<u>Seminar 1</u>

1. Fie $x_n = \frac{1}{n}$, $\forall n \in \mathbb{N}^*$. Arătați folosind def., că limita este 0.

Sol: $\lim\limits_{n \to +\infty} x_n = 0 \iff \forall \varepsilon > 0\ \exists n_\varepsilon \in \mathbb{N}^*$ a.î. $n \geq n_\varepsilon$ avem

$$|x_n - 0| < \varepsilon$$

Fie $\varepsilon > 0$

Căutăm $n_\varepsilon \in \mathbb{N}^*$ a.î. $\forall n \geq n_\varepsilon$ avem $|x_n - 0| < \varepsilon$

$$|x_n - 0| = |x_n| = \left|\frac{1}{n}\right| \underset{n \in \mathbb{N}^*}{=} \frac{1}{n}$$

$\frac{1}{n} < \varepsilon \,/\, (\cdot)^{-1} \iff n_\varepsilon > \frac{1}{\varepsilon}$ Alegem $n_\varepsilon = \left[\frac{1}{\varepsilon}\right] + 1 \in \mathbb{N}^*$

$$n \geq n_\varepsilon \Rightarrow \frac{1}{n} < \varepsilon$$

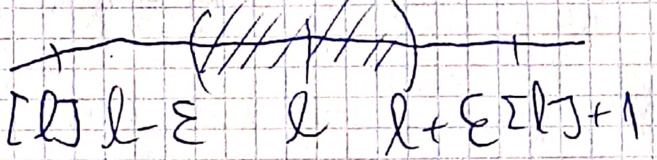
Avem $n_\varepsilon > \frac{1}{\varepsilon}$ și $\forall n \geq n_\varepsilon$ rezultă că $n > \frac{1}{\varepsilon}$

Deci $\forall n \geq n_\varepsilon$ avem $|x_n - 0| < \varepsilon$, i.e. $\lim\limits_{n} x_n = 0$

2. Fie $(x_n)_n \subset \mathbb{Z}$ ~~a~~ și $l \in \mathbb{R}$ a.î. $\lim\limits_{n \to \infty} x_n = l$

Arătați că $l \in \mathbb{Z}$. $\forall \varepsilon > 0, \exists\, n_\varepsilon \in \mathbb{N}$

<u>Sol</u>:



$l \notin \mathbb{Z}$

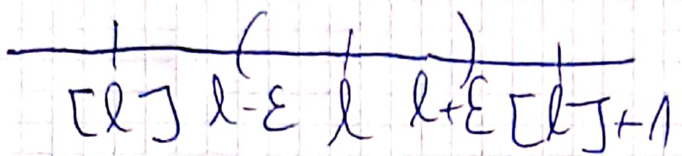
$\lim\limits_{n\to\infty} x_n = l \iff \forall \varepsilon > 0\ \exists\, n_\varepsilon \in \mathbb{N}^*$ a.î.
$\forall n \geq n_\varepsilon,\ |x_n - l| < \varepsilon$ (știm)

(MRA) $l \notin \mathbb{Z}$

$|x_n - l| < \varepsilon \iff -\varepsilon < x_n - l < \varepsilon$

$-\varepsilon + l < x_n < \varepsilon + l$

$x_n \in (l - \varepsilon, l + \varepsilon)$



Alegem $\varepsilon > 0$ a.î. $[l] < l - \varepsilon$, $l + \varepsilon < [l]+1$

Un astfel de ε există deoarece $l - [l] > 0$ ($l \notin \mathbb{Z}$)
și $[l] + 1 - l > 0$

~~Sol~~: $[l] < l - \varepsilon \iff \varepsilon < l - [l]$

$\cancel{\varepsilon < l}$ $\quad l + \varepsilon < [l] + 1 \iff$

Putem alege $\varepsilon \in (0; \min(l - [l], [l] + 1 - l))$

Pt. acest ε $\exists\, n_\varepsilon \in \mathbb{N}$ a.î. $\forall n \geq n_\varepsilon$ avem
$x_n \in (l - \varepsilon, l + \varepsilon)$, dar $x_n \in \mathbb{Z}$ pt oricare $n \in \mathbb{N}$
și $(l - \varepsilon, l + \varepsilon) \cap \mathbb{Z} = \emptyset$ contradicție

Prin urmare, $l \in \mathbb{Z}$

Criteriul raportului pt șiruri cu termeni strict pozitivi

Fie $(x_n)_n \subset (0, +\infty)$ a.î. $\exists \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} \overset{not}{=} l$
$\in [0, \infty] = [0, \infty) \cup \{\infty\}$

1) Dacă $l < 1$, atunci $\lim\limits_{n\to\infty} x_n = 0$

2) Dacă $l > 1$, atunci $\lim\limits_{n\to\infty} x_n = \infty$

3) Dacă $l=1$, atunci aceasta nu decide

3. Fie $a>0$. Det. $\lim_{n\to\infty} n \cdot a^n$

Sol: Fie $x_n = n \cdot a^n$, $n \in \mathbb{N}^*$

Apl. crit. rap. pt. $(x_n)_n$:

