În particular $A \not\sim \mathcal{P}(A)$

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Seminar 2

Relații

O mulțime este un triplet $\alpha = (A, B, \rho)$

unde — A, B = mulțimi $(\neq \emptyset)$

— $\rho \subseteq A \times B = \{(a, b) \mid a \in A, b \in B\}$

O funcție $f = (A, B, \Gamma_f)$ este o relație (între A și B) cu propr. că $\forall a \in A$, $\exists!\, b \in B$ a.î. $(a, b) \in \Gamma_f$

Γ_f = graficul funcției f

Notație $f : A \to B$

$\forall a \longrightarrow \exists!\, f(a)$

Compunerea relațiilor

$\alpha = (A, B, \rho)$, $\beta = (B, C, \rho')$

~~Prod~~ = $\beta \circ \alpha = (A, C, \rho' \circ \rho = \{(a, c) \mid \exists\, b \in B$ a.î. $(a, b) \in \rho, (b, c) \in \rho'\})$

Notație: $(a, b) \in \rho \leftrightarrow a \rho b$

$(a, b) \notin \rho \leftrightarrow a \not\rho b$

T Ex. 1) $\forall \alpha \in (A, B, \rho)$, $\beta = (B, C, \sigma)$, $\gamma = (C, D, \varepsilon)$

2) $\Delta_A = \{(a, a) \mid a \in A\}$ și $1_A = (A, A, \Delta_A)$

$\alpha \cdot 1_A = \alpha$, $\forall \alpha = (A, B, \rho)$

$1_B \cdot \alpha = \alpha$

Ex $A = \{2, 4, 6, 8\}$, $B = \{1, 3, 5, 7\}$

$\rho = \{(x, y) \mid x \geq 6 \vee y \leq 1\}$

$= \{(x, y) \mid x \in \{6, 8\} \vee y = 1\}$

$= \{(6, 1), (6, 3), (6, 5), (6, 7), (8, 1), (8, 3), (8, 5), (8, 7), (2, 1), (4, 1)\}$

Ex $A = B = \mathbb{N}$

$\rho = \{(3, 5), (5, 3), (3, 3), (5, 5)\}$

$\sigma = \{(x, y) \mid x \leq y\} \subseteq \mathbb{N} \times \mathbb{N}$

$\gamma = \{(x, y) \mid y - x = 12\} \subseteq \mathbb{N} \times \mathbb{N}$

$\rho \circ \sigma = \{(a, c) \mid a \in \mathbb{N}, c \in \mathbb{N}, (\exists) b \in \mathbb{N}$ a.î. $(a, b) \in \sigma, (b, c) \in \rho\}$

$\rho \circ \sigma = \{(a, c) \mid a \leq b, (b, c) \in \{(3, 5), (5, 3), (3, 3), (5, 5)\}\} = \{(0, 5), (1, 5), (2, 5), (3, 5), (0, 3), (1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (4, 5), (5, 5)\}$

$\sigma \circ \rho = \{(a, c) \mid a \in \mathbb{N}, c \in \mathbb{N}, (\exists) b \in \mathbb{N}$ a.î. $(a, b) \in \rho, (b, c) \in \sigma\} = \{(a, c) \mid (\exists) b$ a.î.

$= \{(3, c) \mid c \geq 5\} \cup \{(3, 3), (3, 4)\} \cup \{(5, c), c \geq 3\}$

$\rho \circ \gamma = \{(a, c) \mid a \in \mathbb{N}, c \in \mathbb{N}, (\exists) b \in \mathbb{N}$ a.î. $(a, b) \in \gamma, (b, c) \in \rho\}$

$= \{(a, c) \mid \exists b$ a.î. $b - a = 12 \wedge (b, c) \in \rho\}$

$= \emptyset$

$\gamma \circ \rho = \{(a, c) \mid a \in \mathbb{N}, c \in \mathbb{N}, (\exists) b \in \mathbb{N}$ a.î. $(a, b) \in \rho, (b, c) \in \gamma\} = \{(a, c) \mid \exists b$ a.î. $(a, b) \in \rho$

