Supplementary online appendices for "A computational continuum model of poroelastic beds"

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A. Governing micro-scale equations and assumptions

In the current appendix, we explain the non-dimensionalization of the pore-scale governing equations, as well as required assumptions and scale estimates for the derivation of the macroscale governing equations and accompanying pore-scale test problems. The governing equations, presented in main paper (2.1–2.4), are summarized here for convenience:

$$\rho_\mathrm{f}\left(\partial_t u_i + u_j u_{i,j}\right) = \Sigma_{ij,j}, \qquad \Sigma_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij}\left(u\right) \qquad \text{in } \Omega_f, \tag{A 1}$$

$$u_{i,i} = 0, \qquad \text{in } \Omega_f, \tag{A 2}$$

$$u_i = \partial_t v_i, \qquad \text{on } \Gamma_s, \tag{A 3}$$

$$\Sigma_{ij}\hat{n}_j = \sigma_{ij}\hat{n}_j, \qquad \text{on } \Gamma_s, \tag{A 4}$$

$$\rho_\mathrm{s}\partial_t^2 v_i = \sigma_{ij,j}, \qquad \sigma_{ij} = C^\mathrm{sk}_{ijkl}\varepsilon_{kl}\left(v\right) \qquad \text{in } \Omega_s, \tag{A 5}$$

where we have used Σ_{ij} and σ_{ij} to denote fluid and solid stress tensors, respectively. We have also introduced strain rate tensor for the fluid $\varepsilon_{ij}\left(u\right) = 0.5\left(u_{i,j} + u_{j,i}\right)$ and strain tensor for the solid $\varepsilon_{ij}\left(v\right) = 0.5\left(v_{i,j} + v_{j,i}\right)$. Here Ω_f is the fluid domain (both free fluid and fluid between solid skeleton – pore fluid), Ω_s is the solid skeleton domain and Γ_s is boundary between solid and fluid. For convenience, we use the index notation, where summation is carried out over repeating indices and comma indicates a derivative. Now we assume that there is scale separation between two length scales, l and H, with $\epsilon = l/H \ll 1$. Other parameters in this problem are ρ_f, ρ_s, ΔP, μ, $\Delta\tau$ and $\mathbf{C}^\mathrm{sk}$, where ΔP is a characteristic pressure difference, $\Delta\tau$ is a characteristic time scale of the processes being studied and $\mathbf{C}^\mathrm{sk}$ is linear solid skeleton elasticity tensor. Based on these parameters, we can define a velocity scale $U^d \equiv l^2\Delta P/\left(\mu H\right)$ arising from momentum balance at the microscale[42, eq. A 1]. We estimate the scales of flow, pressure, displacement fields and derivatives in the interior as

$$\tilde{u}_i \sim U^d, \qquad \tilde{p} \sim \Delta P, \qquad \tilde{v}_i \sim l, \qquad \left(\right)_{,i} \sim \frac{1}{l}, \qquad \partial_t\left(\right) \sim \frac{1}{\Delta\tau}. \tag{A 6}$$

Based on these estimates, we choose to render the equations dimensionless (we temporarily use "tilde" to denote dimensional quantities), using the relationships

$$\tilde{p} = \Delta P\, p, \quad \tilde{u}_i = U^d u_i, \quad \tilde{x}_i = l\, x_i, \quad \tilde{t} = \Delta\tau\, t, \quad \tilde{v}_i = l\, v_i. \tag{A 7}$$

Note that non-dimensionalization using physical parameters ΔP, μ, l, $\Delta\tau$ and H is one particular choice and different options are possible. With this choice we have the dimensionless order of all terms close to unity

$$u_i = \mathcal{O}\left(1\right), \qquad u_{i,j} = \mathcal{O}\left(1\right), \qquad p = \mathcal{O}\left(1\right), \qquad v_i = \mathcal{O}\left(1\right), \qquad \partial_t\left(\right) = \mathcal{O}\left(1\right). \tag{A 8}$$

