$\overline{III}$ $x-3=-2 \Rightarrow x=1,\ x=-6$ (contradictie)

$\overline{IV}$ $x-3=2 \Rightarrow x=5,\ x-3=6,\ x=9$ $(5,9)$

$\overline{V}$ $x-3=3 \Rightarrow x=6,\ x-3=4,\ x=7$ $(6,7)$

$\Rightarrow |M| = 50-3 = 47$

Mulțimi și funcții

19.10.2022

Curs 2

PROP: $A \underset{f'}{\overset{f}{\rightrightarrows}} B \underset{g'}{\overset{g}{\rightrightarrows}} C$,

A, B, C mulțimi

f, f', g funcții

Atunci: 1) f, g inj $\Rightarrow g \circ f$ inj;

2) f, g surj $\Rightarrow g \circ f$ surj

3) ~~$g \circ f$~~ f, g bij $\Rightarrow g \circ f$ bij

4) $g \circ f$ inj $\Rightarrow f$ inj 5) $g \circ f$ surj $\Rightarrow g$ surj

6) $\left.\begin{array}{l} g \text{ ~~surj~~ inj} \\ g \circ f = g \circ f' \end{array}\right\} \Rightarrow f = f'$

7) $\left.\begin{array}{l} f \text{ surj.} \\ g \circ f = g' \circ f \end{array}\right| \Rightarrow g = g'$

Dem: 4) Def $X \xrightarrow{h} Y$

$X' \xrightarrow{h'} Y'$

$$h = h' \Leftrightarrow \begin{cases} X = X' \\ Y = Y' \\ \forall x \in X,\ h(x) = h'(x) \end{cases}$$

Fie $x, y \in A$ a.î. $f(x) = f(y) \in B$

$\Rightarrow g(f(x)) = g(f(y)) \Leftrightarrow (g \circ f)(x) = (g \circ f)(y)$

$\Downarrow g \circ f$ inj

Deci f inj $x = y$

5) $\forall z \in C\ (\exists)\ x \in A$ a.î. $(g \circ f)(x) = z$

$\Leftrightarrow g(f(x)) = z \left. \begin{array}{l} (\forall)\ z \in C\ (\exists)\ f(x) \in B \\ \text{a.î. } g(f(x)) = z \end{array}\right\} \Leftrightarrow g$ surj

6) $(\forall)\ x \in A\ f(x) = f'(x)$

Fie $x \in A$ arbitrar $\Rightarrow (g \circ f)(x) = (g \circ f')(x)$

$g(f(x)) = g(f'(x)) \underset{\text{inj}}{\overset{''}{\Leftrightarrow}} f(x) = f'(x)$

Temă 7)

Def: Fie A mulțime $id_A = 1_A = \mathbb{1}_A : A \to A$

$id_A(x) = x, (\forall) x \in A$

Def: Fie $A \xrightarrow{f} B$ s.n. funcție inversabilă $\Leftrightarrow (\exists)\ g : B \to A$

$$A \xrightarrow{f} B \xrightarrow{g} A \xrightarrow{f} B$$

$g \circ f = 1_A \quad f \circ g = 1_A$

a.î. $g \circ f = 1_A \Rightarrow \begin{cases} f \text{ inj} \\ g \text{ surj} \end{cases}$

$f \circ g = 1_A \Rightarrow \begin{cases} g \text{ inj} \\ f \text{ surj} \end{cases}$

Teoremă: $f : A \to B$ este bijectivă dc. și numai dacă este inversabilă

OBS: g este unică

Dem „$\Leftarrow$"

OBS id_X este bij $(\forall) X$

$g \circ f = 1_A$ bij implică inj $\overset{PROP}{\Rightarrow}$ f inj, g surj

$f \circ g = 1_B$ $\overset{PROP}{\Rightarrow}$ f surj, g inj

$\Rightarrow f, g$ bij

„$\Rightarrow$" f bij $\Leftrightarrow \forall y \in B\ (\exists)!\ x \in A$ a.î. $f(x) = y$

unic

Def: $g : B \to A \quad g(y) = x$

$(g \circ f)(x) = g(f(x)) = g(y) = x \Rightarrow g \circ f = 1_A$

$(f \circ g)(y) = f(g(y)) = f(x) = y \Rightarrow f \circ g = 1_B$

Exemplu:

