



A century of fluid mechanics: 1870–1970 / Un siècle de mécanique des fluides : 1870–1970

On the developments of Darcy's law to include inertial and slip effects

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ABSTRACT

The empirical Darcy law describing flow in porous media, whose convincing theoretical justification was proposed almost 130 years after its original publication in 1856, has however been extended to account for particular flow conditions. This article reviews historical developments aimed at including inertial and slip effects (respectively, when the Reynolds and Knudsen numbers are not exceedingly small compared to unity). Despite the early empirical extensions to include inertia and slip effects, it is striking to observe that clear formal derivations of physical models to account for these effects were reported only recently.

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1. Introduction

Flows in porous media are of interest in numerous applications ranging from hydrology, hydrocarbon recovery, gas and nuclear waste storage, to drying of wood, transfer in food products or in living tissues to cite but a few. The main characteristic of this particular domain of fluid mechanics lies in the (sometimes extreme) complexity of the geometry of the channels where the flow takes place. Additionally, in many situations, this geometry is unknown in its very details and may vary over more or less long distances characteristics of heterogeneities. Within this context, the physical description of the flow in such materials may appear to be a tremendous challenge.¹ This certainly explains why empiricism remained so strong and lasted longer than in many other fields of fluid mechanics. In many situations, the interest is not in the details of the flow within the pores but rather in the flux-to-force governing laws at length scales including a large numbers of pores, although comprehensive analyses at the pore scale remain the corner stone in any progress towards the derivation of governing laws at larger scales. Clearly, active research in the description of transfer in porous materials was triggered by the publication of Darcy's law in the middle of the 19th century and the emergence of a key macroscopic physical characteristic of a porous medium, namely the ability of a fluid to flow through it, *i.e.* its permeability.

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¹ The one-phase slow flow is probably one of the simplest mechanism one can think about and there is a lot of other tremendously more complex physical processes in porous media of relevance from both scientific and industrial points of view, including multiphase flows, compressible flows, phase change, deformable porous media, reactions in multicomponent systems, etc.

1.1. Darcy's law as an empirical basis

Ever since its empirical formulation in 1856, Darcy's law [1] has been a hallmark in modeling momentum transport through porous media. In this classical publication, there is a section entitled *Détermination des lois d'écoulement de l'eau à travers le sable*, dedicated to the study of water flows through a bed of sand where the following relation is proposed (see page 594 in [1]):

$$q = k \frac{s}{e} (h + e) \quad (1)$$

where q is the volumetric flow rate, s is the cross section of the sand bed, e is the bed width, h is the pressure (or head) difference between the surface and the base of the sand bed, and finally k was proposed as a coefficient that depends on the permeability of the bed² and on the properties of the fluid. For an excellent review about the origin of Darcy's law, the interested reader is referred to the work by Zerner [3]. The use of this simple relation requires that the only resistance to the flow through the porous medium is due to viscous stresses induced by an isothermal, creeping (or laminar) steady flow of a Newtonian fluid within an inert, rigid and homogeneous porous medium. However, the lack of a rigorous upscaling technique prevented a formal derivation of this equation until the late 1960s, as it will be detailed later.

For a very long period of time – around fifty years – this law has been essentially verified experimentally in its global form, but was not considered in a local differential form nor derived on a theoretical basis. One finds a differential expression in the analysis of the flows in aquifers by J. Boussinesq [4] as a result of an analogy with heat transfer in a continuum. This work also reports an extension of the flow-rate-to-head-gradient relationship to non-homogeneous media. A formal derivation of a 1D local expression of this law obtained from the solution to the Stokes equation for a flow parallel to a regular array of infinite parallel cylinders (sufficiently apart from each other, *i.e.* for relatively large porosities) is due to Emersleben in 1925 [5]. A derivation mainly based on dimensional analysis was later proposed by Muskat and Botset in 1931 [2] for a compressible flow in which the pressure difference is recognized to be replaced by the difference of the squares of the pressures.

Substantial literature will then appear during the 1950s, in which many different approaches to demonstrate Darcy's law will be tested (see for instance [6] and references therein). Although these articles helped progressing into the understanding of the applicability of Darcy's law, almost all of them relied on analogies, hypotheses or postulates that left them incomplete. The first extension to three-dimensions and to non-isotropic materials was reported by Hall [7], who introduced a *permeability tensor*, which is also based on some pre-requisites (see in particular Eq. (17) therein and the way the permeability is identified).

