



Université  
de Toulouse

# THÈSE

En vue de l'obtention du

## DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Institut National Polytechnique de Toulouse (INP Toulouse)*

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Présentée et soutenue le 27/02/2018 par :

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**Modeling and simulation of flows over and through fibrous porous media**  
**Modélisation et simulation d'un écoulement autour d'une paroi poreuse et**  
**fibreuse**

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*MEGEP : Dynamique des fluides*

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# Abstract

Any natural surface is in essence non-smooth, consisting of more or less regular roughness and/or mobile structures of different scales. From a fluid mechanics point of view, these natural surfaces offer better aerodynamic performances when they cover moving bodies, in terms of drag reduction, lift enhancement or control of boundary layer separation; this has been shown for boundary layer or wake flows around thick bodies. The numerical simulation of microscopic flows around "natural" surfaces is still out of reach today. Therefore, the goal of this thesis is to study the modeling of the apparent flow slip occurring on this kind of surfaces, modeled as a porous medium, applying Whitaker's volume averaging theory. This mathematical model makes it possible to capture details of the microstructure while preserving a satisfactory description of the physical phenomena which occur.

The first chapter of this manuscript provides an overview of previous efforts to model these surfaces, detailing the most important results from the literature. The second chapter presents the mathematical derivation of the volume-averaged Navier-Stokes equations (VANS) in a porous medium. In the third chapter the flow stability at the interface between a free fluid and a porous medium, formed by a series of rigid cylinders, is studied. The presence of this porous layer is treated by including a drag term in the fluid equations. It is shown that the presence of this term reduces the rates of amplification of the Kelvin-Helmholtz instability over the whole range of wavenumbers, thus leading to an increase of the wavelength of the most amplified mode. In this same context, the difference between the isotropic model and a tensorial approach for the drag term was evaluated, to determine the most consistent approach for the study of the stability of this type of flows. This has led to the conclusion that the most relevant model is the one using the apparent permeability tensor. In the following chapter, based on this last result, the apparent permeability tensor, based on over one hundred direct numerical simulations carried out over microscopic unit cells, has been identified for a three-dimensional porous medium consisting of rigid cylinders. In these configurations the tensor varies according to four parameters: the Reynolds number, the porosity and the orientation of the average pressure gradient (the latter defined by two Euler angles). This parameterization makes it possible to capture local three-dimensional effects. This database has been set up to create, based on a kriging-type approach, a behavioral meta-model for estimating all the components of the apparent permeability tensor.

In the fifth chapter, simulations of the VANS equations are carried out on a macroscopic scale after the implementation of the meta-model, and this allows reasonable computing times. The validation of the macroscopic approach is performed on a flow in a closed cavity covered with a porous layer and a comparison with the results of a very precise DNS, homogenized *a posteriori*, shows a very good agreement and demonstrates the relevance of the approach. The next step has been the study of the passive control of the separation of the flow past a hump which is placed on a porous wall, by the same macroscopic VANS approach. Finally, general conclusions and possible directions of research in the field are presented in the last chapter.

# Résumé

Toute surface naturelle est par essence non lisse, elle est constituée de rugosités plus ou moins régulières et / ou de structures mobiles d'échelles différentes et variées. D'un point de vue mécanique des fluides, ces surfaces naturelles proposent des meilleures performances aérodynamiques en termes de réduction de traînée, d'augmentation de la portance ou de contrôle du décollement lorsqu'elle couvre des corps en mouvement ; et cela a été prouvé pour des écoulements de couches limites ou de sillage, autour de corps épais. La simulation numérique d'écoulements aux échelles microscopiques autour des surfaces « naturelles » demeure de nos jours encore hors de portée. En conséquence, la thèse a pour objet d'étudier la modélisation du glissement apparent de l'écoulement sur ce genre de surface, modélisée comme un milieu poreux, appliquant la théorie de la moyenne-volumique de Whitaker. Ce modèle mathématique permet globalement de présenter les détails de la micro-structure de ses surfaces en moyenne, tout en conservant une description satisfaisante des phénomènes physiques induits par l'écoulement. Le premier chapitre de ce manuscrit dresse un panorama des efforts antérieurs portant sur la modélisation de ces surfaces en précisant les résultats les plus importants issus de la littérature. Le deuxième chapitre présente la dérivation mathématique des équations de Navier-Stokes en moyenne volumique (VANS en anglais) dans un milieu poreux. Dans le troisième chapitre est étudiée la stabilité de l'écoulement à l'interface entre un fluide libre et un milieu poreux, formé par une série de cylindres rigides. La présence de cette couche poreuse est traitée par un terme de traînée dans les équations du fluide. On montre que la présence de ce terme réduit les taux d'amplification de l'instabilité de Kelvin-Helmholtz sur toute la gamme des nombres d'onde et ainsi augmente la longueur d'onde du mode le plus amplifié. Dans ce même contexte a été calculée la différence entre un modèle isotrope et une approche tensorielle pour le terme de traînée, afin de déterminer l'approche la plus consistante pour une étude de stabilité de ce type d'écoulement. Cela a mené à la conclusion que le modèle le plus pertinent est celui utilisant le tenseur de perméabilité apparent. Dans le chapitre suivant, en s'appuyant sur ce dernier résultat, le tenseur de perméabilité apparent est identifié sur la base d'une centaine de simulations numériques directes, pour un milieu poreux tridimensionnel constitué de cylindres rigides, où le problème de fermeture est abordé par la méthode VANS. Dans ces configurations ce tenseur varie en fonction de quatre paramètres : le nombre de Reynolds, la porosité et l'orientation du gradient moyen de pression (définie

par deux angles d’Euler). Cette paramétrisation permet de capturer les effets tridimensionnels locaux. Cette base de données ainsi constituée a permis de créer, sur la base d’une approche de type kriging, un méta-modèle comportemental pour estimer toutes les composantes du tenseur de perméabilité apparente.

Dans le cinquième chapitre sont menées des simulations des équations VANS à l’échelle macroscopique après implémentation du méta-modèle qui autorise des temps de calcul raisonnables. La validation de l’approche à l’échelle macroscopique est effectuée sur un écoulement dans une cavité fermé couverte d’une couche poreuse et une comparaison avec les résultats d’un DNS très précise, homogénéisés a posteriori montre un très bon accord et démontre la pertinence de la démarche. L’étape suivante a consisté en l’étude du contrôle du décollement pour un écoulement autour d’une bosse sur une paroi poreuse par cette même approche VANS macroscopique. Enfin des conclusions générales et des directions de recherche possibles dans le domaine d’étude sont présentées dans le dernier chapitre.

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# Chapter 1

## Poroelastic natural coatings

*Nature is the source of all true knowledge. She has her own logic, her own laws, she has no effect without cause nor invention without necessity*

- , Leonardo Da Vinci

### 1.1 Biomimetics of poroelastic coatings

Usually when we are asked to imagine some "rapid" object as an airplane, a boat or a car, the common sense leads us to think about it as smoothest as possible. However if we look around, nature seems not to agree with the previous statement. In fact most of the surfaces in Nature are not at all smooth, they almost always present more or less regular arrangement of discontinuities at various length scales. Since Nature have had a very large time-span to optimize this kind of surfaces we can be very certain that they are the best possible options. One should pinpoint that the non smoothness of these surfaces can be connected to some other biological functions rather than pure fluid dynamic performance, and of course it can be the case.

With that in mind we want to show the reader some of the most notable examples of "natural" surfaces applied in aerodynamic problems.

Probably the most notable example is the shark skin, in figure 1.1 a segment of the skin is depicted, as if appears to be, under the microscope.

The enlargement shows that the surface is made up by a series of overlapped denticles, and experiment shows that they can move and interact with the flow.

The shark "technology" has somehow been applied by Speedo®; they have fabricated their famous swimming suits with a surface that mimic the roughness of sharks; and they happen to break multiples world records. This controversial swimmers performance was due to the fact that they compress the body giving the swimmer a more and streamlined shape. Even though the company has been publicized their product as if it was a synthetic shark skin, Oeffner and Lauder [106] have shown that the texture of their swimming suits

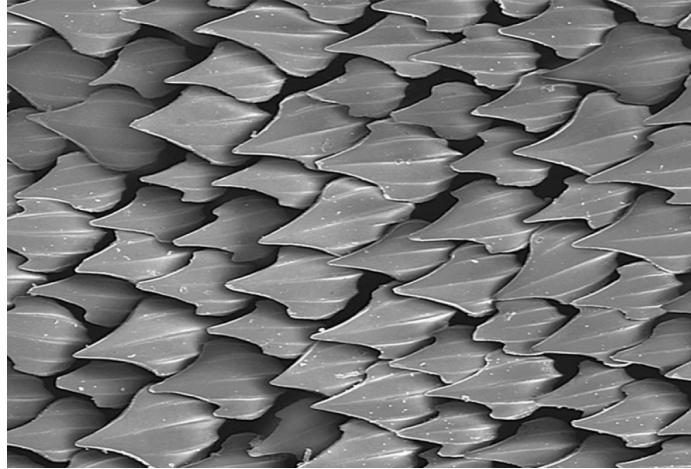


Figure 1.1: Microscope enlarged picture of the shark skin.

is somehow different from the shark dermal structure. In their work the authors have performed swimming experiment of a flat plate with different coatings and they did not find significant speed enhancement with a swimsuit like surface, but the measurements with real shark skin on the contrary give an appreciable improvement in the performance.

Poroelastic surfaces find also applications in aeroacoustics, as a matter of fact owls are well known for their particularly silent flight, especially in the high frequency spectrum. This characteristic is crucial for the owl in order to be able to capture its preys. Obviously it has inspired the scientific community to study their feathers, configuration and shape.



Figure 1.2: Feathers on owl's wing. Left: trailing edge. Right: leading edge. The differences in shape and mechanical properties, as rigidity, between the leading and trailing edge, is a consequence of the different flow regimes in the wing.

Several authors show promising results in characterizing the acoustic properties of the owl's skin and their physical mechanisms. In particular Lilley [86] presents three main characteristics of the owl, which can suppress its airborne noise: the feathers leading edge is shaped like a comb; the trailing edge forms a fringe and also multiple "filaments" in the

bottom surface of wing and legs. According to the same work the authors also present some experimental and empirical evidence on the aeroacoustics mechanism behind the three elements above.

Another example on the same topic is represented by Jaworski and Peake [75] who studied the acoustic scattering problem of a poroelastic half-plane encountering an incident plane wave. This configuration, a simplified owl's wing, explains how the properties of this surface can suppress the noise. They concluded that the combined effects of elasticity and porosity can produce a weaker noise amplification.

Recent computational simulations performed by Chen et al. [35] confirm that the leading edge shape of the feathers truly suppress noise and enhance the lift generation.

Bioinspired aerodynamic surfaces include another peculiar example in the butterflies wings. In figure 1.3 the surface of a "Peacock butterfly" is enlarged in order to show the multiple scales involved; the wing structure present firstly a series of overlapped scales similar to the shark, but if we look closely we can observe that each scale has a complicate permeable structure.

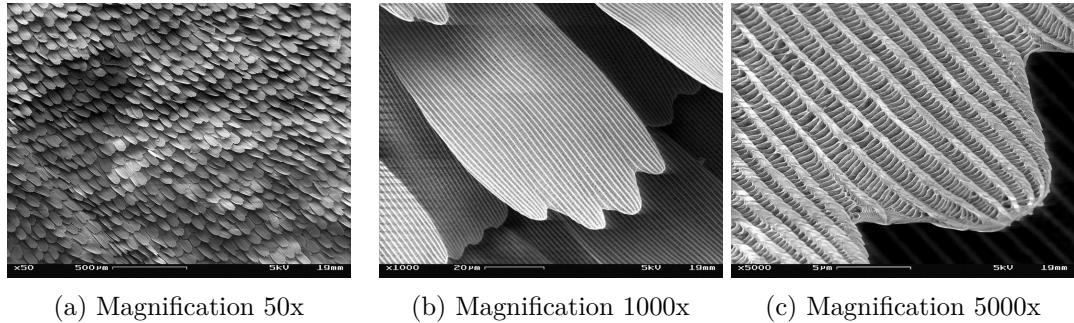


Figure 1.3: Particular of a Peacock butterfly wing, taken with a Scanning Electron Microscope. Images from [wikimedia.org](#)

The work of Slegers et al. [131] studies the effect of such porous structure on the performance in the flight of butterflies. Using cameras to measure the kinematics of their flight, they can compute their efficiency to "climb" (i.e. generate lift) and the stroke amplitude and frequency. The authors conclude that the porous structure of their wing gives a boost in climbing efficiency of 30%; this result clearly stresses out the importance of the poroelastic coating of the wings. Even though the butterfly flight aerodynamic is extremely complex, it is clear that the peculiar structure of the wing's surface is critical for their aerodynamic performances, as also Srygley and Thomas [133] confirm.

The last example concerns super-hydrophobic surfaces; these surfaces are water repellent, i.e. the water can slide over them with much less resistance, resulting in very small values of wettability. This behavior is caused by the microscopic structure which forms the surface (see figure 1.4), in reality the rugosities are arranged in a quasi-regular way, in

order to be able to capture air pockets that rest within the "valleys". These air inclusions provoke an effective slip at the air-liquid interface that cause the drag reduction; they also change the contact angle of droplets. The work of Bottaro et al. [22] summarize the above aspect and their applications.

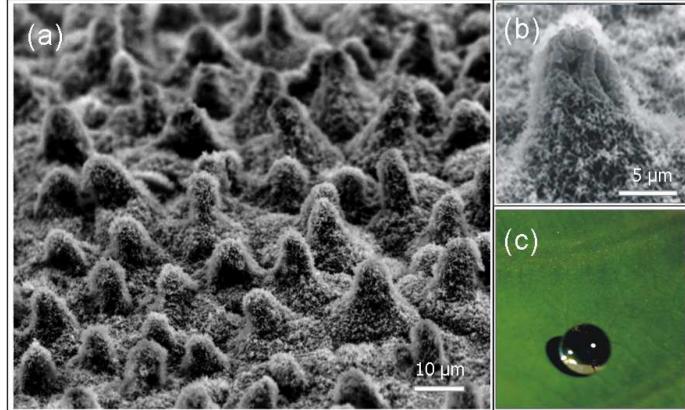


Figure 1.4: (a) Scanning electron microscopy (SEM) image showing the structure of lotus leaf, (b) higher order of magnification on the single protuberance forming the surface and (c) a water drop with high contact angle, attaining an almost spherical shape. Images from Stratakis et al. [134].

Interested readers can find more examples and broaden the above key aspect in Bhushan [19] and Tropea and Bleckmann [136].

### 1.1.1 Riblets and shark-skin surfaces

We have shown that natural surfaces can be an inspiration to find strategies in solving many problems concerning aerodynamics; in the following we focus especially on drag reduction.

It is known that the total drag contribution can be separated into different components and the classical decomposition is between viscous drag (sometimes referred to skin friction) and pressure drag.

$$\int_{A_\sigma} \left[ \underbrace{\left( \frac{p}{\rho} \mathbf{I} \right) \cdot \mathbf{n}_\sigma}_{\text{pressure drag}} + \underbrace{(\nu \nabla \mathbf{v}) \cdot \mathbf{n}_\sigma}_{\text{viscous drag}} \right] dA, \quad (1.1)$$

In (1.1)  $A_\sigma$  is the solid interface of some body where a no-slip condition is applied, and  $\mathbf{n}_\sigma$  is its outward normal. This section talks about the existing ways to reduce the viscous part of the drag since, historically, it has attracted more interest and/or it has made more progress.

In the coming text, we refer the wall shear stress in the turbulent case as:

$$\tau = ((\mu + \mu_t) \nabla \bar{v}) \cdot \mathbf{n}_\sigma = (\mu + \mu_t) \frac{\partial \bar{u}}{\partial y}, \quad (1.2)$$

where  $\mu_t$  is the turbulent viscosity and  $\bar{u}$  is the average velocity stream-wise component; in the laminar case the definition rest the same with the appropriate corrections (there is no turbulent viscosity and no notion of average velocity).

Most of the industrial applications involves turbulent flow; there is a lot of research that aims to reduce the skin-friction in this regime. Table 6.3.1 in the book of McLean [94] includes a wide list of technique already been proposed on the problem.

As the same author pinpoints, the most effective and, probably the most practicable solution are riblets. Riblets are alternating ridges aligned in the stream-wise flow direction and regularly arranged, as figure 1.5 shows.

These surfaces are capable to align the turbulent flow along the mean flow direction, smoothing the fluctuation of the cross-flow in the viscous sublayer. Reducing these fluctuations close to the surface, the turbulent momentum transfer is also reduced and so the shear stress, causing the reduction in skin-friction.

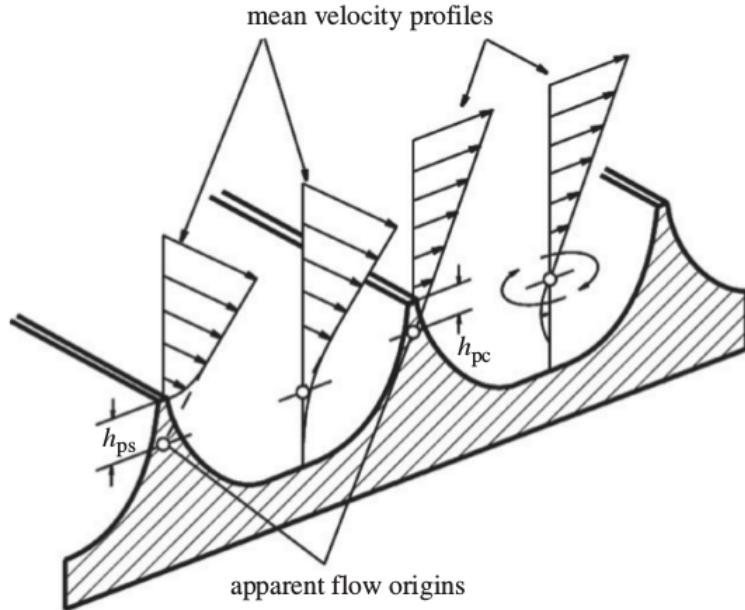


Figure 1.5: Schematics of the *protrusion height* concept. The mean velocity profiles for the stream-wise and cross-flow velocities are shown. In presence of a ridge it is possible to extrapolate the point of zero velocity from the velocity gradient outside the riblet; finding respectively, the *stream-wise protrusion height*  $h_{ps}$  and the *cross-flow protrusion height*  $h_{pc}$ . Image from Bechert et al. [16].

The viscous drag reduction correlates well with the spacing between the ridges expressed in wall units  $s^+$ ; the typical shape of the latter relation is depicted in figure 1.6, where the vertical axis shows the drag reduction against the  $s^+$ . This general shape of the curve, in which the skin friction decreases in certain range of spacing and then increase as the ridge spacing increases, is caused by a competition between the capacity of riblets to obstruct lateral fluid flow and, the increase of penetration of high speed vortices inside this manufactured roughness.

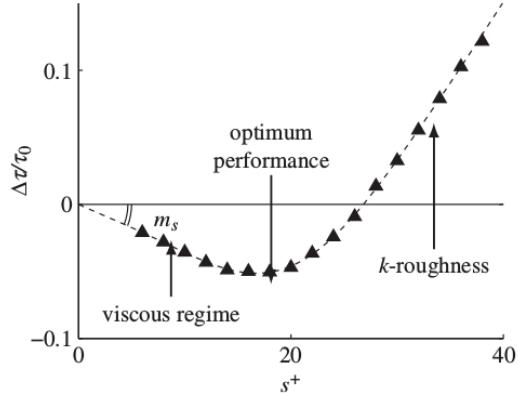


Figure 1.6: Example of drag reduction relation to the ridge spacing. The maximum performance is normally around  $s^+ = 15$ , the picture shows also that when the riblet are really tight one another the laminar case is retrieved. On the contrary when the riblets are far away one another their performance is comparable to rough plate case. Image from Jimenez et al. [76].

This last physical explanation of the riblets performances is presented in the schematics 1.7, where the gray areas show high skin-friction regions caused by the down-wash motion generated by the near-wall vortices. Thus it is clear that, when the riblets are too big, the vortices can penetrate inside the groove and, actually, increase the skin-friction, due to larger area exposed to the local velocity. On the contrary, when the riblets are smaller, the high speed vortex "touches" only the tip of the ridges, so only a small local area of the surface experiences high-shear stresses.

The slope  $m_s$  of the curve in figure 1.6 can be predicted by linear stability theory or by means of empirical correlations García-Mayoral and Jiménez [55].

Computing the performance of such surfaces can be expensive, since the most reliable quantitative theory for such problems consist of direct numerical simulations (DNS) or experiments. However there is only one more theory, besides the already cited expensive ones, that uses the concept of *protrusion height*, shown in figure 1.5, to correlates the shape of these protrusions to the drag reduction (cf. Luchini et al. [87]). In this way the *protrusion height* is defined as the vertical distance between the riblet top ridge and the point of zero

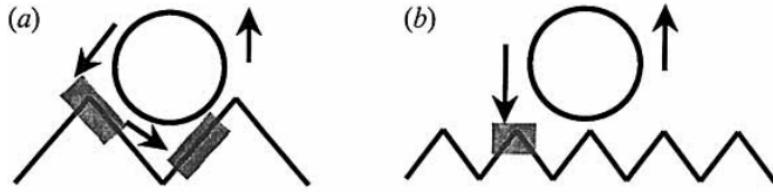


Figure 1.7: Two different sizes of riblets are shown when interacting with a sublayer vortex. In gray it is represented the area where friction is important. Clearly when both sizes are comparable the surface experience a larger friction and the performance is lowered. Image from Choi et al. [36].

velocity, extrapolated from the constant velocity gradient outside above the protrusions. It appears that especially the difference of protrusion heights, computed from the stream-wise  $h_{ps}$  and cross-flow  $h_{pc}$  profile, correlates very well with the drag reduction, and the two quantities can be computed with a simple Stokes problem over the local geometry of the grooves. Further more the last result has been analyzed by Gomez-de Segura et al. [62] and come out with an empirical law for the drag reduction, relating the previous protrusion heights with the permeability expressed in wall units:

$$DR \approx 0.04 \left( \sqrt{K^+_s} - \sqrt{K^+_c} \right), \quad (1.3)$$

where  $K^+_s$  and  $K^+_c$  are the stream-wise and cross-flow permeability tensor components. This law establishes an instrument to estimate the drag reduction from a given geometry of the wall (the permeability tensor can be computed within the porous media homogenization approach as chapter 2 explains).

Another important characteristic of riblets performance is that they are robust in off-design conditions, such as in presence of yaw (misalignment between flow and riblets ridges) and tip ridges erosion (García-Mayoral and Jiménez [54]).

Besides some very specific application such as sailing competitions (the hulls of the USA challengers in the America's Cup 1987 and 2010), the massive utilization of this technology is still in question. Producing such surfaces in a larger area, like the roof of a car or the wing of an airplane, can be an issue for a routine use; because riblets size need to be very little to be effective.

Anyhow riblets like surface has been observed in nature for many years, for example Martin and Bhushan [91] found out that skimmer birds (*Rynchops*) have riblets like grooves in their beak, since they fly with it under the water surface to catch fishes. However, as already introduced, the most clear example of such natural surfaces are shark skin. In his review, Dean and Bhushan [43] present the status of the shape optimization that has been done on the riblets trying to mimic the typical sawtooth shape seen on shark skin, showing that improvements of such geometries over the classical ones has yet to be proven. Shape

optimization on riblets geometry has been studied, the findings show that just a few % can be improved on the base line geometry Bechert et al. [16].

There are, actually, some controversial results in literature stating that surfaces, with actual shark skin replica, can indeed increase drag. Boomsma and Sotiropoulos [21] performed some simulations on actual shark skin denticles using the immersed boundary method; the authors find that in some configuration the actual drag increase up to 40%, but even though their numbers are probably too large (it is known that the immersed boundary method can generate some errors especially in computing forces at high Reynolds number), this can be a clue that the shark skin does not work with the same mechanism as riblets does.

Experiments on such geometries are available in literature (Bechert et al. [15]). The authors have built a synthetic surface, made by artificial shark denticles posed on top of springs, and measure that, even with the introduction of surface elasticity, the actual drag was increased. However, they pinpoint that the actual shark flow regime is not steady as the experiments that he performed, and they speculate that the excellent swimming performance of the shark come from the separation control that flexible denticles can increase in the periodic oscillating flow that the swimming generates.

In addition an experiment using DPIV on a NACA profile covered with actual skin samples of "Isurus oxyrinchus" mako shark, has been performed by Lang et al. [83], confirming that the flexibility of sharks denticles acts as a passive flow control in order to avoid early separation. In fact, the experiment proves that for angles of attack larger than  $15^\circ$  the flow reversal is almost completely avoided. The same author introduce the importance of the different geometries in various part of the body that obviously experience disparate flow conditions. Motta et al. [100] perform a detailed collection of flexibility and scale measurement of different shark species that can be valuable for future studies.

Again, swimming experiments from Oeffner and Lauder [106], who used a flat plate covered with real shark skin, also confirm the previous flow control mechanism and also make some conjectures about possible thrust enhancing controlled by the same movable scales that can move away the leading edge vortex.

Also Itoh et al. [74] show that movable rugosities can outperform riblets, the authors measure the drag reduction of a seal fur (that present fibrous movable surface) against a riblet surface in an experimental channel; their results are shown in figure 1.8.

Compliant surfaces can, in reality, move accordingly to the surface pressure gradients along the boundary layer and so respond to the pressure fluctuation over the surface itself. This mechanism is already known to be beneficial in delaying the transition to turbulence and many authors have presented theoretical and experimental evidence on the effectiveness of this solution Carpenter [34], Bushnell et al. [29].

In conclusion, we have seen that, in order to reduce turbulent skin-friction drag, riblets and natural surfaces use various mechanism such as: sublayer vortices interaction, compliance and separation control. Such solutions have proven to be effective in some cases, but they are related mostly in reducing the viscous component of the total drag. In the

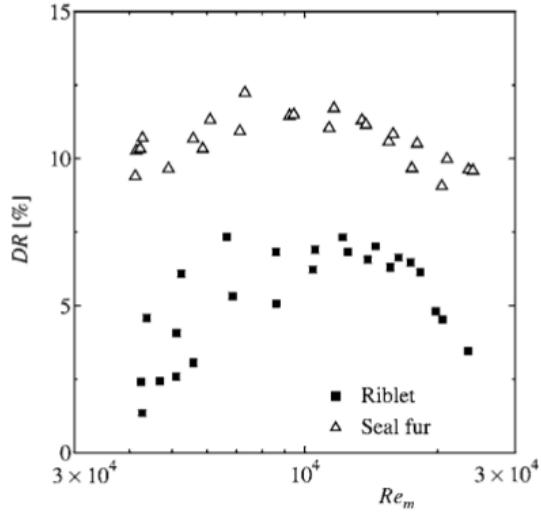


Figure 1.8: Performance comparison between a riblet surface against a seal fur. The drag reduction has been computed as:  $DR\% = \frac{\Delta\tau}{\tau_0}\%$  Image from Itoh et al. [74].

next section we introduce another class of solutions that try to act mostly on the pressure component.

### 1.1.2 Permeable surfaces

As permeable surfaces we indicate permeable coatings that usually have an important thickness; in contrast to riblets, in which the vertical extension outside the wall is limited. In this case the flow can penetrate deep into the porous surface and generate complex interaction mechanism. The next sections presents an overview of the most notable applications of such permeable surfaces.

### 1.1.3 Bluff bodies

There is some experimental evidence that, in laminar regime, generation of some *slip velocity* at the interface between the permeable surface and a fluid, can decrease the skin friction (Beavers and Joseph [14]). However, in the turbulent case it seems that the instabilities developing at the interface can cause an increase in drag up to 40% (Jimenez et al. [76] and Breugem et al. [23]); these instabilities mechanism is further explained in the section 1.3. It is important to precise that the permeable surfaces cited in the above references are all rigid.

The pressure contribution to the resistance is usually the most significant one in bluff

bodies applications, and even in highly streamlined body it is around 10% of the total drag. Researchers tried to find a way to modify the pressure distribution around a bluff body to reduce the associated resistance, and also damp the force oscillation on the body (drag and/or lift).

The pressure drag on a bluff body depends mostly on the difference between the low pressure on the rear part of the body, where there is usually a separated flow region, and the high pressure in the forward part. This idea is sketched in figure 1.9 where two different pressure distributions are shown; the black one represents the classical solid body, and the green one is the one with a porous layer at the back of the body.

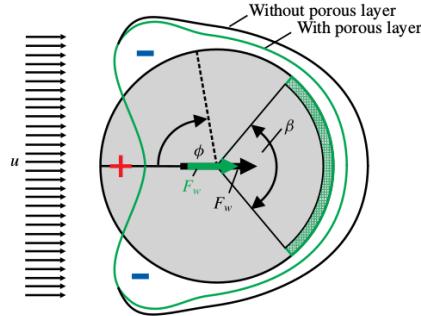


Figure 1.9: Diagram showing an example of angular pressure distribution around a cylinder for viscous flow. The black line is the case of a solid body, the green one is the modified pressure in presence of a porous layer at the rear part. Image from Klausmann and Ruck [78].

The favorable increase of the pressure in the rear point is due to the low speed laminar flow in the porous media that is ejected in the back region where separation takes place. Even in very high speed turbulent flow, the fluid inside permeable surface exhibits a very high energy loss due to the strong dissipation that the medium provides, resulting in a low speed flow ejected downstream of the body.

The permeable interface, producing a slip velocity, can modify the boundary layer that develops above it producing less shear and vorticity, so, it can also modify the stability characteristics of the flow. The instability around a cylinder is due to the shear layer that forms in the top part of the body, when the flow starts to decelerate. This shear layer exhibits a Kelvin–Helmholtz type instability that develops in the classical Von-Karman wake.

This two hypothetical mechanisms has been tested using numerical simulation by multiple authors: Bruneau and Mortazavi [26], Bruneau and Mortazavi [27], Bhattacharyya and Singh [18], Naito and Fukagata [101] and Mimeau et al. [98]. These works study the flow around some classical two dimensional bluff bodies (cylinder, square cylinder, Ahmed body section, 3D hemisphere) with the add of a porous layer.

These works show some very promising results on multiple quantities, like: decrease of enstrophy, lower oscillations in lift signal, drag reduction, regularization of the wake and lower pressure gradients, even if the porous medium is rigid in their case. An example of turbulent flow field downstream to a square cylinder is shown in figure 1.10; the picture display how the porous layer strongly regularize the wake.

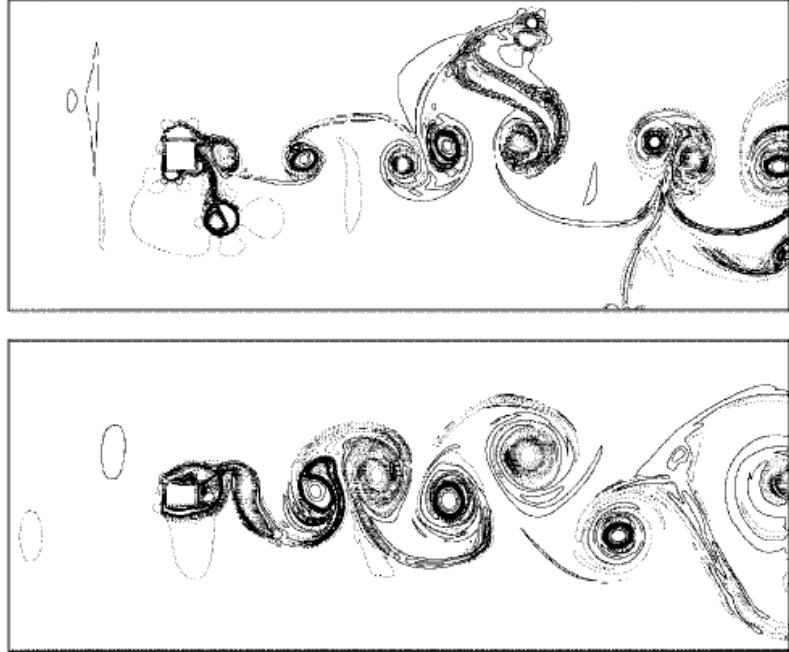


Figure 1.10: Square cylinder vorticity contour for  $Re = 30000$ . Top: solid case. Bottom: porous case with layer extension  $h = 10\%D$ .

The previous works indicate that porous medium parameters, like the medium resistance or its vertical extension, have important effects on the results. The variety of results seems to agree (at least qualitatively) that increasing the porous medium extension over a certain limit is not beneficial, and they also show that the resistance of the medium should not be excessive in order to be effective (high/medium porosity are the best).

In all these works we also observe a reduced pressure drop between the front and the rear of the body, a reduced drag, a delays in the vortex shedding and regularization on both the frequency and amplitude of shredded vorticities oscillations.

These works lead to very optimistic results, but they should be taken with some care; only few cases are three-dimensional, they all use a modeling approach for the porous medium based on a simplified version of the VANS (Volume Average Navier-Stokes equations, see section 1.2.2), without performing any validation of the method. Sometimes they also use the equations outside their field of validity (there is some discussion in the

scientific community in using the previous version of VANS equations for highly turbulent flows).

The lack of validation reflects the fact that reliable experiments of such porous coatings are almost non existent in literature. There is also some confusion in the community on how to compute forces on such bodies surrounded by a porous coating. These differences led some authors (as Naito and Fukagata [101]) to over-estimate the forces and their predictions are not inline with the literature. Moreover Caltagirone [30], with some theoretical bases, establishes that the approach used by Bruneau and Mortazavi [26] is the correct one for that specific version of the VANS used by all the previous authors. **INSERT A REFERENCE TO THE SECTION IN CHAPTER 2 WHERE THERE IS THE FORMULA TO COMPUTE FORCES**

The approach of Favier et al. [49] differentiates itself from the previous works; the authors use a numerical method that includes the dynamic of a moving porous medium made of fibers at the back of a cylinder. Their results in a laminar flow case agree with the prediction of a stabilization of the wake and show some more realistic values of drag reduction, about 15%. However the difficulties in this approach lies in the medium dynamic; it introduce many mechanical parameters that are not easily identifiable for natural surfaces.

