encounters with the Earth, a perturbation, α in its initial velocity, $\boldsymbol{v_0}$ is introduced:

$$\boldsymbol{v_{0,pert}} = (1+\alpha)\boldsymbol{v_0} \tag{6}$$

so both components of the initial velocity are perturbed by the same factor.

3 Results

After integrating the equations of motion with the leapfrog method, the positions of the bodies relative to their common center of mass, from Eq. (3) are used. Also the Sun-Earth L_4 Lagrange point's trajectory is calculated by finding the third edge of the equilateral triangle where the Sun and the Earth are on its other two edges, for every time step. This way we can also calculate how the small object moves around L_4 and when it escapes it.

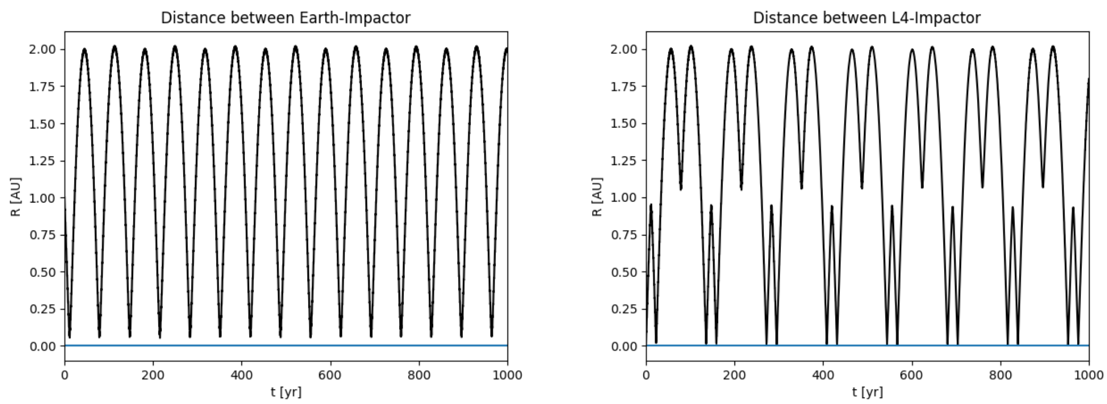

Figure 1: Earth-impactor distance (left) and L_4-impactor distance (right) for $\alpha = 0.005$, as functions of time. The mass increases by 0.01 every 100 yr. The blue horizontal line in the left plot is not indicating zero distance, but Earth's radius, $R_\oplus = 4.26 \times 10^{-5}$ AU.

By integrating the system for an increasing mass of the small object from 0.02 to 0.12, with an increment step of 0.01, and each mass increment happening every 500 yr, with a time step of 0.01 yr, without introducing the perturbation in velocity, the object stays on L_4. If the integration continues for a very long time (by increasing the total time of each mass increment), the objects' orbit can deviate only a little from L_4.

Next, we introduce the perturbation as described in Eq. (6), but try different values for it. If α is small, for example for $\alpha = 0.005$ the impactor stops being on L_4 all the time, but moves with some periodicity ahead or behind it. When moving behind it it approaches Earth, but not enough for its gravitational influence to disturb its trajectory. In Fig. 1 both the distance between the impactor and the Earth, as well as between the impactor and L_4 for this value of a are plotted, with each mass increment happening for 100 yr.

It seems that the increase of the small body's mass doesn't affect its orbit, which deviates from L_4 only because of the perturbation in its initial velocity. Even though the mass increases every 100 yr, this does not affect the encounter distance with the Earth, or how big is the deviation from L_4.