These orders are required later on, when the multi-scale expansion is carried out. Under the chosen normalization (A 7), the governing equations become

$$\epsilon^2 Re_d \left(\bar{\bar{f}}\,\partial_t u_i + u_j u_{i,j}\right) = \Sigma_{ij,j}, \qquad \Sigma_{ij} = -p\delta_{ij} + 2\epsilon\varepsilon_{ij}(u) \qquad \text{in } \Omega_f, \tag{A 9}$$

$$u_{i,i} = 0, \qquad \text{in } \Omega_f, \tag{A 10}$$

$$u_i = \bar{\bar{f}}\,\partial_t v_i, \qquad \text{on } \Gamma_s, \tag{A 11}$$

$$\Sigma_{ij}\hat{n}_j = \frac{1}{\epsilon}\sigma_{ij}\hat{n}_j, \qquad \text{on } \Gamma_s, \tag{A 12}$$

$$\epsilon\bar{\bar{\rho}}\partial_t^2 v_i = \frac{1}{\epsilon}\sigma_{ij,j}, \qquad \sigma_{ij} = \bar{\bar{E}} C^{\mathrm{sk}}_{ijkl}\varepsilon_{kl}(v) \qquad \text{in } \Omega_s. \tag{A 13}$$

This system of non-dimensional equations is valid everywhere and so far there are no additional assumptions employed. The total list of physical parameters in the problem is ρ_{f}, ρ_{s}, ΔP, μ, l, H, $\Delta\tau$, and $\mathbf{C}^{\mathrm{sk}}$. This adds up to $7+n$ parameters, where n depends on the properties of elastic skeleton, and for the current work $n=2$ (we assume that it is isotropic and elasticity is characterized using 2 parameters). According to Buckingham π theorem, we then have to have 6 dimensionless parameters, which are

$$Re_d = \frac{\rho_f U^d H}{\mu}, \quad \bar{\bar{f}} = \frac{1}{\Delta\tau}\frac{l}{U^d}, \quad \epsilon = \frac{l}{L}, \quad \bar{\bar{E}} = \frac{\epsilon E}{\Delta P}, \quad C^{\mathrm{sk}}_{ijkl} = \frac{\tilde{C}^{\mathrm{sk}}_{ijkl}}{E}, \quad \bar{\bar{\rho}} = \frac{\rho_s\, l\, H}{\Delta P\, \Delta\tau^2}. \tag{A 14}$$

Here, we have separated a Young modulus out of the skeleton elasticity and the dimensionless tensor C^{sk}_{ijkl} is characterized by only one scalar – Poisson's ratio ν. If the skeleton is built from an anisotropic material, then the dimensionless tensor C^{sk}_{ijkl} would imply a larger number of non-dimensional parameters. For the isotropic case, this form is useful to describe behaviour of isotropic materials with similar Poisson's ratio but different Young's modulus. In addition, since the resulting equation system in the Stokes limit is linear, in the main paper we plot displacements normalized with dimensionless elasticity parameter $v_i \cdot \bar{\bar{E}}$, which in dimensional setting becomes $\tilde{v}_i \cdot E/(H\Delta P)$. In other words, the values presented in the displacement plots would correspond to simulation with $\bar{\bar{E}} = 1$.

To be able to use the multi-scale expansion, one has to estimate the relative orders of all the terms. We assume the following:

$$\bar{\bar{E}} = \mathcal{O}(1), \qquad \bar{\bar{\rho}} = \mathcal{O}(1), \qquad Re_d \le \mathcal{O}(1), \qquad \bar{\bar{f}} = \mathcal{O}(1). \tag{A 15}$$

Now the relative magnitude between different terms is completely illustrated by the ϵ pre-factors in the governing non-dimensional equations (A 9–A 13) in the interior of the poroelastic material. This, however, does not hold in the free fluid. In other words, in the free fluid region, it is not possible to use the Darcy's law as a governing equation.

