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On the Closure Problem for Darcy's Law

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Abstract. In a previous derivation of Darcy's law, the closure problem was presented in terms of an integro-differential equation for a second-order tensor. In this paper, we show that the closure problem can be transformed to a set of Stokes-like equations and we compare solutions of these equations with experimental data. The computational advantages of the transformed closure problem are considerable.

Key words. Volume averaging, Stokes flow, closure problem, Darcy's law.

0. Nomenclature

Roman Letters

- $\mathscr{A}_{\beta\sigma}$ interfacial area of the β-σ interface contained within the macroscopic system, m^2
- $\mathscr{A}_{\beta e}$ area of entrances and exits for the β-phase contained within the macroscopic system, m^2
- $A_{\beta\sigma}$ interfacial area of the β-σ interface contained within the averaging volume, m^2
- $A_{\beta e}$ area of entrances and exits for the β-phase contained within the averaging volume, m^2
- $\mathbf{B}$ second-order tensor used to respresent the velocity deviation
- $\mathbf{b}$ vector used to represent the pressure deviation, m^{-1}
- $\mathbf{C}$ second-order tensor related to the permeability tensor, m^{-2}
- $\mathbf{D}$ second-order tensor used to represent the velocity deviation, m^2

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$\mathbf{d}$	vector used to represent the pressure deviation, m
$\mathbf{g}$	gravity vector, m/s^2
$\mathbf{I}$	unit tensor
$\mathbf{K}$	$\varepsilon\beta\mathbf{C}^{-1}, -\varepsilon_\beta\langle\mathbf{D}\rangle^\beta$, Darcy's law permeability tensor, m^2
$\mathbf{L}$	characteristic length scale for volume averaged quantities, m
ℓ_β	characteristic length scale for the β-phase, m
ℓ_i	$i = 1, 2, 3$, lattice vectors, m
$\mathbf{n}_{\beta\sigma}$	unit normal vector pointing from the β-phase toward the σ-phase
$\mathbf{n}_{\beta e}$	outwardly directed unit normal vector at the entrances and exits of the β-phase
p_β	pressure in the *β-phase, N/m^2*
$\langle p_\beta\rangle^\beta$	intrinsic phase average pressure, N/m^2
$\tilde{p}_\beta$	$p_\beta - \langle p_\beta\rangle^\beta$, spatial deviation of the pressure in the β-phase, N/m^2
$\mathbf{r}$	position vector locating points in the β-phase, m
$\mathbf{r}_0$	radius of the averaging volume, m
t	time, s
$\mathbf{v}_\beta$	velocity vector in the β-phase, m/s
$\langle\mathbf{v}_\beta\rangle^\beta$	intrinsic phase average velocity in the β-phase, m/s
$\langle\mathbf{v}_\beta\rangle$	phase average or Darcy velocity in the β-phase, m/s
$\tilde{\mathbf{v}}_\beta$	$\mathbf{v}_\beta - \langle\mathbf{v}_\beta\rangle^\beta$, spatial deviation of the velocity in the β-phase m/s
$\mathscr{V}$	averaging volume, m^3
V_β	volume of the β-phase contained in the averaging volume, m^3

Greek Letters

ε_β	$V_\beta/\mathscr{V}$, volume fraction of the β-phase
ρ_β	mass density of the β-phase, kg/m^3
μ_β	viscosity of the β-phase, Nt/m^2

1. Introduction

The boundary-value problem for the point velocity and pressure associated with Stokes flow in rigid porous media can be stated as

$$0 = -\nabla p_\beta + \rho_\beta\mathbf{g} + \mu_\beta\nabla^2\mathbf{v}_\beta, \tag{1.1}$$

$$\nabla\cdot\mathbf{v}_\beta = 0, \tag{1.2}$$

$$\text{B.C.1}\quad \mathbf{v}_\beta = 0, \quad \text{on } \mathscr{A}_{\beta\sigma}, \tag{1.3}$$

$$\text{B.C.2}\quad \mathbf{v}_\beta = \mathbf{f}(\mathbf{r}, \mathbf{t}), \quad \text{on } \mathscr{A}_{\beta\sigma}. \tag{1.4}$$

Here $\mathscr{A}_{\beta\sigma}$ represents the interfacial area for the fluid-solid system illustrated in Figure 1 and $\mathscr{A}_{\beta e}$ represents the area of entrances and exits of the macroscopic system shown in Figure 2. In general, the point velocity will not be known at $\mathscr{A}_{\beta e}$ and Equation 1.4 serves primarily as a reminder of what we *do not know* about the velocity field. It is important to note that we are not interested in solving Equations (1.1) through (1.4) *per se*; rather we are interested in using these equations to