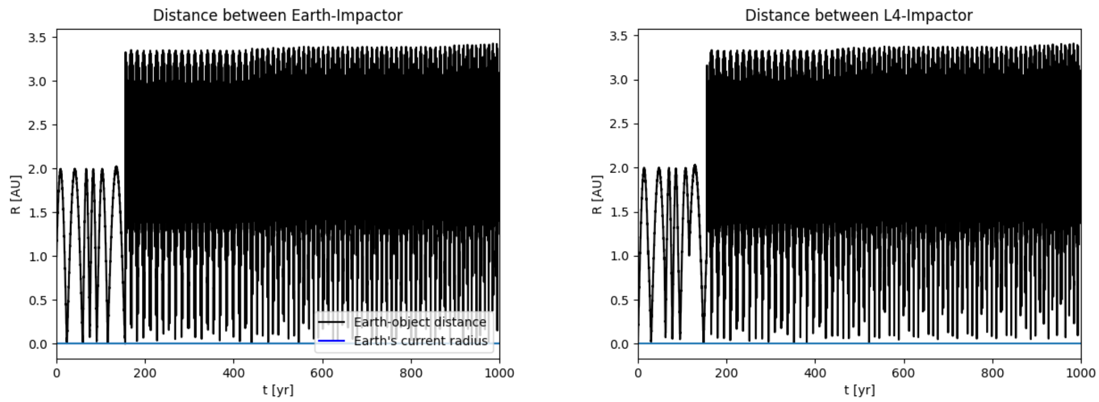

Figure 2: Earth-impactor distance (left) and L_4-impactor distance (right) for $\alpha = -0.0119981$, as functions of time. The object's mass is $0.1\,\mathrm{M}_{\oplus}$. The blue horizontal line in the left plot is indicating the Earth's current radius $R_{\oplus} = 4.26 \times 10^{-5}$ AU and the green one the Moon's current distance from the Earth.

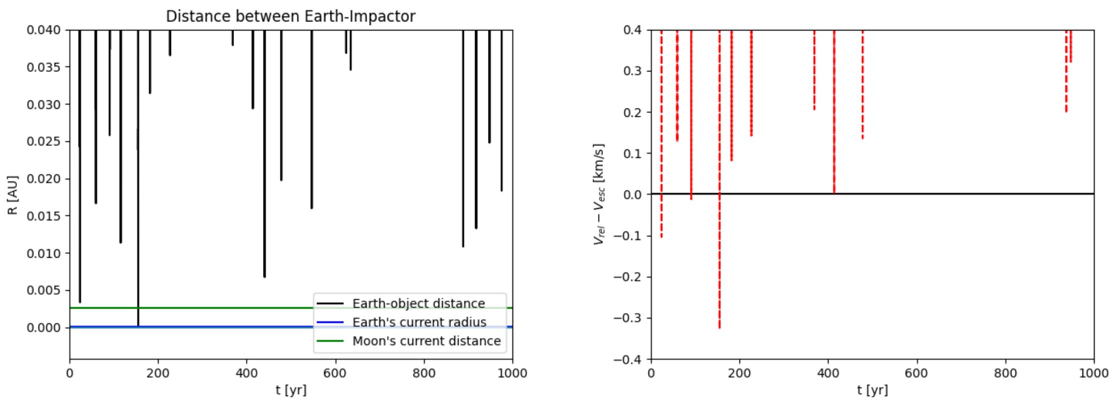

Figure 3: Earth-impactor distance zoomed in close to Eart's radius horizontal line (left) and difference of Earth-impactor relative velocity with Earth's escape velocity (right) for $\alpha = -0.0119981$, as functions of time. The object's mass is $0.1\,\mathrm{M}_{\oplus}$. The blue horizontal line in the left plot is indicating the Earth's current radius $R_{\oplus} = 4.26 \times 10^{-5}$ AU and the green one the Moon's current distance from the Earth.

If we increase the initial velocity perturbation (α), then the impactor passes closer to Earth consecutively. The next thing is to try different values of the perturbation for a constant mass. This mass will be equal to $m_3 = 0.1$, since simulations of the Moon-forming impact suggest that Theia's mass was $\sim 10\% M_{\oplus}$ (Canup 2004). Values of a larger than 0.005 have been tried and also negative values of it. For both positive and negative values, the object's orbit started to deviate from an orbit as the one in Fig. 1, when $|\alpha|$ was greater than 0.01, approximately. One such case is plotted in Fig. 2.

