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The Forchheimer Equation: A Theoretical Development

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Abstract. In this paper we illustrate how the method of volume averaging can be used to derive Darcy's law with the Forchheimer correction for homogeneous porous media. Beginning with the Navier–Stokes equations, we find the volume averaged momentum equation to be given by

$$\langle \mathbf{v}_\beta \rangle = -\frac{\mathsf{K}}{\mu_\beta} \cdot \left(\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g} \right) - \mathsf{F} \cdot \langle \mathbf{v}_\beta \rangle.$$

The Darcy's law permeability tensor, K, and the Forchheimer correction tensor, F, are determined by closure problems that must be solved using a spatially periodic model of a porous medium. When the Reynolds number is small compared to one, the closure problem can be used to prove that F is a linear function of the velocity, and order of magnitude analysis suggests that this linear dependence may persist for a wide range of Reynolds numbers.

Key words: Forchheimer equation, Darcy's law, volume averaging, closure.

Nomenclature

$\mathcal{A}_{\beta\sigma}$	area of the β-σ interface contained with the macroscopic region, m^2.
$\mathcal{A}_{\beta e}$	area of the entrances and exits of the β-phase at the boundary of the macroscopic region, m^2.
$A_{\beta\sigma}$	area of the β-σ interface contained within the averaging volume, m^2.
A_p	surface area of a particle, m^2.
$\mathbf{b}$	the vector field that maps $\mu_\beta \langle \mathbf{v}_\beta \rangle^\beta$ onto $\widetilde{p}_\beta$ when inertial effects are negligible, m^{-1}.
B	tensor that maps $\widetilde{\mathbf{v}}_\beta$ onto $\langle \mathbf{v}_\beta \rangle^\beta$ when inertial effects are negligible.
d_p	$6V_p/A_p$, effective particle diameter, m.
$\mathbf{g}$	gravitational acceleration, m/s^2.
I	unit tensor.
ℓ_β	characteristic length for the β-phase, m.
ℓ_i	$i = 1, 2, 3$, lattice vectors, m.
L	characteristic length for macroscopic quantities, m.
L_ρ	inertial length, m.
$\mathbf{m}$	the vector field that maps $\mu_\beta \langle \mathbf{v}_\beta \rangle^\beta$ onto $\widetilde{p}_\beta$, m^{-1}.
M	tensor that maps $\widetilde{\mathbf{v}}_\beta$ onto $\langle \mathbf{v}_\beta \langle^\beta$.
$\mathbf{n}_{\beta\sigma}$	unit normal vector directed from the β-phase toward the σ-phase.
p_β	total pressure in the β-phase, Pa.
$\langle p_\beta \rangle^\beta$	intrinsic average pressure in the β-phase, Pa.
$\langle p_\beta \rangle$	superficial average pressure in the β-phase, Pa.
$\widetilde{p}_\beta$	$p_\beta - \langle p_\beta \rangle^\beta$, spatial deviation pressure, Pa.
$\mathbf{r}$	position vector, m.

r_0	radius of the averaging volume, m.
t	time, s.
t^*	characteristic process time, s.
$\mathbf{v}_\beta$	velocity in the β-phase, m/s.
$\langle \mathbf{v}_\beta \rangle^\beta$	intrinsic average velocity in the β-phase, m/s.
$\langle \mathbf{v}_\beta \rangle$	superficial average velocity in the β-phase, m/s.
$\tilde{\mathbf{v}}_\beta$	$\mathbf{v}_\beta - \langle \mathbf{v}_\beta \rangle^\beta$, spatial deviation velocity, m/s.
$\mathcal{V}$	local averaging volume, m^3.
V_β	volume of the β-phase contained within the averaging volume, m^3.
V_p	volume of a particle, m^3.
$\mathbf{x}$	position vector locating the centroid of the averaging volume, m.
$\mathbf{y}_\beta$	position vector locating points in the β-phase relative to the centroid of the averaging volume, m.

Greek Symbols

ε_β	$V_\beta/\mathcal{V}$, volume fraction of the β-phase.
ρ_β	density of the β-phase, kg/m^3.
μ_β	viscosity of the β-phase, Pa s.

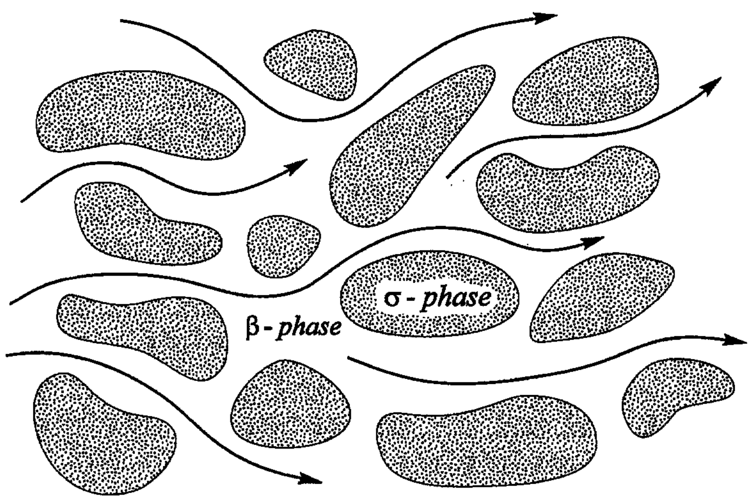

Figure 1. Flow in a Rigid Porous Medium.

1. Volume Averaging

The physical process under consideration is that of single-phase, incompressible flow in a rigid porous medium, such as we have illustrated in terms of the β-σ system shown in Figure 1. The boundary-value problem describing the flow in the macroscopic region shown in Figure 2 is given by

$$\rho_\beta \left(\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{v}_\beta \cdot \nabla \mathbf{v}_\beta \right) = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{v}_\beta,$$

in the β-phase (1.1)

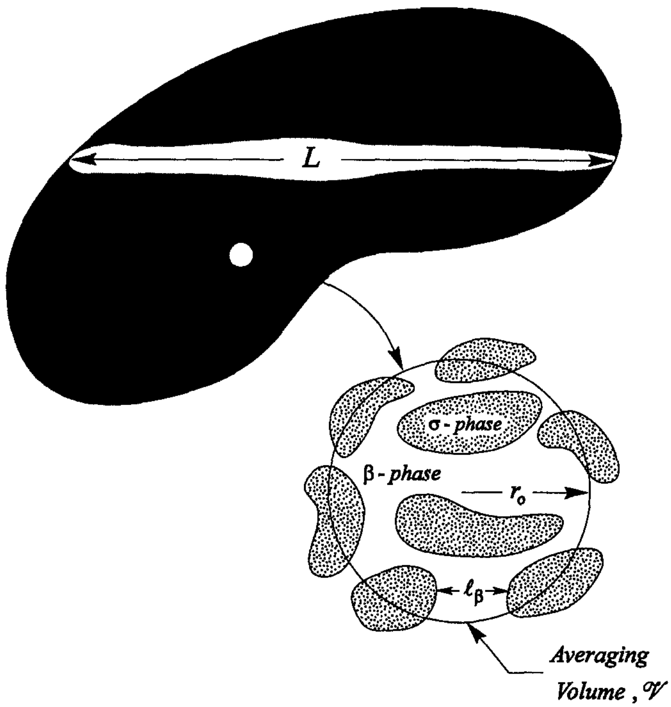

Figure 2. Macroscopic region and local averaging volume.

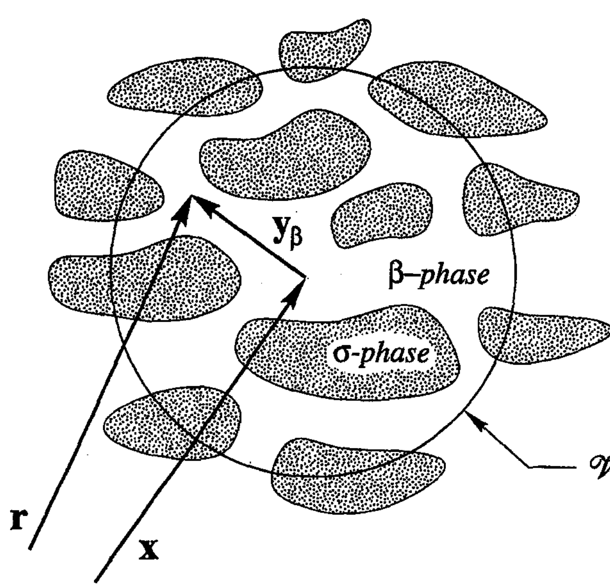

Figure 3. Position vectors associated with the averaging volume.

$$\nabla \cdot \mathbf{v}_\beta = 0, \quad \text{in the } \beta\text{-phase}, \tag{1.2}$$

$$\text{(B.C.1)} \quad \mathbf{v}_\beta = 0, \quad \text{at} \quad \mathcal{A}_{\beta\sigma}, \tag{1.3}$$

$$\text{(B.C.2)} \quad \mathbf{v}_\beta = \mathbf{f}(\mathbf{r}, t), \quad \text{at} \quad \mathcal{A}_{\beta e}. \tag{1.4}$$

Here $\mathcal{A}_{\beta\sigma}$ represents the area of the β-σ interface contained within the macroscopic region illustrated in Figure 2, while $\mathcal{A}_{\beta e}$ represents the β-phase entrances and exits of the macroscopic region. For some function ψ_β associated with the β-phase, the *superficial average* is defined by

$$\langle \psi_\beta \rangle = \frac{1}{\mathcal{V}} \int_{V_\beta} \psi_\beta \, \mathrm{d}V. \tag{1.5}$$

Here V_β represents the volume of the β-phase contained within the averaging volume, and we think of $\langle \psi_\beta \rangle$ as being associated with the centroid of the averaging volume. In Figure 3 we have indicated that the centroid is located by the position vector $\mathbf{x}$, and that points in the β-phase *relative to the centroid* are located by $\mathbf{y}_\beta$. To be more precise about the definition given by Equation (1.5), we could write

$$\langle \psi_\beta \rangle|_x = \frac{1}{\mathcal{V}} \int_{V_\beta(x)} \psi_\beta(\mathbf{x} + \mathbf{y}_\beta) \, \mathrm{d}V_y, \tag{1.6}$$

in order to clearly indicate that $\langle \psi_\beta \rangle$ is associated with the centroid and that integration is carried out with respect to the components of the relative position vector, $\mathbf{y}_\beta$. In general, we will use the simpler nomenclature indicated by Equation (1.5) with the idea that the specific details indicated in Equation (1.6) are understood. In addition to the superficial average, we will need to make use of the *intrinsic average* that is defined according to

$$\langle \psi_\beta \rangle^\beta = \frac{1}{V_\beta} \int_{V_\beta} \psi_\beta \, \mathrm{d}V. \tag{1.7}$$

The superficial and intrinsic averages are related by

$$\langle \psi_\beta \rangle = \varepsilon_\beta \langle \psi_\beta \rangle^\beta, \tag{1.8}$$

in which ε_β is the volume fraction of the β-phase defined as

$$\varepsilon_\beta = \frac{V_\beta}{\mathcal{V}}. \tag{1.9}$$

In some previous publications (see, for example, Whitaker, 1986) the quantities defined by Equations (1.5) and (1.7) have been referred to as the *phase average* and the *intrinsic phase average* respectively. The original choice of words was based on the idea that one should distinguish between the average over *an individual phase* and the average over *multiple phases*. The latter average is typically encountered in *one-equation models* of heat and mass transfer (Nozad *et al.*, 1985; Ochoa-Tapia *et al.*, 1994); however, one must also deal with averages over individual phases in those cases (Quintard and Whitaker, 1993, 1995). A unique set of words does not seem to be available to identify all the types of averages that one encounters in the method of volume averaging; however, it is clear that errors on the order of ε_β have been made because of the failure to distinguish between the averages defined by Equations (1.5) and (1.7). For parameters that are linear in ε_β this means an error of a factor of three, while a *factor of ten* results if the dependence is quadratic. To help avoid such errors the words *superficial* and *intrinsic* have been adopted.

1.1. CONTINUITY EQUATION

We begin the averaging process with the continuity equation to obtain

$$\frac{1}{\mathcal{V}}\int_{V_\beta} \nabla\cdot\mathbf{v}_\beta \,\mathrm{d}V = \langle \nabla\cdot\mathbf{v}_\beta\rangle = 0 \tag{1.10}$$

and make use of the averaging theorem (Howes and Whitaker, 1985)

$$\langle \nabla\psi_\beta\rangle = \nabla\langle\psi_\beta\rangle + \frac{1}{\mathcal{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\psi_\beta \,\mathrm{d}A, \tag{1.11}$$

in order to express Equation (1.10) as

$$\langle \nabla\cdot\mathbf{v}_\beta\rangle = \nabla\cdot\langle\mathbf{v}_\beta\rangle + \frac{1}{\mathcal{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot\mathbf{v}_\beta \,\mathrm{d}A = 0. \tag{1.12}$$

Since the solid phase is impermeable, this result simplifies to

$$\nabla\cdot\langle\mathbf{v}_\beta\rangle = 0. \tag{1.13}$$

The fact that the *superficial average* velocity is solenoidal encourages its use as the *preferred representation* of the macroscopic or volume averaged velocity field; however, we will also have occasion to use the continuity equation in terms of the *intrinsic average velocity* and this form is given by

$$\nabla\cdot(\varepsilon_\beta\langle\mathbf{v}_\beta\rangle^\beta) = 0. \tag{1.14}$$

Here we have made use of Equation (1.8) to represent $\langle\mathbf{v}_\beta\rangle$ in terms of $\varepsilon_\beta\langle\mathbf{v}_\beta\rangle^\beta$.