$\mathbb{R} \xrightarrow{exp} (0, +\infty)$

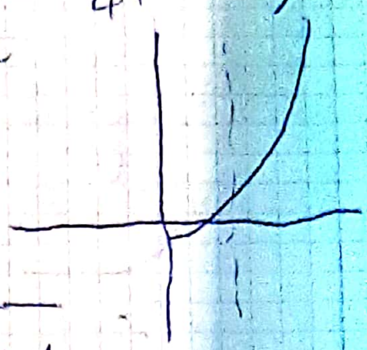
$exp(x) = e^x \quad x \to e^x$

$\ln : (0, +\infty) \to \mathbb{R}, \ y \to \ln y$

$\ln(e^x) = x$ ② $f(x) = x^2 - 3x + 2 = (x-1)(x-2)$

$e^{\ln y} = y$ $f : [\frac{3}{2}, +\infty) \to [-\frac{1}{4}, +\infty)$

$x = \frac{3}{2}$ este axa de simetrie



$g = f^{-1} : [-\frac{1}{4}, +\infty) \to [\frac{3}{2}, +\infty)$

$x^2 - 3x + 2 - y = 0$

$x_{1,2} = \frac{3 \pm \sqrt{1+4y}}{2}$, $x = \frac{3 + \sqrt{1+4y}}{2}$

$g(y) = \frac{3 + \sqrt{1+4y}}{2}$

$g(x) = \frac{3 + \sqrt{1+4x}}{2}$

(T) Dem. $f \circ g = id_{[-\frac{1}{4}, +\infty)}$

Imaginea directă și imaginea (preimaginea) inversă a unei funcții

Fie $X \subseteq A$, $Y \subseteq B$

Def. Img. directă a lui X prin f

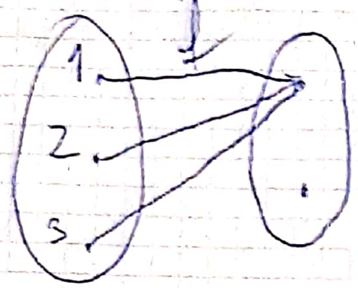
$f(X) = \{ y \in B \mid (\exists) x \in X$ a.î. $f(x) = y \}$

$\cancel{f(x) = y} = \{ f(x) \mid x \in X \}$

Def. $\operatorname{Im} f = f(A)$ = imag. mulț. A prin f (imaginea lui f)

Def. Preimaginea unei submulțimi din B (imaginea inversă) $f^{-1}(Y) = \{ x \in A \mid f(x) \in Y \}$

Exemplu



$A \xrightarrow{f} B$

$\{1,2,3\} \quad \{a,b\}$

$f(1)=f(2)=f(3)=a$

$\operatorname{Im} f = f(\{1,2,3\}) = \{a\} \subset B$

$f^{-1}(\{a\}) = \{1,2,3\} = f^{-1}(B)$

$f^{-1}(\{b\}) = \emptyset$

PROP: Fie $f: A \to B$; $X_1, X_2 \subseteq A$; $Y_1, Y_2 \subseteq B$

1) $X_1 \subseteq X_2 \subseteq A \Rightarrow f(X_1) \subseteq f(X_2) \subseteq B$

2) $Y_1 \subseteq Y_2 \subseteq B \Rightarrow f^{-1}(Y_1) \subseteq f^{-1}(Y_2) \subseteq A$

3) $Y_1, Y_2 \subseteq B \Rightarrow f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$

4) —//— $\Rightarrow f^{-1}(Y_1) \cup f^{-1}(Y_2)$

5) $X_1, X_2 \subseteq A \Rightarrow f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$

6) —//— $\Rightarrow f(X_1 \cap X_2) \subseteq f(X_1) \cap f(X_2)$

Pt. f inj avem $f(X_1 \cap X_2) \supseteq f(X_1) \cap f(X_2)$

Deci pt. f inj avem "="

Dem. 6) Exemplu "$\supseteq$" NU este adevărată.