Despite the lack of formal derivation of Darcy's law, which can be expressed for a 1D flow in the x -direction as [8]

$$q = - \frac{Ks}{\mu} \frac{\partial \langle p_\beta \rangle^\beta}{\partial x} \quad (2)$$

the meaning of the permeability and its relationship to the underlying pore structure focused closed attention. In the above expression, K is the permeability having units of m^2 and μ the fluid viscosity. In addition, $\langle p_\beta \rangle^\beta$ is the intrinsically-averaged pressure in the porous medium, defined as:

$$\langle p_\beta \rangle^\beta = \frac{1}{V_\beta} \int_{\mathcal{V}_\beta} p_\beta \, dV \quad (3)$$

Here, \mathcal{V}_β is the domain (of volume V_β) occupied by the fluid phase β within a representative averaging domain (or REV) (see Fig. 1), and p_β is the pore-scale pressure.

An early estimate of K was inspired by an analysis due to Blake in 1922 [9] of flow over packings of grains of different shapes and a comparison to flows in capillary tubes that resulted in the following estimate

$$K = \frac{1}{k_0 S_0^2} \frac{\varepsilon^3}{(1 - \varepsilon)^2} \quad (4)$$

where ε is the porous medium porosity, S_0 denotes the specific surface of the particles and it is defined as the ratio of the area of the particle to its volume. The coefficient S_0 , related to the effective particle diameter, d_p , was identified from an analogy with spherical particles by

$$S_0 = \frac{6}{d_p} \quad (5)$$

² In [1], H. Darcy indicated that k “depends on the permeability of the sand layer”. In fact, k is the hydraulic conductivity, having the unit of a length per unit time. The intrinsic permeability as a physical quantity, denoted by K (or \mathbf{K} in tensorial form) in the present article, appeared later in the literature. It seems that M. Muskat (see for instance [2]) was the first who used this coefficient.

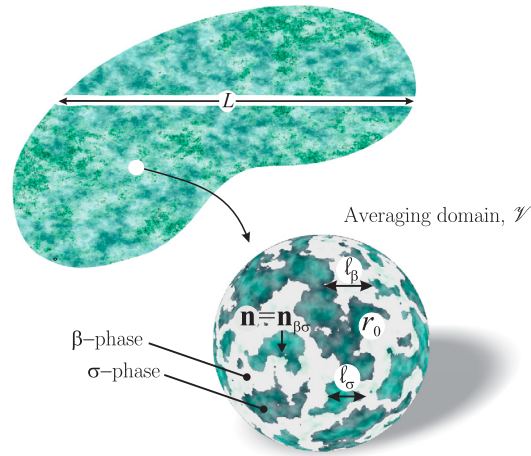


Fig. 1. Sketch of a porous medium including an averaging domain and the phases involved.

Finally, k_0 is an adjustable coefficient that was later known as the Kozeny [10] coefficient. Carman [11] suggested taking $k_0 = 5$, so that equation (4) can now be expressed as

$$K = \frac{d_p^2 \varepsilon^3}{180(1 - \varepsilon)^2} \quad (6)$$

which is usually known as the Kozeny–Carman equation and which is often used, sometimes abusively, to predict the value of permeability. Certainly, a precise correlation for any type of pore structure is out of reach.

1.2. Theoretical foundation of Darcy's law

The lack of convincing formal derivations of Darcy's law that lasted over more than a century is obviously related to the lack of clear upscaling methods allowing one to obtain the *macroscopic* conservation equations from their *microscopic* (i.e. pore-scale) counterparts, as pointed out by Zerner [3]. These methods emerged in the 1970s and a first attempt was proposed by S. Whitaker in 1966 [12], who clearly obtained a generalization of Darcy's law in the following vectorial form:

$$\langle \mathbf{v}_\beta \rangle = -\frac{\mathbf{K}}{\mu} \cdot (\nabla \langle p_\beta \rangle^\beta - \rho \mathbf{g}) \quad (7)$$

where ρ is the fluid density, \mathbf{g} is the gravity acceleration, \mathbf{K} is the permeability tensor and $\langle \mathbf{v}_\beta \rangle$ the seepage velocity, which is defined as the superficial average:

$$\langle \mathbf{v}_\beta \rangle = \frac{1}{V} \int_{V_\beta} \mathbf{v}_\beta \, dV \quad (8)$$

with V being the volume of the REV and \mathbf{v}_β the pore-scale velocity vector. The same result was achieved exactly in the same period by C. Marle [13]. However, in these articles, no clear structural link (i.e. a *closure*) is provided between the micro- and the macroscale allowing one to infer the dependence of the permeability upon the pore-scale geometry. It was not earlier than in the middle of the 1980s that a *closed* rigorous form was achieved by Whitaker [14] using the volume averaging method [15], which included an intrinsic closure scheme allowing one to predict the values of the components of the permeability tensor \mathbf{K} . Some derivations of Darcy's law have used other upscaling techniques such as homogenization [16]. These and other techniques depart from the governing equations at the pore scale and, after the application of an averaging operator (such as the one sketched in Fig. 1) and many mathematical steps, lead to an upscaled model in terms of effective-medium coefficients that capture the essential (i.e. non-redundant) information from the pore scale. In this way, the permeability tensor can be viewed as a signature of the porous medium topology at a scale that is larger than the pore scale.

