AS5040 :: Prof. Dr. Joel George M :: Flight Mechanics :: IIT Madras
Final Assignment (End-Semester) Submission
AE20M002

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Abstract

This document is Final Assignment submission of Roll Number **AE20M002** , named **Appili Vamsi Krishna** via mooodle course page of AS5040.

Contents

1 Question

Submit a summary on 6 DoF equations written in wind axes and the use of it.

2 Wind Axis

i. The wind-fixed axes are anchored at the airplane CG, same as the body-fixed axis system, but the X^W axis is always aligned along the resultant velocity vector (relative wind). The alignment of the Y^W and Z^W axes forming a right-handed orthogonal axis system.

ii. An aircraft flying with inertial velocity V in a crosswind v, the relative wind V_∞ is the vector sum of V and v.

iii. The body X^B axis is aligned along the inertial velocity V, whereas the wind X^W axis is along the relative wind V_∞.

iv. When writing the equations of motion, it is the inertial velocity V that matters, whereas when modeling the aerodynamic forces acting on the airplane, the relative wind V_∞ is of interest.

v. In the absence of wind (still atmosphere), the inertial velocity and the relative wind are numerically identical.

vi. Another point is worth noting: since the X^W axis is always aligned along the relative wind

vii. The orientation and the angular rate of the wind-fixed axes therefore are not necessarily the same as those of the body-fixed axes even though they share a common point of origin.

viii. In simple words, the body-fixed axes reveal where the airplane is pointing, whereas the wind-fixed axes (in still atmosphere) show where the airplane is going-and these are not necessarily the same.

3 Notations

I. In Earth axis,

Z^E axis, angle ψ

Y^E axis, angle θ

X^B axis, angle φ

II. In Wind axis,

Z^W axis angle χ

Y^W axis, angle γ

X^W axis, angle μ

4 Rotations

4.1 General

1. Earth axis to Body axis

$$\begin{bmatrix} X^B \\ Y^B \\ Z^B \end{bmatrix} = R_\varphi^T R_\theta^T R_\psi^T \begin{bmatrix} X^E \\ Y^E \\ Z^E \end{bmatrix} \tag{1}$$

2. Earth axis to Wind axis

$$\begin{bmatrix} X^E \\ Y^E \\ Z^E \end{bmatrix} = \underbrace{\begin{bmatrix} c\chi & -s\chi & 0 \\ s\chi & c\chi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_\chi} \underbrace{\begin{bmatrix} c\gamma & 0 & s\gamma \\ 0 & 1 & 0 \\ -s\gamma & 0 & c\gamma \end{bmatrix}}_{R_\gamma} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\mu & -s\mu \\ 0 & s\mu & c\mu \end{bmatrix}}_{R_\mu} \begin{bmatrix} X^W \\ Y^W \\ Z^W \end{bmatrix} = R_\chi R_\gamma R_\mu \begin{bmatrix} X^W \\ Y^W \\ Z^W \end{bmatrix} \tag{2}$$

3. Wind axis to Earth axis

$$\begin{bmatrix} X^W \\ Y^W \\ Z^W \end{bmatrix} = R_\mu^T R_\gamma^T R_\chi^T \begin{bmatrix} X^E \\ Y^E \\ Z^E \end{bmatrix} \tag{3}$$

4. Body axis to Stability axis

$$\begin{bmatrix} X^S \\ Y^S \\ Z^S \end{bmatrix} = \underbrace{\begin{bmatrix} c\alpha & 0 & s\alpha \\ 0 & 1 & 0 \\ -s\alpha & 0 & c\alpha \end{bmatrix}}_{R_\alpha} \begin{bmatrix} X^B \\ Y^B \\ Z^B \end{bmatrix} \tag{4}$$

5. Body axis to wind axis

$$\begin{bmatrix} X^W \\ Y^W \\ Z^W \end{bmatrix} = \underbrace{\begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_{-\beta}} \begin{bmatrix} X^S \\ Y^S \\ Z^S \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} X^S \\ Y^S \\ Z^S \end{bmatrix} = \underbrace{\begin{bmatrix} c\alpha & 0 & s\alpha \\ 0 & 1 & 0 \\ -s\alpha & 0 & c\alpha \end{bmatrix}}_{R_\alpha} \begin{bmatrix} X^B \\ Y^B \\ Z^B \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} X^W \\ Y^W \\ Z^W \end{bmatrix} = R_{-\beta} R_\alpha \begin{bmatrix} X^B \\ Y^B \\ Z^B \end{bmatrix} \tag{7}$$

