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Momentum transfer within a porous medium. II. Stress boundary condition

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In this paper, we derive a boundary condition at the interface between a free fluid and a porous medium stating that the stress is transferred both to the fluid within the porous medium and to the solid skeleton. A zero stress jump is obtained so that the total stress is preserved at the interface. The boundary condition is obtained with the volume averaging method following the approach of Ochoa-Tapia and Whitaker ["Momentum transfer at the boundary between a porous medium and a homogeneous fluid—I. Theoretical development," Int. J. Heat Mass Transfer **38**(14), 2635–2646 (1995)], but starting from the momentum balances written on the fluid and on the solid of the porous region, the latter was derived in part I of this paper. In the same way, also the boundary condition at the interface between a porous medium and a homogeneous solid is obtained. Both boundary conditions describe the equilibrium of forces at the interface, where part of the stress is carried by the solid skeleton and part by the fluid within the porous medium. With the derived boundary conditions, together with the stress transfer equation within the solid skeleton, it is now possible to satisfy the overall force equilibrium on a shear cell partially filled with a porous medium. © *2014 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4902956]

I. INTRODUCTION

Transport phenomena at the boundary between a porous medium and a fluid layer are ubiquitous both in environmental processes and in industrial applications. In the literature, great attention has been paid to study the configuration where a fluid layer flows over a porous medium. This problem has been investigated with different approaches, and we will focus on the macroscopic one, also referred to as the two-domain approach, where the averaged transport equations of the porous medium domain are coupled to those of the fluid layer through a suitable boundary condition. In 1967, Beavers and Joseph[1] proposed their kinematic boundary condition that assumed a velocity slip at the interface, proportional to the velocity gradient at the interface, through an empirical parameter to be experimentally determined by means of, e.g., a best fit of the observed velocity profile. It has been shown, however, that the slip velocity resulted relevant only for channels with a thin free fluid region.[2] Recently, Jäger *et al.*[3] proposed a "pressure slip" condition at the interface where, in general, the jump is proportional to that imposed by Beavers and Joseph to the velocity, and for isotropic porous media, it results nil. This pressure slip condition was also numerically investigated.[4] Differently from Beavers and Joseph, in 1974, Neale and Nader[5] assumed velocity continuity at the interface

$$v|_{d_i^+} = u|_{d_i^-}. \tag{1}$$

In Eq. (1), v is the velocity of the fluid in the open region, u is the averaged fluid velocity in the porous medium, and the interface between the porous medium and the open region is placed at d_i (Fig. 1). They were also the first to propose a boundary condition, to be coupled with Eq. (1), in terms of stress balance at the interface, that in a shear flow problem, resulted in the fluid tangential stress continuity.

$$\mu \left.\frac{dv}{dy}\right|_{d_i^+} = \mu_{eff} \left.\frac{du}{dy}\right|_{d_i^-}, \tag{2}$$

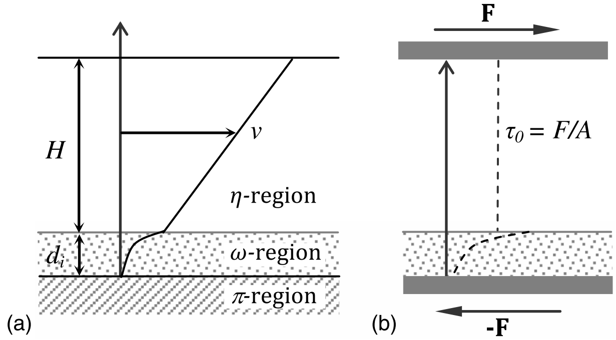

FIG. 1. Schematic representation of the fluid velocity profile (a) and of the fluid stress (b) over and through a porous medium for a shear-driven flow. The overall force equilibrium is sketched.

where the fluid viscosity is μ and μ_{eff} is the so called effective viscosity of the fluid within the porous medium. The authors showed that Beavers and Joseph (henceforth referred to as BJ) boundary condition leads to the same results of Eq. (2) as long as the BJ empirical parameter is set to $\sqrt{(\mu_{eff}/\mu)}$ and thus, showing that several data can be predicted by varying $\sqrt{(\mu_{eff}/\mu)}$ from 0.1 to 4. They, however, suggested using $\mu_{eff} = \mu$, in agreement with Brinkman,[6] who was the first to introduce the effective viscosity in his extension to Darcy's law. Equation (2) was used also by other authors, see, e.g., Vafai and Thiyagaraja,[7] with a choice of μ_{eff} different from μ. This latter point was widely debated in the literature, particularly regarding the dependence of μ_{eff} on the geometry and structure of the porous medium. The simplest assumption, $\mu_{eff} = \mu$ was adopted by several authors, e.g., Vafai and Kim,[8] but different corrections were proposed based on the solid geometry architecture in the limit of very high void fraction, e.g., Altenberger *et al.*[9] and Freed and Muthukumar,[10] and numerical studies led even to non-monotonic dependences of the effective viscosity on the void fraction.[11,12]

In 1995, Ochoa-Tapia and Whitaker[13] theoretically deduced a new stress boundary condition based on the non-local form of the volume averaged Stokes equation. They were aimed at making the volume averages in the open fluid compatible with those in the porous medium. To this end, they considered a large scale averaging volume across the interface that included both the porous region and the free fluid one, landing to the conclusion that the stress shows a jump passing from one region to the other.

$$\mu \left.\frac{dv}{dy}\right|_{d_i^+} = \mu_{eff} \left.\frac{du}{dy}\right|_{d_i^-} - \mu \frac{\beta}{\sqrt{K}} v_i, \tag{3}$$

where K is the intrinsic permeability of the porous medium, v_i is the velocity at the interface, and β is a dimensionless parameter of order one, theoretically known in a non-closed form that must be, in fact, experimentally determined. When $\beta = 0$, Eq. (2) is recovered. In Eq. (3), $\mu_{eff} = \mu\varepsilon^{-1}$, as derived by Whitaker,[14] with ε the porous medium void fraction. A comprehensive comparison of the different boundary conditions can be found in Alazmi and Vafai.[15]

A general agreement is found in the literature on Eq. (1) and, recently, also on Eq. (3), nevertheless, the description of the problem is not yet fully satisfactory. Indeed, there were several attempts to theoretically calculate the stress jump coefficient β starting from Goyeau *et al.*,[16] passing through the work of Chandesris and Jamet[17–19] or those of Valdés-Parada *et al.*[20,21] and many others,[22,23] all of them landing to the introduction of other unknown parameters. Recently, also an attempt to measure the jump coefficient was done by Carotenuto and Minale.[24] It is, however, clear that, among other factors, the location of the interface is very relevant for the determination of β.

The stress balance boundary conditions discussed here, Eqs. (2) and (3) and also BJ one, that were rationalised in terms of a stress balance,[25] share the same basic idea that the stress of the fluid in the open region is transferred only to the fluid within the porous medium, despite the fact that the solid skeleton of the porous medium is capable of supporting stresses. According to the author, only Nield and Bejan[26] suggested to divide the free fluid stress at the interface between that transferred to the fluid and that to the solid skeleton of the porous medium. Something similar was also assumed for the heat flux and different boundary conditions were proposed, as discussed by Yang and Vafai.[27] In addition, as we already discussed in part I,[28] if the stress carried by the solid skeleton is neglected, it becomes impossible to satisfy the equilibrium of the forces acting on a shear flow cell partially occupied by a porous medium. Indeed, Darcy's term in the fluid momentum balance in the porous medium represents a momentum sink that causes the stress dissipation within the porous domain, see Fig. 1. This is not only theoretically crucial but it is also practically relevant in many problems and in particular in those encountered in rheology[29,30] where both the torque transferred to the fluid and its angular velocity are the information required to measure fluid properties like viscosity.

The paper aims at deriving a boundary condition where the stress is transferred both to the fluid within the porous medium and to the solid skeleton. This will be possible thanks to the momentum transfer equation within the solid skeleton developed in part I, which for the homogeneous region of the porous medium actually coincides with Biot's equation.[31] The developed boundary condition will then apply to porous media like porous rocks, brush geometries, foam metals, and sandpapers, i.e., to porous media where the solid phase is somehow continuous. Ochoa-Tapia and Whitaker[13] (henceforth referred to as OTW) approach will be followed. The boundary conditions at the interface between a porous medium and a homogenous solid will also be derived.

