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Transport in Ordered and Disordered Porous Media I: The Cellular Average and the Use of Weighting Functions

MICHEL QUINTARD[1] and STEPHEN WHITAKER[2]
[1] *Laboratoire Energétique et Phénomènes de Transfert, Unité de Recherche Associée au CNRS, URA 873, Université de Bordeaux I, 33405 Talence Cedex, France*
[2] *Department of Chemical Engineering, University of California, Davis, CA 95616, U.S.A.*

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Abstract. In this work we consider transport in ordered and disordered porous media using single-phase flow in rigid porous media *as an example*. We define *order* and *disorder* in terms of geometrical integrals that arise naturally in the method of volume averaging, and we show that dependent variables for ordered media must generally be defined in terms of the *cellular average*. The cellular average can be constructed by means of a weighting function, thus transport processes in both ordered and disordered media can be treated with a single theory based on weighted averages. Part I provides some basic ideas associated with ordered and disordered media, weighted averages, and the theory of distributions. In Part II a generalized averaging procedure is presented and in Part III the closure problem is developed and the theory is compared with experiment. Parts IV and V provide some geometrical results for computer generated porous media.

Key words. Cellular average, weighting functions, ordered media, disordered media.

Nomenclature

Roman Letters

$\mathcal{A}_{\beta\sigma}$	interfacial area of the $\beta - \sigma$ interface contained within the macroscopic region, m^2.
$\mathcal{A}_{\beta e}$	area of entrances and exits for the β-phase contained within the macroscopic system, m^2.
$\mathbf{g}$	gravity vector, m/s^2.
I	unit tensor
K	traditional Darcy's law permeability tensor, m^2.
L	general characteristic length for volume averaged quantities, m.
ℓ_β	characteristic length (pore scale) for the β-phase.
$\tilde{m}(\mathbf{y})$	weighting function.
$m(-\mathbf{y})$	$\tilde{m}(\mathbf{y})$, convolution product weighting function.
$\tilde{m}_{\mathcal{V}}$	special weighting function associated with the traditional averaging volume.
$\mathbf{n}_{\beta\sigma}$	unit normal vector pointing from the β-phase toward the σ-phase.
p_β	pressure in the β-phase, N/m^2.
p_0	reference pressure in the β-phase, N/m^2.
$\langle p_\beta \rangle^\beta$	traditional intrinsic volume averaged pressure, N/m^2.
r_0	radius of a spherical averaging volume, m.

$\mathbf{r}$ position vector, m.
$\mathbf{r}_\beta$ position vector locating points in the β-phase, m.
$\mathcal{V}$ averaging volume, m^3.
V_β volume of the β-phase contained in the averaging volume, m^3.
V_{cell} volume of a unit cell, m^3.
$\mathbf{v}_\beta$ velocity vector in the β-phase, m/s.
$\langle \mathbf{v}_\beta \rangle$ traditional superficial volume averaged velocity, m/s.
$\mathbf{x}$ position vector locating the centroid of the averaging volume or the convolution product weighting function, m.
$\mathbf{y}$ position vector relative to the centroid, m.,
$\mathbf{y}_\beta$ position vector locating points in the β-phase relative to the centroid, m.

Greek Letters
γ_β indicator function for the β-phase.
$\delta_{\beta\sigma}$ Dirac distribution associated with the $\beta - \sigma$ interface.
ϵ_β $V_\beta/\mathcal{V}$, volume average porosity.
ρ_β mass density of the β-phase, kg/m^3.
μ_β viscosity of the β-phase, Ns/m^2.

1. Introduction

For single-phase, quasi-steady, incompressible Stokes flow in the rigid porous medium illustrated in Figure 1, there is general agreement that Darcy's law takes the form (Greenkorn, 1983)

$$\langle \mathbf{v}_\beta \rangle = -\frac{1}{\mu_\beta} \mathsf{K} \cdot (\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g}). \tag{1.1}$$

Sometimes it is important to express this result more precisely as

$$\langle \mathbf{v}_\beta \rangle \Big|_{\mathbf{x}} = -\frac{1}{\mu_\beta} \mathsf{K} \cdot \left(\frac{\partial \langle p_\beta \rangle^\beta}{\partial \mathbf{x}} - \rho_\beta \mathbf{g} \right) \tag{1.2}$$

in which $\mathbf{x}$ represents the position vector locating the centroid of the averaging volume shown in Figure 1. In Equations (1.1) and (1.2) the *superficial* average velocity is defined by

$$\langle \mathbf{v}_\beta \rangle \Big|_{\mathbf{x}} = \frac{1}{\mathcal{V}} \int_{V_\beta(\mathbf{x})} \mathbf{v}_\beta(\mathbf{x} + \mathbf{y}_\beta) \, \mathrm{d}V, \tag{1.3}$$

while the *intrinsic* average pressure is given by

$$\langle p_\beta \rangle^\beta \Big|_{\mathbf{x}} = \frac{1}{V_\beta(\mathbf{x})} \int_{V_\beta(\mathbf{x})} p_\beta(\mathbf{x} + \mathbf{y}_\beta) \, \mathrm{d}V \,. \tag{1.4}$$

