UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part 1A

Saturday 9th June, 2012 13.30 to 16.30

PHYSICS

Attempt **all** *questions from Section A and* **one** *question from* **each** *of Sections B, C, D, and E.*

Numbers in the right hand margins indicate the approximate distribution of the marks available. Note that Section A carries approximately one third of the total marks.

The paper consists of 10 sides including this one and is accompanied by a Mathematical Formulae Handbook giving values of constants and containing mathematical formulae which you may quote without proof.

Answers to **each Section** *must be written in a* **separate script** *booklet. Should you run out of paper please ask the invigilator for another booklet and tie the two together with a tag.*

The cover page of **each booklet** *must be completed before leaving the examination.*

STATIONERY REQUIREMENTS
5 x 8 page script booklets
Rough workpad

SPECIAL REQUIREMENTS
Mathematical Formulae Handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

A

SECTION A

Attempt all questions. Answers should be concise and relevant formulae may be quoted without proof.

1 A sphere of radius $2a$ has a uniform density ρ except for a spherical cavity of radius a at its centre. Find the gravitational field at distances a, $2a$, and $4a$ from the centre. [5]

2 The fundamental frequency of a circular drum skin depends on its radius, the mass per unit area of the skin and the tension (the force per unit length applied at the rim). Use dimensional analysis to predict the effect on the frequency of simultaneously doubling the radius and the tension. [5]

3 A mass of 10 kg is suspended from a vertical spring. It is released from rest with the spring initially at zero extension. The system undergoes lightly damped harmonic motion with frequency 0.5 Hz, and the amplitude of oscillation falls to 1% of its initial value after 30 s. If a mass of 20 kg is used in place of the 10 kg mass, what is the new frequency of oscillation and how long does it take the amplitude of oscillation to fall to 1% of its new initial value? [5]

4 A resistance R is in series with an inductance L. At angular frequency ω the magnitude of the complex impedance Z of this combination is given by

$$|Z|^2 = R^2 + (\omega L)^2.$$

Find $|Z|$, and the error in $|Z|$, given that $R = (100\,\Omega) \pm 1\%$ and $\omega L = 135 \pm 6\,\Omega$. [5]

5 Six very narrow slits with centres $50\,\mu$m apart are illuminated normally by light with wavelength 600 nm; one of the slits is covered by phase-reversing plastic (introducing an additional half wavelength of path). The resulting interference pattern is observed on a distant screen. Find the intensity on the screen at an angle 2×10^{-3} rad to the incident beam in terms of the intensity I which would be produced by any of the slits on its own. [5]

6 Find the maximum charge that can be stored in an air-gap parallel plate capacitor with plates measuring $2\,\mathrm{mm} \times 2\,\mathrm{mm}$, given that the breakdown strength of air is $3 \times 10^6\ \mathrm{V\,m^{-1}}$.

[5]

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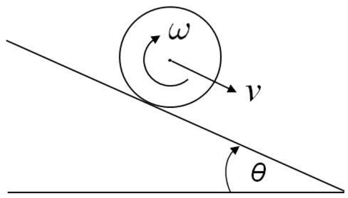
SECTION B

Answer **one** *question only.*

B7 A uniform ball of mass m and radius a rolls without slipping down a plane inclined at an angle θ to the horizontal. At time t its speed v and its angular velocity ω are as shown in the first diagram.

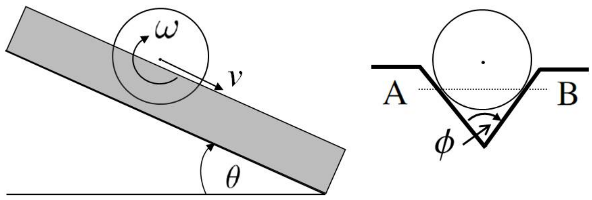


The ball is released from rest at time t=0. Show that its acceleration at time t is:

$$\frac{5}{7}g\sin\theta.$$ [5]

[*The moment of inertia of the ball about an axis through its centre of mass is* $\frac{2}{5}ma^2$.]

Now, instead of a plane ramp, the ball rolls in a deep wedge-shaped groove of angle ϕ as shown in the second diagram so that the ball instantaneously rolls about the axis AB.