II. GOVERNING EQUATIONS

In a shear problem where a fluid flows over and within a porous medium, see Fig. 1, the averaged continuity equation and momentum balance can be written in the homogeneous region of the porous medium[28] (ω–region) as

$$\omega\text{–region} \qquad \nabla \cdot \left\langle \mathbf{v}_\beta \right\rangle_\omega = 0, \tag{4}$$

$$\omega\text{–region} \qquad -\nabla \left\langle p_\beta \right\rangle_\omega^\beta + \rho_\beta \mathbf{g} + \frac{\mu}{\varepsilon_\omega} \nabla \cdot \left(\nabla \left\langle \mathbf{v}_\beta \right\rangle_\omega + \nabla \left\langle \mathbf{v}_\beta \right\rangle_\omega^T \right) - \mu \mathbf{K}^{-1} \cdot \left\langle \mathbf{v}_\beta \right\rangle_\omega = \mathbf{0}. \tag{5}$$

Notice that the nomenclature from now on will be the same of that used in part I, which essentially coincides with that adopted by OTW. Thus, ∇ is the gradient operator, $\cdot$ indicates the inner product, vectors, and tensors are written in bold, p_β and $\mathbf{v}_\beta$ are the pressure and the velocity of the β–phase, i.e., the fluid phase within the porous medium, respectively, ρ_β is the fluid density, $\mathbf{g}$ is the gravity acceleration vector, and $\mathbf{K}$ is the intrinsic permeability tensor defined in part I. The functions associated to the homogeneous region of the porous medium are indicated with the subscript ω. The only novelty is that for the sake of simplicity, where possible, we refer to the void fraction and not to the volume fraction of the generic phase, as done in part I, thus, ε_ω is the void fraction of the homogeneous region of the porous medium. In Eq. (5), two different volume averages appear, the phase average $\langle \psi_\theta \rangle$ and the intrinsic phase average $\langle \psi_\theta \rangle^\theta$,

$$\langle \psi_\theta \rangle = \frac{1}{\mathcal{V}} \int_{V_\theta} \psi_\theta dV, \tag{6}$$

$$\langle \psi_\theta \rangle^\theta = \frac{1}{V_\theta} \int_{V_\theta} \psi_\theta dV = \frac{1}{\varepsilon_\theta^\#} \langle \psi_\theta \rangle, \tag{7}$$

where V_θ is the volume of the generic θ–phase contained in the integration volume $\mathcal{V}$ (Fig. 2) and $\varepsilon_\theta^\# = V_\theta/\mathcal{V}$.

The phase average represents the mean over the entire integration volume, while the intrinsic phase average is the genuine volume average of a generic function with respect to its domain. Equation (5) is Darcy's equation with the first Brinkman's correction,[6] as deduced by Whitaker,[14] and the use of two different averages is understood if it is considered that the velocity in Darcy's equation is typically calculated as the volumetric flow rate crossing the porous medium divided by the cross section area, thus it coincides with $\left\langle \mathbf{v}_\beta \right\rangle_\omega$, while the pressure gradient in Darcy's experiment is that applied at the boundary of the porous medium and thus coincides with the liquid pore pressure gradient, i.e., $\nabla \left\langle p_\beta \right\rangle_\omega^\beta$, as Whitaker correctly pointed out. The latter assumption is implicitly based on the hypothesis that at the boundary, the free fluid normal force is divided between

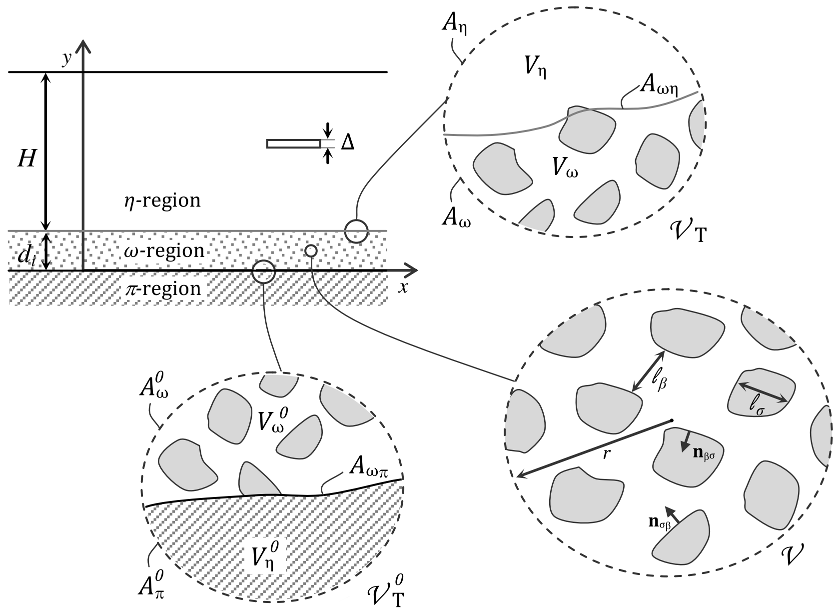

FIG. 2. Sketch of a shear cell partially filled with a porous medium. The different domains are indicated together with the zooms of the different volumes of integration used in the paper: $\mathcal{V}$ in the porous medium (ω–region), $\mathcal{V}_T$ across the interface $A_{\omega\eta}$, and $\mathcal{V}_T^0$ across the interface $A_{\omega\pi}$. The volumes and the areas of the interfaces of each region are also indicated in the zooms together with the characteristic length scales. The unit vectors normal to the solid fluid interface are shown. The disk of thickness Δ is the integration volume in the free fluid (η–region).

that transferred to the fluid within the porous medium and that to the solid skeleton. Indeed, in an approximated way, without paying attention to the transition region across the interface and to the various averages, the equilibrium of normal forces can be written as

$$\text{at } A_{\omega\eta}, \qquad p_\eta A_{\omega\eta} = p_\omega A_{\omega\eta}^\beta + T_\sigma^n A_{\omega\eta}^\sigma, \tag{8}$$

where p_η is the pressure of the free fluid, p_ω is the pressure of the fluid within the porous medium, i.e., the pore pressure, T_σ^n is the normal stress supported by the solid skeleton, $A_{\omega\eta}$ is the area of the interface between ω- and η–region, and $A_{\omega\eta}^\theta$ is its portion occupied by the θ-phase. The free fluid normal force can be now split into that transferred to the fluid and that to the solid

$$p_\eta A_{\omega\eta} = p_\eta A_{\omega\eta}^\beta + p_\eta A_{\omega\eta}^\sigma; \tag{9}$$

thus, Eq. (8) becomes

$$\begin{aligned} a) \qquad & p_\eta A_{\omega\eta}^\beta = p_\omega A_{\omega\eta}^\beta; \\ b) \qquad & p_\eta A_{\omega\eta}^\sigma = T_\sigma^n A_{\omega\eta}^\sigma. \end{aligned} \tag{10}$$

Equations (8)–(10a) demonstrate that imposing $p_\eta = p_\omega$ is in fact equivalent to assuming that the free fluid stress is divided between the fluid and the solid of the porous medium. This supports further the basic idea of this paper to derive a boundary condition where the stress is transferred both to the fluid within the porous medium and to the solid skeleton.

Going back to Eq. (5), Brinkman's correction is not neglected since, e.g., in simple shear, as the one depicted in Fig. 1, it is the only driving force available for the fluid motion. In addition, though it was empirically introduced by Brinkman, as also pointed out by Nield,[32] in 1986, Whitaker[14] theoretically derived it and also introduced the so called second Brinkman's correction, which is here neglected being either zero, for porous media with a constant void fraction in the homogeneous region, or negligible, for system with a slowly variable void fraction. In the homogeneous region of the porous medium also the momentum transfer on the solid skeleton can be written as derived in part I.

$$\omega\text{–region} \qquad \nabla \cdot \langle \mathbf{T}_\sigma \rangle_\omega^\sigma + (\rho_\sigma - \rho_\beta)\,\mathbf{g} + \mu \frac{\varepsilon_\omega}{1-\varepsilon_\omega} \mathbf{K}^{-1} \cdot \langle \mathbf{v}_\beta \rangle_\omega = \mathbf{0}, \tag{11}$$

where $\mathbf{T}_\sigma$ is the stress tensor of the σ–phase, i.e., the solid skeleton, and ρ_σ its density. Equation (11) coincides with Biot's one[31] that is here rationalised in terms of volume averages. Also in this equation, in agreement with Eq. (5), the phase average is used for the velocity, while the intrinsic one is adopted for the stress. It is worthwhile underlining that in Eq. (11) conventionally the tensions are considered positive and the compressions negative, while the contrary holds for the fluid phase, Eq. (5), where the pressures are positives. Thus, in Eq. (11), the classical sign convention of the physics of solids is followed, while in Eq. (5), the convention classically used in fluid dynamics is adopted.