Here $\mathcal{V}$ represents the averaging volume illustrated in Figure 1, and V_β represents the volume of the β-phase contained within the averaging volume. The nomenclature used in Equations (1.3) and (1.4) clearly indicates that the average values are

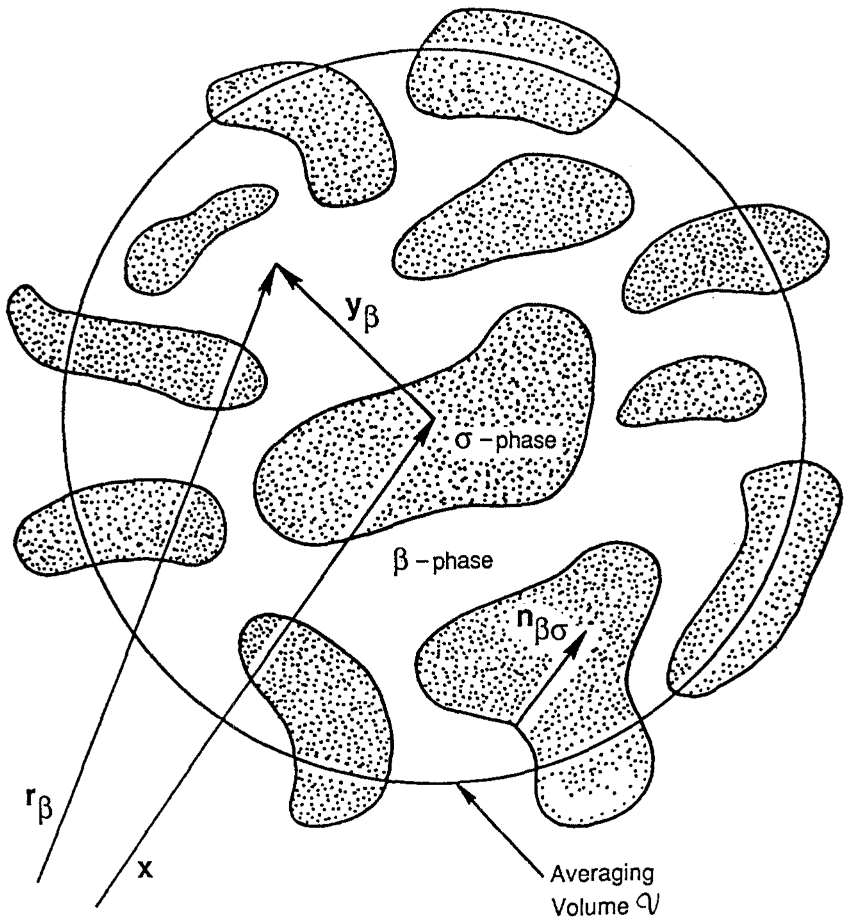

Fig. 1. Fluid-solid system.

assigned to the centroid located by $\mathbf{x}$, while integration is carried out with respect to the position vector $\mathbf{y}_\beta$.

The use of the *intrinsic* average pressure in Darcy's law results from the fact that $\langle p_\beta \rangle^\beta$ is the pressure that one might measure with a typical probe, or that one might be able to specify at a boundary. The use of the *superficial* average velocity follows from the convenient form of the continuity equation for incompressible flow

$$\nabla \cdot \langle \mathbf{v}_\beta \rangle = 0. \tag{1.5}$$

Here it is understood that $\langle \mathbf{v}_\beta \rangle$ is evaluated at $\mathbf{x}$ and that ∇ represents differentiation with respect to $\mathbf{x}$.

The form of Darcy's law expressed by Equation (1.1) has been developed by numerous authors using different approaches (Matheron, 1965; Marle, 1967; Raats and Klutes, 1968; Gray and O'Neil, 1976; Dagan, 1979; Rubinstein and

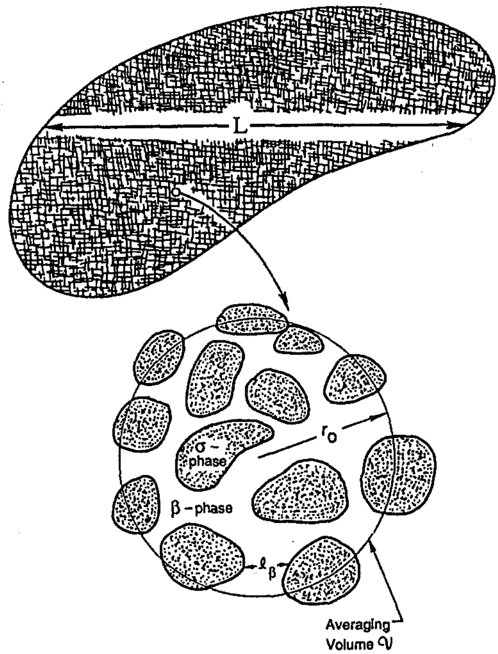

Fig. 2. Macroscopic region of a porous medium and an associated averaging volume.