6. Wind axis to Body axis

$$\begin{bmatrix} X^B \\ Y^B \\ Z^B \end{bmatrix} = R_\alpha^T R_{-\beta}^T \begin{bmatrix} X^W \\ Y^W \\ Z^W \end{bmatrix} = R_{-\alpha} R_\beta \begin{bmatrix} X^W \\ Y^W \\ Z^W \end{bmatrix} \tag{8}$$

4.2 Navigational Equation

$$\begin{bmatrix} \dot{x}^E \\ \dot{y}^E \\ \dot{z}^E \end{bmatrix} = R_\psi R_\theta R_\varphi \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & -s\varphi \\ 0 & s\varphi & c\varphi \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}^E \\ \dot{y}^E \\ \dot{z}^E \end{bmatrix} = \begin{bmatrix} c\psi c\theta & c\psi s\theta s\varphi - s\psi c\varphi & c\psi s\theta c\varphi + s\psi s\varphi \\ s\psi c\theta & s\psi s\theta s\varphi + c\psi c\varphi & s\psi s\theta c\varphi - c\psi s\varphi \\ -s\theta & c\theta s\varphi & c\theta c\varphi \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} c\alpha & 0 & -s\alpha \\ 0 & 1 & 0 \\ s\alpha & 0 & c\alpha \end{bmatrix} \begin{bmatrix} c\beta & -s\beta & 0 \\ s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V\cos\beta\cos\alpha \\ V\sin\beta \\ V\cos\beta\sin\alpha \end{bmatrix} \tag{10}$$

substitute (10) in (9), we get

$$\begin{bmatrix} \dot{x}^E \\ \dot{y}^E \\ \dot{z}^E \end{bmatrix} = R_\psi R_\theta R_\varphi \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & -s\varphi \\ 0 & s\varphi & c\varphi \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}^E \\ \dot{y}^E \\ \dot{z}^E \end{bmatrix} = \begin{bmatrix} c\psi c\theta & c\psi s\theta s\varphi - s\psi c\varphi & c\psi s\theta c\varphi + s\psi s\varphi \\ s\psi c\theta & s\psi s\theta s\varphi + c\psi c\varphi & s\psi s\theta c\varphi - c\psi s\varphi \\ -s\theta & c\theta s\varphi & c\theta c\varphi \end{bmatrix} \begin{bmatrix} V\cos\beta\cos\alpha \\ V\sin\beta \\ V\cos\beta\sin\alpha \end{bmatrix} \tag{11}$$

In the absence of wind, the inertial velocity vector in wind-fixed axes is simply $\begin{bmatrix} V & 0 & 0 \end{bmatrix}^T$. Let us transform this to the Earth-fixed axes using Equation (2) to find the airplane ground velocity components $\dot{x}^E, \dot{y}^E, \dot{\mathrm{z}}^{\mathrm{E}}$

$$\begin{bmatrix} \dot{x}^E \\ \dot{y}^E \\ \dot{z}^E \end{bmatrix} = \begin{bmatrix} c\chi & -s\chi & 0 \\ s\chi & c\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\gamma & 0 & s\gamma \\ 0 & 1 & 0 \\ -s\gamma & 0 & c\gamma \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\mu & -s\mu \\ 0 & s\mu & c\mu \end{bmatrix} \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V\cos\gamma\cos\chi \\ V\cos\gamma\sin\chi \\ -V\sin\gamma \end{bmatrix} \tag{12}$$

4.3 Velocity in Body Axis

$$\begin{bmatrix} X^B \\ Y^B \\ Z^B \end{bmatrix} = R_{-\alpha} R_\beta \begin{bmatrix} X^W \\ Y^W \\ Z^W \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} c\alpha & 0 & -s\alpha \\ 0 & 1 & 0 \\ s\alpha & 0 & c\alpha \end{bmatrix} \begin{bmatrix} c\beta & -s\beta & 0 \\ s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V\cos\beta\cos\alpha \\ V\sin\beta \\ V\cos\beta\sin\alpha \end{bmatrix} \tag{14}$$

