1. Consider the vector field $\vec{F}(x, y) = (x+y^2, -y)$ and its associated continuous dynamical system $(W^t, \mathbb{R}^2)$.

   (a) For each of the following functions, show whether or not it is a flow for $(W^t, \mathbb{R}^2)$.

      i. $\vec{\varphi}(t) = (-e^{-2t}, \sqrt{3}e^{-t})$
      ii. $\vec{\varphi}(t) = (-e^{-4t}, \sqrt{3}e^{-2t})$
      iii. $\vec{\varphi}(t) = (\frac{4}{3}e^{t} - \frac{1}{3}e^{-2t}, e^{-t})$
      iv. $\vec{\varphi}(t) = (e^{t} - e^{-2t}, e^{-t})$

   (b) A *constant flow* is a flow $\vec{\varphi}$ such that $\vec{\varphi}(t_1) = \vec{\varphi}(t_2)$ for all $t_1, t_2 \in \mathbb{R}$. Find all constant flows for $\vec{F}$. (Hint: think about $\vec{\varphi}\,'$ in this situation.)

   (c) Classify all fixed points of $(W^t, \mathbb{R}^2)$ as stable or unstable.

   (d) Find *all* flows of $(W^t, \mathbb{R}^2)$ that pass through the point $(1, 0)$. Remember, if $\vec{\varphi}$ is a flow with this property, it is not a requirement that $\vec{\varphi}(0) = (1, 0)$, only that $\vec{\varphi}(t_0) = (1, 0)$ for some t_0.

   (e) We call the flows $\vec{\varphi}_1$ and $\vec{\varphi}_2$ time shifts of each other if $\vec{\varphi}_1(t) = \vec{\varphi}_2(t + t_0)$ for some t_0. Show that the flows from part (d) are time shifts of each other.

2. Let's explore the question of whether every discrete dynamical system can be described as the time-1 map of a continuous dynamical system.

   Let (W^t, X) be a continuous dynamical system, and let (T, X) be its time-1 map. That is, $T(x) = W^1(x)$.

   The point $x \in X$ is called *periodic* for (W^t, X) if there exists a $t \neq 0$ so that $W^t(x) = x$ and is called *periodic* for (T, X) if there exists an $i \neq 0$ so that $T^i x = x$. In both cases, the minimum t or i such $W^t(x) = x$ or $T^i(x) = x$ is called the *period* of x.

   (a) Suppose $x \in X$ is a *periodic point* for (T, X). Must x be a periodic point for (W^t, X)?

   (b) Suppose $x \in X$ is a *periodic point* for (W^t, X). Must x be a periodic point for (T, X)?

   (c) Show that if x is a point of period at least 2 for (T, X), then there are infinitely many periodic points for (T, X) *of the same period*.

   (d) Consider the *logistic map* $T : [0, 1] \to [0, 1]$ defined by $x \mapsto rx(1 - x)$. Show that when $r = \frac{19}{6}$ the logistic map has exactly two points of period 2. (Hint: use a computer algebra system to solve any nasty equations you come across!)

   (e) Are all discrete dynamical systems time-1 maps to continuous dynamical systems? Justify your answer.

3. The function $G : \mathbb{R}^n \to \mathbb{R}^m$ is called *affine* if there exists a matrix A and a vector $\vec{p}$ so that $G(\vec{x}) = A\vec{x} + \vec{p}$ for all $\vec{x}$.

   Let $F : \mathbb{R}^n \to \mathbb{R}^m$. A *first-order approximation* to F at the point $\vec{w} \in \mathbb{R}^n$ is an affine function $L : \mathbb{R}^n \to \mathbb{R}^m$ satisfying
$$\lim_{\|\vec{x}\| \to 0} \frac{F(\vec{w} + \vec{x}) - L(\vec{w} + \vec{x})}{\|\vec{x}\|} = 0.$$

   (a) Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $x \mapsto x^2$. Find a first-order approximation, L_0, to F at 0, and a first-order approximation, L_2, to F at 2.

   (b) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Define $L_{\vec{w}}$ to be the first-order approximation of F at the point $\vec{w}$. Find $L_{\vec{w}}$.

   (c) Let L be an affine function and let $L_{\vec{w}}$ be the first-order approximation to L at $\vec{w}$. Prove that $L = L_{\vec{w}}$ regardless of $\vec{w}$.

(d) Prove that if $(W^t, \mathbb{R}^n)$ is a continuous dynamical system with velocities given by $V(\vec{x}) = A\vec{x}$ for some matrix A, then the point $\vec{x} \in \mathbb{R}^n$ is stable under W^t if and only if the point $\vec{0}$ is stable under W^t.

(e) From calculus, you know that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then $L(w + x) = f'(w)(x) + f(w)$ is a first-order approximation to f at w, where f' is the derivative of f. Use this knowledge to find a first-order approximation to $F(x, y) = (x+y^2, -y)$ at the points $(1, 1)$ and $(1, 0)$.

(f) Let $(W^t, \mathbb{R}^2)$ be the continuous dynamical system that flows along $F(x, y) = (x+y^2, -y)$. Classify $(1, 1)$ and $(1, 0)$ as stable or unstable. Justify your answer.

4. Stability/instability describes how points behave under a dynamical system. Let's take a moment to think about how *volumes/areas* behave.

   For this problem, you may use any facts you know about the determinant without justification (so long as they're true...).

   (a) The *trace* of a square matrix X, denoted $\text{Tr}(X)$, is the sum of its diagonal matrices. Let $A = [\vec{a}_1|\cdots|\vec{a}_n]$ be a matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$. Let E_i be the identity matrix with the ith column replaced with $\vec{a}_i$. Show that
   $$\text{Tr}(A) = \sum \det(E_i).$$

   (b) Let $(W^t, \mathbb{R}^2)$ be the continuous dynamical system which flows vectors along the vector field given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   Write out the limit definition of the derivative $\frac{\partial W^t}{\partial t}$ at time $t = 0$. How does this derivative relate to A?

   (c) Write down a first-order approximation to W^t *with respect to time* at $t = 0$.
   Hint: the hardest part of this question is figuring out what the previous sentence actually means. Don't get discouraged!

Programming Problems

For the programming problems, please use the Jupyter notebook available at

https://utoronto.syzygy.ca/jupyter/user-redirect/git-pull?repo=https://github.com/siefkenj/2020-MAT-335-webpage&subPath=homework/homework2-exercises.ipynb

Make sure to comment your code and use "Markdown" style cells to explain your answers.

1. xxx