1B Paper 6: Communications

Handout 5: Digital Passband Modulation

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Lent Term 2024

Pulse Amplitude Modulation (recap)

Recall that the PAM signal carrying the information symbols $X_1, X_2, \ldots$ is

$$x_b(t) = \sum_k X_k p(t - kT)$$

- $x_b(t)$ is a *baseband* signal

  (Recall: its bandwidth is the same as that of the pulse $p(t)$)
- If the channel is a baseband channel, e.g., ethernet cable, we can directly transmit $x_b(t)$

But most channels are *passband* — we are only allowed to transmit our signal over a fixed frequency band centred around a carrier frequency f_c.

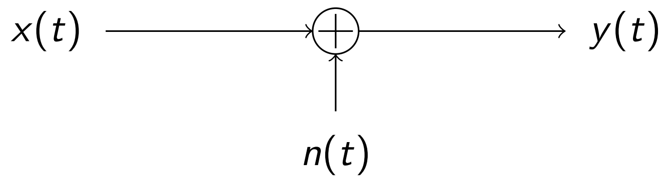
E.g., a wireless channel may have $f_c = 2$ GHz and channel bandwidth $= 10$ MHz

How do you "up-convert" the PAM signal to passband ?

Baseband to Passband

The natural thing to do is to modulate the amplitude of a high-frequency carrier with $x_b(t)$:

$$x(t) = x_b(t)\cos(2\pi f_c t) = \left[\sum_k X_k p(t-kT)\right]\cos(2\pi f_c t)$$



- For lack of a standard name, we'll call this passband modulation scheme up-converted PAM.
- Note the similarity with analogue modulation (DSB-SC): here the information signal is a "digital" waveform, which is determined by the constellation symbols and a pulse.

Up-converted PAM with a rectangular pulse

$$p(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } t \in (0, T] \\ 0 & \text{otherwise} \end{cases}$$

is also called "Amplitude Shift Keying" (ASK).

In this case, the transmitted passband waveform is

$$\begin{aligned} x(t) &= \sum_k X_k p(t-kT)\cos(2\pi f_c t) \\ &= \frac{1}{\sqrt{T}} X_k \cos(2\pi f_c t) \quad \text{for } t \in [kT, (k+1)T) \end{aligned}$$

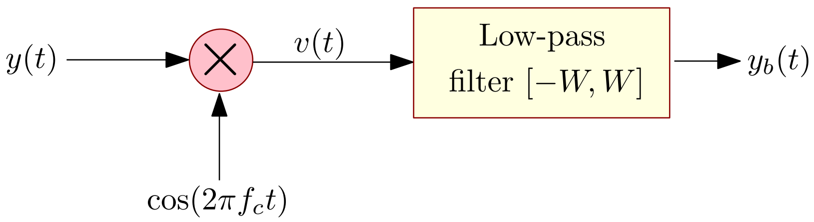
where X_k is chosen from a PAM constellation, say $\{-A, A\}$.

Note that a rectangular pulse (and consequently, ASK) is not bandwidth efficient.

(You may encounter the terminology ASK in past Tripos questions. You will not be examined on it, but recognise that it is just up-converted PAM with a rectangular pulse shape.)

Demodulation of up-converted PAM at the Rx

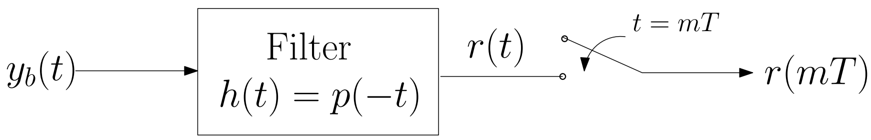
First, **down-convert** via product modulator + low-pass filter



$$y_b(t) = x_b(t) + n_b(t) = \sum_k X_k p(t - kT) + n_b(t)$$

where $n_b(t)$ is baseband noise.

Next demodulate *baseband* waveform $y_b(t)$. We already know how to do this: *pass $y_b(t)$ through matched filter, and then sample at times $\{mT\}_{m\in\mathbb{Z}}$*



Detection

- The sampled output of the matched filter at time mT is

$$Y_m = X_m + N_m$$

  where $N_m = \int n_b(\tau)p(\tau - mT)\, d\tau$ (see prev. handout for details)

- N_m is Gaussian with zero mean and variance σ^2
  (σ^2 can be empirically estimated)

- Maximum-likelihood detection: Choose $\hat{X}_m$ to be the constellation symbol closest to Y_m