$$\alpha = \tan^{-1}\left(\frac{w}{u}\right) \tag{15}$$

$$\beta = \sin^{-1}\left(\frac{v}{V}\right) \tag{16}$$

$$V = \sqrt{(u^2 + v^2 + w^2)} \tag{17}$$

4.4 Body axis and Wind axis :: Euler angles

from reference [1]

$$
\begin{aligned}
&\sin\gamma = \cos\alpha\cos\beta\sin\theta - \sin\beta\sin\varphi\cos\theta - \sin\alpha\cos\beta\cos\varphi\cos\theta \\
&\sin\mu\cos\gamma = \sin\theta\cos\alpha\sin\beta + \sin\varphi\cos\theta\cos\beta - \sin\alpha\sin\beta\cos\varphi\cos\theta \\
&\cos\mu\cos\gamma = \sin\theta\sin\alpha + \cos\alpha\cos\varphi\cos\theta
\end{aligned} \tag{18}
$$

5 Wind-Axis Angular Velocity Components (p_w, q_w, r_w) and Euler Angle Rates $(\dot{\mu}, \dot{\gamma}, \dot{\chi})$

$$
\begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix}_W = \begin{bmatrix} 0 \\ 0 \\ \dot{\chi} \end{bmatrix}_E + \begin{bmatrix} 0 \\ \dot{\gamma} \\ 0 \end{bmatrix}_1 + \begin{bmatrix} \dot{\mu} \\ 0 \\ 0 \end{bmatrix}_W \tag{19}
$$

Thus,

$$
\begin{bmatrix} 0 \\ 0 \\ \dot{\chi} \end{bmatrix}_E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & \sin\mu \\ 0 & -\sin\mu & \cos\mu \end{bmatrix} \begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\chi} \end{bmatrix}_W \tag{20}
$$

and,

$$
\begin{bmatrix} 0 \\ \dot{\gamma} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & \sin\mu \\ 0 & -\sin\mu & \cos\mu \end{bmatrix} \begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\gamma} \\ 0 \end{bmatrix}_W \tag{21}
$$

Hence,

$$
\begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\gamma \\ 0 & \cos\mu & \sin\mu\cos\gamma \\ 0 & -\sin\mu & \cos\gamma\cos\mu \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{\gamma} \\ \dot{\chi} \end{bmatrix} \tag{22}
$$

Also,

$$
\begin{bmatrix} \dot{\mu} \\ \dot{\gamma} \\ \dot{\chi} \end{bmatrix} = \begin{bmatrix} 1 & \tan\gamma\sin\mu & \tan\gamma\cos\mu \\ 0 & \cos\mu & -\sin\mu \\ 0 & \sec\gamma\sin\mu & \sec\gamma\cos\mu \end{bmatrix} \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} \tag{23}
$$

Similarly, for body axis system, we can observe,

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\varphi & \sin\varphi\cos\theta \\ 0 & -\sin\varphi & \cos\varphi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{24}$$

Also,

$$\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \tan\theta\sin\varphi & \tan\theta\cos\varphi \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sec\theta\sin\varphi & \sec\theta\cos\varphi \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{25}$$

5.1 Difference between (p, q, r) and (p_w, q_w, r_w)

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix}_B - \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix}_W = (-\dot{\beta})\hat{k}_W + \dot{\alpha}\hat{j}_B \tag{26}$$

Thus,

$$\begin{bmatrix} p_b^b - p_w^b \\ q_b^b - q_w^b \\ r_b^b - r_w^b \end{bmatrix} = \begin{bmatrix} \dot{\beta}\sin\alpha \\ \dot{\alpha} \\ -\dot{\beta}\cos\alpha \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} p_w^b \\ q_w^b \\ r_w^b \end{bmatrix} = \begin{bmatrix} p_b^b - \dot{\beta}\sin\alpha \\ q_b^b - \dot{\alpha} \\ r_b^b + \dot{\beta}\cos\alpha \end{bmatrix} = \begin{bmatrix} p - \dot{\beta}\sin\alpha \\ q - \dot{\alpha} \\ r + \dot{\beta}\cos\alpha \end{bmatrix} \tag{28}$$

