1.

20.10.2022

Seminar 3

1) $\rho = \{(\frac{a}{b}, \frac{a+1}{b}) \mid a, b \in \mathbb{Z}, b \neq 0\}$

Este ρ funcție?

Nu, avem perechea $(\frac{1}{2}, \frac{2}{2}) \in \rho$ dar și perechea $(\frac{2}{4}, \frac{3}{4}) \in \rho$

$\frac{1}{2} = \frac{2}{4}$ și

2) $\rho = \{(\frac{a}{b}, \frac{a+1}{b}) \mid a, b \in \mathbb{Z}, b \neq 0, (a,b)=1\}$

Este ρ funcție?

Nu, deoarece perechea $(\frac{1}{2}, \frac{2}{2}) \in \rho$, dar și perechea $(\frac{-1}{-2}, \frac{-1+1}{-2}) \in \rho$

$\frac{-1}{-2} = \frac{1}{2}$ și $0 \neq 1$

3) $\rho = \{(\frac{a}{b}, \frac{a+1}{b}) \mid a \in \mathbb{Z}, b \in \mathbb{N}^*, (a,b) = 1\}$

Este ρ funcție

P.p. $\frac{a}{b} = \frac{c}{d} = x$, $a, b, c, d \in \mathbb{Z}^*$
$b, d \in \mathbb{N}^*$

$(a, b) = 1$

$(a', b') = 1$

$\Rightarrow \begin{cases} a = a' \\ b = b' \end{cases} \Rightarrow \frac{a+1}{b} = \frac{a'+1}{b'}$ $\quad ab' = a'b$

$(a, b) = 1$

$\Rightarrow a \mid a'$

$a \mid a'$ (1)

$ab' = a'b \Rightarrow \left.\begin{matrix} a' \mid ab' \\ (a', b') = 1 \end{matrix}\right| \Rightarrow a' \mid a$ (2)

Din (1) și (2) $\Rightarrow \left.\begin{matrix} a = \pm a' \\ b, b' \in \mathbb{N}^* \end{matrix}\right| \Rightarrow a \cdot a' > 0 \Rightarrow a = a'$

$\left.\begin{matrix} a = a' \\ \frac{a}{b} = \frac{a'}{b'} \end{matrix}\right| \overset{a \neq 0}{\Rightarrow} b = b'$

Dacă $a = a' = 0 \Rightarrow \frac{0}{b} = \frac{0}{b'}, \forall b, b' \in \mathbb{N}^*$

f inj?

Fie $f : \mathbb{Q} \to \mathbb{Q}$, $f\left(\frac{a}{b}\right) = \frac{a+1}{b}$, $\forall a \in \mathbb{Z}$, $b \in \mathbb{N}^*$, $(a, b) = 1$

Este f inj, dar surj.

Fie $a, a' \in \mathbb{Z}$, $b, b' \in \mathbb{N}^*$ a.î. $(a, b) = 1$, $(a', b') = 1$

și $f\left(\frac{a}{b}\right) = f\left(\frac{a'}{b'}\right) \Leftrightarrow \frac{a}{b} = \frac{a'}{b'}$

$\Rightarrow \frac{a+1}{b} = \frac{a'+1}{b'}$ $\quad ab' + b' = a'b + b$

$(a+1)b' = (a'+1)b$

$2 \cdot 3 \quad 3 \cdot 2$

$a = 1, b' = 3, a' = 2, b = 2$

$f\left(\frac{1}{2}\right) = f\left(\frac{2}{3}\right) \Rightarrow f$ nu e inj.

~~$x(x-1)$~~

$\frac{3}{4} = \frac{a+1}{b}$, $a \in \mathbb{Z}^*$, $b \in \mathbb{N}^*$, $(a, b) = 1$

$\frac{a+1}{b} = y \overset{b \neq 0}{\Rightarrow} y \cdot b = a + 1 \in \mathbb{Z}^*$, $b \in \mathbb{N}^*$

$\frac{c}{d} \cdot b = a+1$

$c \cdot b = (a+1)d$

~~$\frac{m}{n} = \frac{a+1}{b}$, $m \in \mathbb{Z}, n \in \mathbb{N}^*, a \in \mathbb{Z}, b \in \mathbb{N}^*$~~