Torquato, 1989). Equation (1.1) is generally considered to be valid when the Reynolds number is less than one (Hassanizadeh and Gray, 1987; Prieur du Plessis and Masliyah, 1988) and when certain length scale constraints are satisfied (Dagan, 1979; Sanchez-Palencia, 1980; Tartar, 1980; Whitaker, 1969 and 1986a). These length-scale constraints usually take the form

$$\ell_\beta \ll r_0 \ll L \tag{1.6}$$

where ℓ_β, r_0, and L are illustrated in Figure 2.

1.1. HYDROSTATIC PROBLEM

In order to identify the difficulties that arise with *ordered* or *spatially periodic porous media,* we consider the hydrostatic condition since this allows us to derive an exact representation for the intrinsic phase average pressure and its gradient. The hydrostatic problem associated with the spatially periodic array illustrated in Figure 3 is given by

$$0 = -\nabla p_\beta + \rho_\beta \mathbf{g} \tag{1.7a}$$

$$\text{B.C.1} \quad p_\beta = p_0, \quad \mathbf{r}_\beta = 0 \tag{1.7b}$$

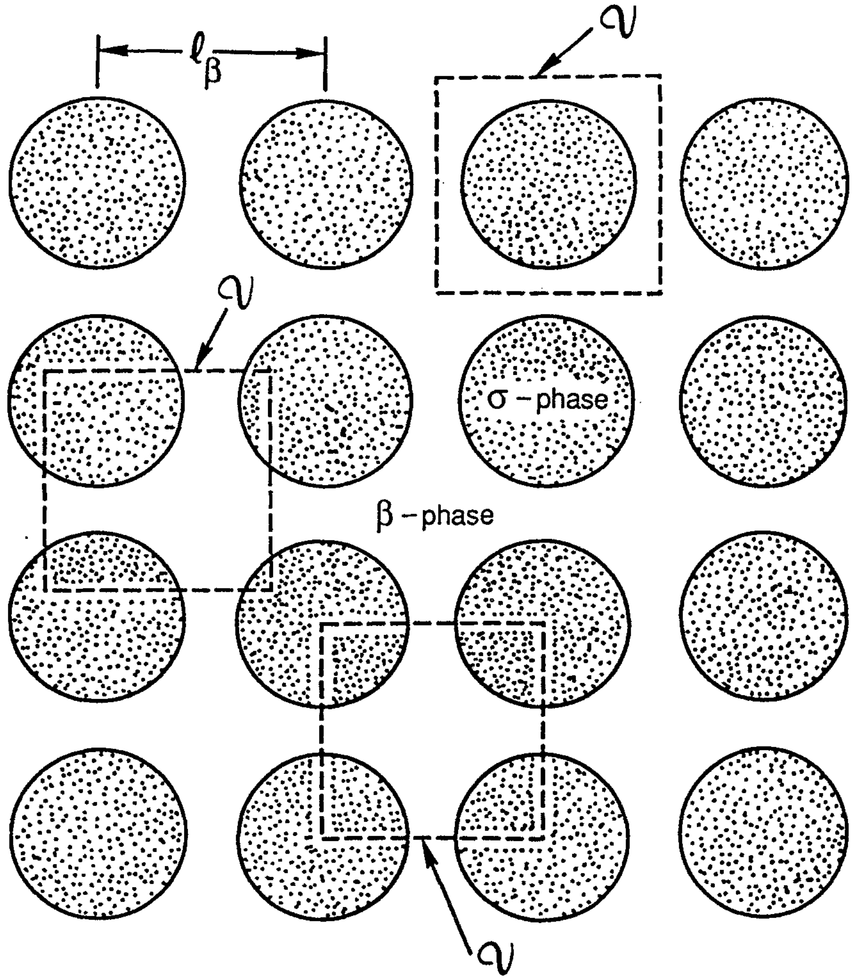

Fig. 3. Spatially periodic porous medium.

in which we have used $\mathbf{r}_\beta$ to denote points in the β-phase. The solution of Equations (1.7) is given by

$$p_\beta = \mathbf{r}_\beta \cdot \rho_\beta \mathbf{g} + p_0 \tag{1.8}$$

and the intrinsic phase average pressure defined by Equation (1.4) takes the form

$$\langle p_\beta \rangle^\beta \Big|_{\mathbf{x}} = \langle \mathbf{r}_\beta \rangle^\beta \Big|_{\mathbf{x}} \cdot \rho_\beta \mathbf{g} + p_0. \tag{1.9}$$

