$\Rightarrow x_n$ convergent, deoarece $l < 1$

Seminar 2

1. Det. $\overline{\lim} x_n$ și $\underline{\lim} x_n$, precizând dacă există $\lim\limits_{n\to\infty} x_n$, unde:

a) $x_n = 1 + 2(-1)^{n+1} + 3\cdot(-1)^{\frac{n(n+1)}{2}}\ \forall n \in \mathbb{N}$

Sol:

I. ~~n = 4k~~

$x_{4n} = 1 + 2\cdot(-1) + 3\cdot(-1)^{4n(4n+1)}\ \forall n \in \mathbb{N}$

Sol: $x_{4n} = 1 + 2\cdot(-1)^{4n+1} + 3(-1)^{2n(4n+1)}$

$= 1 - 2 + 3 = 2 \xrightarrow{n\to\infty} 2$

$x_{4n+1} = 1 + 2(-1)^{4n+2} + 3\cdot(-1)^{\frac{(4n+1)(4n+2)}{2}}$

$= 1 + 2 - 3 = 0 \xrightarrow{n\to\infty} 0$

$x_{4n+2} = 1 + 2(-1)^{4n+3} + 3(-1)^{(2n+1)(4n+3)} = -4 \to -4$

$x_{4n+3} = 1 + 2(-1)^{4n+4} + 3(-1)^{\frac{(4n+3)(2n+2)}{2}}$

$= 1 + 2 + 3 = 6 \xrightarrow[n\to\infty]{} 6$

~~$\lim\limits_{n\to\infty}$~~

$\mathbb{N} = 4\mathbb{N} \cup (4\mathbb{N}+1) \cup (4\mathbb{N}+2) \cup (4\mathbb{N}+3)$

$\mathcal{L}((x_n)_n) = \{2, 0, -4, 6\}$

$\overline{\lim}\, x_n = 6$

$\underline{\lim}\, x_n = -4$

$\overline{\lim}\, x_n \neq \underline{\lim}\, x_n \Rightarrow$ NU EXISTĂ $\lim\limits_{n\to\infty} x_n$

~~lim x ...~~

b) $x_n = \sin \frac{n\pi}{3} \quad \forall n \in \mathbb{N}$

Sol:

$x_{3n} = \sin n\pi = 0 \to 0$

$x_{3n+1} = \sin\left(n\pi + \frac{\pi}{3}\right) = \sin n\pi \cdot \cos\frac{\pi}{3} + \cos n\pi \cdot \sin\frac{\pi}{3}$

$= (-1)^n \cdot \frac{\sqrt{3}}{2}$

$x_{6n+1} = (-1)^{2n} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \to \frac{\sqrt{3}}{2}$

$x_{6n+4} = (-1)^{2n+1} \cdot \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} \to -\frac{\sqrt{3}}{2}$

$x_{3n+2} = \sin\frac{(3n+2)\pi}{2} = \sin\left(n\pi + \frac{2\pi}{3}\right) =$

$\sin(n\pi) \cdot \cos\frac{2\pi}{3} + \sin\frac{2\pi}{3} \cdot \underbrace{\cos(n\pi)}_{=(-1)^n}$

$= (-1)^n \cdot \frac{\sqrt{3}}{2}$

$x_{3\cdot 2n+2} = (-1)^{2n} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \xrightarrow[n\to\infty]{} \frac{\sqrt{3}}{2}$

$x_{3\cdot(2n+1)+2} = (-1)^{2n+1} \cdot \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} \xrightarrow[n\to\infty]{} -\frac{\sqrt{3}}{2}$

$\mathbb{N} = 6\mathbb{N} \cup (6\mathbb{N}+1) \cup \ldots \cup (6\mathbb{N}+5)$

$\mathcal{L}(x_n) = \left\{\pm\frac{\sqrt{3}}{2}, 0\right\}$

c) $x_n = \dfrac{n \cos\frac{n\pi}{2}}{n^2+1} \quad \forall n \in \mathbb{N}$

Sol

Vom arăta că $\lim_{n\to\infty} x_n = 0$

$-1 \le \cos\frac{n\pi}{2} \le 1 \quad / \cdot \frac{n}{n^2+1}$

$$-\frac{n}{n^2+1} \le \cos\frac{n\pi}{2} \le \frac{n}{n^2+1},\ \forall n \in \mathbb{N}$$

$$\searrow 0 \quad \downarrow 0 \quad \swarrow 0$$

Conform criteriului cleștelui, avem $\lim\limits_{n\to\infty} x_n = 0$, i.e. $\overline{\lim}\, x_n = 0$, $\underline{\lim}\, x_n = 0$

2. Det. suma seriei de mai jos și precizați dacă este convergentă sau divergentă!

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$$

Sol: $x_n = \frac{n}{(n+1)!}$, $\forall n \in \mathbb{N}^*$

$$S_n = x_1 + x_2 + \dots + x_n = \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n}{(n+1)!}$$

$$= \frac{1+1-1}{2!} + \frac{2+1-1}{3!} + \dots + \frac{n+1-1}{(n+1)!}$$

$$= \frac{2}{2!} - \frac{1}{2!} + \frac{3}{3!} - \frac{1}{3!} + \dots + \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!}$$

$$= 1 - \cancel{\frac{1}{2!}} + \cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}} + \cancel{\frac{1}{3!}} - \dots + \cancel{\frac{1}{n!}} - \frac{1}{(n+1)!}$$

$$= 1 - \frac{1}{(n+1)!},\ \forall n \in \mathbb{N}^*$$

$\lim\limits_{n\to\infty} s_n = 1 \in \mathbb{R} \Rightarrow \sum\limits_{n=1}^{\infty} x_n$ convergentă