1.2. MOMENTUM EQUATION

The superficial average of the Navier–Stokes equations can be expressed as

$$\left\langle \rho_\beta\frac{\partial\mathbf{v}_\beta}{\partial t}\right\rangle + \langle\rho_\beta\mathbf{v}_\beta\cdot\nabla\mathbf{v}_\beta\rangle = -\langle\nabla p_\beta\rangle + \langle\rho_\beta\mathbf{g}\rangle + \langle\mu_\beta\nabla^2\mathbf{v}_\beta\rangle \tag{1.15}$$

and since we have restricted our study to incompressible flows, we are free to ignore the variations of ρ_β everywhere in the macroscopic region illustrated in Figure 2. This allows us to write Equation (1.15) as

$$\rho_\beta \left\langle \frac{\partial \mathbf{v}_\beta}{\partial t} \right\rangle + \rho_\beta \langle \nabla \cdot (\mathbf{v}_\beta \mathbf{v}_\beta) \rangle = -\langle \nabla p_\beta \rangle + \varepsilon_\beta \rho_\beta \mathbf{g} + \langle \mu_\beta \nabla^2 \mathbf{v}_\beta \rangle, \tag{1.16}$$

in which we have used $\langle 1 \rangle = \varepsilon_\beta$ in the gravitational term. In addition, we have made use of Equation (1.2) in order to arrange the convective inertial term in the form of the dyadic product, $\mathbf{v}_\beta \mathbf{v}_\beta$. In our treatment of the viscosity, we require only that the variations of μ_β be negligible *within the averaging volume* and that allows us to simplify Equation (1.16) to

$$\rho_\beta \left\langle \frac{\partial \mathbf{v}_\beta}{\partial t} \right\rangle + \rho_\beta \langle \nabla \cdot (\mathbf{v}_\beta \mathbf{v}_\beta) \rangle = -\langle \nabla p_\beta \rangle + \varepsilon_\beta \rho_\beta \mathbf{g} + \mu_\beta \langle \nabla^2 \mathbf{v}_\beta \rangle, \tag{1.17}$$

Since the volume of the β-phase contained within the averaging volume is independent of time, we can interchange time differentiation and spatial integration to obtain

$$\left\langle \frac{\partial \mathbf{v}_\beta}{\partial t} \right\rangle = \frac{1}{\mathcal{V}} \int_{V_\beta} \frac{\partial \mathbf{v}_\beta}{\partial t} \, \mathrm{d}V = \frac{\partial}{\partial t} \left\{ \frac{1}{\mathcal{V}} \int_{V_\beta} \mathbf{v}_\beta \, \mathrm{d}V \right\} = \frac{\partial \langle \mathbf{v}_\beta \rangle}{\partial t}. \tag{1.18}$$

The averaging theorem can be used in order to express the convective inertial terms as

$$\langle \nabla \cdot (\mathbf{v}_\beta \mathbf{v}_\beta) \rangle = \nabla \cdot \langle \mathbf{v}_\beta \mathbf{v}_\beta \rangle + \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \mathbf{v}_\beta \mathbf{v}_\beta \, \mathrm{d}A \tag{1.19}$$

and on the basis of Equation (1.3) this simplifies to

$$\langle \nabla \cdot (\mathbf{v}_\beta \mathbf{v}_\beta) \rangle = \nabla \cdot \langle \mathbf{v}_\beta \mathbf{v}_\beta \rangle. \tag{1.20}$$

To eliminate the average of a product, we make use of the velocity decomposition given by Gray (1975)

$$\mathbf{v}_\beta = \langle \mathbf{v}_\beta \rangle^\beta + \tilde{\mathbf{v}}_\beta, \tag{1.21}$$

so that the convective inertial term can be expressed as

$$\langle \mathbf{v}_\beta \mathbf{v}_\beta \rangle = \langle \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta \rangle + \langle \langle \mathbf{v}_\beta \rangle^\beta \tilde{\mathbf{v}}_\beta \rangle + \langle \tilde{\mathbf{v}}_\beta \langle \mathbf{v}_\beta \rangle^\beta \rangle + \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle \tag{1.22}$$

Following the work of Carbonell and Whitaker (1984), we neglect the variation of average quantities within the averaging volume so that Equation (1.22) takes the form

$$\langle \mathbf{v}_\beta \mathbf{v}_\beta \rangle = \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta \langle 1 \rangle + \langle \mathbf{v}_\beta \rangle^\beta \langle \tilde{\mathbf{v}}_\beta \rangle + \langle \tilde{\mathbf{v}}_\beta \rangle \langle \mathbf{v}_\beta \rangle^\beta + \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle \tag{1.23}$$

This simplification requires that we impose the following length-scale constraints

$$\ell_\beta \ll r_0, \quad r_0^2 \ll L^2, \tag{1.24}$$

in which ℓ_β is the characteristic length for the β-phase, r_0 is the radius of the averaging volume, and L is a generic length-scale associated with averaged quantities (Quintard and Whitaker, 1994). These three length-scales are illustrated in Figure 2. It is consistent with Equation (1.24) to set the average of the spatial deviation equal to zero, i.e.,

$$\langle \tilde{\mathbf{v}}_\beta \rangle = 0. \tag{1.25}$$

Under these circumstances, Equation (1.23) simplifies to

$$\langle \mathbf{v}_\beta \mathbf{v}_\beta \rangle = \varepsilon_\beta \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle, \tag{1.26}$$

in which we have again made use of $\langle 1 \rangle = \varepsilon_\beta$. Substitution of Equations (1.18) through (1.26) into Equation (1.17) allows us to express the volume averaged Navier–Stokes equations as

$$\rho_\beta \frac{\partial \langle \mathbf{v}_\beta \rangle}{\partial t} + \rho_\beta \varepsilon_\beta \langle \mathbf{v}_\beta \rangle^\beta \nabla \cdot \langle \mathbf{v}_\beta \rangle^\beta + \rho_\beta \nabla \cdot \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle$$
$$= -\langle \nabla p_\beta \rangle + \varepsilon_\beta \rho_\beta \mathbf{g} + \mu_\beta \langle \nabla^2 \mathbf{v}_\beta \rangle. \tag{1.27}$$

Here we have made use of the fact that the divergence of $\varepsilon_\beta \langle \mathbf{v}_\beta \rangle^\beta$ is zero as indicated by Equation (1.14), and we can also make use of Equation (1.8) and the fact that ε_β is independent of time in order to express the local acceleration in terms of the intrinsic average velocity.

$$\rho_\beta \varepsilon_\beta \frac{\partial \langle \mathbf{v}_\beta \rangle^\beta}{\partial t} + \rho_\beta \varepsilon_\beta \langle \mathbf{v}_\beta \rangle^\beta \nabla \cdot \langle \mathbf{v}_\beta \rangle^\beta + \rho_\beta \nabla \cdot \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle$$
$$= -\langle \nabla p_\beta \rangle + \varepsilon_\beta \rho_\beta \mathbf{g} + \mu_\beta \langle \nabla^2 \mathbf{v}_\beta \rangle. \tag{1.28}$$

The right-hand side of Equation (1.28) has been analyzed in detail by Whitaker (1986) and by Quintard and Whitaker (1994), and we follow those prior studies to arrive at

$$\rho_\beta \varepsilon_\beta \frac{\partial \langle \mathbf{v}_\beta \rangle^\beta}{\partial t} + \rho_\beta \varepsilon_\beta \langle \mathbf{v}_\beta \rangle^\beta \nabla \cdot \langle \mathbf{v}_\beta \rangle^\beta + \rho_\beta \nabla \cdot \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle$$
$$= -\varepsilon_\beta \nabla \langle p_\beta \rangle^\beta + \varepsilon_\beta \rho_\beta \mathbf{g} + \mu_\beta \Big(\varepsilon_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle^\beta +$$
$$+ \nabla \varepsilon_\beta \cdot \nabla \langle \mathbf{v}_\beta \rangle^\beta + \langle \mathbf{v}_\beta \rangle^\beta \nabla^2 \varepsilon_\beta \Big) +$$
$$+ \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta) \, \mathrm{d}A \tag{1.29}$$

Here we have represented the pressure according to

$$p_\beta = \langle p_\beta \rangle^\beta + \tilde{p}_\beta, \tag{1.30}$$

which is analogous to the decomposition for the velocity given by Equation (1.21).

Equation (1.29) represents a *superficial average* form of the Navier–Stokes equations, i.e., each term represents a force *per unit volume of the porous medium* Traditionally, the *intrinsic average* momentum equation is preferred since it provides a form containing $\nabla \langle p_\beta \rangle^\beta$ and this is a key quantity of interest. The *intrinsic average* form is obtained by dividing Equation (1.29) by ε_β to obtain

$$\begin{aligned}
&\rho_\beta \frac{\partial \langle \mathbf{v}_\beta \rangle^\beta}{\partial t} + \rho_\beta \langle \mathbf{v}_\beta \rangle^\beta \nabla \cdot \langle \mathbf{v}_\beta \rangle^\beta + \rho_\beta \varepsilon_\beta^{-1} \nabla \cdot \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle \\
&= -\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} + \mu_\beta \Big(\nabla^2 \langle \mathbf{v}_\beta \rangle^\beta + \\
&\quad + \varepsilon_\beta^{-1} \nabla \varepsilon_\beta \cdot \nabla \langle \mathbf{v}_\beta \rangle^\beta + \varepsilon_\beta^{-1} \langle \mathbf{v}_\beta \rangle^\beta \nabla^2 \varepsilon_\beta \Big) + \\
&\quad + \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta) \, \mathrm{d}A.
\end{aligned} \tag{1.31}$$

When the terms involving $\nabla \varepsilon_\beta$ and $\nabla^2 \varepsilon_\beta$ are *important*, there is no simple representation for the area integral in Equation (1.31), and that means that there is no simple solution to the closure problem to be discussed in the next section. Regions in which the porosity varies rapidly are generally associated with the boundary between a porous medium and either a homogeneous fluid or a homogeneous solid. Those regions can be conveniently treated in terms of a *momentum jump condition*, the development of which is described by Ochoa-Tapia and Whitaker (1995).

In regions where the terms involving $\nabla \varepsilon_\beta$ and $\nabla^2 \varepsilon_\beta$ can be neglected, Equation (1.31) simplifies to

$$\begin{aligned}
&\rho_\beta \frac{\partial \langle \mathbf{v}_\beta \rangle^\beta}{\partial t} + \rho_\beta \langle \mathbf{v}_\beta \rangle^\beta \nabla \cdot \langle \mathbf{v}_\beta \rangle^\beta + \underbrace{\rho_\beta \varepsilon_\beta^{-1} \nabla \cdot \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle}_{\text{volume filter}} \\
&= -\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle^\beta + \\
&\quad \underbrace{+ \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta) \, \mathrm{d}A}_{\text{surface filter}}
\end{aligned} \tag{1.32}$$

Here we have identified the term involving the volume average of $\tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta$ as a *volume filter*, while the last term involving the area integral of $-\mathbf{I}\tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta$ is referred to as a *surface filter*. We use these words because the microscale information that will be available in the closure problem will be *filtered* by these integrals. That is to say that not all the information available at the microscale will appear in the volume averaged momentum equation, and knowing how these filters function is an important aspect of the method of volume averaging.

The third term on the right hand side of Equation (1.32) represents the *Brinkman correction* (Brinkman, 1947) which is often retained to permit the use of boundary conditions involving continuity of the volume averaged velocity; however, such boundary conditions are usually imposed in regions where there are rapid changes in the porosity. Under those circumstances, it is better to make use of a momentum jump condition (Ochoa-Tapia and Whitaker, 1995) to develop the boundary condition. While the Brinkman term is generally negligible, we will retain this term for completeness and move on to the development of the closure problem for $\tilde{p}_\beta$ and $\tilde{\mathbf{v}}_\beta$.

2. Closure

In order to obtain the closed form of Equation (1.32), we need to develop the governing differential equations and boundary conditions for $\tilde{\mathbf{v}}_\beta$ and $\tilde{p}_\beta$. This will lead us to a local closure problem in terms of *closure variables* and a method of predicting the Darcy's law permeability tensor and the Forchheimer correction tensor that appear in the closed form.

2.1. BOUNDARY CONDITION

The no-slip boundary condition plays a key role (Whitaker, 1997) in the closed form of Equation (1.32), thus we begin our analysis with Equation (1.3) and make use of the velocity decomposition represented by Equation (1.20) to obtain

$$\text{(B.C.1)} \qquad \tilde{\mathbf{v}}_\beta = -\underbrace{\langle \mathbf{v}_\beta \rangle^\beta}_{\text{source}}, \quad \text{at } \mathcal{A}_{\beta\sigma}. \tag{2.1}$$

Here we have identified the intrinsic average velocity evaluated at the β-σ interface as a *source*, and we will soon see that it is essentially the only source in the closure problem

2.2. CONTINUITY EQUATION

In order to develop the continuity equation for $\tilde{\mathbf{v}}_\beta$, we subtract Equation (1.14) from Equation (1.2) and follow the previous work of Whitaker (1986) and Quintard and Whitaker (1994) to obtain

$$\nabla \cdot \tilde{\mathbf{v}}_\beta = 0. \tag{2.2}$$

We are now ready to move on to the momentum equation for the spatial deviation velocity, $\tilde{\mathbf{v}}_\beta$, and the spatial deviation pressure, $\tilde{p}_\beta$.

