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Transport in Ordered and Disordered Porous Media II: Generalized Volume Averaging

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Abstract. In this paper we develop the averaged form of the Stokes equations in terms of weighting functions. The analysis clearly indicates at what point one must choose a media-specific weighting function in order to achieve spatially smoothed transport equations. The form of the weighting function that produces the cellular average is derived, and some important geometrical theorems are presented.

Key words. Volume averaging, weighting functions, ordered media, disordered media, Brinkman correction.

Nomenclature

Roman Letters

$A_{\beta\sigma}$	interfacial area of the β–σ interface associated with the local closure problem, m^2.
$A_{\beta e}$	area of entrances and exits for the β-phase contained within the averaging system, m^2.
A_p	surface area of a particle, m^2.
d_p	$6\,V_p/A_p$, effective particle diameter, m.
$\mathbf{g}$	gravity vector, m/s^2.
I	unit tensor
K_m	permeability tensor for the weighted average form of Darcy's law, m^2.
L	general characteristic length for volume averaged quantities, m.
L_p	general characteristic length for volume averaged pressure, m.
L_ϵ	characteristic length for the porosity, m.
L_v	characteristic length for the volume averaged velocity, m.
l_β	characteristic length (pore scale) for the β-phase.
l_i	$i = 1,\ 2,\ 3$ lattice vectors, m.
$\tilde{m}(\mathbf{y})$	weighting function.
$m(-\mathbf{y})$	$\tilde{m}(\mathbf{y})$, convolution product weighting function.
$\tilde{m}_{\mathcal{V}}$	special weighting function associated with the traditional averaging volume.
$m_{\mathcal{V}}$	special convolution product weighting function associated with the traditional averaging volume.
m_g	general convolution product weighting function.
m_V	unit cell convolution product weighting function.
m_C	special convolution product weighting function for ordered media which produces the cellular average.

m_D	special convolution product weighting function for disordered media.
m_M	master convolution product weighting function for ordered and disordered media.
$\mathbf{n}_{\beta\sigma}$	unit normal vector pointing from the β-phase toward the σ-phase.
p_β	pressure in the β-phase, N/m^2.
$\langle p_\beta \rangle_m$	superficial weighted average pressure, N/m^2.
$\langle p_\beta \rangle_m^\beta$	intrinsic weighted average pressure, N/m^2.
$\langle p_\beta \rangle^\beta$	traditional intrinsic volume averaged pressure, N/m^2.
$\tilde{p}_\beta$	$p_\beta - \gamma_\beta \langle p_\beta \rangle_m^\beta$, spatial deviation pressure, N/m^2.
r_0	radius of a spherical averaging volume, m.
r_m	support of the convolution product weighting function, m.
$\mathbf{r}$	position vector, m.
$\mathbf{r}_\beta$	position vector locating points in the β-phase, m.
$\mathcal{V}$	averaging volume, m^3.
V_β	volume of the β-phase contained in the averaging volume, m^3.
V_{cell}	volume of a unit cell, m^3.
$\mathbf{v}_\beta$	velocity vector in the β-phase, m/s.
$\langle \mathbf{v}_\beta \rangle_m$	superficial weighted average velocity, m/s.
$\langle \mathbf{v}_\beta \rangle_m^\beta$	intrinsic weighted average velocity, m/s.
V_σ	volume of the σ-phase contained in the averaging volume, m^3.
V_p	volume of a particle, m^3.
$\langle \mathbf{v}_\beta \rangle$	traditional superficial volume averaged velocity, m/s.
$\tilde{\mathbf{v}}_\beta$	$\mathbf{v}_\beta - \gamma_\beta \langle \mathbf{v}_\beta \rangle_m^\beta$, spatial deviation velocity, m/s.
$\mathbf{x}$	position vector locating the centroid of the averaging volume or the convolution product weighting function, m.
$\mathbf{y}$	position vector relative to the centroid, m.
$\mathbf{y}_\beta$	position vector locating points in the β-phase relative to the centroid, m.

Greek Letters

γ_β	indicator function for the β-phase.
$\delta_{\beta\sigma}$	Dirac distribution associated with the β–σ interface.
ϵ_β	$V_\beta/\mathcal{V}$, volume average porosity.
$\epsilon_{\beta m}$	$m * \gamma_\beta$, weighted average porosity.
ρ_β	mass density of the β-phase, kg/m^3.
μ_β	viscosity of the β-phase, Ns/m^2.
ϵ_σ	$V_\sigma/\mathcal{V}$, volume fraction of the σ-phase.

1. Introduction

In Part I we identified the process under consideration by the following boundary value problem

$$\nabla \cdot \mathbf{v}_\beta = 0, \quad \text{in the } \beta\text{-phase}, \tag{1.1}$$

$$0 = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{v}_\beta, \quad \text{in the } \beta\text{-phase}, \tag{1.2}$$

$$\text{B.C.1} \quad \mathbf{v}_\beta = 0, \quad \text{at } \mathcal{A}_{\beta\sigma}, \tag{1.3}$$

$$\text{B.C.2} \quad \mathbf{v}_\beta = \mathbf{f}(\mathbf{r},\ \mathbf{t}), \quad \text{at } \mathcal{A}_{\beta e}. \tag{1.4}$$

Here $\mathcal{A}_{\beta\sigma}$ represents the area of the β–σ interface contained in the macroscopic region shown in Figure 1, while $\mathcal{A}_{\beta e}$ represents the area of the β-phase entrances and exits associated with the macroscopic system. We wish to spatially smooth Equations (1.1) and (1.2) making use of the convolution product

$$m * \psi = \int_{\mathbb{R}^3} m(\mathbf{x} - \mathbf{r})\psi(\mathbf{r})\, \mathrm{d}V_r \tag{1.5}$$

for the distribution ψ defined by

$$\psi = \begin{cases} \psi_\beta, & \text{in the } \beta\text{-phase}, \\ \psi_\sigma, & \text{in the } \sigma\text{-phase}. \end{cases} \tag{1.6}$$

In Equation (1.5) the position vector $\mathbf{x}$ locates a reference point (the centroid of the averaging volume in the traditional approach) while $\mathbf{r}$ locates any point in $\mathbb{R}^3$. For the special weighting function defined by

$$m = m_{\mathcal{V}} = \begin{cases} \frac{1}{\mathcal{V}}, & |\,\mathbf{x} - \mathbf{r}\,| \leq r_0 \\ 0, & |\,\mathbf{x} - \mathbf{r}\,| > r_0, \end{cases} \tag{1.7}$$

we see that Equation (1.5) takes the form

$$m_{\mathcal{V}} * \psi = \frac{1}{\mathcal{V}} \int_{\mathcal{V}} \psi\, \mathrm{d}V \tag{1.8}$$

and on the basis of Equation (1.6) this can be expressed as

$$m_{\mathcal{V}} * \psi = \frac{1}{\mathcal{V}} \int_{V_\beta} \psi_\beta\, \mathrm{d}V + \frac{1}{\mathcal{V}} \int_{V_\sigma} \psi_\sigma\, \mathrm{d}V. \tag{1.9}$$

In terms of the traditional *superficial* average, Equation (1.9) yields

$$m_{\mathcal{V}} * \psi = \langle \psi_\beta \rangle + \langle \psi_\sigma \rangle, \tag{1.10}$$

while the use of *intrinsic* averages provides

$$m_{\mathcal{V}} * \psi = \epsilon_\beta \langle \psi_\beta \rangle^\beta + \epsilon_\sigma \langle \psi_\sigma \rangle^\sigma. \tag{1.11}$$

The gradient of ψ takes the form (Schwartz, 1978)

$$\nabla\psi = (\nabla\psi)^{\mathrm{u}} + \mathbf{n}_{\beta\sigma}(\psi_\sigma - \psi_\beta)\delta_{\beta\sigma} \tag{1.12}$$

in which $\nabla\psi$ is the gradient in the *sense of a distribution* and $(\nabla\psi)^{\mathrm{u}}$ is the gradient in the *usual sense*. This latter distribution is given by

$$(\nabla\psi)^{\mathrm{u}} = \begin{cases} \nabla\psi_\beta, & \text{in the } \beta\text{-phase}, \\ \nabla\psi_\sigma, & \text{in the } \sigma\text{-phase}, \end{cases} \tag{1.13}$$

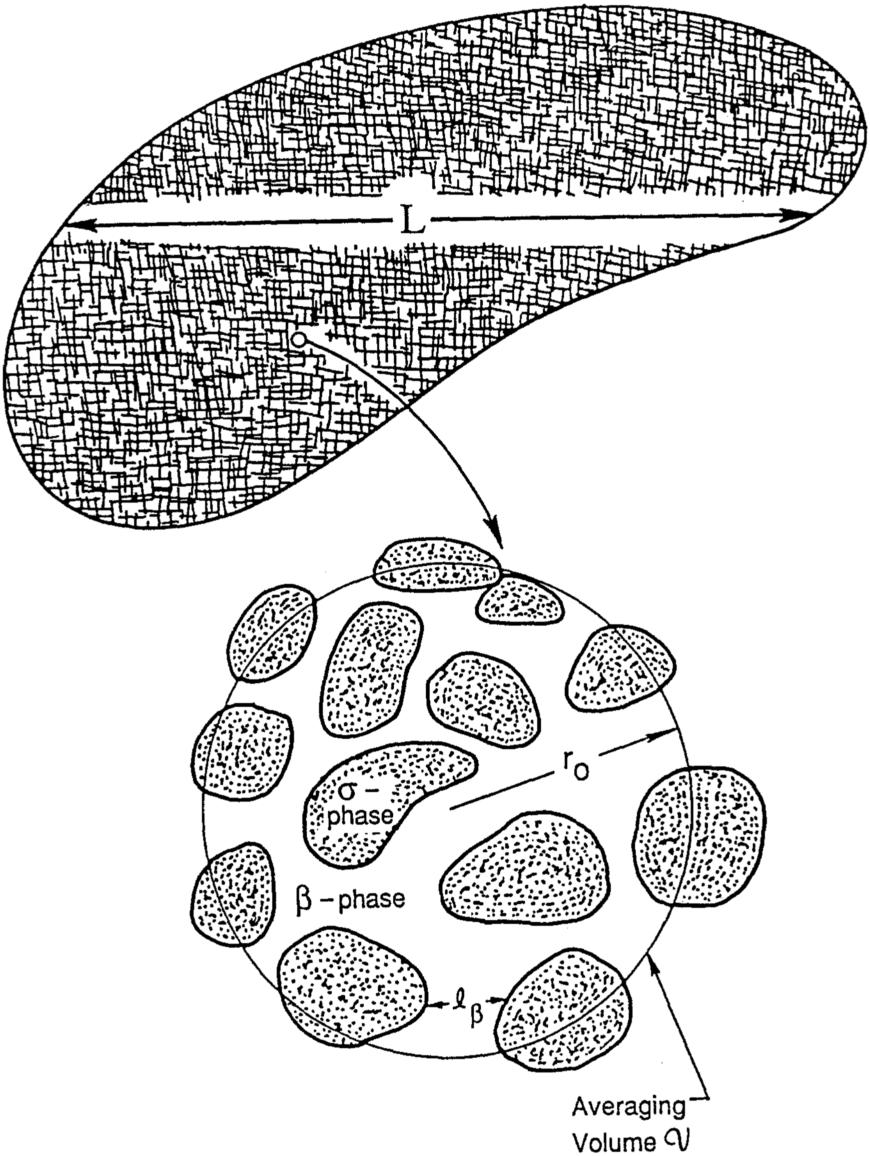

Fig. 1. Macroscopic region of a porous medium.