For the trajectory in Fig. 2 a value of $\alpha = -0.01198$ was used, the same as in Belbruno and Gott 2005, with which they found an orbit leading to direct collision with the Earth. In our plot of Earth-impactor distance the Earth's radius is illustrated as a blue horizontal line. There is no direct collision in this case, but the object encounters the Earth multiple times, some of them closer than the current orbital distance of the Moon (green horizontal line). After a very close approach the object ends in orbits far away from L_4, as shown in the right plot of Fig. 2.

In Fig. 3 the left plot of Fig. 2 is zoomed in close to the horizontal lines that indicate Earth's radius and the current Moon-Earth distance (top), as well as the difference between the relative velocity of the object with the Earth and the escape velocity of the Earth at that distance ($v_{esc} = \sqrt{2Gm_1/|\boldsymbol{r_1} - \boldsymbol{r_3}|}$), for each time (bottom). This way we can see that when this difference of velocities crosses the zero line, the object can be captured in an orbit around the Earth.

4 Discussion

By numerically solving the equations of motion of the three-body problem for the Sun, the Earth and a third small object initially located in L_4, we verified that perturbations in the small body's velocity can drive it in close encounters with the Earth, which will perturb its trajectory even more. The mass of the body was found to not affect its deviation from an orbit around L_4.

The limit in magnitude for this perturbation to provide the aforementioned result is found to be around the same as proposed by Belbruno and Gott 2005. This approach is probably very simplistic, as multiple perturbations from many planetesimals gathered around L_4 might have occurred, although the possibility of a migrating giant planet that influenced its orbit has to be considered too. The direction of this perturbation has to be studied more in detail too, as 'pushes' in different directions can lead to very different orbits.

