Seminar 4 29.10.2022

\# nr. de funcții surj.

$A_i = \{ f : \{1, \dots, m\} \to \{1, \dots, n\} \mid i \notin \text{Im} f \}$

$|A_1 \cup \dots \cup A_n| = \sum_{i=1} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j|$

$+ \dots (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n|$

$= \sum_{i=1}^{n} (n-1)^m$

$- \sum_{1 \le i \le j \le n} (n-2)^m$

$\vdots$

$(-1)^n \sum 1^m$

$= (n-1)^m \cdot C_n^1 - (n-2)^m C_n^2 + \dots (-1)^m \cdot 1^m C_n^{n-1}$

$= \sum_{i=1}^{n-1} (-1)^{i+1} C_n^i (n-i)^m$

$\Rightarrow$ nr. funct. surj este $n^m - \sum_{i=1}^{n-1} (-1)^{i+1} C_n^i (n-i)^m$

$= \sum_{i=0}^{n-1} (-1)^{i+1} C_n^i (n-i)^m$

ex.

1) Temă nr. de perm $\sigma \in S_n$ pt care

$\exists! \; i \in \overline{1,n}, \; \sigma(i) = i$

$\sigma \in A_i, \; \sigma \notin A_j \; \forall j \ne i$

$\sigma \in \bigcup_{i=1} \left(A_i \setminus \bigcup_{j \ne i} A_j \right)$

Fie $f : \{X_1, X_2, \dots, X_n\} \to \{Y_1, Y_2, \dots, Y_n\}$

atunci: f inj $\overset{1}{\Leftrightarrow}$ f surj $\overset{2}{\Leftrightarrow}$ f bij.

„⇒" f inj $\Rightarrow \operatorname{Im} f = \{\underbrace{f(x_1), \dots, f(x_n)}_{n \text{ elem}}\}$
$\subset \underbrace{\{y_1, y_2, \dots, y_n\}}_{n \text{ elem}}$

Deci $\operatorname{Im} f = \{y_1, \dots, y_n\} \Rightarrow f$ surj.

„$\overset{1}{\Leftarrow}$" + „$\overset{2}{\Rightarrow}$"

Pt orice $i = \overline{1,n}$. Fie $A_i = \{1 \le j \le n \mid f(x_j) = y_i\}$

f surj $\Rightarrow A_i \neq \emptyset, \forall i \Rightarrow |A_i| \ge 1, \forall i$

$\forall_{\substack{i,i' \\ i \neq i'}} A_i \cap A_{i'} = \emptyset$

Deci $\cancel{A_1 \cup A_j}$ $|\underbrace{A_1 \cup A_2 \cup \dots \cup A_n}_{\subseteq \text{ o unei mult cu } n \text{ elem, deci are card} \le n}| = |A_1| + |A_2| + \dots + |A_n| \ge n$

$\Rightarrow |A_i| = 1$ pt. $\forall i = \overline{1,n} \Rightarrow$ pt. $\forall y_i \ \exists!$ un j a.î. $f(x_j) = y_i$

Pt. o mult. oarecare M, UASE:

1) M ~~inj~~ infinită
2) $\exists f : M \to M$ funcție inj care nu este surj
3) $\exists g : M \to M$ funcție surj care nu este inj.

2) ⇒ 1) și 3) ⇒ 1) rez. din probl. anterioară

1) ⇒ 2) M inf $\Rightarrow \exists \{x_1, \dots, x_n, \dots\} \subseteq M$
$\mid$ pt $\forall k$, $M \setminus \{x_1, \dots, x_k\} \neq \emptyset$
$\exists x_{k+1}$

Iau $f: M \to M$

$$f(x) = \begin{cases} x_{k+1}, & \text{dacă } x = x_k \\ x, & \text{dacă } x \notin \{x_1, \dots, x_n, \dots\} \end{cases}$$

$x, \forall$ ... $f \Rightarrow f$ surj, dar este inj

(f) ~~$3) \Rightarrow 1)$~~ $1) \Rightarrow 3)$

M inf $\Rightarrow \exists \{x_1, x_2, \dots, x_n, \dots\} \subseteq M$

(nt $\forall k$, $M \setminus \{x_1, \dots, x_k\} \neq \emptyset$

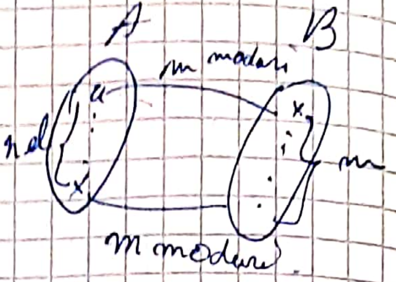
$\Rightarrow \exists x_{k+1}$

Iau $f: M \to M$

$$f(x) = \begin{cases} x_0, & \text{dacă } x = x_0 \\ x_{k-1}, & \text{dacă } x = x_k,\ x \neq x_0 \\ x, & \text{dacă } x \notin \{x_1, \dots, x_n, \dots\} \end{cases}$$