$\forall m \in \mathbb{Z}, n \in \mathbb{N}^*, (m,n)=1, \exists a \in \mathbb{Z}, b \in \mathbb{N}^*$ cu $(a,b)=1$ a.î. $\frac{m}{n} = \frac{a+1}{b}$

$\left.\begin{array}{l} mb = (a+1)n \\ (m,n)=1 \end{array}\right| \Rightarrow m \mid a+1 \Rightarrow \exists\, k \in \mathbb{Z}$ a.î. $a = km-1$

$mb = kmn$

$m=0 \Rightarrow n=1$ $\quad \frac{0}{1} = \frac{-1+1}{1}$

$m \neq 0 \Rightarrow b = kn$

$\exists k \in \mathbb{Z}$ a.î. $(km-1, kn)=1$

$k = mn$

$(m^2n-1, mn^2)=1$

Dacă $\exists p$ prim a.î. $\begin{cases} p \mid km-1 \\ p \mid kn \end{cases} \Leftrightarrow$

$\begin{cases} p \mid m^2n-1 \\ p \mid n^2m \end{cases}$ $\quad p \mid n$ sau $p \mid m$

$\Rightarrow p \mid mn$

$mn \mid m^2n$

$\Rightarrow \text{\}}$ $p \mid 1$

$\emptyset \neq M \supseteq A, B$

2) $f : \mathcal{P}(M) \to \mathcal{P}(A) \times \mathcal{P}(B)$

$f(X) = (X \cap A, X \cap B)$

i) f inj $\Leftrightarrow A \cup B = M$

ii) f surj. $\Leftrightarrow A \cap B = \emptyset$

Sol $f:\mathcal{P}(M)\to\mathcal{P}(A)\times\mathcal{P}(B)$

i) ~~...~~ f i

ii) f = surj $\Leftrightarrow A\cap B=\emptyset$

"$\Rightarrow$"

$\forall Y, Z,\ Y\subseteq A, Z\subseteq B\ \exists X\subseteq M$ a.î.

$$f(X)=(Y,Z)$$

$$(Y,Z)=(X\cap A, X\cap B)$$

$$Y=X\cap A$$
$$Z=X\cap B$$

"$\Leftarrow$" $A\cap B=\emptyset \Leftarrow f$ surj.

$X\subset M$

(MRA) $A\cap B\neq\emptyset \Rightarrow \exists x\in A\cap B$

$(\emptyset,\{x\})\in\mathcal{P}(A)\,\mathcal{P}(B)$

f surj $\Rightarrow \exists X\subseteq M$ a.î.

$f(X)=(\emptyset,\{x\}) \Leftrightarrow \begin{cases} X\cap A=\emptyset \\ X\cap B=\{x\}\end{cases}$

$\left.\begin{array}{l} x\in X \\ x\notin A \\ x\in A\cap B\end{array}\right\} \Rightarrow$ ↯

iii) "$\Leftarrow$" $A\cap B=\emptyset \Rightarrow f$ surj

Fie $X\subseteq A \wedge Y\subseteq B$, $X, Y\subseteq M \wedge Z=X\cup Y\subseteq M$

$f(Z)=(Z\cap A, Z\cap B)=((X\cup Y)\cap A, (X\cup Y)\cap B)=((X\cap A)\cup(Y\cap A), (X\cap B)\cup(Y\cap B))$
$=(X,Y)$

~~iii)~~

$f : \mathbb{N} \to [0,1)$

$f(n) = \{2^n \cdot \sqrt{3}\}, \forall n \in \mathbb{N}$

Este f injectivă?

Fie $m, n \in \mathbb{N}$, $f(m) = f(n) \Leftrightarrow \{2^m\sqrt{3}\} = \{2^n\sqrt{3}\}$

$$2^m\sqrt{3} - [2^m\sqrt{3}] = 2^n\sqrt{3} - [2^n\sqrt{3}]$$

$$\underset{\substack{\in \mathbb{R}\setminus\mathbb{Q} \\ \text{dacă} \\ m \neq n}}{(2^m - 2^n)\sqrt{3}} = \underbrace{[2^m\sqrt{3}] - [2^n\sqrt{3}]}_{\in \mathbb{Z}}$$

$m \neq n \Rightarrow$ ↯

$m = n \Rightarrow f$ injectivă

Este surj? NU $\quad 0 \notin \operatorname{Im} f$