Over the past century, there have been some modifications brought to Darcy's law that extended its applicability to more complicated transport processes than those originally considered by Darcy. Among the extensions to Darcy's law, a non-exhaustive list should include: 1) Forchheimer's modification to allow for the study of non-creeping flow regimes [17]; 2) Brinkman's correction to include macroscopic viscous stress by introducing an effective viscosity coefficient [18]; 3) Klinkenberg's modification of the permeability tensor to study gas slip flows in porous media [19]; 4) application to heterogeneous media by means of large-scale volume averaging [20]; 5) non-isothermal flow of non-Newtonian fluids in porous media (cf. [21] for instance).

In what follows, the focus is laid upon two extensions to Darcy's law that are of major importance, namely the *inertial one-phase flow* and the *gas slip flow* in homogeneous porous media. Our aim is to carefully review these extensions and draw some conclusions and perspectives on these subjects.

2. Inertial one-phase flow in porous media

The analogy with flows in ducts has been widely used in the derivation of empirical flow models in porous media and might have been a source of inspiration for H. Darcy to obtain the filtration law he reported in his book [1]. As mentioned in a remarkable analysis by Zerner [3], H. Darcy dedicated specific experiments to verify Poiseuille's law in the context of slow flow. One of the major motivations was his questioning of the validity of the relationship between the "pressure drop" ΔP and the average velocity u in a tube of length L , which was then of common use, *i.e.*

$$\frac{\Delta P}{L} = au + bu^2 \quad (9)$$

a relationship mainly due to du Buat and Gaspard Riche de Prony (see [3]), with a and b being coefficients that had to be determined experimentally.

2.1. Forchheimer's correction to Darcy's law

It is striking to observe that the form of Eq. (9) corresponds to the relationship proposed by Forchheimer [17] to account for "rapid" flows in porous media with the classical analogy on u taken as the filtration or seepage velocity in 1D:

$$\frac{\Delta P}{L} = \frac{\mu}{K}u + \rho\beta u^2 \quad (10)$$

where ρ is the fluid density and β is the Forchheimer coefficient, also called the coefficient of inertial resistance or inertial resistance factor.

The quadratic correction introduced by Forchheimer about half a century after the empirical validation and publication of Darcy's law was obviously inspired from Eq. (9), despite this remarkable time lapse between Forchheimer's publication and that of Darcy. In 1931, Muskat and Botset [2] reported experimental results of gas flow through different types of porous materials, in which they observed that the gradient of the square of the pressure was proportional to a power of the mass velocity ranging between 1 and 2 (1 when the flow was "completely viscous", and 2 when it was "completely turbulent"). This empirical form of Eqs. (9) and (10) was accepted to account for inertial flows in porous media over an additional half-century during which only few alternative forms were put forth, like for instance [22]

$$\frac{\Delta P}{L} = au + bu^2 + cu^3 \quad (11)$$

with a , b and c being adjustable coefficients. During the fifty years following Forchheimer's publication, some confusion remained about the physical origin of the quadratic correction to Darcy's law as it was often attributed to turbulence, although some references made clear statements on that point [6]. In fact, Irmay [6] argued that there is no reason, in general, to expect a linear solution to the non-linear Navier–Stokes equations. The Poiseuille solution in straight tubes is an exception caused by vanishing curvature, which is not the case in real porous media. Agreement has been quite unanimous on the threshold value of the Reynolds number at which the correction becomes significant, *i.e.* for $1 \leq Re_d \leq 15$, where $Re_d = \frac{\rho \langle v_\beta \rangle d_g}{\mu}$ is based on the filtration velocity $\langle v_\beta \rangle$ and the typical grain size d_g of the porous material. However, it was not before 1962 and the publication by Chauveteau and Thirriot [23], in which a flow regime classification was proposed, that turbulence in this range of Reynolds numbers was dismissed. Turbulence has been confirmed to typically arise for $Re_d \sim 100$ [24–26].

During this period, and even up to the end of the 1970s, comparisons with experimental results were reported, showing quite good agreement for various types of porous structures, including packed beds of grains, bundles of capillary tubes or fibrous media for flows of gases or liquids [27,28]. From a practical point of view, the Forchheimer model has been used for applications in hydrology, petroleum and chemical engineering [29,30]. From a theoretical point of view, the same period was marked by various attempts to justify the form of Eq. (9). This was carried out on the basis of different approaches ranging from simplified derivations from the Navier–Stokes equations or analogies with pressure drop through capillary orifices [6]. The emergence of more formal upscaling techniques during the 1970s has led to further developments in the following years that also attempted to justify the quadratic form of the inertial correction to Darcy's law [13,31–33].