In the homogenous part of the free fluid (η–region), the averaged continuity equation and Stokes equation holds

$$\eta\text{–region} \qquad \nabla \cdot \langle \mathbf{v}_\beta \rangle_\eta = 0, \tag{12}$$

$$\eta\text{–region} \qquad -\nabla \langle p_\beta \rangle_\eta^\beta + \rho_\beta \mathbf{g} + \mu \nabla \cdot \left(\nabla \langle \mathbf{v}_\beta \rangle_\eta + \nabla \langle \mathbf{v}_\beta \rangle_\eta^T \right) = \mathbf{0}, \tag{13}$$

where the functions associated to the homogeneous region of the free fluid are indicated with the subscript η. Notice that in η–region, the phase average coincides with the intrinsic one since the void fraction, ε_η, is 1. Equations (4) and (5), and (11)–(13) are valid under some length constraints clearly discussed by Whitaker[14] and OTW that we here synthesize

$$\omega\text{–region} \qquad \frac{\ell_\beta}{r} << 1; \qquad \left(\frac{r}{L}\right)^2 << 1 \tag{14}$$

$$\eta\text{–region} \qquad \left(\frac{\Delta}{H}\right)^2 << 1, \tag{15}$$

where the characteristic lengths are shown in Fig. 2 together with the integration volumes over which the averages are made. Thus, ℓ_β is the length scale of β–phase within the ω–region, r is the characteristic length of the averaging volume in ω–phase, L is the characteristic length over which the averaged quantities vary, Δ is the thickness of the disk over which the averages are made in η–region, and H is the geometrical length of η–region.

In the transition zone across the interface between ω– and η–region, the following balances hold without any length constraint:

$$-\nabla \langle p_\beta \rangle^\beta + \rho_\beta \mathbf{g} + \frac{\mu}{\varepsilon} \nabla \cdot \left(\nabla \langle \mathbf{v}_\beta \rangle + \nabla \langle \mathbf{v}_\beta \rangle^T \right) - \frac{\mu}{\varepsilon} \nabla \varepsilon \cdot \left[\nabla \left(\frac{\langle \mathbf{v}_\beta \rangle}{\varepsilon} \right) + \nabla \left(\frac{\langle \mathbf{v}_\beta \rangle}{\varepsilon} \right)^T \right] - \mu \Phi_\beta = \mathbf{0}, \tag{16}$$

$$\nabla \cdot \langle \mathbf{T}_\sigma \rangle^\sigma + (\rho_\sigma - \rho_\beta)\,\mathbf{g} + \mu \Phi_\sigma = \mathbf{0}, \tag{17}$$

with

$$\mu \Phi_\beta = -\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \left\{ -\mathbf{I} \left(p_\beta - \langle p_\beta \rangle^\beta \right) + \mu \left[\left(\nabla \mathbf{v}_\beta + \nabla \mathbf{v}_\beta^T \right) - \left(\nabla \langle \mathbf{v}_\beta \rangle^\beta + \left(\nabla \langle \mathbf{v}_\beta \rangle^\beta \right)^T \right) \right] \right\} dA, \tag{18}$$

$$\mu \Phi_\sigma = \frac{1}{V_\sigma} \left[\int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} \cdot \mathbf{T}_\sigma dA - \left(\int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} dA \right) \cdot \langle \mathbf{T}_\sigma \rangle^\sigma \right], \tag{19}$$

where $\mathbf{n}_{\theta\alpha}$ is the unit vector normal to the $\theta\alpha$ interface directed from θ- to α–phase and $\mathbf{I}$ is the second order unit tensor. Notice that the averages in Eqs. (16) and (17) are not labelled with a subscript to indicate that these quantities refer to the transition zone, i.e., the volumes $\mathcal{V}_T$ and $\mathcal{V}_T^0$. In Eq. (16), the first (3rd term of *lhs*) and the second (4th term of *lhs*) Brinkman's corrections are recognised. Equations (16) and (18) can be found in OTW, while Eqs. (17) and (19) are derived in Minale.[28]

III. BOUNDARY CONDITIONS AT $A_{\omega\eta}$

OTW basic idea is to use Eqs. (4) and (5) and Eqs. (11)–(13) also within the transition zone, i.e., inside the averaging volume $\mathcal{V}_T$ across the interface (Fig. 2). In this region, quantities such as $\langle \mathbf{v}_\beta \rangle_\omega$, $\langle p_\beta \rangle_\eta^\beta$, or $\langle \mathbf{T}_\sigma \rangle_\omega^\sigma$ may not accurately predict the local corresponding volume averaged quantities and thus, excess functions are introduced to correct the approximation. In this way, they obtained

$$\text{at } A_{\omega\eta}, \qquad \langle \mathbf{v}_\beta \rangle_\omega = \langle \mathbf{v}_\beta \rangle_\eta. \tag{20}$$

Eq. (20) also holds in our case since the new momentum balance, Eq. (11), does not interfere with the procedure to get to this kinematic condition.

Concerning the stress boundary condition, it is more convenient, as it will be clear in the following, to start from Eqs. (5), (11), (13), (16), and (17), where only the volume phase averages appear, thus, using Eqs. (6) and (7), the momentum balances are rewritten as

$$\omega\text{–region} \quad -\nabla \langle p_\beta \rangle_\omega + \varepsilon_\omega \rho_\beta \mathbf{g} + \mu \nabla \cdot \left(\nabla \langle \mathbf{v}_\beta \rangle_\omega + \nabla \langle \mathbf{v}_\beta \rangle_\omega^T \right) - \mu \varepsilon_\omega \mathbf{K}^{-1} \cdot \langle \mathbf{v}_\beta \rangle_\omega = \mathbf{0}, \tag{21}$$

$$\omega\text{–region} \quad \nabla \cdot \langle \mathbf{T}_\sigma \rangle_\omega + (1 - \varepsilon_\omega) (\rho_\sigma - \rho_\beta) \mathbf{g} + \mu \varepsilon_\omega \mathbf{K}^{-1} \cdot \langle \mathbf{v}_\beta \rangle_\omega = \mathbf{0}. \tag{22}$$

Moreover, in Eqs. (21) and (22), we neglected terms proportional to the void fraction gradient, in agreement with the already discussed length constraints. It is worthwhile underlining, however, that the derivation of the new boundary condition is not modified by the presence of these terms. In η–region, Eq. (13) remains unchanged since in the free fluid the intrinsic phase average coincides with the phase average, being $\varepsilon_\eta = 1$. In the transition region, i.e., in $\mathcal{V}_T$, Eqs. (16) and (17) can be rewritten as

$$\text{in } \mathcal{V}_T, \quad \begin{aligned} &-\nabla \langle p_\beta \rangle + \varepsilon \rho_\beta \mathbf{g} + \mu \nabla \cdot \left(\nabla \langle \mathbf{v}_\beta \rangle + \nabla \langle \mathbf{v}_\beta \rangle^T \right) + \\ &+ \nabla \varepsilon \cdot \left\{ \mathbf{I} \frac{\langle p_\beta \rangle}{\varepsilon} - \mu \left[\nabla \left(\frac{\langle \mathbf{v}_\beta \rangle}{\varepsilon} \right) + \nabla \left(\frac{\langle \mathbf{v}_\beta \rangle}{\varepsilon} \right)^T \right] \right\} - \mu \varepsilon \Phi_\beta = \mathbf{0}, \end{aligned} \tag{23}$$

$$\text{in } \mathcal{V}_T, \quad \nabla \cdot \langle \mathbf{T}_\sigma \rangle + (1 - \varepsilon) (\rho_\sigma - \rho_\beta) \mathbf{g} + \frac{\nabla \varepsilon \cdot \langle \mathbf{T}_\sigma \rangle}{1 - \varepsilon} + \mu (1 - \varepsilon) \Phi_\sigma = \mathbf{0}. \tag{24}$$

Using Eqs. (18) and (19), Eqs. (23) and (24) become without any length constraint

$$\text{in } \mathcal{V}_T, \quad \begin{aligned} &-\nabla \langle p_\beta \rangle + \varepsilon \rho_\beta \mathbf{g} + \mu \nabla \cdot \left(\nabla \langle \mathbf{v}_\beta \rangle + \nabla \langle \mathbf{v}_\beta \rangle^T \right) + \\ &- \frac{1}{\mathcal{V}_T} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \left[\mathbf{I} p_\beta - \mu \left(\nabla \mathbf{v}_\beta + \nabla \mathbf{v}_\beta^T \right) \right] dA = \mathbf{0}, \end{aligned} \tag{25}$$

$$\text{in } \mathcal{V}_T, \quad \nabla \cdot \langle \mathbf{T}_\sigma \rangle + (1 - \varepsilon) (\rho_\sigma - \rho_\beta) \mathbf{g} + \frac{1}{\mathcal{V}_T} \int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} \cdot \mathbf{T}_\sigma dA = \mathbf{0}. \tag{26}$$