We can make use of the nomenclature illustrated in Figure 1 to express the position vector as

$$\mathbf{r}_\beta = \mathbf{x} + \mathbf{y}_\beta \tag{1.10}$$

This allows us to write Equation (1.9) as

$$\langle p_\beta \rangle^\beta \Big|_{\mathbf{x}} = \mathbf{x} \cdot \rho_\beta \mathbf{g} + \langle \mathbf{y}_\beta \rangle^\beta \Big|_{\mathbf{x}} \cdot \rho_\beta \mathbf{g} + p_0. \tag{1.11}$$

Taking the derivative with respect to $\mathbf{x}$ leads to

$$\frac{\partial \langle p_\beta \rangle^\beta}{\partial \mathbf{x}} \Big|_{\mathbf{x}} = \rho_\beta \mathbf{g} + \frac{\partial \langle \mathbf{y}_\beta \rangle^\beta}{\partial \mathbf{x}} \Big|_{\mathbf{x}} \cdot \rho_\beta \mathbf{g} \tag{1.12a}$$

and we can also express this result as

$$\nabla \langle p_\beta \rangle^\beta = \rho_\beta \mathbf{g} + \nabla \langle \mathbf{y}_\beta \rangle^\beta \cdot \rho_\beta \mathbf{g}. \tag{1.12b}$$

Here it is understood that all averaged quantities are evaluated at the centroid. From Equation (1.1) with $\langle \mathbf{v}_\beta \rangle = 0$, we obtain the following version of Equation (1.12)

$$\nabla \langle p_\beta \rangle^\beta = \rho_\beta \mathbf{g}, \quad \text{Darcy's Law}. \tag{1.13}$$

Clearly these two results will only be compatible if $\nabla \langle \mathbf{y}_\beta \rangle^\beta$ is small compared to one, and for the averaging volumes shown in Figure 3 this seems unlikely.

In Part IV we examine the geometry of ordered porous media and there we show that

$$\nabla \langle \mathbf{y}_\beta \rangle^\beta = \mathbf{O}(1 - \epsilon_\beta) \tag{1.14}$$

in which ϵ_β is the porosity. Use of this result in Equation (1.12b) leads to a form for the hydrostatic pressure given by

$$\nabla \langle p_\beta \rangle^\beta = \rho_\beta \mathbf{g} + \mathbf{O}[(1 - \epsilon_\beta)\rho_\beta \mathbf{g}]. \tag{1.15}$$

This result is obviously inconsistent with Darcy's law, and it tells us that the intrinsic phase average pressure defined by Equation (1.4) is not the correct pressure to be used with ordered or spatially periodic porous media.

1.2. CELLULAR AVERAGE

For spatially periodic porous media, quantities such as $\langle \mathbf{y}_\beta \rangle^\beta$, $\nabla \langle \mathbf{y}_\beta \rangle^\beta$, etc. are spatially periodic and have zero average values. This means that we can average Equation (1.12b) over a unit cell to obtain

$$\frac{1}{V_{\text{cell}}} \int_{V_{\text{cell}}} \nabla \langle p_\beta \rangle^\beta \, \mathrm{d}V = \rho_\beta \mathbf{g} \tag{1.16}$$

since

$$\frac{1}{V_{\text{cell}}} \int_{V_{\text{cell}}} \nabla \langle \mathbf{y}_\beta \rangle^\beta \, \mathrm{d}V = \mathrm{O}. \tag{1.17}$$

The averaging theorem can be used to interchange differentiation and integration in Equation (1.16) leading to

$$\nabla\left\{\frac{1}{V_{\text{cell}}}\int_{V_{\text{cell}}}\langle p_\beta\rangle^\beta\,\mathrm{d}V\right\} = \rho_\beta\mathbf{g}. \tag{1.18}$$

This doubly averaged pressure is the *cellular average* pressure and Equation (1.18) suggests that it is the correct pressure to use with Darcy's law for flow in spatially periodic porous media. In reality, the situation is somewhat more complex than is indicated by Equation (1.18), and in subsequent sections we will see that a more generalized averaging procedure is required. This generalized procedure makes use of weighting functions, and in Part II we show that the cellular average can be expressed as a weighted average. This allows us to treat ordered and disordered porous media with a single theory since a general weighting function can be used which contains the cellular average.

1.3. DISORDERED MEDIA

Equations (1.12) and (1.13) will be equivalent when

$$\nabla\langle\mathbf{y}_\beta\rangle^\beta \ll \mathsf{I} \tag{1.19}$$

and this constraint appears to be consistent with what are often referred to as *disordered media.* To be perfectly clear about what we mean by disordered media, we offer the following definition:

A porous medium is disordered *with respect to* an averaging volume $\mathcal{V}$ when $\nabla\langle\mathbf{y}_\beta\rangle^\beta \ll \mathsf{I}$.