Spectrum of up-converted PAM

The transmitted waveform $x(t) = x_b(t)\cos(2\pi f_c t)$ has spectrum

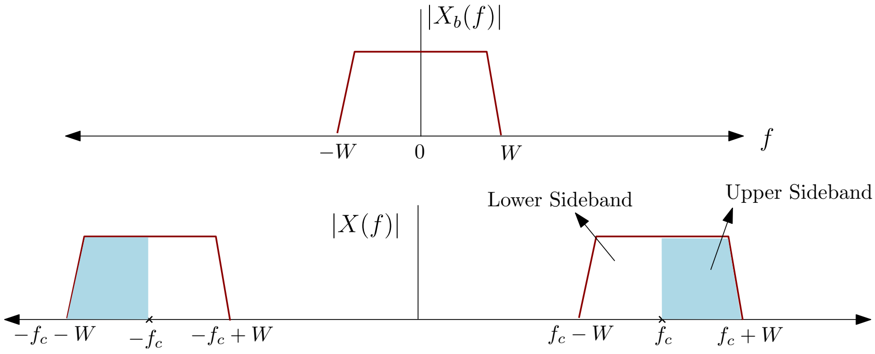
$$X(f) = \frac{1}{2}\left[X_b(f - f_c) + X_b(f + f_c)\right]$$

Note that

$$x_b(t) = \sum_k X_k p(t - kT)$$

is a **real** signal since both the pulse $p(t)$ and the symbols $\{X_k\}$ are real-valued $\Rightarrow$ $X_b(-f) = X_b^*(f)$

- The spectrum $X(f)$ for $f \geq 0$ determines $X(f)$ for $f < 0$



- Sending both sidebands is redundant, since all the information is contained in one

A bandwidth-efficient alternative to up-converted PAM

- One way to save bandwidth is to send only one sideband, just like SSB-SC (see Amplitude Modulation handout)
- But since we are transmitting digital information, there is a better way: *make the information symbols complex-valued*

The baseband waveform is again

$$x_b(t) = \sum_k X_k p(t - kT)$$

but now the constellation from which the symbols X_k are drawn is *complex-valued* (i.e., X_k are now two-dimensional)

$x_b(t)$ is now complex, but the passband signal we transmit has to be *real*. It is generated as

$$\begin{aligned} x(t) &= \mathrm{Re}\left[x_b(t)\, e^{j2\pi f_c t}\right] \\ &= \mathrm{Re}(x_b(t))\cos(2\pi f_c t) - \mathrm{Im}(x_b(t))\sin(2\pi f_c t) \end{aligned}$$

This is called **Quadrature Amplitude Modulation** (QAM)

Quadrature Amplitude Modulation

The upconverted QAM waveform that we transmit is

$$x(t) = \sum_k p(t - kT)\,[\mathbf{Re(X_k)}\cos(2\pi f_c t) - \mathbf{Im(X_k)}\sin(2\pi f_c t)]$$
$$= \sum_k p(t - kT)\,|\boldsymbol{X_k}|\cos(2\pi f_c t + \phi_{\mathbf{k}})$$

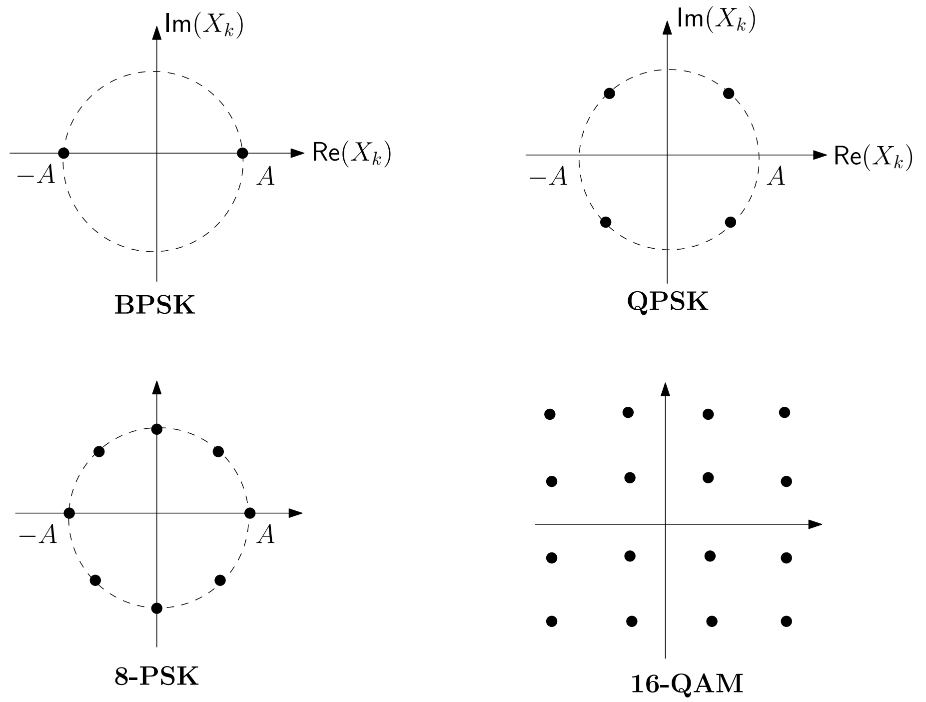
where $|X_k|$ and ϕ_k denote the magnitude and phase of the complex symbol X_k