Equations (25) and (26) hold inside the boundary region and, following OTW, we form their integral equations and require that they are satisfied on average in $\mathcal{V}_T$. Summing the resulting integral equations, we obtain

$$\text{In } \mathcal{V}_T, \quad \begin{aligned} &\int_{\mathcal{V}_T} \left[\nabla \cdot \langle \mathbf{T}_\sigma \rangle - \nabla \langle p_\beta \rangle + \mu \nabla \cdot \left(\nabla \langle \mathbf{v}_\beta \rangle + \nabla \langle \mathbf{v}_\beta \rangle^T \right) \right] dV + \int_{\mathcal{V}_T} \left[(1 - \varepsilon) (\rho_\sigma - \rho_\beta) \mathbf{g} \right. \\ &\left. + \varepsilon \rho_\beta \mathbf{g} \right] dV + \int_{\mathcal{V}_T} \left\{ \frac{1}{\mathcal{V}_T} \int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} \cdot \left[\mathbf{T}_\sigma + \mathbf{I} p_\beta - \mu \left(\nabla \mathbf{v}_\beta + \nabla \mathbf{v}_\beta^T \right) \right] dA \right\} dV = \mathbf{0}. \end{aligned} \tag{27}$$

As discussed in Minale,[28] at β–σ interface, the following boundary condition holds:

$$\text{on } A_{\beta\sigma}, \qquad \mathbf{n}_{\sigma\beta}\cdot\left[\mathbf{T}_\sigma + \mathbf{I}\left(\rho_\sigma-\rho_\beta\right)gz\right] = \mathbf{n}_{\sigma\beta}\cdot\left[\mathbf{I}\left(-p_\beta+\rho_\beta gz\right)+\mu\left(\nabla\mathbf{v}_\beta+\nabla\mathbf{v}_\beta^T\right)\right]. \tag{28}$$

Equation (27) then becomes

$$\begin{aligned}&\int_{\mathcal{V}_T}\left[\nabla\cdot\langle\mathbf{T}_\sigma\rangle-\nabla\langle p_\beta\rangle+\mu\nabla\cdot\left(\nabla\langle\mathbf{v}_\beta\rangle+\nabla\langle\mathbf{v}_\beta\rangle^T\right)\right]dV+\int_{\mathcal{V}_T}\{(1-\varepsilon)\nabla\left[\left(\rho_\sigma-\rho_\beta\right)gz\right]\\&\quad+\varepsilon\nabla\left(\rho_\beta gz\right)\}\,dV+\int_{\mathcal{V}_T}\left[\nabla(1-\varepsilon)\left(\rho_\sigma-\rho_\beta\right)gz+\nabla\varepsilon\rho_\beta gz\right]dV=\mathbf{0},\end{aligned}\tag{29}$$

where we used

$$\begin{aligned}&\int_{\mathcal{V}_T}\left\{\frac{1}{\mathcal{V}_T}\int_{A_{\beta\sigma}}\mathbf{n}_{\sigma\beta}\cdot\mathbf{I}\left[-\left(\rho_\sigma-\rho_\beta\right)gz+\rho_\beta gz\right]dA\right\}dV\\&\quad=\int_{\mathcal{V}_T}\left[\nabla(1-\varepsilon)\left(\rho_\sigma-\rho_\beta\right)gz+\nabla\varepsilon\rho_\beta gz\right]dV.\end{aligned}\tag{30}$$

Equation (30) is obtained considering the potential energy in the surface integral on lhs constant and

$$\nabla\varepsilon=\frac{1}{\mathcal{V}_T}\int_{A_{\theta\alpha}}\mathbf{n}_{\sigma\beta}dA. \tag{31}$$

The divergence theorem is now applied to Eq. (29)

$$\begin{aligned}&\int_{A_\omega}\mathbf{n}_\omega\cdot\left\{\left[\langle\mathbf{T}_\sigma\rangle+\mathbf{I}(1-\varepsilon)\left(\rho_\sigma-\rho_\beta\right)gz\right]-\mathbf{I}\left(\langle p_\beta\rangle-\varepsilon\rho_\beta gz\right)+\mu\left(\nabla\langle\mathbf{v}_\beta\rangle+\nabla\langle\mathbf{v}_\beta\rangle^T\right)\right\}dA+\\&+\int_{A_\eta}\mathbf{n}_\eta\cdot\left\{\left[\langle\mathbf{T}_\sigma\rangle+\mathbf{I}(1-\varepsilon)\left(\rho_\sigma-\rho_\beta\right)gz\right]-\mathbf{I}\left(\langle p_\beta\rangle-\varepsilon\rho_\beta gz\right)+\mu\left(\nabla\langle\mathbf{v}_\beta\rangle+\nabla\langle\mathbf{v}_\beta\rangle^T\right)\right\}dA=\mathbf{0},\end{aligned}\tag{32}$$

where A_ω and A_η, are the surfaces of $\mathcal{V}_T$ in ω– and η–region, respectively, and $\mathbf{n}_\omega$ and $\mathbf{n}_\eta$ are the normal unit vectors to the surfaces A_ω and A_η, directed outward from $\mathcal{V}_T$, respectively. The same can be done for the equations valid in ω-region, Eqs. (21) and (22)

$$\begin{aligned}&\int_{A_\omega}\mathbf{n}_\omega\cdot\left\{\left[\langle\mathbf{T}_\sigma\rangle_\omega+\mathbf{I}(1-\varepsilon)\left(\rho_\sigma-\rho_\beta\right)gz\right]-\mathbf{I}\left(\langle p_\beta\rangle_\omega-\varepsilon\rho_\beta gz\right)+\mu\left(\nabla\langle\mathbf{v}_\beta\rangle_\omega+\nabla\langle\mathbf{v}_\beta\rangle_\omega^T\right)\right\}dA+\\&+\int_{A_{\omega\eta}}\mathbf{n}_{\omega\eta}\cdot\left\{\left[\langle\mathbf{T}_\sigma\rangle_\omega+\mathbf{I}(1-\varepsilon)\left(\rho_\sigma-\rho_\beta\right)gz\right]-\mathbf{I}\left(\langle p_\beta\rangle_\omega-\varepsilon\rho_\beta gz\right)\right.\\&\left.+\mu\left(\nabla\langle\mathbf{v}_\beta\rangle_\omega+\nabla\langle\mathbf{v}_\beta\rangle_\omega^T\right)\right\}dA=\mathbf{0},\end{aligned}\tag{33}$$

where $A_{\omega\eta}$ is the area of the interface between V_ω and V_η, which are the volumes of ω– and η–region, contained in $\mathcal{V}_T$. Analogously, integrating Eq. (13) over V_η and applying the divergence theorem it is obtained

$$\begin{aligned}&\int_{A_\eta}\mathbf{n}_\eta\cdot\left[-\mathbf{I}\left(\langle p_\beta\rangle_\eta-\rho_\beta gz\right)+\mu\left(\nabla\langle\mathbf{v}_\beta\rangle_\eta+\nabla\langle\mathbf{v}_\beta\rangle_\eta^T\right)\right]dA\\&\quad+\int_{A_{\omega\eta}}\mathbf{n}_{\eta\omega}\cdot\left[-\mathbf{I}\left(\langle p_\beta\rangle_\eta-\rho_\beta gz\right)+\mu\left(\nabla\langle\mathbf{v}_\beta\rangle_\eta+\nabla\langle\mathbf{v}_\beta\rangle_\eta^T\right)\right]dA=\mathbf{0}.\end{aligned}\tag{34}$$