and it is this type of notation that leads us to express Equations (1.1) through (1.4) as

$$(\nabla \cdot \mathbf{v}_\beta)^{\mathrm{u}} = 0, \quad \text{in } \mathbb{R}^3, \tag{1.14}$$

$$0 = -(\nabla p_\beta)^{\mathrm{u}} + \gamma_\beta \rho_\beta \mathbf{g} + \mu_\beta (\nabla^2 \mathbf{v}_\beta)^{\mathrm{u}}, \quad \text{in } \mathbb{R}^3, \tag{1.15}$$

$$\text{B.C.1} \quad \mathbf{v}_\beta = 0, \quad \text{at } \mathcal{A}_{\beta\sigma}, \tag{1.16}$$

$$\text{B.C.2} \quad \mathbf{v}_\beta = \mathbf{f}(\mathbf{r}, \mathbf{t}), \quad \text{at } \mathcal{A}_{\beta e}. \tag{1.17}$$

Here γ_β is the β-phase indicator function defined by

$$\gamma_\beta(\mathbf{y}) = \begin{cases} 1, & \text{when } \mathbf{y} \text{ locates a point in the } \beta\text{-phase}, \\ 0, & \text{when } \mathbf{y} \text{ locates a point in the } \sigma\text{-phase}, \end{cases} \tag{1.18}$$

in which the relative position vector $\mathbf{y}$ is defined according to

$$\mathbf{y} = \mathbf{r} - \mathbf{x} \tag{1.20}$$

The form of Equations (1.14) and (1.15) is based on the idea that $\mathbf{v}_\beta$ and p_β are the *special distributions* described by

$$\mathbf{v}_\beta = \begin{cases} \mathbf{v}_\beta, & \text{in the } \beta\text{-phase}, \\ 0, & \text{in the } \sigma\text{-phase}, \end{cases} \tag{1.21}$$

$$p_\beta = \begin{cases} p_\beta, & \text{in the } \beta\text{-phase}, \\ 0, & \text{in the } \sigma\text{-phase}. \end{cases} \tag{1.22}$$

Clearly our nomenclature is ambiguous at this point since we have used $\mathbf{v}_\beta$ and p_β to represent both the distribution and its value in the β-phase. Given the special nature of our velocity and pressure distributions, this should not lead to confusion.

1.1. SPATIAL SMOOTHING

We begin with the continuity equation and form the weighted average of Equation (1.14) to obtain

$$m * (\nabla \cdot \mathbf{v}_\beta)^{\mathrm{u}} = 0. \tag{1.23}$$

The generalized form of the averaging theorem is derived in Appendix A and we list the result for the scalar ψ as

$$m * (\nabla\psi)^{\mathrm{u}} = \nabla(m * \psi) + m * (\mathbf{n}_{\beta\sigma}\psi_\beta\delta_{\beta\sigma}) \tag{1.24}$$

when ψ is zero in the σ-phase. The vector form of this theorem can be used to express Equation (1.23) as

$$\nabla \cdot (m * \mathbf{v}_\beta) + m * (\mathbf{n}_{\beta\sigma} \cdot \mathbf{v}_\beta\delta_{\beta\sigma}) = 0 \tag{1.25}$$

and the no-slip condition given by Equation (1.16) leads to the form

$$\nabla \cdot (m * \mathbf{v}_\beta) = 0. \tag{1.26}$$

For the special case in which $m = m_{\mathcal{V}}$ one can use Equations (1.5) and (1.7) to obtain

$$m_{\mathcal{V}} * \mathbf{v}_\beta = \frac{1}{\mathcal{V}} \int_{V_\beta} \mathbf{v}_\beta \, \mathrm{d}V \tag{1.27}$$

since $\mathbf{v}_\beta$ is zero in the σ-phase. Under these circumstances, Equation (1.26) simplifies to the traditional continuity equation given by

$$\nabla \cdot \langle \mathbf{v}_\beta \rangle = 0, \quad m = m_{\mathcal{V}}. \tag{1.28}$$

Moving on to the Stokes equations, we form the weighted average of Equation (2.21) to obtain

$$0 = -m * (\nabla p_\beta)^{\mathrm{u}} + m * (\rho_\beta \mathbf{g} \gamma_\beta) + m * [\mu_\beta (\nabla^2 \mathbf{v}_\beta)^{\mathrm{u}}]. \tag{1.29}$$

The scalar and vector forms of the weighted averaging theorem provide

$$m * (\nabla p_\beta)^{\mathrm{u}} = \nabla (m * p_\beta) + m * (\mathbf{n}_{\beta\sigma} p_\beta \delta_{\beta\sigma}), \tag{1.30}$$

$$m * (\nabla \cdot \nabla \mathbf{v}_\beta)^{\mathrm{u}} = \nabla \cdot [m * (\nabla \mathbf{v}_\beta)^{\mathrm{u}}] + m * [\mathbf{n}_{\beta\sigma} \cdot (\nabla \mathbf{v}_\beta)^{\mathrm{u}} \delta_{\sigma\beta}], \tag{1.31}$$

$$m * (\nabla \mathbf{v}_\beta)^{\mathrm{u}} = \nabla (m * \mathbf{v}_\beta) + m * (\mathbf{n}_{\beta\sigma} \mathbf{v}_\beta \delta_{\beta\sigma}). \tag{1.32}$$

From the no-slip condition the last term in Equation (1.32) is zero, and when Equations (1.30) through (1.32) are used in Equation (1.29) we obtain

$$\begin{aligned} 0 = & -\nabla (m * p_\beta) + \rho_\beta \mathbf{g} (m * \gamma_\beta) + \mu_\beta \nabla^2 (m * \mathbf{v}_\beta) + \\ & + m * [\mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I} p_\beta + \mu_\beta (\nabla \mathbf{v}_\beta)^{\mathrm{u}}) \delta_{\beta\sigma}]. \end{aligned} \tag{1.33}$$

One can think of this result as the governing differential equation for $m * \mathbf{v}_\beta$ that is analogous to the local volume average form presented by Whitaker (1986, Equation (2.24)). In obtaining Equation (1.33) we have used

$$m * (\rho_\beta \mathbf{g} \gamma_\beta) = \rho_\beta \mathbf{g} (m * \gamma_\beta) \tag{1.34}$$

with the idea that $\rho_\beta \mathbf{g}$ can be treated as a constant. When m is given by Equation (1.7) we see that $m * \gamma_\beta$ represents the traditional volume fraction of the β-phase expressed by

$$m_{\mathcal{V}} * \gamma_\beta = V_\beta / \mathcal{V} = \epsilon_\beta. \tag{1.35}$$

In order to more easily relate Equation (1.33) to the traditional form, we employ the nomenclature given by

$$m * \gamma_\beta = \epsilon_{\beta m}, \tag{1.36a}$$

$$m * p_\beta = \langle p_\beta \rangle_m, \tag{1.36b}$$

$$m * \mathbf{v}_\beta = \langle \mathbf{v}_\beta \rangle_m. \tag{1.36c}$$

This allows us to express Equation (1.33) as

$$\begin{aligned} 0 = & -\nabla \langle p_\beta \rangle_m + \epsilon_{\beta m} \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle_m + \\ & + m * [\mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I} p_\beta + \mu_\beta (\nabla \mathbf{v}_\beta)^{\mathrm{u}}) \delta_{\beta\sigma}]. \end{aligned} \tag{1.37}$$

Here we are confronted by two problems: (1) The weighted-average pressure, $m * p_\beta = \langle p_\beta \rangle_m$, represents a *superficial* average rather than an *intrinsic* average, and (2) the last term in Equation (1.37) involves an area integral of the point values of p_β and $\mathbf{v}_\beta$ and these must be expressed in terms of averaged values in order to obtain a *closed form* of Equation (1.37)

2. Use of Spatial Decompositions

It is easy to extract the intrinsic average pressure from the superficial average, $\langle p_\beta \rangle_m$; however the presence of the point values of p_β and $\mathbf{v}_\beta$ in Equation (1.37) represents a challenging problem. The approach that we follow is comparable to that used in the analysis of turbulent transport processes in which one decomposes a dependent variable into a time-averaged part and a temporal fluctuation. Within the framework of volume averaging, we decompose the point values in terms of average values and *spatial deviations*. We do this in a manner analogous to Gray's (1975) decomposition

$$p_\beta = \gamma_\beta \langle p_\beta \rangle_m^\beta + \tilde{p}_\beta, \tag{2.1}$$

$$\mathbf{v}_\beta = \gamma_\beta \langle \mathbf{v}_\beta \rangle_m^\beta + \tilde{\mathbf{v}}_\beta \tag{2.2}$$

in which the intrinsic weighted averages are defined by

$$\langle p_\beta \rangle_m^\beta = m * p_\beta / m * \gamma_\beta, \tag{2.3}$$

$$\langle \mathbf{v}_\beta \rangle_m^\beta = m * \mathbf{v}_\beta / m * \gamma_\beta. \tag{2.4}$$

At this point we should note that we have said nothing specific about the weighting function other than the three hypotheses that were given in Part I. To be clear we restate these conditions as

$$\text{H1.} \quad m \in \mathrm{C}^\infty, \tag{2.5}$$

$$\text{H2.} \quad m \text{ has a compact support over } \mathbb{R}^3, \tag{2.6}$$

$$\text{H3.} \quad \int_{\mathbb{R}^3} m(\mathbf{x} - \mathbf{r})\, \mathrm{d}V_r = 1. \tag{2.7}$$

Use of Equations (2.1) and (2.2) in Equation (1.37) leads to

$$\begin{aligned} 0 = & -\nabla(\epsilon_{\beta m} \langle p_\beta \rangle_m^\beta) + \epsilon_{\beta m} \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle_m + \\ & + m * [\mathbf{n}_{\beta\sigma} \cdot \{-\mathsf{I} \gamma_\beta \langle p_\beta \rangle_m^\beta + \mu_\beta \nabla(\gamma_\beta \langle \mathbf{v}_\beta \rangle_m^\beta)\} \delta_{\beta\sigma}] + \\ & + m * [\mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I} \tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta) \delta_{\beta\sigma}] \end{aligned} \tag{2.8}$$

in which the last two terms represent weighted area integrals over the $\beta - \sigma$ interface. The last term in Equation (2.8) leads us to the closure problem which will be presented in Part III, while the next to the last term presents us with a potentially very difficult problem. In its present form, Equation (2.8) represents a *nonlocal* transport equation since the next to the last term involves an area integral in which $\langle p_\beta \rangle_m^\beta$ and $\langle \mathbf{v}_\beta \rangle_m^\beta$ are evaluated at points other than the centroid located by $\mathbf{x}$. Quintard and Whitaker (1990a) have explored the nonlocal theory for two-phase flow in heterogeneous porous media and the result is extremely complex. Thus, there is considerable motivation for removing the average values, $\langle p_\beta \rangle_m^\beta$ and $\langle \mathbf{v}_\beta \rangle_m^\beta$, from the area integral in Equation (2.8) in order to develop a local theory. To illustrate how this is done, and to illustrate the key role that the weighting function plays in this operation, we will consider only the terms involving $\langle p_\beta \rangle_m^\beta$.

