Finite Difference Warmup Experiments

Consider the transport equation $u_t + au_x = 0$ on the 2-periodic domain $-1 \leq x \leq 1$ with initial data $u(0, x) = e^{-36x^2}$.

Let u_i^n be a gridfunction on the grid $x_i = -1 + ih, h = 2/N, i = 1, \ldots, N$, at time $t_n = kn$, and consider the schemes

$$u_i^{n+1} = u_i^n - akD_+ u_i^n, \tag{1}$$
$$u_i^{n+1} = u_i^n - akD_- u_i^n, \tag{2}$$
$$u_i^{n+1} = u_i^n - akD_0 u_i^n, \tag{3}$$
$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - akD_0 u_i^n, \tag{4}$$

for $i = 1, \ldots, N$ and complemented with periodic boundary conditions $u_0^n = u_N^n$ and $u_{N+1}^n = u_1^n$. Here

$$D_+ w_i = \frac{w_{i+1} - w_i}{h},$$
$$D_- w_i = \frac{w_i - w_{i-1}}{h},$$
$$D_0 w_i = \frac{w_{i+1} - w_{i-1}}{2h}.$$

are the standard finite difference operators.

1. Take $a = 1$, $N = 50$ and set $k \approx 0.5h$ and evolve the solution until time $t = 2$ for all of the schemes above. Hand in plots of the numerical solutions at the final time and the initial data. Which of the schemes appear to be stable?

2. Repeat but with $k \approx 0.5h^2$. Hand in plots of the numerical solutions at the final time and the initial data. Now, which of the schemes appear to be stable?

3. Next consider a "random choice" method. Precisely, when you are computing u_i^{n+1} then (for each i) select, at random from a uniform distribution, the right hand side to be either (1) or (2). Use the stable scheme 75% of the time (on average). Use $k \approx 0.5h$. Is this scheme stable?

4. For the above scheme and for a scheme where you randomly (50% / 50%) mix the stable scheme of (1) and (2) with (3) compute the maximum error at time $t = 2$ (the exact solution is the initial data). In a log-log plot display the errors as a function of h for $n = 10, 11, \ldots, 1000$. In the same figure also plot h^1 and h^2. Note that (1), (2) and (4) are first order in space and (3) is second order accurate in space and all of the schemes are first order accurate in time. What gives?