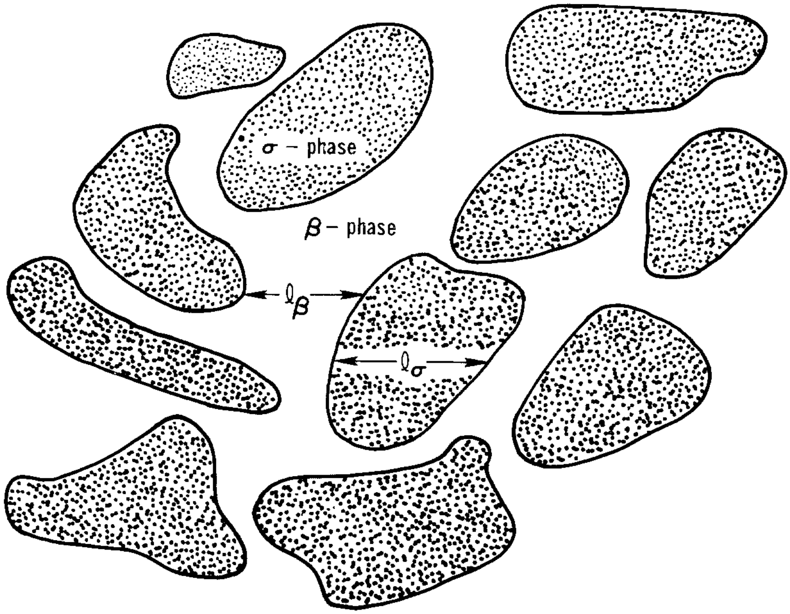

Fig. 1. Solid-fluid system.

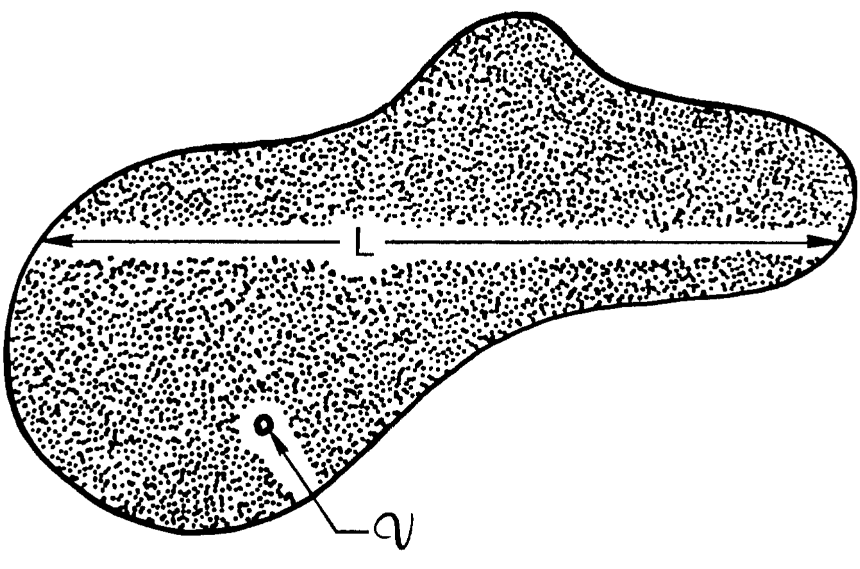

Fig. 2. Macroscopic system.

develop a *closure problem* that can be solved to produce the permeability tensor, **K**, which appears in Darcy's law.

The local volume averaged form of the continuity equation can be expressed as (Whitaker, 1986a, Equation (2.12))

$$\nabla \cdot \langle \mathbf{v}_\beta \rangle = 0 \tag{1.5}$$

and the volume averaged form of Equation (1.1) is given by

$$0 = -\varepsilon_\beta \nabla \langle p_\beta \rangle^\beta - \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \tilde{p}_\beta \, \mathrm{d}A + \varepsilon_\beta \rho_\beta \mathbf{g} + + \mu_\beta \left\{ \nabla^2 \langle \mathbf{v}_\beta \rangle + \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{v}_\beta \, \mathrm{d}A \right\}, \tag{1.6}$$

in which $\mathscr{V}$ is the averaging volume illustrated in Figure 2. This is Equation (2.24) of Whitaker (1986a) and arguments are given in that paper which lead to

$$\nabla^2 \langle \mathbf{v}_\beta \rangle \ll \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{v}_\beta \, \mathrm{d}A. \tag{1.7}$$