2.1. GEOMETRICAL THEOREMS

In order to explore the possibility that $\langle p_\beta \rangle_m^\beta$ can be removed from the area integral in Equation (2.8), we need to develop some results associated with a Taylor series expansion for $\langle p_\beta \rangle_m^\beta$. One can use the definition given by Equation (1.5) to develop the explicit expression

$$m * [\mathbf{n}_{\beta\sigma} \cdot \{-\mathsf{I}\gamma_\beta \langle p_\beta \rangle_m^\beta\}\delta_{\beta\sigma}]$$
$$= \int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})[(-\mathbf{n}_{\beta\sigma}\gamma_\beta(\mathbf{r})\langle p_\beta \rangle_m^\beta \,|_\mathbf{r})\delta_{\beta\sigma}(\mathbf{r})]\,\mathrm{d}V_r \tag{2.9}$$

and a Taylor series expansion of $\langle p_\beta \rangle_m^\beta \,|_\mathbf{r}$ about the centroid $\mathbf{x}$ takes the form

$$\begin{aligned}\langle p_\beta \rangle_m^\beta \,|_\mathbf{r} \;=\; & \langle p_\beta \rangle_m^\beta \,|_\mathbf{x} + (\mathbf{r}-\mathbf{x})\cdot\nabla\langle p_\beta \rangle_m^\beta \,|_\mathbf{x} + \\ & + \tfrac{1}{2}(\mathbf{r}-\mathbf{x})(\mathbf{r}-\mathbf{x}){:}\nabla\nabla\langle p_\beta \rangle_m^\beta \,|_\mathbf{x} + \cdots .\end{aligned} \tag{2.10}$$

Substitution of Equation (2.10) in Equation (2.9) leads to

$$\begin{aligned}& m * [\mathbf{n}_{\beta\sigma} \cdot \{-\mathsf{I}\gamma_\beta \langle p_\beta \rangle_m^\beta\}\delta_{\beta\sigma}] \\ &= \left\{\int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})[-\mathbf{n}_{\beta\sigma}\gamma_\beta(\mathbf{r})\delta_{\beta\sigma}(\mathbf{r})]\,\mathrm{d}V_r\right\}\langle p_\beta \rangle_m^\beta \,|_\mathbf{x} + \\ &\quad + \left\{\int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})[-\mathbf{n}_{\beta\sigma}(\mathbf{r}-\mathbf{x})\gamma_\beta(\mathbf{r})\delta_{\beta\sigma}(\mathbf{r})]\,\mathrm{d}V_r\right\}\cdot\nabla\langle p_\beta \rangle_m^\beta \,|_\mathbf{x} + \\ &\quad + \frac{1}{2}\left\{\int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})[-\mathbf{n}_{\beta\sigma}(\mathbf{r}-\mathbf{x})(\mathbf{r}-\mathbf{x})\gamma_\beta(\mathbf{r})\delta_{\beta\sigma}(\mathbf{r})]\,\mathrm{d}V_r\right\} \\ &\quad {:}\nabla\nabla\langle p_\beta \rangle_m^\beta \,|_\mathbf{x} + \cdots\end{aligned} \tag{2.11}$$

since all terms evaluated at $\mathbf{x}$ are constants with respect to the convolution product. In Appendix B we prove a sequence of geometrical theorems given by

$$\int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})[\mathbf{n}_{\beta\sigma}\gamma_\beta(\mathbf{r})\delta_{\beta\sigma}(\mathbf{r})]\,\mathrm{d}V_r = -\nabla(m*\gamma_\beta), \tag{2.12a}$$

$$\int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})[\mathbf{n}_{\beta\sigma}(\mathbf{r}-\mathbf{x})\gamma_\beta(\mathbf{r})\delta_{\beta\sigma}(\mathbf{r})]\,\mathrm{d}V_r = -\nabla[m*(\gamma_\beta\mathbf{y})], \tag{2.12b}$$

$$\int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})[\mathbf{n}_{\beta\sigma}(\mathbf{r}-\mathbf{x})(\mathbf{r}-\mathbf{x})\gamma_\beta(\mathbf{r})\delta_{\beta\sigma}(\mathbf{r})]\,\mathrm{d}V_r = -\nabla[m*(\gamma_\beta\mathbf{y}\mathbf{y})], \tag{2.12c}$$

$$\cdots\cdots\cdots \tag{2.12d}$$

and this allows us to express our original convolution product as

$$\begin{aligned}& m * [\mathbf{n}_{\beta\sigma} \cdot \{-\mathsf{I}\gamma_\beta \langle p_\beta \rangle_m^\beta\}\delta_{\beta\sigma}] \\ &= \{\nabla(m*\gamma_\beta)\}\langle p_\beta \rangle_m^\beta + \{\nabla[m*(\gamma_\beta\mathbf{y})]\}\nabla\langle p_\beta \rangle_m^\beta + \\ &\quad + \tfrac{1}{2}\{\nabla[m*(\gamma_\beta\mathbf{y}\mathbf{y})]\}{:}\nabla\nabla\langle p_\beta \rangle_m^\beta + \cdots .\end{aligned} \tag{2.13}$$

Here it is understood that $\langle p_\beta \rangle_m^\beta$ and gradients of $\langle p_\beta \rangle_m^\beta$ on the right hand side of Equation (2.13) are evaluated at $\mathbf{x}$.

At this point we wish to consider Equation (2.13) within the context of only the two pressure terms contained in Equation (1.33). Those two terms are represented by the three pressure terms in Equation (2.8), and we can use Equation (2.13) to represent the pressure terms according to

$$\begin{aligned} &-\nabla(m * p_\beta) + m * [\mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I} p_\beta)\delta_{\beta\sigma}] \\ &\quad = -(m * \gamma_\beta)\nabla\langle p_\beta \rangle_m^\beta + m * [\mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I}\tilde{p}_\beta)\delta_{\beta\sigma}] + \\ &\quad\quad + \{\nabla[m * (\gamma_\beta \mathbf{y})]\} \cdot \nabla\langle p_\beta \rangle_m^\beta + \\ &\quad\quad + \tfrac{1}{2}\{\nabla[m * (\gamma_\beta \mathbf{y}\mathbf{y})\}{:}\nabla\nabla\langle p_\beta \rangle_m^\beta + \cdots . \end{aligned} \tag{2.14}$$

Here we have represented $\epsilon_{\beta m}$ in the original form given by $m * \gamma_\beta$ since we now want to focus attention on the role of the weighting function and how it interacts with the structure of the porous medium under consideration. To do so, one must think of Equation (2.14) as being incorporated into the spatially smoothed Stokes equations given by Equations (1.33) and (2.8), and one should remember that we have said nothing special about the weighting function other than the three hypotheses given by Equations (2.5) through (2.7).

2.2. ROLE OF THE WEIGHTING FUNCTION

To begin with, we require of the weighting function that $m * \gamma_\beta$, $m * (\gamma_\beta \mathbf{y})$, $m * (\gamma_\beta \mathbf{y}\mathbf{y})$, etc. be constant or, at the very least, that they *contain no small-scale variations*. If this is not the case, we will be confronted with the type of problem illustrated by Equations (1.12) and (1.13) in Part I. From Equations (1.7) and (1.35) we have seen that the special weighting function, m_V, gives rise to the traditional concept of porosity, thus we could follow the discussion of Marle (1967) and require that the support for m, designated by r_m, be *large enough* so that $m * \gamma_\beta$ be essentially devoid of small-scale variations. This is comparable to Bear's (1972) requirement that an REV exist. Marle's representations of $m * \gamma_\beta$ are reproduced in Figure 2. In Figure 2a the curve represents what one would expect from a typical disordered porous medium, i.e. if r_m is *sufficiently large* the porosity will be devoid of small-scale variations. Figure 2b is Marle's representation for a stratified, heterogeneous porous medium and the existence of two well-behaved domains forms the basis for the method of large-scale averaging (Quintard and Whitaker, 1987; Plumb and Whitaker, 1988) and for the use of the method of spatial homogenization (Douglas and Arbogast, 1990; Auriault and Boutin, 1992) with double porosity models (see Chen, 1989, for a review). The absence of any well-behaved regions leads to Cushman's (1990) concept of evolving heterogeneities and the need for nonlocal theories. Clearly the weighting function must be chosen to match the porous medium under consideration, and to connect this idea with the relativist concept of Baveye and Sposito (1984), it means that the instrument