Looking at the integrals of each equation from Eq. (32) to Eq. (34), we can better clarify our choice to write all the balance equations only in terms of volume phase averages. Indeed, the integrand functions represent the fluid or the solid stress acting on the surface with normal vector $\mathbf{n}_\alpha$ or $\mathbf{n}_{\theta\alpha}$, which once integrated over a surface lead to force vectors. To obtain a physically sensible force, each stress must be integrated over its surface of action and, as discussed above, the surface of action of the volume phase averages is the whole surface, while that of the intrinsic phase average $\langle\psi_\theta\rangle^\theta$ is only the portion of the whole surface occupied by the θ–phase. Let us remind, indeed, that, e.g., the intrinsic phase averaged pressure represents the pore pressure that acts only in the pore region, thus, it is applied only to the portion of the surface occupied by the pore-phase. It is now clear that, by using the phase averages only, the surface integrals we formed have all a direct physical meaning. To derive the boundary condition, let us now require Eq. (32), i.e., the whole momentum balance averaged in $\mathcal{V}_T$ be equal to the sum of Eq. (33) and Eq. (34), i.e., the sum of the averaged integral momentum balances valid in ω- and η-region, respectively,

$$\begin{aligned}
&\int_{A_{\omega\eta}} \mathbf{n}_{\omega\eta} \cdot \Big\{ -\mathbf{I}\left(\langle p_\beta\rangle_\omega - \varepsilon_\omega \rho_\beta g z\right) + \mu\left(\nabla\langle \mathbf{v}_\beta\rangle_\omega + \nabla\langle \mathbf{v}_\beta\rangle_\omega^T\right) + \left[\langle \mathbf{T}_\sigma\rangle_\omega + \mathbf{I}\left(1-\varepsilon_\omega\right)\left(\rho_\sigma - \rho_\beta\right) g z\right] + \\
&\qquad + \mathbf{I}\left(\langle p_\beta\rangle_\eta - \rho_\beta g z\right) - \mu\left(\nabla\langle \mathbf{v}_\beta\rangle_\eta + \nabla\langle \mathbf{v}_\beta\rangle_\eta^T\right)\Big\}\, dA = \\
&= \int_{A_\omega} \mathbf{n}_\omega \cdot \Big\{ -\mathbf{I}\left[\left(\langle p_\beta\rangle - \varepsilon \rho_\beta g z\right) - \left(\langle p_\beta\rangle_\omega - \varepsilon_\omega \rho_\beta g z\right)\right] + \mu\Big[\left(\nabla\langle \mathbf{v}_\beta\rangle + \nabla\langle \mathbf{v}_\beta\rangle^T\right) \\
&\qquad - \left(\nabla\langle \mathbf{v}_\beta\rangle_\omega + \nabla\langle \mathbf{v}_\beta\rangle_\omega^T\right)\Big] + \\
&\qquad + \left[\langle \mathbf{T}_\sigma\rangle + \mathbf{I}\left(1-\varepsilon\right)\left(\rho_\sigma - \rho_\beta\right) g z\right] - \left[\langle \mathbf{T}_\sigma\rangle_\omega + \mathbf{I}\left(1-\varepsilon_\omega\right)\left(\rho_\sigma - \rho_\beta\right) g z\right]\Big\} \\
&+ \int_{A_\eta} \mathbf{n}_\eta \cdot \Big\{ -\mathbf{I}\left[\left(\langle p_\beta\rangle - \varepsilon \rho_\beta g z\right) - \left(\langle p_\beta\rangle_\eta - \rho_\beta g z\right)\right] + \mu\Big[\left(\nabla\langle \mathbf{v}_\beta\rangle + \nabla\langle \mathbf{v}_\beta\rangle^T\right) \\
&\qquad - \left(\nabla\langle \mathbf{v}_\beta\rangle_\eta + \nabla\langle \mathbf{v}_\beta\rangle_\eta^T\right)\Big] + \\
&\qquad + \left[\langle \mathbf{T}_\sigma\rangle + \mathbf{I}\left(1-\varepsilon\right)\left(\rho_\sigma - \rho_\beta\right) g z\right]\Big\}\, dA.
\end{aligned} \tag{35}$$

The integral of the rhs of Eq. (35) represents an excess quantity that tends to zero both in the homogeneous ω– and η–region. The excess stress can be then defined and its general form is derived.

A. Excess surface stress

Following OTW, the excess surface stress, $\langle \mathbf{\Sigma}_s\rangle$, is defined as

$$\begin{aligned}
&\oint_C \mathbf{n}_s \cdot \delta\langle \Sigma_s\rangle\, ds = \int_{A_{\omega\eta}} \nabla_A \cdot \delta\langle \Sigma_s\rangle\, dA = \\
&\int_{A_\omega} \mathbf{n}_\omega \cdot \Big\{ -\mathbf{I}\left[\left(\langle p_\beta\rangle - \varepsilon \rho_\beta g z\right) - \left(\langle p_\beta\rangle_\omega - \varepsilon_\omega \rho_\beta g z\right)\right] \\
&\qquad + \mu\left[\left(\nabla\langle \mathbf{v}_\beta\rangle + \nabla\langle \mathbf{v}_\beta\rangle^T\right) - \left(\nabla\langle \mathbf{v}_\beta\rangle_\omega + \nabla\langle \mathbf{v}_\beta\rangle_\omega^T\right)\right] + \\
&\qquad + \left[\langle \mathbf{T}_\sigma\rangle + \mathbf{I}\left(1-\varepsilon\right)\left(\rho_\sigma - \rho_\beta\right) g z\right] - \left[\langle \mathbf{T}_\sigma\rangle_\omega + \mathbf{I}\left(1-\varepsilon_\omega\right)\left(\rho_\sigma - \rho_\beta\right) g z\right]\Big\} \\
&\qquad + \int_{A_\eta} \mathbf{n}_\eta \cdot \Big\{ -\mathbf{I}\left[\left(\langle p_\beta\rangle - \varepsilon \rho_\beta g z\right) - \left(\langle p_\beta\rangle_\eta - \rho_\beta g z\right)\right] \\
&\qquad + \mu\left[\left(\nabla\langle \mathbf{v}_\beta\rangle + \nabla\langle \mathbf{v}_\beta\rangle^T\right) - \left(\nabla\langle \mathbf{v}_\beta\rangle_\eta + \nabla\langle \mathbf{v}_\beta\rangle_\eta^T\right)\right] + \\
&\qquad + \left[\langle \mathbf{T}_\sigma\rangle + \mathbf{I}\left(1-\varepsilon\right)\left(\rho_\sigma - \rho_\beta\right) g z\right]\Big\}\, dA,
\end{aligned} \tag{36}$$

where C represents a closed curve contained in the surface $A_{\omega\eta}$, ds is the arc length of C, $\mathbf{n}_s$ is the unit vector normal to C, δ is the thickness of the transition zone, and ∇_A is the two dimensional divergence operator acting on $A_{\omega\eta}$ and defined in OTW.

As discussed by OTW, the excess function can be considered proportional to the jump of the function passing from the homogeneous porous medium to the free fluid, thus we hypothesise

$$\begin{aligned}
\nabla_A \cdot \delta\langle \Sigma_s\rangle = \mathbf{n}_{\omega\eta} \cdot \Big\{ &-\mathbf{I}\left(\langle p_\beta\rangle_\omega - \varepsilon_\omega \rho_\beta g z\right) + \mu\left(\nabla\langle \mathbf{v}_\beta\rangle_\omega + \nabla\langle \mathbf{v}_\beta\rangle_\omega^T\right) \\
&+ \left[\langle \mathbf{T}_\sigma\rangle_\omega + \mathbf{I}\left(1-\varepsilon_\omega\right)\left(\rho_\sigma - \rho_\beta\right) g z\right] + \\
&+ \mathbf{I}\left(\langle p_\beta\rangle_\eta - \rho_\beta g z\right) - \mu\left(\nabla\langle \mathbf{v}_\beta\rangle_\eta + \nabla\langle \mathbf{v}_\beta\rangle_\eta^T\right)\Big\} \cdot \mathbf{F},
\end{aligned} \tag{37}$$

where $\mathbf{F}$ is an adjustable dimensionless second order tensor on the order of unity.

B. The boundary condition

Using the definitions of the excess stress given above, and considering that the boundary condition must be valid for any $A_{\omega\eta}$, Eq. (35) becomes

$$\mathbf{n}_{\omega\eta}\cdot\Big\{-\mathbf{I}\left(\langle p_\beta\rangle_\omega-\varepsilon_\omega\rho_\beta g z\right)+\mu\left(\nabla\langle\mathbf{v}_\beta\rangle_\omega+\nabla\langle\mathbf{v}_\beta\rangle_\omega^T\right)+\left[\langle\mathbf{T}_\sigma\rangle_\omega+\mathbf{I}\left(1-\varepsilon_\omega\right)\left(\rho_\sigma-\rho_\beta\right)g z\right]+ \\ +\mathbf{I}\left(\langle p_\beta\rangle_\eta-\rho_\beta g z\right)-\mu\left(\nabla\langle\mathbf{v}_\beta\rangle_\eta+\nabla\langle\mathbf{v}_\beta\rangle_\eta^T\right)\Big\}=\nabla_A\cdot\delta\left\langle\Sigma_s\right\rangle. \tag{38}$$