Thus, one can understand QAM in two ways:

1. QAM has two carriers, the cosine carries $\text{Re}(X_k)$ and the sin carries the $\text{Im}(X_k)$.
2. In QAM, the information symbol modulates both the amplitude *and* phase of the carrier; in up-converted PAM the information symbol is real and only modulates the amplitude

- Pulse shape $p(t)$ is the same as that for PAM
- The main difference between QAM and PAM is the constellation. In PAM, $\text{Im}(X_k) = 0$.

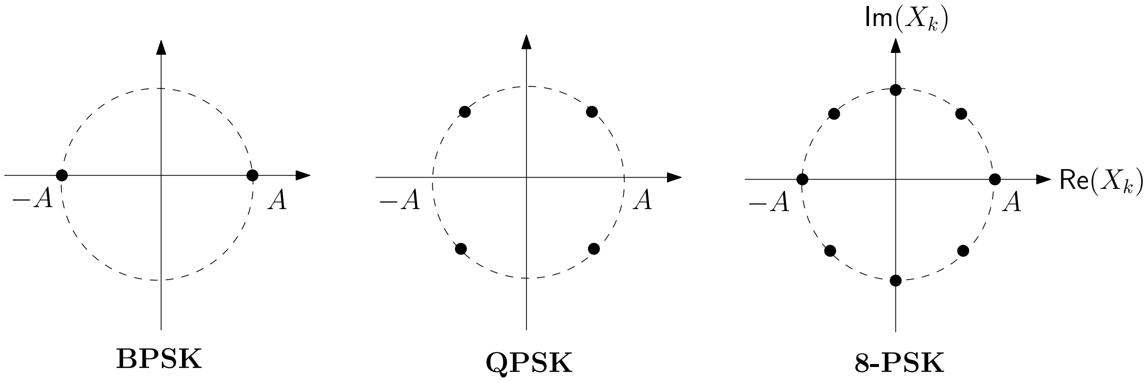
Some typical QAM Constellations



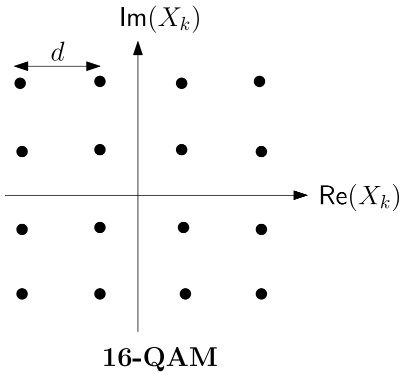
In "Phase Shift Keying" (PSK), the magnitude of X_k is constant, and the information is in the phase of the symbol.

In a constellation with M symbols, each symbol corresponds to $\log_2 M$ bits

Average Energy per Symbol



For all the PSK constellations, average *symbol* energy $E_s = A^2$



Average energy per symbol for 16-QAM

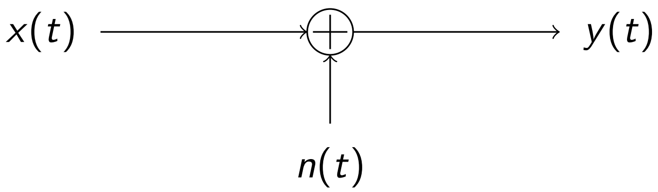
$$E_s = \frac{40d^2}{16} = 2.5d^2$$

Average energy per *bit* $E_b = E_s / \log_2 M$

The transmitted waveform is

$$x(t) = \sum_k p(t - kT) \left[X_k^r \cos(2\pi f_c t) - X_k^i \sin(2\pi f_c t) \right]$$

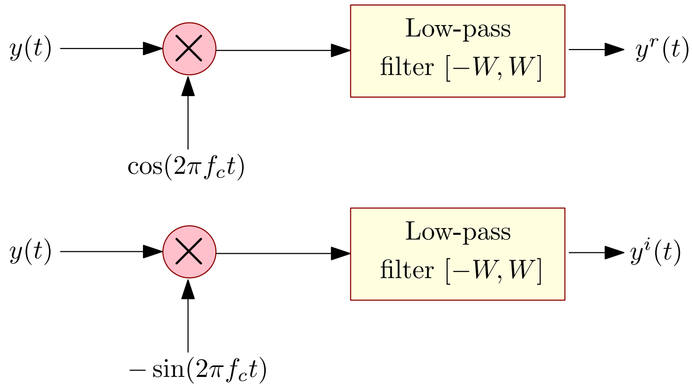
where $X_k^r := \mathrm{Re}(X_k)$ and $X_k^i := \mathrm{Im}(X_k)$