In the main paper section 5 introduced restrictions of this theory has been obtained by relaxing the equality conditions from (A 15). By "relaxing" we mean that the theory should not only be applicable to one value of each of the dimensionless parameter, as set by (A 15), but for a range of dimensionless parameters. The generalization from the equality assumptions (A 15) to the inequality restrictions is done as fallows:

(i) The equality assumption $\bar{\bar{E}} = \mathcal{O}(1)$ is relaxed to the inequality restriction based on numerical tests in the main paper section 4, in which one could see that the theory works also for very stiff materials $\bar{\bar{E}} \gg \mathcal{O}(1)$. Therefore we conclude that the theory works for a range of dimensionless parameters $\bar{\bar{E}} \ge \mathcal{O}(1)$, which includes the theoretical assumption $\bar{\bar{E}} = \mathcal{O}(1)$ and numerical validation at $\bar{\bar{E}} \gg \mathcal{O}(1)$.

(ii) The equality assumption $\bar{\bar{\rho}} = \mathcal{O}(1)$ is relaxed to inequality restriction $\bar{\bar{\rho}} \le \mathcal{O}(1)$ based on physical understanding that the developed method works also in the non-inertial regime; that is, when the solid density is sufficiently small that inertial effects can be neglected, the parameter $\bar{\bar{\rho}}$ can be set to zero.

(iii) Finally, the equality assumption $\bar{\bar{f}} = \mathcal{O}(1)$ is relaxed to inequality restriction $\bar{\bar{f}} \leq \mathcal{O}(1)$ based on numerical tests in the main paper section 4. There we have validated the model equations for an "infinitely slow" or steady test problem, for which $\bar{\bar{f}} = 0$. Uniting the two parameter limits, for which the theory should work, we get $\bar{\bar{f}} \leq \mathcal{O}(1)$.

Although the equality assumptions on the dimensionless parameters (A 15) are strict and in principle one could set them to the specific value in the equations, we retain these coefficients in the equations to facilitate tests away from the outlined assumptions.

B. Derivation of homogenized effective equations

We are now ready to derive the homogenized governing equations for the poroelastic material in macroscale. Additionally, we obtain the microscale test problems for determining the effective material properties. We introduce the macroscale and microscale coordinates

$$X_i = \frac{\tilde{x}_i}{H} \qquad \text{and} \qquad x_i = \frac{\tilde{x}_i}{l},$$

respectively. These coordinates are appropriate to describe the macroscopic and microscopic variations and are related to each other by $X_i = \epsilon x_i$. In the new coordinates, the spatial derivative is given by

$$()_{,i} = ()_{,i_1} + \epsilon\, ()_{,i_0}\,, \tag{B 1}$$

where $()_{,i_0}$ denotes the derivative with respect to X_i and $()_{,i_1}$ with respect to x_i. Then we introduce a multi-scale expansions as

$$u_i(X_i, x_i) = u_i^{(0)}(X_i, x_i) + \epsilon u_i^{(1)}(X_i, x_i) + \epsilon^2 u_i^{(2)}(X_i, x_i) + \mathcal{O}(\epsilon^3), \tag{B 2}$$

$$p(X_i, x_i) = p^{(0)}(X_i, x_i) + \epsilon p^{(1)}(X_i, x_i) + \epsilon^2 p^{(2)}(X_i, x_i) + \mathcal{O}(\epsilon^3), \tag{B 3}$$

$$v_i(X_i, x_i) = v_i^{(0)}(X_i, x_i) + \epsilon v_i^{(1)}(X_i, x_i) + \epsilon^2 v_i^{(2)}(X_i, x_i) + \mathcal{O}(\epsilon^3), \tag{B 4}$$

which we insert into the main equations (A 9–A 13). The first two orders of fluid momentum equation (A 9) after the expansion with corresponding stress tensors are

$$\epsilon^0 :\ \Sigma^{(0)}_{ij,j_1} = 0, \qquad \Sigma^{(0)}_{ij} = -p^{(0)}\delta_{ij}, \tag{B 5}$$