2.3. MOMENTUM EQUATION

Here we recall the point and volume averaged momentum equations given by

$$\rho_\beta\left(\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{v}_\beta \cdot \nabla \mathbf{v}_\beta\right) = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{v}_\beta, \tag{2.3}$$

$$\begin{aligned}
&\rho_\beta \frac{\partial \langle \mathbf{v}_\beta\rangle^\beta}{\partial t} + \rho_\beta \langle \mathbf{v}_\beta\rangle^\beta \nabla \cdot \langle \mathbf{v}_\beta\rangle^\beta + \rho_\beta \varepsilon_\beta^{-1} \nabla \cdot \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta\rangle \\
&\quad = -\nabla \langle p_\beta\rangle^\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta\rangle^\beta + \\
&\qquad + \frac{1}{V_\beta}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta)\,\mathrm{d}A.
\end{aligned} \tag{2.4}$$

Subtracting the second of these from the first provides the spatial deviation momentum equation that takes the form

$$\begin{aligned}
&\rho_\beta \frac{\partial \tilde{\mathbf{v}}_\beta}{\partial t} + \rho_\beta \mathbf{v}_\beta \cdot \nabla \tilde{\mathbf{v}}_\beta + \rho_\beta \tilde{\mathbf{v}}_\beta \cdot \nabla \langle \mathbf{v}_\beta\rangle^\beta \\
&\quad = -\nabla \tilde{p}_\beta + \mu_\beta \nabla^2 \tilde{\mathbf{v}}_\beta - \frac{1}{V_\beta}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta)\,\mathrm{d}A + \\
&\qquad + \rho_\beta \varepsilon_\beta^{-1} \nabla \cdot \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta\rangle.
\end{aligned} \tag{2.5}$$

In this result, the term $\nabla\langle \mathbf{v}_\beta\rangle^\beta$ represents a parameter that must be specified in order to solve the complete closure problem; however, $\rho_\beta \tilde{\mathbf{v}}_\beta \cdot \nabla\langle \mathbf{v}_\beta\rangle^\beta$ is *not* a non-homogeneous term in Equation (2.5), thus $\nabla\langle \mathbf{v}_\beta\rangle^\beta$ is *not* a source in the closure problem.

The closure equation given by Equation (2.5) can be simplified considerably on the basis of the following order of magnitude estimates

$$\tilde{\mathbf{v}}_\beta = \mathbf{O}(\langle \mathbf{v}_\beta\rangle^\beta), \qquad \nabla \tilde{\mathbf{v}}_\beta = \mathbf{O}\frac{\langle \mathbf{v}_\beta\rangle^\beta}{\ell_\beta}, \qquad \nabla\langle \mathbf{v}_\beta\rangle^\beta = \mathbf{O}\frac{\langle \mathbf{v}_\beta\rangle^\beta}{L}. \tag{2.6}$$

The first of these results from the no-slip condition given by Equation (2.1), while the second and third are based on the characteristic lengths associated with $\tilde{\mathbf{v}}_\beta$ and $\langle \mathbf{v}_\beta\rangle^\beta$. Since the length scale associated with the averaged quantity, $\langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta\rangle$, is comparable to that for $\langle \mathbf{v}_\beta\rangle^\beta$, we can simplify Equation (2.5) on the basis of

$$\rho_\beta \mathbf{v}_\beta \cdot \nabla \tilde{\mathbf{v}}_\beta \gg \rho_\beta \tilde{\mathbf{v}}_\beta \cdot \nabla \langle \mathbf{v}_\beta\rangle^\beta, \tag{2.7a}$$

$$\rho_\beta \mathbf{v}_\beta \cdot \nabla \tilde{\mathbf{v}}_\beta \gg \rho_\beta \varepsilon_\beta^{-1} \nabla \cdot \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle. \tag{2.7b}$$

This requires only the imposition of the length-scale constraint given by

$$\ell_\beta \ll L \tag{2.8}$$

and this is entirely consistent with constraints that have already been imposed. For many practical cases, one can make use of the quasi-steady simplification on the basis that

$$\rho_\beta \frac{\partial \tilde{\mathbf{v}}_\beta}{\partial t} \ll \mu_\beta \nabla^2 \tilde{\mathbf{v}}_\beta. \tag{2.9}$$

This means that we will impose the time-scale constraint

$$\frac{\nu_\beta t^*}{\ell_\beta^2} \gg 1, \tag{2.10}$$

in which t^* represents the characteristic process time, and ν_β is the kinematic viscosity.

At this point we have dealt with the closure analogs of Equations (1.1) through (1.3), and we need only construct a boundary condition associated with Equation (1.4) in order to complete the statement of the boundary value problem for the spatial deviation pressure and velocity.

2.4. CLOSURE PROBLEM

We summarize the closure problem in a form analogous to Equations (1.1) through (1.4), and note that it is this boundary-value problem that provides the microscale information for the filters in Equation (1.32).

$$\begin{aligned} \rho_\beta \mathbf{v}_\beta \cdot \nabla \tilde{\mathbf{v}}_\beta &= -\nabla \tilde{p}_\beta + \mu_\beta \nabla^2 \tilde{\mathbf{v}}_\beta - \\ &\quad -\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta)\, \mathrm{d}A, \end{aligned} \tag{2.11}$$

$$\nabla \cdot \tilde{\mathbf{v}}_\beta = 0, \tag{2.12}$$

$$\text{(B.C.1)} \quad \tilde{\mathbf{v}}_\beta = -\underbrace{\langle \mathbf{v}_\beta \rangle^\beta}_{\text{source}}, \quad \text{at } \mathcal{A}_{\beta\sigma}, \tag{2.13}$$

$$\text{(B.C.2)} \quad \tilde{\mathbf{v}}_\beta = -\underbrace{\mathbf{g}(\mathbf{r}, t)}_{\text{source}}, \quad \text{at } \mathcal{A}_{\beta e}. \tag{2.14}$$

The boundary condition given by Equation (2.14) is a reminder of what we *do not* know about the $\tilde{\mathbf{v}}_\beta$-field rather than what we do know. However, we do know

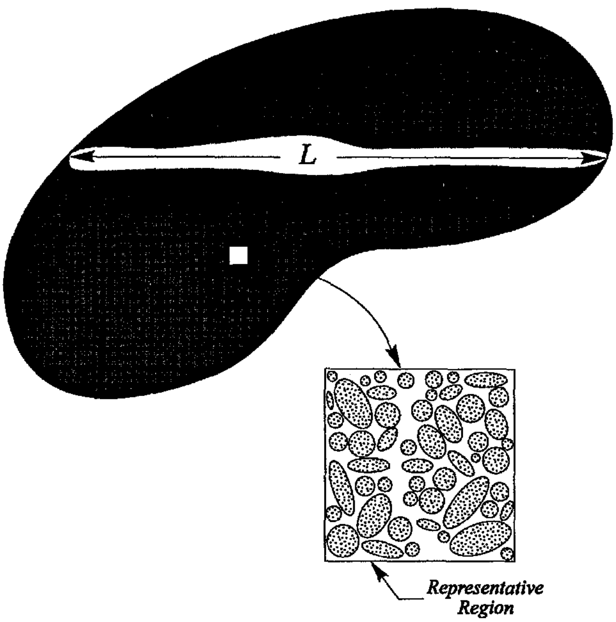

Figure 4. Representative region.

that $\mathbf{g}(\mathbf{r}, t)$ is on the order of $\langle \mathbf{v}_\beta \rangle^\beta$, and we do know that the boundary condition given by Equation (2.14) will influence the spatial deviation fields only in a region of thickness ℓ_β at the boundary of the macroscopic region illustrated in Figure 2. This suggests that the boundary condition at $\mathcal{A}_{\beta e}$ can be ignored if we can find a suitable replacement.

2.5. LOCAL CLOSURE PROBLEM

Obviously we do not want to solve Equations (2.11) through (2.14) in the macroscopic region illustrated in Figure 2; instead we wish to solve the closure problem in some representative region such as the one illustrated in Figure 4. To do so, we must be willing to discard the boundary condition given by Equation (2.14) and replace it with some *local condition* associated with the representative region shown in Figure 4. This naturally leads us to treat the representative region as a unit cell in a spatially periodic model of a porous medium so that our closure problem takes the form

$$\rho_\beta \mathbf{v}_\beta \cdot \nabla \tilde{\mathbf{v}}_\beta = -\nabla \tilde{p}_\beta + \mu_\beta \nabla^2 \tilde{\mathbf{v}}_\beta - \\ -\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta)\, \mathrm{d}A, \qquad (2.15)$$

$$\text{(B.C.1)} \quad \tilde{\mathbf{v}}_\beta = -\underbrace{\langle \mathbf{v}_\beta \rangle^\beta}_{\text{source}}, \qquad \text{at} \quad \mathcal{A}_{\beta\sigma}, \tag{2.17}$$

$$\text{Periodicity}: \quad \tilde{p}_\beta(\mathbf{r} + \ell_i) = \tilde{p}_\beta(\mathbf{r}), \qquad \tilde{\mathbf{v}}_\beta(\mathbf{r} + \ell_i) = \tilde{\mathbf{v}}_\beta(\mathbf{r}),$$
$$i = 1, 2, 3, \tag{2.18}$$

$$\text{Average:} \quad \langle \tilde{\mathbf{v}}_\beta \rangle^\beta = 0. \tag{2.19}$$

Here we require that the average of the spatial deviation velocity be zero, and this condition is necessary in order to evaluate the integral in Equation (2.15). It is important to note that the periodicity condition given by Equation (2.18) is *consistent with* a spatially periodic model *only* if variations of $\langle \mathbf{v}_\beta \rangle^\beta$ can be neglected within the unit cell.

2.6. CLOSURE VARIABLES

Given the single, constant source in the boundary value problem for $\tilde{p}_\beta$ and $\tilde{\mathbf{v}}_\beta$, we propose a solution for the spatial deviation velocity and pressure of the form

$$\tilde{\mathbf{v}}_\beta = \mathbf{M} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{u}, \tag{2.20}$$

$$\tilde{p}_\beta = \mu_\beta \mathbf{m} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \xi, \tag{2.21}$$

in which $\mathbf{u}$ and ξ represent vector and scalar fields respectively. We are free to specify $\mathbf{M}$ and $\mathbf{m}$ in any way we wish, and we specify these two functions by means of the following closure problem

$$(\rho_\beta \mathbf{v}_\beta / \mu_\beta) \cdot \nabla \mathbf{M}$$
$$= -\nabla \mathbf{m} + \nabla^2 \mathbf{M} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\mathbf{m} + \nabla \mathbf{M}) \, \mathrm{d}A, \tag{2.22a}$$

$$\nabla \cdot \mathbf{M} = 0, \tag{2.22b}$$

$$\text{(B.C.1)} \quad \mathbf{M} = -\mathbf{I}, \qquad \text{at} \quad A_{\beta\sigma}, \tag{2.22c}$$

$$\text{Periodicity:} \quad \mathbf{m}(\mathbf{r} + \ell_i) = \mathbf{m}(\mathbf{r}), \quad \mathbf{M}(\mathbf{r} + \ell_i) = \mathbf{M}(\mathbf{r}), \quad i = 1, 2, 3, \tag{2.22d}$$

$$\text{Average:} \quad \langle \mathbf{M} \rangle^\beta = 0. \tag{2.22e}$$

When $\mathbf{M}$ and $\mathbf{m}$ are specified in this manner, one can prove that the vector $\mathbf{u}$ is *zero* and that the scalar ξ is a *constant*, and the proof is given in Appendix A. Since the constant ξ will not pass through the filter represented by the area integral in Equation (1.32), we can express the spatial deviation velocity and pressure as

$$\tilde{\mathbf{v}}_\beta = \mathbf{M} \cdot \langle \mathbf{v}_\beta \rangle^\beta, \tag{2.23}$$

$$\tilde{p}_\beta = \mu_\beta \mathbf{m} \cdot \langle \mathbf{v}_\beta \rangle^\beta. \tag{2.24}$$

These two representations can now be used to develop the closed form of the volume averaged momentum equation.