A similar model has been used by Venkataraman and Bottaro [138], in which they applied a movable porous coating in the top part of NACA airfoil. In this case the synchronization between the oscillations of the structures and the natural frequency of the fluid is responsible for the pressure distribution modification. They have shown the robustness of this solution in a wide range of angle of attack, in the best case they have found some lift enhancement and a drag reduction around 10%.

Later on Rosti et al. [124] worked on a similar configuration with only one movable flap on the low pressure side of the airfoil; the results both numerical and experimental qualitatively agree (on the flow mechanism) with the results in the complete porous case.

**SHOW SOME RESULTS ON THE 3D SPHERE USING HOMOGENIZATION WHEN IT WILL BE PUBLISHED giuseppe ... [61]**

The very few experiments in literature on this porous coatings show less promising results associated to drag reduction.

For example, Heenan and Morrison [69] perform an experiment in which they take a backward facing step with a porous insert in the re-circulation region. Their measurement shows a 13% decrease of the peak of pressure at the wall and a relocation of the detachment point further downstream. A maximum of 9% of drag reduction was observed. The effect of adding a porous surface in this case was to limit the pressure fluctuations that cause the re-circulation bubble unsteadiness.

Later on Klausmann and Ruck [78] studied a 3D cylinder with a porous insert in the back (as in figure 1.9); the authors use a wind tunnel testing with pressure measurements around the body and particle image velocimetry (PIV) flow capture. Their results confirm that the porous layer on the leeward side increases the pressure in that zone, causing the reduction of drag. The drag reduction measured was around 10% over various Reynolds

number (in turbulence range); this value was sensible to the geometrical parameters of the medium as the position and its size. At our knowledge this is the first example of actual measurements of flow quantities using PIV, that can later be used to perform some validation on different numerical models. The above results are partially confirmed by a similar experimental analysis by Grizzetti et al. [66].

Some other experimental data can be found in the case of flow over aquatic canopies (Zhang and Nepf [156], Segalini et al. [126] and Hamed et al. [68]) even though the published data is limited and the experiments show the presence of a free surface that increase the difficulty of the problem and limit the possible use as a simple validation.

From this section the main physical mechanism that is tied to permeable surfaces has been introduced. Even though the different approaches in literature seems to be discordant in the predicted values of some fundamental items such as the forces, a general trend on all data shows that porous coatings can be effectively used in many situations. It is clear that the scientific community needs much more experimental data in order to develop new and improved numerical and theoretical models for such permeable coatings.

#### 1.1.4 Canopy flow

Another important class of flows over poroelastic carpets are the *canopy flows* as named in literature. These type of problems involve flows over flexible slender structure such as trees and aquatic vegetation. The behavior of wind over plants is very important in a large variety of fields, like: the transport of substances as  $CO_2$  and nutrients or preventing agricultural damage (wind-throw of crop fields); also some similarities with urban canopies can be found Ghisalberti [57].

The boundary layer profile over such canopies differs substantially from the rough wall one, as figure 1.11 shows. The vegetation resistance causes the creation of an inflection point in the mean velocity profile that lead to a mixing layer type of instability near the vegetation top. As a consequence of such instabilities Finnigan [51] indicate that the vegetation can heavily modify the turbulence spectra as a result of the interface instabilities and the coherent structures above it. The two pictures at the bottom of figure 1.11 outline the above statements; the spectrum in case of canopy flow presents a larger peak in the frequency of the mixing layer instability, a steeper slope in the energy cascade part due to the larger dissipation inside the permeable layer and possible high frequency peaks associated to the swinging of the pants that can emit or absorb small scales vortices.

Is it clear from literature that the dynamic of the permeable substrate made by vegetation is extremely important and should always be taken into account to fully generalize the physics in such problems involving moving canopies; Nepf [102] shows how the interface between aquatic plants and the free flow can be largely modified due to the movement of the fibers.

In order to discriminate the different behavior of the fibrous structure it is convenient to introduce some important non-dimensional parameters used in fluid structure interaction

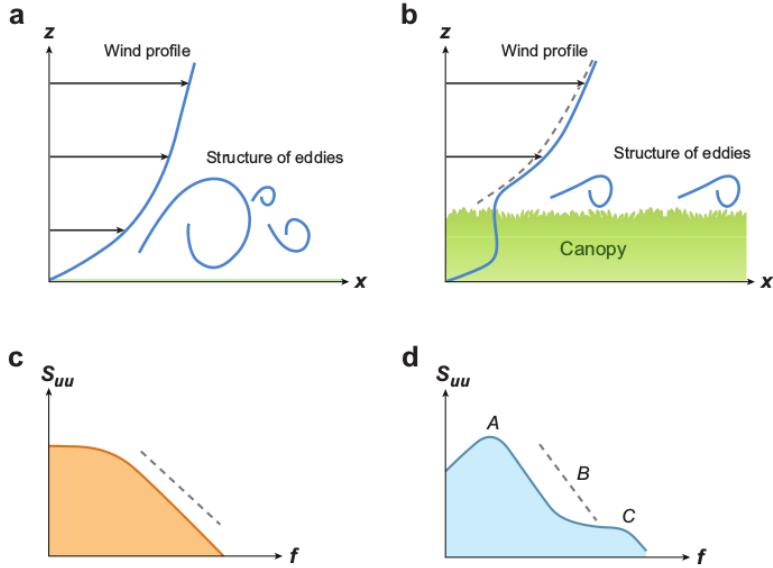


Figure 1.11: Figure **a** and **b** show respectively the schematics of the mean flow over a rough wall and a canopy flow; the difference in the eddy size is clear, also the inflection point in the canopy flow velocity profile is transparent. The figure **c** and **d** instead show the turbulent spectra for the two different flows above, in case of rough wall a Kolmogorov type of spectra can be retrieved; in case of canopy flow it is possible to see a larger peak in the frequency of the mixing layer instability, a steeper slope in the energy cascade part and high frequency peaks at high frequencies. Image from De Langre [42]

problems:

$$m^* = \rho_\beta / \rho_\sigma, \quad C_Y = \rho_\beta U^2 s^3 / E, \quad s = L/\ell,$$

The first one is the *mass ratio*, the second is called *Cauchy number* and the last one is the *slenderness* of the structure. The mass ratio  $m^*$  is a measure of the added mass effects caused by the solid inertia, however these effects are usually negligible in case of fibrous permeable media. The Cauchy number  $C_Y$  defines the static deformation of a fiber caused by the fluid flow ( $E$  is the modulus of elasticity of the solid material); when Cauchy number is greater than unity, important deformations are expected. This last parameter is extremely important since it controls a phenomenon called *reconfiguration* that leads to drag reduction (Gosselin and De Langre [64] and Alvarado et al. [4]). The reconfiguration can be defined as the capability of the structure to adopt a new shape when forced by a flow, usually it becomes more streamlined to reduce its exposed frontal area with the aim to reduce the total drag. When dealing with this phenomenon one should take into account of the frontal area  $A$  and the drag coefficient  $C_D$  as an ensemble, in order to avoid misinterpretation of the drag reduction; in figure 1.12 the ratio of the couple  $AC_D$

has been represented for different natural structures against the Cauchy number and it's evident that for a  $C_Y > 1$  a drastic drag reduction can be observed.

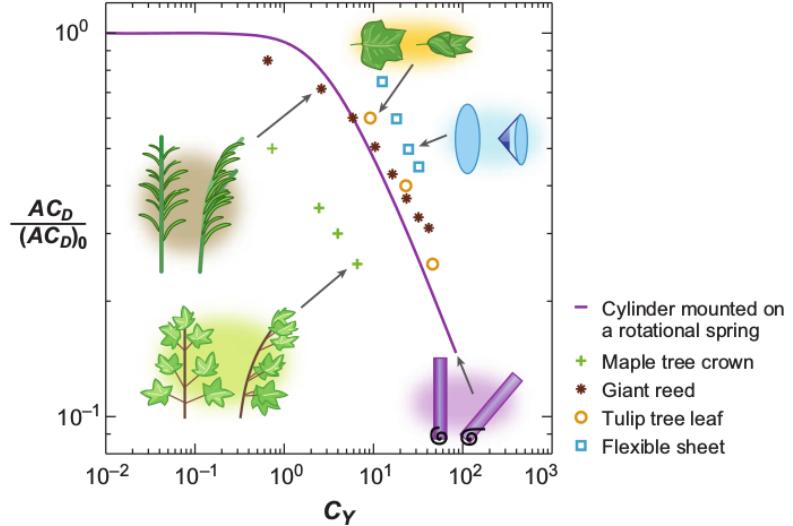


Figure 1.12: The effects of the Cauchy number  $C_Y$  on the drag reduction are presented in the figure. The drag reduction has been represented as the ratio between the frontal area  $A$  and the drag coefficient  $C_D$  at static condition (subscript 0) and the dynamic condition (no subscript). Image from De Langre [42]

The overall reconfiguration of the permeable medium can lead to pressure recovery and a wake regularization applied when applied to a bluff body, as the experiments by Gosselin and De Langre [64] show.

Another non-dimensional number to mention is the *reduced velocity*, that can be derived from the previous ones:

$$U_R = \sqrt{C_Y s/m^*}$$

This number is used dealing with vortex induced vibration of slender structures; when it is near to one, dynamical coupling between the fluid and the structure is expected, such as resonance or lock-in phenomena.

Canopies can also help to prevent separation in presence of adverse pressure gradients, Belcher et al. [17] show an analysis of the flow over an hill covered with canopies using either numerical and experimental data; the authors show how the permeable layer can present a re-circulation region inside the canopy in the decreasing slope side of the hill, this zone move the separation away from the flow over the hill to the internal structure of the canopy.

It is important to pinpoint that the above results are restricted to fibrous or slender structures and they cannot be extrapolated in general for different porous structure and

shapes, even though similar mechanisms are expected.

The research on canopy flow embraces a wide range of configurations and this makes very difficult the comparison of the results since most of the authors use very different models in a lot of various regimes of velocities, using flexible structures with very different shapes. Even if experiments are easier to find, like Segalini et al. [126], Segalini et al. [127], Maza et al. [92], Barsu et al. [10] and Alvarado et al. [4], there is no quantitatively mathematical model established for the fluid and structure equations and almost all models available rely on empirical correlations that fit the data in each different application.

## 1.2 Models for flows through porous surfaces

In this section we want to show some insight of the various models characteristics of flows through poroelastic layers. In order to be as clear as possible we have taken as example a very simple geometry to sketch the problem; the flow over a wall that include flexible multiple filaments. This simple geometrical configuration still has all the characteristic and difficulties of more interesting applications, such as a bluff body with a poroelastic layer.

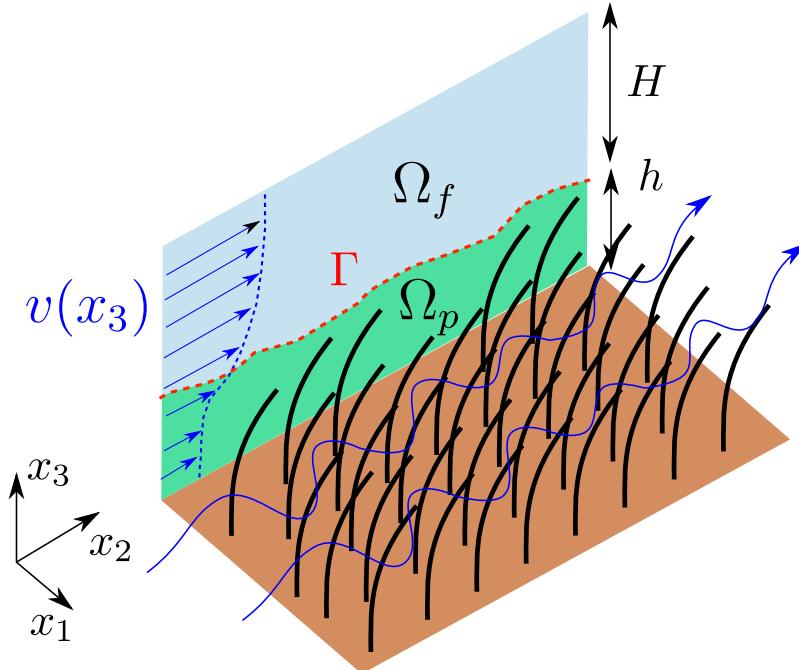


Figure 1.13: Sketch of a fully developed flow over a poroelastic surface made of multiple filaments.

The figure 1.13 shows a graphical schema of such flow; the main fluid direction is aligned with the  $x_2$  axis and the projection of velocity stream-wise component is shown in

the plane  $x_2 - x_3$ . Such flow can bend the filaments and, of course, go by their ensemble. The hypothetical surface that envelops all the filaments lid ( $\Gamma$ ), defines the limit between the flow without obstacle ( $\Omega_f$ ) and the one inside the poroelastic medium ( $\Omega_p$ ), their projection are shown in the  $x_2 - x_3$  plane.

In order to computationally solve this problem there are some key points to address:

- Length scales: the flow presents interaction at multiple scales. The flow can develop Kelvin–Helmholtz type instabilities on the interface (of size  $h$ ) and they can even penetrate inside the medium and brake up to very small scales eddies. In order to resolve this complex dynamic one should use a very fine numerical mesh (highly computational expensive) or come up with a model (like in the context of turbulence modeling). Turbulence dynamic can be also problematic; the hypothesis that pore size eddies can exist deep inside the porous medium is still at debate in the community. How to deal with such small scale dynamic and/or find a model is not an easy task.
- Compliance (fluid structure interaction): if the filaments are flexible, they can bend and swing due to the fluid impact on them. We have to take into account a mechanical solid model for the filaments (for example Bernoulli beam), including also the computation of energy that the swing motion re-inject inside the fluid. This two-way coupling could also be really computational expensive in presence of a large number of filaments, and also, if the flexibility is important, one should in principle take into account the contact and repulsion between the fibers. If the porous medium has more complicated shapes (like the scales in the butterfly wing) to come out with a simplified model for the solid dynamic is even harder and the use of a general finite elements discretization is probably a necessity (increasing also the computational cost of the problem). Another approach consists of considering a "rheology" model for the medium, for example treating it like a porous medium that has elastic properties as an ensemble. Such models are applicable only to porous media where the solid inclusions are connected to each other; in principle they are convenient, computational speaking, but their mathematical description can be difficult.
- Anisotropy: the model used should be capable of treat permeable surfaces that have different responses when stressed in different directions. For instance the geometrical disposition and/or the mechanical properties of the medium can be non-homogeneous, so it appear more permeable in one direction and show a preferential flow path.

Dupont et al. [44] performed a LES simulation introducing a two way coupling for the fluid-structure interaction problem, they validate their code with video recording of a similar experiment and the frequency measurements of the Kelvin–Helmholtz instabilities at the interface agrees very well. They have not specified the computational configuration used, but they have mentioned an important high performance computing center in the acknowledgment which made us suppose that the computational power involved was substantial. Recently also in Marjoribanks et al. [89] a similar approach has been used.

Some other examples that solve the full coupled problem directly are in Pinelli et al. [111], Favier et al. [50] and Revell et al. [122], but in this case the number of filaments is small and so they can be though more as isolated filaments rather than a poroelastic medium.

Due to the computationally expensiveness of solving the problem directly, the scientific community has came out with other approaches that treat the porous domain with a generalized model that does not resolve the fine scale inside the filaments, but instead expressed them as a function of the larger length scales present in the fluid domain  $\Omega_f$ .

These are called homogenization approaches and the key point in such methods are:

- The division of the overall domain in two different parts: the fluid domain  $\Omega_f$  and the porous domain  $\Omega_p$
- Two different fluid models are solved in the two domains, in  $\Omega_f$  the Naver-Stokes equations for incompressible Newtonian fluids are solved. In the porous part there are a number of different models that adds source terms in the former equations to take into account the presence of the porous medium.
- The two domains should be coupled together with a boundary condition at the interface or a special treatment for the equation in the transition region.
- A model for the dynamic of the solid porous part, it can model the structures directly or with an averaged rheology model.

The key points show above are extensively expanded, in chapter 2, for the homogenization method chosen in this thesis. However, in the next sections the two main branches in literature, that take into account the presence of a porous medium layer, are summarized in order to give a panoramic on all possible choices.

### 1.2.1 Isotropic drag models

In the case of flow through vegetation (canopy flows) it is common to use an isotropic drag model (the drag is equal in the three principal directions of the medium) for parameterize the resistance of the canopy. The drag can be a function of the wall normal direction, but in most of the application is taken as a constant. The isotropic hypothesis can be correct in case of dense vegetation; even if the vertical component of the resistance should be smaller. However since these models are mostly applied in channel problems where the mean flow is mostly stream-wise, the resistance in the vertical direction can be approximated in this manner; in applications where the transpiration of the interface is important (wake control of bluff body) the isotropic drag model is, certainly, not the most adequate.

The drag resistance is included in the Navier-Stokes equations as a sink:

$$\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{v}_\beta \cdot \nabla \mathbf{v}_\beta = -\frac{1}{\rho_\beta} \nabla p_\beta + \nu_\beta \nabla^2 \mathbf{v}_\beta - C_{Da} |\mathbf{v}_\beta| \mathbf{v}_\beta, \quad (1.4)$$

Where the subscript  $\beta$  indicate the fluid phase, and  $C_D$  the drag coefficient of the isolated fiber. The parameter  $a$  is the frontal area per unit volume of the vegetation, it is an expression of the porosity of the medium. The sink term is quadratic in the velocity, but there is some evidence that the reconfiguration phenomena can change this relationship as a consequence of the drag reduction (Gosselin and De Langre [64] and Alvarado et al. [4]).

From our point of view this approach lack of strong mathematical formalism, on the basis of which one should be able to derive all the additional terms of the equations; as a consequence, this method heavily relies on empirical relations. The other issue is that the isotropic hypothesis rules out the possibility to model the anisotropic nature of most surfaces in which we are interested (as equation (1.3) suggest the difference in the permeability along each direction can be important for drag reduction).

In the field of flows through vegetation some authors have successfully used this approach, for example Maza et al. [92] and Maza et al. [93] used it to study wave attenuation and Ghisalberti and Nepf [56], Battiato and Rubol [11] developed simple models for the 2D mean flow over a canopy.

### 1.2.2 Homogenization models

In this section we want to introduce the most used approach to derive the equations valid in the porous domain. The fundamental idea is to construct a micro-scale model, either for the fluid and for the solid, and then derive the macro-scale equations using some averaging operator over the micro-scale.

The two most used homogenization methods are the *Volume Averaging* method (Whitaker [146]), and the (*Multiple Scales*) method Mei and Vernescu [96]; they can be more broadly classified as perturbations methods. The key differences and the main results retrieved using these approaches are presented in the following.

#### Volume Averaging

The method of Volume Averaging has been developed to treat transport equations in porous media applications; in this case the presence of two different length scales is obvious, as it can be evinced by figure 1.14.

The core idea of the methods is to firstly define an average operator as

$$\langle \psi \rangle^\beta = \frac{1}{V_\beta} \int_V \psi dV$$

in this case the variable  $\psi$  represents any vector or scalar variable in the system of equations that we want to homogenize, for Navier-Stokes they are the velocity and the pressure.

The average operator has the purpose to "scale up" the equation, in fact the second crucial step of the method is to decompose the variables as proposed by Gray [65]:

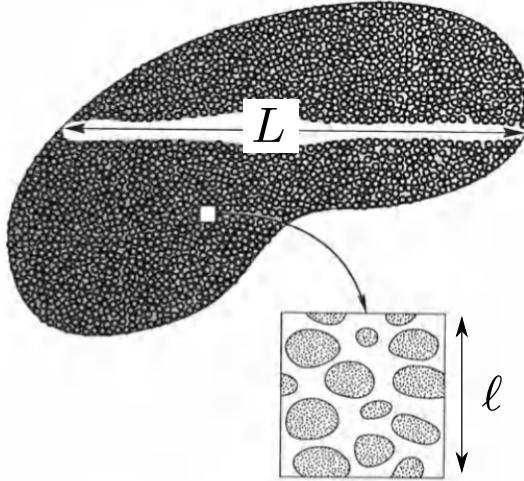


Figure 1.14: Schematics of a porous medium system of size  $L$ , with a zoom on the microscopic structure and its scale  $\ell$ . Image from Whitaker [146].

$$\psi = \underbrace{\langle \psi \rangle}_{O(L)}^\beta + \underbrace{\tilde{\psi}}_{O(\ell)}, \quad (1.5)$$

The equation (1.5) shows how each variable can be decomposed in an averaged part which contains only spatial variations at the macro-scale  $L$  and a *fluctuation* part that contains only the micro-scale  $\ell$  spatial variations.

Also the decomposition can be injected in the transport equations that we want to homogenize, and after some mathematical manipulations it is possible to retrieve the new averaged equations that include only variables of order  $L$ . Since this is the method chosen to develop our work, all the technical details are explained in chapter 2.

To introduce briefly some other aspects about this method, we show as example, how to derive the homogenized version of the Stokes equation. The described problem is a flow inside a rigid porous medium, like the one in figure 1.14. The Stokes equation valid for the fluid phase, indicated with the  $\beta$  subscript, reads:

$$0 = -\frac{1}{\rho_\beta} \nabla p_\beta + \nu_\beta \nabla^2 \mathbf{v}_\beta, \quad (1.6)$$

So, here it is important to precise that equation (1.6) is valid only in the liquid phase and to solve it we have to consider a no-slip boundary condition at the interface with the solid phase, with the difficulties that come to define the complex structure of the solid inclusion. Applying the Averaging Method, we can derive a homogeneous version of (1.6) that is valid in all the porous domain that include the two different phases, the solid

and the liquid one. The homogenized version of the last is the famous Darcy equation  $\langle \mathbf{v}_\beta \rangle^\beta = -\frac{\mathbf{K}}{\varepsilon \mu_\beta} \nabla \langle p_\beta \rangle^\beta$ , developed with this approach by Whitaker [143].

The Darcy equation allows to recognize two additional quantities that arise from the averaging procedure, the first one is a scalar called porosity  $\varepsilon$  that represents the ratio between the volume of the fluid inside a reference volume, over the total volume itself. The second one is the tensor  $\mathbf{K}$  called permeability, it expresses the resistance of the porous medium that affects the flow in its travel. The term  $\mathbf{K}$  plays the same role as  $C_{Da}$  in the isotropic drag model; the main difference is that the permeability tensor can be computed directly from the geometry of the medium (see chapter 2) so it does not rely on empirical relations. Also the tensorial nature of this terms allows us to model porous inclusions that are anisotropic.

Applications of the theory include flow where inertial terms are not negligible (Whitaker [145]), porous media with small deformations (Whitaker [144]) and high deformations (Hus-song et al. [71]), turbulent problems (Soulaine and Quintard [132], Breugem et al. [23]), interface between a permeable medium and a free flow (Beavers and Joseph [14]), multi-phase systems (Whitaker [141]) and heat transfer (Carbonell and Whitaker [32]).

It is impossible to go into detail of the derivation of equation for each specific problem, but we hope that the readers have understood the differences between this method and the isotropic drag model of the previous section.

## Multiple Scales

The multiple scales method is similar to the previous one; and it has also been applied to similar problems in the context of porous media applications.

In this method we start with the assumption of scale separation between  $\ell$ , the micro-scale, and  $L$ , the macro-scale. The scale separation factor can be defined as  $\epsilon = \ell/L \ll 1$ . Using the same examples as the previous section, we show how to compute the homogenized version of the Stokes equation for fluid flow through porous media. We introduce the micro-scale and the macro-scale coordinates defined respectively as:

$$X_i = \frac{\tilde{x}_i}{L}, \quad x_i = \frac{\tilde{x}_i}{\ell},$$

where  $x_i$  are the original eulerian coordinate of the problem. Using the above separation factor it is possible to expand the variables of the problem (pressure and velocity) into different magnitude orders indicated as superscript:

$$\psi(X_i, x_i) = \psi^{(0)}(X_i, x_i) + \epsilon \psi^{(1)}(X_i, x_i) + \epsilon^2 \psi^{(2)}(X_i, x_i) + O(\epsilon^3),$$

Injecting this decomposition inside the equation (1.6) it is possible to derive a set of hierarchical equations, one for each order of the expansion. It can be shown that analyzing each equation in the set lead to the homogenized equation:

$$v_i^{(0)} = -K_{ij} \frac{\partial p^{(0)}}{\partial X_j}, \quad (1.7)$$

In which either the pressure or the velocity fields appears only at the order zero, and the equation depends only on the macro-scale length.

The same permeability tensor  $\mathbf{K}$  as before is retrieved, with the same definition and interpretation. It is clear that for this toy problem we end up with the same set of homogenized equation; the point that has changed is the starting hypotheses of the method and the mathematical development to compute them.

A full analysis of the dualism of the two approaches can be found in the work by Davit et al. [41].

The multiple scales method has also been used to study many other problems: inertial effects (Mei and Auriault [95], Skjetne and Auriault [130]), coupling between a free fluid and a porous media (Mikelic and Jäger [97]), porous media with small deformations (Auriault and Sanchez-Palencia [7]), heat conduction in composites (Auriault [8]), rigid and moving permeable layers (Zampogna and Bottaro [151], Lācis et al. [82] and Zampogna and Bottaro [153]).

There is also another homogenization based method called *Mixture Theory* that is also worth a mention. It is based on a similar approach as the previous ones and Rajagopal [120] showed that it is possible to retrieve the same equation as the previous two methods in case of porous media flow.

### 1.3 Stability of flows over permeable surfaces

Flows through submerged aquatic plants exhibit large scale vortices at the top of the vegetation, advecting along the flow direction and causing a periodic waving of the plants, referred to as monami (if the fluid is air) and honami (in case of water) Inoue [73], Ackerman and Okubo [1]. The effect of the onset of the monami is depicted in figure 1.17.

Vortices arise from the nonlinear amplification of a Kelvin-Helmholtz instability mode, related to the presence of an inflection point in the base flow profile (Asaeda et al. [6]); the profile itself is inflectional because the fluid is slowed down by the drag exerted by the canopy, whose modeling has recently been addressed (Py et al. [112], Singh et al. [129], Zampogna et al. [155] and Tilton and Cortelezzi [135]). The correct prediction of the onset and characteristics of the Kelvin-Helmholtz instability are important to assessing the effects of turbulence (Finnigan [51], Jimenez et al. [76]) in particular to:

- understand how the vertical exchange of momentum occurs Ikeda and Kanazawa [72].
- clarify how the transport of CO<sub>2</sub> and dissolved nutrients or sediments takes place. This exchange takes place between the obstructed vegetation flow and the free overflow motion (Gambi et al. [53], Eckman [45], Grizzle et al. [67]).

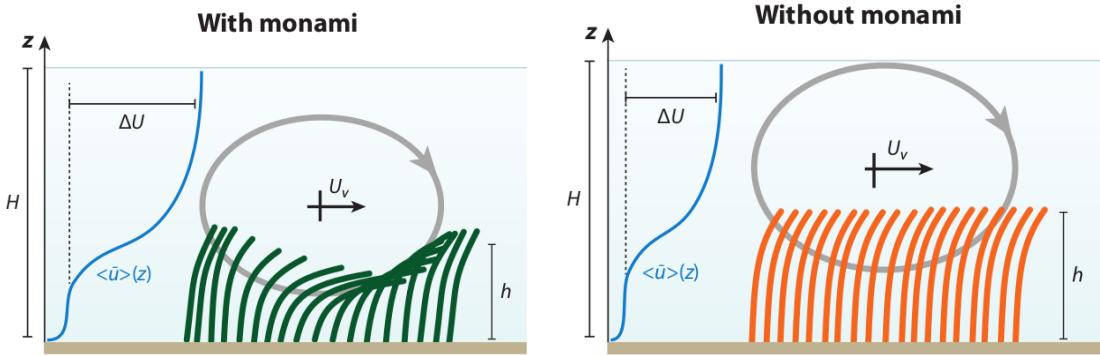


Figure 1.15: Left: when the drag of the canopy is high enough it generates canopy-scale vortices by Kelvin-Helmholtz instability. These vortices may interact with the flexible vegetation and generate a waving motion called monami. Right: when this interaction is too weak, the canopy only bend. Image from Nepf [102].

- assess the changes in the morphology of the vegetation in inland or coastal wetlands in response to continuous periodic forcing Asaeda et al. [6], Patil and Singh [108].

One of the possible approaches to study how and when these instabilities start is the linear stability analysis; in the following section we briefly introduce the key assumption and simplifications of the method over other possible choices, and in the next section some results in the context of permeable surfaces are presented.

### 1.3.1 Stability theory generalities

Stability theory in general covers the modeling of transition of fluid systems towards unsteady states and at least turbulence. In its most basic form the theory gives us a fast and robust method to compute the frequency and grow rate of the unstable modes, if there is any, in the base flow.

The linear stability relies on the decomposition of the flow variables  $\mathbf{q}$  into a steady-state part  $\bar{\mathbf{q}}$ , called base flow, and an unsteady part  $\tilde{\mathbf{q}}$ :

$$\mathbf{q}(\mathbf{x}, t) = \bar{\mathbf{q}}(\mathbf{x}) + \eta \tilde{\mathbf{q}}(\mathbf{x}, t)$$

Here  $\eta$  is a small amplitude parameter. Then the unsteady part is simplified by the hypothesis to have a general wave form

$$\tilde{\mathbf{q}} = \hat{\mathbf{q}}(\mathbf{x}) e^{i\Theta(\mathbf{x}, t)},$$

where  $\hat{\mathbf{q}}$  is the amplitude and  $\Theta$  is the phase of the perturbation. The choice made to determine the time and space dependence of either the phase function and the amplitude

determine a certain hierarchy inside the stability theories; for example based on the number of directions not supposed periodic in the amplitude function, we move from a local approach (1 direction non-periodic) to a global one (all the 3 direction are non-periodic). Figure 1.16 below present each possible choice in literature and the theory that derives from it:

	Denomination	Basic state assumption	Amplitude function	Phase function $\Theta$
Global	TriGlobal PSE-3D	$\partial_1 \bar{\mathbf{q}} \ll \partial_2 \bar{\mathbf{q}}, \partial_3 \bar{\mathbf{q}}$	$\bar{\mathbf{q}}(x_1^*, x_2, x_3)$ $\bar{\mathbf{q}}(x_1^*, x_2, x_3)$	$\hat{\mathbf{q}}(x_1, x_2, x_3)$ $\hat{\mathbf{q}}(x_1^*, x_2, x_3)$
	BiGlobal	$\partial_1 \bar{\mathbf{q}} = 0$	$\bar{\mathbf{q}}(x_2, x_3)$	$\hat{\mathbf{q}}(x_2, x_3)$
Nonlocal	PSE	$\partial_1 \bar{\mathbf{q}} \ll \partial_2 \bar{\mathbf{q}}, \partial_3 \bar{\mathbf{q}} = 0$	$\bar{\mathbf{q}}(x_1^*, x_2)$	$\hat{\mathbf{q}}(x_1^*, x_2)$
Local	OSE	$\partial_1 \bar{\mathbf{q}} = \partial_3 \bar{\mathbf{q}} = 0$	$\bar{\mathbf{q}}(x_2)$	$\hat{\mathbf{q}}(x_2)$

Figure 1.16: Classification of modal linear stability theories. Table reported from Juniper et al. [77].

In our case we have limited our study to a local approach build on mode decomposition, *local stability theory* (LST, also known as *ordinary stability equations* OSE in the denomination of figure 1.16). In the LST we make the hypothesis that the amplitude and the base flow depend only on the wall normal spatial coordinate (parallel flow) and the phase function take into account the periodicity in time and in the stream-wise and cross-flow directions. The last hypothesis should not be seen as a simplification since there are some class of problems (such as canopy flows) in which two of the three directions are really homogeneous. The complete formulation is in the following equation:

$$\tilde{\mathbf{q}}(\mathbf{x}, t) = \hat{\mathbf{q}}(x_2) e^{i(\alpha x_1 + \beta x_3 - \omega t)}$$

where  $\alpha$  is the stream-wise wave-number,  $\beta$  is the cross-flow wave-number and  $\omega$  is the wave phase.

Casting this form for the pressure and velocity inside the Navier-Stokes equation and neglecting second order terms ( $\eta^2$ ) or higher, the equations at order one are linear and describe the evolution of the perturbations, taking the base flow as an input of the problem. In order to study the stability of the perturbations in their time evolution, problem known as *temporal stability*, we fix the space perturbation form imposing  $\alpha$  and  $\beta$  as real numbers (inputs of the problem) and solving for  $\omega$  as a complex number. With such choices the problem became a generalized eigenvalue one for the wave phase:

$$A\hat{\mathbf{q}} = \omega B\hat{\mathbf{q}}$$

The solution gives the frequency (real part of the eigenvalues) and the growth-rate (imaginary part) of the perturbation modes (eigenvectors) of the flow.

The above introduction of the method is quite condensed, however, in the last years a lot of literature has been developed on the subject, (Juniper et al. [77], Criminale et al. [38], Schmid and Henningson [125]); the problem has also been extensively studied in his computational aspect by Canuto et al. [31].

### Monami/Honami and Kelvin-Helmholtz rolls

We have already highlight that the above framework concerning the stability problem has been applied in some porous media flow (canopy) configurations, also including the vegetation movement. Because of the flexibility of the vegetation, some theoretical studies have focused on the modeling of the stems of the aquatic plants and their displacement in response to the forcing by the water flow (Py et al. [112], Patil and Singh [108], Gosselin and De Langre [63] and Py et al. [113]).

It has been studied in Finnigan [51] that these large coherent structures control turbulence dynamics over the canopy. Movements of the latter generate sweeps (and ejections) of fluids that generates the counter-rotating stream-wise eddy evolving as Kelvin-Helmholtz rolls. The complex evolution of vortices is shown in figure 1.17.