$$\epsilon^1 :\ \Sigma^{(1)}_{ij,j_1} + \Sigma^{(0)}_{ij,j_0} = 0, \qquad \Sigma^{(1)}_{ij} = -p^{(1)}\delta_{ij} + 2\varepsilon^1_{ij}\left(u^{(0)}\right), \tag{B 6}$$

where we have introduced the strain rate tensor in the micro-scale $\varepsilon^1_{ij}(u) = 0.5\left(u_{i,j_1} + u_{j,i_1}\right)$. The first two orders of the fluid continuity equation (A 10) after the expansion are

$$\epsilon^0 :\ u^{(0)}_{i,i_1} = 0, \tag{B 7}$$

$$\epsilon^1 :\ u^{(1)}_{i,i_1} + u^{(0)}_{i,i_0} = 0. \tag{B 8}$$

The first three orders of the solid displacement equation (A 13) after the expansion with corresponding stress tensors are

$$\epsilon^{-1} :\ \bar{\bar{E}}\sigma^{(-1)}_{ij,j_1} = 0, \qquad \sigma^{(-1)}_{ij} = C^{\mathrm{sk}}_{ijkl}\varepsilon^1_{kl}\left(v^{(0)}\right), \tag{B 9}$$

$$\epsilon^0 :\ \bar{\bar{E}}\sigma^{(0)}_{ij,j_1} + \bar{\bar{E}}\sigma^{(-1)}_{ij,j_0} = 0, \qquad \sigma^{(0)}_{ij} = C^{\mathrm{sk}}_{ijkl}\varepsilon^1_{kl}\left(v^{(1)}\right) + C^{\mathrm{sk}}_{ijkl}\varepsilon^0_{kl}\left(v^{(0)}\right), \tag{B 10}$$

$$\epsilon^1 :\ \bar{\bar{E}}\sigma^{(1)}_{ij,j_1} + \bar{\bar{E}}\sigma^{(0)}_{ij,j_0} = \bar{\bar{\rho}}\ddot{v}^{(0)}_i, \qquad \sigma^{(1)}_{ij} = C^{\mathrm{sk}}_{ijkl}\varepsilon^1_{kl}\left(v^{(2)}\right) + C^{\mathrm{sk}}_{ijkl}\varepsilon^0_{kl}\left(v^{(1)}\right). \tag{B 11}$$

Here we have used double-dot to denote second derivative in time $\ddot{v}_i = \partial_t^2 v_i$. Note that the orders of the solid equations are shifted to one order lower value to match the orders of the stress

boundary condition between solid structure and pore geometry (A 12). The boundary conditions (A 11–A 12) for all orders $n \in (0, \infty)$ take the form

$$u_i^{(n)} = \bar{\bar{f}}\, \dot{v}_i^{(n)}, \tag{B 12}$$

$$\Sigma_{ij}^{(n)} \hat{n}_j = \bar{\bar{E}} \sigma_{ij}^{(n)} \hat{n}_j, \tag{B 13}$$

where the time derivative is denoted using dot notation $\dot{v}_i = \partial_t v_i$ and there are no lower order values from the fluid stresses $\sigma_{ij}^{(-1)} \hat{n}_j = \Sigma_{ij}^{(-1)} \hat{n}_j = 0$.

Solving $\mathcal{O}(1)$-problem for fluid and $\mathcal{O}\left(\epsilon^{-1}\right)$-problem for elasticity, gives us

$$p^{(0)} = p^{(0)}(X_i), \qquad v_i^{(0)} = v_i^{(0)}(X_i), \tag{B 14}$$

that is, the leading order pressure and displacement depend only on the macroscale. The fluid $\mathcal{O}(\epsilon)$-problem is

$$-p_{,i_1}^{(1)} + u_{i,j_1 j_1}^{(0)} = p_{,i_0}^{(0)}, \tag{B 15}$$

$$u_{i,i_1}^{(0)} = 0, \tag{B 16}$$

$$u_i^{(0)}\Big|_{\Gamma} = \bar{\bar{f}}\, \dot{v}_i^{(0)}. \tag{B 17}$$

This problem can be solved using the ansatz

$$u_i^{(0)} = -\mathcal{K}_{ij} p_{,i_0}^{(0)} + \bar{\bar{f}}\, \dot{v}_i^{(0)}, \tag{B 18}$$