This simplification is associated with the length-scale constraint

$$\left(\frac{\ell_\beta}{L} \right)^2 \ll 1, \tag{1.8}$$

and it allows us to write Equation (1.6) as

$$0 = -\varepsilon_\beta \nabla \langle p_\beta \rangle^\beta - \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \tilde{p}_\beta \, \mathrm{d}A + \varepsilon_\beta \rho_\beta \mathbf{g} + + \mu_\beta \left\{ \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \nabla \mathbf{v}_\beta \, \mathrm{d}A \right\}. \tag{1.9}$$

In deriving this result the pressure was decomposed (Gray, 1975) according to

$$p_\beta = \langle p_\beta \rangle^\beta + \tilde{p}_\beta, \tag{1.10}$$

and in the previous development (Whitaker, 1986a) a similar decomposition was used for the velocity. In this treatment we seek a solution to the closure problem directly in terms of $\tilde{p}_\beta$ and $\mathbf{v}_\beta$ rather than in terms of $\tilde{p}_\beta$ and $\tilde{\mathbf{v}}_\beta$. This is the type of approach used by Sanchez-Palencia (1980) in a treatment based on the asymptotic analysis described in detail by Bensoussan *et al.* (1978). A recent example of this approach is given by Allaire (1989), and an extensive discussion is available in the work of Nassik (1979).

2. Closure

Use of Equation (1.10) in Equation (1.1) leads to the following closure problem.

$$-\nabla \tilde{p}_\beta + \mu_\beta \nabla^2 \mathbf{v}_\beta = \nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g} \tag{2.1}$$

$$\nabla \cdot \mathbf{v}_\beta = 0, \tag{2.2}$$

$$\text{B.C.1} \quad \mathbf{v}_\beta = 0, \quad \text{on } \mathscr{A}_{\beta\sigma}, \tag{2.3}$$

$$\text{B.C.2} \quad \mathbf{v}_\beta = \mathbf{f}(\mathbf{r}, t), \quad \text{on } \mathscr{A}_{\beta e}. \tag{2.4}$$

At this point one proposes (Sanchez-Palencia, 1980, Equation (2.25), Chapter 7) that $\tilde{p}_\beta$ and $\mathbf{v}_\beta$ can be expressed in terms of $\nabla\langle p_\beta\rangle^\beta - \rho_\beta \mathbf{g}$. This leads to the following representations for $\mathbf{v}_\beta$ and $\tilde{p}_\beta$:

$$\mathbf{v}_\beta = \frac{1}{\mu_\beta}\mathbf{D} \cdot (\nabla\langle p_\beta\rangle^\beta - \rho_\beta \mathbf{g}), \tag{2.5}$$

$$\tilde{p}_\beta = \mathbf{d} \cdot (\nabla\langle p_\beta\rangle^\beta - \rho_\beta \mathbf{g}), \tag{2.6}$$

and Tartar (1980) has proved that the formal asympotic expansion for $\mathbf{v}_\beta$ and $\tilde{p}_\beta$ converges to Equations (2.5) and (2.6) when

$$\left(\frac{\ell_\beta}{L}\right) \to 0. \tag{2.7}$$

The expressions given by Equations (2.5) and (2.6) are based on the idea that the nonhomogeneous term on the right hand side of Equation (2.1) represents the *only* source for the $\tilde{p}_\beta$ and $\mathbf{v}_\beta$-fields. This means that the nonhomogeneous term $\mathbf{f}(\mathbf{r}, t)$ in Equation (2.4) is assumed to have no influence on the $\tilde{p}_\beta$ and $\mathbf{v}_\beta$-fields.

In the method of volume averaging (Carbonell and Whitaker, 1984; Whitaker, 1986b; Whitaker, 1989), the boundary condition given by Equation (2.4) poses a *serious problem* since one can easily think of physical processes in which the entire pressure and velocity fields are controlled by the boundary condition imposed at $\mathscr{A}_{\beta e}$. This becomes especially clear if we express the boundary condition as

$$\text{B.C.1} \quad \langle \mathbf{v}_\beta\rangle^\beta + \tilde{\mathbf{v}}_\beta = \mathbf{f}(\mathbf{r}, t), \quad \text{at } \mathscr{A}_{\beta e}. \tag{2.8}$$