Appendix A

Below is the code used for this project. The code is written in Python and edited in JupyterLab.

```
import numpy as np
import scipy
import matplotlib as mplt
from matplotlib import pyplot as plt
import sympy as sp
from sympy import *
from scipy.optimize import fsolve
from IPython.display import display, Latex
from mpl_toolkits.mplot3d import Axes3D
from matplotlib import animation
from ipywidgets import interact, interactive, fixed, interact_manual
import ipywidgets as widgets

"""Setting three-body system"""

G=1.184E-4 #au^3(M_earth * yr^2)
kms_to_AUyr=0.2108 #velocity convertion from km/s to AU/yr

dt=0.001
tmax=1000 # duration of each mass increment
time=np.arange(0,tmax,dt) #In yr
#
    ------------------------------------------------------------------------

m1=1 #Earth's mass

r1_0=np.array([0, 1, 0]) #Earth's initial position relative to CM
v1_0=np.array([-29.78*kms_to_AUyr, 0, 0]) #Earth's initial velocity

b1_0={'pos':r1_0, 'vel':v1_0, 'mass':m1}
#
    ------------------------------------------------------------------------

m2=333E+3 #Sun's mass in earth masses

r2_0=np.array([0, 0, 0]) #Sun's initial position and velocity relative to
    CM
v2_0=np.array([0, 0, 0])

b2_0={'pos':r2_0, 'vel':v2_0, 'mass':m2}
#
    ------------------------------------------------------------------------

m3_list=np.arange(0.02,0.12,0.01)
m3_list=[0.1]
def rot(p):

    p=np.deg2rad(p)
    rotation_p=[[np.cos(p), -np.sin(p), 0],
                [np.sin(p),  np.cos(p), 0],
                [        0,          0, 1]]
    return rotation_p
```

```
r_L4= np.dot(rot(60), [r1_0[0], r1_0[1], r1_0[2]])
r3_0= r_L4
pert=-0.0119981
v3_0= np.dot(rot(60), [v1_0[0], v1_0[1], v1_0[2]])
v3_0= (1+pert)*v3_0
b3_0={'pos':r3_0, 'vel':v3_0, 'mass':m3_list[0]}

"""Three-body problem gravitational acceleration calculation"""

def three_body(b1, b2, b3, G):
    r1=b1['pos']; r2=b2['pos']; r3=b3['pos']
    m1=b1['mass']; m2=b2['mass']; m3=b3['mass']

    r12= np.linalg.norm(r1-r2) #magnitude of r=r1-r2
    r13= np.linalg.norm(r1-r3)
    r23= np.linalg.norm(r2-r3)

    a1= -G*m2*(r1-r2)/r12**3 - G*m3*(r1-r3)/r13**3

    a2= -G*m1*(r2-r1)/r12**3 - G*m3*(r2-r3)/r23**3

    a3= -G*m1*(r3-r1)/r13**3 - G*m2*(r3-r2)/r23**3

    return np.array([a1, a2, a3])

"""Leapfrog for 3 bodies"""

def leapfrog_int(a, v, r, m, G, dt):
    rnew=r+v*dt+1/2*a*dt**2
    b1_lf={'pos':rnew[0], 'mass':m[0]}
    b2_lf={'pos':rnew[1], 'mass':m[1]}
    b3_lf={'pos':rnew[2], 'mass':m[2]}

    anew= three_body(b1_lf, b2_lf, b3_lf, G)

    vnew=v+1/2*(a+anew)*dt

    return np.array([rnew, vnew, anew])


"""Numerical Solver"""

%%time
b1={'mass':m1}; b2={'mass':m2}; b3={'mass':m3_list[0]}
b1['pos']= b1_0['pos']
b2['pos']= b2_0['pos']
b3['pos']= b3_0['pos']

b1['vel']= b1_0['vel']
b2['vel']= b2_0['vel']
b3['vel']= b3_0['vel']
R_rel=np.zeros(shape=(len(m3_list)*len(time)))
r3_L4_rel=np.zeros(shape=(len(m3_list)*len(time)))
a_i_plus_1= three_body(b1, b2, b3, G)

r1_CM_traj= np.zeros(shape=(3,len(m3_list)*len(time)))
r2_CM_traj= np.zeros(shape=(3,len(m3_list)*len(time)))
```

```
r3_CM_traj= np.zeros(shape=(3,len(m3_list)*len(time)))

r_L4_traj=np.zeros(shape=(3,len(m3_list)*len(time)))
r_L4_traj[:,0]=r_L4

h1=np.zeros(shape=(len(m3_list)*len(time)))
h2=np.zeros(shape=(len(m3_list)*len(time)))

h3=np.zeros(shape=(len(m3_list)*len(time)))

V_rel=np.zeros(shape=(len(m3_list)*len(time)))

for j, m3 in enumerate(m3_list):
    for i in range(len(time)):

        if i==0 and j==0:
            r_CM= (m1*b1['pos']+m2*b2['pos']+m3*b3['pos'])/(m1+m2+m3)