Fie A, B mult finite

$|A| = n$, $|B| = m$



$\Rightarrow \underbrace{m \cdot \ldots \cdot m}_{n \text{ ori fct}}$

(1) nr fct de la A la B $= m^n$

(2) nr fct inj de la A la B

$$= \begin{cases} 0, & m < n \\ A_m^n, & m \geq n \end{cases}$$

(3) nr fct st cresc de la A la B

$$= \begin{cases} 0, & m < n \\ C_m^n, & m \geq n \end{cases}$$

(4). nr fct cresc de la A la B

$$= \begin{cases} 0 & , m = 0 \\ C_{m+n+1} & m \geq 1 \end{cases}$$

$f : \{1, 2 \ldots n\} \to \{1, \ldots \overset{m+n-1}{m}\}$ cresc

$g : \{1 \ldots n\} \to \{1 \ldots ,$

$g = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ f(1) & f(2)+1 & \ldots & & f(n)+n-1 \end{pmatrix}$ st cresc

Reciproc $g : \{1 \ldots n\} \to \{1 \ldots m+n-1\}$ st cresc

$f : \{1 \ldots n\} \to \{1 \ldots m\}$

$$f = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ g(1) & g(1)-1 & - - - & \ldots & g(n)-n+1 \end{pmatrix}$$

(5) $A_1, \ldots A_n$ mult finite

Atunci $|A_1 \cup A_2 \cup \ldots A_n| = \sum_{i=1}^{n} |A_i|$

$$- \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k|$$

$$- - - - (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Pf $n=2$ $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Pp ~~at n $|A_1 \cup \ldots \cup A_n|$~~ = adev pt n.

$|A_1 \cup \ldots \cup A_n \cup A_{n+1}| = |\underbrace{A_1 \cup \ldots A_n}_{X}| \cup A_{n+1}$

$= |X| + |A_{n+1}| - |(A_1 \cap \ldots A_n) \cap A_{n+1}| =$

$= \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| \ldots (-1)^{n+1}$

$|A_1 \cap A_2 \ldots A_n| + |A_{n+1}| - |\bigcup_{i=1}^{n} (A_i \cap A_{n+1})|$

$- \sum_{i=1}^{n} |A_i \cap A_{n+1}|$

$= \sum_{1 \le i < j \le n} |(A_i \cap A_{n+1}) \cap (A_j \cap A_{n+1})| - \ldots (-1)^{n+1} |(A_1 \cap A_{n+1})$

$\cap \ldots \cap (A_n \cap A_{n+1})|$

$= \sum_{i=1}^{n+1} |A_i| - \sum_{1 \le i < j \le n+1} |A_i \cap A_j| + \sum_{1 \le i < j \le n+1} |A_i \cap A_j \cap A_k| \ldots (-1)^{n+2}$

$|A_1 \cap A_2 \cap \ldots A_{n+1}|$

⑥ m fct nurj de la A la B

$f: \{1, 2 \ldots n\} \to \{1, 2 \ldots m\}$

$(\forall) i = \overline{1,m}$ $A_i = \{ f: \{1, \ldots, n\} \to \{1 \ldots m\} /$

$i \notin Im(f)\}$

f nu ete surj $\Leftrightarrow f \in |A_1 \cup \dots \cup A_m|$

nr f ct nesurj este $|A_1 \cup \dots \cup A_m|$

$$|A_1 \cup \dots \cup A_m| = \sum_{i=1}^{m} |A_i| - \sum_{1 \le i < j \le m} |A \dots + \dots (-1)^{m+1} |A_1 \cap A_2 \dots A_m|$$

$$= \sum_{i=1}^{m} (m-1)^n - \sum_{1 \le i < j \le n} (m-2)^n \dots (-1)^m \sum 1^n$$

$$= (m-1)^n \cdot C_m^1 - (m-2)^n C_m^2 \dots \dots (-1)^m \cdot C_m^{m-1}$$

$$= \sum_{i=1}^{m-1} (-1)^{i+1} C_m^i (m-i)^n$$

$\Rightarrow$ nr fct surjective $= m^n - \sum_{i=1}^{m-1} (-1)^{i+1} C_m^i (m-i)^n$

$$\sum_{i=0} (m-n)^n C_m^i \cdot (-1)^i$$