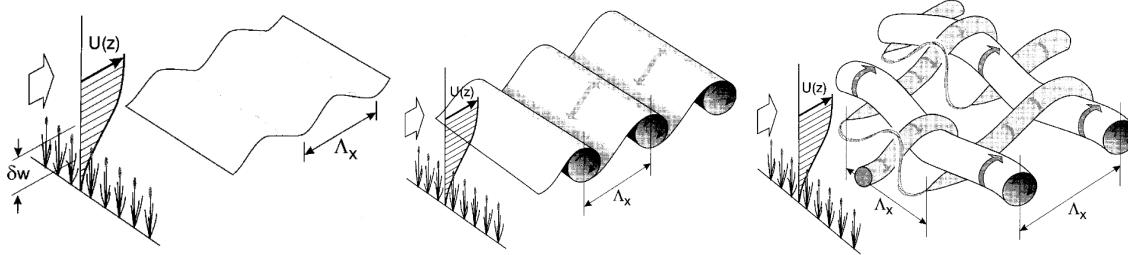


Figure 1.17: Left: first emergence of the Kelvin-Helmholtz instability. The growth-rate is proportional to the shear magnitude at the inflection point. Center: the instability evolves in rollers consisting of high vorticity that are spaced with a similar wave-length  $\Lambda_x$  as the previous stage. Right: secondary instabilities in the rollers lead to their kinking and pairing, coherent structures appear in the transverse and stream-wise dimensions. Image from Finnigan [51].

However, Kelvin-Helmholtz vortices occur whether or not, the plants bend and to ascertain causes and effects to first order it is acceptable to focus on rigid porous structures. The flow over and through a submerged array of rigid, cylindrical pillars has been the basis of the approach of Ghisalberti and Nepf [59], Ghisalberti and Nepf [56] and Ghisalberti and Nepf [58], who have conducted a series of careful experiments. Their results have often been used by fluid dynamicists to put forth and test theoretical hypotheses to predict the frequency and wavelength of the large scale vortical motion, for a variety of conditions. The configuration studied consists of a regular grid of rigid pillars, orthogonal to the surface, of

identical height  $h$ . In some of the theoretical models proposed to analyze the stability of this system, the Rayleigh equation is used throughout the water channel, with or without a drag term in correspondence of the canopy Raupach et al. [121], Py et al. [112], Singh et al. [129], Zampogna et al. [155] and Luminari et al. [88] have recently demonstrated that the addition of a drag term through the vegetation reduces the amplification factor of the Kelvin-Helmholtz instability throughout the whole range of wave-numbers and increases mildly the wavelength of the fastest growing mode. In chapter 3 we study how the perturbation of the base flow affects the predicted amplification factor and wave-length, we test the difference between the isotropic drag model and the tensorial approach, in order to show which is the most robust approach for stability computations.

## Chapter 2

# Volume Average Method

*Do not worry about your difficulties in mathematics; I can assure you that mine are still greater.*

- Letter to junior high school student Barbara Wilson,  
January 7, 1943, *Albert Einstein*

### 2.1 Introduction

We have already said in the previous chapter that we use the volume averaging method to scale up the fluid equations, valid in the microscopic spaces of the porous medium, in order to find a macroscopic description that is valid everywhere in the porous medium domain (not only in the fluid phase). Theoretical aspect of the volume averaging method applied to fluid in porous media can be found in Whitaker [146], Whitaker [143], Whitaker [145], Quintard and Whitaker [114], Quintard and Whitaker [115], Quintard and Whitaker [116], Quintard and Whitaker [117], Quintard and Whitaker [118] and many others contributions that are introduced in the next chapter. In the following paragraph we describe the various stages necessary in calculating the local average version of equations (2.1).

### 2.2 Homogenization procedure

The mathematical method of volume averaging is based on some fundamental stages that one should follow in order to retrieve the homogenized version of the equations. The main steps are:

- Definition of the averaging operator
- Use of theorems that permits to interchange the derivation and averaging operations
- Decompose the fields in an "average plus perturbation" manner

- Assume (based on the problem characteristics) some length-scales constraint that help to simplify and define a local closure problem

Such schema is graphically resumed in Paéz-García et al. [107] and Davit et al. [41]; a similar flowchart of the complete overall procedure is showed in figure 2.1.

## 2.3 Derivation of VANS equations for 3D incompressible fluids

The dynamic of the fluid phase (indicated with the subscript  $\beta$ ), inside and above the porous medium, is governed by the Navier-Stokes equation for incompressible Newtonian fluid:

$$\begin{cases} \frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{v}_\beta \cdot \nabla \mathbf{v}_\beta = -\frac{1}{\rho_\beta} \nabla p_\beta + \nu_\beta \nabla^2 \mathbf{v}_\beta \\ \nabla \cdot \mathbf{v}_\sigma = 0 \\ \mathbf{v}_\beta = \mathbf{v}_\sigma \quad \text{at } A_{\beta\sigma} \end{cases} \quad (2.1)$$

where  $\mathbf{v}_\beta$ ,  $p_\beta$ ,  $\rho_\beta$  and  $\nu_\beta$  stand, respectively, for the velocity, the pressure, the density and the kinematic viscosity of the fluid. The interface between the fluid and the solid is indicated as  $A_{\beta\sigma}$ , in which the no-slip condition for the velocity apply. In the above boundary condition  $\mathbf{v}_\sigma$  is velocity of the solid phase. Initial condition should also be specified in order to solve the system, but they do not take active part in the homogenization procedure. The next sections shows how to average this equations using the volume average method.

### 2.3.1 Definition of the averaging operators

Figure 2.2 show the schematics of the internal structure of a fibrous porous medium; all the quantities needed in order to develop our mathematical approach are also indicated. The continuous lines in the figure are used to indicate the shape of the volume used in the average operators,  $V|_x$  indicate the overall volume with centroid  $\mathbf{x}$  and  $V_\beta|_x$  indicate the volume occupied of the only fluid phase inside the same volume. The coordinate  $\mathbf{r} = \mathbf{x} + \mathbf{y}$  represent the centroid of a possible volume in which one can compute the average quantities, the boundaries of the same volume are indicated with dotted lines.

Let  $\psi_\beta$  be a an arbitrary order tensors (scalar, vector or second order tensor) defined in the fluid phase of the volume  $V$  with  $x$  as centroid coordinates. We define two different volume averaging operators; the *intrinsic average* indicated as  $\langle . \rangle^\beta$  reads:

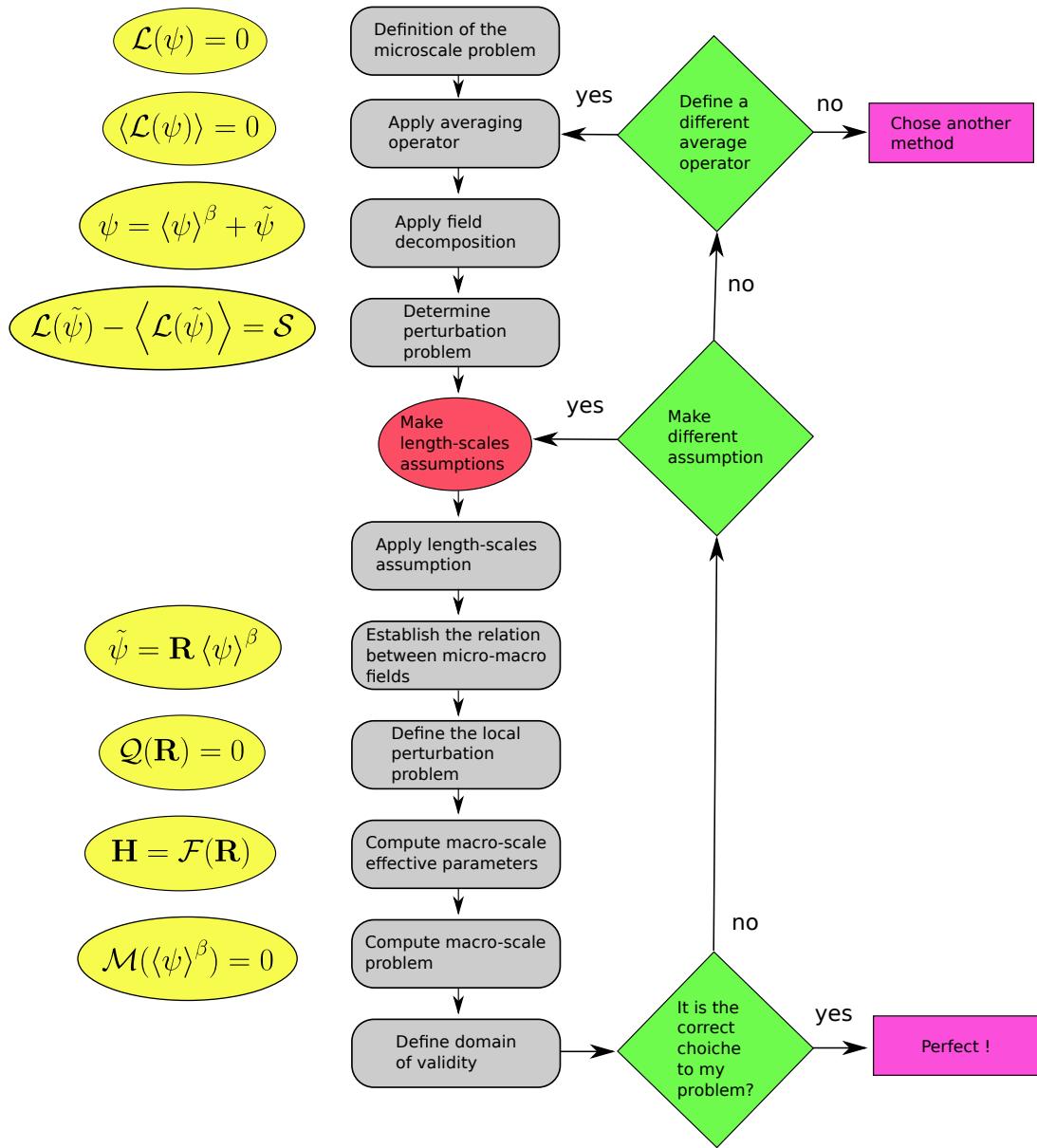


Figure 2.1: Illustration of the volume average homogenization procedure. Image adapted from Davit et al. [41]

$$\langle \psi_\beta \rangle^\beta |_{\mathbf{x}} = \frac{1}{V_\beta(\mathbf{x})} \int_{V_\beta(\mathbf{x})} m(\mathbf{y}) \psi_\beta(\mathbf{x} + \mathbf{y}, t) dV_\beta, \quad (2.2)$$

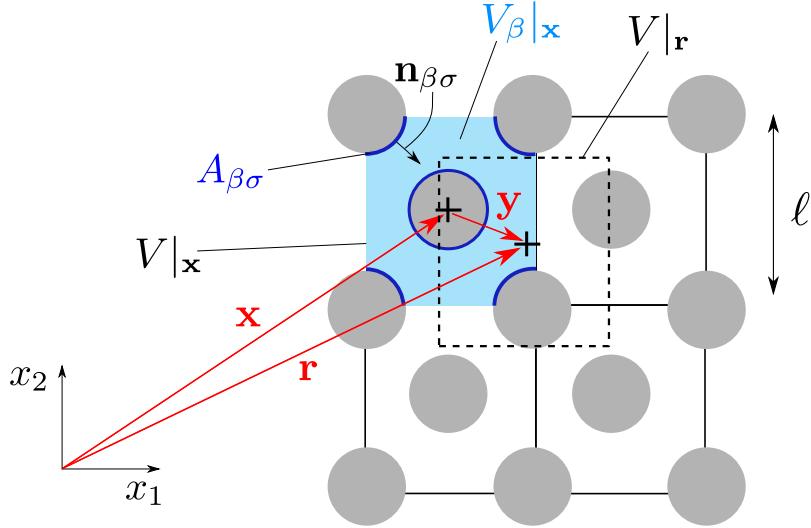


Figure 2.2: A graphic representation of the averaging volumes and interfaces in case of fibrous (ordered) porous media. In this example the fibers are in staggered arrangement. The edges of the volumes that have centroid position  $\mathbf{x}$  are shown in continuous lines and the ones with centroid  $\mathbf{r}$  are shown in dotted lines.

where  $m$  is a weight function defined on  $V$  and  $\mathbf{y}$  is the relative position vector with respect to the centroid  $\mathbf{x}$  of the averaging volume  $V_\beta$ .

The second one is the *superficial average* indicated with  $\langle \rangle$ :

$$\langle \psi_\beta \rangle |_{\mathbf{x}} = \frac{1}{V} \int_{V_\beta(\mathbf{x})} m(\mathbf{y}) \psi_\beta(\mathbf{x} + \mathbf{y}, t) dV_\beta, \quad (2.3)$$

The difference between the two formulation is that the former takes into account the actual fluid phase in averaging the filed instead of the pure size of the total volume.

In order to use a less heavy notation, the subscript  $|_{\mathbf{x}}$  is dropped in the following procedure, but we keep in mind that the volume averaged quantities are explicitly dependent on the center position of the volume. Either averaging operators are defined as a volume integral in which the integration domain size and shape should be chosen carefully in each different application; some more details on this choice are presented in the next paragraphs.

Inside the average operators we can further use a weight function ( $m$ ), that has the aim to guarantee smooth volume averaged fields. However the choice of its shape depends on the porous media geometry, and its size is connected to the chosen average volume.

In the sake of simplify the notation we consider the filter  $m$  to be constant ( $1/V$ ) inside the volume so we could drop it from the formulation of the average operators. However any shape of the function  $m$  can be used without formally change the final form of the averaged equation.

The porosity of a porous medium cell is defined as:

$$\varepsilon = \frac{V_\beta}{V} \quad (2.4)$$

which represent of how much fluid is actually present inside the averaging volume, and it also gives an indication of how packed are the fibers of our porous media.

Using the above definition is possible to express a relationship between the two averaging operators:

$$\langle \psi_\beta \rangle = \varepsilon \langle \psi_\beta \rangle^\beta \quad (2.5)$$

### 2.3.2 Choice of shape and size of averaging volume and filter function

The problem of choosing the right filter, for a give geometry of the porous medium, has been extensively studied by the series of works Quintard and Whitaker [114], Quintard and Whitaker [115], Quintard and Whitaker [116], Quintard and Whitaker [117], Quintard and Whitaker [118] and more recently generalized by Davit and Quintard [40].

The authors above differentiate their results for ordered and disordered porous medium; and they show that in each case a specific size and shape of the filter (and the volume) is needed. Sometimes in the following text we refer to the volume in which the average procedure is applied as REV, *reference elementary volume*. For disordered porous media a spherical volume is the most appropriate, and the REV size ( $\ell_V$ ) should satisfy the length scale constraint:

$$\ell_\beta \ll \ell_v \ll L$$

Instead for ordered porous media the most appropriate shape is a cube with side:

$$O(\ell_\beta) = \ell_V \ll L$$

that can be reinterpreted as the separation of scale parameter in multiple scale analysis  
 $\epsilon = \frac{\ell_\beta}{L} \ll 1$ .

Ochoa-Tapia and Whitaker [104] confirm the same length-scales constraints even in case of an interface between a free fluid and a porous medium.

The size of it ( $\ell_v$ ) should be chosen with the above specifications, these length scales constraints assure that the volume is large enough that periodic boundary condition can be applied in the exterior of the volume but still be able to capture all the phenomena that take place at the micro-scale (scale  $\ell_\beta$ ). If the REV size is the correct one, increasing or decreasing its size, of a certain amount, should not change the averages quantities.

We have already said that the function can help to attenuate variation of the averaged fields due to geometrical fluctuations of the porous medium. It has in fact the property of a low-pass filter for the perturbations fields.

The filter function can play an important role also in the interpretation of the averaged equation in fact (as showed later) in order to retrieve a local form of the VANS equations we have to be able to make the statements:

$$\langle\langle\psi_\beta\rangle|_{\mathbf{r}}\rangle|_{\mathbf{x}} = \langle\psi_\beta\rangle|_{\mathbf{x}} \quad (2.6)$$

that is equivalent to say that the averaged fields contains small variations of at the micro-scale (inside the averaging volume  $V$ ). In order to satisfy this requirement certain filter choices can perform better than others, although the same conclusion can be derived from the length-scales constraints. In appendix 2.7 at the end of this chapter we further explain the details of the approximation needed to derive equations 2.6.

For disordered porous medium the *hat function* filter  $m^\square$  which has the form:

$$m^\square(\mathbf{y}) \begin{cases} \frac{1}{V} & |\mathbf{y}| \leq r_0 \\ 0 & |\mathbf{y}| > r_0 \end{cases} \quad (2.7)$$

can be used to produce smooth averaged fields.

Instead for ordered porous medium the literature show that triangle shaped functions called *cellular filter*  $m^\triangle$  performs better:

$$m^\triangle(\mathbf{y}) \begin{cases} (\ell_{beta}/2 - |\mathbf{y}|) & |\mathbf{y}| \leq r_0 \\ 0 & |\mathbf{y}| > r_0 \end{cases} \quad (2.8)$$

Whichever is the choice on the  $m$  shape, Davit and Quintard [40] has re-formulated recently the required hypothesis that a  $m$  function should satisfy; without entry in the details (that is beyond the aim of this paragraph) we present some of the most important generalizations from the above works that should apply to any choice of the function  $m$ :

- be normalized in as:  $\int_{V_\beta} m(\mathbf{y}) dV_\beta = 1$
- have compact support
- $m * \psi_\beta \in C^k$  where  $k$  represent the order of the closure
- $(m y^j) * \psi_\beta \begin{cases} 0 & \text{if } j \text{ odd} \\ const & \text{if } j \text{ even} \end{cases}$

In the last requirement we have used the fact that the average operation can also be defined as a convolution product between the filter function and the flow field quantities, Marle [90]:

$$\langle\psi_\beta\rangle|_{\mathbf{x}} = \frac{1}{V} \int_{V_\beta(\mathbf{x})} m(\mathbf{y}) \psi_\beta(\mathbf{x} + \mathbf{y}, t) dV_\beta = m * \psi_\beta$$

Even though we have shown the importance of the filter function shape, it should be clear that older literature works that had implicitly used the  $m^\square$  are not wrong, even in case of periodic ordered porous media. In fact if we assume that the field are well behaved (even without checking if the filter or the REV size are the good one) we assume that the equation (2.6) olds and that some simplifications of the averaging operators are permitted. This permits us to finally find a closed local form of the equation and possibly use the good filter a posteriori to smooth the fields. However neglecting the use of the proper filter can induce some problem on the interpretations of the averaged fields (as Quintard and Whitaker [114] show for the example case of hydrostatic pressure); so particular care should be used especially when making comparison to experiments.

In the following derivation of the equation we do not formally include any filter function inside the averaged operators in order to not make the notation heavier. However we indicate in the following text where this hypothesis on the filter is formally needed to derive the equation.

### 2.3.3 Theorems involving derivatives of spatial averaged quantities

In this paragraph we report some the theorems related to the volume averaging technique. The purpose to these theorems is to invert the sequence of the derivative and volume average operation.

**Theorem 2.3.1** (Spatial averaging theorem). *Let  $\psi_\beta$  be a scalar quantity defined in the fluid phase  $\beta$ , then:*

$$\langle \nabla \psi_\beta \rangle = \nabla \langle \psi_\beta \rangle + \frac{1}{V} \int_{A_{\beta\sigma}} \psi_\beta \mathbf{n}_{\sigma\beta} dA \quad (2.9)$$

In the above  $\langle \psi_\beta \rangle$  is evaluated at  $\mathbf{x}$  and the operator  $\nabla$  express the differentiation operation in respect to  $\mathbf{x}$  also.

**Corollary 2.3.1** (Vector form of (2.9)). *The vector form of the spatial averaging theorem is given by:*

$$\langle \nabla \cdot \boldsymbol{\psi}_\beta \rangle = \nabla \cdot \langle \boldsymbol{\psi}_\beta \rangle + \frac{1}{V} \int_{A_{\beta\sigma}} \boldsymbol{\psi}_\beta \cdot \mathbf{n}_{\sigma\beta} dA \quad (2.10)$$

**Corollary 2.3.2.** *Applying the theorem (2.9) to a constant field  $\psi_\beta = 1$  we obtain:*

$$\nabla \varepsilon = - \int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta} dA, \quad (2.11)$$

**Theorem 2.3.2** (Reynolds transport theorem). *Let  $\psi_\beta$  be a scalar quantity defined in the fluid phase  $\beta$ , then:*

$$\frac{\partial}{\partial t} \int_{V_\beta(t)} \psi_\beta \, dV = \int_{V_\beta(t)} \frac{\partial \psi_\beta}{\partial t} \, dV + \int_{A_{\beta\sigma}(t)} \psi_\beta (\mathbf{v}_\sigma \cdot \mathbf{n}_{\beta\sigma}) \, dA, \quad (2.12)$$

where  $\mathbf{v}_\sigma$  is the point velocity of the solid-fluid interface  $A_{\beta\sigma}$ .

The three theorems and the corollary are indispensable to develop the closed form of the equations. One interesting thing to pay attention is that the theorems switch the average and derivative operation but always introduces a non local integral term.

### 2.3.4 Averaged continuity equations

We start by finding the averaged version of the continuity equation in (2.1):

$$\langle \nabla \cdot \mathbf{v}_\beta \rangle = 0 \quad (2.13)$$

Applying theorem (2.9) to the previous equation we get:

$$\langle \nabla \cdot \mathbf{v}_\beta \rangle = \nabla \cdot \langle \mathbf{v}_\beta \rangle + \frac{1}{V} \int_{A_{\beta\sigma}} \mathbf{v}_\beta \cdot \mathbf{n}_{\sigma\beta} \, dA$$

The boundary condition at the interface ( $\mathbf{v}_\sigma = \mathbf{v}_\beta$ ) imply that the integral above can be modified:

$$= \nabla \cdot \langle \mathbf{v}_\beta \rangle + \frac{1}{V} \int_{A_{\beta\sigma}} \mathbf{v}_\sigma \cdot \mathbf{n}_{\sigma\beta} \, dA$$

Now we rewrite the last term as if it was a result of the Reynolds transport theorem applied to a constant unitary scalar field:

$$= \nabla \cdot \langle \mathbf{v}_\beta \rangle + \frac{\partial}{\partial t} \frac{1}{V} \int_{V_\beta} dV - \frac{1}{V} \int_{V_\beta} \frac{\partial 1}{\partial t} \, dV$$

This results can be simplified; in fact the last integral is zero due to the time derivation of a constant field. The first term can be further developed, obtaining finally the averaged continuity equation (2.14).

$$\nabla \cdot \langle \mathbf{v}_\beta \rangle + \frac{\partial \varepsilon}{\partial t} = 0 \quad (2.14)$$

### 2.3.5 Averaged momentum equations

In this paragraph we compute the average version of the momentum equation in (2.1):

$$\frac{\partial \mathbf{v}_\beta}{\partial t} + \nabla \cdot (\mathbf{v}_\beta \mathbf{v}_\beta) = -\frac{1}{\rho_\beta} \nabla p_\beta + \nu_\beta \nabla^2 \mathbf{v}_\beta \quad (2.15)$$

In order to keep the procedure readable we show the development of each term separately, in the same order as they appear in equation (2.15).

### Temporal derivative term

Using theorem (2.12) we can rewrite the first term of the equation as:

$$\left\langle \frac{\partial \mathbf{v}_\beta}{\partial t} \right\rangle = \frac{\partial \langle \mathbf{v}_\beta \rangle}{\partial t} - \frac{1}{V} \int_{A_{\beta\sigma}(t)} (\mathbf{v}_\sigma \cdot \mathbf{n}_{\sigma\beta}) \mathbf{v}_\beta \, dA, \quad (2.16)$$

### Convective term

Theorem (2.10) applied to the convective term gives us:

$$\langle \nabla \cdot (\mathbf{v}_\beta \mathbf{v}_\beta) \rangle = \nabla \cdot \langle \mathbf{v}_\beta \mathbf{v}_\beta \rangle + \frac{1}{V} \int_{A_{\beta\sigma}} (\mathbf{v}_\beta \mathbf{v}_\beta) \cdot \mathbf{n}_{\sigma\beta} \, dA, \quad (2.17)$$

The boundary condition at the interface ( $\mathbf{v}_\sigma = \mathbf{v}_\beta$ ) imply that the integrals inside the convective and temporal part are equals, so the total left end side of the momentum equation became:

$$\frac{\partial \langle \mathbf{v}_\beta \rangle}{\partial t} + \nabla \cdot \langle \mathbf{v}_\beta \mathbf{v}_\beta \rangle \quad (2.18)$$

### Pressure term

The pressure term is also expanded using theorem 2.9:

$$\left\langle -\frac{1}{\rho_\beta} \nabla p_\beta \right\rangle = -\frac{1}{\rho_\beta} \nabla \langle p_\beta \rangle - \frac{1}{V} \int_{A_{\beta\sigma}} \frac{p_\beta}{\rho_\beta} \mathbf{n}_{\sigma\beta} \, dA, \quad (2.19)$$

### Diffusion term

Here we fist use the identity  $\nabla^2 = \nabla \cdot (\nabla)$  (laplacian = div(grad)), then we apply the theorem 2.10 directly to this expansion to get:

$$\langle \nu_\beta \nabla^2 \mathbf{v}_\beta \rangle = \langle \nu_\beta \nabla \cdot \nabla \mathbf{v}_\beta \rangle = \nabla \cdot \langle \nu_\beta \nabla \mathbf{v}_\beta \rangle + \frac{1}{V} \int_{A_{\beta\sigma}} \nu_\beta \nabla \mathbf{v}_\beta \cdot \mathbf{n}_{\sigma\beta} \, dA, \quad (2.20)$$

Now re-using the theorem 2.10 on  $\langle \nabla \mathbf{v}_\beta \rangle$ :

$$\begin{aligned}
(2.20) \quad &= \nabla \cdot \nu_\beta \nabla \langle \mathbf{v}_\beta \rangle + \nabla \cdot \left( \frac{1}{V} \int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} \cdot \nu_\beta \mathbf{v}_\beta \, dA \right) + \frac{1}{V} \int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} \cdot \nu_\beta \nabla \mathbf{v}_\beta \, dA = \\
&= \nu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle + \nabla \cdot \left( \frac{1}{V} \int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} \cdot \nu_\beta \mathbf{v}_\beta \, dA \right) + \frac{1}{V} \int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} \cdot \nu_\beta \nabla \mathbf{v}_\beta \, dA = \\
&\quad \text{using Gauss theorem on the second term} \\
&= \nu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle + \nabla \cdot \left( \frac{1}{V} \int_{V_\beta} \nabla \cdot \nu_\beta \mathbf{v}_\beta \, dV_\beta \right) + \frac{1}{V} \int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} \cdot \nu_\beta \nabla \mathbf{v}_\beta \, dA = \\
&\quad \text{which is zero due to the continuity equation} \\
&= \nu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle + \frac{1}{V} \int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} \cdot \nu_\beta \nabla \mathbf{v}_\beta \, dA
\end{aligned} \tag{2.21}$$

The above use of the Gauss theorem, to prove that the first integral term is zero, can be bypassed in case of rigid porous media. Which due to the b.c. at the fluid-solid is zero.

Before continue the development we group all the various terms together:

$$\begin{aligned}
&\frac{\partial \langle \mathbf{v}_\beta \rangle}{\partial t} + \nabla \cdot \langle \mathbf{v}_\beta \mathbf{v}_\beta \rangle = -\frac{1}{\rho_\beta} \nabla \langle p_\beta \rangle + \nu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle + \\
&+ \frac{1}{V} \int_{A_{\beta\sigma}} \left( -\frac{p_\beta}{\rho_\beta} \mathbf{I} + \nu_\beta \nabla \mathbf{v}_\beta \right) \cdot \mathbf{n}_{\sigma\beta} \, dA
\end{aligned} \tag{2.22}$$

This is still not the averaged version of the momentum equation, since it has the presence of the non-homogeneous term  $\langle \mathbf{v}_\beta \mathbf{v}_\beta \rangle$  and the integral term still has the point variables inside. In the next section these we show how to thereat these two terms in order make them function of the only averaged quantities.

### 2.3.6 Length scale decomposition

In order to finally get the average version of the problem (2.1) we make use of the decomposition proposed by Gray [65]:

$$\psi_\beta(\mathbf{r}, t) = \langle \psi_\beta \rangle^\beta|_{(\mathbf{r}, t)} + \tilde{\psi}_\beta(\mathbf{r}, t) \tag{2.23}$$

where  $\tilde{\psi}_\beta$  is the microscopic scale contribution to add at the intrinsic volume average quantity  $\langle \psi_\beta \rangle^\beta$  (that is representative of the macro-scale  $L$ ) to obtain the point value of the considered quantity  $\psi_\beta$ . This decomposition has been introduced in order to separate the different scales of the spatial variation of the fields, and so separate the low frequencies from the high one.

If the hypothesis of that this division holds, it is possible to demonstrate that the average value of the perturbation field is null (the paragraph 2.7 specifically address the hypothesis behind this result):

$$\langle \tilde{\psi}_\beta \rangle = \langle \psi_\beta \rangle - \langle \langle \psi_\beta \rangle^\beta \rangle \approx \langle \psi_\beta \rangle - \varepsilon \langle \psi_\beta \rangle^\beta = 0$$

Using the above results we can modify the non-homogeneous term in equation (2.22):

$$\langle \mathbf{v}_\beta \mathbf{v}_\beta \rangle = \langle \langle \mathbf{v}_\beta \rangle \langle \mathbf{v}_\beta \rangle^\beta \rangle + 2 \langle \langle \mathbf{v}_\beta \rangle^\beta \tilde{\mathbf{v}}_\beta \rangle + \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle = \varepsilon \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle \quad (2.24)$$

For each integral term of 2.22 we also apply the same field decomposition:

$$\begin{aligned} \frac{1}{V} \int_{A_{\beta\sigma}} -\frac{p_\beta}{\rho_\beta} \mathbf{n}_{\sigma\beta} dA &= \frac{1}{V} \int_{A_{\beta\sigma}} -\frac{1}{\rho_\beta} (\langle p_\beta \rangle^\beta + \tilde{p}_\beta) \mathbf{n}_{\sigma\beta} dA = \\ &= +\frac{1}{\rho_\beta} \nabla \varepsilon \langle p_\beta \rangle^\beta - \frac{1}{V} \int_{A_{\beta\sigma}} \frac{\tilde{p}_\beta}{\rho_\beta} \mathbf{n}_{\sigma\beta} dA \end{aligned} \quad (2.25)$$

$$\begin{aligned} \frac{1}{V} \int_{A_{\beta\sigma}} \nu_\beta \nabla \mathbf{v}_\beta \cdot \mathbf{n}_{\sigma\beta} dA &= \frac{1}{V} \int_{A_{\beta\sigma}} \nu_\beta \nabla (\langle \mathbf{v}_\beta \rangle^\beta + \tilde{\mathbf{v}}_\beta) \cdot \mathbf{n}_{\sigma\beta} dA = \\ &= -\nu_\beta \nabla \varepsilon \nabla \langle \mathbf{v}_\beta \rangle^\beta + \frac{1}{V} \int_{A_{\beta\sigma}} \nu_\beta \nabla \tilde{\mathbf{v}}_\beta \cdot \mathbf{n}_{\sigma\beta} dA \end{aligned} \quad (2.26)$$

The momentum equation now reads:

$$\begin{aligned} \frac{\partial \langle \mathbf{v}_\beta \rangle}{\partial t} + \nabla \cdot (\varepsilon \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta) + \nabla \cdot (\langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle) &= -\frac{1}{\rho_\beta} \nabla \langle p_\beta \rangle + \nu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle + \\ &- \nu_\beta \nabla \varepsilon \nabla \langle \mathbf{v}_\beta \rangle^\beta + \frac{1}{\rho_\beta} \nabla \varepsilon \langle p_\beta \rangle^\beta + \frac{1}{V} \int_{A_{\beta\sigma}} \left( -\frac{\tilde{p}_\beta}{\rho_\beta} \mathbf{I} + \nu_\beta \nabla \tilde{\mathbf{v}}_\beta \right) \cdot \mathbf{n}_{\sigma\beta} dA \end{aligned} \quad (2.27)$$

At this stage the momentum equation is not closed since it has the mixed presence of surface and intrinsic averaged quantities, and also perturbation fields. In order to overcome this problems in the next section the intrinsic version of these equation is finally computed.

### 2.3.7 Intrinsic average form

In order to get the intrinsic average formulation we use the relation (2.5) to express surface averaged quantities in terms of intrinsic ones.