2.2. Refinements on the inertial correction to Darcy's law

Impulsed by the development of both computational resources and numerical methods, the velocity dependence of the corrective term on Darcy's law regained much attention from the early 1990s on. It was during this period that the quadratic velocity dependence of this correction was questioned and analyzed in depth. Numerical simulations carried out

through networks of parallel cylinders of circular cross-section for a flow orthogonal to the cylinders showed that the correction scales as a 3rd power of the filtration velocity [34] instead of a quadratic term. This behavior was theoretically confirmed quasi simultaneously, regardless the geometry at the pore-scale, from formal derivations based on a rigorous upscaling procedure using double-scale homogenization with a closure process involving periodic representations of the porous medium [35–37]. The exponent 3 was confirmed for a Reynolds number, Re_p , based on the characteristic pore size ranging between $\delta^{1/2}$ and 1, where δ is the micro-to-macroscale ratio. This result was further emphasized later on [38] and, from its original evidence, led to identify this regime as the so-called “*weak inertia regime*”, formalized for homogeneous isotropic and periodic media. Additional numerical works over larger intervals of the Reynolds number for flows in many different structures confirmed the existence of weak inertia and extended the classification of flows under (at least) three distinct regimes with crossovers, namely [39] i) the weak inertia regime occurring at the onset of non-linearity in the flow-rate-to-pressure-drop relationship for $\delta^{1/2} \leq Re_p \leq 1$; ii) the “*strong inertia regime*” characterized by a correction to Darcy’s law that scales as the square of the filtration velocity, i.e. leading to a Forchheimer type of model for Reynolds numbers in the range 1 to 10; iii) the turbulent regime appearing for Reynolds numbers typically of the order of 100.

Nevertheless, a detailed physical explanation of a cubic or quadratic correction to Darcy’s law in order to account for inertia still leaves much to be desired and remains a widely open question [40], while the description of the non-linearity is essentially qualitative. Obviously, inertial *macroscopic* forces cannot be invoked as they remain negligible compared to viscous forces [41], and this can be proved to hold as long as $Re_p \ll \delta^{-1}$. Consequently, the nature of the non-linearity in the relationship between the macroscopic drag force and filtration velocity must certainly be explained from the signature of viscous and inertial forces at the microscale. On the one hand, several mechanisms suggesting that inertia alone, at the pore-scale, can explain the macroscale behavior may be put forth such as: i) streamlines bending due to the tortuosity of the structure and to local converging-diverging flow patterns; ii) backflows and separations resulting from form drag; iii) pore networks actively involved in the flow that are velocity-dependent as a consequence of i) and ii) yielding variations in the dissipation of the kinetic energy [42]. On the other hand, microscale viscous drag effects may be considered to contribute to the non-linearity when boundary layers at the solid–fluid interfaces, which become thicker when the Reynolds number increases (see an experimental observation in [24]), are taken into account. With this mechanism, the inertial core flow (in the center of the pores) may be easily understood as being strongly dependent upon the Reynolds number, partly explaining the different regimes.

2.3. Further developments

Although the existence of the two regimes (weak and strong inertia) has been widely accepted, a lot still needs to be understood regarding the universal existence and dependence of these regimes (and the associated crossovers) upon many parameters such as porosity, structural order, anisotropy, etc. In addition, most of the experimental or numerical characterizations of the macroscale inertial correction have been carried out in 1D in a scalar form although many references pointed out that tensorial coefficients must be involved [31,36,41,32]. Even if the development did not allow for the formal identification of the above-mentioned regimes, a convincing derivation, relying on rigorous upscaling, of a macroscopic model for momentum transport of one-phase flowing in homogeneous porous media with inertia is certainly due to Whitaker [41]. This model, subject to time ($\mu t^* \gg \rho \ell_\beta^2$), length-scales ($\ell_\beta \ll r_0 \ll L$) and Reynolds number ($\frac{\rho \langle \mathbf{v}_\beta \rangle \ell_\beta}{\mu} \left(\frac{\ell_\beta}{L} \right) \ll 1$) constraints that were clearly formulated, provides the macroscale momentum equation, which reads

$$\langle \mathbf{v}_\beta \rangle = -\frac{\mathbf{H}}{\mu} \cdot (\nabla \langle p_\beta \rangle^\beta - \rho \mathbf{g}) \quad (12)$$