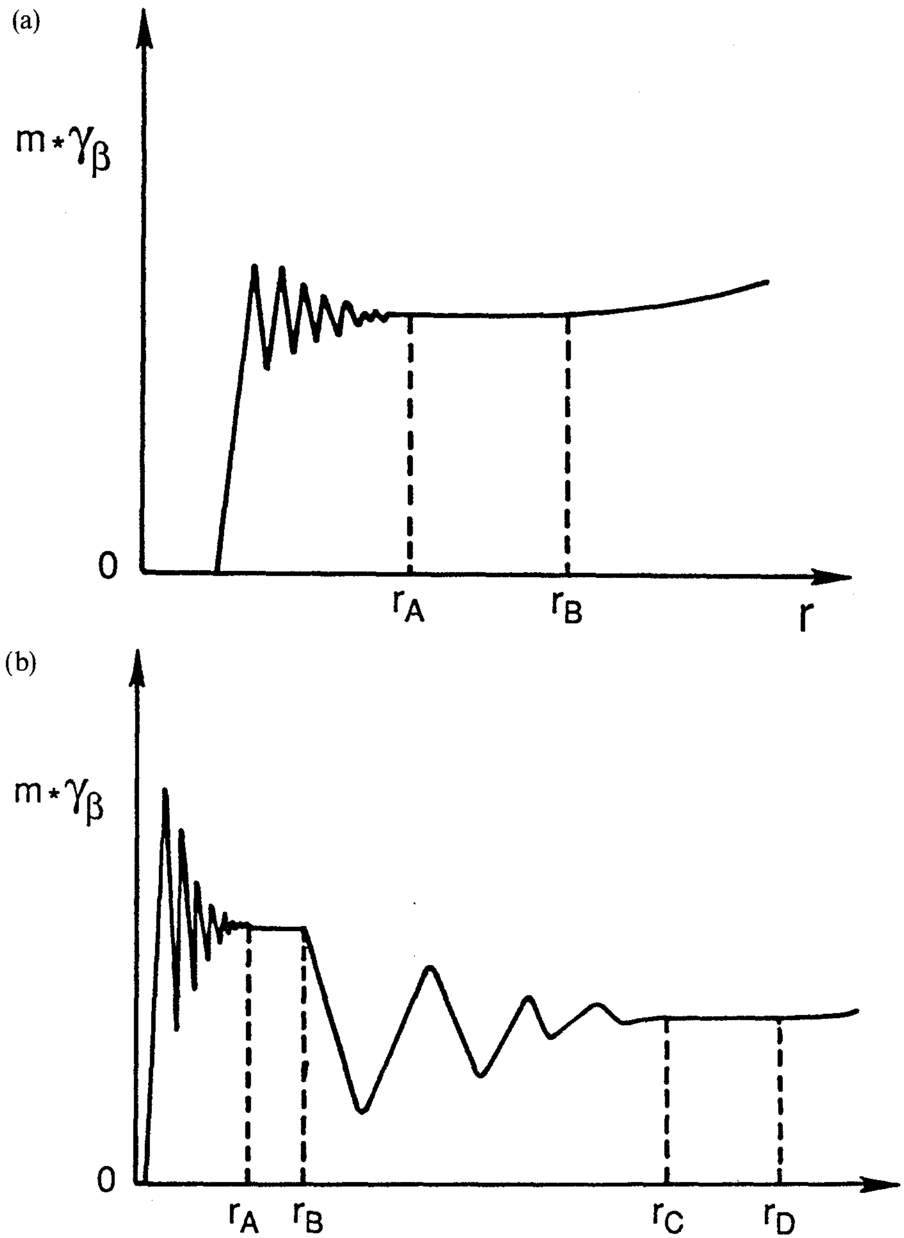

Fig. 2. Porosity in terms of the support for a weighting function.

weighting function (Cushman, 1984; Maneval *et al.*, 1990) should be chosen to match the porous medium. When numerical experiments are carried out, this is easily accomplished (Barrère, 1990; Quintard and Whitaker, 1990b).

2.3. DISORDERED MEDIA

For *disordered porous media* we generally think of r_m as being constrained by

$$l_\beta \ll r_m \tag{2.15}$$

and evidence for this result is given in Part IV where we show that Equation (2.15) leads to

$$\nabla[m * (\gamma_\beta \mathbf{y})] \ll \mathsf{I}. \tag{2.16}$$

This result is equivalent to our definition of a disordered porous medium given by Equation (1.19) in Part I, and it allows us to simplify Equation (2.14) by imposing the condition

$$\{\nabla[m * (\gamma_\beta \mathbf{y})]\} \cdot \nabla\langle p_\beta\rangle_m^\beta \ll (m * \gamma_\beta)\nabla\langle p_\beta\rangle_m^\beta. \tag{2.17}$$

Concerning the last term in Equation (2.14), one can easily show (Carbonell and Whitaker, 1984, Sec. 2) that

$$[m * (\gamma_\beta \mathbf{yy})] = \mathbf{O}(\epsilon_{\beta m} r_m^2) \tag{2.18}$$

and that the gradient of this term can be estimated as

$$\nabla[m * (\gamma_\beta \mathbf{yy})] = \mathbf{O}(\epsilon_{\beta m} r_m^2 / L_\epsilon). \tag{2.19}$$

Here L_ϵ represents the characteristic length associated with $\epsilon_{\beta m}$ and if the porous medium is homogeneous L_ϵ will be infinite. In order to discard the last term in Equation (2.14), we require that

$$\{\nabla[m * (\gamma_\beta \mathbf{yy})]\}:\nabla\nabla\langle p_\beta\rangle_m^\beta \ll (m * \gamma_\beta)\nabla\langle p_\beta\rangle_m^\beta. \tag{2.20}$$

This leads to the following length-scale constraint

$$r_m^2 \ll L_\epsilon L_p \tag{2.21}$$

in which L_p is the characteristic length associated with $\nabla\langle p_\beta\rangle_m^\beta$. Thus if $\nabla\langle p_\beta\rangle_m^\beta$ is a constant, L_p is infinite and Equation 2.21 is automatically satisfied.

When Equations (2.17) and (2.20) are satisfied, we can simplify Equation (2.14) to

$$\begin{aligned} &-\nabla(m * p_\beta) + m * [\mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I} p_\beta)\delta_{\beta\sigma}] \\ &\quad = -\epsilon_{\beta m}\nabla\langle p_\beta\rangle^\beta + m * [\mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I}\tilde{p}_\beta)\delta_{\beta\sigma}]. \end{aligned} \tag{2.22}$$

For disordered porous media this requires that the support of the weighting function be constrained by Equations (2.15) and (2.21). It is important to note that the first of these is a purely *geometrical constraint* while the second involves the *process length scale*, L_p. To be precise, we impose the following constraints on m for disordered porous media

H4. For disordered porous media the support for m is constrained by

$$r_m^2 \gg l_\beta, \tag{2.23a}$$

$$r_m^2 \ll L_\epsilon L_p \tag{2.23b}$$

in addition to the constraints given earlier by Equations (2.5) through (2.7).

2.4. ORDERED MEDIA

In considering the right hand side of Equation (2.14) for *ordered systems* we find a much different situation than we encountered with disordered systems. To begin with, ordered or spatially periodic systems are homogeneous and this means that we need only find a weighting function that will produce constant values for $m * \gamma_\beta$, $m * (\gamma_\beta \mathbf{y})$, $m * (\gamma_\beta \mathbf{y}\mathbf{y})$, etc. This can be stated as

H.5 For ordered porous medium m is chosen so that

$$m * \underbrace{(\gamma_\beta \mathbf{y} \ldots \mathbf{y})}_{n-\text{times}} = \begin{cases} \text{constant, } n \text{ is even,} \\ 0, \ n \text{ is odd,} \end{cases} \tag{2.24}$$

and the weighting function that permits us to accomplish this is given by

$$m = m_g * m_V * m_V. \tag{2.25}$$

Here m_V is similar to $m_\mathcal{V}$ and the former is defined explicitly as

$$m_V = \begin{cases} \frac{1}{V_{\text{cell}}}, & \mathbf{y} \in V_{\text{cell}}, \\ 0, & \mathbf{y} \notin V_{\text{cell}} \end{cases} \tag{2.26}$$

in which V_{cell} is the volume of a unit cell. In Equation (2.25) we have used m_g to represent a general weighting function that only needs to satisfy the conditons identified by H1, H2 and H3. While m_V *does not* satisfy H1, the weighting function represented by Equation (2.25) *does* satisfy H1 since $m_g \in C^\infty$. It is important to note that H5 is satisfied by the weighting function given by Equation (2.25) because of the presence of the double convolution, $m_V * m_V$, and it is this double convolution that represents the cellular average. To be absolutely clear about this point we note that for spatially periodic porous media

$$m_V * m_V * \underbrace{(\gamma_\beta \mathbf{y} \ldots \mathbf{y})}_{n-\text{times}} = \begin{cases} \text{constant, } n \text{ is even,} \\ 0, \ n \text{ is odd.} \end{cases} \tag{2.27}$$

The proof of this is given in Appendix C. The length-scale constraint *associated with* H5 could be expressed as

$$r_m = 0(l_\beta), \quad \text{ordered media} \tag{2.28}$$

and this should be contrasted with the constraints for disordered porous media given by Equations (2.23).

One-dimensional representations of the weighting functions for disordered and ordered porous media are shown in Figure 3. One way of thinking about these representations is that the *proper* weight function is *media specific.* On the other hand, one can construct a *master* weight function, such as the one shown in Figure 4, which is a *proper* weight function for *both* disordered and ordered porous media. One should remember that in order to construct a C^∞ weight function one uses

$$m = m_g * m_\text{D}, \qquad m = m_g * m_\text{C}, \qquad m = m_g * m_\text{M}, \tag{2.29}$$

since m_D, m_C, and m_M are only C°. The nomenclature used here is chosen with the idea that D refers to *disorder*, C represents the *cellular* average to be used with ordered media, and M identifies the *master* weighting function that can be used with either ordered or disordered porous media. Here one can see that the proposed average method removes all questions, such as those raised by Veverka (1981) and M1š (1987), concerning the differentiability of volume averaged quantities. This particular point has been made repeatedly by Marle (1984); however, we do not believe that differentiability is a serious practical matter. For example, one need only choose m_g so that the C^1 discontinuities illustrated in Figure 3 are removed and this can be done without changing the essential features of the averages computed in terms of m_D and m_C.

At this point we are in a position to clarify some of our earlier vague comments concerning the cellular average. To illustrate the characteristics of the weighting function given by Equation (2.25), we consider the following three functions associated with the β-phase.

1. A constant, C_β.
2. A periodic function, f_β.
3. A linear function, $\mathbf{h}_\beta \cdot \mathbf{r}_\beta$, in which $\mathbf{h}_\beta$ is a constant vector.