It seems clear that this definition would require that the averaging volume be *sufficiently large* and this would appear to be satisfied by the first part of Equation (1.6). For example, if we used the averaging volume shown in Figure 3 to determine $\nabla\langle\mathbf{y}_\beta\rangle^\beta$ for a disordered porous media we would recover the result given by Equation (1.14) as opposed to that given by Equation (1.19). Obviously we have proposed an operational definition that is especially well-suited for the method of volume averaging, and in other studies of transport processes in *random media* (Strieder and Aris, 1973; Torquato, 1986; Rubinstein and Torquato, 1989; Shah and Ottino, 1987) one does not find this type of definition. However, it would appear to be consistent with those studies as we have indicated in Parts IV and V. In terms of our definition of a disordered porous medium, we see that Equation (1.4) is the *correct* definition of the average pressure for a disordered media, while the cellular average pressure given by Equation (1.18) appears to be the *correct* average pressure to be used with ordered media. We will be more precise about this situation in Part II.

Concerning ordered and disordered porous media, we need to point out that neither of these two special cases exist in terms of real porous media. However, real porous media often contain elements of order and disorder, thus studies of both models will improve our understanding of real systems.

2. Weighted Averages

In order to develop a general averaging procedure that will be valid for both ordered and disordered systems, we will make use of weighting functions as originally proposed by Matheron (1965), used by Anderson and Jackson (1967) in an independent study of fluidized beds, and employed extensively by Marle (1965, 1967, 1982, 1984) for the study of a variety of transport processes in porous media. Baveye and Sposito (1984) and Cushman (1984) have demonstrated the importance of *instrument* weighting functions for the interpretation of experimental observations, thus the development presented here has both theoretical and experimental applications.

Our starting point for any transport process in porous media will be the governing point equations and boundary conditions. In this case we begin with

$$\nabla \cdot \mathbf{v}_\beta = 0, \quad \text{in the } \beta\text{-phase}, \tag{2.1}$$

$$0 = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{v}_\beta, \quad \text{in the } \beta\text{-phase}, \tag{2.2}$$

$$\text{B.C.1} \quad \mathbf{v}_\beta = 0, \quad \text{at } \mathcal{A}_{\beta\sigma}, \tag{2.3}$$

$$\text{B.C.2} \quad \mathbf{v}_\beta = \mathbf{f}(\mathbf{r},\, \mathbf{t}), \quad \text{at } \mathcal{A}_{\beta e} \tag{2.4}$$

Here we have used $\mathcal{A}_{\beta\sigma}$ to represent the interfacial area between the fluid and solid phases, while $\mathcal{A}_{\beta e}$ has been used to represent the area of entrances and exits for the macroscopic system illustrated in Figure 2. In general, the boundary condition at $\mathcal{A}_{\beta e}$ is known only in terms of averages (Prat, 1989; 1990; 1992), thus Equation (2.4) serves as a reminder of what we *do not know* about the boundary value problem rather than what we do know.

2.1. WEIGHTING FUNCTIONS

In order to illustrate the use of weighting functions with Equations (2.1) and (2.2), we consider the pressure p_β and the weighting function $\tilde{m}(\mathbf{y})$ associated with the point $\mathbf{x}$. A one-dimensional example of $\tilde{m}(\mathbf{y})$ is presented in Figure 4, and we can express the three-dimensional version as

$$\tilde{m}(\mathbf{y}) = \begin{cases} \mathrm{C} \exp\left(-\frac{1}{\mathrm{r}_0^2 - \mathbf{y}\cdot\mathbf{y}}\right), & |\,\mathbf{y}\,| \le r_0. \\ 0 \qquad\qquad\qquad\quad, & |\,\mathbf{y}\,| > r_0 \end{cases} \tag{2.5}$$