Since the characteristic length scale for $\tilde{\mathbf{v}}_\beta$ is ℓ_β, we are certain that the influence of $\tilde{\mathbf{v}}_\beta$ at the entrances and exits, $\mathscr{A}_{\beta e}$, can only penetrate into the macroscopic region a distance on the order of ℓ_β. On the other hand, the influence of $\langle \mathbf{v}_\beta\rangle^\beta$ specified at $\mathscr{A}_{\beta e}$ will be felt throughout the entire macroscopic region illustrated in Figure 2. For this reason, it is prudent to set up the closure problem entirely in terms of spatial deviations (Crapiste *et al.*, 1986); however, it will be instructive to proceed with the representations given by Equations (2.5) and (2.6). When these results are substituted into Equations (2.1) through (2.4), one obtains

$$-\nabla\mathbf{d} + \nabla^2\mathbf{D} = \mathbf{I}, \tag{2.9}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{2.10}$$

$$\text{B.C.1} \quad \mathbf{D} = 0, \quad \text{on } \mathscr{A}_{\beta\sigma}, \tag{2.11}$$

$$\text{B.C.2} \quad \mathbf{D} = \mathbf{G}(\mathbf{r}, t), \quad \text{on } \mathscr{A}_{\beta e}. \tag{2.12}$$

Here we have made repeated use of simplifications of the type

$$\nabla\mathbf{d} \cdot (\nabla\langle p_\beta\rangle^\beta - \rho_\beta \mathbf{g}) \gg \mathbf{d} \cdot (\nabla(\nabla\langle p_\beta\rangle^\beta - \rho_\beta \mathbf{g}))^T, \tag{2.13a}$$

$$\nabla\mathbf{D} \cdot (\nabla\langle p_\beta\rangle^\beta - \rho_\beta \mathbf{g}) \gg \mathbf{D} \cdot (\nabla(\nabla\langle p_\beta\rangle^\beta - \rho_\beta \mathbf{g}))^T, \tag{2.13b}$$

on the basis of Equation (2.7). Once again we note that the boundary condition given by Equation (2.12) reminds us of what we *do not know* about the **D**-field.

For the typical averaging volume associated with the development of Equation (1.9), we impose the constraint

$$\ell_\beta \ll \mathbf{r}_0, \tag{2.14}$$

where $\mathbf{r}_0$ is the radius of the spherical averaging volume represented by $\mathscr{V}$. Given Equation (2.14), we know that

$$A_{\beta e} \ll A_{\beta\sigma}, \tag{2.15}$$

where $A_{\beta e}$ represents the entrances and exits of the β-phase contained within the averaging volume. This allows us to express Equation 1.9 as

$$0 = \nabla\langle p_\beta\rangle^\beta + \rho_\beta \mathbf{g} + \frac{1}{V_\beta}\int_{A_{\beta\sigma}+A_{\beta e}} \mathbf{n}_\beta \cdot [-\mathbf{I}\tilde{p}_\beta + \mu_\beta \nabla\mathbf{v}_\beta]\,\mathrm{d}A, \tag{2.16}$$

which, in turn, takes the form

$$0 = -\nabla\langle p_\beta\rangle^\beta + \rho_\beta \mathbf{g} + \frac{1}{V_\beta}\int_{V_\beta} [-\nabla\tilde{p}_\beta + \mu_\beta \nabla^2\mathbf{v}_\beta]\,\mathrm{d}V. \tag{2.17}$$

This form of the volume averaged Stokes equations can also be obtained directly from Equation (2.1), thus it should not be surprising that Equation (2.17) is identically satisfied by Equations (2.5) and (2.6).

In order to obtain the intrinsic phase average form of Darcy's law, one simply forms the average of Equation (2.5) leading to

$$\langle\mathbf{v}_\beta\rangle^\beta = \frac{1}{V_\beta}\int_{V_\beta} \mathbf{v}_\beta\,\mathrm{d}V = \frac{\langle\mathbf{D}\rangle^\beta}{\mu_\beta}\cdot(\nabla\langle p_\beta\rangle^\beta - \rho_\beta\mathbf{g}). \tag{2.18}$$

The phase average form of Darcy's law is given by

$$\langle\mathbf{v}_\beta\rangle = -\frac{\mathbf{K}}{\mu_\beta}\cdot(\nabla\langle p_\beta\rangle^\beta - \rho_\beta\mathbf{g}), \tag{2.19}$$

in which the permeability tensor is defined as

$$\mathbf{K} = -\varepsilon_\beta\langle\mathbf{D}\rangle^\beta. \tag{2.20}$$

It should be clear that the closure problem given by Equations (2.9) through (2.12) will *not be solved* in the macroscopic region illustrated in Figure 2. Instead, it will be solved in some *representative* elementary volume (Bear, 1972). Such a representative elementary volume is illustrated in Figure 3, and if we wish to solve the *local problem* associated with Equations (2.9) through (2.11) we must be willing to discard the boundary condition given by Equation (2.12) and replace it with a spatially periodic condition (Sanchez-Palencia, 1980). This leads to the local closure