5.2 p_w, q_w, r_w

$$\begin{bmatrix} p_w^w \\ q_w^w \\ r_w^w \end{bmatrix} = R_{-\beta}R_\alpha \begin{bmatrix} p_w^b \\ q_w^b \\ r_w^b \end{bmatrix} = \underbrace{\begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_{-\beta}} \underbrace{\begin{bmatrix} c\alpha & 0 & s\alpha \\ 0 & 1 & 0 \\ -s\alpha & 0 & c\alpha \end{bmatrix}}_{R_\alpha} \begin{bmatrix} p - \dot{\beta}\sin\alpha \\ q - \dot{\alpha} \\ r + \dot{\beta}\cos\alpha \end{bmatrix} \tag{29}$$

Thus, we can have,

$$\begin{aligned} p_w &= (p\cos\alpha + r\sin\alpha)\cos\beta + (q - \dot{\alpha})\sin\beta \\ q_w &= -(p\cos\alpha + r\sin\alpha)\sin\beta + (q - \dot{\alpha})\cos\beta \\ r_w &= (-p\sin\alpha + r\cos\alpha) + \dot{\beta} \end{aligned} \tag{30}$$

6 Transitional Equations of Motion :: Wind Axis

6.1 Basics

$$\left.\frac{d\underline{A}}{dt}\right|_I = \left.\frac{d\underline{A}}{dt}\right|_W + (\underline{\omega}_W \times \underline{A}) \tag{31}$$

6.2 Force Equation

$$\underline{F} = m\left.\frac{d\underline{V}_C}{dt}\right|_W + m\,(\underline{\omega}_W \times \underline{V}_C) \tag{32}$$

6.3 $\underline{F}_W$

$\underline{\omega}_W = [p_w q_w r_w]^T$

The cross product of vector $\underline{\omega}_W$ can be expressed in matrix form as

$$\underline{\omega}_W \times = \begin{bmatrix} 0 & -r_w & q_w \\ r_w & 0 & -p_w \\ -q_w & p_w & 0 \end{bmatrix} \tag{33}$$

$$\underline{F}_W = m\begin{bmatrix} \dot{V} \\ 0 \\ 0 \end{bmatrix} + m\begin{bmatrix} 0 & -r_w & q_w \\ r_w & 0 & -p_w \\ -q_w & p_w & 0 \end{bmatrix}\begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} = m\begin{bmatrix} \dot{V} \\ r_w V \\ -q_w V \end{bmatrix} \tag{34}$$

6.4 $\underline{F}_W$: applied

$$\underline{F}_W = \underline{F}_W^G + \underline{F}_W^A + \underline{F}_W^P \tag{35}$$

6.4.1 Gravity

$$\underline{F}_W^G = \begin{bmatrix} -mg\sin\gamma \\ mg\sin\mu\cos\gamma \\ mg\cos\mu\cos\gamma \end{bmatrix} \tag{36}$$

6.4.2 Aerodynamic

$$\underline{F}_W^A = \underbrace{\begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_{-\beta}} \begin{bmatrix} -D \\ Y \\ -L \end{bmatrix}_S = \begin{bmatrix} -D\cos\beta + Y\sin\beta \\ D\sin\beta + Y\cos\beta \\ -L \end{bmatrix} \tag{37}$$

6.4.3 Propulsion

$$\underline{F}_W^T = \underbrace{\begin{bmatrix} c\beta & s\beta & 0 \\ -s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_{-\beta}} \underbrace{\begin{bmatrix} c\alpha & 0 & s\alpha \\ 0 & 1 & 0 \\ -s\alpha & 0 & c\alpha \end{bmatrix}}_{R_\alpha} \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix}_B = T \begin{bmatrix} \cos\beta\cos\alpha \\ -\sin\beta\cos\alpha \\ -\sin\alpha \end{bmatrix} \tag{38}$$

6.5 Assembled Wind axis force equation

$$\begin{aligned} &\begin{bmatrix} -mg\sin\gamma \\ mg\sin\mu\cos\gamma \\ mg\cos\mu\cos\gamma \end{bmatrix} + \begin{bmatrix} -D\cos\beta + Y\sin\beta \\ D\sin\beta + Y\cos\beta \\ -L \end{bmatrix} + \begin{bmatrix} T\cos\beta\cos\alpha \\ -T\sin\beta\cos\alpha \\ -T\sin\alpha \end{bmatrix} \\ &= m \begin{bmatrix} \dot{V} \\ ((-p\sin\alpha + r\cos\alpha) + \dot{\beta})V \\ ((p\cos\alpha + r\sin\alpha)\sin\beta - (q-\dot{\alpha})\cos\beta)V \end{bmatrix} \end{aligned} \tag{39}$$