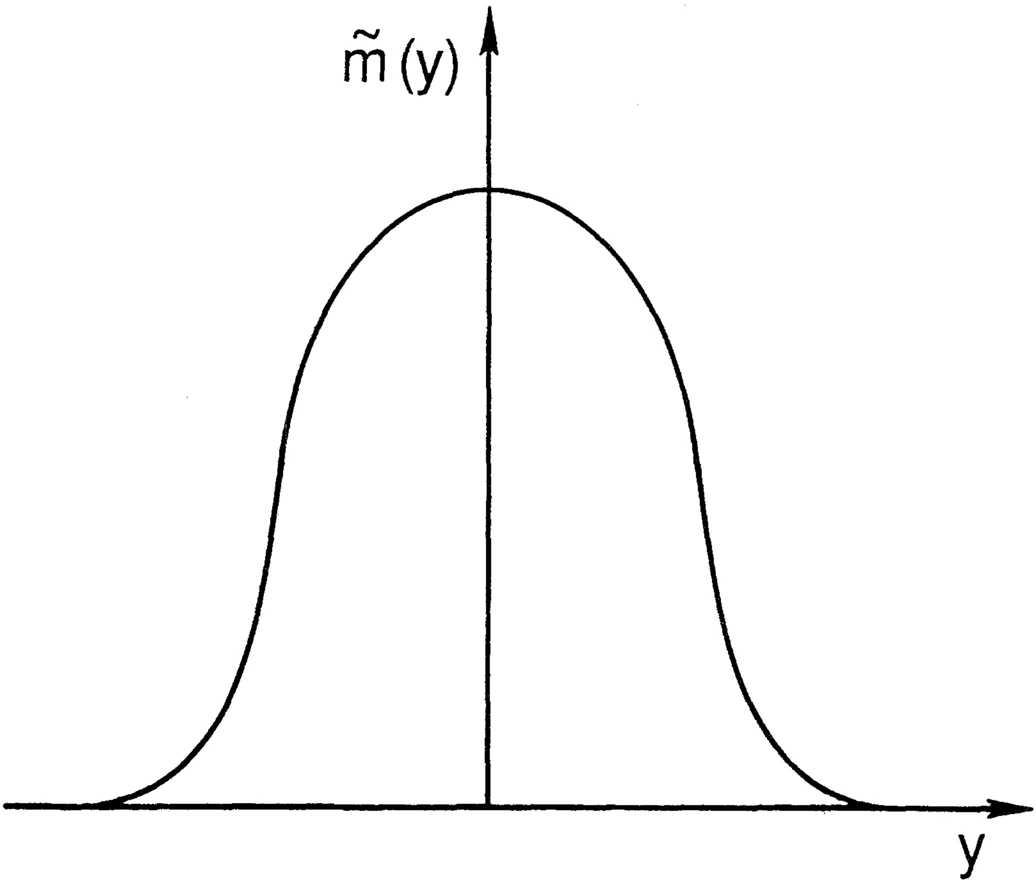

Fig. 4. Illustrative weighting function.

In this case $\tilde{m}(\mathbf{y}) \in C^{\infty}$ and $\tilde{m}(\mathbf{y})$ has a compact support, thus our weighting function has the characteristics of a *test function* in the theory of distributions (Schwartz, 1978; Richards and Youn, 1990). Since $\tilde{m}(\mathbf{y})$ is defined in both the β and σ-phases, we find it convenient to think of p_β as being defined in the *usual sense* in the β-phase and *as being zero* in the σ-phase. This allows us to express a superficial weighted averaged as

$$\left\{ \begin{array}{c} \text{superficial} \\ \text{weighted} \\ \text{average} \\ \text{pressure} \end{array} \right\} = \int_{\mathbb{R}^3} \tilde{m}(\mathbf{y})\, p_\beta(\mathbf{x}+\mathbf{y})\, \mathrm{d}V_y \tag{2.6}$$

in which $\mathrm{d}V_y$ represents a differential volume element expressed in terms of the components of $\mathrm{d}\mathbf{y}$. When $\tilde{m}(\mathbf{y})$ takes on the special form given by

$$\tilde{m}_V(\mathbf{y}) = \begin{cases} \frac{1}{V}, & |\,\mathbf{y}\,| \leq r_0 \\ 0, & |\,\mathbf{y}\,| > r_0 \end{cases} \tag{2.7}$$

Equation 2.6 reduces to

$$\left\{\begin{array}{c}\text{superficial}\\ \text{weighted}\\ \text{average}\\ \text{pressure}\end{array}\right\} = \frac{1}{\mathcal{V}}\int_{V_{\beta(\mathbf{x})}} p_\beta(\mathbf{x}+\mathbf{y}_\beta)\,\mathrm{d}V_y \tag{2.8}$$

since p_β is zero in the σ-phase. Equation (2.8) indicates that $\tilde{m}_{\mathcal{V}}$ is the weighting function that produces the traditional superficial volume average.