$$p^{(1)} = -\mathcal{A}_j p_{,j_0}^{(0)}, \tag{B 19}$$

where the second term in the velocity ansatz is added to satisfy the velocity boundary condition at the surface with the solid skeleton. In principle, one could introduce a second proportionality tensor, but the resulting problem would always be trivial to solve, yielding factor δ_{ij} before the velocity of the solid skeleton. Exactly the same conclusion holds also for the derivation of the velocity boundary condition [42], therefore the time derivative of the displacement appears in the boundary condition in the main paper (2.12) and (2.13). The solid $\mathcal{O}(1)$-problem is

$$\left[C_{ijkl}^{\mathrm{sk}} \varepsilon_{kl}^{1}\left(v^{(1)}\right)\right]_{,j_1} = 0, \tag{B 20}$$

$$\bar{\bar{E}}\left[C_{ijkl}^{\mathrm{sk}} \varepsilon_{kl}^{1}\left(v^{(1)}\right) + C_{ijkl}^{\mathrm{sk}} \varepsilon_{kl}^{0}\left(v^{(0)}\right)\right] \hat{n}_j\Big|_{\Gamma} = -\, p^{(0)} \hat{n}_i\Big|_{\Gamma}. \tag{B 21}$$

This problem can be solved using the ansatz

$$v_i^{(1)} = \chi_{ikl} \varepsilon_{kl}^{0}\left(v^{(0)}\right) - \frac{\eta_i}{\bar{\bar{E}}} p^{(0)}, \tag{B 22}$$

where division by $\bar{\bar{E}}$ is done in order to arrive with test problem for η_i, which is independent from the dimensionless parameter $\bar{\bar{E}}$.

Inserting these ansatzes into the governing equations, one can group different terms together and form a particular solution by setting individual group contributions to zero. For the fluid problem, only one Stokes system has to be solved as a closure problem, which is

$$\mathcal{A}_{j,i_1} - \mathcal{K}_{ij,k_1 k_1} = \delta_{ij}, \tag{B 23}$$

$$\mathcal{K}_{ij,i_1} = 0, \tag{B 24}$$

$$\mathcal{K}_{ij}\big|_{\Gamma} = 0, \tag{B 25}$$

where Γ is the boundary with solid skeleton. This problem is solved in one unit-cell using periodic boundary conditions at all sides for both K_{ij} and A_j fields. This test problem is explained in the main paper near equations (3.9–3.10) in dimensional setting. For the elasticity, two quantities have

to be solved for. The first one is for the displacement tensor χ_{ijk}, which is governed by

$$\left[C^{\mathrm{sk}}_{ijkl}\left\{\varepsilon^1_{kl}\left(\chi^{mn}\right)+\delta_{km}\delta_{ln}\right\}\right]_{,j_1}=0, \tag{B 26}$$

$$\left[C^{\mathrm{sk}}_{ijkl}\left\{\varepsilon^1_{kl}\left(\chi^{mn}\right)+\delta_{km}\delta_{ln}\right\}\right]\hat{n}_j=0, \tag{B 27}$$

where elasticity $\bar{\bar{E}}$ has been cancelled out as a common factor. In this case, the displacement field is also exposed to periodic boundary conditions at all unit-cell sides. In order to ensure uniqueness of the solution, one can constrain the average values of the displacement tensors or enforce point constraints. Finally, there is a problem for proportionality vector before pressure

$$\left[C^{\mathrm{sk}}_{ijkl}\varepsilon^1_{kl}\left(\eta\right)-\delta_{ij}\right]_{,j_1}=0, \tag{B 28}$$

$$\left[C^{\mathrm{sk}}_{ijkl}\varepsilon^1_{kl}\left(\eta\right)-\delta_{ij}\right]\hat{n}_j=0, \tag{B 29}$$

where $\bar{\bar{E}}$ has been cancelled in the product of the elasticity tensor with the pre-factor $\eta/\bar{\bar{E}}$ in the displacement ansatz. Also this problem needs to be fixed using periodic boundary conditions and constraints on average values. The elasticity test problems are explained in the main paper near equations (3.3–3.3) and (3.6–3.7) in dimensional setting.