These functions are all defined in the *usual sense* in the β-phase and they are zero in the σ-phase. We begin with the first function and form the convolution product to obtain

$$m * C_\beta = m_g * m_V * (m_V * C_\beta). \tag{2.30}$$

Here the term in parenthesis is a constant given by the porosity times C_β, and this leads to

$$m_V * (m_V * C_\beta) = (m_V * C_\beta) \tag{2.31}$$

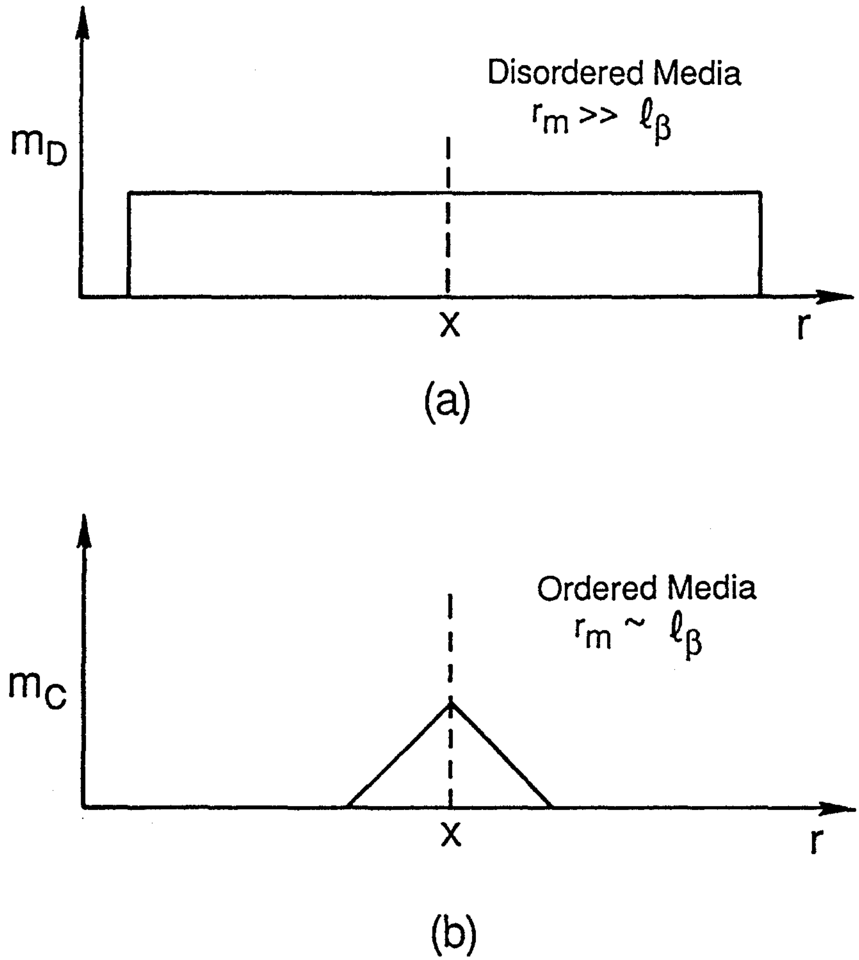

Fig. 3. Weighting functions for disordered and ordered porous media.

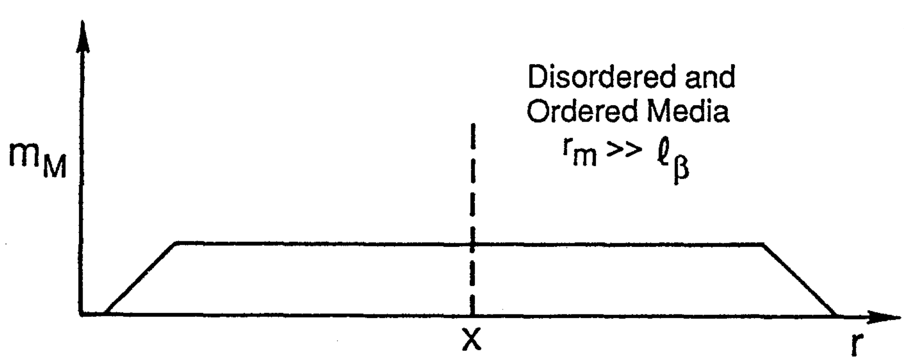

Fig. 4. Master weighting function for disordered and ordered porous media.

since m_V satisfies H3, the normalization condition given by Equation (2.7). Under these circumstances, Equation (2.30) simplifies to

$$m * \mathbf{C}_\beta = m_g * (m_V * \mathbf{C}_\beta). \tag{2.32}$$

Once again we note that $m_V * \mathbf{C}_\beta$ is a constant, and since m_g satisfies H3 we have

$$1. \quad m * \mathbf{C}_\beta = m_V * \mathbf{C}_\beta. \tag{2.33}$$

Moving on to the spatially periodic functon, f_β, we immediately note that $(m_V * f_\beta)$ is a constant, thus we can repeat the procedure given above to arrive at

$$2. \quad m * f_\beta = m_V * f_\beta. \tag{2.34}$$

The analysis of $\mathbf{h}_\beta \cdot \mathbf{r}_\beta$ begins with

$$m * (\mathbf{h}_\beta \cdot \mathbf{r}_\beta) = m_g * m_V * m_V * (\mathbf{h}_\beta \cdot \mathbf{r}_\beta) \tag{2.35}$$

and since $\mathbf{h}_\beta$ is a constant this takes the form

$$m * (\mathbf{h}_\beta \cdot \mathbf{r}_\beta) = \mathbf{h}_\beta \cdot [m_g * (m_V * m_V * \mathbf{r}_\beta]. \tag{2.36}$$

The position vector can be represented according to

$$\mathbf{r}_\beta = \mathbf{x} + \mathbf{y}_\beta$$

as illustrated in Figure 1 of Part I, and this leads to

$$m_V * m_V * \mathbf{r}_\beta = m_V * m_V * \mathbf{x} + m_V * m_V * \mathbf{y}_\beta. \tag{2.37}$$

Our discussion in Section 1 of Part I indicated that $m_V * \mathbf{y}_\beta$ was *not zero*; however, we did not indicate that the cellular average was indeed zero and the details are given in Part IV. Under these circumstances $m_V * m_V * \mathbf{y}_\beta$ is zero and Equation (2.37) indicates that

$$m_V * m_V * \mathbf{r}_\beta = \text{constant}. \tag{2.38}$$

This allows us to write Equation (2.36) as

$$m * (\mathbf{h}_\beta \cdot \mathbf{r}_\beta) = \mathbf{h}_\beta \cdot [(m_V * m_V * \mathbf{r}_\beta)] \tag{2.39}$$

and we again use the fact that $\mathbf{h}_\beta$ is a constant to obtain

$$3. \quad m * (\mathbf{h}_\beta \cdot \mathbf{r}_\beta) = m_V * m_V * (\mathbf{h}_\beta \cdot \mathbf{r}_\beta). \tag{2.40}$$

In each of these three cases we have followed our previous treatment of the pressure and velocity, i.e., these functions are defined in the *usual* sense in the β-phase and they are zero in the σ-phase. For these three functions we see that m_g does not enter into the evaluation of the average quantities which are entirely determined by m_V as defined by Equation (2.26). In the case of nonlinear functions, *which can not be linearized over the support of m*, the general weighting function will indeed come into play and the problem of precisely specifying average values becomes more difficult.

2.5. VELOCITY TERMS

Given either H4 for disordered porous media or H5 for ordered porous media, we obtain Equation (2.22). This can be substituted into Equation (1.33) and we can follow the analysis to Equation (2.8) in order to obtain

$$\begin{aligned} 0 = {} & -\epsilon_{\beta m}\nabla\langle p_\beta\rangle_m^\beta + \epsilon_{\beta m}\rho_\beta \mathbf{g} + \mu_\beta\nabla^2\langle \mathbf{v}_\beta\rangle_m + \\ & + m * [\mathbf{n}_{\beta\sigma}\cdot\{\mu_\beta\nabla(\gamma_\beta\langle\mathbf{v}_\beta\rangle_m^\beta)\}\delta_{\beta\sigma}] + \\ & + m * [\mathbf{n}_{\beta\sigma}\cdot(-\mathsf{I}\tilde{p}_\beta + \mu_\beta\nabla\tilde{\mathbf{v}}_\beta)\delta_{\beta\sigma}]. \end{aligned} \tag{2.41}$$

At this point one can follow the development from Equation (2.9) to (2.13) so that a Taylor series expansion for $\nabla(\gamma_\beta\langle\mathbf{v}_\beta\rangle_m^\beta)$ leads to

$$\begin{aligned} 0 = {} & -\epsilon_{\beta m}\nabla\langle p_\beta\rangle_m^\beta + \epsilon_{\beta m}\rho_\beta \mathbf{g} + \\ & + \int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})[\mathbf{n}_{\beta\sigma}\cdot(-\mathsf{I}\tilde{p}_\beta + \mu_\beta\nabla\tilde{\mathbf{v}}_\beta)\delta_{\beta\sigma}]\,\mathrm{d}V_r + \\ & + \mu_\beta\nabla^2\langle\mathbf{v}_\beta\rangle_m^\beta - \mu_\beta\{\nabla(m*\gamma_\beta)\}\cdot\nabla\langle\mathbf{v}_\beta\rangle_m^\beta - \\ & - \mu_\beta\{\nabla[m*(\gamma_\beta\mathbf{y})]\}{:}\nabla\nabla\langle\mathbf{v}_\beta\rangle_m^\beta - \tfrac{1}{2}\mu_\beta\{\nabla[m*(\gamma_\beta\mathbf{y}\mathbf{y})]\} \\ & \vdots\nabla\nabla\nabla\langle\mathbf{v}_\beta\rangle_m^\beta - \cdots. \end{aligned} \tag{2.42}$$

Here we have arranged Equation (2.42) so that the three most important terms are listed first, the fourth term is the Brinkman correction, and the remaining terms result from the Taylor series expansion of $\nabla(\gamma_\beta\langle\mathbf{v}_\beta\rangle_m^\beta)$ and the geometrical theorems given by Equations (2.12).

2.6. ORDERED MEDIA

In this case we invoke the condition indicated by H5 in order to conclude that

$$m * \gamma_\beta = \text{constant}, \tag{2.43a}$$

$$m * (\gamma_\beta\mathbf{y}) = 0, \tag{2.43b}$$

$$m * (\gamma_\beta\mathbf{y}\mathbf{y}) = \text{constant}. \tag{2.43c}$$

Under these circumstances Equation (2.42) immediately reduces to

$$\begin{aligned} 0 = {} & -\epsilon_{\beta m}\nabla\langle p_\beta\rangle_m^\beta + \epsilon_{\beta m}\rho_\beta \mathbf{g} + \\ & + \int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})[\mathbf{n}_{\beta\sigma}\cdot(-\mathsf{I}\tilde{p}_\beta + \mu_\beta\nabla\tilde{\mathbf{v}}_\beta)\delta_{\beta\sigma}]\,\mathrm{d}V_r + \\ & + \mu_\beta\nabla^2\langle\mathbf{v}_\beta\rangle_m^\beta \end{aligned} \tag{2.44}$$

in which the Brinkman (1947) correction is the only remaining term containing gradients of the average velocity. It is very important to note that the Brinkman correction cannot be used at an impermeable surface in order to satisfy a no-slip condition. In the neighborhood of an impermeable surface a porous medium such as the one illustrated in Figure 3 of Part I is *not spatially periodic* and the results given by Equations (2.43) *are not correct*. In order for Equations (2.43) to be valid the centroid $\mathbf{x}$ must be located at least one unit cell away from an impermeable surface.