In expressing the weighted average by Equation (2.6), we have followed the development of Marle (1967); however, other approaches are possible. Anderson and Jackson (1967) found it convenient to form the weighted average by integrating $\tilde{m}(\mathbf{y})p_\beta(\mathbf{x}+\mathbf{y})$ over the β-phase contained in $\mathbb{R}^3$, and a similar approach was employed by Whitaker (1986b) in a study of dispersion. In their studies of multiphase transport phenomena, Gray and Lee (1977) and Gray and Hassanizadeh (1989) used the indicator function defined by

$$\gamma_\beta(\mathbf{y}) = \begin{cases} 1, & \text{when } \mathbf{y} \text{ locates a point the } \beta\text{-phase} \\ 0, & \text{when } \mathbf{y} \text{ locates a point in the } \sigma\text{-phase} \end{cases} \tag{2.9}$$

in their definition of averaged quantities. For example, we could use $\gamma_\beta(\mathbf{y})$ to express Equation (2.6) as

$$\left\{\begin{array}{c}\text{superficial}\\ \text{weighted}\\ \text{average}\\ \text{pressure}\end{array}\right\} = \int_{\mathbb{R}^3} \tilde{m}(\mathbf{y})\,\gamma_\beta(\mathbf{x}+\mathbf{y})\,p_\beta(\mathbf{x}+\mathbf{y})\,\mathrm{d}V_y \tag{2.10}$$

however, this still demands that one say *something* about p_β in the σ-phase in order that $\gamma_\beta p_\beta$ have any significance in the σ-phase. At the very least one must say that p_β is *bounded* in the σ-phase in order to use $\gamma_\beta p_\beta$ in the evaluation of the weighted average. Our situation concerning the development of weighted averages for the β-phase is this: Either we form the integral of $\tilde{m}(\mathbf{y})p_\beta(\mathbf{x}+\mathbf{y})$ over *only* the β-phase contained in $\mathbb{R}^3$, or we are forced to make some statement about p_β in the σ-phase (such as p_β is bounded). Our motivation for defining the weighted average by Equation (2.6), with p_β being zero in the σ-phase, is based on Marle's (1967) observation that weighted averages in $\mathbb{R}^3$ can be viewed as convolution products and this leads to considerable mathematical simplification.

2.2. CONVOLUTION PRODUCT

In subsequent parts of this study it will be necessary to arrange Equation (2.6) in the form of a convolution product. This can be accomplished by noting that (see Figure 1)

$$\mathbf{r} = \mathbf{x} + \mathbf{y} \tag{2.11}$$

and making use of the *definition*

$$m(-\mathbf{y}) = \tilde{m}(\mathbf{y}) \tag{2.12}$$

so that Equation (2.6) takes the form

$$\left\{ \begin{array}{c} \text{superficial} \\ \text{weighted} \\ \text{average} \\ \text{pressure} \end{array} \right\} = \int_{\mathbb{R}^3} m(\mathbf{x} - \mathbf{r}) p_\beta(\mathbf{r}) \, \mathrm{d}V_r. \tag{2.13}$$

At this point we make use of the traditional nomenclature associated with the convolution product (Richards and Youn, 1990) and express the right hand side of Equation (2.13) as

$$\int_{\mathbb{R}^3} m(\mathbf{x} - \mathbf{r}) p_\beta(\mathbf{r}) \, \mathrm{d}V_r = m * p_\beta \Big|_{\mathbf{x}} . \tag{2.14}$$

Obviously the convolution product is a weighted average

$$m * p_\beta \Big|_{\mathbf{x}} = \left\{ \begin{array}{c} \text{superficial} \\ \text{weighted} \\ \text{average} \\ \text{pressure} \end{array} \right\} \tag{2.15}$$

and it is directly related to the superficial volume average as indicated by Equation (2.8).

In our development of a generalized averaging procedure, we will need to make several specific statements about the weighting function, m. The first three of these are given by

H1. $m \in \mathrm{C}^\infty$

H2. m has a compact support over $\mathbb{R}^3$

H3. m satisfies the following normalization condition

$$\int_{\mathbb{R}^3} m(\mathbf{x} - \mathbf{r}) \, \mathrm{d}V_r = 1. \tag{2.16}$$

Among these requirements, the first condition is particularly important since it gives rise to average quantities that are infinitely differentiable. This avoids the problem posed by Veverka (1981) and Mlš (1987) concerning the differentiability of averages, and in subsequent sections we shall see that H1 does not place any severe constraints on the problem of developing precise definitions of average quantities.

In the averaging procedure to be described in Part II we will encounter the convolution product of the gradient of the pressure, thus we need to be precise about derivatives in the *usual sense* and in the *sense of a distribution.* We consider a general distribution defined by

$$\psi = \begin{cases} \psi_\beta \ , & \text{in the } \beta\text{-phase} \\ \psi_\sigma \ , & \text{in the } \sigma\text{-phase} \end{cases} . \tag{2.17}$$

If ψ were the pressure, we would think of ψ_β as the pressure defined in the *usual sense* in the β-phase and we would take ψ_σ to be zero. The gradient of the distribution defined by Equation (2.17) is given by (Schwartz, 1978)

$$\nabla\psi = (\nabla\psi)^{\mathrm{u}} + \mathbf{n}_{\beta\sigma}(\psi_\sigma - \psi_\beta)\delta_{\beta\sigma} \tag{2.18}$$

in which $\delta_{\beta\sigma}$ represents the Dirac distribution associated with the singular surface $\mathcal{A}_{\beta\sigma}$, and $\mathbf{n}_{\beta\sigma}$ is the unit normal vector illustrated in Figure 1. Given any *function* φ, the Dirac distribution is defined according to

$$\int_{\mathbb{R}^3} \varphi\delta_{\beta\sigma}\, \mathrm{d}V = \int_{\mathcal{A}_{\beta\sigma}} \varphi\, \mathrm{d}A. \tag{2.19}$$