To derive the governing equations, the volume average operator, as defined in the main paper, equation (3.2), is used. We employ the volume averaging on the ansatz of the fluid velocity and arrive to

$$\langle u^{(0)}_i\rangle-\theta\bar{\bar{f}}\dot{v}^{(0)}_i=-\langle K_{ij}\rangle p^{(0)}_{,i_0}, \tag{B 30}$$

where $\theta=V_f/l^3$ is the volume fraction of the fluid (porosity). Note that $\dot{v}^{(0)}_i$ is independent of the microscale, therefore averaging only results in θ pre-factor. In order to continue derivations, we express the volume average of the first order velocity as

$$\langle u^{(1)}_{i,i_1}\rangle=\frac{1}{l^3}\oint u^{(1)}_i\hat{n}_i\,dS=\frac{1}{l^3}\int_\Gamma u^{(1)}_i\hat{n}_i\,dS+\frac{1}{l^3}\int_\Pi u^{(1)}_i\hat{n}_i\,dS, \tag{B 31}$$

where the closed surface integral has been divided into a part over the boundary between solid and fluid Γ and a part over the periodic boundary Π. Due to periodicity, the second term is zero, and the first one can be rewritten using boundary condition (A 3) at the solid skeleton as

$$\begin{aligned}\frac{1}{l^3}\int_\Gamma u^{(1)}_i\hat{n}^f_i\,dS=\frac{\bar{\bar{f}}}{l^3}\int_\Gamma \dot{v}^{(1)}_i\hat{n}^f_i\,dS=-\frac{\bar{\bar{f}}}{l^3}\int_\Gamma \dot{v}^{(1)}_i\hat{n}^s_i\,dS-\frac{\bar{\bar{f}}}{l^3}\int_\Pi \dot{v}^{(1)}_i\hat{n}^s_i\,dS=\\ =-\frac{\bar{\bar{f}}}{l^3}\oint \dot{v}^{(1)}_i\hat{n}^s_i\,dS=-\bar{\bar{f}}\langle\dot{v}^{(1)}_{i,i_1}\rangle,\end{aligned} \tag{B 32}$$

where we have added an integral of zero value, taken over the periodic boundaries of the solid. We have also transferred between adjacent domains with $\hat{n}^s_i=-\hat{n}^f_i$. Hence in this example we clearly see that the application of Gauss theorem in the microstructure relies on the assumption of periodicity. Nevertheless, the resulting model could also be used if the periodicity assumption is satisfied only approximately. Using the linear expression for the first order displacement (B 22), we arrive with the continuity equations for the macroscale

$$\langle u^{(0)}_i\rangle_{,i_0}=\bar{\bar{f}}\langle\chi^{kl}_{i,i_1}\rangle\varepsilon^0_{kl}\left(\dot{v}^{(0)}\right)-\frac{\bar{\bar{f}}}{\bar{\bar{E}}}\langle\eta_{i,i_1}\rangle\dot{p}^{(0)}, \tag{B 33}$$

where we have used the assumption that the porosity is uniform, and thus the derivative and volume averaging commute. To obtain the final equation in the macroscale, we introduce a stress

tensor, which is defined over the whole volume as

$$T_{ij}^{(0)} = \begin{cases} \Sigma_{ij}^{(0)} & \text{in } \Omega_f, \\ \bar{\bar{E}}\sigma_{ij}^{(0)} & \text{in } \Omega_s, \end{cases} \tag{B 34}$$

which can be differentiated with respect to the macroscale, and the result is

$$T_{ij,j_0}^{(0)} = \begin{cases} \Sigma_{ij,j_0}^{(0)} & \text{in } \Omega_f, \\ \bar{\bar{E}}\sigma_{ij,j_0}^{(0)} & \text{in } \Omega_s. \end{cases} \tag{B 35}$$

