Seminar 4

1. a) Arătați că $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

b) Stud. conv. seriei $\sum_{n=1}^{\infty} \left(1-\cos\frac{1}{n}\right) x^n$, $x>0$

Sol:

a) $\lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}$.

$\lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}$

b) Fie $x_n = \left(1-\cos\frac{1}{n}\right) x^n$, $\forall n \in \mathbb{N}^*$

Apl. crit. rap.

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\left(1-\cos\frac{1}{n+1}\right)\cdot x}{1-\cos\frac{1}{n}} = \lim_{n\to\infty} \frac{1-\cos\frac{1}{n+1}}{\left(\frac{1}{n+1}\right)^2}$$

$$= \lim_{n\to\infty} x\cdot \frac{\frac{1-\cos\frac{1}{n+1}}{\left(\frac{1}{n+1}\right)^2}\cdot\left(\frac{1}{n+1}\right)^2}{\frac{1-\cos\frac{1}{n}}{\left(\frac{1}{n}\right)^2}\cdot\left(\frac{1}{n}\right)^2} = x\cdot\lim_{n\to\infty} \frac{\frac{1-\cos\frac{1}{n+1}}{\frac{1}{(n+1)^2}}}{\frac{1-\cos\frac{1}{n}}{\left(\frac{1}{n}\right)^2}}\cdot\frac{n^2}{(n+1)^2}$$

$= x$

1) Dacă $x<1$, $\sum_n x_n$ conv.

2) Dacă $x>1$, $\sum_n x_n$ diverg.

3) Pt. $x=1$, crit. nu decide

$$\sum_{n=1}^{\infty}\left(1-\cos\frac{1}{n}\right)$$

$$x_n = 1-\cos\frac{1}{n}$$

$\lim\limits_{n\to\infty}$ Fie $y_n = \frac{1}{n^2}, \forall n\in\mathbb{N}^*$

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}} \underset{a)}{=} \frac{1}{2}\in\mathbb{R}$$

$$\Rightarrow \sum_n x_n \sim \sum_n y_n$$

Conform crit. de comp. cu limită, avem aceeași natură $\Rightarrow \sum_n x_n \sim \sum_n y_n$

$\sum_n y_n = \sum_n \frac{1}{n^2}$ - conv (serie armonică gen. cu $\alpha = 2$)

Deci $\sum_n x_n$ conv.

2. St. convergența (natura) seriilor de mai jos:

a) $$\sum_{n=1}^{\infty}\frac{\sin\frac{1}{n(n+1)}}{\left(\cos\frac{1}{n}\right)\left(\cos\frac{1}{n+1}\right)}$$

$$\frac{\sin x}{x}\to 1$$

b) $$\sum_{n=1}^{\infty}\frac{x^n}{n^2},\ x\in[-1,1]$$

Studiem conv. SERIEI din enunț

$$x_n = \frac{x^n}{n^2},\ \forall n\in\mathbb{N}^*$$

$$|x_n| = \frac{|x^n|}{n^2} = \frac{|x|^n}{n^2},\ \forall n\in\mathbb{N}^*$$

$$|x_n| \le \frac{1}{n^2},\ \forall n\in\mathbb{N}^*$$

Fie $y_n = \frac{1}{n^2}, \forall n \in \mathbb{N}^*$

$\sum_n y_n = \sum_n \frac{1}{n^2}$ conv (serie armonică generalizată $\lambda = 2$)

Conform crit comp. cu leg. avem că

$\sum_n |x_n| \sim \sum_{n=1} y_n$ conv.

Deci, $\sum_n |x_n|$ conv, i.e. $\sum_n x_n$ abs conv

atunci $\sum_n x_n$ conv.

c) $\sum_n \frac{\cos nx}{n^\lambda}, x \in \mathbb{R}, \lambda > 0$

Sol: $x_n = \frac{1}{n^\lambda}, y_n = \cos(nx)$

$|y_1 + \dots + y_n| \leq M$

$z = \cos x + i \sin x$

Sol: Apl. crit. Abel-Dirichlet (I)

Fie $x_n = \frac{1}{n^\lambda}, \forall n \in \mathbb{N}^*$

Fie $y_n = \cos nx, \forall n \in \mathbb{N}^*$

$(x_n)_n$ descrescătoare și $\lim_{n\to\infty} x_n = 0$ (1)

? $\exists M > 0$ a.î. $\forall n \in \mathbb{N}^*$ să avem

$|y_1 + \dots + y_n| \leq M$?