Show that the acceleration is now given by:

$$\frac{5g\sin\theta}{5+\left[2/\sin^2(\phi/2)\right]}$$ [4]

Such a ramp with θ=30° and ϕ=90° is used in a bowling alley to aim a 5 kg ball at the pins.

The ball is released from rest at the top of the ramp and takes 1s to roll down its entire length. Find the speed of the ball and its total kinetic energy when it reaches the end of the ramp. [3]

The ball then exits the ramp and rolls without slipping on the horizontal floor towards the pins. If the ramp is designed in such a way that the ball exits the ramp without changing its total kinetic energy, how long will it take to reach the pins which are 20 m from the bottom of the ramp? [3]

[*You may neglect air resistance throughout this question.*]

B8 Give expressions for the total energy E and the momentum p of a relativistic particle with mass m and speed u. Use these to show that $E^2 = p^2c^2 + m^2c^4$. State the form this takes for photons, which have zero mass, and show that this is consistent with the de Broglie relations $E = hf$ and $p = h/\lambda$. [4]

An electron (mass m) is at rest in a laboratory, when a photon with energy E_0, equal to the rest energy mc^2 of the electron, collides with it. Find the initial momenta of the photon and the electron in a frame moving with speed v in the same direction as the photon. Hence show that the speed of the zero momentum (ZM) frame is $c/2$. [4]

In the ZM frame, after the collision, the photon is observed to leave at 90° to its original direction. Find the speed and direction of the electron in the ZM frame. By transforming both velocity components of the electron back to the lab frame, find the speed and direction of the electron in the lab frame. [5]

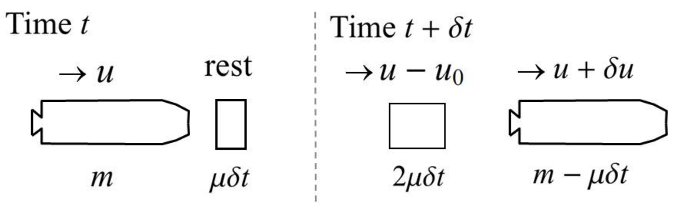
Find also the final energy of the photon in the lab frame. [2]

[*You may use without proof the expressions for frequency and velocity transformations between frames in standard configuration*

$$f' = \sqrt{\frac{1-v/c}{1+v/c}}\,f, \quad u'_x = \frac{u_x - v}{1 - u_x v/c^2}, \quad u'_y = \frac{u_y}{\gamma\,(1 - u_x v/c^2)} \,.$$
]

B9 State clearly the principle of conservation of linear momentum for a set of interacting bodies, and the conditions under which it applies. Outline how it follows from Newton's Laws of Motion. [3]

At time t a (non-relativistic) spacecraft has total mass $m(t)$ including fuel and is travelling at speed $u(t)$; it takes in mass μ per unit time from the interstellar medium in front of it, combines this mass with an equal mass of a fuel carried internally, and ejects the products at a relative speed u_0 backwards. The two parts of the diagram below illustrate the situation at two successive instants of time t and $t + \delta t$.



By considering the total momentum, show that the acceleration of the craft is given by

$$m\frac{\mathrm{d}u}{\mathrm{d}t} = 2\mu u_0 - \mu u \,.$$ [4]

Deduce an expression for the maximum speed u_{max} the spacecraft can achieve relative to the interstellar medium. Explain in words why the craft is subject to a maximum speed, while a conventional rocket engine (taking in nothing from the medium in front) is not. [3]

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In a particular spacecraft, starting from rest at $t = 0$ with a total mass m_i, the engine is designed so that μ is constant whatever the speed of the craft. Write down an expression for its mass m at time t. By integrating the equation of motion, find the speed u of the spacecraft at time t, and show that its acceleration is in fact constant. Explain why the speed u_{max} cannot actually be achieved in practice. [5]

SECTION C

Answer **one** *question only.*

C10 A pendulum consists of a point mass M suspended on a light rigid rod of length L which swings in a vertical plane under gravity. At time t the rod makes an angle $\theta(t)$ to the vertical. Draw a diagram indicating the forces on the mass. Hence, assuming $\theta(t) \ll 1$ rad at all times, show that the equation of motion is given approximately by

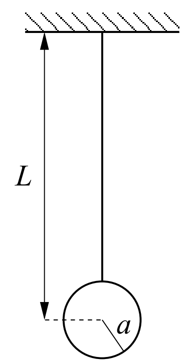
$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\omega_0^2\theta.$$