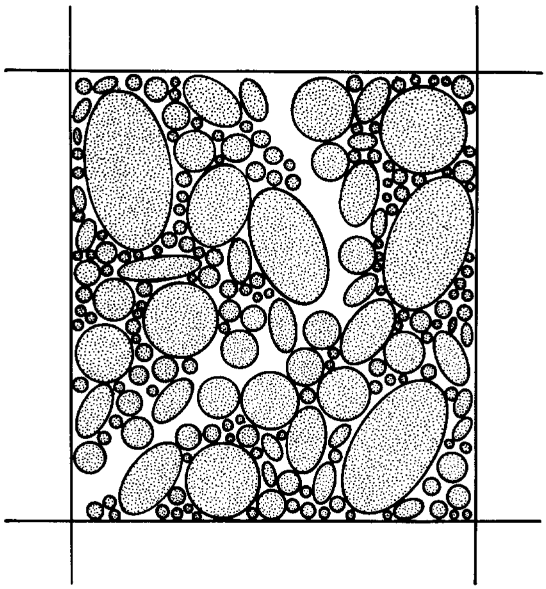

Fig. 3. Representative region of a porous medium.

problem given by

$$-\nabla \mathbf{d} + \nabla^2 \mathbf{D} = \mathbf{I}, \tag{2.21}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{2.22}$$

$$\text{B.C.1} \quad \mathbf{D} = 0, \quad \text{on } A_{\beta\sigma}. \tag{2.23}$$

Periodicity:

$$\mathbf{D}(\mathbf{r} + \ell_i) = \mathbf{D}(\mathbf{r}), \qquad \mathbf{d}(\mathbf{r} + \ell_i) = \mathbf{d}(\mathbf{r}), \quad i = 1, 2, 3. \tag{2.24}$$

This closure problem is equivalent to that developed by Whitaker (1986a) and the proof is given in Appendix A. While the two closure problems are equivalent, the computational advantages of Equations (2.12) through (2.24) are enormous. The computational procedure required for Whitaker's closure problem can be described as:

(1) An integro-differential equation is solved to produce the **b** and **B**-fields (see Equations (A.1) through (A.5)).
(2) The **B**-field must be differentiated in order to evaluate the area integral that defines a tensor **C** (see Equation (A.7)).
(3) The inverse of **C** must be developed to determine **K** (see Equation (A.6)).

By comparison, the closure problem given by Equations (2.21) through (2.24) represents a set of Stokes-like boundary value problems, the solution of which is described in Appendix B. Once the solution for the **D**-field is obtained, it is integrated (*not differentiated*) in order to produce the permeability tensor defined by Equation (2.20).

In making use of Equations (2.21) through (2.24) rather than Equations (2.9) through (2.12), we must keep three important points in mind:

(1) The length scale associated with **D** is the small length scale, ℓ_β. Under these circumstances, the boundary condition represented by Equation (2.12) can only influence the **D**-field in a region of thickness ℓ_β. Because of this, it has a negligible influence on the **D**-field within the macroscopic region shown in Figure 2.
(2) The periodic conditions indicated by Equation (2.24) are extremely weak conditions and should have only a minor influence on the **D**-field within the representative region illustrated in Figure 3.
(3) The **D**-field generated by Equations (2.21) through (2.24) is *filtered* (Cushamn, 1984) by the integration indicated in Equation 2.20. To predict a satisfactory value of **K**, one need only include in the representative region those geometrical characteristics that pass through the filter.

One must also keep in mind that the objective of the closure problem is to predict **K** and not to predict the details of the pressure and velocity fields. The details of these fields for flow in spatially periodic porous media are available in the work of Barrere (1990), and the role of the closure problem in the analysis of flow in both ordered and disordered systems is discussed by Quintard and Whitaker (1992).

3. Comparison Between Theory and Experiment

The closure problem represented by Equations (2.21) through (2.24) has been solved by Eidsath (1981) and by Zick and Homsy (1982), among others, as indicated in Appendix B. Theory and experiment can be compared in terms of a permeability, K, which is made dimensionless by the characteristic length $d_p\varepsilon_\beta/(1-\varepsilon_\beta)$. Here d_p is the effective particle diameter and the use of the factor $\varepsilon_\beta/(1-\varepsilon_\beta)$ provides a characteristic length that is proportional to the hydraulic diameter (Whitaker, 1983, Section 7.8). For fully three-dimensional flows the comparison between theory and experiment is shown in Figure 4. The theoretical results are those of Zick and Homsy (1982) for three different arrays of spheres, and the comparison with the three different experimental studies is very good. In addition we have shown in Blake–Kozeny correlation (Bird *et al.*, 1960, p. 199) with a Kozeny constant equal to five. This provides the following correlation for a wide range of experimental data.