We first start with the continuity equation that became:

$$\nabla \cdot (\varepsilon \langle \mathbf{v}_\beta \rangle^\beta) + \frac{\partial \varepsilon}{\partial t} = 0$$

For the momentum equation we start from the temporal derivative term that became:

$$\frac{\partial \langle \mathbf{v}_\beta \rangle}{\partial t} = \frac{\partial (\varepsilon \langle \mathbf{v}_\beta \rangle^\beta)}{\partial t} = \frac{\partial \varepsilon}{\partial t} \langle \mathbf{v}_\beta \rangle^\beta + \varepsilon \frac{\partial \langle \mathbf{v}_\beta \rangle^\beta}{\partial t}$$

Applying the same relation to the viscous term we get:

$$\nabla^2 \langle \mathbf{v}_\beta \rangle = \nabla^2 (\varepsilon \langle \mathbf{v}_\beta \rangle^\beta) = \varepsilon \nabla^2 \langle \mathbf{v}_\beta \rangle^\beta + \langle \mathbf{v}_\beta \rangle^\beta \nabla^2 \varepsilon + 2 \nabla \varepsilon \nabla \langle \mathbf{v}_\beta \rangle^\beta \quad (2.28)$$

and the pressure one also transform into:

$$\nabla \langle p_\beta \rangle = \nabla (\varepsilon \langle p_\beta \rangle^\beta) = \varepsilon \nabla \langle p_\beta \rangle^\beta + \langle p_\beta \rangle^\beta \nabla \varepsilon \quad (2.29)$$

Putting now altogether:

$$\begin{aligned} & \frac{\partial \varepsilon}{\partial t} \langle \mathbf{v}_\beta \rangle^\beta + \varepsilon \frac{\partial \langle \mathbf{v}_\beta \rangle^\beta}{\partial t} + \nabla \cdot (\varepsilon \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta) + \nabla \cdot (\langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle) = \\ &= -\varepsilon \nabla \left( \frac{\langle p_\beta \rangle^\beta}{\rho_\beta} \right) - \frac{\langle p_\beta \rangle^\beta}{\rho_\beta} \nabla \varepsilon + \nu_\beta \varepsilon \nabla^2 \langle \mathbf{v}_\beta \rangle^\beta + \nu_\beta \langle \mathbf{v}_\beta \rangle^\beta \nabla^2 \varepsilon + 2 \nabla \varepsilon \nabla \langle \mathbf{v}_\beta \rangle^\beta \\ &+ \frac{1}{\rho_\beta} \nabla \varepsilon \langle p_\beta \rangle^\beta - \frac{1}{V} \int_{A_{\beta\sigma}} \frac{\tilde{p}_\beta}{\rho_\beta} \mathbf{n}_{\sigma\beta} dA \\ &- \nu_\beta \nabla \varepsilon \nabla \langle \mathbf{v}_\beta \rangle^\beta + \frac{1}{V} \int_{A_{\beta\sigma}} \nu_\beta \nabla \tilde{\mathbf{v}}_\beta \cdot \mathbf{n}_{\sigma\beta} dA \end{aligned} \quad (2.30)$$

After the proper simplification we have the final versions of the Navier-Stokes system of equations (2.1) using intrinsic quantities:

$$\left\{ \begin{aligned} & \frac{\partial \varepsilon}{\partial t} \langle \mathbf{v}_\beta \rangle^\beta + \varepsilon \frac{\partial \langle \mathbf{v}_\beta \rangle^\beta}{\partial t} + \nabla \cdot (\varepsilon \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta) + \nabla \cdot (\langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle) = \\ &= -\varepsilon \nabla \left( \frac{\langle p_\beta \rangle^\beta}{\rho_\beta} \right) + \nu_\beta \varepsilon \nabla^2 \langle \mathbf{v}_\beta \rangle^\beta + \nabla \varepsilon \nabla \langle \mathbf{v}_\beta \rangle^\beta + \nu_\beta \langle \mathbf{v}_\beta \rangle^\beta \nabla^2 \varepsilon \\ &+ \frac{1}{V} \int_{A_{\beta\sigma}} \left( -\frac{\tilde{p}_\beta}{\rho_\beta} \mathbf{I} + \nu_\beta \nabla \tilde{\mathbf{v}}_\beta \right) \cdot \mathbf{n}_{\sigma\beta} dA \\ & \nabla \cdot (\varepsilon \langle \mathbf{v}_\beta \rangle^\beta) + \frac{\partial \varepsilon}{\partial t} = 0 \end{aligned} \right. \quad (2.31)$$

First of all is important to pinpoint that the intrinsic momentum equation represent the force per unit volume, so it explicitly depends on the porosity of the medium that can change the unit volume size (this is why it has terms involving gradients of the porosity). In application where porosity can vary spatially (like the interface of a porous medium) this formulation has the advantage to treat explicitly the interface non-homogeneities (further discussion of the interface treatment is presented in paragraph 2.5).

The equation (2.31) is also *non-local* since it has volume average quantities and surface integrals. These terms need some explicit manipulation in order to get a close formulation of the above system. In the next paragraphs we develop closure formulations of these terms. In order to do be clear we name these terms as *sub-filter stresses*  $\zeta$  and *microscopic force* at the fluid-solid interface  $\mathbf{F}^m$ :

$$\zeta = \nabla \cdot (\langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle)$$

$$\mathbf{F}^m = \frac{1}{V} \int_{A_{\beta\sigma}} \left( -\frac{\tilde{p}_\beta}{\rho_\beta} \mathbf{I} + \nu_\beta \nabla \tilde{\mathbf{v}}_\beta \right) \cdot \mathbf{n}_{\sigma\beta} \, dA$$

## 2.4 Closure problems

### 2.4.1 Microscopic force $\mathbf{F}^m$

It's a surface filter. The term  $\mathbf{F}^m$  is referred as *surface filter* since it act as a filter for the micro-scale fields that even if they were available that are going to be filtered by the integral over the fluid-solid interface.

In theory there is no simple representation for  $\mathbf{F}^m$  when the terms that includes the terms including the gradient of  $\nabla \varepsilon$ . Although since we are interested in developing a **local** closure problem that will depend on the geometry of one REV is possible to simply these terms. This means that the closure problems are not correct in the interface between a porous medium and a free fluid; but in the last chapter we show that even though we are committing an error we can still use the same closure problems and obtain good results.

In order to develop a closure problem we recall the decomposition by Gray [65]:

$$\psi_\beta = \langle \psi_\beta \rangle^\beta + \tilde{\psi}_\beta \quad (2.32)$$

From the continuity equation valid at the microscopic scale of the fluid phase we subtract the continuity equation valid for the intrinsic average velocity:

$$\nabla \cdot \mathbf{v}_\beta = 0$$

$$\nabla \cdot \langle \mathbf{v}_\beta \rangle^\beta = 0$$

we obtain the continuity equation for the perturbations:

$$\nabla \cdot \tilde{\mathbf{v}}_\beta = 0 \quad (2.33)$$

With the same procedure we first dived the momentum equation in system 2.31 by the permeability  $\varepsilon$  and then subtract them from the point equations obtaining:

$$\begin{aligned}
& \frac{\partial \tilde{\mathbf{v}}_\beta}{\partial t} + \mathbf{v}_\beta \cdot \nabla \tilde{\mathbf{v}}_\beta + \tilde{\mathbf{v}}_\beta \cdot \nabla \langle \mathbf{v}_\beta \rangle^\beta = \\
&= -\nabla \left( \frac{\tilde{p}_\beta}{\rho_\beta} \right) + \nu_\beta \nabla^2 \tilde{\mathbf{v}}_\beta - \varepsilon^{-1} \nabla \cdot (\tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta) + \varepsilon^{-1} \nabla \varepsilon \nabla \langle \mathbf{v}_\beta \rangle^\beta + \varepsilon^{-1} \nu_\beta \langle \mathbf{v}_\beta \rangle^\beta \nabla^2 \varepsilon \\
&\quad - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \left( -\frac{\tilde{p}_\beta}{\rho_\beta} \mathbf{I} + \nu_\beta \nabla \tilde{\mathbf{v}}_\beta \right) \cdot \mathbf{n}_{\sigma\beta} \, dA
\end{aligned} \tag{2.34}$$

Now in order to simplify the above equations we introduce these length-scale estimates:

$$\tilde{\mathbf{v}}_\beta = O(\langle \mathbf{v}_\beta \rangle^\beta), \quad \nabla \tilde{\mathbf{v}}_\beta = O\left(\frac{\langle \mathbf{v}_\beta \rangle^\beta}{\ell}\right), \quad \nabla \langle \mathbf{v}_\beta \rangle^\beta = O\left(\frac{\langle \mathbf{v}_\beta \rangle^\beta}{L}\right), \quad \varepsilon = O(\delta)$$

the last length scale estimates is based on the fact that the porosity varies on scale  $\delta$  that is much greater than  $\ell$ , this means  $\ell \ll \delta$ . Valdés-Parada et al. [137] and Ochoa-Tapia and Whitaker [104] make this argument considering that  $\delta$  in their case is the size of the zone in which the porosity change because we have an interface between a porous medium and a free fluid. However is important to state that this assumption does not holds at the interface of all the porous media geometry (for ordered porous media  $\varepsilon = O(\ell)$ ) but even in these cases eliminate the terms containing the gradient of the porosity is a necessity to formulate a closure problem. Whitaker [145] state clearly that there is no easy way to define a *local* closure problem when the above approximations does not holds, here we drop these terms to fulfill the derivation of the closure problem although we keep in mind this problem holds only far from region where the porosity field change (even if in the last chapter we relax this hypothesis).

Continuing with the order estimates we can neglect some of the terms in which the order of magnitude is order :

$$\mathbf{v}_\beta \cdot \nabla \tilde{\mathbf{v}}_\beta \gg \tilde{\mathbf{v}}_\beta \cdot \nabla \langle \mathbf{v}_\beta \rangle^\beta \Rightarrow O\left(\frac{\langle \mathbf{v}_\beta \rangle^\beta}{\ell}\right) \gg O\left(\frac{\langle \mathbf{v}_\beta \rangle^\beta}{L}\right) \tag{2.35}$$

$$\mathbf{v}_\beta \cdot \nabla \langle \tilde{\mathbf{v}}_\beta \rangle^\beta \gg \varepsilon^{-1} \nabla \cdot (\langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle) \Rightarrow O\left(\frac{(\langle \mathbf{v}_\beta \rangle^\beta)^2}{\ell}\right) \gg O\left(\frac{(\langle \mathbf{v}_\beta \rangle^\beta)^2}{\delta}\right) \tag{2.36}$$

$$\frac{\partial \tilde{\mathbf{v}}_\beta}{\partial t} \ll \nu_\beta \nabla^2 \tilde{\mathbf{v}}_\beta \Rightarrow O\left(\frac{(\langle \mathbf{v}_\beta \rangle^\beta)^2}{\ell}\right) \gg O\left(\frac{(\langle \mathbf{v}_\beta \rangle^\beta)^2}{L}\right) \tag{2.37}$$

In the last estimation we have assumed that the time scale associated respectively with the micro and macro-scale are  $t = \frac{\ell}{\langle \mathbf{v}_\beta \rangle^\beta}$  and  $T = \frac{L}{\langle \mathbf{v}_\beta \rangle^\beta}$ . This assumption imply the *quasi-stationary* of the perturbation problem, and it is physically held by the fact that the perturbation problem can be considered steady from the macroscopic evolution

perspective Davit et al. [41] and Zhu et al. [157]. The readers familiars with the multiples scales methods can notice that the above simplification we have neglected terms of order  $\epsilon$  or bigger, that is coherent with the same theory in which only zero order terms are used in the local closure formulation.

The order of magnitude estimation leave use with:

$$\begin{cases} \mathbf{v}_\beta \cdot \nabla \tilde{\mathbf{v}}_\beta = -\nabla \left( \frac{\tilde{p}_\beta}{\rho_\beta} \right) + \nu_\beta \nabla^2 \tilde{\mathbf{v}}_\beta - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \left( -\frac{\tilde{p}_\beta}{\rho_\beta} \mathbf{I} + \nu_\beta \nabla \tilde{\mathbf{v}}_\beta \right) \cdot \mathbf{n}_{\sigma\beta} dA \\ \nabla \cdot \tilde{\mathbf{v}}_\beta = 0 \\ \tilde{\mathbf{v}}_\beta = -\langle \mathbf{v}_\beta \rangle^\beta \quad \text{at } A_{\beta\sigma} \end{cases} \quad (2.38)$$

that is the transport equations system for the perturbation fields. The problem of the above system is that it is defined on all the macroscopic porous region and so we would like to find a way to reduce the size of the domain and still obtain the same results. The problem above is also depend on macroscopic terms that act as source terms, and because if not this micro-macro scale couplings would constraint use to solve on the overall domain for either the averaged and the fluctuations field we no practical advantage over the point equations. Although is possible to use Green function to solve the problem in this form Wood and Valdés-Parada [147], we are interested in a *local* closure problem.

This can be done restricting the solution region to a single REV, enforcing periodic boundary condition at the exterior of such volume. Such hypothesis is consistent with periodic order porous media because we have already stated that the variations connected to mean averaged quantities can be neglected inside the REV (see paragraph 2.7). The problem stated in these manner became:

$$\begin{cases} \mathbf{v}_\beta \cdot \nabla \tilde{\mathbf{v}}_\beta = -\nabla \left( \frac{\tilde{p}_\beta}{\rho_\beta} \right) + \nu_\beta \nabla^2 \tilde{\mathbf{v}}_\beta - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \left( -\frac{\tilde{p}_\beta}{\rho_\beta} \mathbf{I} + \nu_\beta \nabla \tilde{\mathbf{v}}_\beta \right) \cdot \mathbf{n}_{\sigma\beta} dA \\ \nabla \cdot \tilde{\mathbf{v}}_\beta = 0 \\ \tilde{\mathbf{v}}_\beta = -\langle \mathbf{v}_\beta \rangle^\beta \quad \text{at } A_{\beta\sigma} \\ \tilde{p}_\beta(\mathbf{x} + \ell_i) = \tilde{p}_\beta(\mathbf{x}), \quad \tilde{\mathbf{v}}_\beta(\mathbf{x} + \ell_i) = \tilde{\mathbf{v}}_\beta(\mathbf{x}), \quad i = 1, 2, 3 \\ \langle \tilde{\mathbf{v}}_\beta \rangle^\beta = 0 \end{cases} \quad (2.39)$$

In the above problem the averaged condition  $\langle \tilde{\mathbf{v}}_\beta \rangle^\beta = 0$  is used to fix te above problem that otherwise will be defined up to a constant.

Now we introduce the closure variables  $\mathbf{R}$ ,  $\mathbf{r}$ , for which we made the hypothesis that:

$$\tilde{\mathbf{v}}_\beta(\mathbf{x}) = \mathbf{R}(\mathbf{x}) \cdot \langle \mathbf{v}_\beta \rangle^\beta(\mathbf{x}) + \boldsymbol{\xi}(\mathbf{x}) \quad (2.40)$$

$$\tilde{p}_\beta(\mathbf{x}) = \mu_\beta \mathbf{r}(\mathbf{x}) \cdot \langle \mathbf{v}_\beta \rangle^\beta(\mathbf{x}) + \gamma(\mathbf{x}) \quad (2.41)$$

the above choice is just dictate from convenience. Is very important to pinpoint that the above one is a crucial assumption since we imply that exist a linear correlation between the micro and macro-scale fields. Even if the closure variables can be space dependent.

Since we are free to define the tensor  $\mathbf{R}$  and the vector  $\mathbf{r}$  as we wish we chose to define it by means of the closure problems:

$$\begin{cases} \frac{\mathbf{v}_\beta}{\nu_\beta} \cdot \nabla \mathbf{R} = -\nabla \mathbf{r} + \nabla^2 \mathbf{R} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} (-\mathbf{r}\mathbf{I} + \nabla \mathbf{R}) \cdot \mathbf{n}_{\sigma\beta} dA \\ \nabla \cdot \mathbf{R} = 0 \\ \mathbf{R} = \mathbf{I} \quad \text{at } A_{\beta\sigma} \\ \mathbf{r}(\mathbf{x} + \ell_i) = \mathbf{r}(\mathbf{x}), \quad \mathbf{R}(\mathbf{x} + \ell_i) = \mathbf{R}(\mathbf{x}), \quad i = 1, 2, 3 \\ \langle \mathbf{R} \rangle^\beta = 0 \end{cases} \quad (2.42)$$

With  $\mathbf{R}$  and  $\mathbf{r}$  specified as above we can prove that  $\xi$  is zero and  $\gamma$  a constant. The above problem is a closed formulation for the two fields  $\mathbf{R}$  and  $\mathbf{r}$  but it is difficult to solve computationally since it is an integral-differential equation. In order to simplify it we can decompose the problem in two parts, the first one give us the *permeability tensor* and the second one the *Forchheimer tensor*.

We represent  $\mathbf{R}$  and  $\mathbf{r}$  as:

$$\mathbf{R} = \mathbf{B} + \mathbf{C}, \quad \mathbf{r} = \mathbf{b} + \mathbf{c}$$

So the previous decomposition became:

$$\tilde{\mathbf{v}}_\beta = \mathbf{B} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{C} \cdot \langle \mathbf{v}_\beta \rangle^\beta \quad (2.43)$$

$$\tilde{p}_\beta = \mu_\beta \mathbf{b} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mu_\beta \mathbf{c} \cdot \langle \mathbf{v}_\beta \rangle^\beta \quad (2.44)$$

so the first problem became:

$$\begin{cases} 0 = -\nabla \mathbf{b} + \nabla^2 \mathbf{B} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} (-\mathbf{b}\mathbf{I} + \nabla \mathbf{B}) \cdot \mathbf{n}_{\sigma\beta} dA \\ \nabla \cdot \mathbf{B} = 0 \\ \mathbf{B} = -\mathbf{I} \quad \text{at } A_{\beta\sigma} \\ \mathbf{b}(\mathbf{x} + \ell_i) = \mathbf{b}(\mathbf{x}), \quad \mathbf{B}(\mathbf{x} + \ell_i) = \mathbf{B}(\mathbf{x}), \quad i = 1, 2, 3 \\ \langle \mathbf{B} \rangle^\beta = 0 \end{cases} \quad (2.45)$$

and the second one:

$$\begin{cases} \frac{\mathbf{v}_\beta}{\nu_\beta} \cdot \nabla \mathbf{B} + \frac{\mathbf{v}_\beta}{\nu_\beta} \cdot \nabla \mathbf{C} = -\nabla \mathbf{c} + \nabla^2 \mathbf{C} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} (-\mathbf{c}\mathbf{I} + \nabla \mathbf{C}) \cdot \mathbf{n}_{\sigma\beta} dA \\ \nabla \cdot \mathbf{C} = 0 \\ \mathbf{C} = 0 \quad \text{at } A_{\beta\sigma} \\ \mathbf{c}(\mathbf{x} + \ell_i) = \mathbf{c}(\mathbf{x}), \quad \mathbf{C}(\mathbf{x} + \ell_i) = \mathbf{C}(\mathbf{x}), \quad i = 1, 2, 3 \\ \langle \mathbf{C} \rangle^\beta = 0 \end{cases} \quad (2.46)$$

Injecting the decompositions 2.41 inside the surface filter  $\mathbf{F}^m$  we get:

$$\mathbf{F}^m = \frac{1}{V_\beta} \int_{A_{\beta\sigma}} (-\mathbf{r}\mathbf{I} + \nabla \mathbf{R}) \cdot \mathbf{n}_{\sigma\beta} dA$$

Diving then the closure variable as in 2.44 is possible to define the *permeability tensor*  $\mathbf{K}$ :

$$\frac{1}{V_\beta} \int_{A_{\beta\sigma}} (-\mathbf{b}\mathbf{I} + \nabla \mathbf{B}) \cdot \mathbf{n}_{\sigma\beta} dA = -\varepsilon \mathbf{K}^{-1}$$

and the *Forchheimer tensor*  $\mathbf{F}$ :

$$\frac{1}{V_\beta} \int_{A_{\beta\sigma}} (-\mathbf{c}\mathbf{I} + \nabla \mathbf{C}) \cdot \mathbf{n}_{\sigma\beta} dA = -\varepsilon \mathbf{K}^{-1} \cdot \mathbf{F}$$

using this definition to make the changing of variables proposed by Barrere et al. [9]:

$$\mathbf{d} = \varepsilon^{-1} \mathbf{b} \cdot \mathbf{K}, \quad \mathbf{D} = \varepsilon^{-1} (\mathbf{B} + \mathbf{I}) \cdot \mathbf{K}$$

making the problem (2.45):

$$\begin{cases} 0 = -\nabla \mathbf{d} + \nabla^2 \mathbf{D} + \mathbf{I} \\ \nabla \cdot \mathbf{D} = 0 \\ \mathbf{D} = 0 \quad \text{at } A_{\beta\sigma} \\ \mathbf{d}(\mathbf{x} + \ell_i) = \mathbf{d}(\mathbf{x}), \quad \mathbf{D}(\mathbf{x} + \ell_i) = \mathbf{D}(\mathbf{x}), \quad i = 1, 2, 3 \\ \langle \mathbf{D} \rangle^\beta = \varepsilon^{-1} \mathbf{K} \end{cases} \quad (2.47)$$

and the following for the second problem:

$$\mathbf{m} = \varepsilon^{-1} \mathbf{n} \cdot \mathbf{H}, \quad \mathbf{M} = \varepsilon^{-1} (\mathbf{N} + \mathbf{I}) \cdot \mathbf{H}$$

where  $\mathbf{H}$  is called *effective permeability tensor* and is so defined:

$$\mathbf{H}^{-1} = \mathbf{K}^{-1} (\mathbf{I} + \mathbf{F})$$

so the problem became:

$$\begin{cases} \frac{\mathbf{v}_\beta}{\nu_\beta} \nabla \mathbf{M} = -\nabla \mathbf{m} + \nabla^2 \mathbf{M} + \mathbf{I} \\ \nabla \cdot \mathbf{M} = 0 \\ \mathbf{M} = 0 \quad \text{at } A_{\beta\sigma} \\ \mathbf{m}(\mathbf{x} + \ell_i) = \mathbf{m}(\mathbf{x}), \quad \mathbf{M}(\mathbf{x} + \ell_i) = \mathbf{M}(\mathbf{x}), \quad i = 1, 2, 3 \\ \langle \mathbf{M} \rangle^\beta = \varepsilon^{-1} \mathbf{H} \end{cases} \quad (2.48)$$

However exist in literature simplified regression that permits to by-pass the *local* closure problem computation and get directly the tensors  $\mathbf{K}$  and  $\mathbf{F}$ . One of the most famous is the Kozeny-Carman Kozeny [80] relation and the modified Ergun equation [119]. Extended version of this empirical formulation can be found in Zampogna and Bottaro [151] and [148]. The above relationship are always based on some regression from experiments or computation, and they are parameterized usually with the porosity and some geometrical characteristic of the medium. Using this formulation has the downside that range of application can be restricted to very small Reynolds number and that the geometry assumed is most of the time very simple (spheres, or 2D regular arranged cylinders); this is why the closure problem rest the main reliable source of computation of the two tensor.

**explain that tis is the fermature**

$$\mathbf{F}^M = -\nu_\beta \varepsilon \mathbf{H}^{-1} \cdot \langle \mathbf{v}_\beta \rangle^\beta$$

#### 2.4.2 Sub-filter stresses $\zeta$

We have already seen that another term that needs a closure came out from the VANS equation and it is the *sub-filter stresses* term.

$$\zeta = \nabla \cdot (\langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle)$$

This term act as a volume filter for the perturbation fields product that appear inside the volume averaging operator. Although we have already seen in the previous paragraph that this term can be neglected, based on some length-scale argument, here we want to explain briefly what this term represent and possibly when it can be important.

Breugem et al. [23] and Nepf [103] divides the nature of sub-filter stresses in two different components:

- *mechanical diffusion*: inside the pore the fluid has to move around the solid structure (for examples fibers) causing an augmentation of diffusion inside the VANS momentum equations. This mechanism is so caused by the tortuosity of the flow path of the different particles.

- *turbulent dispersion*: is caused by the subfilter scales eddies that creates at the pore scale. This type of additional diffusivity can be anisotropic reflecting the anisotropicity of the medium; for example in fibers case the vertical penetration and breakdown of eddies is much higher than the horizontal one.

However comparing some simple formulation of such terms to the VANS equations shows that even if the two different components are equally important they are negligible for the volume averaged field equations.

Although we speculate that this term can become important in situations involving elastic porous media such as canopy flow where when the interface experience sweep and injection of fluid at the interface. In fact Finnigan [51] and De Langre [42] have shown that the turbulence spectrum is modified in the above cases and we argue that the sub-filter stresses term can, probably, model this high frequency vortices.

In order to better study this term we need reliable full DNS inside the porous media at high pore Reynolds number, but such simulations are very expensive and almost un-existent in literature. Experimental measurements inside the porous structure can be another way to study this volume filter even though such measurements are very hard to perform even with the latest technique.

## 2.5 Interface treatment

The problem of the interface condition between a porous medium and a free fluid has been approached by many different methods; Ehrhardt [47] has given a concise but very clear introduction on the problem, however the field is still evolving and in the last couple of years many new works has been presented Minale [99], Angot et al. [5], Lācis and Bagheri [81] and cite new work of Giuseppe. The problem is per se very complex and our work was not focused on developing a new condition; besides in this paragraph we want to explain our choice for the interface treatment over the many possible ones. The interface conditions can be classified in two groups, the *one domain approach* (ODA) and the *two domain approach* (TDA). In the TDA the porous domain is splitted from the free fluid domain and a boundary condition at the interface is specified; the necessity to do so was mainly caused by the fact that the first fluid model to coupled were the Stokes equations and the Darcy ones that have different orders so were incompatible at the interface. Some later work introduce the Brinkmann model to adjust the order of the porous media equations but the validity of this correction deep inside the porous medium was questionable. The principal works that follow this approach are Beavers and Joseph [14], Mikelic and Jäger [97], Ochoa-Tapia and Whitaker [104] and Le Bars and Worster [85]. These works all have in common a formulation in which a certain slip is specified at the interface, for example the condition specified in the first of the above references reads:

$$\langle \mathbf{v}_\beta \rangle^\beta(x, \Gamma^+) = \frac{\sqrt{\mathbf{K}}}{\alpha} \frac{\partial \langle \mathbf{v}_\beta \rangle^\beta(x, \Gamma^-)}{\partial y}$$

where  $\Gamma^+$  and  $\Gamma^-$  represent vertical coordinate above and below the interface,  $\mathbf{K}$  is the permeability tensor and  $\alpha$  is a coefficient based on the porous medium geometry. The other cited proposition change and extend this formulation but basically the all impose a velocity jump at the interface that is function of a parameter (here  $\alpha$ ) that is basically a fit to experimental data.

On the contrary the ODA approach presume that the final averaged equation are valid in all the domain (porous and free fluid) but the quantities that define the presence of the porous media (porosity and permeability) vanished in the free fluid region. This method is also know as *penalization method*, one of the first porous media application was Caltagirone [30] after that it was used by many other authors, like Bruneau and Mortazavi [26], Bruneau and Mortazavi [27], Bruneau et al. [28], Hussong et al. [71]. We think that using a boundary condition at the interface is not a superior approach nor physically neither mathematically. As a matter of fact either methods require a parameter to close the formulation; with the advantage, using penalization method, that the parameter is the spatial distribution of the porosity field that is trivial to compute known the geometry of the medium. Although how to vary the permeability in the transition zone is not clear, in fact for most of the authors the jump from the porous media value and the free fluid one is sharp; so neglecting the variation of permeability at the transition zone appear to be acceptable, even though example of linear variation of this terms exists Caltagirone [30]. Hussong et al. [71] made a direct comparison (with a DNS simulation) for the porosity treatment concluding that the variation of the permeability so the VANS terms that include the gradient of the porosity are very important to have a good comparison with high fidelity computation.

Direct comparison between the ODA and TDA is presented in Cimolin and Discacciati [37] concluding that the macroscopic description of the interface provided by the two different methods is similar and the penalization method has the advantage to be easily implemented in a Navier-Stokes solver and it does not present sensible convergence properties of the TDA.

Also there is evidence in literature (Ochoa-Tapia et al. [105]) that exist a transition zone with the size of the pore scale in which the velocity and pressure have a continuous variation and not a steep one, also it has been demonstrated, by the same author, that the same transition zone is physical and not a results of the averaging procedure.

Our final decision on the interface treatment is to adopt the penalization approach with the porosity variation computed directly from the geometry of our fibrous medium and a steep variation of the effective permeability at the interface; even if in chapter 5 ([add actual reference to the chapter](#)) we have tried a variation based on an hyperbolic function smoothing that has not given better results.

## 2.6 Elastic porous media: hybrid homogenization approach

Elastic porous media has been firstly studied by Biot [20] which developed a model for the stresses wave propagation inside the solid matrix of a porous medium. His model was used in porous media with almost the same density as the solid, as for the case of saturated soil.

Whitaker [142] has also approached the problem using the volume average method to homogenize either the solid and the fluid part, however the closure problems for the tensor that came out from the averaging of the solid equation are very difficult to solve and there are no following studies that neither clarify the problem nor confirm the formulation.

In our derivation of the VANS the only term that has to be changed in order to account to the porous media that can move if the microscopic force  $\mathbf{F}^m$ , in any of the others development we have used the hypothesis of rigid porous media. This term should be modified to take into account that the force that the fluid exert on the solid part has to take into account the solid velocity:

$$\mathbf{F}^M = -\nu_\beta \varepsilon \mathbf{H}^{-1} \cdot [\langle \mathbf{v}_\beta \rangle^\beta - \langle \mathbf{v}_\sigma \rangle^\beta]$$

This same closure was also adopted in the works of Hussong et al. [71] and Wang et al. [139]. Now we want to present what we can an *hybrid homogenization approach*; as indicated in paragraph 1.2 in order to describe the dynamic of a poroelastic layer we need an fluid model, a solid model and an interface condition. In the above sections we have presented the fluid model (VANS equations) and the interface treatment; here we propose an approach that is transparent to whatever solid model one want to use. The pseudo-code fo the macroscopic algorithm can be sketched like the one in 1.

```

Data:  $(\langle \mathbf{v}_\beta \rangle^\beta)^n, (\langle \mathbf{v}_\sigma \rangle^\beta)^n, (\langle p_\beta \rangle^\beta)^n$ 
1 initialization;
2 while  $t < T$  do
3    $n \rightarrow 0;$ 
4   while  $n < n_{max}$  do
5     Fluid solver VANS:  $(\langle \mathbf{v}_\beta \rangle^\beta)^{n+1}, (\langle p_\beta \rangle^\beta)^{n+1}$  Solid solver:  $\mathbf{v}_\sigma^{n+1}$  Solid
        averaging:  $\varepsilon^{n+1}(x), (\langle \mathbf{v}_\sigma \rangle^\beta)^{n+1}$  Relaxation ?? if
         $\|(\langle \mathbf{v}_\beta \rangle^\beta)^{n+1} - (\langle \mathbf{v}_\beta \rangle^\beta)^n\| < \epsilon$  then
        | break;
6     else
7       |  $n + 1 \rightarrow n;$ 
8     end
9   end
10  end
11   $t \rightarrow t + \Delta t;$ 
12 end
```

**Algorithm 1:** Macroscopic algorithm for fluid-structure interaction of homogenized poroelastic medium

Discuss the solid model equation and the a posteriori homogenization of the same are beyond the scope of this section. The only idea that we want to express is that with just some slight modification in the fluid equation is possible to integrate them in a macroscopic algorithm that can take into account moving fibrous media.

## 2.7 Note on the average of an average field

In the above sections have briefly talked about the results in equation 2.6 that we recall here:

$$\langle\langle\psi_\beta\rangle|_{\mathbf{r}}\rangle|_{\mathbf{x}} = \langle\psi_\beta\rangle|_{\mathbf{x}}$$

And introducing the decomposition 2.23 the above results can be used to state that the perturbation fields has null averaged:

$$\langle\tilde{\psi}_\beta\rangle = 0$$

But let's recall what the average operator really does when applied to an averaged quantities:

$$\langle\langle\psi_\beta\rangle|_{\mathbf{r}}\rangle|_{\mathbf{x}} = \frac{1}{V} \int_{V_\beta(\mathbf{x})} \langle\psi_\beta\rangle|_{\mathbf{r}}(\mathbf{r}) dV$$

The above equation can be described as the average computed over the volume  $V$  with centroid  $\mathbf{x}$ , of the averaged field  $\langle\psi_\beta\rangle|_{\mathbf{r}}$  that can vary, because of the change of  $\mathbf{r}$ .

In order to show how the above expression can be simplified we expand the averaged quantity  $\langle\psi_\beta\rangle|_{\mathbf{r}}$  over the centroid  $\mathbf{x}$  using Taylor's polynomial:

$$\langle\psi_\beta\rangle|_{\mathbf{r}} = \langle\psi_\beta\rangle|_{\mathbf{x}} + \mathbf{y} \cdot \nabla \langle\psi_\beta\rangle|_{\mathbf{x}} + \frac{1}{2} \mathbf{y} \mathbf{y} : \cdot \nabla \nabla \langle\psi_\beta\rangle|_{\mathbf{x}} + O(\mathbf{y}^3)$$

Now if we put this expansion inside the averaging operator we get:

$$\langle\langle\psi_\beta\rangle|_{\mathbf{r}}\rangle|_{\mathbf{x}} = \langle\psi_\beta\rangle|_{\mathbf{x}} + \langle\mathbf{y}\rangle|_{\mathbf{x}} \cdot \nabla \langle\psi_\beta\rangle|_{\mathbf{x}} + \frac{1}{2} \langle\mathbf{y} \mathbf{y}\rangle|_{\mathbf{x}} : \cdot \nabla \nabla \langle\psi_\beta\rangle|_{\mathbf{x}} + O(\mathbf{y}^3)$$

The term  $\langle\mathbf{y}\rangle$  is zero for REV used in ordered porous media since they are always chosen to be symmetric around the REV centroid. The second term can be shown to be negligible either with the same length-scale constraint used in the REV definition, in fact Ochoa-Tapia and Whitaker [104], Paéz-García et al. [107] showed that this term is order  $O(\epsilon^2)$ . Although there is possible to choose an appropriate filter function that enforce  $m * \mathbf{y} \mathbf{y} = 0$  strictly, these function are impractical Davit and Quintard [40]. As we recall from section 2.3.2 the triangle shaped filter almost satisfy this hypothesis, if fact the triangle function guarantee a second order closure and it means that doing the second term convolution product only the constant part of the periodic function is kept and this is also negligible since it is  $O(\epsilon^2)$ .

# Chapter 3

## Drag-model sensitivity of Kelvin-Helmholtz waves in canopy flows

*While knowledge can create problems, it is not through ignorance that we can solve them.*

- Asimov's New Guide to Science, 1984, Isaac Asimov

Gomez-de Segura et al. [62] state that their results agree with yours ??

