CM12004: Problem Sheet

1. (i)

X	Y	P	Q	$P \to Q$
0	0	0	1	1
0	1	1	1	1
1	0	1	1	1
1	1	1	0	0

conclusion: neither

(ii)

X	Y	P	Q	$P \to Q$
0	0	0	1	1
0	1	1	0	0
1	0	1	0	0
1	1	1	0	0

conclusion: neither

(iii)

X	Y	P	Q	$P \to Q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	1
1	1	1	1	1

conclusion: tautology

(iv)

X	Y	P	Q	$P \to Q$
0	0	1	1	1
0	1	1	1	1
1	0	1	1	1
1	1	0	0	1

conclusion: tautology

(v)

X	Y	Z	P	Q	$P \to Q$
0	0	0	0	0	1
0	0	1	0	0	1
0	1	0	0	0	1
0	1	1	0	1	1
1	0	0	0	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

conclusion: tautology

(vi)

X	Y	P	Q	$P \to Q$
0	0	1	1	1
0	1	1	1	1
1	0	0	1	1
1	1	1	0	0

conclusion: neither

(vii)

X	Y	P	Q	$P \to Q$
0	0	1	0	0
0	1	1	1	1
1	0	0	0	1
1	1	1	0	0

conclusion: neither

(viii)

X	Y	Z	P	Q	$P \to Q$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	0	1	1
0	1	0	0	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	0	1	0	1	1
1	1	1	0	0	1
1	1	1	1	1	1

conclusion: tautology

2. (a) binary logical connectives operate on 2 binary inputs. each binary input can have 2 values, 1 or 0.
$\therefore$ there are 4 possible combinations of inputs that can be operated on by a connective.
$\implies$ each logical connective can have 4 possible binary outputs, one corresponding to each possible input combination.
having 4 possible binary outputs $\implies$ there are 16 possible combinations of outputs a connective can have.
$\therefore$ **answer: there are 16 possible binary logical connectives.**

(b) (i) $\neg(\neg(P \wedge \neg Q) \wedge \neg(\neg P \wedge Q))$
(ii) $\neg(\neg P \vee Q) \vee \neg(P \vee \neg Q)$
(iii) $(P \to Q) \to \neg(Q \to P)$

(c)

$$\bar{\wedge} = \neg(X \wedge Y) \quad (1)$$
$$\neg = X \bar{\wedge} X \quad (2)$$
$$\wedge = \neg(\neg(X \wedge Y)) = (X \bar{\wedge} Y) \bar{\wedge} (X \bar{\wedge} Y) \quad (3)$$
$$\vee = \neg(\neg X \wedge \neg Y) = (X \bar{\wedge} X) \bar{\wedge} (Y \bar{\wedge} Y) \quad (4)$$
$$\to = \neg X \vee Y = \neg(X \wedge \neg Y) = X \bar{\wedge} (Y \bar{\wedge} Y) \quad (5)$$

(d)

$$\bar{\vee} = \neg(X \vee Y) \quad (6)$$
$$\neg = X \bar{\vee} X \quad (7)$$
$$\vee = \neg(\neg(X \vee Y)) = (X \bar{\vee} Y) \bar{\vee} (X \neg Y) \quad (8)$$
$$\wedge = \neg(\neg X \vee \neg Y) = (X \bar{\vee} X) \bar{\vee} (Y \bar{\vee} Y) \quad (9)$$
$$\to = \neg X \vee Y = ((X \bar{\vee} X) \bar{\vee} Y) \bar{\vee} ((X \bar{\vee} X) \bar{\vee} Y) \quad (10)$$

3. (a) (i) **question:** $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z}(x^2 < y + 1)$
true
Proof.

$$x^2 < y + 1 \quad (11)$$
$$\implies x^2 - 1 < y \quad (12)$$
$$\implies y > x^2 - 1 \quad (13)$$

if $x \in \mathbb{Z}$ and $y \in \mathbb{Z} \implies \exists y \mid y > x^2 - 1 \because \mathbb{Z}$ has no upper bound, so there will always be a bigger y $\forall x$. □

(ii) **question:** $\exists x \in \mathbb{Z} \forall y \in \mathbb{Z}(x^2 < y + 1)$
false
Proof.

$$x^2 < y + 1 \quad (14)$$
$$\implies x^2 - 1 < y \quad (15)$$

we know this to be false, as for it to be true, $x^2 - 1$ would have to be smaller than every integer, which is impossible. □

(iii) **question:** $\exists x \in \mathbb{Z} \forall x \in \mathbb{Z}(x^2 < y + 1)$
false
Proof.

$$x^2 < y + 1 \quad (16)$$
$$\implies x^2 - 1 < y \quad (17)$$

for a similar reason to (ii) we know this to also be false, as for it to be true, y would have to be bigger than every integer, which is impossible. □

(iv) **question:** $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z}((x < y) \to (x^2 < y^2))$
true
Proof. let $y = x + 1$.
$x < y$ is true $\forall x, y \because x < x + 1$
$\implies (x < y) \to (x^2 < y^2)$ is true if $(x^2 < y^2)$ is true.
$y^2 = (x + 1)^2 = x^2 + 2x + 1 \implies y^2 > x^2$
$\therefore (x < y) \to (x^2 < y^2)$ is true when $y = x + 1$.
$\therefore \exists y \in \mathbb{Z}(y = x + 1) \forall x \in \mathbb{Z}((x < y) \to (x^2 < y^2))$ □

(b) (i) let $x = 6, y = 5. \implies x^2 = 36,\ y + 1 = 6.\ 36 > 6 \therefore$ **false.**
(ii) let $x = 2, y = 5. \implies x^2 = 4,\ y + 1 = 6.\ 4 < 6 \therefore$ **true.**
(iii) let $y = -1. \implies y + 1 = 0. \forall x \in \mathbb{Z}, x^2 \geq 0 \therefore \nexists x \mid x^2 < 0 \therefore$ **false.**
(iv) let $x = 0, \implies x^2 = 0.$ if $y > 0, \implies y > x$ but also $y^2 > 0 \therefore (x^2 < y^2)$ is true. if $y <= 0, \implies y < x \therefore ((x < y) \to (x^2 < y^2))$ is true. $\therefore$ **true.**

4. (a)
    - $2^{\emptyset} = 1$
    - $2^{\{0\}} = 2$
    - $2^{\{0\} \cup \{1\}} = 4$
    - $2^{\{\emptyset, 0, 1\}} = 2^3 = 8$
    - $2^{2^{2^{\{0,1\}}}} = 2^{2^{2^2}} = 2^{256} = 65536$

   (b) for $A, |A| = 2^n$. therefore, there are 2^n values of S. for each value of s, either $x \in S$, or $x \notin S$. $\implies$ there are 2^{n-1} values of S where $x \in S$. $x \in A$, $\therefore$ there are n values of x. $\implies$ there are $n \cdot 2^{n-1}$ pairs $\{(x, S) \mid x \in S, \S \in 2^A\}$.

5. (a)

   (b)

   (c)

6. (a)
    - $2^{\emptyset}$
    - $2^{\{0\}}$
    - $2^{\{0\} \cup \{1\}}$
    - $2^{\{\emptyset, 0, 1\}}$
    - $2^{2^{2^{\{0,1\}}}}$

   (b) (i)

   (ii)