2.7. CLOSED FORM

To obtain the closed form of the volume averaged momentum equation, we first recall Equation (1.32)

$$\begin{aligned} &\rho_\beta \frac{\partial \langle \mathbf{v}_\beta \rangle^\beta}{\partial t} + \rho_\beta \langle \mathbf{v}_\beta \rangle^\beta \nabla \cdot \langle \mathbf{v}_\beta \rangle^\beta + \rho_\beta \varepsilon_\beta^{-1} \nabla \cdot \langle \tilde{\mathbf{v}}_\beta \tilde{\mathbf{v}}_\beta \rangle \\ &\quad = -\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle^\beta + \\ &\quad + \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \tilde{p}_\beta + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta) \, \mathrm{d}A \end{aligned} \tag{2.25}$$

and make use of Equations (2.23) and (2.24) to obtain

$$\begin{aligned} &\rho_\beta \frac{\partial \langle \mathbf{v}_\beta \rangle^\beta}{\partial t} + \rho_\beta \langle \mathbf{v}_\beta \rangle^\beta \nabla \cdot \langle \mathbf{v}_\beta \rangle^\beta + \\ &\quad + \rho_\beta \varepsilon_\beta^{-1} \nabla \cdot \left(\langle \mathbf{v}_\beta \rangle^\beta \cdot \langle \mathbf{M}^T \mathbf{M} \rangle \cdot \langle \mathbf{v}_\beta \rangle^\beta \right) \\ &\quad = -\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle^\beta + \\ &\quad + \mu_\beta \left\{ \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\mathbf{m} + \nabla \mathbf{M}) \, \mathrm{d}A \right\} \cdot \langle \mathbf{v}_\beta \rangle^\beta. \end{aligned} \tag{2.26}$$

While the *microscopic inertial effects* that appear in the closure problem given by Equations (2.22) *cannot be neglected* when the Reynolds number is large compared to one, the *macroscopic inertial effects* in Equation (2.26) are generally negligible and we wish to demonstrate this in the following paragraphs.

2.8. MACROSCOPIC INERTIAL EFFECTS

In order to demonstrate that the macroscopic inertial terms in Equation (2.26) are negligible, we need estimates of the inertial terms on the left-hand side and of the dominant viscous term on the right-hand side. From the closure problem given by Equations (2.22) we note that

$$\mathbf{M} = \mathbf{O}(1) \tag{2.27}$$

and that the characteristic length for $\mathbf{M}$ is the small lengthscale, ℓ_β. Directing our attention specifically to Equation (2.22a), we obtain

$$\left\{ \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{Im} + \nabla \mathbf{M}) \, \mathrm{d}A \right\} = \mathbf{O}(\ell_\beta^{-2}) + \mathbf{O}\left(\frac{\rho_\beta \langle v_\beta \rangle^\beta}{\mu_\beta \ell_\beta} \right), \tag{2.28}$$

since $\nabla \mathbf{m}$ cannot be significantly larger than the largest term in Equation (2.22a). This result can be used to estimate the last term in Equation (2.26) as

$$\mu_\beta \left\{ \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{Im} + \nabla \mathbf{M} \, \mathrm{d}A \right\} \cdot \langle \mathbf{v}_\beta \rangle^\beta =$$

$$= \mathbf{O}\left(\frac{\mu_\beta \langle v_\beta \rangle^\beta}{\ell_\beta^2} \right) + \mathbf{O}\left(\frac{\rho_\beta \langle v_\beta \rangle^\beta \langle v_\beta \rangle^\beta}{\ell_\beta} \right) \tag{2.29}$$

and we need to compare this with estimates of the inertial terms in Equation (2.26).

We begin with the first term on the left-hand side of Equation (2.26) and estimate the local acceleration as

$$\rho_\beta \frac{\partial \langle \mathbf{v}_\beta \rangle^\beta}{\partial t} = \mathbf{O}\left(\frac{\rho_\beta \langle v_\beta \rangle^\beta}{t^*} \right). \tag{2.30}$$

Moving on to the second term, we represent the volume averaged velocity in terms of the magnitude and the unit tangent vector

$$\langle \mathbf{v}_\beta \rangle^\beta = \langle v_\beta \rangle^\beta \boldsymbol{\lambda}, \tag{2.31}$$

in order to obtain

$$\rho_\beta \langle \mathbf{v}_\beta \rangle^\beta \cdot \nabla \langle \mathbf{v}_\beta \rangle^\beta = \rho_\beta \langle v_\beta \rangle^\beta \boldsymbol{\lambda} \cdot \nabla \langle \mathbf{v}_\beta \rangle^\beta = \rho_\beta \langle v_\beta \rangle^\beta \frac{\mathrm{d} \langle \mathbf{v}_\beta \rangle^\beta}{\mathrm{d}s}. \tag{2.32}$$

Here we have made use of the fact that $\boldsymbol{\lambda} \cdot \nabla$ is the directional derivative, and we have used s to represent the arclength measured along a volume averaged streamline. We can use the definition of the *inertial length* (Whitaker, 1982) to express our estimate of the convective inertial term as

$$\rho_\beta \langle \mathbf{v}_\beta \rangle^\beta \cdot \nabla \langle \mathbf{v}_\beta \rangle^\beta = \mathbf{O}\left[\frac{\rho_\beta (\langle v_\beta \rangle^\beta)^2}{L_\rho} \right]. \tag{2.33}$$

Our estimate of the third term on the left-hand side of Equation (2.26) is given by

$$\rho_\beta \varepsilon_\beta^{-1} \nabla \cdot (\langle \mathbf{v}_\beta \rangle^\beta \cdot \langle \mathbf{M}^T \mathbf{M} \rangle \cdot \langle \mathbf{v}_\beta \rangle^\beta) = \mathbf{O}\left(\frac{\rho_\beta \varepsilon_\beta^{-1} \langle v_\beta \rangle^\beta \langle v_\beta \rangle^\beta}{L} \right). \tag{2.34}$$

and a little thought will indicate that $L \leqslant L_\rho$, thus we seek only the constraints associated with the following two restrictions

$$\rho_\beta \frac{\partial \langle \mathbf{v}_\beta \rangle^\beta}{\partial t} \ll \mu_\beta \left\{ \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{Im} + \nabla \mathbf{M} \, \mathrm{d}A \right\} \cdot \langle \mathbf{v}_\beta \rangle^\beta, \tag{2.35}$$

$$\rho_\beta \varepsilon_\beta^{-1} \nabla \cdot (\langle \mathbf{v}_\beta \rangle^\beta \cdot \langle \mathbf{M}^T \mathbf{M} \rangle \cdot \langle \mathbf{v}_\beta \rangle^\beta)$$

$$\ll \mu_\beta \left\{ \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{Im} + \nabla \mathbf{M}) \, \mathrm{d}A \right\} \cdot \langle \mathbf{v}_\beta \rangle^\beta. \tag{2.36}$$

Directing our attention to the first of these, we use the estimates given by Equations (2.29) and (2.30) and require only that the local acceleration be small compared to the viscous effect in order to conclude that the macroscopic flow is quasi-steady when

$$\frac{\nu_\beta t^*}{\ell_\beta^2} \gg 1. \tag{2.37}$$

This result is identical to that given by Equation (2.10) which was imposed in the development of the closure problem. Moving on to the second restriction given by Equation (2.36), we employ the estimates given by Equations (2.29) and (2.34) and to obtain

$$\frac{\rho_\beta \langle v_\beta \rangle^\beta \ell_\beta}{\mu_\beta} (\ell_\beta / L) \ll 1, \qquad \ell_\beta \ll L. \tag{2.38}$$

There are a wide variety of processes for which the constraints given by both Equations (2.37) and (2.38) will be satisfied and the closed form of the momentum equation will simplify to

$$\begin{aligned} 0 \; = \; & -\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle^\beta + \\ & + \mu_\beta \left\{ \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{Im} + \nabla \mathbf{M} \, \mathrm{d}A \right\} \cdot \langle \mathbf{v}_\beta \rangle^\beta. \end{aligned} \tag{2.39}$$

Here we have what appears to be Darcy's law with the Brinkman correction; however, the Forchheimer correction term is contained in the area integral as pointed out by Ruth and Ma (1992) and by Ma and Ruth (1993).

2.9. DECOMPOSITION OF THE CLOSURE PROBLEM

Rather than attack the general closure problem given by Equations (2.22), it is of interest to decompose the problem into two parts. The first part will produce the Darcy's law (Darcy, 1856) permeability tensor that depends only on the geometry of the porous medium under consideration, and the second part will lead to an inertial correction, i.e., the Forchheimer equation (Forchheimer, 1901). To accomplish this decomposition, we represent $\mathbf{m}$ and $\mathbf{M}$ as

$$\mathbf{m} = \mathbf{b} + \mathbf{c}, \qquad \mathbf{M} = \mathbf{B} + \mathbf{C} \tag{2.40}$$

and replace the representations given by Equations (2.23) and (2.24) with

$$\tilde{\mathbf{v}}_\beta = \mathbf{B} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{C} \cdot \langle \mathbf{v}_\beta \rangle^\beta, \tag{2.41}$$

$$\tilde{p}_\beta = \mu_\beta \mathbf{b} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mu_\beta \mathbf{c} \cdot \langle \mathbf{v}_\beta \rangle^\beta. \tag{2.42}$$

At this point we *define* the vector $\mathbf{b}$ and the tensor $\mathbf{B}$ by the following closure problem Whitaker (1986).

PROBLEM I

$$0 = -\nabla \mathbf{b} + \nabla^2 \mathbf{B} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\mathbf{b} + \nabla \mathbf{B})\, dA, \tag{2.43a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.43b}$$

$$\text{(B.C.1)} \qquad \mathbf{B} = -\mathbf{I}, \quad \text{at } A_{\beta\sigma}, \tag{2.43c}$$

$$\text{Periodicity:} \qquad \mathbf{b}(\mathbf{r} + \ell_i) = \mathbf{b}(\mathbf{r}), \qquad \mathbf{B}(\mathbf{r} + \ell_i) = \mathbf{B}(\mathbf{r}), \quad i = 1, 2, 3, \tag{2.43d}$$

$$\text{Average:} \qquad \langle \mathbf{B} \rangle^\beta = 0. \tag{2.43e}$$

Given this definition of $\mathbf{b}$ and $\mathbf{B}$, we can substitute Equations 2.41 and 2.42 into the closure problem given by Equations (2.15) through (2.19) in order to obtain the following closure problem for $\mathbf{c}$ and $\mathbf{C}$.

PROBLEM II

$$\underbrace{(\rho_\beta \mathbf{v}_\beta / \mu_\beta) \cdot \nabla \mathbf{B}}_{\text{source}} + \left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta} \right) \cdot \nabla \mathbf{C}$$

$$= -\nabla \mathbf{c} + \nabla^2 \mathbf{C} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\mathbf{c} + \nabla \mathbf{C})\, dA, \tag{2.44a}$$

$$\nabla \cdot \mathbf{C} = 0, \tag{2.44b}$$

$$\text{(B.C.1)} \qquad \mathbf{C} = 0, \quad \text{at } A_{\beta\sigma}, \tag{2.44c}$$

$$\text{Periodicity:} \qquad \mathbf{c}(\mathbf{r} + \ell_i) = \mathbf{c}(\mathbf{r}), \quad \mathbf{C}(\mathbf{r} + \ell_i) = \mathbf{C}(\mathbf{r}), \quad i = 1, 2, 3, \tag{2.44d}$$

$$\text{Average:} \qquad \langle \mathbf{C} \rangle^{\beta} = 0. \tag{2.44e}$$

In order to determine $\mathbf{c}$ and $\mathbf{C}$ from this boundary-value problem, one needs to first solve Equations (2.43) for the $\mathbf{B}$-field and then calculate the velocity field on the basis of the Navier–Stokes equations. Methods for solving the Navier-Stokes equations in spatially periodic systems are described in the literature, and in Section 5 we will show that these methods apply directly to the solution of the closure problems.

In order to clarify the structure of the closed form of the volume averaged Navier–Stokes equations, we make use of Equations (2.40) in Equation (2.39) to obtain

$$\begin{aligned} 0 = & -\nabla \langle p_\beta \rangle^{\beta} + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle^{\beta} + \\ & + \mu_\beta \left\{ \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{Ib} + \nabla \mathbf{B}) \, \mathrm{d}A \right\} \cdot \langle \mathbf{v}_\beta \rangle^{\beta} \\ & + \mu_\beta \left\{ \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{Ic} + \nabla \mathbf{C}) \, \mathrm{d}A \right\} \cdot \langle \mathbf{v}_\beta \rangle^{\beta}. \end{aligned} \tag{2.45}$$

At this point it is convenient to define the *Darcy's law permeability tensor* by

$$\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot [-\mathbf{Ib} + \nabla \mathbf{B}] \, \mathrm{d}A = -\varepsilon_\beta \mathbf{K}^{-1} \tag{2.46}$$

and the *Forchheimer correction tensor* by

$$\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot [-\mathbf{Ic} + \nabla \mathbf{C}] \, \mathrm{d}A = -\varepsilon_\beta \mathbf{K}^{-1} \cdot \mathbf{F}. \tag{2.47}$$

Here we note that the definitions of $\mathbf{K}$ and $\mathbf{F}$ have been deliberately chosen to produce a momentum equation containing the *superficial average* velocity rather than the *intrinsic average* velocity that appears in Equation (2.45). Use of these two definitions in Equation (2.45) leads to a result

$$\begin{aligned} 0 = & -\nabla \langle p_\beta \rangle^{\beta} + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle^{\beta} - \\ & - \mu_\beta \mathbf{K}^{-1} \cdot \langle \mathbf{v}_\beta \rangle - \mu_\beta \mathbf{K}^{-1} \cdot \mathbf{F} \cdot \langle \mathbf{v}_\beta \rangle \end{aligned} \tag{2.48}$$

that can be arranged in the form

$$\langle \mathbf{v}_\beta \rangle = -\frac{\mathbf{K}}{\mu_\beta} \left(\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g} - \underbrace{\mu_\beta \nabla^2 \langle \mathbf{v}_\beta \rangle^\beta}_{\substack{\text{Brinkman}\\ \text{correction}}} \right) - \underbrace{-\mathbf{F} \cdot \langle \mathbf{v}_\beta \rangle}_{\substack{\text{Fochheimer}\\ \text{correction}}} \tag{2.49}$$

It is important to recognize that the Brinkman correction is expressed in terms of the *intrinsic average* velocity and not the *superficial average* velocity that appears in Darcy's law and the Forchheimer correction. On the basis of the lengthscale constraints that have already been imposed, we can ignore the Brinkman correction to obtain the Forchheimer equation.

$$\langle \mathbf{v}_\beta \rangle = -\frac{\mathbf{K}}{\mu_\beta} (\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g}) - \mathbf{F} \cdot \langle \mathbf{v}_\beta \rangle. \tag{2.50}$$

At this point we can provide some qualitative and quantitative information about the tensor $\mathbf{F}$ on the basis of the closure problems given by Equations (2.43) and (2.44).