Include this paper Sharma et al. [128]

### 3.1 Introduction

modify the introduction to not repeat the same parts that you have presented in the intro

Flows through submerged aquatic plants exhibit large scale vortices at the top of the vegetation, advecting along the flow direction and causing a periodic waving of the plants, referred to as monami [1]. Vortices arise from the nonlinear amplification of a Kelvin-Helmholtz instability mode, related to the presence of an inflection point in the base flow profile; [6] the profile itself is inflectional because the fluid is slowed down by the drag exerted by the canopy, whose modeling has recently been addressed [112] [129] [155]. The correct prediction of the onset and characteristics of the Kelvin-Helmholtz instability is important for assessing the effects of turbulence, in particular to

- understand how the vertical exchange of momentum occurs, 6
- clarify how the transport of  $CO_2$ , dissolved nutrients or sediments takes place between the obstructed vegetation flow and the free overflow motion, 7–10 and also

- assess the changes in the morphology of the vegetation in inland or coastal wetlands in response to continuous periodic forcing [6] [108]

Because of the flexibility of the vegetation, some theoretical studies have focussed on the modeling of the stems of the aquatic plants and their displacement in response to the forcing by the water flow. [112] [108] However, Kelvin-Helmholtz vortices occur whether or not the plants bend and—to ascertain causes and effects to first order—it is acceptable to focus on the flow over and through a submerged array of rigid, cylindrical pillars. This has been the basis of the approach by Ghisalberti and Nepf [59] [56] [58] who have conducted a series of careful experiments; their results have often been used by fluid dynamicists to put forth and test theoretical hypotheses to predict the frequency and wavelength of the large scale vortical motion, for a variety of conditions. The configuration studied consists of a regular grid of rigid pillars, orthogonal to the surface, of identical height  $h$ ; in some of the theoretical models proposed to analyze the stability of this system, the Rayleigh equation is used throughout the water channel, with or without a drag term in correspondence of the canopy. [121] [112] [129] [155] have recently demonstrated that the addition of a drag term through the vegetation reduces the amplification factor of the Kelvin-Helmholtz instability throughout the whole range of wavenumbers and increases mildly the wavelength of the fastest growing mode; further unpublished work by the same authors shows that the addition of a mixing length turbulence model in the stability equations has but a negligible influence on the leading instability mode. Questions remain, however, on the accuracy of the drag model and on its sensitivity. A partial answer to these questions is provided in [155]: there, a different model, applicable within the vegetated layer and based on the equations ruling the behavior of a transversely isotropic porous medium, has been developed and the stability results appear to better match experimental correlations. This conclusion is, however, not consolidated yet, and further studies are needed to assess the influence of the model of the drag force through the vegetation, both in setting up a particular (inflectional) mean flow and on the onset and growth of Kelvin-Helmholtz waves. The present work addresses the points above through an adjoint-based sensitivity analysis along the lines of [22] the direct stability equations are written with account of viscosity, and the adjoint equations are found and solved in the temporal framework. Results in the spatial setting are discussed in Appendix B, where a digression is made on the computation of the group velocity of the instability waves by the use of the adjoint fields. The sensitivity functions to both mild modifications in the base shear layer and in the drag coefficient are computed and discussed. Finally, a different sensitivity analysis is developed on the basis of the recent anisotropic model by [155] and the results qualitatively compared to those obtained with the more conventional isotropic-drag-force model.

## 3.2 Model of the canopy flow

### 3.2.1 The mean flow

To obtain the mean flow on top of which small amplitude perturbations are superimposed, the procedure outlined by [56] and recently closely followed by [155] is used. For the sake of conciseness, the procedure which relies on several empirical correlations is not repeated here, aside from a few brief comments. A mildly inclined water channel is considered, with a canopy formed by rigid cylindrical dowels of height  $h$  equal to  $13.8\text{ cm}$  and diameter  $d = 0.64\text{ cm}$ . The frontal area of the vegetation per unit volume, i.e., the packing density of the elements, is either  $a = 0.04\text{ cm}^{-1}$  or  $0.08\text{ cm}^{-1}$ ; the free surface is positioned at a level  $H = 46.7\text{ cm}$  from the bottom plate and the flow velocity at the free surface,  $U_2$ , varies from  $4.4$  to  $13.7\text{ cm/s}$ . The Froude number,  $F_r = \frac{U_2}{gH}$  is thus very low and water surface fluctuations can be ignored [24]. To a good approximation the mean flow can be taken as steady and parallel, with the streamwise velocity varying from the value  $U_1$  at the bottom wall (not accounting for the thin bottom boundary layer) to the value  $U_2$  at the top, near the free surface (3.1). The slope of the bottom surface is very small; it is denoted as  $S$  and, in the experiments by [56] varies from  $1.8 \times 10^{-6}$

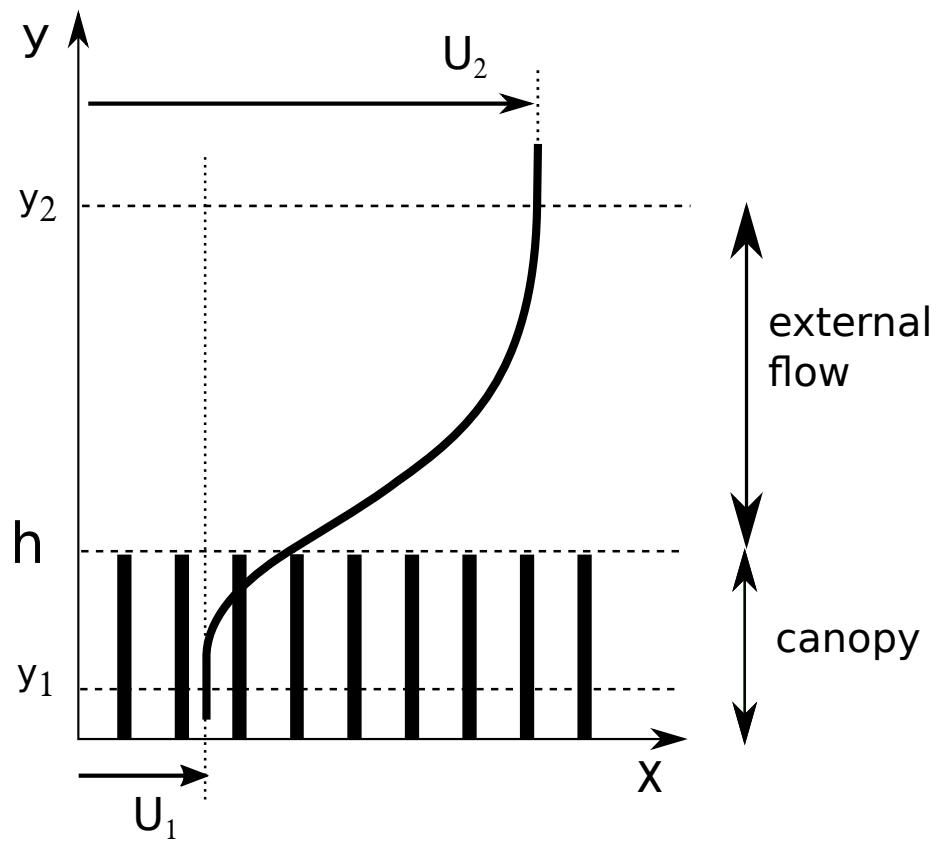


Figure 3.1: Configuration studied with main notations

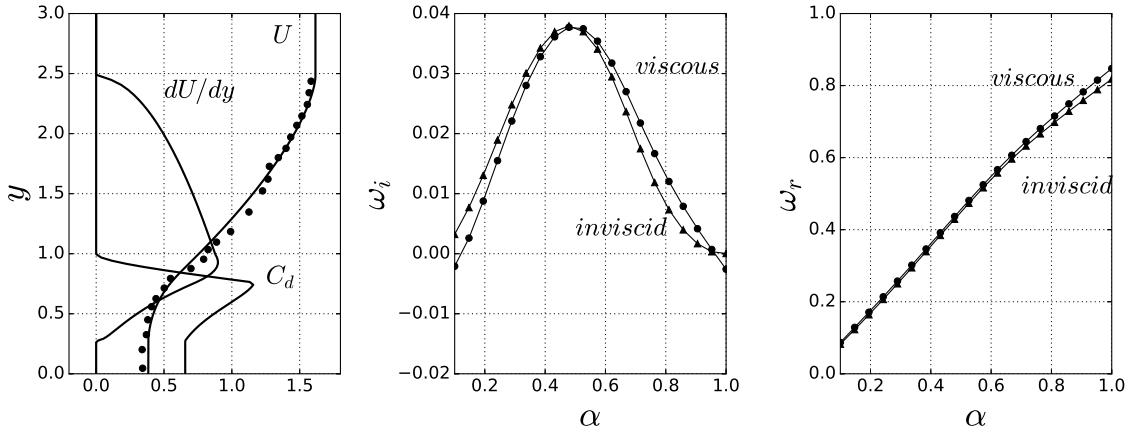


Figure 3.2: Left frame: mean flow  $U$ , together with experimental data points [56], its first derivative, and drag coefficient distribution (case G). Center: viscous and inviscid growth rates,  $\omega_i$ , as a function of the streamwise wavenumber  $\alpha$ . Right: corresponding frequencies,  $\omega_r$

to  $10^{-4}$ ; such a slope provides the driving force for the motion. The viscous term is small compared to the turbulent diffusion term, so that the mean streamwise momentum equation can be approximated by

$$gS = \frac{\partial \bar{u}'v'}{\partial y} + \frac{1}{2}C_D(y)aU(y)^2 \quad (3.1)$$

with  $g$  the acceleration of gravity and  $C_d$  an isotropic drag function available from the experiments, variable across the canopy and equal to zero when  $y \geq h$ . The Reynolds stress  $\bar{u}'v'$  is modelled with the Boussinesq assumption, introducing a turbulent viscosity which depends on a mixing length and on the gradient of the mean velocity  $U$ . Referring to [56] for details of the empirical correlations used to close the equations and the solution method, we limit ourselves here to stating that the results obtained for the mean flow are very close to those reported in [155] (cf. their Figure 3) and closely match experimental points for the cases G, H, I, and J considered (we use the same terminology of [59] [56] [58] to indicate the different flow configurations). An example of mean flow is reported in 3.2 (left frame). There, one can observe the computed flow (against discrete measurement points), its first derivative, and the drag coefficient distribution for one representative case (experiment G), used below also to discuss stability and sensitivity results. Other procedures have been employed in the past to calculate the mean flow, with satisfactory results. For example, [129] have considered a constant value of  $C_d$  through the canopy, while [155] have coupled, at a fictitious interface, the fluid equations outside the canopy to Darcy's law within the vegetation. Thus, for the purposes of the present paper, the mean flow is assumed as given; it could be, for example, simply a fit through experimental data.

Nonetheless, in Appendix A we provide some considerations on how  $C_d$  affects the mean flow in the model used here.

### 3.2.2 Stability and sensitivity equations

A temporal linear stability analysis is carried out, with the generic perturbation  $q'(x, y, t)$  of the form

$$q'(x, y, z, t) = \tilde{q}(y)e^{i(\alpha x - \omega t)} \quad (3.2)$$

with  $\alpha$  the real streamwise wavenumber and  $\omega$  a complex number whose real part,  $\omega_r$ , is the frequency of the mode and the imaginary part,  $\omega_i$ , is the growth rate. The dimensionless linear stability equations in primitive variables read

$$\begin{aligned} i\alpha u + Dv &= 0, & D &= d/dy \\ \left[ i(\alpha U - \omega) - \frac{D^2 - \alpha^2}{Re} + aC_d U \right] u + U' v + i\alpha p &= 0, & U' &= \frac{dU}{dy} \\ \left[ i(\alpha U - \omega) - \frac{D^2 - \alpha^2}{Re} \right] v + Dp &= 0 \end{aligned} \quad (3.3)$$

with the perturbation velocity components which vanish when  $y = 0$  and  $y_\infty$ . The upper boundary of the computational domain is taken far enough away from the lower boundary to ensure that the results do not vary upon modifications of  $y_\infty$ . All the terms in the equations are dimensionless; the mean speed through the shear layer,  $U_m = \frac{U_1 + U_2}{2}$ , is used to scale the disturbance velocity components, pressure is scaled with  $\rho U_m^2$ , distances with  $h$ , and time with  $h/U_m$ . The Reynolds number in the equations above is thus defined as  $Re = \rho U_m / \mu h$ , with  $\rho$  and  $\mu$  the fluid's density and dynamic viscosity, respectively. The computations are performed both at the  $Re$  values of the experiments and in the inviscid limit ( $Re^{-1} \rightarrow 0$ ), for comparison purposes. In the latter case, the boundary conditions are simply  $v = 0$  at  $y = 0$  and  $y_\infty$ . System 3.3 above and its boundary conditions are, in the following, also written in short notation as  $\mathcal{L}q = 0$ . The eigenvalues of the system are those complex values of  $\omega$  which yield non-trivial solutions for  $u$ ,  $v$ , and  $p$ . Two numerical collocation codes are written, and successfully compared; one is based on the equations in primitive variables form, the second solves an Orr-Sommerfeld-like equation (with the addition of the drag term) along the lines of [129]. In both cases, a spectral scheme based on N Chebyshev polynomials is used (N is typically equal to 300 to ensure grid-converged results), with an algebraic mapping between the physical and the spectral domains ([70]). Viscous and inviscid stability results for case G are shown in 3.2 (center and right frames); differences are small, in consideration of the fact that the Reynolds number of the viscous case is relatively large ( $Re = 3450$ ). The viscous wavenumber of largest amplification is

found for  $\alpha = 0.4790$ ; the waves are weakly dispersive, particularly at low wavenumbers (an original interpretation of phase and group velocities is proposed in Appendix B). The wavelength of largest growth is smaller than that found by [155] which was 0.73; this is related to the slightly different base flow in the two cases (in the present contribution a smoothing has been applied to the  $U$  velocity distribution to render  $dU/dy$  continuous across  $y$ ) and highlights the sensitivity of this stability problem to base flow variations. Following [22] it is assumed that small variations in base flow and drag coefficient entail infinitesimal variations in the system's eigenvalues and eigenfunctions. We stress here the fact that  $C_d$  is identically equal to zero outside of the canopy, and this implies that there are no possible variations in  $C_d$  for  $y \geq 1$ . The sensitivity functions to variations in  $U$  and  $C_d$  are obtained by using the properties of the adjoint system which is defined from the Lagrange identity

$$0 = \delta\langle q^\dagger, \mathcal{L}q \rangle = \langle q^\dagger, \mathcal{L}\delta q \rangle + \langle q^\dagger, \frac{\partial \mathcal{L}}{\partial U} q \delta U \rangle + \langle q^\dagger, \frac{\partial \mathcal{L}}{\partial C_d} q \delta C_d \rangle + \langle q^\dagger, \frac{\partial \mathcal{L}}{\partial \omega} q \rangle \delta \omega \quad (3.4)$$

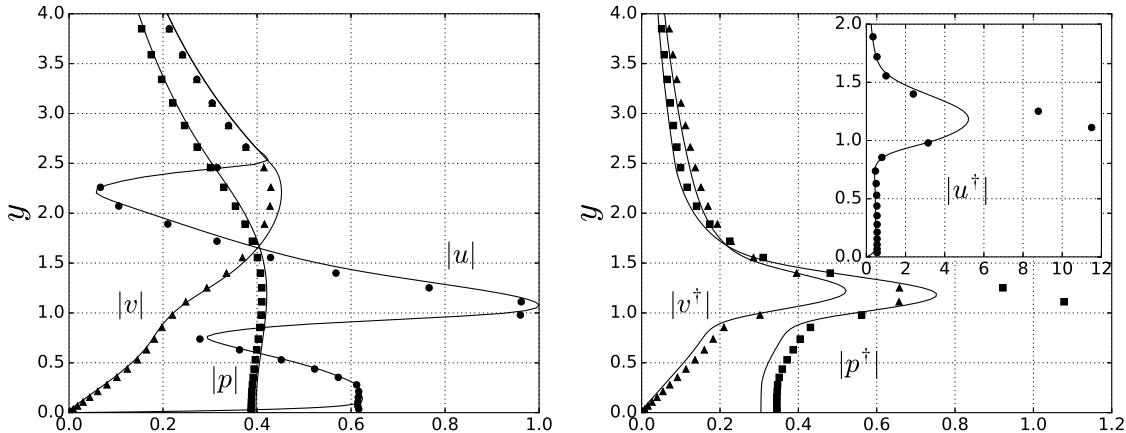


Figure 3.3: Moduli of direct (left frame) and adjoint (right frame) eigenfunctions for the viscous (continuous lines,  $Re = 3450$ ) and the inviscid (symbols) case, in correspondence to the wavenumber of largest amplification.

and considering the effect of independent variations of  $U$  and  $C_d$  onto  $q$  and  $\omega$ . It is found that

$$\delta\omega = \delta\omega_r + i\delta\omega_i = \int_0^{y_\infty} G_U(y)\delta U(y)dy + \int_0^1 G_{C_D}(y)\delta C_D(y)dy \quad (3.5)$$

with

$$\begin{aligned} G_U &= \alpha \left[ \bar{v^\dagger v} + \bar{u^\dagger u} \right] + i(\bar{u^\dagger v})' - iaC_d \bar{u^\dagger u} \\ G_{C_d} &= -iaU \bar{u^\dagger u} \end{aligned} \quad (3.6)$$

the required sensitivity functions; the real parts of  $G_U$  and  $G_{C_d}$  express sensitivities to variations in the frequency of the mode while the imaginary parts are sensitivities to variations in the growth rate. Direct and adjoint eigenfunctions are normalized so that  $N_\omega = 1$ , with

$$N_\omega = \int_0^{y_\infty} \left[ \bar{v^\dagger v} + \bar{u^\dagger u} \right] dy \quad (3.7)$$

An example of direct and adjoint eigenfunctions is provided in 3.3, both in the viscous case ( $Re = 3450$ ) and in the inviscid limit, for  $\alpha = 0.4790$ . It is interesting to observe that while the direct eigenfunctions are almost overlapped, the same is not the case for the adjoint eigenfunctions, with the inviscid mode (drawn with symbols) which has a larger amplitude than the viscous one. The shapes of the direct eigenfunctions are very close to those reported in [155]. The adjoint modes reveal that the flow is most sensitive to streamwise forcing, particularly when it occurs slightly above the edge of the canopy. Source terms in the mass conservation and in the vertical momentum equations are much less effective.

### 3.3 SENSITIVITY RESULTS FOR THE ISOTROPIC DRAG MODEL

Some representative sensitivity functions are plotted in 3.4; viscous and inviscid results concur in showing that the largest sensitivities to variations of  $U$  are found right above the vegetation's edge, where there are peaks in the adjoint eigenfunctions and where  $d^2U/dy^2$  vanishes. The  $U$ -sensitivities are negligible within the vegetated layer and for values of  $y$  larger than twice the canopy's height. The  $C_d$ -sensitivities are non-negligible only in close proximity of the interface. It is interesting to observe that real and imaginary parts of the  $U$ -sensitivity functions are shifted in  $y$  with respect to one another; this means that, for example, a localized perturbation at a given  $y$  position (above the canopy) might have a strong repercussion on the growth rate but not on the frequency of the most unstable Kelvin-Helmholtz mode, or vice versa. Comparing left and right frames of the figure, it is seen that inviscid  $G_U$  sensitivity functions display sharper peaks and steeper gradients, and yield larger variations in  $\omega$  than their viscous counterparts in the proximity of the  $U$  inflection point, a clear consequence of the inviscid mechanism ruling the instability. In both the viscous and the inviscid models, the sensitivity to base flow variations is typically one order of magnitude larger than the sensitivity to changes in the drag coefficient.

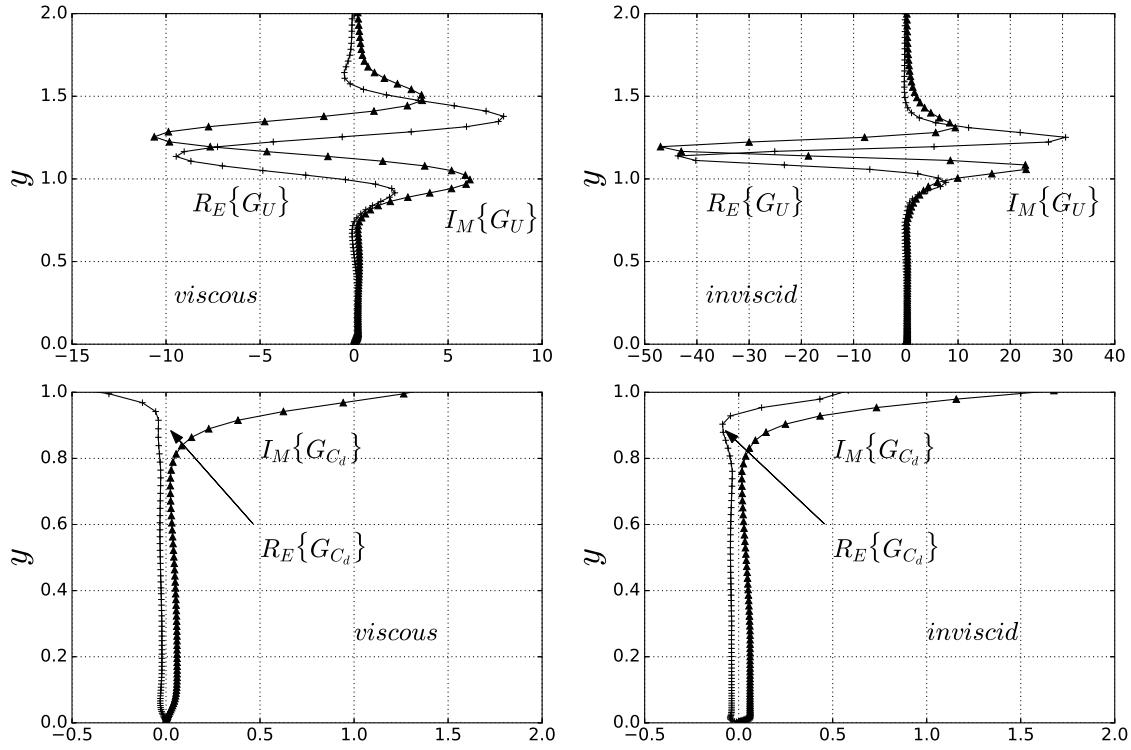


Figure 3.4: Real and imaginary parts of the sensitivities to mean flow variations (top) and to variations in the drag distribution function (bottom), for the parameters of 3.3

The infinite norm of the sensitivities for the four cases studied (G, H, I, and J) is reported in 3.5; the main result found is that  $|G_U|_\infty$  grows monotonically with  $\alpha$  (and more so in the inviscid case) whereas  $|G_{C_d}|_\infty$  does not. It is consistently found that  $|G_U|_\infty$  of case H is larger than that of case I, which exceeds the corresponding value of case J, in turn larger than  $|G_U|_\infty$  of case G. This is not unexpected in view of the values of the mean shear  $\frac{U_2 - U_1}{H}$  which are, going from H to G, equal

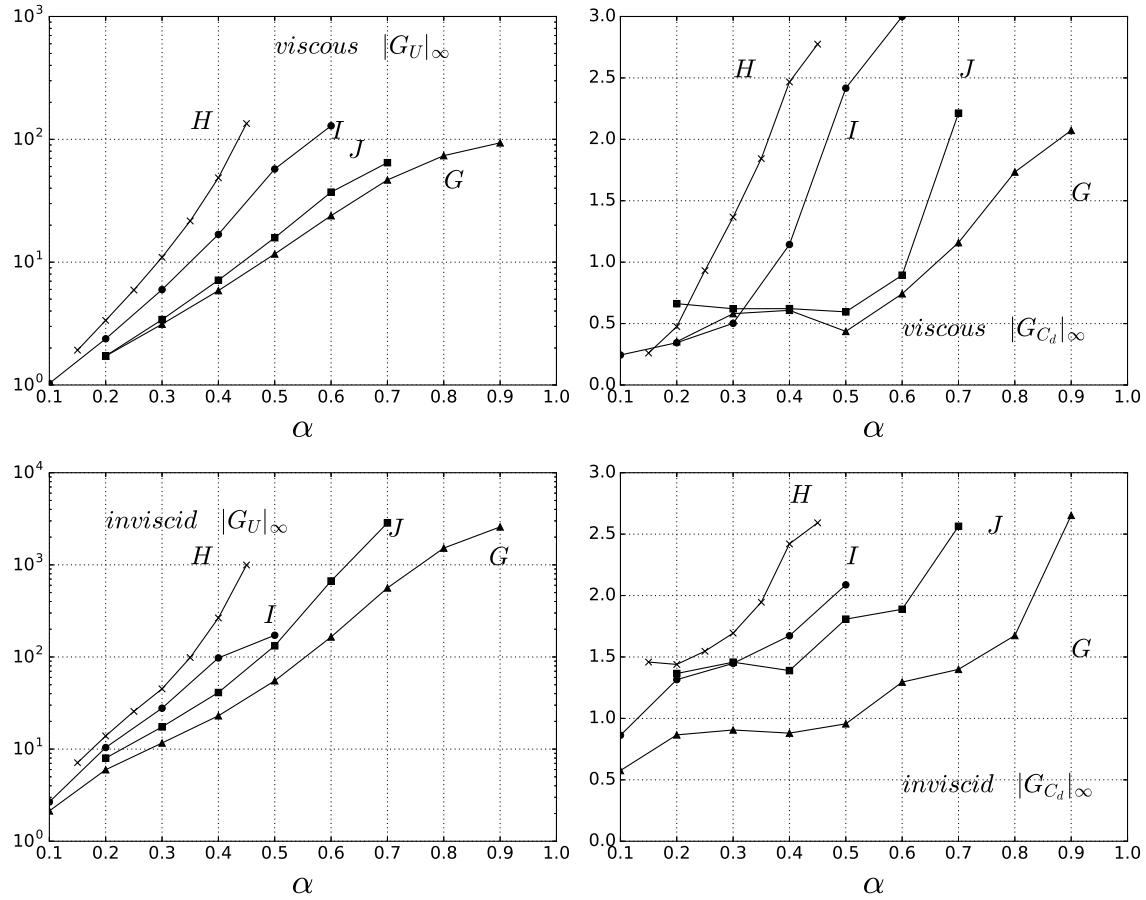


Figure 3.5: Infinite norms of the sensitivity functions for varying  $\alpha$

to  $0.236$ ,  $0.158$ ,  $0.084$ , and  $0.071 s^{-1}$ , respectively. The sensitivity of the eigenvalue  $\omega$  to variations in the mean flow is generally stronger than the corresponding sensitivity to variations in the drag coefficient (aside for the long wave limit, where they are comparable). This might be interpreted positively, considering that the use of a scalar coefficient  $C_d$  to represent the drag within the canopy is but a crude approximation. An alternative model to represent the flow throughout a network of rigid, cylindrical dowels has recently been proposed by [155] The sensitivity results for such a new model are discussed next.

### 3.4 AN ALTERNATIVE SENSITIVITY MODEL: ACCOUNTING FOR THE CANOPY ANISOTROPICITY

The stability problem in this section is based on the coupling between two regions, one outer region dominated by inertia and ruled by the inviscid equations and an inner one dominated by viscosity and ruled by Darcy's law, with account of the canopy geometry through a tensorial permeability, as described by [155] Normalizing the disturbance equation which couples pressure and velocity in the inner region with the same scales as previously, we obtain

$$u_i' = -Re \frac{d}{ah^2} \mathcal{K}_{ij} \frac{\partial p'}{\partial x_j}, \quad (x_1, x_2) = (x, y) \quad (3.8)$$

with  $\mathcal{K}_{ij}$  the dimensionless permeability. The effective interface between the inertial region and the slow, viscosity-dominated region does not coincide with the edge of the canopy; in fact, the rapid outer flow penetrates through the upper part of the vegetation and an effective matching between outer and inner flows must be enforced some distance  $\delta$  below the canopy's edge [85]. This distance, a penetration depth, has been successfully computed by [151] for a few cases and is found to increase with the Reynolds number of the flow; for experiment G discussed below it is  $\delta = 0.40$  [152]. On account of the results shown in 3.4, with the sensitivities which are negligible for  $y \approx 0.60$ , we expect that the exact position of the effective interface will not affect the results significantly. Using the fact that the velocity within the orthotropic porous medium is divergence free, the interface condition to be applied at  $y_{itf} = 1 - \delta$  is found to be 3.9

$$v|_{itf} + B(\alpha)p|_{itf} = 0 \quad (3.9)$$

with

$$B(\alpha) = Re \frac{d}{ah^2} \sqrt{\mathcal{K}_{11} \mathcal{K}_{22}} \alpha \tanh(\theta), \quad \theta = \alpha \sqrt{\frac{\mathcal{K}_{11}}{\mathcal{K}_{22}}} y_{itf}$$

The second boundary condition that the Rayleigh stability equation must satisfy at  $y_\infty$  is simply  $v = 0$ . Thus, we solve only for the inviscid flow in the outer region, and the permeability of the inner domain enters the equations only through the interface condition 3.9.  $\mathcal{K}_{ij}$  is a two- by-two diagonal tensor;  $\mathcal{K}_{11}$  is the component of the dimensionless permeability along  $x$  and  $\mathcal{K}_{22}$  is the  $y$  component. For case G considered here, the packing density of the elements is  $a = 0.04\text{cm}^{-1}$ ; it is also found that  $\mathcal{K}_{11} = 0.0512$  and  $\mathcal{K}_{22} = 0.0575$  [152], so that the function  $B(\alpha)$  reads  $B = 15.727\alpha \tanh(0.566\alpha)$ .

### 3.4.1 The sensitivity equations

The adjoint equations in this case are the same as system 3.3, without the terms containing  $1/Re$  and  $C_d$ , and the boundary conditions are

$$v^\dagger|_{itf} - B(\alpha)p^\dagger|_{itf} = 0, \quad v^\dagger|_{y_\infty} = 0 \quad (3.10)$$

The variation in the complex frequency is related to variations in the mean flow and in the permeability components through the equation

$$\delta\omega = \int_{y_{itf}}^{y_\infty} G_U(y)\delta U(y)dy + G_{\mathcal{K}_{11}}\delta\mathcal{K}_{11} + G_{\mathcal{K}_{22}}\delta\mathcal{K}_{22}$$

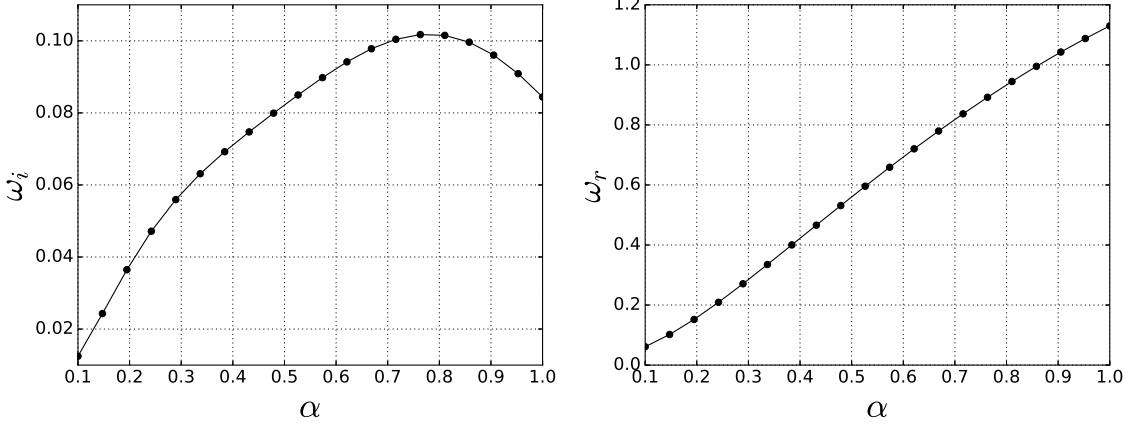


Figure 3.6: Amplification factor (left) and frequency of the most unstable mode as a function of  $\alpha$ , for the anisotropic drag model

with

$$\begin{aligned} G_U &= \alpha \left[ \overline{v^\dagger v} + \overline{u^\dagger u} \right] + i(\overline{u^\dagger v})' \\ G_{\mathcal{K}_{11}} &= -\frac{i}{2}\alpha Re \frac{d}{ah^2} \left[ \overline{p^\dagger p} \right] |_{itf} \sqrt{\frac{\mathcal{K}_{22}}{\mathcal{K}_{11}}} \left\{ \tanh \theta + \frac{\theta}{\cosh^2 \theta} \right\} \\ G_{\mathcal{K}_{22}} &= -\frac{i}{2}\alpha Re \frac{d}{ah^2} \left[ \overline{p^\dagger p} \right] |_{itf} \sqrt{\frac{\mathcal{K}_{11}}{\mathcal{K}_{22}}} \left\{ \tanh \theta - \frac{\theta}{\cosh^2 \theta} \right\} \end{aligned} \quad (3.11)$$

the required sensitivities, with the normalization  $\int_{y_{itf}}^{y_\infty} [\overline{v^\dagger v} + \overline{u^\dagger u}] = 1$ . In writing  $\delta\omega$  above, we have made the assumption that the mean flow  $U$  does not vary at the two extreme points of the integration domain. The stability results (for the same parameters as in 3.2)

are displayed in 3.6. As already observed in [155], both the growth rate and the frequency are slightly larger with this model than with the isotropic resistance model, for all  $\alpha$ 's, and the most unstable mode is found at a larger value of  $\alpha$  (here  $\alpha \approx 0.8$ ) in better agreement with experimental correlations [155] [121]. Also in this case the waves are found to be only weakly dispersive. Eigenfunctions are plotted in 3.7, together with the real and imaginary parts of the  $G_U$  sensitivity function. As in 3.3, the modulus of the  $u$  eigenfunction peaks near the edge of the canopy ( $y = 1$ ), whereas the adjoint eigenfunctions have a maximum value slightly above. As a general remark, the shapes of the direct and adjoint modes are quite similar to those found with the isotropic resistance model; as reported at the end of 3.2.2, it is found that the flow is most sensitive to streamwise momentum forcing. Also, real and imaginary parts of  $G_U$  have a double-peak structure, like in the isotropic-drag model, but now the largest absolute value of  $G_U$  is

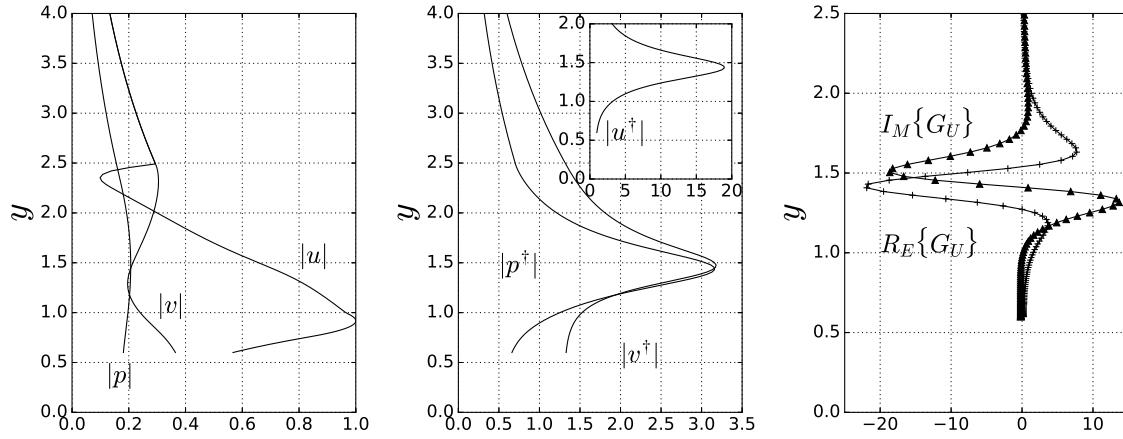


Figure 3.7: Left and center frames: moduli of direct and adjoint eigenfunctions; pressure and “adjoint pressure” are drawn with dashed lines. Right: real and imaginary parts of the sensitivity function  $G_U$  ( $\alpha = 0.4790$ )

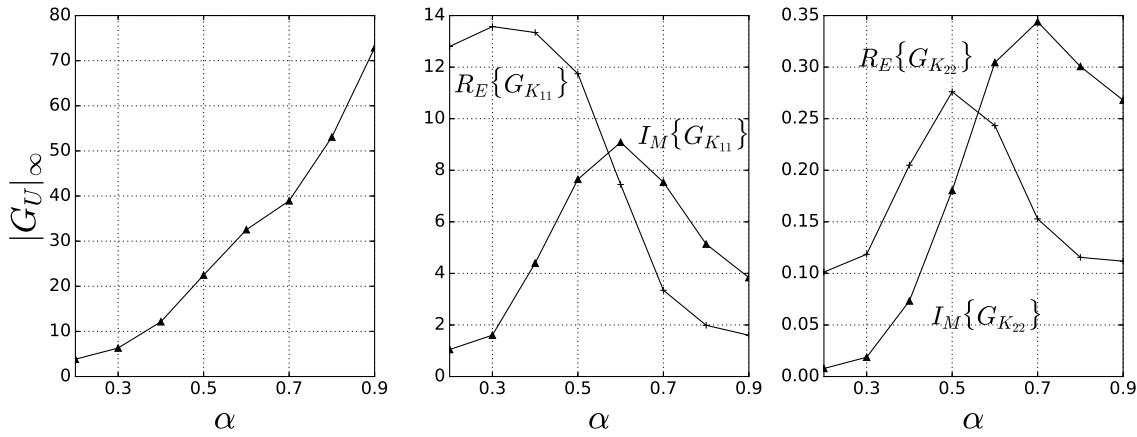


Figure 3.8: Case G. Left: infinite norm of  $G_U$  for varying  $\alpha$ . Center and right frames: real and imaginary parts of the sensitivity coefficients to variations in the permeability components

smaller and shifted towards a larger  $y$  than in the previous inviscid case (cf.3.4, top-right frame). This can also be appreciated by the inspection of 3.8 (left);  $|GU|_\infty$  still grows monotonically with  $\alpha$ , but the sensitivity is smaller than that computed earlier (cf. 3.5) with either the viscous or inviscid model (it is actually closer to the viscous sensitivity, as an effect of the interface condition). Furthermore, it is interesting to observe that both real and imaginary parts of  $G_U$  vanish for  $y = y|_{itf}$  (cf.3.7, right), and this supports the statement made previously that a small shift in the position of the effective interface has but a minor influence on the most unstable mode. The sensitivity coefficients for the two components of the permeability tensors are displayed in 3.8 (center and right frames): the present model is more effective to variations in  $\mathcal{K}_{11}$  than to  $\mathcal{K}_{22}$  as far as modifying the complex eigenfrequency. Significantly, different ranges of wavenumbers behave differently as far as the variation in  $\omega$  is concerned. The frequency  $\omega_r$  of long waves (around  $\alpha \approx 0.3$ ) is more easily modified by acting on  $\mathcal{K}_{11}$  (with an almost negligible effect on the growth rate of the wave); conversely, the growth rate of modes with large values of  $\alpha$  is affected efficiently by variations in the first component of the permeability tensor.