Find an expression for ω_0^2 and deduce the period T_0. [5]

The motion of the pendulum can be described by $\theta(t) = \theta_0 \cos(\omega_0 t + \phi)$. Deduce expressions for the potential energy and the kinetic energy of the pendulum as a function of time and show explicitly that the total energy is constant. At $t = 0$ the angular displacement of the pendulum is α and its angular speed is β; find expressions for θ_0 and ϕ in terms of α and β. [5]

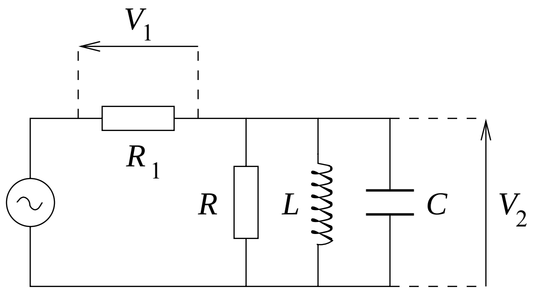
The point mass is now replaced by a bob, consisting of a uniform solid sphere of radius a with the same mass M, as shown in the diagram below. Find an expression for the period T of this system, and show that for the case when $a \ll L$ the period is given by $T \approx T_0(1 + a^2/5L^2)$. The bob is made of aluminium (density 2700 kg m^{-3}) and has a mass $M = 0.200$ kg. If $L = 1.00$ m find the percentage difference between T and T_0. [5]

[*The moment of inertia of the bob about an axis through its centre of mass is* $\frac{2}{5}Ma^2$.]



C11 (a) Explain what is meant by complex impedance. Assuming that for a capacitor $q = CV$ (where q is the charge on the capacitor and V the voltage across it) deduce an expression for the complex impedance of a capacitor. [3]

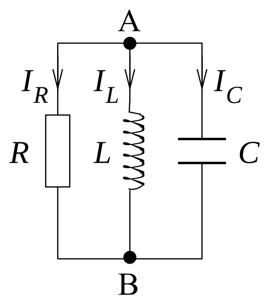
A circuit consists of a resistor R_1 in series with a parallel combination of a resistor R, an inductor L and a capacitor C as shown in the first diagram. A sinusoidal voltage of angular frequency ω is applied to the circuit. Find a general expression for V_2/V_1. Draw clearly labelled sketches showing how the amplitude and phase of V_2/V_1 vary as a function of angular frequency ω. Give physical reasons for the behaviour as a function of frequency. [5]



(b) For the parallel resonant circuit shown in the second diagram, the current I_L through the inductor is such that

$$V = L\frac{dI_L}{dt} \quad i.e. \quad I_L = \frac{1}{L}\int V dt,$$

where V is the voltage across AB.



Write down the corresponding expressions for I_C and I_R and hence show that [2]

$$C\frac{d^2V}{dt^2} + \frac{1}{R}\frac{dV}{dt} + \frac{1}{L}V = 0.$$

Show by substitution, or otherwise, that a general solution of this equation is given by

$$V = Ae^{-\gamma t}e^{i(\omega t+\phi)},$$

and find expressions for γ and ω. [*You may assume* $4R^2 > L/C$.] [5]

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SECTION D
Answer **one** *question only.*

D12 Consider a string which lies along the x axis and is clamped at $x = 0$ and $x = \mathrm{L}$. The string is under tension T, and has a mass per unit length ρ.

Assuming the angular frequency ω is related to the wave number k for the string by $\omega = k\sqrt{T/\rho}$, find the lowest frequency of vibration ω_1 and the next higher frequency ω_2. [3]

The transverse displacement of the string is given by

$$y(x,t) = A_1 \sin(k_1 x)\cos(\omega_1 t) + A_2 \sin(k_2 x)\cos(\omega_2 t),$$

where k_1 and k_2 are the wavenumbers corresponding to ω_1 and ω_2, respectively. For the case $A_1 = A_2$ draw clearly labelled sketches of the displacement of the string as a function of x at time $t = 0$ and at time $t_1 = \pi/(2\omega_1)$. [4]

Find an expression for the times at which the displacement of the string will be the same as at $t = 0$. [2]