7 Rotational Equations of Motion :: Wind Axis

$$\underline{M} = \left.\frac{d\underline{h}}{dt}\right|_I = \left.\frac{dh}{dt}\right|_W + \underline{\omega}_W \times \underline{h} \tag{40}$$

$$\begin{aligned} \underline{h} &= \int dm\{(x\hat{i} + y\hat{j} + z\hat{k}) \times [(p\hat{i} + q\hat{j} + \hat{k}) \times (x\hat{i} + y\hat{j} + r\hat{k})] \\ &= \int dm\{(x\hat{i} + y\hat{j} + z\hat{k}) \times [(qz - yr)\hat{i} + (rx - pz)\hat{j} + (py - qx)\hat{k}] \\ &= \left[p\int \left(y^2 + z^2\right) dm - q\int xy\delta m - r\int xz\delta m\right]\hat{i} \\ &\quad + \left[-p\int yxdm + q\int \left(x^2 + z^2\right) dm - r\int yzdm\right]\hat{j} \\ &\quad + \left[-p\int zxdm - q\int zydm + r\int \left(x^2 + y^2\right) dm\right]\hat{j} \\ &= [pI_{xx} - qI_{xy} - rI_{xz}]\,\hat{i} + [-pI_{yx} + qI_{yy} - rI_{yz}]\,\hat{j} \\ &\quad + [-pI_{zx} - qI_{zy} + rI_{zz}]\,\hat{k} \end{aligned} \tag{41}$$

$$\begin{aligned} &I_{xx} = \int \left(y^2 + z^2\right) dm, \quad I_{yy} = \int \left(x^2 + z^2\right) dm, \quad I_{zz} = \int \left(x^2 + y^2\right) dm \\ &I_{xy} = I_{yx} = \int xydm, \quad I_{xz} = I_{zx} = \int xzdm, \quad I_{yz} = I_{zy} = \int yzdm \end{aligned} \tag{42}$$

$$\underline{h} = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} = \underline{I} \cdot \underline{\omega}_B \tag{43}$$

where, $\underline{\mathrm{I}}$ is Inertial tensor

$$\begin{aligned} \underline{M} = \left.\frac{d\underline{h}}{dt}\right|_B + \underline{\omega}_W \times \underline{h} = \left.\frac{d}{dt}\right|_B \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} + \underline{\omega}_W \times \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} \\ = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \left.\frac{d}{dt}\right|_W \begin{bmatrix} p_w \\ q_w \\ r_W \end{bmatrix} + \underline{\omega}_W \times \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} \end{aligned} \tag{44}$$

$\underline{\omega}_W = [p_w q_w r_w]^T$

The cross product of vector $\underline{\omega}_W$ can be expressed in matrix form as

$$\underline{\omega}_W \times = \begin{bmatrix} 0 & -r_w & q_w \\ r_w & 0 & -p_w \\ -q_w & p_w & 0 \end{bmatrix} \tag{45}$$

$$
\begin{aligned}
&\underline{\mathrm{M}} \\
&= \frac{dh}{dt}\Big|_B + \underline{\omega}_W \times \underline{h} \\
&= \frac{d}{dt}\Big|_B \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} + \underline{\omega}_W \times \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} \\
&= \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \frac{d}{dt}\Big|_W \begin{bmatrix} p_w \\ q_w \\ r_W \end{bmatrix} + \underline{\omega}_W \times \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} \\
&= \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{p_w} \\ \dot{q_w} \\ \dot{r_w} \end{bmatrix} + \begin{bmatrix} 0 & -r_w & q_w \\ r_w & 0 & -p_w \\ -q_w & p_w & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} \\
&= \begin{bmatrix} I_{xx} \cdot \dot{p_w} - I_{xy} \cdot \dot{q_w} - I_{xz} \cdot \dot{r_w} + p_w \cdot (I_{yx} \cdot r_w - I_{zx} \cdot q_w) + q_w \cdot (-I_{yy} \cdot r_w - I_{zy} \cdot q_w) + r_w \cdot (I_{yz} \cdot r_w + I_{zz} \cdot q_w) \\ p_w \cdot (I_{xx} \cdot r_w + I_{zx} \cdot p_w) + q_w \cdot (I_{zy} \cdot p_w - I_{xy} \cdot r_w) + r_w \cdot (-I_{xz} \cdot r_w - I_{zz} \cdot p_w) - I_{yx} \cdot \dot{p_w} + I_{yy} \cdot \dot{q_w} - I_{yz} \cdot \dot{r_w} \\ p_w \cdot (-I_{xx} \cdot q_w - I_{yx} \cdot p_w) + q_w \cdot (I_{xy} \cdot q_w + I_{yy} \cdot p_w) + r_w \cdot (I_{xz} \cdot q_w - I_{yz} \cdot p_w) - I_{zx} \cdot \dot{p_w} - I_{zy} \cdot \dot{q_w} + I_{zz} \cdot \dot{r_w} \end{bmatrix}
\end{aligned}
\tag{46}
$$