or equivalently

$$\langle \mathbf{v}_\beta \rangle = -\frac{\mathbf{K}}{\mu} \cdot (\nabla \langle p_\beta \rangle^\beta - \rho \mathbf{g}) - \mathbf{F} \cdot \langle \mathbf{v}_\beta \rangle \quad (13)$$

where $\langle \mathbf{v}_\beta \rangle$ is the filtration (or seepage) velocity as defined in Eq. (8), \mathbf{H} is the apparent permeability, \mathbf{K} the intrinsic permeability and \mathbf{F} the inertial correction tensor (which is a function of the filtration velocity); $\nabla \langle p_\beta \rangle^\beta$ is the macroscopic pressure gradient, $\langle p_\beta \rangle^\beta$ being defined in Eq. (3). Details on the averaging method employed to obtain this result are given in [15]. In addition to the macroscopic model, the upscaling provides the means to determine the associated macroscopic coefficients (\mathbf{K} , \mathbf{H} and \mathbf{F} contained in Eqs. (12) and (13)) from the solution to the ancillary problems (so-called closure problems). Undoubtedly, while completing the physical description, this contribution concludes some previous derivations on the same problem performed with homogenization [36,40] and opens new perspectives to more in-depth investigation of the inertial correction to Darcy’s law.

In this spirit, closure problems were solved in order to compute \mathbf{H} and \mathbf{F} for different ordered and disordered model structures so as to investigate the existence of the regimes and their dependence upon the porosity and the microscale pore structure [43]. The main important conclusions deriving from this analysis can be summarized as follows. The two tensors \mathbf{H} and \mathbf{F} are generally dense and non-symmetric for ordered structures, even if the medium is isotropic at the macroscale in the Darcy regime. The non-symmetry coincides with a macroscopic drag force, which is not necessarily aligned with the macroscopic velocity. Symmetry is recovered for specific pressure gradient orientations along symmetry axes of the structure

(when present). Dissymmetry of these tensors decreases when structural disorder increases. Whatever the structure under concern, the weak inertia regime (a cubic inertial correction to Darcy's law) is always observed. For ordered structures, the strong inertia regime (*i.e.* a quadratic (Forchheimer) correction to Darcy's law) does not necessarily exist and it is otherwise restricted to a narrow interval of the Reynolds number. For disordered structures, the Forchheimer type of correction is a robust approximation over a very significant range of values of the Reynolds number. Moreover, the crossover value of the Reynolds number at which the quadratic correction becomes relevant decreases with increasing structural disorder, the weak inertia regime being restricted to a small range of Reynolds numbers where the correction is not very significant. This certainly explains why this regime is most of the time overlooked in experimental investigations on porous media having a random pore structure. In any situation, using a model such as that proposed in Eq. (9) implies that the permeability in the linear term in the filtration velocity differs from the intrinsic permeability. Even if some progress has been achieved in the physical explanation and the theoretical derivation of formal models to account for inertia effects for one-phase flows in porous media, much work remains to be done to understand fully the non-linearities associated with this type of process. For instance, most of the analyses so far were dedicated to the laminar steady regime and the occurrence of unsteadiness remains widely unexplored (this problem was barely outlined recently in [44]), as well as the turbulent regime; however, further discussion of these topics is out of the scope of the present review.

3. Slip flow in porous media

Gas flows in porous media differ considerably from liquid-phase flows, in particular for situations in which the pore sizes are comparable to the mean free path of the gas molecules. This is the case in many practical applications including micro- and nano-fluidic systems such as MEMS and nano-porous media, transport in fibrous media, gas flow during soil remediation, long-term nuclear waste disposal, among many others. Due to the current high relevance of this type of transport, the evolution from the pioneering works performed in the nineteenth century to some of the current developments are briefly summarized in this section.

3.1. Slip flow background

A one-phase flow in confined systems of dimensions comparable to the mean free path leads to rarefaction effects that give rise to many interesting contributions in transport phenomena. In his classical study of the stresses that rarefied gases experience, J.C. Maxwell [45] proposed that, close to the surface of a solid, there should be a sliding of the gas in contact with the solid in the direction of a tangential stress. Maxwell proposed that the velocity should be proportional to the tangential stress and inversely proportional to the viscosity of the fluid. Under isothermal conditions, the sliding velocity for a 1D flow in the x -direction is expressed as:

$$v = G \frac{dv}{dx} \quad (14)$$

where the coefficient G is the coefficient of slipping, defined as

$$G = \frac{2}{3} \left(\frac{2}{f} - 1 \right) \lambda_\beta \quad (15)$$

with λ_β being the mean free path and f the fraction of gas molecules that are diffusely scattered at the surface. Hence, if the solid surface is wholly absorbent, $G = 2/3\lambda_\beta$. The proposal from Maxwell is consistent with a previous study by Navier [46], and we shall thus refer to Eq. (14) as the Navier–Maxwell equation. It should be noted that Maxwell obtained Eq. (14) considering a single-component gas; the extension of this equation to multicomponent mixtures is found in Jackson [47].