2.7. DISORDERED MEDIA

In this case we make use of Equation (1.36a) to obtain

$$\nabla(m * \gamma_\beta) \cdot \nabla\langle \mathbf{v}_\beta\rangle_m^\beta = \nabla\epsilon_{\beta m} \cdot \nabla\langle \mathbf{v}_\beta\rangle_m^\beta, \tag{2.45a}$$

then impose our definition of disordered media given by Equation (2.16) to generate the restriction,

$$\nabla[m * (\gamma_\beta \mathbf{y})]\!:\!\nabla\nabla\langle \mathbf{v}_\beta\rangle_m^\beta \ll \nabla\nabla\langle \mathbf{v}_\beta\rangle_m^\beta \tag{2.45b}$$

and use Equation (2.19) to develop the estimate

$$\nabla[m * (\gamma_\beta \mathbf{y}\mathbf{y})]\vdots\nabla\nabla\nabla\langle \mathbf{v}_\beta\rangle^\beta = 0\left[\left(\frac{\epsilon_{\beta m} r_m^2}{L_\epsilon L_v}\right)\nabla\nabla\langle \mathbf{v}_\beta\rangle_m^\beta\right], \tag{2.45c}$$

in which L_v is the characteristic length for $\langle \mathbf{v}_\beta\rangle_m^\beta$. To begin with, we can compare these viscous terms to the Brinkman correction to conclude that the inequalities

$$\mu_\beta\nabla[m * (\gamma_\beta \mathbf{y})]\!:\!\nabla\nabla\langle \mathbf{v}_\beta\rangle_m^\beta \ll \mu_\beta\nabla^2\langle \mathbf{v}_\beta\rangle_m, \tag{2.46a}$$

$$\mu_\beta\nabla[m * (\gamma_\beta \mathbf{y}\mathbf{y})]\vdots\nabla\nabla\nabla\langle \mathbf{v}_\beta\rangle_m^\beta \ll \mu_\beta\nabla^2\langle \mathbf{v}_\beta\rangle_m \tag{2.46b}$$

are satisfied whenever the following length-scale constraints are in effect

$$r_m \gg l_\beta \tag{2.47a}$$

$$r_m^2 \ll L_\epsilon L_v. \tag{2.47b}$$

This is nothing more than H4 written for the velocity field instead of the pressure field and when Equations (2.47) are in force Equation (2.42) simplifies to

$$\begin{aligned} 0 = &-\epsilon_{\beta m}\nabla\langle p_\beta\rangle_m^\beta + \epsilon_{\beta m}\rho_\beta \mathbf{g} + \\ &+ \int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})[\mathbf{n}_{\beta\sigma}\cdot(-\mathsf{I}\tilde{p}_\beta + \mu_\beta\nabla\tilde{\mathbf{v}}_\beta)\delta_{\beta\sigma}]\,\mathrm{d}V_r + \\ &+ \mu_\beta\nabla^2\langle \mathbf{v}_\beta\rangle_m - \mu_\beta\nabla\epsilon_{\beta m}\cdot\nabla\langle \mathbf{v}_\beta\rangle_m^\beta. \end{aligned} \tag{2.48}$$

Before going on to develop the closure problem for $\tilde{p}_\beta$ and $\tilde{\mathbf{v}}_\beta$, we need to think about situations in which the last two terms in Equation (2.48) might be nonnegligible.

2.8. BRINKMAN CORRECTION

We need to be able to compare the last three terms in Equation (2.48) on a consistent basis, thus we use Equation (2.4) to express the superficial velocity as

$$\langle \mathbf{v}_\beta \rangle_m = \epsilon_{\beta m} \langle \mathbf{v}_\beta \rangle_m^\beta \tag{2.49}$$

which provides

$$\begin{aligned} \nabla^2 \langle \mathbf{v}_\beta \rangle_m &= \epsilon_{\beta m} \nabla^2 \langle \mathbf{v}_\beta \rangle_m^\beta + 2\nabla \epsilon_{\beta m} \cdot \nabla \langle \mathbf{v}_\beta \rangle_m^\beta + \\ &\quad + (\nabla^2 \epsilon_{\beta m}) \langle \mathbf{v}_\beta \rangle_m^\beta . \end{aligned} \tag{2.50}$$

This allows us to express Equation (2.48) as

$$\begin{aligned} 0 &= -\epsilon_{\beta m} \nabla \langle p_\beta \rangle_m^\beta + \epsilon_{\beta m} \rho_\beta \mathbf{g} + \\ &\quad + \int_{\mathbb{R}^3} m(\mathbf{x} - \mathbf{r}) [\mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I} \tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta) \delta_{\beta\sigma}] \, \mathrm{d}V_r + \\ &\quad + \mu_\beta [\epsilon_{\beta m} \nabla^2 \langle \mathbf{v}_\beta \rangle_m^\beta + \nabla \epsilon_{\beta m} \cdot \nabla \langle \mathbf{v}_\beta \rangle^\beta + (\nabla^2 \epsilon_{\beta m}) \langle \mathbf{v}_\beta \rangle_m^\beta] \end{aligned} \tag{2.51}$$

and our order of magnitude estimates of the last three terms are given by

$$\epsilon_{\beta m} \nabla^2 \langle \mathbf{v}_\beta \rangle_m^\beta = 0 \left[\frac{\epsilon_{\beta m} \Delta \langle \mathbf{v}_\beta \rangle_m^\beta}{L_v^2} \right], \tag{2.52a}$$

$$\nabla \epsilon_{\beta m} \cdot \nabla \langle \mathbf{v}_\beta \rangle_m^\beta = 0 \left[\frac{\Delta \epsilon_{\beta m} \Delta \langle \mathbf{v}_\beta \rangle_m^\beta}{L_\epsilon L_v} \right], \tag{2.52b}$$

$$(\nabla^2 \epsilon_{\beta m}) \langle \mathbf{v}_\beta \rangle_m^\beta = 0 \left[\frac{\Delta \epsilon_{\beta m} \langle \mathbf{v}_\beta \rangle_m^\beta}{L_\epsilon^2} \right]. \tag{2.52c}$$

Near an impermeable solid the changes in $\epsilon_{\beta m}$ and $\langle \mathbf{v}_\beta \rangle_m^\beta$ are given by

$$\Delta \epsilon_{\beta m} \sim \epsilon_{\beta m}, \qquad \Delta \langle \mathbf{v}_\beta \rangle_m^\beta \sim \langle \mathbf{v}_\beta \rangle_m^\beta \tag{2.53}$$

and the characteristic lengths can be estimated as

$$L_\epsilon = 0(r_0), \qquad L_v = 0(r_0). \tag{2.54}$$

Under these circumstances all three of the Brinkman-like terms in Equation (2.51) are the same order of magnitude. On the other hand, all of the analysis leading from Equation (2.8) to Equation (2.50) is of *questionable validity* since neither H4 nor H5 is satisfied in the region near an impermeable solid. To be very specific, we note that the constraint given by Equation (2.47b) must surely fail for any averaging volume within a distance r_0 of an impermeable solid surface.

Since Equation (2.51) is not a reliable representation of the volume averaged Stokes equations near an impermeable solid surface, we move away from that region and search for conditions under which the last three terms might be non-negligible. This requires that we estimate the magnitude of the integral term in Equation (2.51), and we begin by considering an array of particles having an effective diameter d_p and interparticle distance l_β. These two length scales are related by

$$d_p \sim l_\beta(1 - \epsilon_{\beta m})^{1/3}. \tag{2.55}$$

In order to simplify our analysis of the integral in Equation (2.51), we make use of the weighting function given by Equation (1.7) to obtain

$$\int_{\mathbb{R}^3} m(\mathbf{x} - \mathbf{r})[\mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I}\tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta)\delta_{\beta\sigma}]\,\mathrm{d}V_r$$

$$= \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I}\tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta)\,\mathrm{d}A. \tag{2.56}$$

Since the interfacial area per unit volume can be estimated as

$$\frac{A_{\beta\sigma}}{\mathcal{V}} \sim \frac{d_p^2}{l_\beta^3} \sim \frac{(1 - \epsilon_{\beta m})^{2/3}}{l_\beta} \tag{2.57}$$

we can express Equation (2.56) as

$$\int_{\mathbb{R}^3} m(\mathbf{x} - \mathbf{r})[\mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I}\tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta)\delta_{\beta\sigma}]\,\mathrm{d}V_r$$

$$= 0\left[\frac{(1 - \epsilon_\beta)^{2/3}}{l_\beta}\,\tilde{p}_\beta,\ \mu_\beta \nabla \tilde{\mathbf{v}}_\beta)\right]. \tag{2.58}$$

In order to generate an estimate of $\nabla \tilde{\mathbf{v}}_\beta$ at the β–σ interface, we note that the no-slip condition provides

$$\tilde{\mathbf{v}}_\beta = -\langle \mathbf{v}_\beta \rangle_m^\beta, \quad \text{at } \mathcal{A}_{\beta\sigma} \tag{2.59}$$

and this leads to

$$\mathbf{n}_{\beta\sigma} \cdot \nabla \tilde{\mathbf{v}}_\beta = 0\left(\frac{\langle \mathbf{v}_\beta \rangle_m^\beta}{d_p}\right), \quad \text{at } \mathcal{A}_{\beta\sigma}. \tag{2.60}$$

This estimate assumes that the particles are essentially three-dimensional in nature, and on the basis of Equation (2.60) we express Equation (2.58) as

$$\int_{\mathbb{R}^3} m(\mathbf{x} - \mathbf{r})[\mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I}\tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta)\delta_{\beta\sigma}]\,\mathrm{d}V_r$$

$$= 0\left[\frac{\mu_\beta(1-\epsilon_{\beta m})^{1/3}\langle\mathbf{v}_\beta\rangle_m^\beta}{l_\beta^2}\right]. \tag{2.61}$$

Here we have made the plausible assumption that the contribution of $\tilde{p}_\beta$ to the integral is no greater than the contribution of $\mu_\beta \mathbf{n}_{\beta\sigma} \cdot \nabla\tilde{\mathbf{v}}_\beta$.

We are now in a position to compare the integral term in Equation (2.51) with the three Brinkman-like terms for which order of magnitude estimates are given by Equations (2.52). Before making this comparison, it is important to recognize that the averaging process will produce large length scales that are always greater than or equal to r_m, and we express this idea, along with Equation (2.15) as,

$$L \geq r_m \gg l_\beta, \quad \text{disordered media} \tag{2.62a}$$

while for ordered media we draw upon Equation (2.28) to write

$$L \geq r_m = 0(l_\beta), \quad \text{ordered media.} \tag{2.62b}$$

To compare Equations (2.52) with Equation (2.61) we consider two cases: A typical porous medium with $\epsilon_{\beta m} \sim 0.4$ and a high porosity porous medium with $(1-\epsilon_{\beta m}) \ll 1$.