In Equation (2.18) we have used $\nabla\psi$ to represent the gradient in the *sense of a distribution* while $(\nabla\psi)^{\mathrm{u}}$ is the gradient in the *usual sense.* This means that the *distribution* $(\nabla\psi)^{\mathrm{u}}$ is given by

$$(\nabla\psi)^{\mathrm{u}} = \begin{cases} \nabla\psi_\beta \ , & \text{in the } \beta\text{-phase} \\ \nabla\psi_\sigma \ , & \text{in the } \sigma\text{-phase.} \end{cases} \tag{2.20}$$

The problem that we have chosen to study in this investigation of ordered and disordered systems is quasi-steady as indicated by Equations (2.1) through (2.4). However, there are many unsteady transport processes in porous media, and in these processes we will be concerned with the *distribution* $(\partial\psi/\partial t)^{\mathrm{u}}$ which takes the form

$$\left(\frac{\partial\psi}{\partial t}\right)^{\mathrm{u}} = \begin{cases} \frac{\partial\psi_\beta}{\partial t} \ , & \text{in the } \beta\text{-phase} \\ \frac{\partial\psi_\sigma}{\partial t} \ , & \text{in the } \sigma\text{-phase.} \end{cases} \tag{2.21}$$

Two important theorems associated with the convolution product of the derivatives represented by Equations (2.20) and (2.21) are given by (Marle, 1982)

$$m * (\nabla\psi)^{\mathrm{u}} = \nabla(m * \psi) + m * [\mathbf{n}_{\beta\sigma}(\psi_\beta - \psi_\sigma)\delta_{\beta\sigma}], \tag{2.22}$$

$$m * \left(\frac{\partial\psi}{\partial t}\right)^{\mathrm{u}} = \frac{\partial}{\partial t}(m * \psi) - m * [\mathbf{n}_{\beta\sigma} \cdot \mathbf{w}(\psi_\beta - \psi_\sigma)\delta_{\beta\sigma}]. \tag{2.23}$$

The first of these represents the generalized *spatial averaging theorem* while the second can be thought of as the weighting function version of the *general transport theorem* in which $\mathbf{n}_{\beta\sigma} \cdot \mathbf{w}$ is the speed of displacement of the singular surface (Whitaker, 1992).

At this point we are ready to return to Equations (2.1) through (2.4) in order to reformulate the boundary value problem so that it is suitable for use with the convolution product. This leads to

$$(\nabla \cdot \mathbf{v}_\beta)^{\mathrm{u}} = 0, \quad \text{in } \mathbb{R}^3 \tag{2.24}$$

$$0 = -(\nabla p_\beta)^{\mathrm{u}} + \gamma_\beta \rho_\beta \mathbf{g} + \mu_\beta (\nabla^2 \mathbf{v}_\beta)^{\mathrm{u}}, \quad \text{in } \mathbb{R}^3 \tag{2.25}$$

$$\text{B.C.1} \quad \mathbf{v}_\beta = 0, \quad \text{at } \mathcal{A}_{\beta\sigma} \tag{2.26}$$

$$\text{B.C.2} \quad \mathbf{v}_\beta = \mathbf{f}(\mathbf{r}, \mathbf{t}), \quad \text{at } \mathcal{A}_{\beta e} \tag{2.27}$$

Here it is convenient to use the indicator function with the body force term since $\rho_\beta \mathbf{g}$ is a constant and not a function. In this formulation of the boundary value problem it is understood that p_β and $\mathbf{v}_\beta$ are taken to be zero in the σ-phase. As was pointed out by Marle (1967), this allows us to avoid a variety of algebraic complexities with no loss of generality. Given Equations (2.24) through (2.27), along with the definition of a weighted average expressed by Equations (2.14) and (2.15), we are ready to move on to the problem of spatial smoothing which is described in Part II. To complete our development of the spatially smoothed transport equations, we require the closure problem described in Part III. Solution of the local version of that problem allows us to compare theory with experiment, and this is done in Part III.

3. Conclusions

In this introduction we have illustrated the need for cellular averages in the study of transport processes in ordered porous media, and we have suggested, but not proved, that the cellular average can be expressed in terms of a weighting function. Weighted averages can be conveniently written as convolution products if one makes use of the theory of distributions, and we have illustrated how a typical transport process can be described in terms of distributions.

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