$$\frac{K(1-\varepsilon_\beta)^2}{d_p^2\varepsilon_\beta^3} = \frac{1}{180}\,. \tag{3.1}$$

It is of some interest to note that this result is identical to the modified Ergun equation (with the Reynolds number equal to zero) proposed by Macdonald *et al.* (1979), and they estimate that Equation (3.1) can predict experimental results for

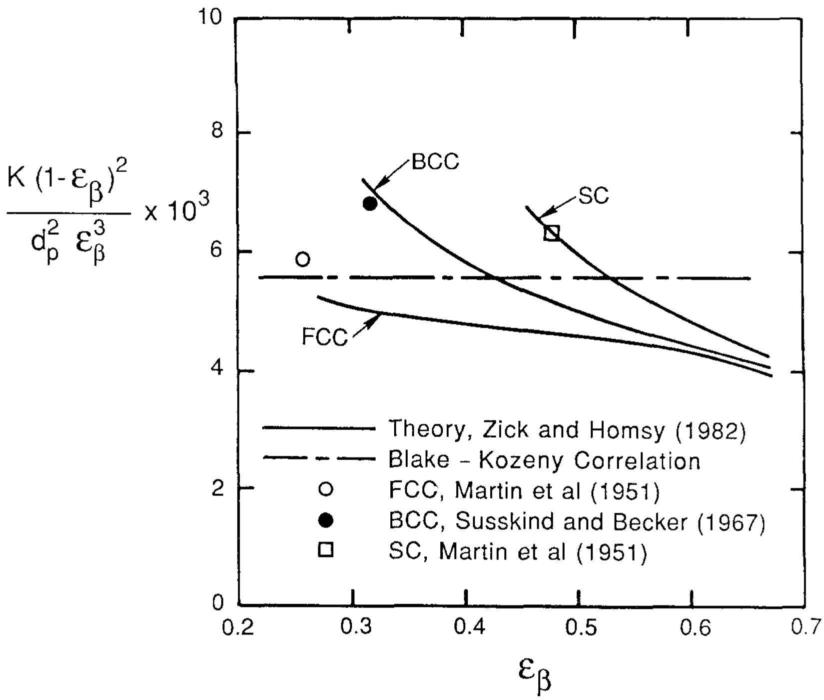

Fig. 4. Comparison between theory and experiment.

unconsolidated porous media with an accuracy of ±50%. The closure calculations of Zick and Homsy (1982) are in reasonably good agreement with the Blake–Kozeny correlation, in excellent agreement with three different experimental studies, and in excellent agreement with the theoretical studies of Snyder and Stewart (1966), Sorensen and Stewart (1974), and Sangani and Acrivos (1982).

For two-dimensional array of cylinders, Eidsath (1981) has carried out a direct comparison with the experimental work of Bergelin *et al.* (1950). The results are listed in Table I and they indicate very good agreement between theory and experiment.

Table I. Comparison between theory and experiment for flow through arrays of cylinders $\frac{K(1-\varepsilon_\beta)^2}{d_p^2\varepsilon_\beta^3} \times 10^3$

Model	Experiment Bergelin *et al.* (1950)	Theory Eidsath (1981)
Triangular array $\varepsilon_\beta = 0.418$	3.28	3.68
Inclined square array, $\varepsilon_\beta = 0.497$	2.31	2.37
Staggered square array, $\varepsilon_\beta = 0.497$	2.31	2.27

4. Conclusions

The integro-differential closure problem originally developed by Whitaker (1986a) can be transformed to a set of Stokes-like equations which can be solved with a considerable reduction in computational effort. Motivation for the transformation is based on the analysis of Sanchez-Palencia (1980) and the method of spatial homogenization (Bensoussan *et al.*, 1978). Available solutions of the closure problem are in good agreement with a variety of experimental studies.