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot a = a$$

I Dacă $a>1 \Rightarrow \lim_n x_n = \infty$

II Dacă $a<1 \Rightarrow \lim_n x_n = 0$

III. Dacă $a=1$, nu se poate decide

$a=1 \Rightarrow \lim_{n\to\infty} x_n = \lim_{n\to\infty} n = +\infty$

Am obținut $\lim_{n\to\infty} x_n = \begin{cases} 0, & a \in (0,1) \\ +\infty, & a \in [1,+\infty) \end{cases}$

<u>Criteriul radicalului pt. șiruri cu termeni pozitivi</u>

Fie $(x_n)_n \subset [0,+\infty)$ a.î. $\exists \lim_{n\to\infty} \sqrt[n]{x_n} \overset{not}{=} l \in [0,+\infty)$

1) Dacă $l<1$, atunci $\lim_{n\to\infty} x_n = 0$

2) Dacă $l>1$, atunci $\lim_{n\to\infty} x_n = +\infty$

3) Dacă $l=1$, atunci crit. dat nu se aplică

4. Fie $a, b \in (0,+\infty)$. Det. $\lim_{n\to\infty} \left(\frac{an^2+5n+1}{bn^2+3n-1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{x_n} = \lim_{n\to\infty}\frac{an^2+5n+3}{bn^2+3n+1} = \frac{a}{b}$$

1) Dacă $\frac{a}{b} < 1 \Rightarrow \lim\limits_{n\to\infty} x_n = 0$

2) Dacă $\frac{a}{b} > 1 \Rightarrow \lim\limits_{n\to\infty} x_n = \infty$

3) Dacă $\frac{a}{b} = 1$:

$$\Rightarrow x_n = \left(\frac{an^2+5n+3}{\cancel{bn}\,an^2+3n+1}\right)^n$$

$$= \left(1 + \frac{2n+2}{an^2+3n+1}\right)^n$$

$$= \left(1 + \frac{2n+2}{an^2+3n+1}\right)^{\frac{2(an^2+3n+1)(n+1)}{n}\cdot\frac{2n+2}{an^2+3n+1}}$$

$$\Rightarrow \lim_{x_n\to\infty} x_n = e^{\lim\limits_{n\to\infty}\frac{1}{\frac{(an^2+3n+1)\cancel{(n+1)}}{2n(n+1)}}} = \cancel{+\infty}\,\frac{1}{a}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(1 + \frac{an^2+5n+3}{an^2+3n+1} - 1\right)^n$$

$$\lim_{n\to\infty} x_n = \begin{cases} 0, & a<b \\ \infty, & a>b \\ e^{\frac{2}{a}}, & a=b \end{cases}$$

<u>Prop</u> Fie $(x_n)_n \subset (0,\infty)$ a.î. $\exists \lim\limits_{n\to\infty}\frac{x_{n+1}}{x_n} \overset{not}{=} l \in [0,\infty]$

Atunci $\exists \lim\limits_{n\to\infty}\sqrt[n]{x_n} = l$

5. Det. $\lim\limits_{n\to\infty}\sqrt[n]{n}$

<u>Sol</u>: Fie $(x_n)_n$, $x_n = n$, $\forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{n+1}{n} = 1 = l \Rightarrow \exists \lim_{n\to\infty} \sqrt[n]{x_n} = l = 1$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{n} = 1$$

$\uparrow$ Prop.

6. Fie $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n \ \forall n \in \mathbb{N}$

Sol: Arătați că $(x_n)_n$ este monotonă și mărginit

Monotonie

Fie $n \in \mathbb{N}^*$ $x_{n+1} - x_n = \frac{1}{n+1} + \ln n - \ln(n+1)$

$= \frac{1}{n+1} - (\ln(n+1) - \ln n)$

$= \frac{1}{n+1} - \ln\frac{n+1}{n}$

$\frac{1}{n+1} > \ln\frac{n+1}{n}$

$\ln\left(\frac{n+1}{n}\right)^{n+1} > 0$

Fie ~~$f(n)$~~ $f_n : [n, n+1] \to \mathbb{R}$

$f_n(x) = \ln x$

f_n cont. pe $[n, n+1]$

f_n deriv. pe $(n, n+1)$

$\Rightarrow \exists c_n \in (n, n+1)$ (Lagrange)

a.î. $f_n'(c_n) = \frac{f_n(n+1) - f_n(n)}{n+1-n} \Rightarrow \exists c_n \in (n, n+1)$

~~a.î. $f_n'(c_n) =$~~

$c_n \in (n, n+1) \Rightarrow n < c_n < n+1$

$\Rightarrow \frac{1}{n+1} < \frac{1}{c_n} < \frac{1}{n}$

$x_{n+1} - x_n = \frac{1}{n+1} - \frac{1}{c_n} < 0$

$\Rightarrow (x_n)_n$ str. $\searrow$ (1)

Mărginire: Cont. voi! $\square$

Cu Lagrange