### 3.5 CONCLUDING REMARKS

We have considered two different models of the flow through a vegetated layer experiencing Kelvin-Helmholtz destabilization. One model is based on the use of a single drag coefficient to express the force exerted by the vegetation on the fluid, the second considers the canopy as an orthotropic porous medium and is based on Darcy's equation with a tensorial permeability [151]. Both models have advantages and drawbacks. The main

advantage of the first model is that the drag coefficient can be taken to vary across the canopy; whether this positive consideration, based on macroscopic experimental measurements [59] [56] [58], carries over to the stability problem remains to be established. The second model, applicable to dense porous media, considers two independent parameters to express the disturbance flow perpendicular and parallel to the rigid dowels forming the canopy. Such parameters and components of the transversely isotropic permeability tensor  $K_{ij}$  arise from the solution of a local Oseen problem [151]. The drawback of the second model is the fact that an interface (whether real or effective) appears, and adequate matching conditions must be enforced there. Despite much work since the seminal contribution by [13], a consensus on the “best” interface conditions between a pure fluid region and a porous medium has not yet emerged. The models have been put to test through a classical sensitivity analysis [22]. Beyond displaying stability results which correspond better to those to be expected from available experimental correlations [121] [155], the anisotropic model is less sensitive to variations in the base flow (with potentially larger variations in frequency and growth rate of the instability mode for the case of shorter waves). As far as a direct comparison between  $G_{C_d}$  and  $G_{K_{ii}}$  is concerned, this can hardly be made since the variables represent different objects; in particular, the pressure drop through the canopy depends directly on  $C_d$  and inversely on the permeability. The present results indicate that the anisotropic model depends significantly on the value of the apparent [151] permeability component  $K_{11}$ , whose evaluation must thus be conducted carefully. This model is also of interest for further developments, in particular for the study of instabilities developing over waving canopies. Darcy’s law in this latter case would need to be modified, as described in [96] and [154].

## Acknowledgment

The authors would like to thank the IDEX Foundation of the University of Toulouse for the financial support granted to the last author under the project “Attractivity Chairs.” The computations have been conducted at the CALMIP center, Grant No. P1540. The referees are gratefully acknowledged for their comments leading, in particular, to the correct interpretation of the sensitivity of the drag coefficient and to the material in Appendix A.

## APPENDIX A: EFFECT OF $C_d$ ON THE MEAN FLOW

In 3.2 of the paper it is described how the eigenvalue  $\omega$  varies as an effect of independent variations of  $U$  and  $C_d$ . However, since  $C_d$  is not zero within the canopy and it is used to compute the mean flow profile  $U$ , we should in principle have expressed  $\delta U$  as  $\delta U = \frac{dU}{dC_d} \delta C_d$  and considered a single sensitivity function  $G^*_{C_d} = G_{C_d} + \frac{dU}{dC_d} G_U$ , instead of the two sensitivities given in 3.6. This would have certainly been the appropriate line

of action if the mean flow equation were issued from exact equations, in which case we should have considered also the adjoint of the base flow equation in our variational problem. However, the mean flow model by [56] contains empirical approximations and parameters, and alternative models [129], [155]—including very different ones—have been used successfully in the past to predict the mean field; we have thus made the choice, in both 3.3 and 3.4, of considering the mean flow as given, and to take independent variations of  $U$  and  $C_d$  in the stability analysis to assess the effect of modifications in either variable. If we were to find how much the base flow depends on the drag coefficient in this particular problem, we would need to determine the function  $U(C_d)$  and take its derivative. Since both  $U$  and  $C_d$  are functions of the space coordinate  $y$ , the implicit dependence can be found, and we have plotted it for one case on the left frame of 3.9. Clearly, the function  $U = f(C_d)$  is not single-valued and therefore the derivative can be calculated only over two separate  $U$  (or, equivalently,  $y$ ) intervals. We have carried out the derivation numerically over each interval, within the range  $0.3 \leq y \leq 1$ , and the result is reported on the right frame of 3.9. The filled triangle and circle symbols indicate the two  $y$  intervals within the canopy. We first observe that both the location where  $C_d$  is maximum and the shape of the function  $U = f(C_d)$  are strongly correlated to the drag law  $C_d(y)$ , modeled by [56]

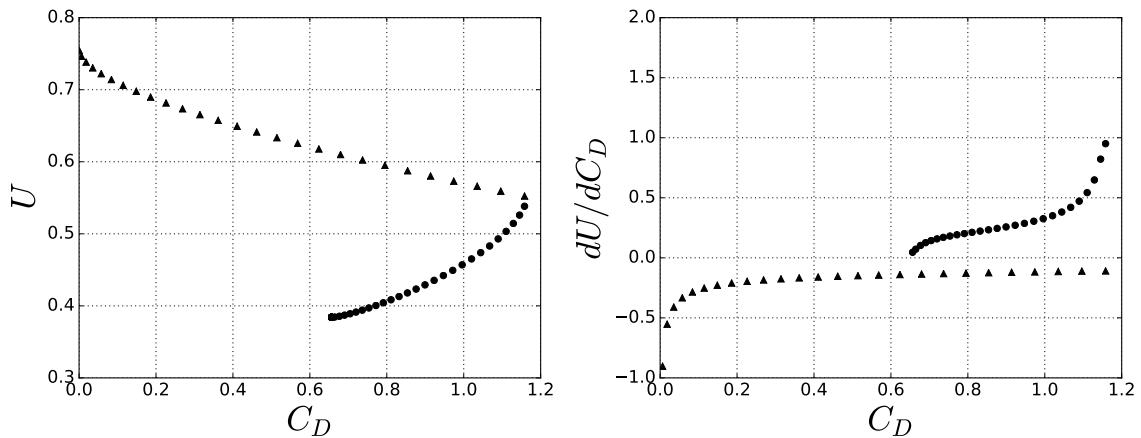


Figure 3.9: Case G. Left: mean velocity profile,  $U$ , versus the drag coefficient,  $C_d$ . Right: first derivative,  $dU/dC_d$ . The triangles denote the region  $y \in [0.76, 1]$ , the filled circles denote the region  $y \in [0.3, 0.76]$ .

through their measurement data (cf. their Figure 7 and Equation (18)). We also notice that the derivative  $dU/dC_d$  is reasonably small except locally at the point where the derivative of the function is not continuous, where it is of order 1. The discontinuity there is however artificial since the function  $C_d(y)$  given in Equation (18) of [56], where  $C_d$  is divided into a parabolic and a linear part, can be easily modified to yield a continuous first derivative at  $y = 0.76$  if required, still maintaining a mean flow very close to the measured

one.

## APPENDIX B: A DIGRESSION ON SPATIAL STABILITY THEORY AND GROUP VELOCITY

Stability problems such as the first one considered in this paper can be approached with the spatial theory framework, with the wavenumber  $\alpha$  complex, its imaginary part being a growth rate, and the circular frequency  $\omega$  a real constant parameter. Let us generalize the sensitivity analysis by considering, as a first step,  $\alpha$  and  $\omega$  as complex numbers which can vary. Equation 3.4 contains one additional term and reads:

$$0 = \delta\langle q^\dagger, \mathcal{L}q \rangle = \langle q^\dagger, \mathcal{L}\delta q \rangle + \langle q^\dagger, \frac{\partial \mathcal{L}}{\partial U} q\delta U \rangle + \langle q^\dagger, \frac{\partial \mathcal{L}}{\partial C_d} q\delta C_d \rangle + \langle q^\dagger, \frac{\partial \mathcal{L}}{\partial \omega} q \rangle \delta \omega + \langle q^\dagger, \frac{\partial \mathcal{L}}{\partial \alpha} q \rangle \delta \alpha \quad (3.12)$$

To obtain the sensitivities in the spatial problem (for which  $\delta\omega = 0$ ) we now have to solve an adjoint system similar to 3.3, where  $\omega^\dagger$  is replaced by  $\omega$  and  $\alpha$  by  $\alpha^\dagger$ . The variation of the wavenumber  $\delta\alpha = 0$  is thus given by:

$$\delta\alpha = \delta\alpha_r + i\delta\alpha_i = \int_0^{y_\infty} G_U(y)\delta U(y)dy + \int_0^1 G_{C_d}(y)\delta C_d(y)dy$$

the functions  $G_U$  and  $G_{C_d}$  maintain the same form as in the temporal theory 3.6, with the direct and adjoint eigenfunctions which are now normalized by imposing that  $N_\alpha = -1$ , with

$$N_\alpha = \int_0^{y_\infty} \left[ \left( U - \frac{2i\alpha}{Re} \right) (\bar{v}^\dagger v + \bar{u}^\dagger u) + \bar{p}^\dagger u + \bar{u}^\dagger p \right] d y$$

Let us now consider a problem in which  $U$  and  $C_d$  are not allowed to vary, but  $\alpha$  and  $\omega$  are. With reference to Equation 3.12, with any choice of normalization of direct and adjoint modes, it is found that  $N_\omega\delta\omega = N_\alpha\delta\alpha$ . Thus, once the adjoint problem is solved, it is possible to accurately compute the group velocity  $c_g$  of any stability problem using the value of  $N_\omega$  and  $N_\alpha$ , i.e.,

$$c_g := \frac{d\omega_r}{d\alpha_r} \approx \frac{\text{real}(N_\alpha)}{\text{real}(N_\omega)} \quad (3.13)$$

Note that  $c_g$  above is different from the “complex group velocity”  $C_g := \frac{d\omega}{d\alpha} \approx \frac{N_\alpha}{N_\omega}$ , and it is also  $c_g \neq \text{real}(C_g)$ . Relation 3.13 can be employed in either a spatial or temporal stability analysis and some representative results (for case G) are provided in Table I with the phase velocity  $c_r := \omega_r/\alpha_r$  and the group velocity determined from Equation 3.13. The temporal or spatial amplification factors,  $\omega_i$  or  $-\alpha_i$ , respectively, are also given for all

cases using Gaster's transformation:  $\omega_i = -\alpha_i c_g$ . Two types of errors on the calculation of the group velocity (noted *err*) are given in the table; the top four values, relative to the temporal theory, are defined as

$$err = \frac{|c_g|_{3.13} - c_g|_{FD}|}{c_g|_{3.13}}$$

with  $c_g|_{FD}$  arising from a first-order finite difference approximation of the group velocity. The bottom four values are defined by the formula

$$err = \frac{|c_g|_{temporal} - c_g|_{spatial}|}{c_g|_{temporal}}$$

The relative difference on  $c_g$  between temporal and spatial theory is rather low. It has to be kept in mind, however, that a stability analysis in the spatial framework yields a nonlinear eigenvalue problem, with a consequent larger numerical system than in the temporal framework; therefore, by inverting matrices of the same size, the accuracy is expected to be slightly lower. The accuracy of the growth rate approximated through Gaster's relationship is also found to be acceptable.

Theory	<i>Re</i>	$\alpha_r$	$\omega_r$	$-\alpha_i$	$\omega_i$	$c_r$	$c_g$	<i>err</i> (%)
Temporal	500	<b>0.5</b>	0.4778	<i>0.0248</i>	0.0254	0.9556	1.0245	0.54
	3450	<b>0.5</b>	0.4601	<i>0.0413</i>	0.0404	0.9202	0.9797	0.06
	$10^5$	<b>0.5</b>	0.4514	<i>0.0436</i>	0.0421	0.9028	0.9661	0.63
	$10^9$	<b>0.5</b>	0.4508	<i>0.0451</i>	0.0425	0.9016	0.9427	2.90
Spatial	500	0.4993	<b>0.4778</b>	0.0248	0.0250	0.9569	1.0100	1.41
	3450	0.4990	<b>0.4601</b>	0.0427	0.0404	0.9220	0.9471	3.30
	$10^5$	0.4996	<b>0.4514</b>	0.0449	0.0416	0.9109	0.9371	3.46
	$10^9$	0.4993	<b>0.4508</b>	0.0450	0.0411	0.9028	0.9143	3.01

Table 3.1: Temporal versus spatial stability, Case G. The model employed here is based on a modified Orr-Sommerfeld equation—rather than a system based on primitive variables as done in the bulk of the paper—which is why the temporal results have slightly larger growth rates  $\omega_i$  than those displayed in Fig. 3.2; this is related to the need of computing numerically  $d^2U/dy^2$  and  $dC_d/dy$  in the Orr-Sommerfeld-like equation. In italics, the growth rates obtained from Gaster's transformation are reported; the parameters imposed in each simulation are indicated with bold characters. The solutions for  $Re = 10^9$  coincide with those found using the inviscid equations.

The amplitude of the sensitivity functions,  $|G_U(y)|$  and  $|G_{C_d}(y)|$ , in the spatial and temporal stability frameworks is of same order of magnitude (not shown here) since they are related through temporal spatial the complex group velocity  $C_g$ . It is found that

$|G_U^{temporal}| \approx |C_g| |G_U^{spatial}|$  with  $|C_g| \approx c_g \approx 1$  in the present case. Obtaining and comparing results in the temporal and spatial stability frameworks, such as in Table I, is a good means to validate the sensitivity functions and to verify the accuracy of the computations of the adjoint stability equations.

## APPENDIX C: correction to compare continuous and discrete adjoint eigenfunctions

The discretization operation transform the operator  $\mathcal{L}$  into a matrix  $\mathbf{A}$  and of course do the same things to the unknown functions that becomes vectors.

continuous	discrete
$\mathcal{L}$	$\mathbf{A}$
$q$	$\hat{q}$

This has a serious and most often hidden repercussion in th approach to solve the adjoint equations.

As above stated the derivation of the adjoint equation start with the enforcing of the Lagrange identity:

$$\langle q; \mathcal{L}q \rangle = \langle \mathcal{L}^\dagger q^\dagger; q \rangle \quad (3.14)$$

where the scalar product  $\langle ; \rangle$  is defined in our case as:

$$\langle a; b \rangle = \int_0^{y_\infty} \bar{a} \cdot b dy \approx \sum_{i=1}^N \sum_{j=1}^N \hat{a}_i^T w_{i,j} \hat{b}_j = \hat{a}_i^T \mathbf{M} \hat{b}_j = \langle a; b \rangle_{\mathbf{M}} \quad (3.15)$$

Is it clear from equation 3.15 that the scalar product takes two different forms in the continuous and in the discrete case. In fact in the discrete case is mandatory to introduce the quadrature rule weights  $w_{i,j}$  of the chosen discretization.  $\mathbf{M}$  is the matrix representation of the weights and is symmetric and positive defined.

In order to compute and solve the adjoint equation one could proceed as follow:

- The direct problem is defined in the continuous space as  $\mathcal{L}q = 0$
- Chose a discretization (FEM, FD, Chebychev polynomials...) and transform the above problem in a discrete one  $\mathbf{A}\hat{q}$
- Solve it to obtain the discrete version of the eigenfunctions  $\hat{q}$

For the adjoint problem on should at first compute the adjoint operator, this can be done using the Lagrangian identity at a continuous level:

$$\begin{aligned} \langle q; \mathcal{L}q \rangle &= \langle \mathcal{L}^\dagger q^\dagger; q \rangle \\ \Rightarrow \int_0^{y_\infty} \overline{q^\dagger} \mathcal{L}q dy &= \int_0^{y_\infty} \overline{\mathcal{L}^\dagger q^\dagger} q dy \end{aligned} \quad (3.16)$$

From the last equation starting from the left part is it possible after some manipulation to retrieve the form on the right part and so find the formulation of the adjoint operator.

It is important to pinpoint that in the above equation the scalar product  $\langle a; b \rangle$  is enforced at a continuous level.

And now to solve the adjoint system the procedure 3.5 can be used changing the direct system with the adjoint one. The above way of computing the adjoint and solve the system is called **continuous approach**.

To summarize this approach one can straight forward solve the direct problem computationally, mathematically find the adjoint operator using the continuous scalar product and the Lagrange identity and then discretize the adjoint problem and solve it computationally. This is why the **continuous approach** is sometimes known as derive than discretize. And the stability and accuracy problems derive directly from the fact that we discretize the problem two times (the direct first and than the adjoint).

On the contrary in the **discrete approach** the scalar product 3.15 is enforced at the discrete level in order to use the already discretized direct equation to retrive the adjoint system at a discrete level, to limit the computational errors.

$$\begin{aligned} \langle q^\dagger; \mathcal{L}q \rangle &= \langle \mathcal{L}^\dagger q^\dagger; q \rangle \\ \Rightarrow \overline{\hat{q}^\dagger}^T \mathbf{M} \mathbf{A} \hat{q} &= \left( \overline{\mathbf{A}^\dagger \hat{q}^\dagger} \right)^T \mathbf{M} \hat{q} \\ \Rightarrow \mathbf{M} \mathbf{A} &= \overline{\mathbf{A}^\dagger}^T \mathbf{M} \\ \Rightarrow \mathbf{A}^\dagger &= \mathbf{M}^{-1} \overline{\mathbf{A}}^T \mathbf{M} \end{aligned} \quad (3.17)$$

## Chapter 4

# Effect of geometrical parameters and inertia on the apparent permeability tensor in fibrous porous media

*It is far better to foresee even without certainty than not to foresee at all.*

- The Foundations of Science, *Henri Poincare*

### 4.1 Introduction

Include this paper Penha et al. [110]

The flow through porous media is a problem of importance for several natural and technological applications. Since Darcy's original formulation [39], which relates the flow rate through a porous bed to the pressure drop across the bed's sides, many corrections have been made to account, for example, for viscous effects [25] or for the consequences of inertia [52]. All of the cited works are of empirical nature, but homogenisation has been able to recover all of these formulations rigorously starting from the Navier-Stokes equations [146]. This latter approach is sometimes defined VANS, for Volume-Averaged Navier-Stokes.

The theory requires the knowledge of a number of terms, most notably, in the case of an isotropic porous bed, a permeability coefficient and a Forchheimer coefficient. Initial efforts in defining these terms were based on a combination of physical reasoning and measurements, leading to expressions known as the Kozeny-Carman [80, 33] and the Ergun [48] correlations. The first provides the permeability for the laminar flow of a single-phase fluid through a packed bed of sand grains, as function of the porosity and the diameter of the

grains, while the second extends Darcy's law to let the pressure drop depend on two terms, one proportional to the velocity and the second to its square, thus accounting for inertia. These approaches do not consider microstructural or geometrical features of the porous bed, which can render the permeability a tensorial quantity, and are often restricted to simple unidirectional flows. In the present work we are concerned with a transversely isotropic material composed by parallel fibers of circular cross-section, with one axis of symmetry,  $(O, x_3)$ ; in such materials the permeability is a diagonal tensor with the component in the direction parallel to the fibers greater than those along the transverse axes. For such an arrangement we will investigate the effects of both the direction of the forcing pressure gradient and inertia. When the latter effect is present, embodied by a Reynolds number  $Re_d$ , based on mean velocity through the medium and fibers' diameter, exceeding an order one threshold, the permeability is no more simply defined upon geometrical properties. This new permeability, which arises from a well-defined closure problem, is then called *apparent permeability*.

The influence of the geometry of the solid inclusions has been addressed previously by Yazdchi et al. [149] for arrays of cylinders in both square and hexagonal (or staggered) patterns, with the cylinders' section which can vary in shape. The results, in the two-dimensional and low Reynolds number limits, demonstrate the dependence of the permeability component along the flow direction to both the porosity and the direction of the macroscopic pressure gradient. The direction of the pressure gradient is found to have a weak effect for beds of medium-high porosity ( $\varepsilon > 0.7$ ) and a stronger dependence appears upon the geometry of the solid inclusions.

The influence of the Reynolds number on the permeability and on the Forchheimer correction has been presented in a number of papers. One of the contributions most relevant here is due to Edwards et al. [46]. These authors show that, for arrays of fibers, the apparent permeability decreases with the increase of the Reynolds number, and the rate of this decrease depends on the geometry of the array; also, the Reynolds number is found to have a stronger influence on the apparent permeability when the medium is highly porous. The results of the work by Edwards et al. [46] agree with those by Zampogna and Bottaro [150] and with our own work (as shown later), all for the case of cylindrical fibers, although some issues remain on the persistence of steady solutions in the simulations by Edwards et al. [46] in cases for which a limit cycle should have set in. A fully three-dimensional porous medium, more complex than those discussed so far, has been considered by Soulaine and Quintard [132], confirming the decreasing trend of the apparent permeability with the Reynolds number.

Another contribution which deserves mention is that by Lasseux et al. [84]; they have computed the permeability tensor for various Reynolds numbers, in a two-dimensional geometry with cylinders of square cross-section. Forcing the flow along the main symmetric directions of the fiber, Lasseux et al. [84] have identified different regimes:

- a creeping flow regime for  $0 < Re_d < 10^{-3}$ , without Forchheimer terms;

- a weak inertia regime for  $10^{-3} < Re_d < 1$ , with the Forchheimer correction quadratic in  $Re_d$ ;
- a strong inertia regime for  $1 < Re_d < 10$ , where the Forchheimer correction is linear with the Reynolds number;
- a turbulent regime, for  $Re_d > 10$ , with the Forchheimer correction again quadratic with the Reynolds number.

The boundaries between the different regimes are specific to the geometrical arrangements and to the porosities being considered; a step forward in rendering (some of) these boundaries rigorous and independent of the arrangement of the pores, through the definition of a Reynolds number which accounts for a "topological" coefficient, has been recently made by Pauthenet et al. [109]. For the purposes of the present paper, we must retain that Lasseux et al. [84] have parametrized the Forchheimer correction with the Reynolds number, and have found that the inertial correction is orders of magnitude smaller than the Darcy's term, at least before the turbulent regime sets in. Moreover, Lasseux et al. [84] have studied how a Forchheimer tensor,  $\mathbf{F}$ , depends upon the direction of the macroscopic forcing term with respect to the orientation of the square cross-section of the fibers, for  $Re_d$  up to 30. It is concluded that a deviation angle,  $\gamma$ , exists between the direction of the pressure gradient and that of the mean flow, because of the fibers' geometry. Finally, the inertial correction is strongly influenced by the orientation of the driving pressure gradient, and the tensor  $\mathbf{F}$  is not symmetric (in fact the off-diagonal components are found to be inversely proportional to the diagonal terms, and symmetric with respect to rotations about the diagonal axis of the square, i.e. the direction at  $45^\circ$  in the  $x_1 - x_2$  plane).

The effect of variations in the forcing angle, with restrictions to angles in the  $x_1 - x_2$  plane, is also examined by Soulaine and Quintard [132] with conclusions in qualitative agreement with those of both the contribution just cited and our results described further below. In all cases, the off-diagonal components of the apparent permeability tensor are small and the diagonal components display but a small variation upon rotation of the driving pressure gradient.

As already anticipated, this work investigates how the direction of the macroscopic pressure gradient, the porosity and the Reynolds number can modify the Darcy and Forchheimer closures arising from a VANS model of a fibrous porous medium. We will consider a three-dimensional unit cell for the microscopic model (such a unit cell is sometimes denoted REV, for Representative Elementary Volume), with a generic forcing whose direction is defined by two Euler angles. Given the formidable space of parameters, some representative results are first shown and discussed. Response surfaces in the space of parameters are then identified by the use of a metamodel based on kriging interpolation. For the sake of space, only the first diagonal component of the apparent permeability tensor is discussed in detail in the paper; however, all components have been computed. They represent an extremely

useful data base which we are now in the process of using in macroscopic simulations of flows through bundles of fibers of varying orientation and density.

## 4.2 The Volume-Averaged Navier-Stokes (VANS) method

### 4.3 Validation and setup

In this section the numerical methodology, the parameters, the setup and the validation for some reference cases are given.

#### 4.3.1 Computational domain

The geometry used for the base REV is shown in figure 4.1: a cylindrical inclusion is present at the centre of the REV and four quarters of cylinders are situated at the corners. The lateral length of the cubic envelop is  $\ell$ , which is used as length scale for the microscopic problem; the diameter  $d$  of the cylinders is adapted as a function of the desired porosity  $\varepsilon$ , ratio between the fluid volume over the total REV volume ( $\ell^3$ ).

The forcing term  $\mathbf{f}$  of the DNS is a vector whose direction is defined by two Euler angles, with rotations of the form:  $\theta \mathbf{e}_3 + \phi \mathbf{e}_2^T$  (cf. figure 4.1). Its amplitude is set a priori and is connected to the Reynolds number,  $Re_d$ , defined with the mean velocity over the REV and the fiber diameter,  $d$ .  $Re_d$  is a result of the calculations, once the mean velocity is evaluated.

#### 4.3.2 Numerical setup

The simulations have been carried out with the open-source code OpenFOAM [140], based on a finite volume discretization with a colocated arrangement for the unknowns. The standard solver icoFoam (incompressible Navier-Stokes) has been modified in order to include a constant pressure gradient acting as a forcing term  $\mathbf{f}$  in equation (??). The coupling between the velocity and the pressure equations is based on the pressure implicit split operator referred to as the PISO algorithm. The time derivative term is discretized using the second order backward Euler scheme and all the spatial terms use a second-order central difference stencil based on Gauss finite volume approach. The velocity system is solved with a preconditioned bi-conjugate gradient (PBiCG) iterative solver with the tolerance on the velocity residuals set to  $10^{-8}$ , associated to a diagonal incomplete lower upper preconditioner (DILU). The pressure equation is solved with a geometric-algebraic multigrid (GAMG) algorithm associated to a Gauss-Seidel smoother and the tolerance on the pressure residuals is here equal to  $10^{-6}$ . Cyclic boundary conditions are applied to all fields on all fluid boundaries along the three directions, and the no-slip condition is imposed on the surface of the solid inclusions. The time step  $\Delta t$  is automatically determined to ensure that the maximum Courant number,  $Co$ , respects the condition:  $Co = ||v_\beta|| \Delta t / \Delta x < 1/2$ ,

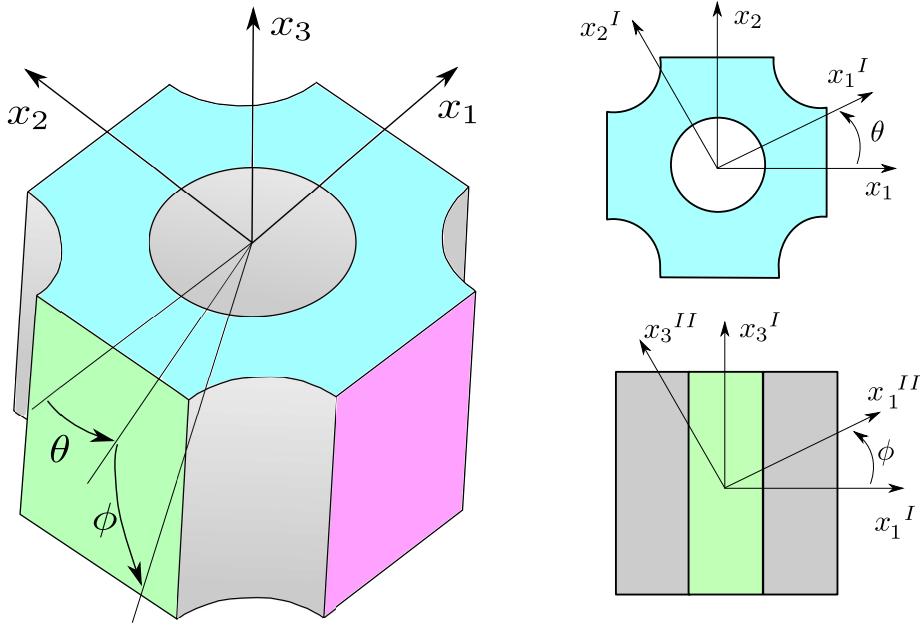


Figure 4.1: REV for the fiber geometry investigated.

in which  $\|v_\beta\|$  is the local velocity magnitude in the REV and  $\Delta x$  is the local grid spacing.  $Co$  is basically the ratio between the fluid speed and the velocity to propagate information through the mesh and the condition  $Co < 1/2$  is found to be sufficient to have a stable solver.

### 4.3.3 Mesh convergence analysis

The mesh has been computed using the internal OpenFOAM mesher named *snappyHexMesh*. The final grid is mainly composed by hexahedral cells with a refined regular grid in the boundary layer regions next to the solid surfaces. Three different mesh sizes, with  $0.65 \times 10^6$ ,  $10^6$  and  $1.5 \times 10^6$  elements, have been tested in order to demonstrate spatial convergence. This has been assessed using the Grid Convergence Index (*GCI*) introduced by Roache [123].

Details of the coarsest mesh used are shown in figure 4.2. On the right frame a close up of the grid in the neighbourhood of the fiber's boundary is displayed: twenty points are used in the structured portion of the mesh along the wall-normal direction.

The GCI method is based upon a grid refinement error estimator derived from the theory of generalized Richardson extrapolation. It measures the ratio between the computed value of a quantity over the asymptotic numerical value, thus indicating how far the solution is from the asymptotic ("exact") value. The procedure is simple and provides a method to estimate the order of the spatial convergence, based on two or three different

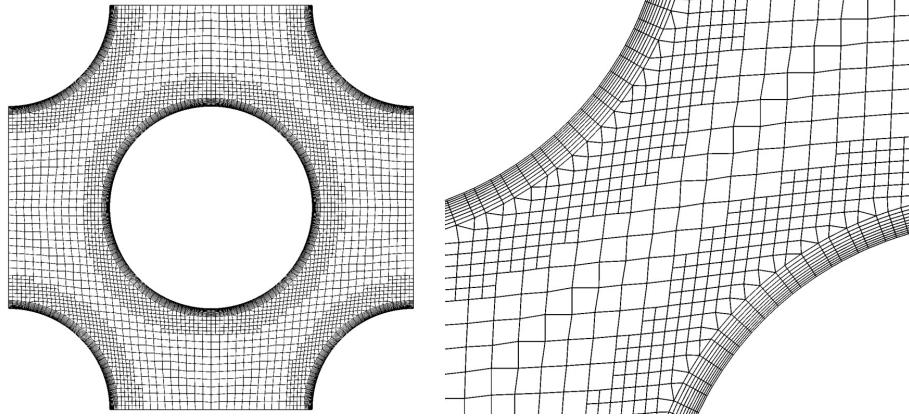


Figure 4.2: Mesh used for the computation; top view (left) and zoom in the boundary layer region (right).  $\varepsilon = 0.6$ .

grid sizes. First of all, the grids must be generated with the same algorithm and they must have the same final quality. In each simulation a physical scalar quantity representative of the physical phenomenon must be sampled. The method follows the following four steps:

1. Estimate the order of convergence of the procedure, defined as  $p = \ln \left( \frac{f_3 - f_2}{f_2 - f_1} \right) / \ln r$ , where  $r$  is the grid refinement ratio between each grid (it is computed as the ratio between the number of elements of two consecutive grids; the approach imposes that  $r$  should remain constant between any couple of consecutive grids and be larger than 1.1), and  $f_i$  represents the quantity of interest in each grid (1=coarse, 2=medium and 3=fine).
2. Compute the relative error between grid  $i$  and  $j$ :  $|\epsilon|_{ij} = \frac{f_j - f_i}{f_i}$ , for  $(i, j) \in \{(1, 2), (2, 3)\}$ .
3. Compute  $GCI_{ij} = \frac{F_s |\epsilon|_{ij}}{r^p - 1}$ , with  $F_s$  a safety factor equal to 1.25 if the grids are three, and equal to 3 if the grids are only two [123].
4. Check whether each grid level yields a solution that is in the asymptotic range of convergence; this means that the quotient  $AC = \frac{GCI_{23}}{GCI_{12}} \frac{1}{r^p}$  should be as close as possible to one.