During the last quarter of the nineteenth century, rarefaction effects were shown to increase significantly the flow rate with respect to that predicted from Poiseuille's law [48]. This motivated several investigations during the early years of the twentieth century, in particular those by M. Knudsen. The scattering of gas molecules from solid walls was fundamental in Knudsen's theory [49]. Knudsen studied molecular flows in tubes and determined the dependence on tube dimensions. He discussed the transition from Poiseuille's flow in terms of the ratio of the mean free path of the gas molecules to the characteristic size of the apparatus (say, ℓ_β). The reason for considering this ratio is due to the fact that, in the continuum approach, the slip velocity may be understood as the average flow velocity of the molecules at a distance from the wall that is equal to the mean free path [50]. Hence, as the mean free path becomes a bigger fraction of the tube diameter, the slip velocity increases with respect to the bulk velocity. This important ratio between the mean free path and the tube diameter is nowadays known as the Knudsen number in his honor (*i.e.* $Kn = \lambda_\beta/\ell_\beta$). In the comprehensive review by Steckelmacher [51], the historical development and relevance of Knudsen's works are presented in detail.

Later on, Adzumi [52–54], published a series of papers dedicated to the study of gas flow through capillaries. He considered three cases: 1) when λ_β is very small in comparison to the diameter of the capillary (*i.e.* $Kn \ll 1$); 2) when λ_β is large compared to the diameter (*i.e.* $Kn > 1$), and 3) when λ_β is comparable to the capillary diameter (*i.e.* $Kn \sim 1$). In the first case, he found Poiseuille's law to be quite suitable, and the flow is inversely proportional to the gas viscosity [52]. In the second case, the flow rate was found to be independent of the viscosity, but inversely proportional to the square root of the

gas molecular weight, M , [53]. The flow characteristics in the third case were found by Adzumi to be a combination of the two first ones [54]. Nowadays, there is some consensus that the following bounds are identifiable: for $Kn < 10^{-3}$, the laws of continuum mechanics are safely applicable, and non-slip can be assumed at the solid boundaries; for $10^{-3} < Kn < 10^{-1}$, the flow regime corresponds to slip flow, and the Navier–Maxwell equation must be considered at the solid–fluid interface; for $10^{-1} < Kn < 10$, there is a transition regime in which the laws of continuum mechanics are likely to fail because the continuum hypothesis is no longer satisfied. For porous media applications in the transition regime, Maxwell [55] proposed that the action of the porous material over the gas was similar to a number of *dust* particles of the moving system, hence giving rise to the well-known *dusty gas model*. This model has the nice feature that the interactions of gas molecules with the dust molecules simulate their interaction with the rigid porous matrix, thus avoiding the problem of flux variations across the sections of the pores [47]. Finally, for $Kn > 10$ the gas kinetic theory must be considered because description in terms of particle–wall collision operators is required. This limiting situation is called molecular streaming or Knudsen flow, and it is characterized by the fact that the flow takes place by diffusion, instead of viscous, mechanisms [50]. Here, the term diffusion means that the flow results from creep at the wall rather than from molecule-to-molecule collisions.

3.2. Pore-scale slip flow and its consequence on Darcy's law

Few years after Adzumi's works, in his study of the permeability of gas flows in porous media in the slip regime, Klinkenberg [19] found that this coefficient is almost a linear function of the reciprocal mean pressure, $\langle p_\beta \rangle^\beta$. Using an idealized porous medium representation consisting in an array of capillaries, Klinkenberg was able to deduce the following relation between the apparent permeability, K_s , and the intrinsic permeability, K :

$$K_s = K \left(1 + \frac{b}{\langle p_\beta \rangle^\beta} \right) \quad (16)$$

with b being a surface- and gas-dependent constant. Evidently, at sufficiently large gas pressures, K_s approaches K . Klinkenberg used the above equation to predict the values of K_s/K in different experimental conditions with reliable accuracy, thus showing the need to consider slip effects in the determination of the permeability. In this way, the Darcy–Klinkenberg model is (gravity is omitted here):

$$\langle v_\beta \rangle = -\frac{K}{\mu} \left(1 + \frac{b}{\langle p_\beta \rangle^\beta} \right) \frac{\partial \langle p_\beta \rangle^\beta}{\partial x} \quad (17)$$

for an average one-dimensional flow in the x -direction, while μ is the fluid dynamic viscosity.