$$
\underline{M} = \begin{bmatrix} \mathcal{L} \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx} \cdot \dot{p_w} - I_{xy} \cdot \dot{q_w} - I_{xz} \cdot \dot{r_w} + p_w \cdot (I_{yx} \cdot r_w - I_{zx} \cdot q_w) + q_w \cdot (-I_{yy} \cdot r_w - I_{zy} \cdot q_w) + r_w \cdot (I_{yz} \cdot r_w + I_{zz} \cdot q_w) \\ p_w \cdot (I_{xx} \cdot r_w + I_{zx} \cdot p_w) + q_w \cdot (I_{zy} \cdot p_w - I_{xy} \cdot r_w) + r_w \cdot (-I_{xz} \cdot r_w - I_{zz} \cdot p_w) - I_{yx} \cdot \dot{p_w} + I_{yy} \cdot \dot{q_w} - I_{yz} \cdot \dot{r_w} \\ p_w \cdot (-I_{xx} \cdot q_w - I_{yx} \cdot p_w) + q_w \cdot (I_{xy} \cdot q_w + I_{yy} \cdot p_w) + r_w \cdot (I_{xz} \cdot q_w - I_{yz} \cdot p_w) - I_{zx} \cdot \dot{p_w} - I_{zy} \cdot \dot{q_w} + I_{zz} \cdot \dot{r_w} \end{bmatrix} \tag{47}
$$

Equation (47) is the equation in wind axis reference frame., also the most general rotational equations of motion Here, there is a trick, ie., Inertial matrix changes with time., we need to keep updating all values in Inertial matrix, when we are in dynamic motion. In steady state, (**??**) works well. In dynamic state, (47) need to be updated with incorporation of certain sensors or algorithms in systems, during the flight.

Thus, for calculation of moments, we use, body fixed frame to reduce complexity in calculations, the final form of the airplane rotational dynamics Equation with the moment components in body fixed frame is given as (in reference [1])

$$\begin{bmatrix} \mathcal{L} \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{p} - I_{xz}\dot{r} \\ I_{yy}\dot{q} \\ -I_{zx}\dot{p} + I_{zz}\dot{r} \end{bmatrix} + \begin{bmatrix} -pqI_{zx} - qr\left(I_{yy} - I_{zz}\right) \\ \left(p^2 - r^2\right) I_{zx} - pr\left(I_{zz} - I_{xx}\right) \\ qrI_{xz} - pq\left(I_{xx} - I_{yy}\right) \end{bmatrix} \tag{48}$$

8 Summary of equations of motion :: 6 DoF

8.1 Position (wind-axis)

Equation, (12)

$$\begin{bmatrix} \dot{x}^E \\ \dot{y}^E \\ \dot{z}^E \end{bmatrix} = \begin{bmatrix} c\chi & -s\chi & 0 \\ s\chi & c\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\gamma & 0 & s\gamma \\ 0 & 1 & 0 \\ -s\gamma & 0 & c\gamma \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\mu & -s\mu \\ 0 & s\mu & c\mu \end{bmatrix} \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V\cos\gamma\cos\chi \\ V\cos\gamma\sin\chi \\ -V\sin\gamma \end{bmatrix} \tag{49}$$

8.2 Orientation (wind-axis)

Equation, (22)

$$\begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\gamma \\ 0 & \cos\mu & \sin\mu\cos\gamma \\ 0 & -\sin\mu & \cos\gamma\cos\mu \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{\gamma} \\ \dot{\chi} \end{bmatrix} \tag{50}$$