Since $\nabla_A\cdot\delta\langle\Sigma_S\rangle$, Eq. (37), has the same form of the *lhs* of Eq. (38), its effect on the boundary condition is lost because of the adjustable nature of the coefficients $\mathbf{F}$ in Eq. (37), we then obtain

$$\text{on } A_{\omega\eta},\quad \mathbf{n}_{\omega\eta}\cdot\Big\{-\mathbf{I}\left(\langle p_\beta\rangle_\omega-\varepsilon_\omega\rho_\beta g z\right)+\mu\left(\nabla\langle\mathbf{v}_\beta\rangle_\omega+\nabla\langle\mathbf{v}_\beta\rangle_\omega^T\right)+\left[\langle\mathbf{T}_\sigma\rangle_\omega+\mathbf{I}\left(1-\varepsilon_\omega\right)\left(\rho_\sigma-\rho_\beta\right)g z\right]+ \\ +\mathbf{I}\left(\langle p_\beta\rangle_\eta-\rho_\beta g z\right)-\mu\left(\nabla\langle\mathbf{v}_\beta\rangle_\eta+\nabla\langle\mathbf{v}_\beta\rangle_\eta^T\right)\Big\}=\mathbf{0}. \tag{39}$$

IV. BOUNDARY CONDITIONS AT $A_{\omega\pi}$

To derive the boundary condition at the interface between a porous medium and a homogeneous solid, we can follow the same procedure of Sec. III, we then identify a volume across the interface, V_T^0 (Fig. 2), where Eqs. (23)–(26) hold together with the continuity equation; in ω–region, Eqs. (21) and (22) are used together with the continuity equation, and in π–region, we write the following averaged momentum balance:

$$\pi\text{–region}\qquad \nabla\cdot\langle\mathbf{T}_\sigma\rangle_\pi^\sigma+\rho_\sigma\mathbf{g}=\nabla\cdot\langle\mathbf{T}_\sigma\rangle_\pi+\rho_\sigma\mathbf{g}=\mathbf{0}. \tag{40}$$

Let us underline that in π–region, as in η–region, the intrinsic averages coincide with the phase averages because the void fraction is either nil (i.e., the solid fraction is unity) or unity.

The kinematic boundary condition can be derived slavishly following OTW and neglecting the obtained jump also in this case. The latter assumption is clearly acceptable for the porous media we are dealing with in this paper (brush geometries, foam metals, etc.), while it may be wrong for granular soils. In the latter case, indeed, the porous medium void fraction necessarily increases while approaching a solid boundary and this may lead to an apparent velocity slip at the interface. Thus, in the case under investigation, the boundary condition is

$$\text{at } A_{\omega\pi},\qquad \langle\mathbf{v}_\beta\rangle_\omega=\langle\mathbf{v}_\beta\rangle_\pi=\mathbf{0}. \tag{41}$$

Concerning the stress boundary condition, as done above, we require Eqs. (21), (22), (25), (26), and (40) to hold on average. Using, then, the divergence theorem and subtracting the integrated Eqs. (25) and (26) to the integrated Eq. (21), (22), and (40), it is obtained

$$\begin{aligned}&\int_{A_{\omega\pi}}\mathbf{n}_{\omega\pi}\cdot\Big\{-\mathbf{I}\left(\langle p_\beta\rangle_\omega-\varepsilon_\omega\rho_\beta g z\right)+\mu\left(\nabla\langle\mathbf{v}_\beta\rangle_\omega+\nabla\langle\mathbf{v}_\beta\rangle_\omega^T\right)+ \\ &\qquad+\left[\langle\mathbf{T}_\sigma\rangle_\omega+\mathbf{I}\left(1-\varepsilon_\omega\right)\left(\rho_\sigma-\rho_\beta\right)g z\right]-\left(\langle\mathbf{T}_\sigma\rangle_\pi+\mathbf{I}\rho_\sigma g z\right)\Big\}\,dA= \\ &=\int_{A_\omega^O}\mathbf{n}_\omega\cdot\Big\{-\mathbf{I}\left(\langle p_\beta\rangle-\varepsilon\rho_\beta g z\right)+\mathbf{I}\left(\langle p_\beta\rangle_\omega-\varepsilon_\omega\rho_\beta g z\right) \\ &\qquad+\mu\left[\left(\nabla\langle\mathbf{v}_\beta\rangle+\nabla\langle\mathbf{v}_\beta\rangle^T\right)-\left(\nabla\langle\mathbf{v}_\beta\rangle_\omega+\nabla\langle\mathbf{v}_\beta\rangle_\omega^T\right)\right]+ \\ &\qquad+\left[\langle\mathbf{T}_\sigma\rangle+\mathbf{I}\left(1-\varepsilon\right)\left(\rho_\sigma-\rho_\beta\right)g z\right]-\left[\langle\mathbf{T}_\sigma\rangle_\omega+\mathbf{I}\left(1-\varepsilon_\omega\right)\left(\rho_\sigma-\rho_\beta\right)g z\right]\Big\}\,dA+ \\ &+\int_{A_\pi^O}\mathbf{n}_\pi\cdot\Big\{-\mathbf{I}\left(\langle p_\beta\rangle-\varepsilon\rho_\beta g z\right)+\mu\left(\nabla\langle\mathbf{v}_\beta\rangle+\nabla\langle\mathbf{v}_\beta\rangle^T\right)+ \\ &\qquad+\left[\langle\mathbf{T}_\sigma\rangle+\mathbf{I}\left(1-\varepsilon\right)\left(\rho_\sigma-\rho_\beta\right)g z\right]-\left[\langle\mathbf{T}_\sigma\rangle_\pi+\mathbf{I}\rho_\sigma g z\right]\Big\}\,dA,\end{aligned} \tag{42}$$

where A_ω^0 and A_π^0 are the surfaces of V_T^0 in ω– and π–region, respectively. As for the stress boundary condition at $A_{\omega\eta}$, the integrals of the *rhs* of Eq. (46) define an excess surface stress that can be considered proportional to the jump of the function passing from the homogeneous porous medium to the homogeneous solid region, i.e., it can be expressed with the same mathematical form of the *lhs*. So, the boundary condition finally becomes

$$\text{on } A_{\omega\pi}, \qquad \begin{aligned} &-\mathbf{I}\left(\langle p_\beta\rangle_\omega - \varepsilon_\omega \rho_\beta g z\right) + \mu\left(\nabla\langle \mathbf{v}_\beta\rangle_\omega + \nabla\langle \mathbf{v}_\beta\rangle_\omega^T\right) + \\ &+\left[\langle \mathbf{T}_\sigma\rangle_\omega + \mathbf{I}\left(1-\varepsilon_\omega\right)\left(\rho_\sigma - \rho_\beta\right) g z\right] - \left(\langle \mathbf{T}_\sigma\rangle_\pi + \mathbf{I}\rho_\sigma g z\right) = \mathbf{0}. \end{aligned} \tag{43}$$

The boundary conditions (43) and (39), together with the momentum balances of Eqs. (5) and (11) guarantee that the total stress in the porous medium is preserved and that it is fully transferred at the interface with a solid boundary and with a free fluid. This solves the paradox discussed in the Introduction.

V. ON THE EXISTENCE OF A STRESS JUMP

To obtain the solution of the shear problem described in Sec. I, which motivated the paper, the momentum balances equations expressed in Eqs. (5), (11), and (13) must be solved with the boundary conditions here developed. In the simple shear case depicted in Fig. 1, the tangential component of momentum balances simplifies to

$$\omega\text{–region}: \; y \in [0, d_i] \qquad \frac{\partial^2 \left\langle v_\beta^x \right\rangle_\omega}{\partial y^2} = \frac{\varepsilon_\omega}{K}\left\langle v_\beta^x \right\rangle_\omega, \tag{44}$$

$$\omega\text{–region}: \; y \in [0, d_i] \qquad \frac{\partial \left\langle T_\sigma^{xy} \right\rangle_\omega^\sigma}{\partial y} = -\frac{\mu}{K}\frac{\varepsilon_\omega}{1-\varepsilon_\omega}\left\langle v_\beta^x \right\rangle_\omega, \tag{45}$$

$$\eta\text{–region}: \; y \in [d_i, H + d_i] \qquad \frac{\partial^2 \left\langle v_\beta^x \right\rangle_\eta}{\partial y^2} = 0, \tag{46}$$

where x is the direction of the velocity, y that of the velocity gradient, and the interface between the porous medium and the free fluid is placed at d_i, while the solid boundary at 0. It is now convenient to express the stress boundary conditions in terms of the same variables of Eqs. (44)–(46) and to extract their components tangential and normal to the interface by multiplying them by the unit vector tangential and normal to the surface, respectively,

$$\text{at } y = d_i, \qquad (1-\varepsilon_\omega)\left\langle T_\sigma^{xy}\right\rangle_\omega^\sigma + \mu\frac{\partial\left\langle v_\beta^x\right\rangle_\omega}{\partial y} = \mu\frac{\partial\left\langle v_\beta^x\right\rangle_\eta}{\partial y}, \tag{47}$$

$$\text{at } y = d_i, \qquad (1-\varepsilon_\omega)\left[\left\langle T_\sigma^{yy}\right\rangle_\omega^\sigma + \left(\rho_\sigma - \rho_\beta\right) g y\right] + \varepsilon_\omega\left(\left\langle p_\beta\right\rangle_\omega^\beta - \rho_\beta g y\right) = \left\langle p_\beta\right\rangle_\eta^\beta - \rho_\beta g y, \tag{48}$$

$$\text{at } y = 0, \qquad (1-\varepsilon_\omega)\left\langle T_\sigma^{xy}\right\rangle_\omega^\sigma + \mu\frac{\partial\left\langle v_\beta^x\right\rangle_\omega}{\partial y} = \left\langle T_\sigma^{xy}\right\rangle_\pi^\sigma, \tag{49}$$

$$\text{at } y = 0, \qquad (1-\varepsilon_\omega)\left[\left\langle T_\sigma^{yy}\right\rangle_\omega^\sigma + \left(\rho_\sigma - \rho_\beta\right) g y\right] + \varepsilon_\omega\left(\left\langle p_\beta\right\rangle_\omega^\beta - \rho_\beta g y\right) = \left\langle T_\sigma^{yy}\right\rangle_\pi^\sigma + \rho_\sigma g y. \tag{50}$$