An important point of discussion in the recent literature related to gas transport in porous media is about the pertinence of the linear and first-order interfacial boundary condition given in Eq. (14). Shen et al. [56] derived a first-order slip condition from the Chapman–Enskog solution to the Boltzmann equation. This approach consists in linearizing the Boltzmann equation using a perturbation expansion for the probability function in terms of the Knudsen number. The resulting condition includes an additional term due to the pressure gradient along the flow's direction. The success of first-order models has been argued to be constrained to slightly rarefied gas flows by Deissler [57]. According to this author, as the pressure in the gas becomes smaller, the velocity profiles may be nonlinear over a distance from the solid surface corresponding to the mean free path and the jumps at the interface may be expected to be functions of higher-order normal and tangential derivatives. Deissler thus proposed to use a second-order boundary condition that matches the Navier–Stokes equations for slip flows, finding good agreement with experimental results. Another early proposal of second-order models is the scalar one by Cercignani [58] on the basis of the Bhatnagar–Gross–Krook (BGK) approximation of the Boltzmann equation. However, these extensions consist of modifications to the scalar equation (14), in other words, they are constrained to simple geometries where one-dimensional flow is applicable. Unfortunately, extensions to more complicated geometries are not straightforward and they remain a challenge. Furthermore, since the Navier–Stokes equations are first-order accurate in the Knudsen number, it is not easy to justify the use of higher-order boundary conditions. Hence, an alternative is to use higher order momentum transport equations, such as the Burnett equation [59], which are the result of keeping the second-order terms in the Chapman–Enskog approach to approximate the Boltzmann equation.

As mentioned above, one of the limitations of the Navier–Maxwell boundary condition is that it is restricted to one-dimensional flows. This limitation is also shared by the Klinkenberg model. To address this issue, Einzel et al. [60] proposed a generalized version of the slip boundary condition, which can be expressed as follows:

$$\mathbf{v}_\beta = - \underbrace{\left(\frac{2 - \sigma_v}{\sigma_v} \right)}_{\xi} \lambda_\beta \mathbf{n} \cdot \left(\nabla \mathbf{v}_\beta + \nabla \mathbf{v}_\beta^T \right) \cdot (\mathbf{I} - \mathbf{nn}) \quad (18)$$

where σ_v is the tangential-momentum accommodation coefficient, \mathbf{n} is the unit normal vector directed from the fluid toward the solid phase and \mathbf{I} is the identity tensor. The coefficient σ_v accounts for the average tangential momentum exchange between the molecules and the fluid, and can vary from zero to one.

3.3. Theoretical macroscopic slip flow models in porous media

In the same line of thoughts as those mentioned for inertial flows, the Klinkenberg model, although widely used in the literature, was formally derived only more than half a century after its publication. Using the homogenization method, Skjetne and Auriault [61] considered the steady-state, low-velocity Navier–Stokes equations for compressible flows in the slip regime, *i.e.* $Re \ll Kn \ll 1$ (for $Re = \mathbf{O}(\delta = \ell/L)$, with ℓ and L being the characteristic length scales at the microscale and macroscale, respectively). In this work, for the first time, a vectorial form of the Klinkenberg model was rigorously deduced for conditions in which both local compressibility and inertial effects are smaller than the wall-slip effects. The apparent permeability tensor was found to be positive-definite and non-symmetric, in general. This study was subsequently expanded by Chastanet et al. [62] to derive the corresponding upscaled model for low-pressure gas flows in dual-porosity media including fractures. Their study highlighted that the Knudsen number should be considered in addition to the separation of characteristic length scales in the system in order to assess the domains of validity of upscaled models for gas flow.

The derivation of the effective-medium equation corresponding to slightly-compressible slip-flow conditions using the volume averaging method has been carried out recently by Lasseux et al. [63]. This work completes the previous derivations by Skjetne and Auriault on the following points: 1) the compressibility effects are taken into account in the framework of slightly compressible flows restricted to small Reynolds numbers and small frequency number; 2) the vectorial form of the slip boundary condition is considered in its complete form, *i.e.* including the complete shear-rate at the fluid-solid interface as shown in Eq. (18); 3) it is derived for a barotropic fluid without any assumption on the equation of state of the gas. For an ideal gas, the resulting upscaled model is

$$\langle v_\beta \rangle = -\frac{1}{\mu} \mathbf{K} \cdot \left(\mathbf{I} + \frac{\xi \mu}{\langle p_\beta \rangle^\beta} \sqrt{\frac{\pi R \langle T_\beta \rangle^\beta}{2M}} \mathbf{S} \right) \quad (19)$$