Equation, (23)

$$\begin{bmatrix} \dot{\mu} \\ \dot{\gamma} \\ \dot{\chi} \end{bmatrix} = \begin{bmatrix} 1 & \tan\gamma\sin\mu & \tan\gamma\cos\mu \\ 0 & \cos\mu & -\sin\mu \\ 0 & \sec\gamma\sin\mu & \sec\gamma\cos\mu \end{bmatrix} \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} \tag{51}$$

8.3 Velocity (wind-axis)

Equation, (39)

$$\begin{bmatrix} -mg\sin\gamma \\ mg\sin\mu\cos\gamma \\ mg\cos\mu\cos\gamma \end{bmatrix} + \begin{bmatrix} -D\cos\beta + Y\sin\beta \\ D\sin\beta + Y\cos\beta \\ -L \end{bmatrix} + \begin{bmatrix} T\cos\beta\cos\alpha \\ -T\sin\beta\cos\alpha \\ -T\sin\alpha \end{bmatrix}$$
$$= m\begin{bmatrix} \dot{V} \\ ((-p\sin\alpha + r\cos\alpha) + \dot{\beta})V \\ ((p\cos\alpha + r\sin\alpha)\sin\beta - (q-\dot{\alpha})\cos\beta)V \end{bmatrix} \tag{52}$$

8.4 Angular Velocity (wind-axis)

Equation, (47)

$$\underline{M} = \begin{bmatrix} \mathcal{L} \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\cdot \dot{p_w} - I_{xy}\cdot \dot{q_w} - I_{xz}\cdot \dot{r_w} + p_w\cdot(I_{yx}\cdot r_w - I_{zx}\cdot q_w) + q_w\cdot(-I_{yy}\cdot r_w - I_{zy}\cdot q_w) + r_w\cdot(I_{yz}\cdot r_w + I_{zz}\cdot q_w) \\ p_w\cdot(I_{xx}\cdot r_w + I_{zx}\cdot p_w) + q_w\cdot(I_{zy}\cdot p_w - I_{xy}\cdot r_w) + r_w\cdot(-I_{xz}\cdot r_w - I_{zz}\cdot p_w) - I_{yx}\cdot \dot{p_w} + I_{yy}\cdot \dot{q_w} - I_{yz}\cdot \dot{r_w} \\ p_w\cdot(-I_{xx}\cdot q_w - I_{yx}\cdot p_w) + q_w\cdot(I_{xy}\cdot q_w + I_{yy}\cdot p_w) + r_w\cdot(I_{xz}\cdot q_w - I_{yz}\cdot p_w) - I_{zx}\cdot \dot{p_w} - I_{zy}\cdot \dot{q_w} + I_{zz}\cdot \dot{r_w} \end{bmatrix} \tag{53}$$

8.5 Angular Velocity (body-axis)

Equation, (48)

$$\begin{bmatrix} \mathcal{L} \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{p} - I_{xz}\dot{r} \\ I_{yy}\dot{q} \\ -I_{zx}\dot{p} + I_{zz}\dot{r} \end{bmatrix} + \begin{bmatrix} -pqI_{zx} - qr\left(I_{yy} - I_{zz}\right) \\ \left(p^2 - r^2\right)I_{zx} - pr\left(I_{zz} - I_{xx}\right) \\ qrI_{xz} - pq\left(I_{xx} - I_{yy}\right) \end{bmatrix} \tag{54}$$

9 Conclusion

* Wind axis system has drag, lift, always orientated parallel and perpendicular to its X axis respectively. Thus, It has some advantages.
* Body axis has some advantages like, easy calculation of moments, since symmetry in ZX plane
* Stability axis has some advantages, in thrust calculations, control-surface reaction force calculations.
* Each system has its advantages, here, we calculate all 6 DoF equations in wind axis, so that we can do steady analysis., and incorporating body-fixed axis moment equations instead of wind-axis moment equations, we can calculate trajectory paths, and respective parameters and can do some performance analysis for optimisation of some critical parameters
* We derived all 6 DoF equations in Wind-Axis reference frame, and understood that its concepts.

10 References

References

[1] Nandan K Sinha and N Ananthkrishnan. *Advanced flight dynamics with elements of flight control.* CRC Press, 2017.