3. Forchheimer Correction

From Equation (2.47) we have the following representation for the Forchheimer correction tensor

$$\mathbf{F} = \mathbf{K} \cdot \left\{ \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot [-\mathbf{Ic} + \nabla \mathbf{C}] \, \mathrm{d}A \right\} \tag{3.1}$$

and in order to learn something about $\mathbf{F}$ we need an estimate of the Darcy's law permeability tensor and an estimate of the area integral that appears in the closure problem given by Equations (2.44). Since $\nabla \mathbf{c}$ and $\nabla^2 \mathbf{C}$ have the characteristics of spatial deviations, one can argue that

$$\langle -\nabla \mathbf{c} + \nabla^2 \mathbf{C} \rangle = \mathbf{O}(-\nabla \mathbf{c} + \nabla^2 \mathbf{C}), \tag{3.2}$$

represents and *overestimate* of the superficial average represented by $\langle -\nabla \mathbf{c} + \nabla^2 \mathbf{C} \rangle$. Use of the divergence theorem and the periodicity conditions for $\mathbf{c}$ and $\mathbf{C}$ allows us to express Equation (3.2) as

$$\frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot [-\mathbf{Ic} + \nabla \mathbf{C}] \, \mathrm{d}A = \mathbf{O}(-\nabla \mathbf{c} + \nabla^2 \mathbf{C}) \tag{3.3}$$

and this suggests that the Forchheimer correction tensor is *overestimated* by

$$\mathbf{F} = \ell_\beta^2 \mathbf{O}(-\nabla \mathbf{c} + \nabla^2 \mathbf{C}). \tag{3.4}$$

Here we have estimated the Darcy's law permeability tensor according to

$$\mathbf{K} = \mathbf{O}(\ell_\beta^2). \tag{3.5}$$

Returning to Equation (2.44a), we make use of Equation (3.3) to express the order of magnitude form of that equation as

$$\underbrace{(\rho_\beta \mathbf{v}_\beta / \mu_\beta) \cdot \nabla \mathbf{B}}_{\text{source}} + (\rho_\beta \mathbf{v}_\beta / \mu_\beta) \cdot \nabla \mathbf{C} = \mathbf{O}(-\nabla \mathbf{c} + \nabla^2 \mathbf{C}). \tag{3.6}$$

Since the tensor $\mathbf{B}$ is a dimensionless quantity of order one and has a characteristic length of ℓ_β, an estimate of the *source* is given by

$$\underbrace{(\rho_\beta \mathbf{v}_\beta / \mu_\beta) \cdot \nabla \mathbf{B}}_{\text{source}} = \mathbf{O}\left(\frac{\rho_\beta \langle v_\beta \rangle^\beta}{\mu_\beta \ell_\beta}\right) \tag{3.7}$$

and Equation (3.6) takes the form

$$\mathbf{O}\left(\frac{\rho_\beta \langle v_\beta \rangle^\beta}{\mu_\beta \ell_\beta}\right) + \mathbf{O}\left[\left(\frac{\rho_\beta \langle v_\beta \rangle^\beta}{\mu_\beta}\right) \cdot \nabla \mathbf{C}\right] = \mathbf{O}(-\nabla \mathbf{c} + \nabla^2 \mathbf{C}). \tag{3.8}$$

About the term $\nabla \mathbf{c}$, we can say that it cannot be *much larger* than any of the other three terms in Equation (3.8), while it could be much smaller that any of those terms. Since it cannot be much larger, we can express Equation (3.8) as

$$\mathbf{O}\left(\frac{\rho_\beta \langle v_\beta \rangle^\beta}{\mu_\beta \ell_\beta}\right) + \mathbf{O}\left[\left(\frac{\rho_\beta \langle v_\beta \rangle^\beta}{\mu_\beta}\right) \cdot \nabla \mathbf{C}\right] = \mathbf{O}(\nabla^2 \mathbf{C}). \tag{3.9}$$

We now construct the estimates

$$\nabla \mathbf{C} = \mathbf{O}\left(\frac{\mathbf{C}}{\delta_\beta}\right), \qquad \nabla^2 \mathbf{C} = \mathbf{O}\left(\frac{\mathbf{C}}{\delta_\beta^2}\right), \tag{3.10}$$

in which δ_β represents the hydrodynamic boundary-layer thickness that is constrained by

$$\delta_\beta = \mathbf{O}(\ell_\beta), \quad \mathrm{Re} \ll 1, \tag{3.11a}$$

$$\delta_\beta = \mathbf{O}\left(\frac{\ell_\beta}{\sqrt{\mathrm{Re}}}\right), \quad \mathrm{Re} \gg 1, \tag{3.11b}$$

and Re represents a Reynolds number that is loosely defined by

$$\mathrm{Re} = \frac{\rho_\beta \langle v_\beta \rangle^\beta \ell_\beta}{\mu_\beta}. \tag{3.12}$$

Use of the estimates given by Equations (3.10) in Equation (3.9) provides

$$\mathbf{C} = \mathbf{O}\left\{ \frac{(\delta_\beta/\ell_\beta)^2 \mathrm{Re}}{1 + \mathbf{O}[(\delta_\beta/\ell_\beta)\mathrm{Re}]} \right\}. \tag{3.13}$$

We now return to the left-hand side of Equation (3.8) and use this estimate of $\mathbf{C}$ in order to obtain

$$\mathbf{O}\left(\frac{\rho_\beta \langle v_\beta \rangle^\beta}{\mu_\beta \ell_\beta} \right) + \mathbf{O}\left[\left(\frac{\rho_\beta \langle v_\beta \rangle^\beta}{\mu_\beta \delta_\beta} \right) \left\{ \frac{\mathbf{O}(\delta_\beta/\ell_\beta)^2 \mathrm{Re}}{1 + \mathbf{O}[(\delta_\beta/\ell_\beta)\mathrm{Re}]} \right\} \right] =$$

$$= \mathbf{O}(-\nabla \mathbf{c} + \nabla^2 \mathbf{C}) \tag{3.14}$$

and when this estimate of $-\nabla \mathbf{c} + \nabla^2 \mathbf{C}$ is used in Equation (3.4) the following estimate of the Forchheimer correction tensor results.

$$\mathbf{F} = \left\{ \mathbf{O}(\mathrm{Re}) + \mathbf{O}\left(\frac{(\delta_\beta/\ell_\beta)\mathrm{Re}^2}{1 + \mathbf{O}[(\delta_\beta/\ell_\beta)\mathrm{Re}]} \right) \right\}. \tag{3.15}$$

Making use of the representations for δ_β given by Equation (3.11) leads to

$$\mathbf{F} = \mathbf{O}(\mathrm{Re}), \quad \mathrm{Re} \ll 1, \tag{3.16}$$

$$\mathbf{F} = \{\mathbf{O}(\mathrm{Re}) + \mathbf{O}(\mathrm{Re})\}, \quad \mathrm{Re} \gg 1. \tag{3.17}$$

Under these circumstances, the quadratic dependence of the Forchheimer correction seems entirely plausible; however, a complete solution of Equations (2.44) is needed to determine the precise functional dependence of $\mathbf{F}$.

For small Reynolds numbers we can say something more specific about the behavior of $\mathbf{C}$ than the order of magnitude estimates given in the previous paragraphs. Returning to the closure problem given by Equations (2.44), we note that $\mathbf{C} \to 0$ as $\mathbf{v}_\beta \to 0$ and this suggests an asymptotic analysis with the Reynolds number being the small parameter. When

$$\mathrm{Re} \ll 1, \tag{3.18}$$

we know that $\mathbf{C} \ll \mathbf{B}$ and we can use Equations (1.21) and (2.41) to express the velocity as

$$\mathbf{v}_\beta = (\mathbf{I} + \mathbf{B}) \cdot \langle \mathbf{v}_\beta \rangle^\beta. \tag{3.19}$$

This result, along with $\nabla\mathbf{C} \ll \nabla\mathbf{B}$, allows us to express Equations (2.44) as

$$\frac{\rho_\beta \langle \mathbf{v}_\beta \rangle^\beta}{\mu_\beta} \cdot (\mathbf{I} + \mathbf{B})^T \cdot \nabla \mathbf{B} = -\nabla \mathbf{c} + \nabla^2 \mathbf{C} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\mathbf{c} + \nabla \mathbf{C}) \, \mathrm{d}A, \tag{3.20a}$$

$$\nabla \cdot \mathbf{C} = 0, \tag{3.20b}$$

$$\text{(B.C.1)} \quad \mathbf{C} = 0, \quad \text{at } A_{\beta\sigma}, \tag{3.20c}$$

$$\text{Periodicity}: \quad \mathbf{c}(\mathbf{r} + \ell_i) = \mathbf{c}(\mathbf{r}), \qquad \mathbf{C}(\mathbf{r} + \ell_i) = \mathbf{C}(\mathbf{r}), \quad i = 1, 2, 3, \tag{3.20d}$$

$$\text{Average}: \quad \langle \mathbf{C} \rangle^\beta = 0. \tag{3.20e}$$

This is essentially the form of the closure problem for the first term in a perturbation expansion, and on the basis of Equations (3.20) it becomes clear that both $\mathbf{c}$ and $\mathbf{C}$ will be *linear functions* of $\langle \mathbf{v}_\beta \rangle^\beta$. This means that $\mathbf{F}$, as defined by Equation (3.1), will also be a *linear function* $\langle \mathbf{v}_\beta \rangle^\beta$.

The problem stated by Equations (1.1) through (1.4) has also been studied by means of the method of spatial homogenization (Bensoussan *et al.*, 1978; Sanchez-Palencia, 1980; Ene and Poliševski, 1987). Using that technique, both Mei and Auriault (1991) and Wodie and Levy (1991) found that the first inertial correction to Darcy's law is a *cubic function* of the velocity rather than the *quadratic dependence* illustrated by Equations (3.20) and the definition of the Forchheimer correction tensor given by Equation (3.1).

4. Solution of Closure Problems

The closure problems given by Equations (2.43) and (2.44) would appear to be rather difficult to solve numerically; however, we can follow the procedure given by Barrère *et al.* (1992) in order to develop rather simple forms for these problems. We begin with Problem I given by Equations (2.43), and make use of the definition given by Equation (2.46) so that the closure problem can be expressed as

$$0 = -\nabla \mathbf{b} + \nabla^2 \mathbf{B} + \varepsilon_\beta \mathbf{K}^{-1}, \tag{4.1a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4.1b}$$

$$\text{(B.C.1)} \quad \mathbf{B} = -\mathbf{I}, \quad \text{at } A_{\beta\sigma}, \tag{4.1c}$$

$$\text{Periodicity}: \quad \mathbf{b}(\mathbf{r}+\ell_i)=\mathbf{b}(\mathbf{r}), \quad \mathbf{B}(\mathbf{r}+\ell_i)=\mathbf{B}(\mathbf{r}), \quad i=1,2,3, \tag{4.1d}$$

$$\text{Average}: \quad \langle\mathbf{B}\rangle^\beta = 0. \tag{4.1e}$$

This boundary value problem can be solved numerically using an *iterative scheme* to search for a value of $\mathbf{K}$ that will produce a $\mathbf{B}$-field that satisfies Equation (4.1e), and Quintard and Whitaker (1993) have used this type of approach in the solution of heat conduction closure problems. However, one can develop a direct solution to this problem by the use of a simple transformation (Barrère *et al.*, 1992). We define a new vector $\mathbf{d}$ and a new tensor $\mathbf{D}$ according to

$$\mathbf{d} = \varepsilon_\beta^{-1}\mathbf{b}\cdot\mathbf{K}, \qquad \mathbf{D} = \varepsilon_\beta^{-1}(\mathbf{B}+\mathbf{I})\cdot\mathbf{K}, \tag{4.2}$$

so that Equations (4.1) can be expressed as

$$0 = -\nabla\mathbf{d} + \nabla^2\mathbf{D} + \mathbf{I}, \tag{4.3a}$$

$$\nabla\cdot\mathbf{D} = 0, \tag{4.3b}$$

$$\text{(B.C.1)} \quad \mathbf{D} = 0, \quad \text{at } A_{\beta\sigma}, \tag{4.3c}$$

$$\text{Periodicity}: \quad \mathbf{d}(\mathbf{r}+\ell_i)=\mathbf{d}(\mathbf{r}), \qquad \mathbf{D}(\mathbf{r}+\ell_i)=\mathbf{D}(\mathbf{r}), \quad i=1,2,3, \tag{4.3d}$$

$$\text{Average}: \quad \langle\mathbf{D}\rangle^\beta = \varepsilon_\beta^{-1}\mathbf{K}. \tag{4.3e}$$

Here we see that the constraint on the average of $\tilde{\mathbf{v}}_\beta$, given by Equation (2.19), is required required in order to determine the permeability tensor as indicated by Equation (4.3e). The solution of Equations (4.3) is relatively straightforward and can be carried out with any numerical code that is capable of solving Stokes equations.