Now we average the divergence of the total stress tensor

$$\langle T_{ij,j_0}^{(0)} \rangle = \langle \Sigma_{ij,j_0}^{(0)} \rangle + \langle \bar{\bar{E}}\sigma_{ij,j_0}^{(0)} \rangle = -\langle \Sigma_{ij,j_1}^{(1)} \rangle + \langle \bar{\rho}\ddot{v}_i^{(0)} \rangle - \langle \bar{\bar{E}}\sigma_{ij,j_1}^{(1)} \rangle, \tag{B 36}$$

where we have used equations (B 6) and (B 11) to rewrite the divergence of zeroth order stress tensors. In the same way as in equations (B 31 – B 32), we can use Gauss theorem, periodicity and the boundary conditions, to write:

$$\langle \Sigma_{ij,j_1}^{(1)} \rangle = -\langle \bar{\bar{E}}\sigma_{ij,j_1}^{(1)} \rangle. \tag{B 37}$$

On the other hand, the zeroth order stress tensors can be averaged using explicit expressions, for the fluid stress we get

$$\langle \Sigma_{ij,j_0}^{(0)} \rangle = -\theta p_{,j_0}^{(0)} \delta_{ij}, \tag{B 38}$$

whereas for the solid stress we find

$$\langle \bar{\bar{E}}\sigma_{ij,j_0}^{(0)} \rangle = \langle \bar{\bar{E}} C_{ijkl}^{\mathrm{sk}} \left[\varepsilon_{kl}^1 \left(\chi^{mn} \right) + \delta_{km}\delta_{ln} \right] \rangle \left[\varepsilon_{mn}^0 \left(v^{(0)} \right) \right]_{,j_0} - \langle C_{ijkl}^{\mathrm{sk}} \varepsilon_{kl}^1 (\eta) \rangle p_{,j_0}. \tag{B 39}$$

To sum up, the final equation in the macroscale is

$$(1-\theta)\,\bar{\rho}\,\ddot{v}_i^{(0)} = \left[\bar{\bar{E}} C_{ijmn} \varepsilon_{mn}^0 \left(v^{(0)} \right) \right]_{,j_0} - \alpha_{ij} p_{,j_0}^{(0)}, \tag{B 40}$$

where the effective elasticity tensor is

$$C_{ijmn} = \langle C_{ijkl}^{\mathrm{sk}} \varepsilon_{kl}^1 \left(\chi^{mn} \right) \rangle + (1-\theta)\, C_{ijmn}^{\mathrm{sk}}, \tag{B 41}$$

and the tensor in front of the pore pressure is

$$\alpha_{ij} = \theta\delta_{ij} + \langle C_{ijkl}^{\mathrm{sk}} \varepsilon_{kl}^1 (\eta) \rangle. \tag{B 42}$$

The equations to be solved for the homogenized macroscale model expressed using relative Darcy's law (B 30) in the conservation of mass law (B 33) are

$$\left[\frac{\mathcal{E}}{E} \right] \dot{p}^{(0)} - \left[\left(\mathcal{K}_{ij} \frac{l^2}{\mu} \right) p_{,j}^{(0)} \right]_{,i} = -\alpha_{kl} \varepsilon_{kl} \left(\dot{v}^{(0)} \right), \tag{B 43}$$

$$(1-\theta)\,\rho_s \ddot{v}_i^{(0)} = \left[\left(E C_{ijmn} \right) \varepsilon_{mn} \left(v^{(0)} \right) \right]_{,j} - \alpha_{ij} p_{,j}^{(0)}, \tag{B 44}$$

