# CM12004: Problem Sheet

1. (i)

| $X$ | $Y$ | $P$ | $Q$ | $P \to Q$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 |

**conclusion:** neither

(ii)

| $X$ | $Y$ | $P$ | $Q$ | $P \to Q$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 |

**conclusion:** neither

(iii)

| $X$ | $Y$ | $P$ | $Q$ | $P \to Q$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 |

**conclusion:** tautology

(iv)

| $X$ | $Y$ | $P$ | $Q$ | $P \to Q$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 |

**conclusion:** tautology

(v)

| $X$ | $Y$ | $Z$ | $P$ | $Q$ | $P \to Q$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

**conclusion:** tautology

(vi)

| $X$ | $Y$ | $P$ | $Q$ | $P \to Q$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 |

**conclusion:** neither

(vii)

| $X$ | $Y$ | $P$ | $Q$ | $P \to Q$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | 0 |

**conclusion:** neither

(viii)

| $X$ | $Y$ | $Z$ | $P$ | $Q$ | $P \to Q$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

**conclusion:** tautology

2. (a) binary logical connectives operate on 2 binary inputs. each binary input can have 2 values, 1 or 0.
$\therefore$ there are 4 possible combinations of inputs that can be operated on by a connective.
$\implies$ each logical connective can have 4 possible binary outputs, one corresponding to each possible input combination.
having 4 possible binary outputs $\implies$ there are 16 possible combinations of outputs a connective can have.
$\therefore$ **answer: there are 16 possible binary logical connectives.**

(b) (i) $\neg(\neg(P \wedge \neg Q) \wedge \neg(\neg P \wedge Q))$
(ii) $\neg(\neg P \vee Q) \vee \neg(P \vee \neg Q)$
(iii) $(P \to Q) \to \neg(Q \to P)$

(c)

$$\bar{\wedge} = \neg(X \wedge Y) \tag{1}$$
$$\neg = X \bar{\wedge} X \tag{2}$$
$$\wedge = \neg(\neg(X \wedge Y)) = (X \bar{\wedge} Y) \bar{\wedge} (X \bar{\wedge} Y) \tag{3}$$
$$\vee = \neg(\neg X \wedge \neg Y) = (X \bar{\wedge} X) \bar{\wedge} (Y \bar{\wedge} Y) \tag{4}$$
$$\to = \neg X \vee Y = \neg(X \wedge \neg Y) = X \bar{\wedge} (Y \bar{\wedge} Y) \tag{5}$$

(d)

$$\bar{\vee} = \neg(X \vee Y) \tag{6}$$
$$\neg = X \bar{\vee} X \tag{7}$$
$$\vee = \neg(\neg(X \vee Y)) = (X \bar{\vee} Y) \bar{\vee} (X \neg Y) \tag{8}$$
$$\wedge = \neg(\neg X \vee \neg Y) = (X \bar{\vee} X) \bar{\vee} (Y \bar{\vee} Y) \tag{9}$$
$$\to = \neg X \vee Y = ((X \bar{\vee} X) \bar{\vee} Y) \bar{\vee} ((X \bar{\vee} X) \bar{\vee} Y) \tag{10}$$

3. (a) (i) **question:** $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z}(x^2 < y + 1)$
**true**
*Proof.*

$$x^2 < y + 1 \quad (11)$$
$$\implies x^2 - 1 < y \quad (12)$$
$$\implies y > x^2 - 1 \quad (13)$$

if $x \in \mathbb{Z}$ and $y \in \mathbb{Z} \implies \exists y \mid y > x^2 - 1 \because \mathbb{Z}$ has no upper bound, so there will always be a bigger $y$ $\forall x$. □

(ii) **question:** $\exists x \in \mathbb{Z} \forall y \in \mathbb{Z}(x^2 < y + 1)$
**false**
*Proof.*

$$x^2 < y + 1 \quad (14)$$
$$\implies x^2 - 1 < y \quad (15)$$

we know this to be false, as for it to be true, $x^2 - 1$ would have to be smaller than every integer, which is impossible. □

(iii) **question:** $\exists x \in \mathbb{Z} \forall x \in \mathbb{Z}(x^2 < y + 1)$
**false**
*Proof.*

$$x^2 < y + 1 \quad (16)$$
$$\implies x^2 - 1 < y \quad (17)$$

for a similar reason to (ii) we know this to also be false, as for it to be true, $y$ would have to be bigger than every integer, which is impossible. □

(iv) **question:** $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z}((x < y) \to (x^2 < y^2))$
**true**
*Proof.* let $y = x + 1$.
$x < y$ is true $\forall x, y \because x < x + 1$
$\implies (x < y) \to (x^2 < y^2)$ is true if $(x^2 < y^2)$ is true.
$y^2 = (x + 1)^2 = x^2 + 2x + 1 \implies y^2 > x^2$
$\therefore (x < y) \to (x^2 < y^2)$ is true when $y = x + 1$.
$\therefore \exists y \in \mathbb{Z}(y = x + 1) \forall x \in \mathbb{Z}((x < y) \to (x^2 < y^2))$ □

(b) (i)

(ii)

(iii)

4. (i)

(ii)

(iii)

(iv)

(v)

(vi)

(vii)

(viii)

5. (a)

   (b)

   (c)

6. (a)
   - $2^{\emptyset}$
   - $2^{\{0\}}$
   - $2^{\{0\}\cup\{1\}}$
   - $2^{\{\emptyset,0,1\}}$
   - $2^{2^{2^{\{0,1\}}}}$

   (b) (i)

   (ii)