In order to determine the Forchheimer correction tensor, we could solve the closure problem given by Equations (2.44); however, it is more convenient to solve the original closure problem given by Equations (2.22) and then extract the tensor $\mathbf{F}$ from the final result. We can express Equations (2.22) as

$$\left(\frac{\rho_\beta\mathbf{v}_\beta}{\mu_\beta}\right)\cdot\nabla\mathbf{M} = -\nabla\mathbf{m} + \nabla^2\mathbf{M} + \varepsilon_\beta\mathbf{H}^{-1}, \tag{4.7a}$$

$$\nabla\cdot\mathbf{M} = 0, \tag{4.7b}$$

$$\text{(B.C.1)} \quad \mathbf{M} = -\mathbf{I}, \quad \text{at } A_{\beta\sigma}, \tag{4.7c}$$

$$\text{Periodicity}: \quad \mathbf{m}(\mathbf{r}+\ell_i) = \mathbf{m}(\mathbf{r}), \qquad \mathbf{M}(\mathbf{r}+\ell_i) = \mathbf{M}(\mathbf{r}),$$
$$i = 1, 2, 3, \tag{4.7d}$$

$$\text{Average}: \quad \langle \mathbf{M} \rangle^\beta = 0, \tag{4.7e}$$

in which $\mathbf{H}$ is a constant tensor defined by

$$\mathbf{H}^{-1} = \mathbf{K}^{-1} \cdot (\mathbf{I} + \mathbf{F}). \tag{4.8}$$

Here we follow the approach used with Problem I and define the new variables

$$\mathbf{m}_0 = \varepsilon_\beta^{-1} \mathbf{m} \cdot \mathbf{H}, \qquad \mathbf{M}_0 = \varepsilon_\beta^{-1} (\mathbf{M} + \mathbf{I}) \cdot \mathbf{H}, \tag{4.9}$$

so that the closure problem given by Equations (4.7) takes the form

$$\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta} \right) \cdot \nabla \mathbf{M}_0 = -\nabla \mathbf{m}_0 + \nabla^2 \mathbf{M}_0 + \mathbf{I}, \tag{4.10a}$$

$$\nabla \cdot \mathbf{M}_0 = 0, \tag{4.10b}$$

$$\text{(B.C.1)} \quad \mathbf{M}_0 = 0, \quad \text{at } A_{\beta\sigma}, \tag{4.10c}$$

$$\text{Periodicity}: \quad \mathbf{m}_0(\mathbf{r}+\ell_i) = \mathbf{m}_0(\mathbf{r}), \qquad \mathbf{M}_0(\mathbf{r}+\ell_i) = \mathbf{M}_0(\mathbf{r}),$$
$$i - 1, 2, 3, \tag{4.10d}$$

$$\text{Average}: \quad \langle \mathbf{M}_0 \rangle^\beta = \varepsilon_\beta^{-1} \mathbf{H}. \tag{4.10e}$$

If we set the left-hand side of Equation (4.10a) equal to zero, this closure problem reduces to the closure problem given by Equations (4.1), and the computed value of $\mathbf{H}$ would be equal to the Darcy's law permeability tensor, $\mathbf{K}$.

In the closure problem given by Equations (4.10), we see a boundary value problem that is essentially equivalent to the Navier–Stokes equations for steady, incompressible flow in a spatially periodic system. This means that Equations (4.10) can be solved using any of the codes that have been developed for the solution of the Navier–Stokes equations. Knowing the tensor $\mathbf{H}$ on the basis of Equations (4.10), and the tensor $\mathbf{K}$ on the basis of Equations (4.3), we can determine the Forchheimer correction tensor according to

$$\mathbf{F} = \mathbf{K} \cdot \mathbf{H}^{-1} - \mathbf{I}. \tag{4.11}$$

The advantage of solving two closure problems in order to determine $\mathbf{K}$ and $\mathbf{F}$ is that there is a great deal of information available concerning the Darcy's law permeability tensor, and the solution of the first closure problem allows one to make use of that information. Concerning these two tensor coefficients, one can prove that **K** is symmetric while **F** is not and this is done in Appendix B.

5. Comparison with Experiment

For a uniform flow in the absence of gravitational effects, Equation (2.50) reduces to

$$\langle v_\beta \rangle = -\frac{K_{zz}}{\mu_\beta} \frac{\partial \langle p_\beta \rangle^\beta}{\partial z} - F_{zz} \langle v_\beta \rangle, \tag{5.1}$$

in which we have used $\langle v_\beta \rangle$ to represent the single, nonzero component of $\langle \mathbf{v}_\beta \rangle$. In Section 3 we presented arguments suggesting that the form for F_{zz} is given by

$$F_{zz} = \mathbf{O}(\mathrm{Re}), \tag{5.2}$$

and this leads to

$$\langle v_\beta \rangle = -\frac{K_{zz}}{\mu_\beta} \frac{\partial \langle p_\beta \rangle^\beta}{\partial z} - \mathbf{O}(\mathrm{Re}) \langle v_\beta \rangle. \tag{5.3}$$

This is certainly consistent with experimental observation (Ergun, 1952), the theoretical considerations of Joseph *et al.* (1982), and recent calculations (Ruth and Ma, 1993; Du Plessis, 1994).

The Ergun equation, as modified by Macdonald *et al.* (1979), can be expressed as

$$\langle v_\beta \rangle = -\frac{1}{\mu_\beta} \frac{d_p^2 \varepsilon_\beta^3}{180(1-\varepsilon_\beta)^2} \frac{\partial \langle p_\beta \rangle^\beta}{\partial z} - \frac{1}{100(1-\varepsilon_\beta)} \frac{\rho_\beta \langle v_\beta \rangle d_p}{\mu_\beta} \langle v_\beta \rangle \tag{5.4}$$

and this empirical result has the same form as the theoretical development expressed by Equation (5.3). In Equation (5.4) the effective particle diameter is used as the characteristic length associated with the β-phase, and this quantity is defined by

$$d_p = \frac{6V_p}{A_p}. \tag{5.5}$$

Here V_p and A_p represent the volume and surface area respectively of the particles that make up the porous medium, and from this definition it should be clear that Equation (5.4) only applies to nonconsolidated porous media. Macdonald *et al.* (1979) have compared this result with a wide range of experimental data, and they indicate that agreement between theory and experiment is generally within ±50 percent.

On the basis of the modified Ergun equation, the components of the Darcy's law permeability tensor and the Forchheimer correction tensor are given by

$$K_{zz} = \frac{d_p^2 \varepsilon_\beta^3}{180(1-\varepsilon_\beta)^2}, \tag{5.6}$$

$$F_{zz} = \frac{1}{100(1-\varepsilon_\beta)} \frac{\rho_\beta \langle v_\beta \rangle^\beta d_p}{\mu_\beta}. \tag{5.7}$$

These results indicate that Equations (3.5) and (3.15) contain the proper functional dependence for both **K** and **F**; however, both Equation (3.5) and Equation (3.15) *overestimate* the values of these two tensors.

The closure problem represented by Equations (4.3) has been solved by Snyder and Stewart (1966), Sorenson and Stewart (1974), Eidsath (1981), Sangani and Acrivos (1982), Zick and Homsy (1982), Edwards *et al.* (1991), Quintard and Whitaker (1995), and others for arrays of spheres and cylinders. For uniform arrays of spheres, the Darcy's law permeability tensor determined by Equation (4-3e) is isotropic, thus there is a single distinct component that can be compared with Equation (5.6) arranged in the form

$$\frac{K(1-\varepsilon_\beta)^2}{d_p^2 \varepsilon_\beta^3} = \frac{1}{180}. \tag{5.8}$$

In Figure 5 we have shown the theoretical calculations of Zick and Homsy (1982) for three different arrays of spheres, along with the experimental studies of Martin *et al.* (1951) for a face-centered cubic array and a simple cubic array, and the results of Susskind and Becker (1967) for a body-centered cubic array. The theoretical results are in excellent agreement with the experimental results for regular arrays of spheres and are consistent with the Ergun correlation. The latter does not illustrate the same porosity dependence as the theoretical results; however, the correlation is based on simple scaling arguments and experimental data having a limited range of porosity. The experimental result of Cioulachtjian *et al.* (1992) for randomly packed spheres is in perfect agreement with the theory for the face-centered cubic array; however, one could hardly have predicted this a priori. Obviously we need to develop methods of characterizing systems that are more sophisticated than simply the use of the porosity, ε_β, and the characteristic length, d_p. The good agreement between the solutions of Zick and Homsy (1982) and the ordered systems studied by Martin *et al.* (1951) and by Susskind and Becker (1967) should not be surprising since the solution of the closure problem given by Equations (4.3) is equivalent to the solution of Stokes equations (Barrère *et al.*, 1992). This can be made clear by demonstrating that the general closure problem given by Equations (4.10) is identical to the steady form of the Navier–Stokes equations for flow in a spatially periodic porous medium.

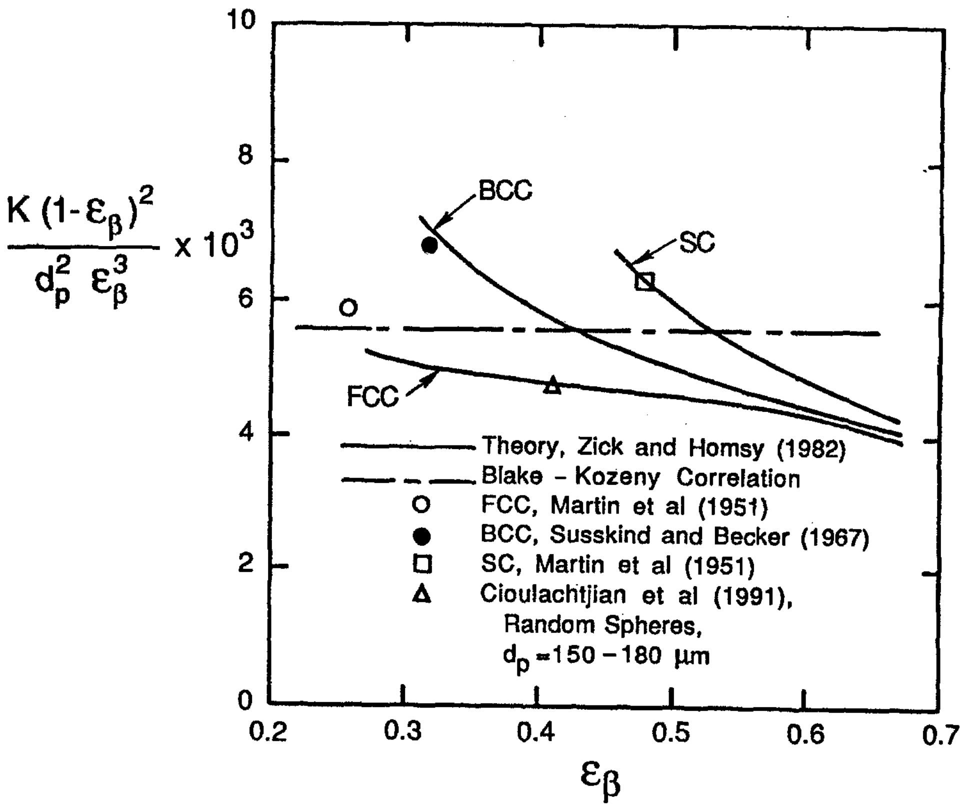

Figure 5. Comparison between theory and experiment.