3. Studiați conv. (sau natura) seriilor:

a) $\sum\limits_{n=1}^{\infty} \frac{a^n}{\sqrt{n}}$, $a > 0$

Sol: $x_n = \frac{a^n}{\sqrt{n}}$, $\forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a \cdot \sqrt[n+1]{n+1}}{\sqrt[n]{n}}$$

$$= \lim_{n\to\infty} \frac{a \cdot \sqrt[n+1]{n+1}}{\sqrt[n]{n}} = a \cdot \frac{1}{1} = a$$

Conform crit. rap. :

1) $a < 1 \Rightarrow$ serie este conv.

2) $a > 1 \Rightarrow$ serie este diverg.

3) $a = 1$

$\Rightarrow x_n = \frac{1}{\sqrt[n]{n}}$

$\Rightarrow \lim_{n\to\infty} x_n = +\infty$

$$\lim_{n\to\infty} x_n = 1 \neq 0 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} \text{ diverg.}$$

$\uparrow$

criteriul suf. de divergență

Sol: Am obținut

$$\sum \frac{a^n}{\sqrt[n]{n}} \begin{cases} \cancel{a<1} \text{ conv}, a < 1 \\ \cancel{a=1} \text{ diverg}, a = 1 \\ \text{diverg}, a > 1 \end{cases}$$

b) $\sum_{n=0}^{\infty} \frac{1}{2^n + n}$

$X_n = \frac{1}{2^n + n}, \forall n \in \mathbb{N}$

$\frac{1}{2^n + n} \leq \frac{1}{2^n}, \forall n \in \mathbb{N}$

Fie $y_n = \frac{1}{2^n}, \forall n \in \mathbb{N}$

$x_n \le y_n, \forall n \in \mathbb{N}$

$\sum_{n=0}^{\infty} y_n = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \Rightarrow$ conv.
(serie geometrică)

Conform criteriului de comp. cu ineg. rezultă că $\sum_{n=0}^{\infty} x_n$ e conv.

c) $\sum_{n=1}^{\infty} \left(\frac{a n^2+5n+3}{2n^2+3n+1}\right)^n, a>0$

Fie $x_n = \left(\frac{a n^2+5n+3}{2n^2+3n+1}\right)^n, \forall n \in \mathbb{N}^*$

Apl. criteriul rad.:

$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{a n^2+5n+3}{2n^2+3n+1} = \frac{a}{2}$

1) $\frac{a}{2} < 1 \Rightarrow a<2$ pt. $a<2 \Rightarrow$ conv.

2) $\frac{a}{2} > 1 \Rightarrow$ pt. $a>2 \Rightarrow$ diverg.

3) $\frac{a}{2} = 1 \Rightarrow$ crit. nu det. natura

$x_n = \left(\frac{2n^2+5n+3}{2n^2+3n+1}\right)^n = \left(1+\frac{2n+2}{2n^2+3n+1}\right)^n$

$= \left(1+\frac{2n+2}{2n^2+3n+1}\right)^{\frac{2n^2+3n+1}{2n+2}\cdot\frac{n(2n+2)}{2n^2+3n+1}}$

$\Rightarrow \lim_{n\to\infty} x_n = e^{\alpha}$

$\alpha = \lim_{n\to\infty} \frac{n(2n+2)}{2n^2+3n+1} = 1$

$\lim_{n\to\infty} x_n = e \ne 0 \Rightarrow$ seria este divergentă

$\uparrow$

crit. suf. de diverg.

d) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}$

$$\frac{\sqrt{n^2+1}}{\sqrt{n^3+1}} < \frac{n}{n^{\frac{3}{2}}}$$

$$\sqrt{n^2+1} \cdot n^{\frac{3}{2}} < n \cdot \sqrt{n^3+1}$$

$$\frac{\sqrt{n^2+1}}{n} \Leftrightarrow \frac{\sqrt{n^2+1}}{n^{\frac{3}{2}}}$$

$$\sum_{n=1}^{\infty} \sqrt{\frac{n^2+1}{n^3+1}}$$

Sol. $x_n = \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}, \forall n \in \mathbb{N}^*$

$y_n = \sqrt{\frac{n^2}{n^3}} = \frac{1}{\sqrt{n}}, \forall n \in \mathbb{N}^*$

~~$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$~~

~~y_n divergente $\Rightarrow x_n$ diverg~~

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}} \cdot \frac{\sqrt{n^3}}{\sqrt{n^2}} = 1$$

Cf. criteriului de comp. cu limită avem că $\sum_{n=1}^{\infty} x_n \simeq \sum_{n=1}^{\infty} y_n$

$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \frac{\sqrt{n^2}}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$, div (este serie armonică generalizată cu $\alpha = \frac{1}{2} < 1$

Deci $\sum_{n=1}^{\infty} x_n$ este divergentă.