In our case the quantity of interest chosen is the intrinsic average velocity inside the porous medium, and the results are summarized in table 4.1. From the table it can be seen that the intrinsic velocity difference is very small from one grid to the next and the coarse grid provides results close to the expected asymptotic value. This is taken as a sufficiently

mesh index	mesh identifier	average velocity REV	metric	value
3	fine	1.11	$GCI_{23}$	0.366%
2	medium	1.07	$GCI_{12}$	1.11%
1	coarse	1.09	AC	1.006

Table 4.1: Convergence analysis. Left: average velocity within the REV, normalized with  $\frac{K_{11}}{\nu_\beta} \|\mathbf{f}\|$ . Right: grid convergence metrics. The REV has  $\varepsilon = 0.6$ , the motion is along  $x_1$ , i.e.  $\theta = \phi = 0$  and  $Re_d \rightarrow 0$ .

convincing argument to carry out all the computations in the following with a grid density equal to that of grid 1.

#### 4.3.4 Validation on two different configurations

The results published in the literature by Zampogna and Bottaro [150] and Yazdchi et al. [149] are now used to validate both the methodology and our choices of the computational parameters. In the cited papers, three-dimensional computations of the permeability components in different cells geometries are presented.

Figure 4.3 displays the comparison for a cell with a square arrangements of the fibers; here the permeability is evaluated along the two principal directions,  $x_1$  and  $x_3$ . A good agreement is found with the published results. Figure 4.4 shows a similar comparison for a staggered arrangement of the inclusions in the unit cell. In this case the section of the cell is rectangular. The agreement for the only permeability component available in the literature is again satisfactory.

Finally, to check the correct implementation of the closure model (??) it is important to verify the equality (??) between the amplitude  $F^M$  of the macroscopic force and its microscopic counterpart obtained through an integration of the DNS fields over the solid boundaries of the inclusions in the REV. Figure 4.5 shows a plot of the relative error between these two forces, i.e.  $\frac{\|F^M - F^m\|}{\|F^m\|}$ , as function of the Reynolds number. We consider the successful comparison displayed in figure 4.5 as the conclusive demonstration of the validity of the approach described here. We have nonetheless carried out the same verification displayed in figure 4.5 for each one of the simulations described in the following, to our satisfaction.

#### 4.3.5 Tests with larger REV's

Since the Reference Elementary Volume (REV) is the unit cell within the porous medium over which average quantities of the VANS are computed, it is important to choose its

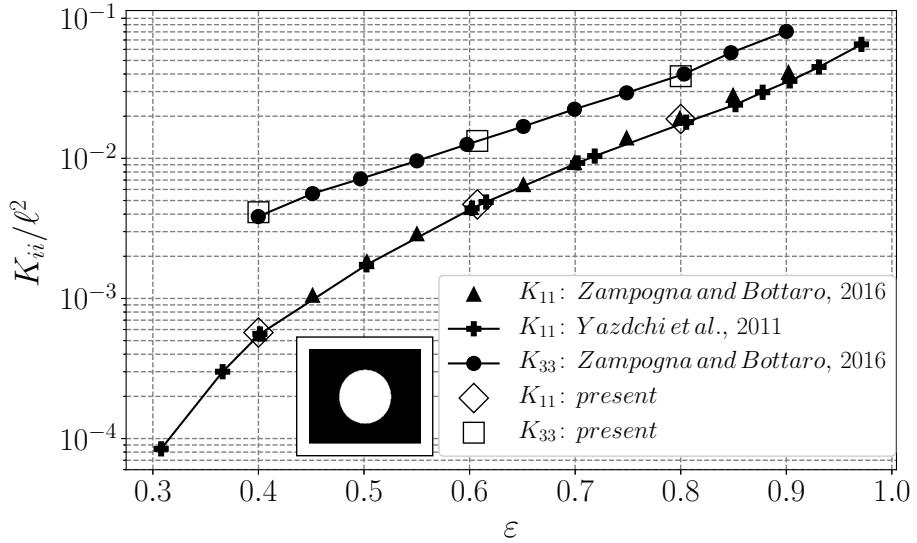


Figure 4.3: Permeability versus porosity for a square arrangement of cylinders. The scaling of the permeability is  $\ell^2$  and is explicitly indicated in the vertical axis.

dimensions appropriately in the inertial regime for, if the REV is too small, it might be easy to miss crucial features of the wakes. For example, to predict the critical Reynolds number,  $Re_c$ , of the first Hopf bifurcation, a REV containing at least three solid inclusions in the direction of the mean pressure gradient is necessary in the simulations by Agnaou et al. [3]. Among the results reported, it is found that, for a fixed REV size, the error committed in the evaluation of the critical Reynolds number increases with the porosity. This same error is considerably reduced when the mean pressure gradient angle is  $\theta = 45^\circ$ . Thus, the choice of the number of inclusions in a REV is a task not to be overlooked, and the final choice must account for the porosity, the direction of the pressure gradient and the microscopic Reynolds number.

Here, the influence of the numbers of inclusions present in a REV is assessed by focussing only on the velocity components after averaging over the REV. The unit cubic cell of side  $\ell$  is used as reference: starting from this, two additional REV's are built, as shown in figure 4.6. The first one is doubled in both the  $x_1$  and  $x_2$  directions and the case tested numerically is characterised by  $\theta = 0$ ,  $\phi = 0$  (i.e. the forcing pressure gradient is directed along  $x_1$ ), porosity  $\varepsilon = 0.6$  and  $Re_d = 50$ . The second REV configuration is a composition of 3 reference REVs on top of one another along  $x_3$ , with the parameters set to  $\theta = 45^\circ$ ,  $\phi = 45^\circ$ ,  $\varepsilon = 0.6$  and  $Re_d = 100$ .

For both these test cases, no appreciable differences, neither in the mean velocity nor in the forces on the fibers, have been observed, with relative errors on the mean velocity with respect to the reference case which remain below 2%. We take this as sufficient

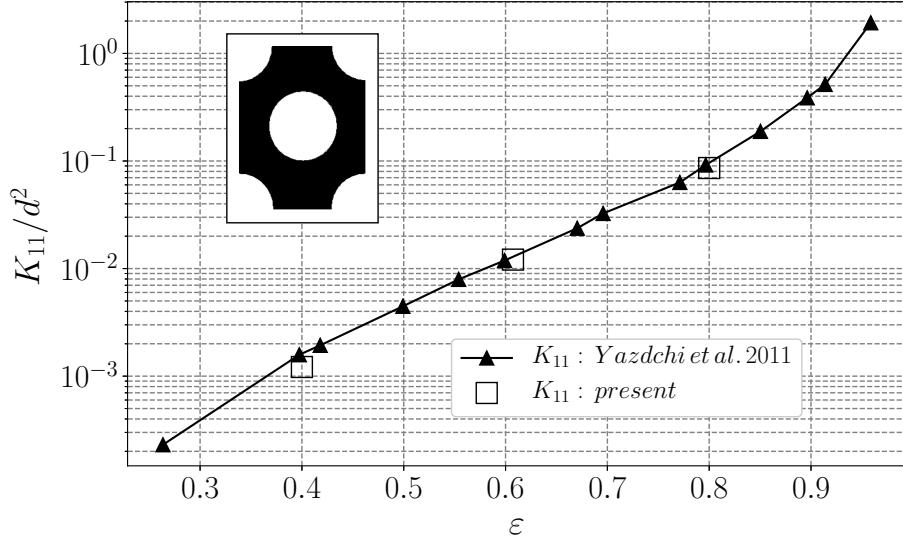


Figure 4.4: Permeability versus porosity for a staggered arrangement of cylinders. The permeability component is here scaled with  $d^2$  (and not  $\ell^2$ ), with  $d$  the diameter of the inclusions.

evidence to use, in the following, only the reference cubic REV of side equal to  $\ell$ , with the understanding that only configurations with  $Re_d$  up to around 100 can be considered.

#### 4.4 Microscopic solutions

In this section, some local microscopic fields computed with direct numerical simulations are shown, together with components of the intermediate tensor  $\mathbf{M}$  coming from the numerical solution of the closure equations (??).

In figure 4.7 (top row) the local  $x_1$  velocity component is drawn for the two-dimensional flow when  $\varepsilon = 0.6$ , for three Reynolds numbers, to cover the transition from the Stokes to the inertial regime. In all plots, the velocities are rendered non-dimensional by the corresponding value of  $\frac{K_{11}}{\nu_\beta} \|\mathbf{f}\|$ . When inertia is absent, the flow has a central symmetry; by increasing the Reynolds number, only the symmetry with respect to the  $x_1$  axis is maintained ( $x_1$  is the direction of the forcing pressure gradient), with the wake's length which increases with  $Re_d$ . When  $Re_d$  is of order 100 the wake spreads to the downstream boundary of the REV, re-entering, because of periodicity, at the upstream side. This  $Re_d$  represents the upper limit of validity for the cubic unit cell of side  $\ell$ ; larger values of  $Re_d$  could only be investigated with longer/larger/thicker REV's.

The non-dimensional local  $M_{11}$  fields for the same parameters are displayed in figure

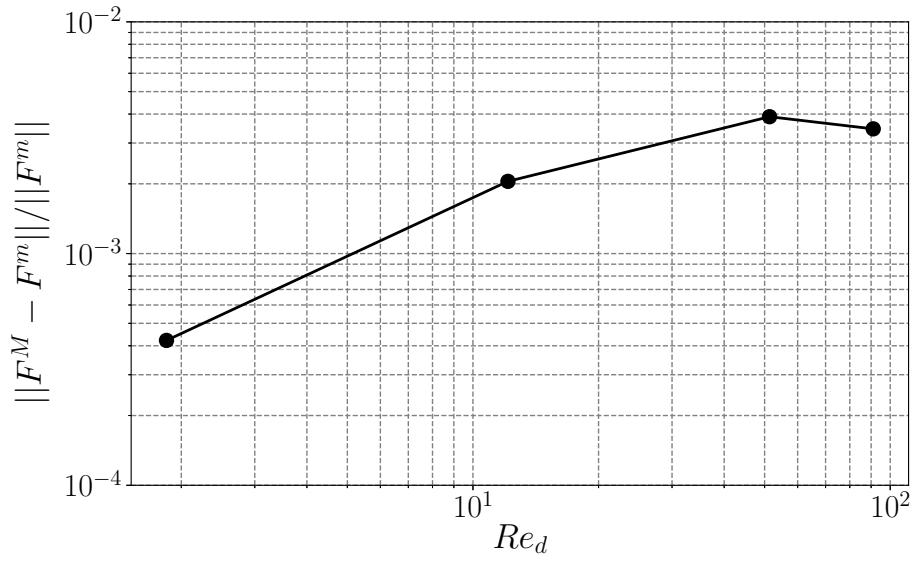


Figure 4.5: Relative error between the microscopically computed forces along the  $x_1$  direction and those arising from the Darcy-Forcheimber model;  $\varepsilon = 0.8$  for the REV in the staggered arrangement of Yazdchi et al. [149].

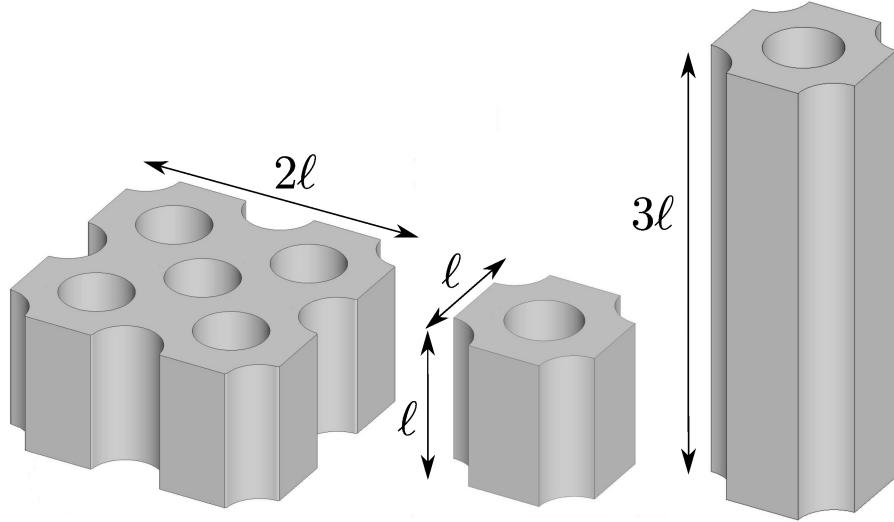


Figure 4.6: REV configurations. Left:  $2 \times 2 \times 1$  arrangement; centre:  $1 \times 1 \times 1$  arrangement (reference); right  $1 \times 1 \times 3$  arrangement.

4.7 (mid row). All values in the figures arise from scaling  $\mathbf{M}$  with  $\ell^2$ . Visually, these local

fields are strongly correlated to the local streamwise velocity component in the whole  $Re_d$  range. This is not unexpected since the local velocity drives the convective term of system (??). The central symmetry of all components of  $\mathbf{M}$  in the Stokes regime is coupled to the rotational invariance of the apparent permeability tensor in two-dimensional flows.

The effect of varying the porosity is shown in figure 4.7 (bottom row) where  $\varepsilon$  is taken equal to 0.4. Even at such a low porosity the stretching of the wake can be noticed, and it increases with  $Re_d$ . Interestingly, this effect is milder when the forcing is inclined by an angle  $\phi$ , since the tighter packing of the inclusions causes a strong deviation of the mean flow along the axis of the fiber. In this case,  $M_{11}$  and  $M_{22}$  behave very similarly to the case  $\phi = 90^\circ$ .

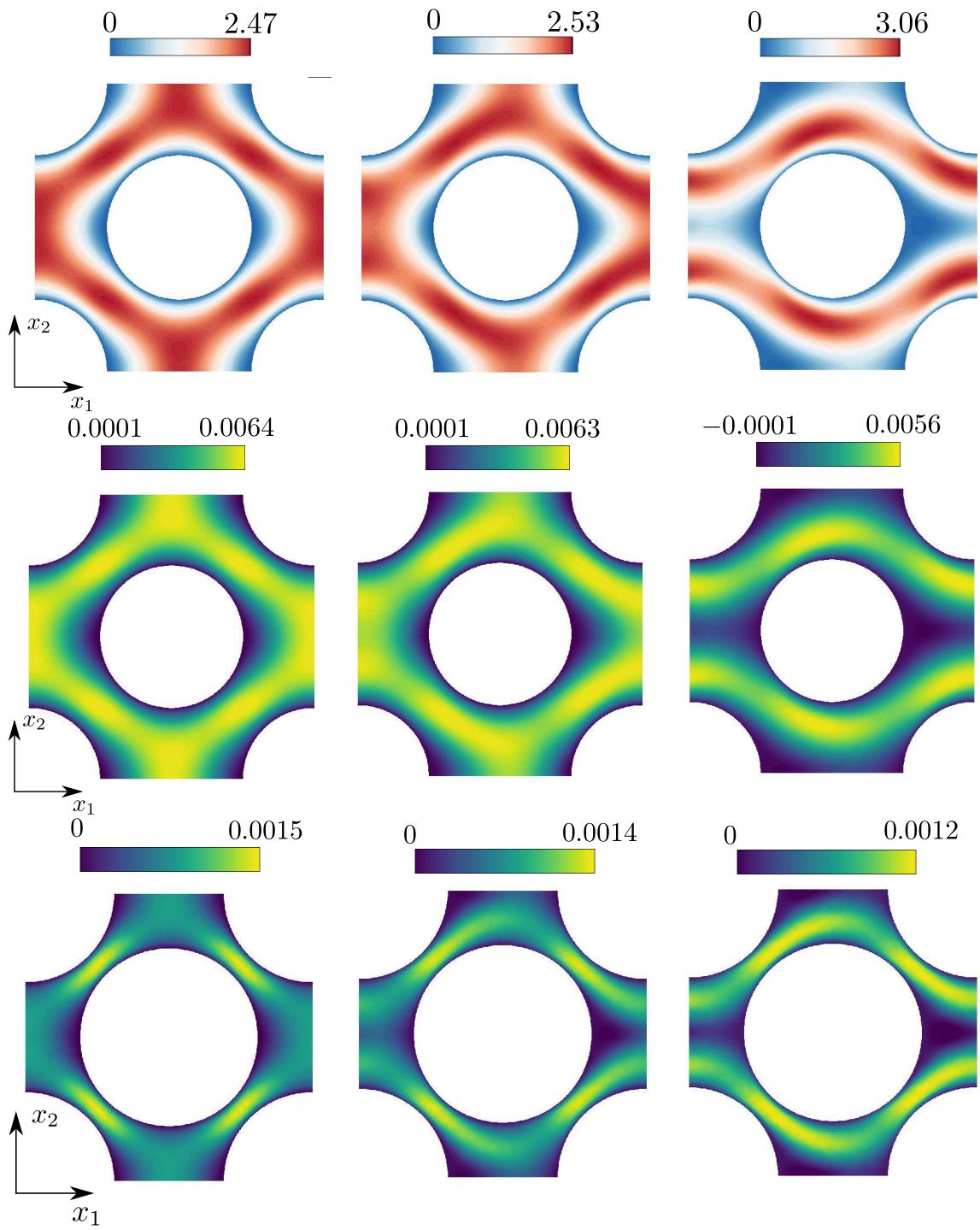


Figure 4.7: Top row: plane view of the dimensionless  $x_1$  component of the local velocity field  $\mathbf{v}_\beta$  for the case  $\theta = 0$ ,  $\phi = 0$ ,  $\varepsilon = 0.6$  and for three Reynolds numbers  $Re_d = 0, 10, 50$ , from left to right. Mid row: microscopic  $M_{19}$  fields corresponding to the images in the top row. Bottom row:  $M_{11}$  fields for the same Euler angles and Reynolds number as in the top two rows, and smaller porosity ( $\varepsilon = 0.4$ ).

Another interesting point emerges by inspection of figure 4.8 where two off-diagonal components of  $\mathbf{M}$  are shown for two porosity values; the first image (left frame) represents a plane flow in the Stokes regime while the second is the plane cut of a three-dimensional solution in the inertial regime. Positive and negative values of the microscopic fields can be seen in both images but, once averaging is applied over the REV, the resulting permeability component is very close to zero (in fact, exactly equal to zero in the Stokes case). This same features occurs for all off-diagonal terms in all cases examined, so that, within the current range of Reynolds numbers, the apparent permeability tensor is, to a good approximation, diagonal<sup>1</sup>.

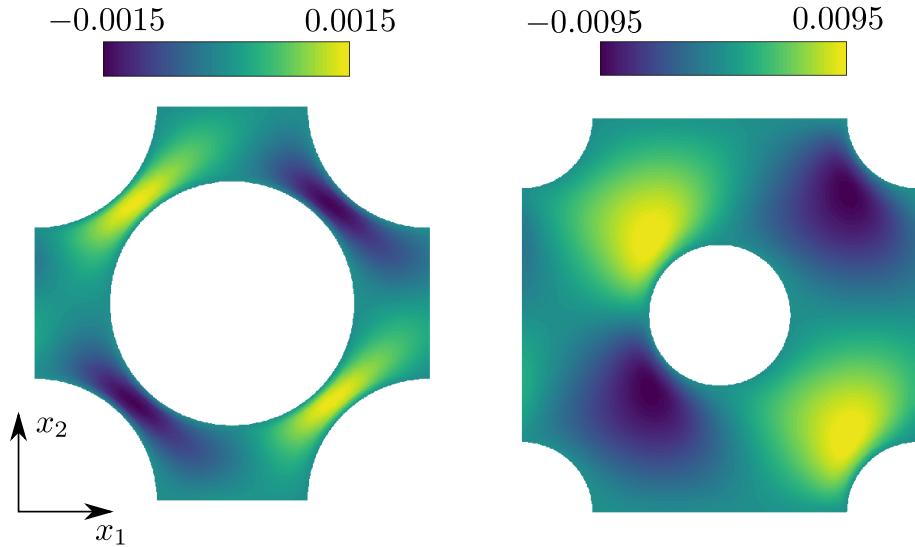


Figure 4.8: right: Non-dimensional  $M_{21}$  field for  $\theta = 0, \phi = 0, Re_d = 10, \varepsilon = 0.8$ , left: Non-dimensional  $M_{12}$  field for  $\theta = 22.5^\circ, \phi = 45^\circ, Re_d = 50, \varepsilon = 0.4$ .

A three-dimensional case is shown in figure 4.9, where all the non-zero terms of the  $\mathbf{M}$  tensor are plotted for a porous structure with  $\varepsilon = 0.6$ . The components shown are  $M_{11}, M_{22}, M_{33}, M_{12}$  and  $M_{21}$ , while  $M_{i3}$  and  $M_{3j}$  are not plotted because they are identically zero to machine accuracy. Distinct features are visible in each image; in particular, in the last frame the  $M_{33}$  microscopic component displays a low wavelength structure along the cylinder's axis. Increasing the dimensions of the REV along  $x_3$  does not alter such a structure, i.e. the  $\ell^3$  domain chosen with its periodic boundary conditions does not filter out significant high wave-numbers of the flow. We further note that the tensor  $\mathbf{M}$  is not symmetric in this case since each off-diagonal component represents the solution of the

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<sup>1</sup>In fact, there are always at least two orders of magnitude differences between the diagonal and the off-diagonal components. While the latter should not, in principle, be ignored, we will focus attention here only on the dominant terms of the permeability tensor.

closure problem in a specific direction (first index of the field) and the forcing term acts orthogonally to it (second index of the field). Once averaged over the REV it is found that both  $H_{12}$  and  $H_{21}$  are very close to zero.

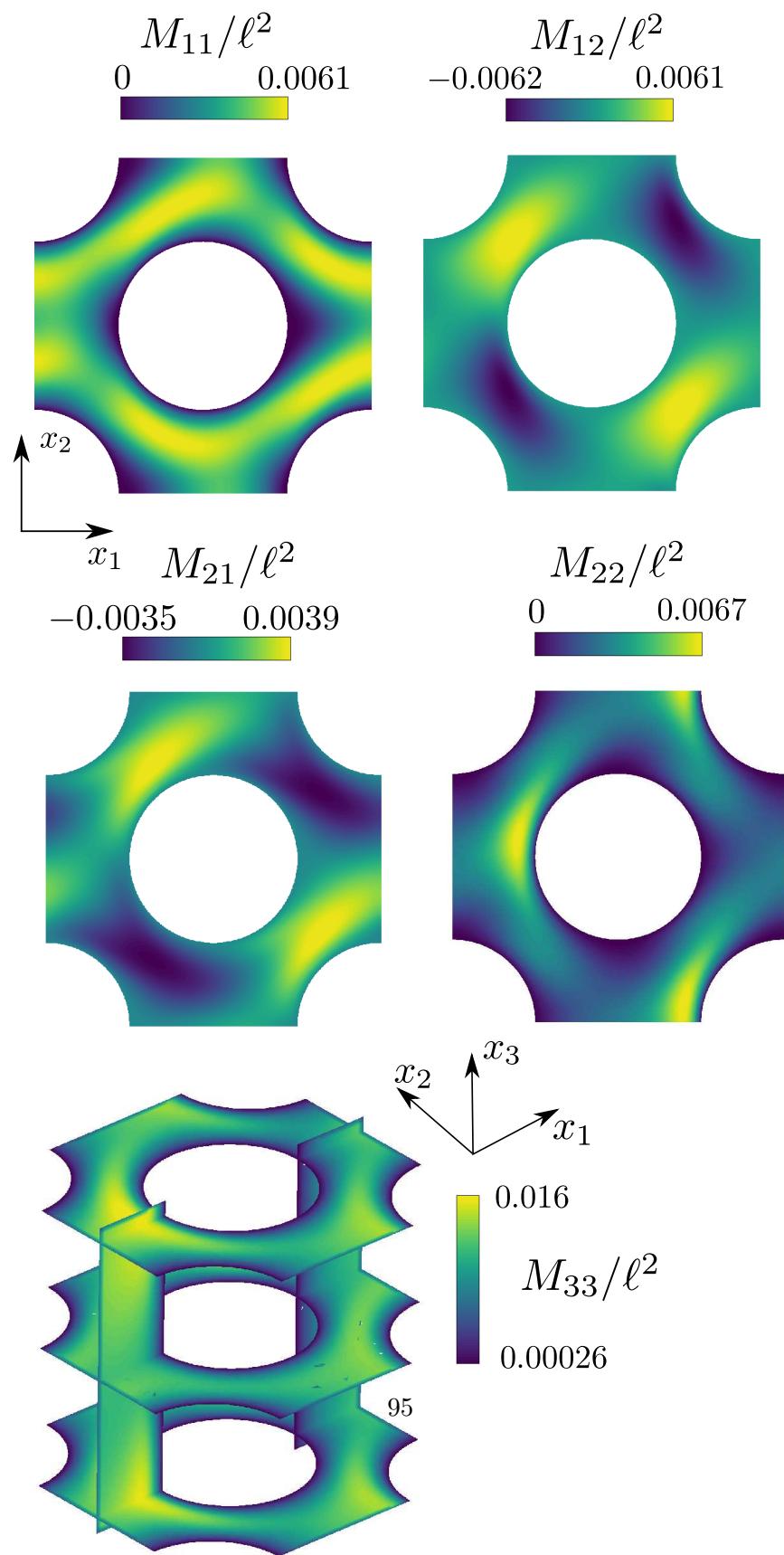


Figure 4.9: Non-dimensional  $\mathbf{M}$  components fields for the case  $\theta = 22.5^\circ, \phi = 45^\circ, Re_d = 50, \varepsilon = 0.6$ .

index	$\theta$	$\phi$	field properties
1	0°	0°	2D symmetric
2	22.5°	0°	2D non-symmetric
3	0°	45°	3D symmetric
4	22.5°	45°	3D non-symmetric
5	–	90°	3D symmetric

Table 4.2: Directions of the forcing tested and property of the solutions.

## 4.5 The apparent permeability tensor

In this section the variations of the diagonal components of the permeability tensor  $\mathbf{H}$  are discussed as function of the direction of the mean forcing, the Reynolds number and the porosity. As stated previously, the Reynolds number ranges from 0 to approximately 100 in order to capture phenomena associated with inertia; the cases considered never lead to unsteady signals. The porosity parameter  $\varepsilon$  is set to either 0.4 (low porosity), 0.6 (medium) or 0.8 (high). The forcing direction is defined by the Euler angles and all the configurations considered in this section are summarized in table 4.2; the choice has been made to explore a reasonably large range of parameters, with both two-dimensional and three-dimensional flows characterized by symmetric and asymmetric patterns.

Let us briefly recall the methodology. First, a DNS is carried out to compute the microscopic flow. Then the closure problem is solved for the tensor  $\mathbf{M}$ . Finally, each component of the apparent permeability  $\mathbf{H}$  is obtained by averaging (equation (2.2)). The results are collected in figures 4.10, 4.11 and 4.12, showing the variation of the diagonal components of  $\mathbf{H}$ .

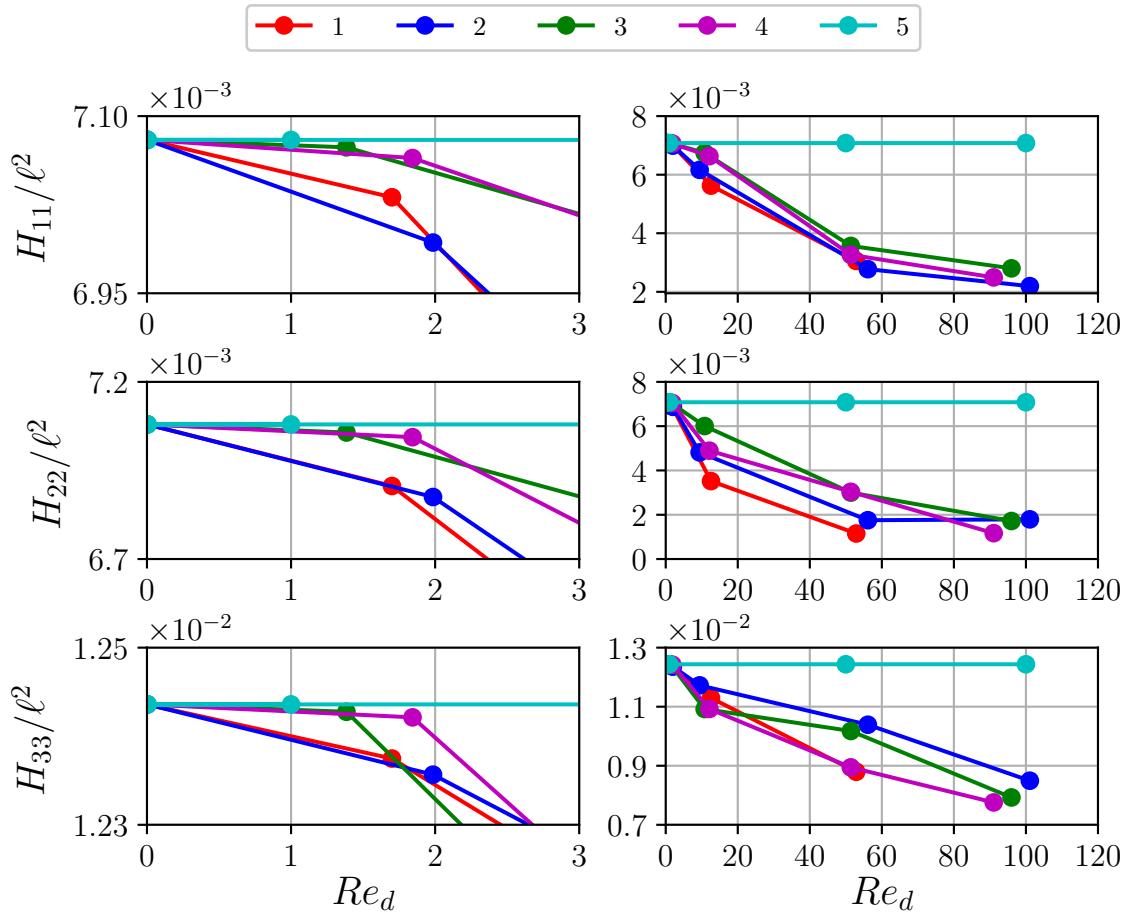


Figure 4.10: Diagonal elements of the apparent permeability  $\mathbf{H}$  as function of the Reynolds number for porosity  $\varepsilon = 0.8$ . The forcing direction is represented through the couple of Euler angles  $(\theta, \phi)$  (cf. table 4.2 for the case index). Left column: low- $Re_d$  regime; right column: inertial regime.

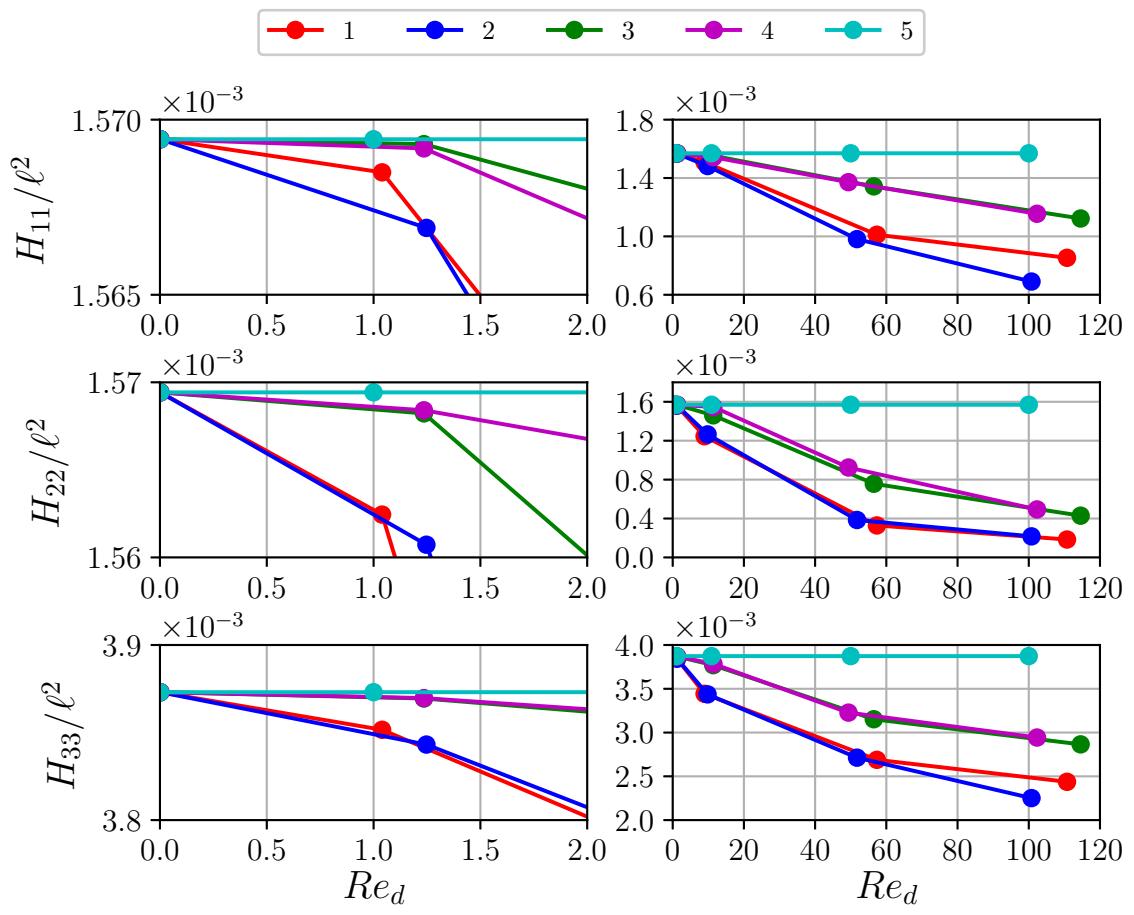


Figure 4.11: Same as figure 4.10 with porosity  $\varepsilon = 0.6$ .