5.1. EXACTNESS

In order to demonstrate that the closure problem given by Equations (4.10) is exact, within the framework of a spatially periodic model of a porous medium, we need to show that Equation (4.10)a is identical to Equation (1.1) for a steady flow. We first recall Equation (4.10a)

$$\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta}\right) \cdot \nabla \mathbf{M}_0 = -\nabla \mathbf{m}_0 + \nabla^2 \mathbf{M}_0 + \mathbf{I} \tag{5.9}$$

and make use of Equations (4.9) in order to develop the following relations

$$\varepsilon_\beta \mathbf{m}_0 \cdot \mathbf{M}^{-1} = \mathbf{m}, \quad \varepsilon_\beta \mathbf{M}_0 \cdot \mathbf{H}^{-1} = (\mathbf{M} + \mathbf{I}). \tag{5.10}$$

If we form the scalar product of these results with $\langle \mathbf{v}_\beta \rangle^\beta$, and make use of the representations given by Equations (2.23) and (2.24), we obtain the following relations for the pressure deviation and the velocity

$$\varepsilon_\beta \mathbf{m}_0 \cdot \mathbf{H}^{-1} \cdot \langle \mathbf{v}_\beta \rangle^\beta = \frac{\tilde{p}_\beta}{\mu_\beta}, \quad \varepsilon_\beta \mathbf{M}_0 \cdot \mathbf{H}^{-1} \cdot \langle \mathbf{v}_\beta \rangle^\beta = \mathbf{v}_\beta. \tag{5.11}$$

Forming the scalar product of $\varepsilon_\beta \mathbf{H}^{-1} \cdot \langle \mathbf{v}_\beta \rangle^\beta$ with Equations (5.9) provides

$$\rho_\beta \mathbf{v}_\beta \cdot \nabla \mathbf{v}_\beta = -\nabla \tilde{p}_\beta + \mu_\beta \nabla^2 \mathbf{v}_\beta + \mu_\beta \mathbf{H}^{-1} \cdot \langle \mathbf{v}_\beta \rangle, \tag{5.12}$$

in which we have replaced $\varepsilon_\beta \langle \mathbf{v}_\beta \rangle^\beta$ with $\langle \mathbf{v}_\beta \rangle$. At this point we can make use of Equation (2.50) in the form

$$\mu_\beta \mathbf{K}^{-1} \cdot (\mathbf{F} + \mathbf{I}) \cdot \langle \mathbf{v}_\beta \rangle = -(\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g}) \tag{5.13}$$

and on the basis of Equation (4.11) this result reduces to

$$\mu_\beta \mathbf{H}^{-1} \cdot \langle \mathbf{v}_\beta \rangle = -(\nabla \langle p_\beta \rangle^\beta - \rho_\beta \mathbf{g}). \tag{5.14}$$

Substitution of this form of the volume averaged momentum equation into Equation (5.12) leads to

$$\rho_\beta \mathbf{v}_\beta \cdot \nabla \mathbf{v}_\beta = -\nabla \tilde{p}_\beta + \mu_\beta \nabla^2 \mathbf{v}_\beta - \nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} \tag{5.15}$$

and on the basis of the pressure decomposition given by Equation (1.29) we have the steady form of the Navier–Stokes equations.

$$\rho_\beta \mathbf{v}_\beta \cdot \nabla \mathbf{v}_\beta = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{v}_\beta. \tag{5.16}$$

This indicates that the general closure problem given by Equations (4.10) is exact, subject to the following conditions

1. The quasi-steady constraint indicated by Equation (2.10) is valid.
2. The various lengthscale constraints are satisfied. These can be summarized by $\ell_\beta \ll r_0 \ll L$.
3. The model of a spatially periodic porous medium is employed.

Although solutions of the Navier–Stokes equations for periodic systems have been carried out (Launder and Massey, 1978; Eidsath *et al.*, 1983; Barrère, 1990; Edwards *et al.*, 1991; Ruth and Ma, 1993; Sahraoui and Kaviany, 1994), they have not been used to directly compute the components of $\mathbf{F}$ for systems comparable to those that are represented in Figure 5. Typical numerical studies for uniform flow in spatially periodic systems would provide values of K_{zz} and F_{zz}, and these could be compared directly with experiment or with the empirical correlations of experimental data.

6. Conclusions

In this work we have derived the volume averaged form of the Navier–Stokes equations along with the closure problems that can be used to determine the Darcy's law permeability tensor and the Forchheimer correction tensor. The closure problems can be considered as exact, subject to the stated time and lengthscale constraints and the model of a spatially periodic porous medium. The theory clearly indicates that the Forchheimer correction is quadratic in the velocity for small values of the Reynolds number, and order of magnitude analysis suggests that this functional dependence should not change greatly with Reynolds number.

Appendix A. Uniqueness

In order to demonstrate that the representations given by Equations (2.23) and (2.24) are unique to within arbitrary constants, we need to prove that the vector $\mathbf{u}$ and the scalar ξ in Equations (2.20) and (2.21) are constants. If they are, they will make no contribution to the filters in Equation (1.32) and this means that $\widetilde{p}_\beta$ and $\widetilde{\mathbf{v}}_\beta$ can be represented by Equations (2.23) and (2.24). If we substitute Equations (2.20) and (2.21) into the local closure problem given by Equations (2.15) through (2.19), and impose the constraints on $\mathbf{m}$ and $\mathbf{M}$ dictated by Equation (2.22), we obtain the following boundary-value problem for $\mathbf{u}$ and ξ

$$\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta}\right) \cdot \nabla \mathbf{u}$$

$$= -\nabla \xi + \nabla^2 \mathbf{u} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta si} \cdot (-\mathbf{I}\xi + \nabla \mathbf{u})\, \mathrm{d}A, \tag{A.1a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{A.1b}$$

$$\text{(B.C.1)} \quad \mathbf{u} = 0, \quad \text{at } A_{\beta\sigma}, \tag{A.1c}$$

$$\text{Periodicity:} \quad \xi(\mathbf{r} + \ell_i) = \xi(\mathbf{r}), \qquad \mathbf{u}(1 + \ell_i) = \mathbf{u}(\mathbf{r}), \quad i = 1, 2, 3, \tag{A.1d}$$

$$\text{Average:} \quad \langle \mathbf{u} \rangle^\beta = 0. \tag{A.1e}$$

If we form the scalar product of Equation (A.1a) with the vector $\mathbf{u}$, and integrate over the volume contained in a unit cell, we obtain

$$\int_{V_\beta} \left[\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta}\right) \cdot \nabla \mathbf{u}\right] \cdot \mathbf{u}\, \mathrm{d}V$$

$$= -\int_{V_\beta} (\nabla \xi) \cdot \mathbf{u}\, \mathrm{d}V - + \int_{V_\beta} (\nabla^2 \mathbf{u}) \cdot \mathbf{u}\, \mathrm{d}V. \tag{A.2}$$

Here we have made use of Equation (A.1e) in order to eliminate the area integral in Equation (A.1a) by means of

$$\int_{V_\beta} \left[\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\xi + \nabla \mathbf{u})\, \mathrm{d}A\right] \cdot \mathbf{u}\, \mathrm{d}V$$

$$= \left[\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\xi + \nabla \mathbf{u})\, \mathrm{d}A\right] \cdot \int_{V_\beta} \mathbf{u}\, \mathrm{d}V = 0. \tag{A.3}$$

Since both $\mathbf{v}_\beta$ and $\mathbf{u}$ are solenoidal, we have

$$\left[\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta}\right) \cdot \nabla \mathbf{u}\right] \cdot \mathbf{u} = \nabla \cdot \left[\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta}\right) \tfrac{1}{2}\mathbf{u} \cdot \mathbf{u}\right], \tag{A.4}$$

$$(\nabla \xi) \cdot \mathbf{u} = \nabla \cdot (\xi \mathbf{u}). \tag{A.5}$$

In addition, we can manipulate the integrand of the last integral in Equation (A.2) to obtain

$$(\nabla^2 \mathbf{u}) \cdot \mathbf{u} = [\nabla \cdot (\nabla \mathbf{u}) \cdot \mathbf{u}] - (\nabla \mathbf{u})^T : (\nabla \mathbf{u}) \tag{A.6}$$

and when these results are substituted into Equation (A.2) we obtain

$$\begin{aligned}
&\int_{V_\beta} \nabla \cdot \left[\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta}\right) \tfrac{1}{2}\mathbf{u} \cdot \mathbf{u}\right] \mathrm{d}V \\
&\quad = - \int_{V_\beta} \nabla \cdot (\xi \mathbf{u}) \, \mathrm{d}V + \int_{V_\beta} [\nabla \cdot (\nabla \mathbf{u}) \cdot \mathbf{u}] \, \mathrm{d}V - \\
&\qquad - \int_{V_\beta} (\nabla \mathbf{u})^T : (\nabla \mathbf{u}) \, \mathrm{d}V.
\end{aligned} \tag{A.7}$$

Directing our attention to the first integral, we make use of the divergence theorem to obtain

$$\begin{aligned}
&\int_{V_\beta} \nabla \cdot \left[\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta}\right) \tfrac{1}{2}\mathbf{u} \cdot \mathbf{u}\right] \mathrm{d}V \\
&\quad = \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \left[\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta}\right) \tfrac{1}{2}\mathbf{u} \cdot \mathbf{u}\right] \mathrm{d}A + \\
&\qquad + \int_{A_{\beta e}} \mathbf{n}_{\beta\sigma} \cdot \left[\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta}\right) \tfrac{1}{2}\mathbf{u} \cdot \mathbf{u}\right] \mathrm{d}A,
\end{aligned} \tag{A.8}$$

in which $A_{\beta e}$ represents the area of entrances and exits for a unit cell and $\mathbf{n}_{\beta e}$ represents the outwardly directed unit normal vector at the entrances and exits. The first integral on the right-hand side of Equation (A.8) will be zero because of the no-slip condition imposed on both $\mathbf{v}_\beta$ and $\mathbf{u}$. Since a spatially periodic model is used to solve the closure problem, we require that the velocity $\mathbf{v}_\beta$ along with the vector $\mathbf{u}$, be spatially periodic, and this means that the second integral on the right hand side of Equation (A.8) will be zero. Under these circumstances, Equation (A.7) simplifies to

$$0 = -\int_{V_\beta} \nabla\cdot(\xi\mathbf{u})\,\mathrm{d}V + \int_{V_\beta} [\nabla\cdot(\nabla\mathbf{u})\cdot\mathbf{u}]\,\mathrm{d}V - \\ - \int_{V_\beta} (\nabla\mathbf{u})^T : (\nabla\mathbf{u})\,\mathrm{d}V. \tag{A.9}$$

We can use the same line of reasoning with the first two integrals in this result to conclude that they are zero, thus we see that Equation (A.2) reduces to

$$0 = \int_{V_\beta} (\nabla\mathbf{u})^T : (\nabla\mathbf{u})\,\mathrm{d}V. \tag{A.10}$$

A little thought will indicate that the integrand in this result consists entirely of squared terms, thus the only solution to Equation (A.10) is given by

$$\nabla\mathbf{u} = 0. \tag{A.11}$$

This means that $\mathbf{u}$ must be a constant vector; however, the constraint given by Equation (A.1e) requires that the constant be zero and we have the proof that

$$\mathbf{u} = 0. \tag{A.12}$$

This proves that the representation given by Equation (2.23) can be used in place of that given by Equation (2.20). Substitution of Equation (A.12) into Equation (A.1) provides

$$0 = -\nabla\xi - \frac{1}{V_\beta}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot(-\mathbf{I}\xi)\,\mathrm{d}A \tag{A.13}$$

and use of the divergence theorem leads us to

$$0 = -\nabla\xi + \frac{1}{V_\beta}\int_{V_\beta} \nabla\xi\,\mathrm{d}V. \tag{A.14}$$

Within the framework of a local closure problem, the integral in this result must be a constant, thus the solution to Equation (A.14) is given by

$$\nabla\xi = \mathbf{c}_1 \tag{A.15}$$

and the scalar ξ has the form

$$\xi = \mathbf{r}\cdot\mathbf{c}_1 + c_2. \tag{A.16}$$

The periodicity condition given by Equation (A.1d) requires that $\mathbf{c}_1 = 0$ and we conclude that ξ is given by

$$\xi = c_2. \tag{A.17}$$

Since this constant cannot contribute to the surface filter in Equation (1.32), we can make use of the representation given by Equation (2.24) instead of that given by Equation (2.21). In this section we have proved that the representations given by Equations (2.23) and (2.24) are unique; however, one must always be aware that this proof is based on the model of a spatially periodic porous medium and on the assumption that the velocity field, $\mathbf{v}_\beta$, is spatially periodic.

Appendix B. Symmetry

We can explore the symmetry conditions for the Darcy's law permeability tensor, K, and the Forchheimer correction tensor, F, directly in terms of the closure problem given by Equations (4.10). We begin by forming the scalar product between Equations (4.10) and an arbitrary constant vector, $\boldsymbol{\lambda}$, and defining the new scalar and vector fields according to

$$s_0 = \mathbf{m}_0 \cdot \boldsymbol{\lambda}, \qquad \mathbf{w}_0 = \mathsf{M}_0 \cdot \boldsymbol{\lambda}. \tag{B.1}$$

This provides us with the following problem

$$\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta}\right) \cdot \nabla \mathbf{w}_0 = -\nabla s_0 + \nabla^2 \mathbf{w}_0 + \boldsymbol{\lambda}, \tag{B.2a}$$

$$\nabla \cdot \mathbf{w}_0 = 0, \tag{B.2b}$$

$$\text{(B.C.1)} \quad \mathbf{w}_0 = 0, \quad \text{at } A_{\beta\sigma}, \tag{B.2c}$$

$$\text{Periodicity:} \quad s_0(\mathbf{r} + \boldsymbol{\ell}_i) = s_0(\mathbf{r}), \qquad \mathbf{w}_0(\mathbf{r} + \boldsymbol{\ell}_i) = \mathbf{w}_0(\mathbf{r}),$$
$$i = 1, 2, 3, \tag{B.2d}$$

$$\text{Average:} \quad \langle \mathbf{w}_0 \rangle^\beta = \varepsilon_\beta^{-1} \mathsf{H} \cdot \boldsymbol{\lambda}. \tag{B.2e}$$