For a more realistic model of a stretched string

$$\omega = k\sqrt{\frac{T}{\rho} + \alpha k^2}\,,$$

where α is a positive constant with appropriate units. Assuming $\alpha k^2 \ll T/\rho$, find an approximate relationship between ω and k to first order in α. Explain qualitatively how this change in the relationship between ω and k will affect your sketches of the string at times $t = 0$ and $t = t_1$. [4]

How will the use of the more realistic model above alter your conclusions about the times at which the displacement of the string will be the same as at t=0? [2]

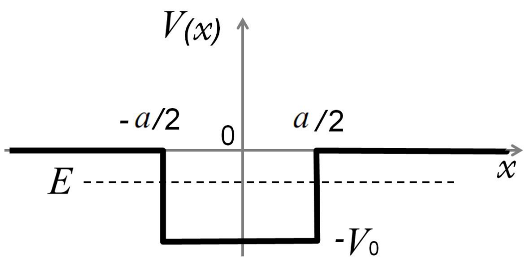
D13 The time-independent Schrödinger equation has the form

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

How are the symbols $V(x)$, E, and $\psi(x)$ to be interpreted? [3]

An electron moving in one dimension is bound in a potential well for which the potential energy is $-V_0$ in the region $|x| < a/2$ and zero elsewhere as shown in the diagram below. The normalised ground state wavefunction has the form

$$\psi(x) = \begin{cases} A\cos(kx) & \text{if} \quad |x| < a/2 \\ C\exp(-q|x|) & \text{elsewhere.} \end{cases}$$



Explain how the Schrödinger equation leads to this mathematical form for the wavefunction, and give expressions for k and q in terms of E and V_0. Make a rough sketch of the wavefunction as a function of position. [3]

Sketch the wavefunction for the first excited bound state, assuming such a well has one, and deduce that its wavenumber k_1 must satisfy $k_1 a > \pi$. Hence show that the well has no excited bound state if

$$V_0 < \frac{\hbar^2\pi^2}{2ma^2}.$$ [2]

For a particular well, $a = 0.2$ nm, $A = 7.29\times10^4$ m$^{-1/2}$, $C = 1.41\times10^5$ m$^{-1/2}$, $q = 11.4$ nm^{-1}and $k = 9.00$ nm^{-1}. Show that the depth of the well V_0 is 8.0 eV, and find the ground state energy E, giving your answer in eV. Find also the probability that the electron is outside the classically accessible region. Does this particular well have any excited bound states? [7]

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SECTION E

Answer **one** *question only.*

E14 State Ampère's Theorem. Use it to show that the magnetic flux density $\boldsymbol{B}$ inside a very long solenoid with n turns per unit length, carrying current I, is approximately $\mu_0 nI$ everywhere inside the solenoid. [3]

Write down a vector expression for the force acting on an electron (charge e, mass m) moving at velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{B}$. Suppose that the velocity $\boldsymbol{v}$ is perpendicular to $\boldsymbol{B}$; show that the particle will move in a circular orbit with angular velocity eB/m and deduce the radius of the orbit. [3]

A tube with length $l = 8$ cm and radius $a = 0.5$ mm, is wound with wire carrying a current such that B is 40 mT everywhere inside it. Electrons with kinetic energy $E_0 = 1$ keV enter the tube at its axis, travelling at an angle $\theta = 3°$ to the axis. Explain why they travel in a helical path, and make a rough sketch of the path looking along the axis. [2]

Find the speed v of the electrons. Assuming that they do not strike the inside wall of the tube, find the time taken for them to reach the far end, and how many times they cross the axis again before getting there. Show that the largest value of θ for which the electrons do not strike the inside wall is

$$\theta_m = \sin^{-1}\left(\frac{eBa}{2mv}\right),$$

and calculate this value. [5]

The tube forms part of an energy filter which accepts only electrons passing through the tube in a time equal to that taken by the 1 keV electrons at $\theta = 0$. If electrons enter the tube on axis as before, but with a range of energies and directions, what are the lowest and highest energies amongst the electrons accepted by the filter? [2]

E15 Write ***short notes*** on two of the following:

a) dipoles; [7½]

b) orbits in a gravitational field; [7½]

c) electric fields and charges in the presence of conductors; [7½]

d) Maxwell's equations and electromagnetic waves. [7½]

END OF PAPER