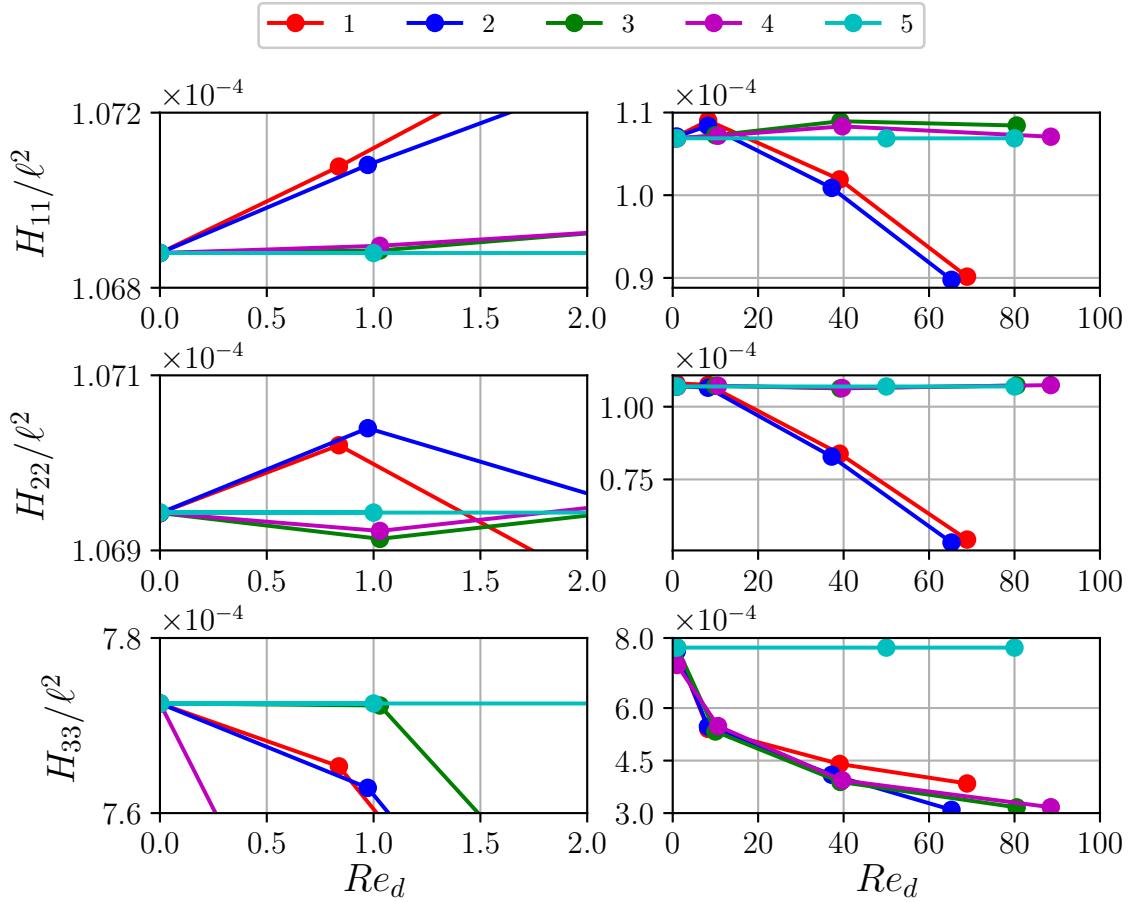


Figure 4.12: Same as figure 4.10 with porosity  $\varepsilon = 0.4$ .

In the left column of each figure we focus on the low- $Re_d$  regime ( $0 < Re_d < 2$ ), while in the right column the effect of inertia can be assessed. As expected, when  $Re_d$  is small the apparent permeability is quasi-Reynolds-number-independent (and can be approximated well by the true permeability). As the Reynolds number increases above a few units, inertial effects grow in importance yielding typically a monotonic decrease of all components of  $\mathbf{H}$ , aside from case indexed 5 ( $\phi = 90^\circ$ ) for which the flow remains aligned with the cylinder's axis. In case 5 the microscopic flow solution is invariant with  $x_3$  and does not change with  $Re_d$  in the range considered, so that  $\mathbf{H}$  is a constant tensor.

When the porosity is large all components show a similar behaviour irrespective of the forcing angle (except, clearly, case 5). Differences start appearing at  $\varepsilon = 0.6$ ; the two cases with  $\phi = 0^\circ$  (index 1 and 2) behave similarly, and so do the two cases indexed 3 and 4 (with  $\phi = 45^\circ$ ). This seems to suggest a weaker effect of  $\theta$  on the permeability components.

For even smaller porosity ( $\varepsilon = 0.4$ ), the blockage which the inclusions cause to the flow produces the unexpected behaviour displayed in figure 4.12. When the flow is purely two-dimensional (cases 1 and 2), variations in the Reynolds number affect  $\mathbf{H}$  significantly; when a pressure gradient along  $x_3$  is present the strong packing of the fibers constrain the fluid to flow prevalently along the fibers' axis, and the apparent permeability is almost  $Re_d$ -independent. When assessing variations in  $H_{jj}$  for this case, attention should also be paid to the fact that the permeability is now at least one order of magnitude smaller than in the previous cases so that variations of the diagonal components shown in figure 4.12 are tiny in absolute terms. This is related to the fact that the inverse of the permeability plays the role of a drag coefficient in the macroscopic expression of the force (cf. equation (??)). In other words, materials with higher porosity (larger space between solid inclusions) offer lower resistance to the motion of the fluid.

Applying the intrinsic average operator to the non-diagonal component of the tensor  $\mathbf{M}$  results in terms that are negligible with respect to their diagonal counterparts, and these results are true for all the parameters considered. This means that there is a very weak coupling between the principal directions of the fiber. The directional decoupling and the diagonal property of the apparent permeability tensor has also been computationally demonstrated on a completely different REV geometry by Soulaine and Quintard [132]. Conversely, Lasseux et al. [84] have carried out a two-dimensional study with fibers of square cross-section, finding that the off-diagonal terms are non-negligible and only about one order of magnitude smaller than the diagonal components. This result is a consequence of the non-rotationally-invariant geometry considered. The present work and the two articles just cited suggest that the diagonal property of the tensor  $\mathbf{H}$  is closely related to the geometry of the porous material, more than to the flow regime.

## 4.6 A metamodel for $\mathbf{H}$

The previous sections has shown how the apparent permeability depends on the two Euler angles, the Reynolds number and the porosity. The space of parameters is formidable and the results found so far are not sufficient to treat, for example, cases characterized by multiple inclusions' sizes and orientations in different regions of the domain, or cases involving a poroelastic medium, with temporally and spatially varying porosity, flow direction and local Reynolds number. The complete solution of the closure problem for a single set of parameters takes approximately 4 CPU hours on our two-processor Intel(r) IVYBRIDGE 2.8Ghz, each with 10 cores and 64 GB of RAM, so that a complete parametric study is, to say the least, unpractical. In view of this, the construction of a metamodel capable to provide a full characterisation of the permeability as a function of all parameters is a worthy endeavor. We have tested several surrogate models, before eventually settling on the kriging approach [79] described in the following.

parameter	values			
$\theta$	0°	22.5°	45°	
$\phi$	0°	22.5°	45°	67.5°
$Re_d$	0	10	50	100
$\varepsilon$	0.4	0.6	0.8	

Table 4.3: Sampling parameters.

#### 4.6.1 DACE sampling

The first step to build a metamodel is the collection of relevant samples. The quality of the final metamodel strongly depends on the samples collected and their number and distribution is of primary importance. The apparent permeability tensor,  $\mathbf{H}$ , depends on four independent variables; the samples have been generated starting from the set of parameters given in table 4.3.

One of the best options to generate the relevant database would be to use a full factorial design approach in which all the combinations of the four variables from table 4.3 are computed. Because of the large number of computations required, this approach has not been retained. We have resorted to the methodology known as DACE (Design and Analysis of Computer Experiments), a technique to fill in the best possible way the space of the parameters of the problem. The Dakota library [2] has been selected for the purpose and the Monte-Carlo incremental random sampling algorithm [60] has been chosen, in order to make efficient use of the cases already computed. This incremental approach selects in a quasi-random way the new samples to generate, starting from the existing ones. In the end, the set of samples comprises 118 cases.

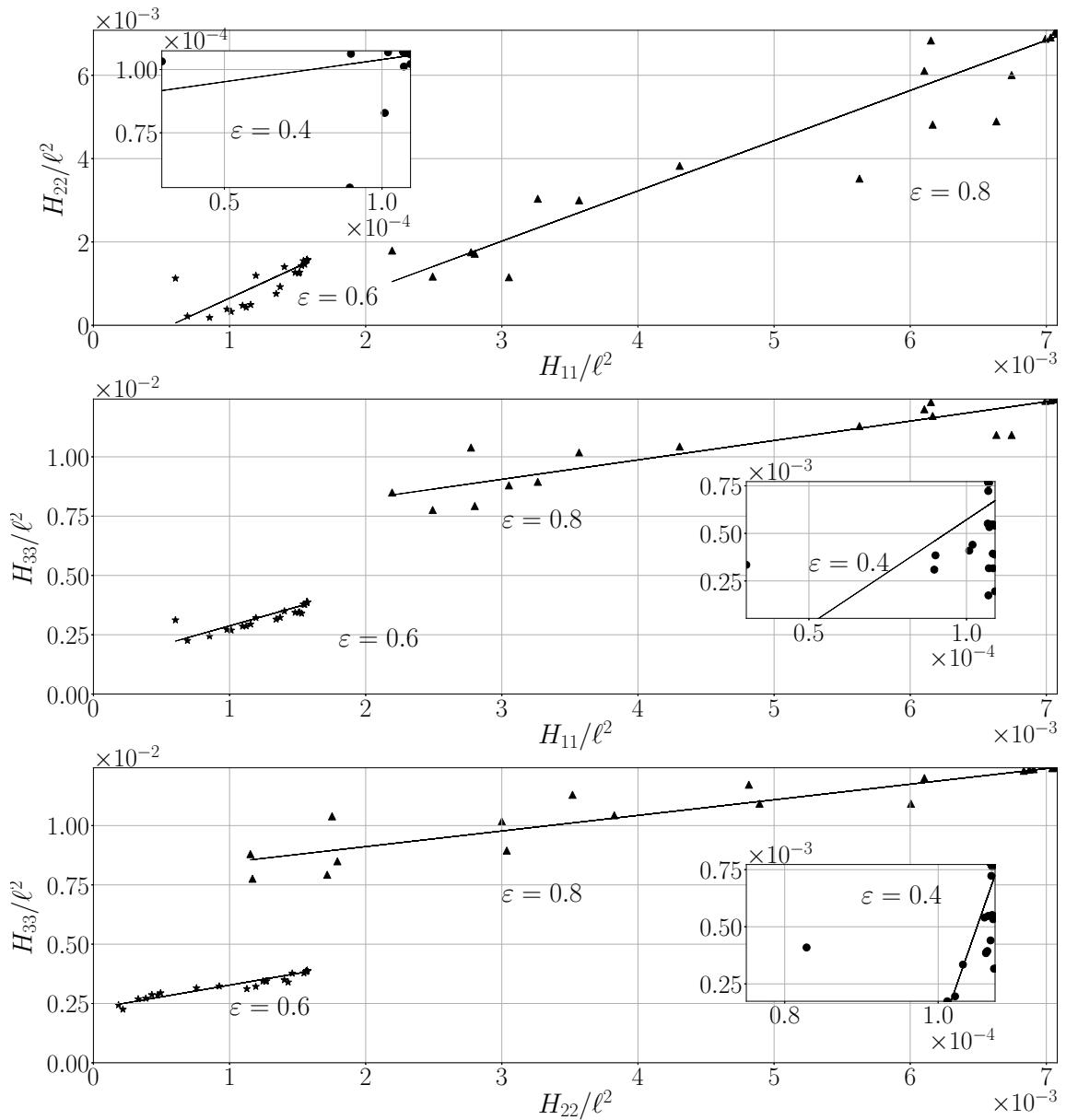


Figure 4.13: Scatter matrix plot for the collected numerical data of the apparent permeability tensor.

In the scatter plot of figure 4.13 the three diagonal components of the permeability tensor are shown as function of one another. The three porosities are separately considered in each of the above plot, and the permeability points are represented with their linear

regression on top. This kind of plot is common in statistical analysis to determine if correlations in the data are present. The permeability components show some correlation with the data points which lie reasonably well on a straight line. This result has a physical implication. Remembering the diagonal dominance of the permeability tensor, we have in the low  $Re_d$  limit:

$$\left( \langle u_\beta \rangle^\beta, \langle v_\beta \rangle^\beta, \langle w_\beta \rangle^\beta \right) \sim \left( H_{11} \frac{\partial p}{\partial x_1}, H_{22} \frac{\partial p}{\partial x_2}, H_{33} \frac{\partial p}{\partial x_3} \right). \quad (4.1)$$

It is then possible to compute the angle between the forcing term,  $\nabla p$ , and the average velocity vector, represented in figure 4.14 for the two-dimensional case,  $\phi = 0$ . This is achieved by taking the ratio between the first two components of Darcy's equation, calling  $\gamma$  the flow deviation with respect to the mean forcing. We thus have:

$$\tan(\theta + \gamma) = \frac{H_{22}}{H_{11}} \tan \theta. \quad (4.2)$$

If the ratio between the two permeability components is equal to one, the angle  $\gamma$  vanishes. The correlation between  $H_{11}$  and  $H_{22}$  controls the deviation of the flow in the  $(x_1, x_2)$  plane, and the argument can easily be extended to  $H_{11}/H_{33}$  and  $H_{22}/H_{33}$  for deviation angles in three-dimensions.

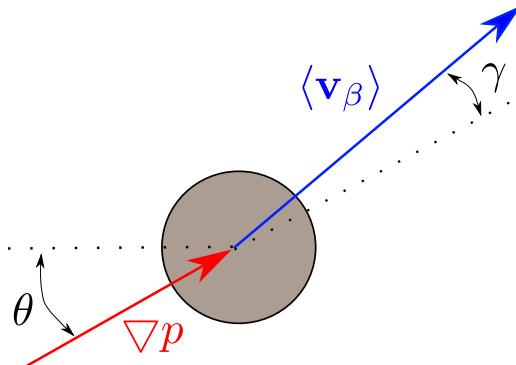


Figure 4.14: Explanatory sketch for the relation between mean pressure gradient and mean velocity field.

Using a linear correlation such as that shown in table 4.4 and figure 4.13, it is observed that in the low porosity case ( $\varepsilon = 0.4$ ) the ratio can become very large indicating a strong deviation of the flow from the forcing direction, because of the strong constraint provided by the inclusions. As the porosity increases, the ratio does not differ much from unity, which means that the deviation remains limited. It is simple to see that the deviation

$\varepsilon$	$H_{11}/H_{22}$	$H_{11}/H_{33}$	$H_{22}/H_{33}$
0.4	1.57	11.06	96.03
0.6	1.50	1.62	0.99
0.8	1.20	0.82	0.66

Table 4.4: Permeability components ratio for three values of the porosity. The permeability ratios here are given by the angular coefficients of the linear correlations displayed in figure 4.13.

angle, for example in the  $(x_1, x_2)$  plane, satisfies the approximate relation

$$\tan \gamma = \frac{\left(1 - \frac{H_{11}}{H_{22}}\right) \tan \theta}{\frac{H_{11}}{H_{22}} + \tan^2 \theta},$$

so that for  $\frac{H_{11}}{H_{22}}$  equal to, say, 1.5, the largest deviation remains always below  $12^\circ$  for any  $\theta$ . It should however be kept in mind that trends based on these ratios are valid only as long as Darcy's law and linear correlations are acceptable. Cases exists for which such trends are violated; for example, a flow with  $\theta = 45^\circ$  and  $\phi = 0^\circ$  has deviation angle  $\gamma$  equal to zero, for whatever porosity. In this case  $H_{11}/H_{22}$  is equal to one and such a point is an outlier in the regression plots of figure 4.13.

#### 4.6.2 Kriging interpolation method

The kriging approach is a linear interpolation/extrapolation method that aims to build a predictor field based on a set of observations  $(\mathbf{x}_i, y(\mathbf{x}_i))$ , for  $i = 1, \dots, n$ .

The predictor  $\hat{f}(\mathbf{x})$  is a sum of a trend function  $t(\mathbf{x})$  and a Gaussian process error model  $e(\mathbf{x})$ :

$$\hat{f}(\mathbf{x}) = t(\mathbf{x}) + e(\mathbf{x}). \quad (4.3)$$

The aim of the error model is to make adjustments on the trend function so that, for any point of the sampling the predictor is exactly equal to the sample, i.e.  $\hat{f}(\mathbf{x}_i) = y(\mathbf{x}_i)$ . This property represents one of the main qualities of this approach. In addition, when the model parameters are conveniently set, the trend function and the covariance model can take into account both smooth and steep variations in the data set.

The trend function defined here is based on a second order least-square regression, with the coefficients found from the solution of the associated linear system. The Gaussian process error model has zero-mean and its covariance between two generic data-points,  $x_i$  and  $x_j$ , is written as

$$\text{Cov}(y(\mathbf{x}_i), y(\mathbf{x}_j)) = \sigma^2 r(\mathbf{x}_i, \mathbf{x}_j).$$

The coefficient  $\sigma$  is an amplitude parameter and  $r(x^i, x^j)$  is a correlation function, based on the Matérn covariance model that reads:

$$r(\mathbf{x}_i, \mathbf{x}_j) = \frac{2^{1-\nu}}{\Gamma(\nu)} \left( \frac{\sqrt{2\nu}|\mathbf{x}_i - \mathbf{x}_j|}{|\boldsymbol{\lambda}|} \right)^\nu K_\nu \left( \frac{\sqrt{2\nu}|\mathbf{x}_i - \mathbf{x}_j|}{|\boldsymbol{\lambda}|} \right), \quad (4.4)$$

where  $K_\nu(\cdot)$  is a modified Bessel function and  $\Gamma(\cdot)$  is the gamma function. The parameters that can be used to tune the metamodel are the amplitude parameter  $\sigma$ , the exponent  $\nu$  and the scale vector  $\boldsymbol{\lambda}$ . The kriging metamodel outputs can show different behaviours for different selections of the above three parameters and their setting is thus crucial. The amplitude parameter  $\sigma$  is chosen to be equal to 1; larger value lead to steeper gradients and undesirable local extrema around the data points. The vector  $\boldsymbol{\lambda} = (\lambda_\theta, \lambda_\phi, \lambda_{Re_d}, \lambda_\varepsilon)$  is a scaling parameter for the distance  $|\mathbf{x}_i - \mathbf{x}_j|$ . In this study, through systematic variations of the parameters it is found that the choice  $\boldsymbol{\lambda} = (1.2, 1, 1, 1)$  yields acceptable results; in particular, the weight along  $\theta$  is mildly larger than in the other directions in order to obtain smoother metamodel surfaces in this direction. The exponent  $\nu$  controls the covariance function and more especially its gradients. When  $\nu = 1/2$  the covariance can be approximated by a negative exponential,  $\exp(-\alpha x)$  and when  $\nu$  goes to infinity it behaves as  $\exp(-\alpha x^2)$ . In the present study, the best (i.e. smoother) results are obtained for  $\nu$  equal to 1.9. The above parameters have been chosen in order to avoid unphysical or unrealistic behaviour of the apparent permeability such as, for instance, negative values or steep, spurious local maxima/minima. The method above is implemented in OpenTURNS and full details are provided by Baudin et al. [12].

In order to prove the robustness of the metamodel we have performed a procedure called cross-validation. This s

and that the number of points choose for the database are enough

The metamodel provides a scalar function (for each term of the  $\mathbf{H}$  tensor) defined in a four-dimensional space. In each of the following figures two parameters are fixed and the response surface is displayed as function of the remaining two, focussing on the  $H_{11}$  component. The other diagonal components of the apparent permeability tensor behave in a similar fashion and will not be shown for brevity. All the results of the metamodel are, however, available from the authors upon request.

In figure 4.15 the angle  $\phi$  is fixed to zero, and the isolines display  $H_{11}$  as function of the angle  $\theta$  and of the Reynolds number,  $Re_d$ , for three values of porosity. The white square symbols indicate the samples used to build the metamodel. The maximum value of each surface is always found for  $Re_d$  equal to zero and  $H_{11}$  typically decreases with  $Re_d$ , when the porosity is sufficiently large. As seen previously, for a porosity approximately greater or equal to 0.6 the variation of the apparent permeability with the angle  $\theta$  is weak in this two-dimensional configuration. For the lowest porosity studied (left frame) the permeability has very small values and the isolines display an irregular behaviour; this is a feature common to all plots relative to the smaller value of  $\varepsilon$ , signaling that it is probably

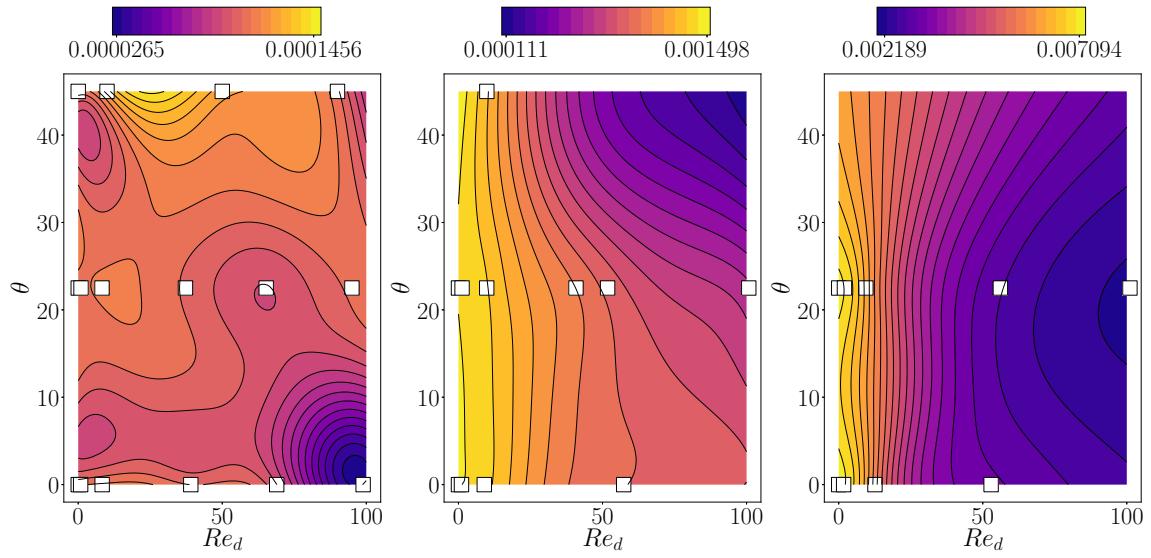


Figure 4.15: Response surfaces of  $H_{11}$  with  $\phi = 0^\circ$  for porosity  $\varepsilon = 0.4, 0.6, 0.8$ , from left to right.

necessary, in this specific case, to insert additional sample points in building the response surfaces.

In figure 4.16 the parameter  $\theta$  is set to  $0^\circ$  and the response surface is displayed in the  $Re_d - \phi$  plane. As already indicated, the results confirm that an increase of the Reynolds number is generally associated to a decrease of the first diagonal component of the apparent permeability tensor. However, the  $H_{11}$  variations with respect to  $\phi$  are more pronounced than those found with respect to  $\theta$  and are due to a real three-dimensionalization of the flow. This conclusion remains to be verified in the lower porosity case (left frame) where the variations are very tiny and more irregular.

In figure 4.17 the Reynolds number is set to the inertial range value of 40 and the response surface is displayed in the  $\theta - \phi$  plane. For the two highest porosity values, 0.6 and 0.8, the results confirm that  $H_{11}$  has a much stronger dependence on  $\phi$  than on  $\theta$ , suggesting that the real test of permeability models must include three-dimensional effects. As seen earlier, the behaviour of the permeability when the porosity is low (left frame in the figure) is not intuitive, with a significant effect of the angle  $\phi$  and a minor influence of  $\theta$ . Again this occurs from the constraint provided to the flow by the inclusions, and from the occurrence of a large deviation  $\gamma$  in these cases.

The response surface is shown in the  $Re_d - \varepsilon$  plane of figure 4.18 for three sets of  $\theta - \phi$  angles. Here a significant effect of the porosity with respect to the Reynolds number is observable. In fact the surface gradient is almost aligned with the porosity direction, i.e. a quasi-Reynolds independence is demonstrated in this plane, and the apparent permeability

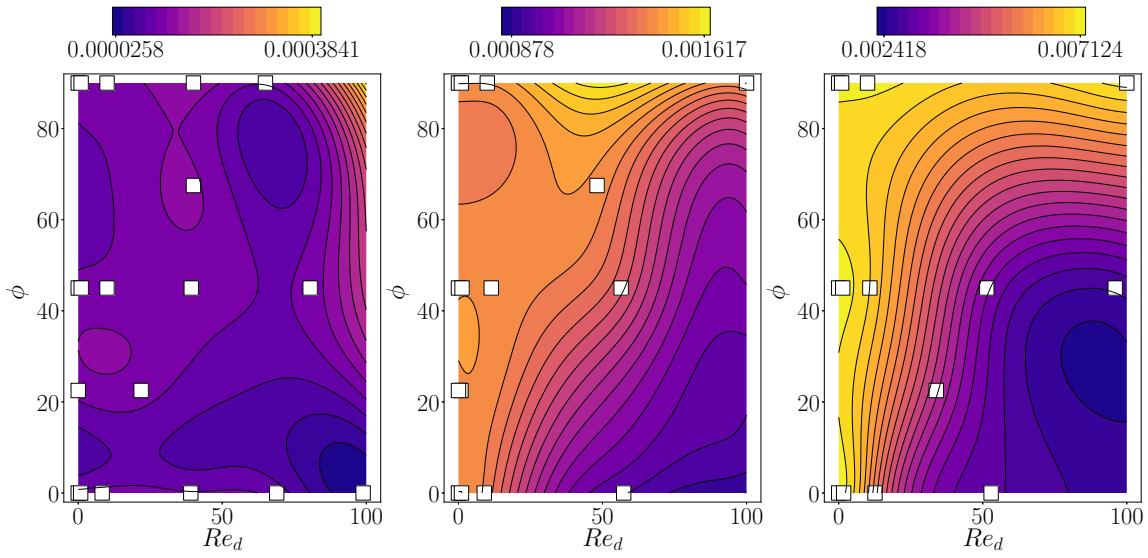


Figure 4.16: Response surfaces of  $H_{11}$  with  $\theta = 0^\circ$  for porosity  $\varepsilon = 0.4, 0.6, 0.8$ , from left to right.

can change by one order of magnitude in the range of the analysed porosity.

Some relatively small Reynolds number effects are visible at porosity equal to 0.8, when the wake of the flow has more space to develop in the inertial regime. In the central figure the flow is aligned with the direction of the fibers and, as expected, it shows practically no dependence with respect to the Reynolds number.

The response surface analysis has confirmed the qualitative trends which had been reached earlier on the basis of a few selected flow cases, yielding at the same time much more detailed information on the behaviour of the apparent permeability with the parameters of the problem. The data base which has been built will be used in future work which will focus, via the VANS approach, on configurations for which neither the porosity nor the local Reynolds number are constant in space or time.

## 4.7 Concluding remarks

The components of the permeability tensor are essential ingredients for any solution of flow through anisotropic porous media. When the flow through the pores resents of significant acceleration effects, the permeability must be modified (it is then called *apparent*) by the presence of a second tensor, the Forchheimer tensor  $\mathbf{F}$ , defined by

$$\mathbf{F} = \mathbf{K}\mathbf{H}^{-1} - \mathbf{I}.$$

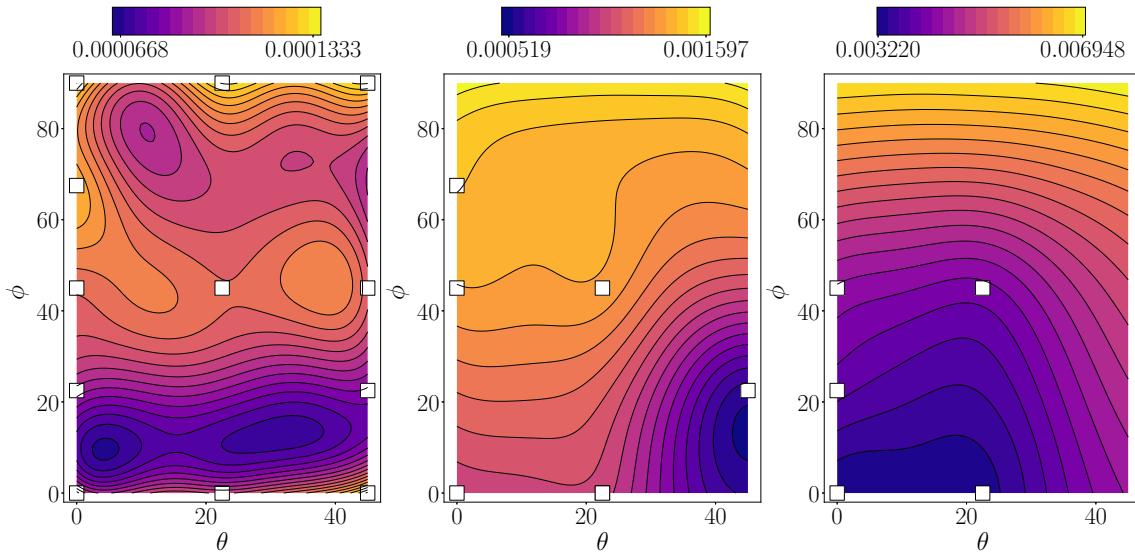


Figure 4.17: Response surfaces of  $H_{11}$  with  $Re = 40$  for porosity  $\varepsilon = 0.4, 0.6, 0.8$ , from left to right.

The permeability,  $\mathbf{K}$ , and the apparent permeability,  $\mathbf{H}$ , can be formally deduced by two closure problems which have been briefly recalled in section 4.2. The real obstacle to the solution of the problem for  $\mathbf{H}$  is the need to know the microscopic velocity fields through the pores. We have solved for such fields in a unit cell (the REV), varying the forcing amplitude and direction, treating over one hundred different cases of flows through arrangements of parallel fibers. From this, we have thus been able to solve the linear system (4.1) for all the unknown elements of the intermediate tensor  $\mathbf{M}$ , from which, through averaging, we have computed the apparent permeability. Such a tensor is indispensable to evaluate accurately the drag force caused by the presence of the fibers, for a macroscopic solution of the flow on the basis of equations [146] when inertial effects are present.

It has been found that the apparent permeability tensor is strongly diagonally dominant for whatever forcing direction and porosity, provided the local Reynolds number remains below a value approximately equal to 100; this results – which is a direct consequence of the transverse isotropy of the material which has been considered here – can be used to compute  $\mathbf{H}$  rapidly, approximating it as a diagonal tensor.

Finally, a metamodel has been used to produce results so as to cover the whole space of parameters, and this has allowed the construction of a complete data base. This data base is now being used in simulations of poroelastic media based on the VANS approach.

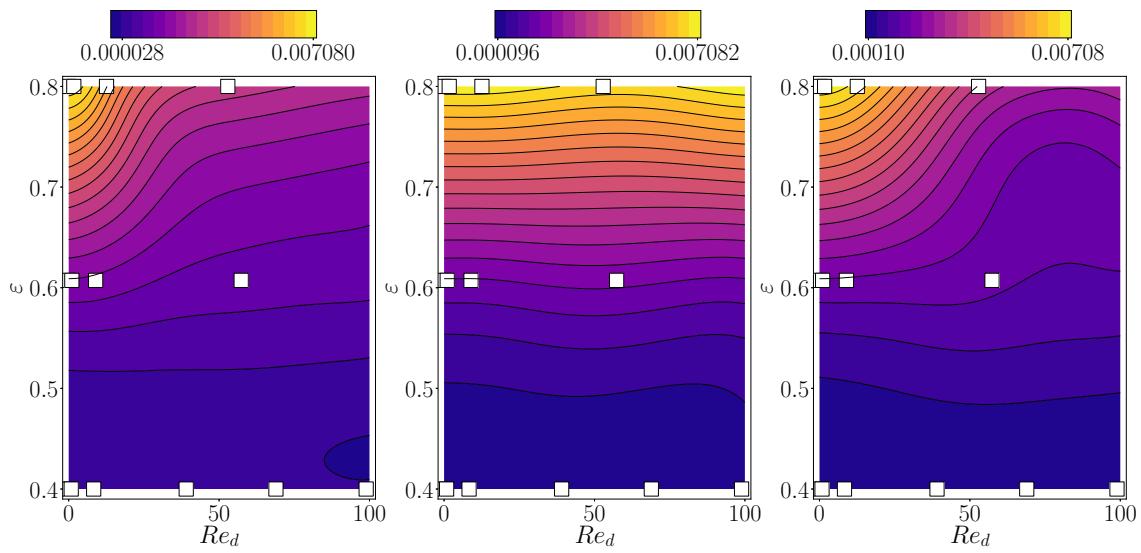


Figure 4.18: Response surface of  $H_{11}$ ; in the left frame  $\phi = \theta = 0$ , in the centre frame  $\phi = 90^\circ$ ,  $\theta = 0$  and on the right  $\phi = 45^\circ$ ,  $\theta = 22.5^\circ$ .

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