where all variables and coordinates are now dimensional. These are the equations explained in the main paper (2.5) and (2.8). We note that if the solid density is very small, as allowed by the restriction (5.5) in the main paper, then the inertial term in solid displacement field can be neglected. Here we have also used the equality

$$\alpha_{kl} = \theta\delta_{kl} - \langle \chi_{i,i_1}^{kl} \rangle, \tag{B 45}$$

which has been derived by Mei & Vernescu[13, p. 258–259] and confirmed within the current work numerically. Now we can relate the parameters given by dimensionless unit-cell problems

to the parameters used in main paper (denoted using tilde) as

$$\tilde{\mathcal{E}} = \frac{\mathcal{E}}{E}, \qquad \tilde{K}_{ij} = K_{ij}\, l^2 \qquad \text{and} \qquad \tilde{C}_{ijkl} = EC_{ijkl}. \tag{B 46}$$

The pore pressure contribution tensor α_{ij} is dimensionless in both settings. Note that one can formulate the unit-cell problems also in dimensional setting, as written in the main paper. The microscale problems in the associated open-source software repository[46] are dimensionless, and one should use the relationships above in order to obtain coefficients for the dimensional equations.

C. Obtaining all model parameters using elasticity tensors

In the current section, we demonstrate how to arrive to all necessary elastic properties using only elasticity of solid skeleton and effective elasticity of solid skeleton, similarly as employed by Gopinath & Mahadevan[28]. First, we define an inverse of fourth-rank tensor $\mathbf{A}$ in index notation as

$$A^{-1}_{ijkl} A_{klmn} = \delta_{im}\delta_{jn}. \tag{C 1}$$

In order to obtain pore pressure contribution tensors using elasticity matrices, we start by multiplying effective elasticity tensor (B 41) by the inverse of the skeleton elasticity and obtain

$$C^{-1(\mathrm{sk})}_{ijkl} C_{klmn} = \left\langle \frac{1}{2}\left(\chi^{mn}_{i,j_1} + \chi^{mn}_{j,i_1}\right)\right\rangle + (1-\theta)\delta_{im}\delta_{jn}, \tag{C 2}$$

which we then multiply by the identity matrix from left to get

$$\delta_{ij} C^{-1(\mathrm{sk})}_{ijkl} C_{klmn} = \left\langle \chi^{mn}_{i,i_1}\right\rangle + (1-\theta)\delta_{mn}. \tag{C 3}$$

Now, in order to match the expression for the tensor α_{ij} (B 45), we subtract the obtained result from the identity matrix

$$\delta_{mn} - \delta_{ij} C^{-1(\mathrm{sk})}_{ijkl} C_{klmn} = \theta\delta_{mn} - \left\langle \chi^{mn}_{i,i_1}\right\rangle, \tag{C 4}$$

which due to symmetry in elasticity tensors $C_{ijkl} = C_{klij}$ can be rewritten as

$$\delta_{mn} - C_{mnkl} C^{-1(\mathrm{sk})}_{klij} \delta_{ij} = \theta\delta_{mn} - \left\langle \chi^{mn}_{i,i_1}\right\rangle, \tag{C 5}$$

which is the final expression for the pore-pressure contribution tensor. In tensorial notation, this expression can be written using double contraction operator as

$$\boldsymbol{\alpha} = \boldsymbol{\delta} - \mathbf{C} : \mathbf{C}^{-1}_{\mathrm{sk}} : \boldsymbol{\delta}. \tag{C 6}$$

This expression is exactly the same one as reported by Gopinath & Mahadevan[28, eq. 2.4]. Carrying out similar derivation for the coefficient $\mathcal{E}$, one can obtain

$$\mathcal{E} = \boldsymbol{\delta} : \left(\mathbf{C}^{-1}_{\mathrm{sk}} : [\boldsymbol{\alpha} - \theta\boldsymbol{\delta}]\right). \tag{C 7}$$

Therefore if one is successful in finding the effective elasticity tensor by other means, the pore-pressure contribution tensor α and coefficient $\mathcal{E}$ can be recovered using expressions (C 6–C 7).