Appendix A

The closure problem for a spatially periodic model of a porous medium was given by Whitaker (1986a, Section 3) as

$$-\nabla \mathbf{b} + \nabla^2 \mathbf{B} = \frac{1}{V_\beta}\int_{V_\beta} [-\nabla \mathbf{b} + \nabla^2 \mathbf{B}]\, \mathrm{d}V, \tag{A.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{A.2}$$

$$\text{B.C.1} \quad \mathbf{B} = -\mathbf{I}, \quad \text{on } A_{\beta\sigma}. \tag{A.3}$$

Periodicity:

$$\mathbf{B}(\mathbf{r} + \ell_i) = \mathbf{B}(\mathbf{r}), \qquad \mathbf{b}(\mathbf{r} + \ell_i) = \mathbf{b}(\mathbf{r}), \quad i = 1, 2, 3, \tag{A.4}$$

$$\langle \mathbf{B} \rangle^\beta = 0. \tag{A.5}$$

The permeability tensor was given by

$$\mathbf{K} = \varepsilon_\beta \mathbf{C}^{-1}, \tag{A.6}$$

where the second order tensor $\mathbf{C}$ was defined by

$$\mathbf{C} = -\frac{1}{V_\beta}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\mathbf{b} + \nabla \mathbf{B})\, \mathrm{d}A. \tag{A.7}$$

Since $\mathbf{B}$ and $\mathbf{b}$ are spatially periodic functions, Equation (A.7) can be written as

$$\mathbf{C} = -\frac{1}{V_\beta}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\mathbf{b} + \nabla \mathbf{B})\, \mathrm{d}A - \frac{1}{V_\beta}\int_{A_{\beta e}} \mathbf{n}_{\beta e} \cdot (-\mathbf{I}\mathbf{b} + \nabla \mathbf{B})\, \mathrm{d}A. \tag{A.8}$$

Use of the divergence theorem leads to

$$\mathbf{C} = -\frac{1}{V_\beta}\int_{V_\beta} (-\nabla \mathbf{b} + \nabla^2 \mathbf{B})\, \mathrm{d}V, \tag{A.9}$$

and this allows us to express Equation (A.1) as

$$-\nabla \mathbf{b} + \nabla^2 \mathbf{B} = -\mathbf{C}. \tag{A.10}$$

We now recognize that $\mathbf{C}$ is a constant tensor so that the multiplication of Equations (A.1) through (A.3) by $\mathbf{C}^{-1}$ leads to

$$-\nabla(\mathbf{b}\cdot\mathbf{C}^{-1})+\nabla^2(\mathbf{B}\cdot\mathbf{C}^{-1})=-\mathbf{I}, \tag{A.11}$$

$$\nabla\cdot(\mathbf{B}\cdot\mathbf{C}^{-1})=0, \tag{A.12}$$

$$\text{B.C.1}\quad (\mathbf{B}\cdot\mathbf{C}^{-1}+\mathbf{I}\cdot\mathbf{C}^{-1})=0,\quad \text{on } A_{\beta\sigma}. \tag{A.13}$$

We also recognize that $\mathbf{I}\cdot\mathbf{C}^{-1}$ is a constant tensor, and this leads to

$$-\nabla(\mathbf{b}\cdot\mathbf{C}^{-1})+\nabla^2(\mathbf{B}\cdot\mathbf{C}^{-1}+\mathbf{I}\cdot\mathbf{C}^{-1})=-\mathbf{I}, \tag{A.14}$$

$$\nabla\cdot(\mathbf{B}\cdot\mathbf{C}^{-1}+\mathbf{I}\cdot\mathbf{C}^{-1})=0, \tag{A.15}$$

$$\text{B.C.1}\quad (\mathbf{B}\cdot\mathbf{C}^{-1}+\mathbf{I}\cdot\mathbf{C}^{-1})=0,\quad \text{on } A_{\beta\sigma}. \tag{A.16}$$

Use of the definitions

$$\mathbf{D}=-(\mathbf{B}\cdot\mathbf{C}^{-1}+\mathbf{I}\cdot\mathbf{C}^{-1}),\qquad \mathbf{d}=-\mathbf{b}\cdot\mathbf{C}^{-1}, \tag{A.17}$$

leads to the following boundary-value problem:

$$-\nabla\mathbf{d}+\nabla^2\mathbf{D}=\mathbf{I}. \tag{A.18}$$

$$\nabla\cdot\mathbf{D}=0, \tag{A.19}$$

$$\text{B.C.1}\quad \mathbf{D}=0,\quad \text{on } A_{\beta\sigma} \tag{A.20}$$

Periodicity:

$$\mathbf{D}(\mathbf{r}+\ell_i)=\mathbf{D}(\mathbf{r}),\qquad \mathbf{d}(\mathbf{r}+\ell_i)=\mathbf{d}(\mathbf{r}),\quad i=1,2,3. \tag{A.21}$$

The average of $\mathbf{D}$ is given by

$$\langle\mathbf{D}\rangle^\beta=-\langle\mathbf{B}\rangle^\beta\cdot\mathbf{C}^{-1}-\mathbf{C}^{-1}, \tag{A.22}$$

and on the basis of Equation (A.5) this simplifies to

$$\langle\mathbf{D}\rangle^\beta=-\mathbf{C}^{-1}. \tag{A.23}$$

Under these circumstances, the permeability tensor given by Equation (A.6) takes the form

$$\mathbf{K}=-\varepsilon_\beta\langle\mathbf{D}\rangle^\beta. \tag{A.24}$$

This was given earlier as Equation (2.20).