$f: \{1,2,3\} \to \{a,b\}$ $f(1)=f(2)=f(3)=a$

$X_1 = \{1,2\}$; $X_2 = \{3\}$ & $X_1 \cap X_2 = \emptyset$

$\Rightarrow f(\emptyset) = \emptyset$

$f(X_1) = \{a\}$, $f(X_2) = \{a\} \neq \emptyset$

6) $X_1 \cap X_2 \subseteq X_1 \Rightarrow f(X_1 \cap X_2) \subseteq f(X_1)$
$\subseteq X_2 \quad f(X_1 \cap X_2) \subseteq f(X_2)$

Dem: f inj

Fie $y \in f(X_1) \cap f(X_2) \Leftrightarrow y \in f(X_1) \wedge y \in f(X_2) \Leftrightarrow (\exists)\ x_1 \in X_1$ a.î. $f(x_1) = y \wedge (\exists)\ x_2 \in X_2$ a.î. $f(x_2) = y$

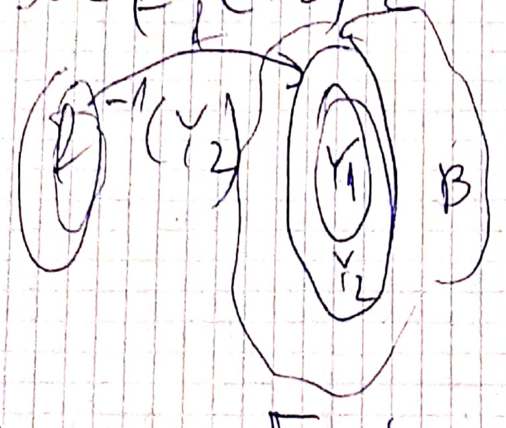
$f(x_1) = f(x_2) = (y) \xRightarrow{f \text{ inj}} x_1 = x_2$

$\underset{X_1}{x_1} = \underset{X_2}{x_2} = x \in X_1 \cap X_2$ și $f(x) = y \Rightarrow y \in f(X_1 \cap X_2)$

7) $f^{-1}(f(X)) \supseteq X$ cu egalitate pt. f inj.

8) $f(f^{-1}(Y)) \supseteq Y$ cu egalitate pt. f surj.

Dem Fie $x \in f^{-1}(Y_1 \cap Y_2) \Leftrightarrow f(x) \in Y_1 \cap Y_2 \Leftrightarrow f(x) \in Y_1 \wedge f(x) \in Y_2 \Leftrightarrow x \in f^{-1}(Y_1) \wedge x \in f^{-1}(Y_2) \Leftrightarrow x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$



Teoremă: Fie A mulțime, U.a.s.e.
1) $|A| < \infty$
2) $(\forall)$ $f: A \to A$ injectivă $\Rightarrow$ f surj (bij)

3) ($\forall$) $f: A \to A$ surj $\Rightarrow$ f injectivă (bij)

Fie $A = \mathbb{N} = \{0, 1, 2, \dots\}$

$f: \mathbb{N} \to \mathbb{N}$ $\quad f(x) = x+1$

$g: \mathbb{N} \to \mathbb{N}$ $\quad g(x) = \begin{cases} 0 & x = 0 \\ x-1 & x \geq 1 \end{cases}$

$g(0) = g(1) = 0$
$\Rightarrow$ nu este injectivă

$(g \circ f)(x) = x$, $(\forall) x \in \mathbb{N}$

$(g \circ f) = \underset{bij}{id_{\mathbb{N}}}$

$\Rightarrow f$ inj, nu este surj $0 \notin Im(f) = f(\mathbb{N})$

g surj.

<u>Def</u>: Două mulț. s.n. <u>echipotente</u> dc. și numai dcă $\exists f: A \to B$ bij.