Repeating this process with the arbitrary constant vector $\boldsymbol{\nu}$, and defining the new scalar and vector fields according to

$$s_1 = \mathbf{m}_0 \cdot \boldsymbol{\nu}, \qquad \mathbf{w}_1 = \mathsf{M}_0 \cdot \boldsymbol{\nu}, \tag{B.3}$$

leads to the following boundary-value problem

$$\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta}\right) \cdot \nabla \mathbf{w}_1 = -\nabla s_1 + \nabla^2 \mathbf{w}_1 + \boldsymbol{\nu}, \tag{B.4a}$$

$$\nabla \cdot \mathbf{w}_1 = 0, \tag{B.4b}$$

$$\text{(B.C.1)} \quad \mathbf{w}_1 = 0, \quad \text{at } A_{\beta\sigma}, \tag{B.4c}$$

$$\text{Periodicity:} \quad s_1(\mathbf{r} + \ell_i) = s_1(\mathbf{r}), \qquad \mathbf{w}_1(\mathbf{r} + \ell_i) = \mathbf{w}_1(\mathbf{r}), \quad i = 1, 2, 3, \tag{B.4d}$$

$$\text{Average:} \quad \langle \mathbf{w}_1 \rangle^\beta = \varepsilon_\beta^{-1} \mathbf{H} \cdot \boldsymbol{\nu}. \tag{B.4e}$$

We now form the scalar product of Equation (B.2a) with the vector field $\mathbf{w}_1$ in order to obtain

$$\nabla \cdot \left[\left(\frac{\rho_\beta \mathbf{v}_\beta}{\mu_\beta} \right) (\mathbf{w}_0 \cdot \mathbf{w}_1) \right] - \frac{1}{\nu_\beta} (\mathbf{w}_0 \mathbf{v}_\beta) : \nabla \mathbf{w}_1$$
$$= -\nabla \cdot (\mathbf{w}_1 s_0) + \nabla \cdot [(\nabla \mathbf{w}_0) \cdot \mathbf{w}_1] - (\nabla \mathbf{w}_0)^T : (\nabla \mathbf{w}_1) + \boldsymbol{\lambda} \cdot \mathbf{w}_1. \tag{B.5}$$

Taking into account the information available in Equations (B.2) and (B.4), we can form the intrinsic average of Equation (B.5) to obtain

$$\frac{1}{\nu_\beta} \langle (\mathbf{w}_0 \mathbf{v}_\beta) : \nabla \mathbf{w}_1 \rangle^\beta = -\langle (\nabla \mathbf{w}_0)^{\mathrm{T}} : (\nabla \mathbf{w}_1) \rangle^\beta + \varepsilon_\beta^{-1} \boldsymbol{\lambda} \cdot \mathbf{H} \cdot \boldsymbol{\nu}. \tag{B.6}$$

If we form the scalar product of Equation (B.4a) with the vector $\mathbf{w}_0$, we can repeat this process to develop the result

$$\frac{\rho_\beta}{\mu_\beta} \langle (\mathbf{w}_1 \mathbf{v}_\beta) : \nabla \mathbf{w}_0 \rangle^\beta = -\langle (\nabla \mathbf{w}_1)^{\mathrm{T}} : (\nabla \mathbf{w}_0) \rangle^\beta + \varepsilon_\beta^{-1} \boldsymbol{\nu} \cdot \mathbf{H} \cdot \boldsymbol{\lambda}. \tag{B.7}$$

One can make use of the orthogonality relations for the irreducible parts of second order tensors to show

$$(\nabla \mathbf{w}_0)^T : (\nabla \mathbf{w}_1) = (\nabla \mathbf{w}_1)^T : (\nabla \mathbf{w}_0) \tag{B.8}$$

and use of this result with Equations (B.6) and (B.7) leads to

$$\boldsymbol{\lambda} \cdot \mathbf{H} \cdot \boldsymbol{\nu} = \boldsymbol{\nu} \cdot \mathbf{H} \cdot \boldsymbol{\lambda} + \frac{\rho_\beta \varepsilon_\beta}{\mu_\beta} [\langle (\mathbf{w}_0 \mathbf{v}_\beta) : \nabla \mathbf{w}_1 \rangle^\beta - \langle (\mathbf{w}_1 \mathbf{v}_\beta) : \nabla \mathbf{w}_0 \rangle^\beta]. \tag{B.9}$$

If the last term in this result were zero, we would have a proof that $\mathbf{H}$ was symmetric; however, this term can only be discarded when the inertial terms are negligible. This means that the Darcy's law permeability tensor, as determined by Equations (4.3), is symmetric

$$\mathbf{K} = \mathbf{K}^T, \tag{B.10}$$

while the Forchheimer correction tensor, **F**, as determined by Equations (4.10) and (4.11), is not.

References

Barrère, J.: 1990, Modélisation des équations de Stokes et Navier–Stokes en milieux poreux, Thèse de l'Université de Bordeaux I.

Barrère, J., Olivier, G., and Whitaker, S.: 1992, On the closure problem for Darcy's law, *Transport in Porous Media* **7**, 209–222.

Bensoussan, A., Lions, J. L. and Papanicolaou, G.: 1978, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam.

Brinkman, H. C. 1947, A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles, *Appl. Sci. Res.* **A1**, 27–34.

Carbonell, R. G. and Whitaker, S.: 1984, Heat and mass transfer in porous media, pages 123–198 in: J. Bear and M.Y. Corapcioglu (eds), *Fundamentals of Transport Phenomena in Porous Media*, Martinus Nijhoff, Dordrecht.

Cioulachtjian, S., Tadrist, L., Occelli, R., *et al.*: 1992, Experimental analysis of heat transfer with phase change in porous media crossed by a fluid flow, *Exp. Thermal Fluid Sci.* **5**, 533–547.

Darcy, H.: 1856, *Les fontaines publiques de la Ville de Dijon*, Dalmont, Paris.

Du Plessis, J. P.: 1994, Analytical quantification of coefficients in the Ergun equation for fluid friction in a packed bed, *Transport in Porous Media* **16**, 189–207.

Edwards, D. A., Shapiro, M., Brenner, H., and Shapira, M.: 1991, Dispersion of inert solutes in spatially periodic, two-dimensional model porous media, *Transport in Porous Media* **6**, 337–358.

Ene, H.I. and Poliševski, D.: 1987, *Thermal Flow in Porous Media*, D. Reidel, Dordrecht.

Eidsath, A. B.: 1981, Flow and dispersion in spatially periodic porous media: A finite element study, MS Thesis, Department of Chemical Engineering, University of California at Davis.

Eidsath, A. B., Carbonell, R. G., Whitaker, S., and Herrmann, L. R.: 1983, Dispersion in pulsed systems III: Comparison between theory and experiments for packed beds, *Chem. Engng. Sci.* **38**, 1803–1816.

Ergun, S.: 1952, Fluid flow through packed columns, *Chem. Eng. Progr.* **48**, 88–94

Forchheimer, P.: 1901, *Wasserbewegung durch Boden*, Z. Ver. Deutsch. Ing. **45**, 1782–1788.

Gray, W.G.: 1975, A derivation of the equations for multiphase transport, *Chem. Engng. Sci.* **30**, 229–233.

Greenkorn, R. A.: 1983, *Flow Phenomena in Porous Media: Fundamentals and Applications in Petroleum, Water, and Food Production*, Marcel Dekker, New York.

Howes, F. A. and Whitaker, S.: 1985, The spatial averaging theorem revisited, *Chem. Engng. Sci.* **40**, 1387–1392.

Joseph, D. D., Nield, D. A. and Papanicolaou, G.: 1982, Nonlinear equation governing flow in saturated porous media, *Water Resour. Res.* **18**, 1049–1052.

Kannbuk, W. G. and Martin, L. H.: 1933, *Proc. Roy. Soc. Lond.* **A141**, 144–153.

Launder, B. E. and Massey, T. H.: 1978, The numerical prediction of viscous flow and heat transfer in tube banks, *J. Heat Trans.* **100**, 565–571.

Ma, H. and Ruth, D. W.: 1993, The microscopic analysis of high Forchheimer number flow in porous media, *Transport in Porous Media* **13**, 139–160.

Macdonald, I. F., El-Sayed, M. S., Mow, K., and Dullien, F. A. L.: 1979, Flow through porous media: The Ergun equation revisited, *Ind. Eng. Chem. Fundam.* **18**, 199–208.

Martin, J. J., McCabe, W. L., and Mourad, C. C.: 1951, Pressure drop through stacked spheres – effect of orientation, *Chem. Engng. Prog.* **47**, 91–98.

Mei, C. C. and Auriault, J.-L.: 1991, The effect of weak inertia on flow through a porous medium, *J. Fluid Mech.* **222**, 647–663.

Nozad, I., Carbonell, R. G., and Whitaker, S.: 1985, Heat conduction in multiphase systems I: Theory and experiment for two-phase systems, *Chem. Engng. Sci.* **40**, 843–855.

Nozad, I., Carbonell, R. G., and Whitaker, S.: 1985, Heat conduction in multiphase systems II: Experimental method and results for three-phase systems, *Chem. Engng. Sci.* **40**, 857–863.

Ochoa-Tapia, J. A., Stroeve, P., and Whitaker, S.: 1994, Diffusive transport in two-phase media: Spatially periodic models and Maxwell's theory for isotropic and anisotropic systems, *Chem. Engng. Sci.* **49**, 709–726.

Ochoa-Tapia, J. A. and Whitaker, S.: 1995, Momentum transfer at the boundary between a porous medium and a homogeneous fluid I: Theoretical development, *Int. J. Heat Mass Trans.* **38**, 2635–2646.

Ochoa-Tapia, J. A. and Whitaker, S.: 1995, Momentum transfer at the boundary between a porous medium and a homogeneous fluid II: Comparison with experiment, *Int. J. Heat Mass Trans.* **38**, 2647–2655.

Quintard, M. and Whitaker, S.: 1993, One and two-equation models for transient diffusion processes in two-phase systems, in: *Advances in Heat Transfer*, Vol. 23, Academic Press, New York, pp. 369–465.

Quintard, M. and Whitaker, S.: 1994, Transport in ordered and disordered porous media I: The cellular average and the use of weighting functions, *Transport in Porous Media* **14**, 163–177.

Quintard, M. and Whitaker, S.: 1994, Transport in ordered and disordered porous media II: Generalized volume averaging, *Transport in Porous Media* **14**, 179–206.

Quintard, M. and Whitaker, S.: 1994, Transport in ordered and disordered porous media III: Closure and comparison between theory and experiment, *Transport in Porous Media* **15**, 31–49.

Quintard, M. and Whitaker, S.: 1994, Transport in ordered and disordered porous media IV: Computer generated porous media, *Transport in Porous Media* **15**, 51–70.

Quintard, M. and Whitaker, S.: 1994, Transport in ordered and disordered porous media V: Geometrical results for two-dimensional systems, *Transport in Porous Media* **15**, 183–196.

Quintard, M. and Whitaker, S.: 1995, Local thermal equilibrium for transient heat conduction: Theory and comparison with numerical experiments, *Int. J. Heat Mass Trans.* **38**, 2779–2796.

Quintard, M. and Whitaker, S.: 1995, Aerosol filtration: An analysis using the method of volume averaging, *J. Aerosol Sci.* **26**, 1227–1255.

Ruth, D. W. and Ma, H.: 1993, Numerical analysis of viscous, incompressible flow in a diverging-converging RUC, *Transport in Porous Media* **13**, 161–177.

Ruth, D. W. and Ma, H.: 1992, On the derivation of the Forchheimer equation by means of the averaging theorem, *Transport in Porous Media* **7**, 255–264.

Sahroui, M. and Kaviany, M.: 1994, Slip and no-slip temperature boundary conditions at the interface of porous, plain media: Convection, *Int. J. Heat Mass Trans.* **37**, 1029–1044.

Sanchez-Palencia, E.: 1980, *Non-homogeneous Media and Vibration Theory*, Lecture Notes in Phys. 127, Springer-Verlag, New York.

Sangani, A. S. and Acrivos, A.: 1982, Slow flow past periodic arrays of cylinders with application to heat transfer, *Int. J. Multiphase Flow* **8**, 193–206.

Snyder, L. J. and Stewart, W. E.: 1966, Velocity and pressure profiles for Newtonian creeping flow in regular packed beds of spheres, *AIChE J.* **12**, 167–173.

Sorensen, J. P. and Stewart, W. E.: 1974, Computation of forced convection in slow flow through ducts and packed beds II: Velocity profile in a simple array of spheres, *Chem. Engng. Sci.* **29**, 817–819.

Susskind, H. and Becker, W.: 1967, Pressure drop in geometrically ordered packed beds of spheres, *AIChE J.* **13**, 1155–1163.

Tennekes, H. and Lumley, J. L.: 1972, *A First Course in Turbulence*, MIT Press, Cambridge, Massachusetts.

Whitaker, S.: 1982, Laws of continuum physics for single-phase, single-component systems, in G. Hetsroni (ed.), *Handbook of Multiphase Systems*, Hemisphere Publ., New York, pp. 1–5 to 1–35.

Whitaker, S.: 1986, Flow in porous media I: A theoretical derivation of Darcy's law, *Transport in Porous Media* **1**, 3–25.

Whitaker, S.: 1997, Volume averaging of transport equations, Chap. 1 in J. P. Du Plessis (ed.), *Fluid Transport in Porous Media*, Computational Mechanics Publications, Southampton, United Kingdom.

Wodie, J.-C. and Levy, T.: 1991, Correction non linéaire de la loi de Darcy, *C.R. Acad. Sci. Paris Série II* **312**, 157–161.

Zick, A. A. and Homsy, G. M.: 1982, Stokes' flow through periodic arrays of spheres, *J. Fluid Mech.* **115**, 13–26.