OBS. M nu depinde de n, dar poate depinde de x

$|y_1 + \dots + y_n| = |\cos x + \dots + \cos nx| = \cancel{\cos x}$
$\cancel{+ \cos}$

Fie $\cancel{x} = \cos x + i \sin x$

$z^2 = \cos 2x + i\sin 2x$

$z^3 = \cos 3x + i\sin 3x$

$\vdots$

$z^n = \cos nx + i\sin nx$

$|y_1 + \dots + y_n|$

$= |Re(z + z^2 + \dots + z^n)|$

P.p. că $z \neq 1$, i.e. $\cos x + i\sin x \neq 1$

$\Rightarrow x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\}$

$$z + z^2 + \dots + z^n = z \cdot \frac{z^n - 1}{z - 1} = \frac{z^{n+1} - z}{z - 1}$$

$$= \frac{\cos((n+1)x) + i\sin((n+1)x) - \cos x - i\sin x}{\cos x + i\sin x - 1}$$

$$= \frac{\cos((n+1)x) - \cos x + i(\sin((n+1)x) - \sin x)}{(\cos x - 1) + i\sin x}$$

$$= \frac{-2\sin\left(\frac{n}{2}x\right)\cdot\sin\left(\frac{n+2}{2}\cdot x\right) + i\cdot 2\cos\left(\frac{n+2}{2}\cdot x\right)\cdot\sin\frac{nx}{2}}{-2\sin^2\frac{x}{2} + i\cdot 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}$$

$$= \frac{\sin\left(\frac{n}{2}\cdot x\right)}{\sin\frac{x}{2}} \cdot \frac{-\sin\frac{n+2}{2}x + i\cos\left(\frac{n+2}{2}x\right)}{-\sin\frac{x}{2} + i\sin\frac{x}{2}}$$

$$\cos\left(2\cdot\frac{x}{2}\right) = 1 - 2\sin^2\frac{x}{2}$$

$$= \frac{\sin\left(\frac{n}{2}x\right)}{\sin\frac{x}{2}} \cdot \frac{\cos\left(\frac{n+2}{2}x\right) + i\sin\left(\frac{n+2}{2}x\right)}{\cos\frac{x}{2} + i\sin\frac{x}{2}}$$

$$= \frac{\sin\left(\frac{n}{2}\cdot x\right)}{\sin\left(\frac{x}{2}\right)} \cdot \cos\left(\frac{(n+1)}{2}x\right) + i\sin\left(\frac{(n+1)}{2}x\right)$$

$$\text{Deci } Re(z + \dots + z^n) = \frac{\sin\left(\frac{n}{2}x\right)}{\sin\left(\frac{x}{2}\right)} \cdot \cos\left(\frac{(n+1)}{2}x\right)$$

$$|y_1 + \dots + y_n| = \left|\frac{\sin\left(\frac{n}{2}x\right)}{\sin\left(\frac{x}{2}\right)} \cdot \cos\left(\frac{n+1}{2}x\right)\right|$$

$\leq \left| \frac{1}{\sin(\frac{x}{2})} \right|$

Alegem $M = \left| \frac{1}{\sin(\frac{x}{2})} \right|$

Avem, $\forall n \in \mathbb{N}^*$, $|y_1 + \ldots + y_n| \leq$

Din ① și ② rezultă, conform crit. Abel-Dirichlet (I), că $\sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} \frac{\cos nx}{n^\lambda}$ conv.

$\boxed{II}$,

Am rez. acest caz, nu uităm că $x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\}$

Fie $x \in \{2k\pi \mid k \in \mathbb{Z}\}$

$$\sum_n \frac{\cos nx}{n^\lambda} = \sum_n \frac{\cos(n \cdot 2k\pi)}{n^\lambda} = \sum_{n=1}^{\infty} \frac{1}{n^\lambda}$$

$\rightarrow$ conv. dacă $\lambda \in (1, +\infty)$

$\rightarrow$ div. dacă $\lambda \in (0, 1]$ □