Notație $A \simeq B$

<u>OBS</u> 1) $A \simeq A$ $\exists$ $id_A : A \to A$ bij (reflex)

2) $A \simeq B$ atunci $B \simeq A$ (~~tranzitivitate~~) (simetrie)

$A \underset{bij}{\xrightarrow{f}} B$ atunci $g = f^{-1}: B \to A$
g bij

3) $A \underset{\substack{f \\ bij}}{\simeq} B \underset{\substack{g \\ bij}}{\simeq} C \Rightarrow A \underset{\substack{g \circ f \\ bij}}{\simeq} C$ (tranzitivitate)

$\mathbb{N}$ = mulțime numărabilă - putem număra elt.

Def. A s.n. numărabilă dacă și numai dacă $A \simeq \mathbb{N}$

$\exists f: \mathbb{N} \to A$ bij

$$A = \{\underset{f(0)}{\underset{\parallel}{a_0}}, \underset{f(1)}{\underset{\parallel}{a_1}}, \underset{f(2)}{\underset{\parallel}{a_2}}, a_3, \dots\}$$

PROP: fie A, B mulțimi numărabile $\Rightarrow A \cup B$ este numărabilă

Dem: $A = \{a_0, a_1, a_2, \dots\}$

$B = \{b_0, b_1, b_2, \dots\}$

$$A \cap B = \{a_0, b_0, a_1, b_1, \dots\dots\}$$

COROLARE = consecință

Fie $A_1, \dots, A_k$ mult. numărabile $\Rightarrow A_1 \cup \dots A_k$ numărabil

Exemplu: $\mathbb{N}$

$\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \dots\}$

OBS: Orice submulțime inf. a unei mult. numărabile este numărabilă.

$A = \{a_0, a_1, a_2, \dots\}$

$B = \{b_{i_0}, \dots\dots\}$

PROP: $\mathbb{N} \times \mathbb{N}$ numărabilă

$\{(a, b) \mid a, b \in \mathbb{N}\}$

Dem. $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$f(a, b) = 2^a(2b+1) - 1$$

Temă inj

f este bij

$f(a_1, b_1) = f(a_2, b_2)$

$\Rightarrow 2^{a_1}(2b_1+1) - 1 = 2^{a_2}(2b_2+1) - 1$

$\Rightarrow \underbrace{2b_1+1}_{\text{impar}} = 2^{a_2-a_1}(2b_2+1)$

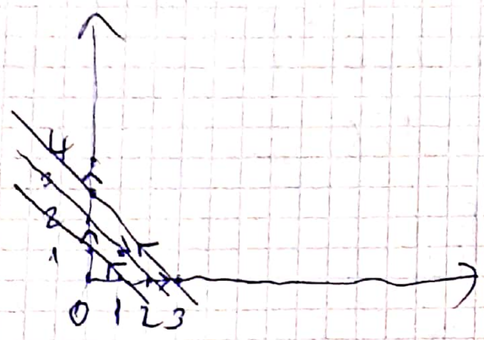
$\Downarrow$

$2^{a_1-a_2} = 1 \Rightarrow a_1 = a_2$

$\Rightarrow 2b_1+1 = 2b_2+1 \Rightarrow b_1 = b_2$

$\Rightarrow (a_1, b_1) = (a_2, b_2)$, f inj

Gen: $\mathbb{N}\times\mathbb{N} = \{(0,0), (1,0), (0,1), (0,2), (1,1), (2,0), (3,0), (2,1), \dots\}$



PROP: Dc. A, B numărabile atunci $A\times B$ este numărabilă.

Def: $A \xrightarrow{f} \mathbb{N}$ bij

$B \xrightarrow{g} \mathbb{N}$

$\mathbb{N}\times\mathbb{N} \to 2^{a}(2b+1)-1$

$(x, y) \longmapsto (f(x), g(y))$

Teorema 1 (Cantor)

$\mathbb{R}$ este o submulțime nenumărabilă

Teorema 2 (Cantor)

Dc. $(\forall)$ A mulțime

$(\not\exists)$ $f : A \to \mathcal{P}(A)$ surjectivă

părțile lui A