Case I

$$\epsilon_{\beta m} \sim 0.4 \tag{2.63a}$$

$$\Delta\epsilon_{\beta m} \sim \epsilon_{\beta m}. \tag{2.63b}$$

All Brinkman-like terms are negligible.

Case II

$$(1-\epsilon_{\beta m}) \ll 1, \tag{2.64a}$$

$$\Delta\epsilon_{\beta m} \sim (1-\epsilon_{\beta m}). \tag{2.64b}$$

Only the first Brinkman-like term is non-negligible.

Here we have *overestimated* $\Delta\epsilon_{\beta m}$, thus we are absolutely certain that the terms involving $\Delta\epsilon_{\beta m}$ and $\nabla^2\epsilon_{\beta m}$ will be negligible. This leads us to express Equation (2.51) as

$$\begin{aligned} 0 = & -\nabla\langle p_\beta\rangle_m^\beta + \rho_\beta \mathbf{g} + \\ & +\epsilon_m^{-1}\int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})[\mathbf{n}_{\beta\sigma}\cdot(-\mathsf{I}\tilde{p}_\beta + \mu_\beta\nabla\tilde{\mathbf{v}}_\beta)\Delta_{\beta\sigma}]\,\mathrm{d}V_r + \\ & +\mu_\beta\nabla^2\langle\mathbf{v}_\beta\rangle_m^\beta \end{aligned} \tag{2.65}$$

with the reminder that:

(i) The Brinkman correction is only nonnegligible when $(1 - \epsilon_{\beta m}) \ll 1$.
(ii) the volume averaged form of the Stokes equations given by Equation (2.65) is not valid in the neighborhood of a homogeneous solid, nor in the neighborhood of a homogeneous fluid when $\epsilon_{\beta m} \sim 0.4$.

For the special case of $\epsilon_{\beta m} = 1$, it is of some interest to note that Equation (1.2) can be volume averaged to obtain

$$0 = -\nabla\langle p_\beta\rangle_m^\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta\rangle_m^\beta, \quad \epsilon_{\beta m} = 1, \tag{2.66}$$

thus we expect that Equations (2.65) and (2.66) can be matched in the neighborhood of a *high porosity* porous medium and a homogeneous fluid.

The role of the Brinkman correction has been discussed by many workers and a survey is provided by Nield and Bejan (1992). The consensus would appear to be that a *Brinkman viscosity* (Kim and Russel, 1985) should be associated with the last term in Equation (2.65); however, this type of result does not appear in our analysis. Levy (1983) has studied this problem from the point of view of homogenization theory and concludes that the Brinkman correction (multiplied by the β-phase viscosity) should be important when the length scales are related by

$$\left(\frac{d_p}{L}\right) \sim \left(\frac{l_\beta}{L}\right)^3 . \tag{2.67}$$

If we compare $\epsilon_{\beta m} \nabla^2 \langle \mathbf{v}\rangle_m^\beta$ as given by Equation (2.52a) with the integral estimated by Equation (2.61), we conclude that they are the same order of magnitude when

$$\frac{\epsilon_{\beta m} \Delta\langle \mathbf{v}_\beta\rangle_m^\beta}{L_v^2} = 0\left[\frac{(1 - \epsilon_{\beta m})^{1/3}\langle \mathbf{v}_\beta\rangle_m^\beta}{l_\beta^2}\right] . \tag{2.68}$$

One can use the length-scale relation given by Equation (2.55) to arrange the above result in the form

$$\epsilon_{\beta m}\Delta\langle \mathbf{v}_\beta\rangle_m^\beta \left(\frac{l_\beta}{L_v}\right) = 0\left[\left(\frac{d_p}{L_v}\right)\langle \mathbf{v}_\beta\rangle_m^\beta\right] . \tag{2.69}$$

Since $\epsilon_{\beta m}$ is order one, we need only assume that

$$\Delta\langle \mathbf{v}_\beta\rangle_m^\beta = 0(\langle \mathbf{v}_\beta\rangle_m^\beta) \tag{2.70}$$

in order to obtain

$$\left(\frac{l_\beta}{L_v}\right)^3 = 0\left(\frac{d_p}{L_v}\right) . \tag{2.71}$$

The correspondence with the work of Levy (1983) simply requires that we interpret L in Equation (2.67) as the characteristic length associated with the velocity field,

L_v. Levy (1983) also discusses the delicate problem of two-dimensional porous media, such as fibrous filters, and in that case the following relation is obtained

$$\left(\frac{l_\beta}{L_v}\right) \ln\left(\frac{l_\beta}{L_v}\right) = 0(1). \tag{2.72}$$

In Part III we will prove that the integral in Equation (2.65) can be expressed as

$$\begin{aligned} &\epsilon_\beta^{-2} \int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})[\mathbf{n}_{\beta\sigma} \cdot (-\mathsf{I}\tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta)\delta_{\beta\sigma}]\, \mathrm{d}V_r \\ &\quad = -\mu_\beta \mathsf{K}_m^{-1} \cdot \langle \mathbf{v}_\beta \rangle_m \end{aligned} \tag{2.73}$$

and this will lead to Darcy's law and the Brinkman correction as given by

$$\langle \mathbf{v}_\beta \rangle_m = -\frac{1}{\mu_\beta} \mathsf{K}_m \cdot (\nabla \langle p_\beta \rangle_m^\beta - \rho_\beta \mathbf{g}) + \mathsf{K}_m \cdot \nabla^2 \langle \mathbf{v}_\beta \rangle_m^\beta. \tag{2.74}$$

Often one finds the Brinkman correction written in terms of the *superficial average* velocity instead of $\langle \mathbf{v}_\beta \rangle_m^\beta$ as we see in Equation (2.74). Because the Brinkman correction is only valid for $(1 - \epsilon_{\beta m}) \ll 1$, this error is of no consequence since the two velocities are essentially equal as indicated by Equation (2.49).

3. Conclusions

The process of spatial smoothing in terms of weighted averages has been examined for single-phase flow in rigid porous media. The point at which media-specific weighting functions must be introduced in the analysis has been identified, and we have shown that the cellular average is required in order to produce spatially smoothed equations for ordered media. The weighting function that produces the cellular average is a triangular-shaped, C° function. This can be made C^∞ (without changing the essential features of the cellular average) by means of a general weighting function, m_g, that removes the *corners* from the cellular weighting function, m_{C}.

Appendix A: Spatial Averaging Theorem

Our objective in this appendix is to derive the spatial averaging theorem when the weighted average is defined by a convolution product. We consider the distribution ψ defined by Equation (1.6), and note that the weighting superficial average, $\langle \psi \rangle_m$, is given by the convolution product

$$\langle \psi \rangle_m = m * \psi_\beta. \tag{A.1}$$

The mathematical background associated with our definition of ψ can be found in discussions of distribution theory (Schwartz, 1978; for a nontechnical introduction, see Richards and Youn, 1990). The theory of distributions provides theorems that can be used to derive the spatial averaging theorem in a straightforward manner and this has already been done by Marle (1967). A key theorem in the theory of distributions is (Boccara, 1990, Chap. 4)

$$\nabla(m * \psi) = (\nabla m) * \psi = m * (\nabla \psi). \tag{A.2}$$

The gradient of ψ is given by

$$\nabla \psi = (\nabla \psi)^{\mathrm{u}} - \mathbf{n}_{\beta\sigma}(\psi_\sigma - \psi_\beta)\delta_{\beta\sigma} \tag{A.3}$$

and use of this result in the theorem given by Equation (A.2) leads to

$$m * (\nabla \psi)^{\mathrm{u}} = \nabla(m * \psi) + m * [\mathbf{n}_{\beta\sigma}(\psi_\beta - \psi_\sigma)\delta_{\beta\sigma}] \tag{A.4}$$

which was given in Part I by Equation (2.22). For the special case in which ψ_σ is zero we obtain

$$m * (\nabla \psi)^{\mathrm{u}} = \nabla(m * \psi) + m * (\mathbf{n}_{\beta\sigma}\psi_\beta\delta_{\beta\sigma}). \tag{A.5}$$

In our analysis of the continuity equation and the Stokes equations, we have not used the unambiguous notation of Equation (1.6) but the ambiguous notation of Equations (1.21) and (1.22). This leads to averaging theorems for $\mathbf{v}_\beta$ and p_β of the form

$$m * (\nabla \mathbf{v}_\beta) = \nabla(m * \mathbf{v}_\beta) + m * (\mathbf{n}_{\beta\sigma}\mathbf{v}_\beta\delta_{\beta\sigma}), \tag{A.6}$$

$$m * (\nabla p_\beta) = \nabla(m * p_\beta) + m * (\mathbf{n}_{\beta\sigma}p_\beta\delta_{\beta\sigma}). \tag{A.7}$$

The advantage of this notation is that it facilitates the correspondences between convolution products and the traditional averages. For example, it produces the following results for the velocity

$$m * \mathbf{v}_\beta = \langle \mathbf{v}_\beta \rangle_m, \tag{A.8a}$$

$$\langle \mathbf{v}_\beta \rangle_m = \langle \mathbf{v}_\beta \rangle, \quad m = m_V. \tag{A.8b}$$

When m is given by Equation (1.7), the generalized averaging theorem given by Equation (A.4) reduces to the theorem derived by Whitaker (1967), and when $\psi_\sigma = 0$ and $m = -m_V$ it reduces to the form presented independently by Slattery (1967). Equation A.4 corresponds exactly to the result derived by Marle (1967) and when $\psi_\sigma = 0$ it provides the theorem developed independently by Anderson and Jackson (1967).

Appendix B: Geometrical Theorems

In this appendix we wish to demonstrate the following theorem

$$\int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})\delta_{\beta\sigma}(\mathbf{r})\mathbf{n}_{\beta\sigma}(\mathbf{r})[(\mathbf{r}-\mathbf{x})\ldots n\text{-times}\ldots(\mathbf{r}-\mathbf{x})]\,\mathrm{d}V_r$$
$$= -\nabla\{m * [\gamma_\beta(\mathbf{y}\ldots n\text{-times}\ldots\mathbf{y})]\} \tag{B.1}$$

which is essential to the process of removing average quantities from area integrals and thus reducing nonlocal transport equation to local transport equations. The first of the n theorems represented by Equation (B.1) has been known for some time (Carbonell and Whitaker, Sec. 2, 1984) and in terms of traditional volume averaging it takes the form

$$\frac{1}{\mathcal{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\,\mathrm{d}A = -\nabla\epsilon_\beta. \tag{B.2}$$

The value of Equations (B.1) and (B.2) is that they allow one to evaluate volume integrals instead of area integrals and this is a key point concerning the geometrical results presented in Parts IV and V.