$\wedge\ c-b=12) = \{(3,17),(5,17),(3,15),(5,15)\}$

<u>Def.</u> $\alpha = (A, B, \rho)$ relatie

$\alpha^{-1} = (B, A, \rho^{-1})$ cu $\rho^{-1} = \{(b,a) \mid (a,b) \in \rho\}$

$\rho^n = \rho, \forall n \in \mathbb{Z}^*$

(T) $\nabla \circ \rho$

Ex) $\rho = \{(2a, 3b) \mid a, b \in \mathbb{Z}\}$

$\rho^n = ?, n \in \mathbb{Z}$

$\rho \circ \rho = \{(x, z) \mid \exists y \in \mathbb{Z}$ a.î. $(x,y) \in \rho \wedge (y,z) \in \rho\}$

$= \{(x,z) \mid \exists a, b \in \mathbb{Z}$ a.î. $(2a, 3b) \in \rho \wedge (3b, z) \in \rho\}$

$= \{(2a, z) \mid a \in \mathbb{Z}, \exists b \in \mathbb{Z}$ a.î. $(3b, z) \in \rho\} = \{(2a, 3b') \mid a, b' \in \mathbb{Z}\} = \rho$

$\Rightarrow \rho^n = \rho, \forall n \in \mathbb{N}^*$

$\rho^{-1} = \{(3a, 2b) \mid a, b \in \mathbb{Z}\}$

$\rho^{-1} \circ \rho^{-1} = \{(x, z) \mid \exists y \in \mathbb{Z}$ a.î. $(x,y) \in \rho^{-1} \wedge (y,z) \in \rho^{-1}\} = \{(3a, z) \mid a \in \mathbb{Z}, \exists b \in \mathbb{Z}$ a.î. $(2b, z) \in \rho^{-1}\} = \{(3a, 2b') \mid a, b' \in \mathbb{Z}\} = \rho^{-1}$

$\Rightarrow \rho^{-n} = \rho^{-1}, \forall n \in \mathbb{N}^*$

$$\rho^n = \begin{cases} \rho^{-1}, & n \in \mathbb{Z} \setminus \mathbb{N} \\ \rho, & n \in \mathbb{N}^* \\ 1_{\mathbb{Z}}, & n = 0 \end{cases}$$

(T) $(\rho^n)^{-1} = (\rho^{-1})^n$

$(\alpha \circ \beta)^{-1} = \beta^{-1} \circ \alpha^{-1}$

Ex) $\rho = \{(a, a+3) \mid a \in \mathbb{Z}\}$

$\rho \circ \rho = \{(x, z) \mid x, z \in \mathbb{Z}, \exists\, y \in \mathbb{Z}$ a.î. $(x, y) \in \rho, (y, z) \in \rho\} = \{(x, z) \mid \exists\, y$ a.î. $y = x+3 \wedge z = y+3\} = \{(x, x+6) \mid x \in \mathbb{Z}\}$

$P(n): \rho^n = \{(a, a+3n) \mid a \in \mathbb{Z}\}, \forall n \in \mathbb{N}^*$

$\rho^{-1} = \{(a, a-3) \mid a \in \mathbb{Z}\}$

$\Rightarrow \rho^{-n} = \{(a, a-3n) \mid a \in \mathbb{Z}\}, \forall n \in \mathbb{N}^*$

$$\rho^n = \begin{cases} (a, a+3n), & n \in \mathbb{N}^* \\ (a, a-3n), & n \in \mathbb{N}^* \\ 1_{\mathbb{Z}}, & n = 0 \end{cases}$$

$P(n) \rightarrow P(n+1)$

$\rho^n \cdot \rho = \{(a, c) \mid a, c \in \mathbb{Z}, \exists\, b$ a.î. $(a, b) \in \rho \wedge (b, c) \in \rho^n\}$

$= \{(a, c) \mid \exists\, b$ a.î. $a+3 = b \wedge b+3n = c\}$

$= \{(a, b+3n) \mid a, b \in \mathbb{Z}\} = \{(a, a+3(n+1)) \mid a \in \mathbb{Z}\} = \rho$

$\rho^{-1} = \{(a+3, a) \mid a \in \mathbb{Z}\} = \{(a, a-3) \mid a \in \mathbb{Z}\}$

$\rho^{-n} = \{(a, a-3n) \mid a \in \mathbb{Z}\}, \forall n \in \mathbb{N}^*$

$\rho \cup \rho^{-1} = \{(a, b) \mid 3 = |a-b|\} \subseteq \mathbb{Z} \times \mathbb{Z}$

ef. $(\rho \cup \rho^{-1})^m$