These equations must be solved together with the velocity continuities

$$\text{at } y = d_i, \qquad \left\langle v_\beta^x\right\rangle_\omega = \left\langle v_\beta^x\right\rangle_\eta, \tag{51}$$

$$\text{at } y = 0, \qquad \left\langle v_\beta^x\right\rangle_\omega = 0. \tag{52}$$

Moreover, the following boundary condition for Eq. (46) must be fulfilled

$$\text{at } y = H + d_i, \qquad \tau_0 = \mu\frac{\partial\left\langle v_\beta^x\right\rangle_\eta}{\partial y}, \tag{53}$$

where τ_0 is the imposed stress at the upper boundary placed at a distance H from the interface with the porous region.

Stokes' equation, Eq. (46), and Eq. (53) state that the shear stress is constant within the fluid layer in the open region and is equal to τ_0. The velocity profile and the solid stress within the porous medium can be now obtained solving Eqs. (44) and (45) with Eqs. (47), (51), and (52) and imposing $\langle T_\sigma^{xy}\rangle_\pi^\sigma = \tau_0$ in Eq. (49), in agreement with the equilibrium of forces discussed in the Introduction. A constitutive equation for T_σ^{xy} is not required under the hypothesis of rigid solid skeleton since the solid skeleton velocity will be consequently nil. The following undetermined non-trivial solution is found:

$$\left\langle v_\beta^x \right\rangle_\omega = C_1 \sqrt{\frac{k}{\varepsilon_\omega}} \sinh\left(y \sqrt{\frac{\varepsilon_\omega}{k}} \right), \tag{54}$$

$$\langle T_\sigma^{xy}\rangle_\omega^\sigma = \frac{\tau_0}{1-\varepsilon_\omega} - \frac{C_1 \mu}{1-\varepsilon_\omega} \cosh\left(y \sqrt{\frac{\varepsilon_\omega}{k}} \right), \tag{55}$$

where sinh(x) and cosh(x) are the hyperbolic sine and cosine functions, and C_1 is an undetermined integration constant. A particular solution fully describing the problem physics can be found by dividing the free fluid stress at the boundary with the porous region between that transferred to the fluid and that to the solid. In this way, the fluid-dynamics problem within the porous medium is decoupled from the stress transfer within the solid skeleton. In particular, the physics of the problem suggests rewriting Eqs. (47) and (48) as

$$\text{at } y = d_i, \qquad \begin{aligned} &a) \quad \mu \frac{\partial \left\langle v_\beta^x \right\rangle_\omega}{\partial y} = \varphi\, \mu \frac{\partial \left\langle v_\beta^x \right\rangle_\eta}{\partial y}, \\ &b) \quad (1-\varepsilon_\omega) \langle T_\sigma^{xy}\rangle_\omega^\sigma = (1-\varphi)\, \mu \frac{\partial \left\langle v_\beta^x \right\rangle_\eta}{\partial y}; \end{aligned} \tag{56}$$

$$\text{at } y = d_i, \qquad \begin{aligned} &a) \quad \varepsilon_\omega \left(\langle p_\beta \rangle_\omega^\beta - \rho_\beta g\, y \right) = \varphi \left(\langle p_\beta \rangle_\eta^\beta - \rho_\beta g\, y \right), \\ &b) \quad (1-\varepsilon_\omega) \left[\langle T_\sigma^{yy}\rangle_\omega^\sigma + (\rho_\sigma - \rho_\beta)\, g\, y \right] = (1-\varphi) \left(\langle p_\beta \rangle_\eta^\beta - \rho_\beta g\, y \right); \end{aligned} \tag{57}$$

where φ is a splitting parameter of the fluid stress tensor varying between zero and one. We already discussed that the pore pressure must be equal to the free fluid pressure, thus, Eq. (57a) immediately imposes $\varphi = \varepsilon_\omega$.

Brinkman's equation, Eq. (44), can be now solved with the boundary conditions ((52) and (56a)), and subsequently the stress balance on the solid skeleton, Eq. (45), can be solved with the boundary condition (56b). Equation (49) guarantees that the whole stress is also transferred to the solid boundary, $\langle T_\sigma^{xy}\rangle_\pi^\sigma = \tau_0$ as imposed by the force equilibrium. A non-trivial solution is found

$$\left\langle v_\beta^x \right\rangle_\omega = \frac{\tau_0}{\mu} \frac{\sqrt{k\, \varepsilon_\omega}}{\cosh\left(d_i \sqrt{\varepsilon_\omega / k} \right)} \sinh\left(y \sqrt{\frac{\varepsilon_\omega}{k}} \right), \tag{58}$$

$$\langle T_\sigma^{xy}\rangle_\omega^\sigma = \frac{\tau_0}{1-\varepsilon_\omega} \left[1 - \frac{\varepsilon_\omega}{\cosh\left(d_i \sqrt{\varepsilon_\omega / k} \right)} \cosh\left(y \sqrt{\frac{\varepsilon_\omega}{k}} \right) \right]. \tag{59}$$

Comparing Eqs. (58) and (59) with Eqs. (54) and (55), one straightforwardly gets

$$C_1 = \frac{\varepsilon_\omega \tau_0}{\mu \cosh\left(d_i \sqrt{\varepsilon_\omega / k} \right)}. \tag{60}$$

One may argue, however, that the assumption to divide the stress at interface between that carried by the fluid and that by the solid as postulated in Eqs. (56) and (57) is arbitrary and that more appropriates boundary conditions for the fluid and for the solid can be obtained following OTW and thus, introducing a stress jump in the fluid boundary condition and an equal and opposite stress jump in the solid skeleton boundary condition, so to fulfil Eq. (39) that guarantees the equilibrium of the forces applied on the shear cell. OTW stress jump might be thus interpreted as the fraction of the free fluid stress transferred to the solid skeleton more than a real jump due to the fluid flow perturbation in the vicinity of the interface caused by the porous medium. In this way, the basic idea of this paper that, within the porous medium, part of the stress is carried by the fluid and part by the solid skeleton is accomplished. OTW boundary condition, Eq. (3), was obtained imposing that the sum of the fluid momentum balances in ω- and η-region were on average equal to the momentum balance in $\mathcal{V}_T$. OTW started from Eqs. (5), (13), and (16) where both intrinsic averages and volume averages appear and, as already discussed in Sec. III, we would prefer to start from the momentum balances where only volume averages appear, and doing so, we would obtain a boundary condition very similar to OTW's one showing an analogous stress jump. For the sake of brevity, since the boundary condition obtained with the procedure depicted in this paper does not add anything physically new to OTW's one, we will here discuss the use of OTW boundary condition. OTW boundary condition can be written as

$$\text{on } A_{\omega\eta},\quad \begin{aligned} \mathbf{n}_{\omega\eta}\cdot\Big\{&-\mathbf{I}\left\langle p_\beta\right\rangle_\omega^\beta+\frac{\mu}{\varepsilon_\omega}\left(\nabla\left\langle\mathbf{v}_\beta\right\rangle_\omega+\nabla\left\langle\mathbf{v}_\beta\right\rangle_\omega^T\right)+\\ &+\mathbf{I}\left\langle p_\beta\right\rangle_\eta^\beta-\mu\left(\nabla\left\langle\mathbf{v}_\beta\right\rangle_\eta+\nabla\left\langle\mathbf{v}_\beta\right\rangle_\eta^T\right)\Big\}=\mu\mathbf{H}\cdot\left(\mathbf{K}^{-1}\cdot\left\langle\mathbf{v}_\beta\right\rangle_\omega\right),\end{aligned} \tag{61}$$

where the tensor $\mathbf{H}$ is easily obtained from the dimensionless drag vector $\mathbf{d}$ defined by OTW. Using this paper notation, the tangential (Eq. (3)) and normal component of Eq. (61) become

$$\text{at } y=d_i,\qquad \begin{aligned} &a)\quad \mu\frac{\partial\left\langle v_\beta^x\right\rangle_\eta}{\partial y}=\frac{\mu}{\varepsilon_\omega}\frac{\partial\left\langle v_\beta^x\right\rangle_\omega}{\partial y}-\mu\frac{\beta}{\sqrt{K}}\left\langle v_\beta^x\right\rangle_\omega,\\ &b)\quad \left\langle p_\beta\right\rangle_\eta^\beta=\left\langle p_\beta\right\rangle_\omega^\beta. \end{aligned} \tag{62}$$