Appendix B

In a numerical study of Stokes flow in spatially periodic porous media, Eidsath *et al.* (1983) expressed Equation (1.1) as

$$0=\nabla\tilde{p}_\beta-(\nabla\langle p_\beta\rangle^\beta-\rho_\beta\mathbf{g})+\mu_\beta\nabla^2\mathbf{v}_\beta, \tag{B.1}$$

and represented the pressure gradient term as

$$\nabla\langle p_\beta\rangle^\beta - \rho_\beta \mathbf{g} = |\nabla\langle p_\beta\rangle^\beta - \rho_\beta \mathbf{g}|\lambda. \tag{B.2}$$

This allowed Equation (B.1) to be expressed as

$$-\nabla u + \nabla^2 U = \lambda, \tag{B.3}$$

where u and $\mathbf{U}$ were defined as

$$u = \tilde{p}_\beta / |\nabla\langle p_\beta\rangle^\beta - \rho_\beta \mathbf{g}|, \qquad \mathbf{U} = \mu_\beta \mathbf{v}_\beta / |\nabla\langle p_\beta\rangle^\beta - \rho_\beta \mathbf{g}|. \tag{B.4}$$

Equation (B.3) was solved subject to

$$\nabla \cdot \mathbf{U} = 0, \tag{B.5}$$

$$\text{B.C.1} \quad \mathbf{U} = 0, \quad \text{on } A_{\beta\sigma}. \tag{B.6}$$

Periodicity:

$$\mathbf{U}(\mathbf{r} + \ell_i) = \mathbf{U}(\mathbf{r}), \qquad \mathbf{U}(\mathbf{r} + \ell_i) = \mathbf{U}(\mathbf{r}), \quad i = 1, 2, 3. \tag{B.7}$$

The $\mathbf{U}$-field was then integrated to provide

$$\langle \mathbf{v}_\beta \rangle = \frac{K}{\mu_\beta}(|\nabla\langle p_\beta\rangle^\beta - \rho_\beta \mathbf{g}|), \tag{B.9}$$

in which the permeability, K, was given by

$$K = -\lambda \cdot \langle \mathbf{U} \rangle. \tag{B.10}$$

The results were in good agreement with laboratory experiments. To reproduce the calculations of Eidsath *et al.* (1983) in terms of the closure problem given by Equations (2.21) through (2.24), one simply forms the scalar product with λ to obtain

$$-\nabla(\mathbf{d} \cdot \lambda) + \nabla^2(\mathbf{D} \cdot \lambda) = \lambda, \tag{B.11}$$

$$\nabla \cdot (\mathbf{D} \cdot \lambda) = 0, \tag{B.12}$$

$$\text{B.C.1.} \quad \mathbf{D} \cdot \lambda = 0, \quad \text{on } A_{\beta\sigma}. \tag{B.13}$$

Periodicity:

$$\mathbf{D} \cdot \lambda(\mathbf{r} + \ell_i) = \mathbf{D} \cdot \lambda(\mathbf{r}), \qquad \mathbf{d} \cdot \lambda(\mathbf{r} + \ell_i) = \mathbf{d} \cdot \lambda(\mathbf{r}), \quad i = 1, 2, 3. \tag{B.14}$$

One need only use the definitions

$$\mathbf{u} = \mathbf{d} \cdot \lambda, \qquad \mathbf{U} = \mathbf{D} \cdot \lambda, \tag{B.15}$$

to see that Equation (2.20) can be used to produce

$$K = \lambda \cdot \mathbf{K} \cdot \lambda = -\lambda \cdot \langle \mathbf{U} \rangle, \tag{B.16}$$

which is Equation (B.10). From this, one can see that the closure problem given by Equations (2.21) through (2.24) can be easily solved by traditional means. Zick and

Homsy (1982) have also solved Equations (B.11) through (B.14) in order to determine the permeability K. In their work the pressure was decomposed into a periodic part and a linear part, thus they expressed the pressure as (see their Equation (2.23))

$$p(\mathbf{x}) = -\frac{1}{\tau_0} F_j x_j + p'(\mathbf{x}), \tag{B.17}$$

in which $p'(\mathbf{x})$ is periodic and F_j represents the components of a constant vector. This decomposition is equivalent to that given by Equation (1.10) which is used in the Stokes equations to obtain Equation (B.1).

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