which involves two tensors, namely, the intrinsic permeability tensor, \mathbf{K} , and a slip-flow correction tensor, \mathbf{S} . The parallelism between Eqs. (17) and (19) is obvious. The ancillary closure problem required to predict the values of the effective-medium coefficients was derived and formally solved for simple porous medium geometries in two- and three-dimensional unit cells. Furthermore, the dependence of s ($\mathbf{S} = s\mathbf{I}$) on K ($\mathbf{K} = K\mathbf{I}$) was found to obey a power-law relationship, with the value of the exponent depending on the geometrical configuration. In a subsequent work by Lasseux et al. [64], the slip correction was more accurately described by considering an expansion in the Knudsen number at the closure level. This leads to a reformulation of the closure problem as a differential (instead of an integro-differential) boundary-value problem, which was solved in more complicated unit cells in order to predict the apparent permeability tensor. Furthermore, with this expansion, the slip-flow correction tensor was shown to be the sum of slip corrections at the successive orders of the Knudsen number. The consideration of the complete form of the boundary condition at the solid–fluid interface was found to be crucial for the prediction of the slip corrections at the different orders of Kn . Their analysis evidenced a nonlinear relationship between the apparent permeability and the Knudsen number. This relationship motivates further theoretical and experimental research on the subject, in particular for highly porous structures.

4. Conclusions

This work has been dedicated to the analysis of the evolution of two major modifications to Darcy's law, namely the inclusion of inertial and slip effects. The review carried out for both extensions suggests the following conclusions and prospects.

- The empirical introduction of a quadratic correction in terms of the filtration velocity to Darcy's law by Forchheimer in 1901 for 1D flows was certainly inspired by a model that was commonly used for flows in pipes prior to the publication of Darcy's law. This correction has been widely supported by empiricism for more than 90 years, a period after which a first theoretical 3D model was achieved by upscaling (homogenization), showing that the onset of deviation from Darcy's law due to inertia involves a correction that rather scales as a 3rd power of the filtration velocity in a weak-inertia regime. For larger Reynolds number values, a so-called strong inertia regime, where the Forchheimer correction is due to hold, was accepted. An upscaled 3D complete model, obtained by volume averaging five years later, was used to highlight the fact that, if the weak inertia regime always exists, the quadratic correction does not hold in some particular situations of pore-scale ordered structures and that, however, in the presence of disorder, the Forchheimer-type correction is a robust one. At this point, a quite different observation from that indicated below for slip flows may be pointed out regarding corrections made to Darcy's law in order to include inertial effects. Indeed, if the underlying physics at the pore scale does not rise any particular question, issues are mainly related to the understanding of the different flow regimes at the macroscale, the physical mechanisms that trigger the transition from one regime to another and the associated range of Reynolds numbers, together with the possible occurrence of unsteadiness.
- The original identification of the slip effect for gas flows in porous media, when the Knudsen number is not exceedingly small compared to unity, by Klinkenberg (1941), which led this author to propose a correction to the intrinsic permeability (in the 1D case) inversely proportional to the mean pressure, has been accepted with empirical justifications for almost 60 years. After this period, a first theoretical 3D model was derived, followed by more refined ones 15 years later

using rigorous upscaling techniques. While these upscaling tools are now at hand for such theoretical developments, it clearly appears that the main issues lie in an appropriate pore-scale description of the physics in that case, and in particular, in the slip-boundary condition, which still requires some efforts to better capture, and possibly extend, its domain of application in terms of the Knudsen number. In the same spirit, despite some attempts, macroscale models for transitional and strongly rarefied flow regimes still require important efforts. In parallel, numerical and experimental investigations are necessary to highlight the understanding of the detailed physics.

As a prospect, one may finally conclude with the following. From the evolution of two modifications to Darcy's law investigated in the present work, it is important to remark that, after one-dimensional corrections were proposed, followed by extensive experimental analyses, there was a long time period before rigorous deductions were presented. This delay can be explained by the fact that the different necessary theoretical frameworks for performing upscaling from the pore-scale to the macroscale are relatively new (about 30–40 years old) and also by the fact that, in many situations, the one-dimensional versions remained relatively satisfactory. However, with recent applications directed to micro- and nanofluidic devices, among others, there is a need for more accurate macroscale models, that can be validated through comparison with reliable experimental data and direct numerical simulations. In this last issue, there is a strong tendency in current research towards understanding macroscale phenomena through imaging and direct simulations. This has been made possible by recent advances in computational capabilities. With this perspective in mind, it is relevant to pose the question of which direction future advances of flows in porous media will take. Probably, experimental and numerical studies would still evolve very significantly. However, strong efforts should certainly be dedicated to more sophisticated upscaling approaches applicable to more challenging situations than those studied in the current state of the art.

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