The boundary conditions for $\langle\mathbf{T}_\sigma\rangle_\omega^\sigma$ are Eqs. (47) and (48). The fluid problem can be now solved independently from the solid skeleton momentum balance and thus, Eqs. (44) and (46) are integrated with the boundary conditions ((51)–(53) and (62a)) to obtain

$$\left\langle v_\beta^x\right\rangle_\omega=\frac{\tau_0}{\mu}\frac{\sqrt{\varepsilon_\omega k}}{\left[\cosh\left(d_i\sqrt{\varepsilon_\omega/k}\right)-\beta\sqrt{\varepsilon_\omega}\sinh\left(d_i\sqrt{\varepsilon_\omega/k}\right)\right]}\sinh\left(y\sqrt{\frac{\varepsilon_\omega}{k}}\right). \tag{63}$$

With Eq. (63), Eq. (45) can be integrated with the boundary condition (47) to obtain

$$\left\langle T_\sigma^{xy}\right\rangle_\omega^\sigma=\frac{\tau_0}{1-\varepsilon_\omega}\left[1-\frac{\varepsilon_\omega}{\cosh\left(d_i\sqrt{\varepsilon_\omega/k}\right)-\beta\sqrt{\varepsilon_\omega}\sinh\left(d_i\sqrt{\varepsilon_\omega/k}\right)}\cosh\left(y\sqrt{\frac{\varepsilon_\omega}{k}}\right)\right]. \tag{64}$$

The approach of OTW can be naturally extended to the ω–π interface thus requiring that also across this interface, the averaged fluid momentum balance valid in $\mathcal{V}_T^0$ be equal to the averaged fluid momentum balance valid in V_ω. Since the fluid momentum balance equations in $\mathcal{V}_T^0$ are the same of those valid in $\mathcal{V}_T$, while in π–region no fluid momentum balance can be written, the procedure developed by OTW can be slavishly followed and the equations for the case at hand coincide with that of OTW by simply substituting the subscript η with the subscript π and setting to zero both $\left\langle\mathbf{v}_\beta\right\rangle_\pi$ and $\left\langle p_\beta\right\rangle_\pi^\beta$, being these two variables not defined in π–region. At the end, one obtains Eq. (61) that, with the substitution rules here dictated, eventually becomes

$$\text{on } A_{\omega\pi},\qquad \mathbf{n}_{\omega\pi}\cdot\left\{-\mathbf{I}\left\langle p_\beta\right\rangle_\omega^\beta+\frac{\mu}{\varepsilon_\omega}\left(\nabla\left\langle\mathbf{v}_\beta\right\rangle_\omega+\nabla\left\langle\mathbf{v}_\beta\right\rangle_\omega^T\right)\right\}=\mu\mathbf{F}_1\cdot\left(\mathbf{K}^{-1}\cdot\left\langle\mathbf{v}_\beta\right\rangle_\omega\right), \tag{65}$$

whose tangential component, using Eq. (52), simplifies to

$$\text{at } y=0,\qquad \frac{\mu}{\varepsilon_\omega}\frac{\partial\left\langle v_\beta^x\right\rangle_\omega}{\partial y}=0. \tag{66}$$

Consequently, since Eq. (49) must hold, the solid skeleton stress must obey the following condition:

$$\text{at } y=0,\qquad (1-\varepsilon_\omega)\left\langle T_\sigma^{xy}\right\rangle_\omega^\sigma=\left\langle T_\sigma^{xy}\right\rangle_\pi^\sigma=\tau_0. \tag{67}$$

Equations (66) and (67) represent a kinematic constraint to the obtained velocity profile (Eq. (63)) and to the stress carried by the solid skeleton (Eq. (64)), respectively, and it is clear that they can be satisfied only for an infinitely thick porous medium. Conversely, for finite porous media, the velocity and stress profiles do not fulfil these kinematic conditions. This allows concluding that there is a lack of generality in OTW approach and so we believe that the phenomenological stress partition introduced by Eq. (57) must be preferred.

VI. FINAL REMARKS

In this paper, we have derived a boundary condition at the interface between a free fluid and a porous medium stating that the stress is transferred both to the fluid within the porous medium and to the solid skeleton. Similarly, a boundary condition at the interface between a porous medium and a homogeneous solid is also obtained. With these boundary conditions, together with the stress transfer equation within the solid skeleton, derived in part I, we have been able to satisfy the overall force equilibrium on a shear cell partially filled with a porous medium. This solves the apparent paradox discussed in the Introduction and in part I.

The boundary conditions are obtained following the approach of OTW but considering not only the momentum balance written on the fluid within the porous medium but also that written on the solid skeleton. To derive the boundary conditions, differently from OTW, we have preferred to express all the stresses within the momentum balances in terms of volume phase averages, and consequently, we did not use the intrinsic phase averages for terms like the pressure. Our choice is justified because at the interface each phase-averaged stress is applied on the same surface and it must be, then, integrated over the same area to obtain the associated force. The use of the same averages thus simplifies the derivation of the boundary condition. The derived boundary condition is essentially a force balance at the interface and a nil stress jump has been found both at the interface with the free fluid and at the interface with the homogeneous solid. The final stress boundary conditions, expressed in terms of the same phase averages of the momentum balances, then become

$$\text{on } A_{\omega\eta},\quad \begin{aligned} \mathbf{n}_{\omega\eta}\cdot\Big\{&-\mathbf{I}\varepsilon_\omega\left(\langle p_\beta\rangle_\omega^\beta-\rho_\beta g z\right)+\mu\left(\nabla\langle\mathbf{v}_\beta\rangle_\omega+\nabla\langle\mathbf{v}_\beta\rangle_\omega^T\right)\\ &+(1-\varepsilon_\omega)\left[\langle\mathbf{T}_\sigma\rangle_\omega^\sigma+\mathbf{I}\left(\rho_\sigma-\rho_\beta\right)gz\right]+\\ &+\mathbf{I}\left(\langle p_\beta\rangle_\eta^\beta-\rho_\beta g z\right)-\mu\left(\nabla\langle\mathbf{v}_\beta\rangle_\eta+\nabla\langle\mathbf{v}_\beta\rangle_\eta^T\right)\Big\}=\mathbf{0}, \end{aligned} \tag{68}$$

$$\text{on } A_{\omega\pi},\quad \begin{aligned} &-\mathbf{I}\varepsilon_\omega\left(\langle p_\beta\rangle_\omega^\beta-\rho_\beta g z\right)+\mu\left(\nabla\langle\mathbf{v}_\beta\rangle_\omega+\nabla\langle\mathbf{v}_\beta\rangle_\omega^T\right)+\\ &+(1-\varepsilon_\omega)\left[\langle\mathbf{T}_\sigma\rangle_\omega^\sigma+\mathbf{I}\left(\rho_\sigma-\rho_\beta\right)gz\right]-\left(\langle\mathbf{T}_\sigma\rangle_\pi^\sigma+\mathbf{I}\rho_\sigma g z\right)=\mathbf{0}. \end{aligned} \tag{69}$$

The first three terms of Eq. (68) represent the sum of the stresses carried by the fluid and the solid skeleton of the porous medium, while the fourth and fifth ones are the stresses carried by the free fluid. Analogously, in Eq. (69), the first three terms of the *lhs* are the sum of the stresses carried by the fluid and the solid skeleton of the porous medium, while the fourth one is the stress carried by the homogeneous solid.

We also showed that in simple shear flow, the use of Eqs. (68) and (69) leads to an undetermined solution that can be specified if the equation for the fluid dynamics in the porous medium is decoupled from the stress transfer equation within the solid skeleton. This can be done if the free fluid stress at the interface is phenomenologically divided between that transferred to the fluid and that to the solid skeleton through to the void fraction. Conversely, if the fluid dynamics problem is decoupled by using OTW boundary condition, the solution found may be valid only for infinitely thick porous media.

As future work, we plan to compare the results of our model with both experimental data available in the literature and new ones.

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