The convolution product on the right-hand side of Equation (B.1) can be expressed explicitly as

$$m * [\gamma_\beta(\mathbf{y}\ldots n\text{-times}\ldots\mathbf{y})]$$
$$= \int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})\gamma_\beta(\mathbf{r})[(\mathbf{r}-\mathbf{x})\ldots n\text{-times}\ldots(\mathbf{r}-\mathbf{x})]\,\mathrm{d}V_r \tag{B.3}$$

and the change of variable

$$\mathbf{u} = \mathbf{r}-\mathbf{x} \tag{B.4}$$

leads to

$$m * [\gamma_\beta(\mathbf{y}\ldots n\text{-times}\ldots\mathbf{y})]$$
$$= \int_{\mathbb{R}^3} m(-\mathbf{u})\gamma_\beta(\mathbf{x}+\mathbf{u})[\mathbf{u}\ldots n\text{-times}\ldots\mathbf{u}]\,\mathrm{d}V_u. \tag{B.5}$$

Keeping in mind that the gradient operator on the right-hand side of Equation (B.1) represents the derivative with respect to $\mathbf{x}$, we form the gradient of Equation (B.5) to obtain

$$\frac{\partial}{\partial\mathbf{x}}\{m * [\gamma_\beta(\mathbf{y}\ldots n\text{-times}\ldots\mathbf{y})]\}$$
$$= \int_{\mathbb{R}^3} m(-\mathbf{u})\left[\frac{\partial\gamma_\beta(\mathbf{x}+\mathbf{u})}{\partial\mathbf{x}}\right][\mathbf{u}\ldots n\text{-times}\ldots\mathbf{u}]\,\mathrm{d}V_u. \tag{B.6}$$

Here we remind the reader that γ_β is a distribution and its derivative is taken as indicated by Equation (1.12). From Equation (B.4) we have that $\mathbf{r} = \mathbf{x} + \mathbf{u}$ and this yields

$$\frac{\partial \gamma_\beta(\mathbf{x}+\mathbf{u})}{\partial \mathbf{x}} = \frac{\partial \gamma_\beta(\mathbf{r})}{\partial \mathbf{r}} \tag{B.7}$$

leading us to express Equation (B.6) as

$$\frac{\partial}{\partial \mathbf{x}}\{m * [\gamma_\beta(\mathbf{y} \ldots n\text{-times} \ldots \mathbf{y})]\}$$

$$= \int_{\mathbb{R}^3} m(-\mathbf{u}) \left[\left(\frac{\partial \gamma_\beta}{\partial \mathbf{r}}\right)_{\mathbf{r}=\mathbf{x}+\mathbf{u}}\right] [\mathbf{u} \ldots n\text{-times} \ldots \mathbf{u}]\, \mathrm{d}V_u. \tag{B.8}$$

We again use Equation (B.4) to arrive at

$$\frac{\partial}{\partial \mathbf{x}}\{m * [\gamma_\beta(\mathbf{y} \ldots n\text{-times} \ldots \mathbf{y})]\}$$

$$= \int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r}) \left(\tfrac{\partial \gamma_\beta}{\partial \mathbf{r}}\right) [(\mathbf{r}-\mathbf{x}) \ldots n\text{-times} \ldots (\mathbf{r}-\mathbf{x})\, \mathrm{d}V_r]. \tag{B.9}$$

This is generally written as

$$\nabla\{m * [\gamma_\beta(\mathbf{y} \ldots n\text{-times} \ldots \mathbf{y})]\}$$

$$= \int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})(\nabla\gamma_\beta)[(\mathbf{r}-\mathbf{x}) \ldots n\text{-times} \ldots (\mathbf{r}-\mathbf{x})]\, \mathrm{d}V_r \tag{B.10}$$

where it is understood that the gradient operator inside the integral represents a derivative with respect to $\mathbf{r}$, whereas outside the integral ∇ represents a derivative with respect to $\mathbf{x}$.

At this point we make use of Equation (1.12) for the indicator function and that yields

$$\nabla\gamma_\beta = -\mathbf{n}_{\beta\sigma}\delta_{\beta\sigma}. \tag{B.11}$$

Use of this result in Equation (B.10) gives us

$$\nabla\{m * [\gamma_\beta(\mathbf{y} \ldots n\text{-times} \ldots \mathbf{y})]\}$$

$$= -\int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})\delta_{\beta\sigma}(\mathbf{r})\mathbf{n}_{\beta\sigma}(\mathbf{r})$$

$$[(\mathbf{r}-\mathbf{x}) \ldots n\text{-times} \ldots (\mathbf{r}-\mathbf{x})]\, \mathrm{d}V_r \tag{B.12}$$

and the n theorems given by Equation (B.1) are proved. When $n = 0$ we have

$$\int_{\mathbb{R}^3} m(\mathbf{x}-\mathbf{r})\mathbf{n}_{\beta\sigma}(\mathbf{r})\delta_{\beta\sigma}(\mathbf{r})\, \mathrm{d}V_r = -\nabla(m * \gamma_\beta) \tag{B.13}$$

which is the generalized form of Equation (B.2).

Appendix C: The Cellular Average

In our analysis of the hydrostatic pressure field in a spatially periodic porous medium, we found that the cellular average could be used to regularize the pressure. We suggested that the cellular average could be represented by a weighting function according to

$$\frac{1}{V_{\text{cell}}}\int_{V_{\text{cell}}}\langle p_\beta\rangle^\beta\,\mathrm{d}V = \frac{m_C * p_\beta}{m_V * \gamma_\beta}. \tag{C.1}$$

Here m_C represents the cellular average weighting function which is given by the double convolution

$$m_C = m_V * m_V \tag{C.2}$$

in which m_V is defined by Equation (2.26). In order to develop an explicit representation for m_C analogous to the one-dimensional version illustrated in Figure 6b, we consider a periodic system defined by the lattice vectors l_i, $i = 1, 2, 3$, and we impose the restriction that

$$l_i \cdot l_j = 0,\ i \neq j. \tag{C.3}$$

Introducing the notation

$$\mathrm{g}(x,\ l) = \begin{cases} 1, & |\,x\,| \leq l/2 \\ 0, & |\,x\,| > l/2 \end{cases} \tag{C.4}$$

allows us to express $m_V(\mathbf{x})$ as

$$m_V(\mathbf{x}) = \frac{1}{l_1 l_2 l_3}[\mathrm{g}(x_1,\ l_1)\mathrm{g}(x_2,\ l_2)\mathrm{g}(x_3,\ l_3)]. \tag{C.5}$$

From Equation (C.2) we have

$$m_C(\mathbf{x}) = \int_{\mathbb{R}^3} m_V(\mathbf{x}-\mathbf{r})m_V(\mathbf{r})\,\mathrm{d}V_r \tag{C.6}$$

which can be expressed as

$$\begin{aligned} m_C(\mathbf{x}) &= \frac{1}{l_1 l_2 l_3}\int_{\mathbb{R}^3} m_V(\mathbf{x}-\mathbf{r}) \\ &\quad [\mathrm{g}(r_1,\ l_1)\mathrm{g}(r_2,\ l_2)\mathrm{g}(r_3,\ l_3)]\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3. \end{aligned} \tag{C.7}$$

On the basis of Equation (C.4) we see that this result takes the form

$$\begin{aligned} m_C(\mathbf{x}) &= \frac{1}{l_1 l_2 l_3}\int_{-\frac{l_1}{2}}^{+\frac{l_1}{2}}\int_{-\frac{l_2}{2}}^{+\frac{l_2}{2}}\int_{-\frac{l_3}{2}}^{+\frac{l_3}{2}} \\ &\quad m_V(x_1-r_1,\ x_2-r_2,\ x_3-r_3)\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3. \end{aligned} \tag{C.8}$$

With the change of variable, $\mathbf{u} = \mathbf{x} - \mathbf{r}$, we can express Equation (C.8) as

$$m_{\mathrm{C}}(\mathbf{x}) = \frac{1}{l_1 l_2 l_3} \int_{x_1-\frac{l_1}{2}}^{x_1+\frac{l_1}{2}} \int_{x_2-\frac{l_2}{2}}^{x_2+\frac{l_2}{2}} \int_{x_3-\frac{l_3}{2}}^{x_3+\frac{l_3}{2}} m_V(u_1,\, u_2,\, u_3)\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3. \tag{C.9}$$

We now use Equation (C.5) leading to

$$m_{\mathrm{C}}(\mathbf{x}) = \frac{1}{l_1 l_2 l_3} \int_{x_1-\frac{l_1}{2}}^{x_1+\frac{l_1}{2}} \mathrm{g}(u_1,\, l_1)\, \mathrm{d}u_1 \int_{x_2-\frac{l_2}{2}}^{x_2+\frac{l_2}{2}} \mathrm{g}(u_2,\, l_2)\, \mathrm{d}u_2 \int_{x_3-\frac{l_3}{2}}^{x_3+\frac{l_3}{2}} \mathrm{g}(u_3,\, l_3)\, \mathrm{d}u_3. \tag{C.10}$$

Each integral in this result has the form given by

$$G(x,\, l) = \int_{x-\frac{l}{2}}^{x+\frac{l}{2}} \mathrm{g}(u,\, l)\, \mathrm{d}u \tag{C.11}$$

in which $G(x,\, \ell)$ takes the form

$$G(x,\, l) = \begin{cases} l + x, & -l \leq x < 0 \\ l - x, & 0 < x \leq l. \\ 0, & |\, x \,| > l \end{cases} \tag{C.2}$$

Use of this general representation in Equation (C.10) provides

$$m_{\mathrm{C}}(\mathbf{x}) = \frac{1}{l_1^2 l_2^2 l_3^2} [G(x_1,\, l_1) G(x_2,\, l_2) G(x_3,\, l_3)] \tag{C.13}$$

and the one-dimensional version of this result that is illustrated in Figure 3b can be written as

$$m_{\mathrm{C}}(\mathbf{x}) = \frac{1}{l_1^2} \begin{cases} l_1 + x, & -l_1 \leq x < 0 \\ l_1 - x, & 0 < x \leq l_1. \\ 0, & |\, x \,| > l_1 \end{cases} \tag{C.14}$$

A little thought will indicate that m_{C} satisfies the normalization condition indicated by Equation (2.7) and that $m_{\mathrm{C}} \in \mathrm{C}^{\circ}$.

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