            r1_CM= -b1['pos']+r_CM
            r2_CM= -b2['pos']+r_CM
            r3_CM= -b3['pos']+r_CM
            r12_CM=(m1*b1['pos']+m2*b2['pos'])/(m1+m2)
        else:
            new= leapfrog_int(a_i_plus_1, np.array([b1['vel'], b2['vel'],
                 b3['vel']]), np.array([b1['pos'], b2['pos'], b3['pos']])
                , np.array([m1, m2, m3]), G, dt)
            b1['pos']= new[0][0]; b1['vel']= new[1][0]
            b2['pos']= new[0][1]; b2['vel']= new[1][1]
            b3['pos']= new[0][2]; b3['vel']= new[1][2]
            a_i_plus_1=np.array(new[2])

            r_CM=(m1*b1['pos']+m2*b2['pos']+m3*b3['pos'])/(m1+m2+m3)
            r12_CM=(m1*b1['pos']+m2*b2['pos'])/(m1+m2)
        r1_CM= -b1['pos']+r_CM
        r2_CM= -b2['pos']+r_CM
        r3_CM= -b3['pos']+r_CM

        r1_CM_traj[:,i+len(time)*j]= r1_CM
        r2_CM_traj[:,i+len(time)*j]= r2_CM
        r3_CM_traj[:,i+len(time)*j]= r3_CM

        R_rel[i+len(time)*j]= np.linalg.norm(r3_CM-r1_CM)

        L4_new=r12_CM-(np.dot(rot(60), [b1['pos'][0], b1['pos'][1], b1['
            pos'][2]]))
        r_L4_traj[:,i+len(time)*j]= np.array([L4_new[0], L4_new[1],
            L4_new[2]])
        r3_L4_rel[i+len(time)*j]= np.linalg.norm(r3_CM-L4_new)

        r1=[b1['pos'][0], b1['pos'][1]]
        v1=[b1['vel'][0], b1['vel'][1]]
        h1[i+len(time)*j]=np.linalg.norm(np.cross(r1,v1))

        r2=[b2['pos'][0], b2['pos'][1]]
        v2=[b2['vel'][0], b2['vel'][1]]
        h2[i+len(time)*j]=np.linalg.norm(np.cross(r2,v2))

        r3=[b3['pos'][0], b3['pos'][1]]
        v3=[b3['vel'][0], b3['vel'][1]]
```

```
        h3[i+len(time)*j]=np.linalg.norm(np.cross(r3,v3))

        V_rel[i+len(time)*j]= np.linalg.norm(np.matrix(v3)-np.matrix(v1))

"""Plots"""

new_t=np.arange(0,tmax*len(m3_list),dt)
plt.plot(new_t, h1, 'b.',label='L of Earth')
plt.ylabel('L [M_e* AU^2 /yr]')
#plt.xlim([0,300])
plt.xlabel('t [yr]')
plt.title('Angular momentum of Earth')
plt.legend()
plt.show()
plt.plot(new_t, h3, 'r.', label='L of Impactor')
plt.title('Angular momentum of Impactor')

#plt.xlim([0,300])
plt.ylabel('L [M_e* AU^2 /yr]')
plt.xlabel('t [yr]')

plt.legend()
plt.show()
plt.plot(new_t, h2, 'k.', label='L of Sun')
plt.title('Angular momentum of the Sun')

#plt.xlim([0,300])
plt.xlabel('t [yr]')

plt.ylabel('L [M_e* AU^2 /yr]')

plt.legend()
plt.show()
plt.plot(new_t, R_rel,'k-',label='Earth-object distance' )
plt.axline([0,4.26352E-5], slope=0, color='b', label= "Earth's current 
    radius")
#plt.axline([0,0.0025695686589742], slope=0, color='g', label="Moon's
    current distance" )
plt.ylabel('R [AU]')
plt.xlabel('t [yr]')
plt.title('Distance between Earth-Impactor')
plt.axline([0,0], slope=0)
plt.ylim([-4.26352E-3,0.04])
plt.xlim([0,1000])
plt.legend()
plt.savefig('zoom_large_pert_Earth-imp')
plt.show()

plt.plot(new_t, r3_L4_rel,'k-')
#plt.axline([0,4.26352E-5], slope=0)
plt.ylabel('R [AU]')
plt.xlabel('t [yr]')
plt.title('Distance between L4-Impactor')
plt.axline([0,0], slope=0)
plt.xlim([0,1000])
plt.savefig('large_pert_L4-imp')

ve=np.sqrt(2*G*m1/R_rel)
```

```
plt.axline([0,0], slope=0, color='k')
plt.plot(new_t, (V_rel-ve)/kms_to_AUyr, 'r--')
plt.ylim([-0.4,0.4])
plt.xlim([0,1000])
plt.xlabel('t [yr]')
plt.ylabel(r'$V_{rel}-V_{esc}$'+r' [km/s]')
plt.savefig('large_pert_vel')
```

References

Edward Belbruno and J. Richard Gott. Where did the moon come from? *The Astronomical Journal*, 129(3):1724–1745, mar 2005. doi: 10.1086/427539.

Robin M. Canup. Simulations of a late lunar-forming impact. , 168(2):433–456, April 2004. doi: 10.1016/j.icarus.2003.09.028.