- The energy per symbol is important because the power of $x(t) \propto \frac{E_s}{T}$
- At the Rx, we have to down-convert $y(t)$ to a baseband waveform via product modulators and low-pass filter
- For QAM, we need *two* product modulators, one for the cosine and the other for sine

At the receiver, we have

$$y(t) = \sum_k p(t - kT)\ \left[X_k^r \cos(2\pi f_c t) - X_k^i \sin(2\pi f_c t)\right] + n(t)$$



After down-converting, we get

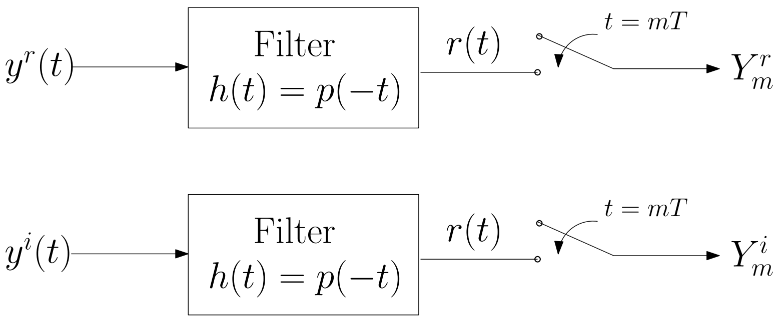
$$y^r(t) = \sum_k X_k^r p(t - kT) + n^r(t)$$

$$y^i(t) = \sum_k X_k^i p(t - kT) + n^i(t)$$

where $n^r(t)$ and $n^i(t)$ are filtered (baseband) versions of $n(t)$.

Next perform matched-filter demodulation of $y^r(t)$ and $y^i(t)$

Demodulation



- The sampled outputs of the matched filters for $m = 0, 1, \ldots$ are:

$$Y_m^r = X_m^r + N_m^r$$

$$Y_m^i = X_m^i + N_m^i$$

- It can be shown that N_m^r and N_m^i are each independent Gaussians distributed as $\mathcal{N}(0, \sigma^2)$ for each m
- For each m, we now have to *detect* the complex-valued constellation symbol $X_m = (X_m^r, X_m^i)$ from $Y_m = (Y_m^r, Y_m^i)$

Detection

The discrete-time channel is

$$Y = X + N$$

where X, Y, N are now two-dimensional vectors, i.e., complex numbers with $X = (X^r, X^i)$ and $Y = (Y^r, Y^i)$

The optimal detection rule (assuming all constellation symbols are equally likely) is the maximum-likelihood detection rule

$$\hat{X} = \arg\max_{x \in \mathcal{C}} f(Y|x)$$

"*Choose the symbol from which y is most likely to have occurred*"

The conditional distribution of $Y = (Y^r, Y^i)$ given $x = (x^r, x^i)$ is

$$\begin{aligned} f(Y|x) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(Y^r - x^r)^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(Y^i - x^i)^2/2\sigma^2} \\ &= \frac{1}{2\pi\sigma^2} e^{-[(Y^r - x^r)^2 + (Y^i - x^i)^2]/2\sigma^2} \end{aligned}$$

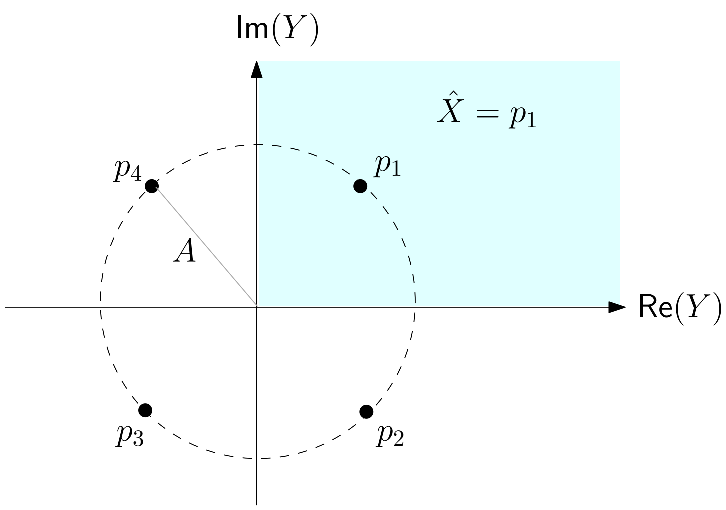
The optimal detector is therefore

$$\hat{X} = \arg\min_{x \in \mathcal{C}} \; (Y^r - x^r)^2 + (Y^i - x^i)^2 = \arg\min_{x \in \mathcal{C}} |Y - x|^2$$

Choose the constellation symbol x closest to observed output Y

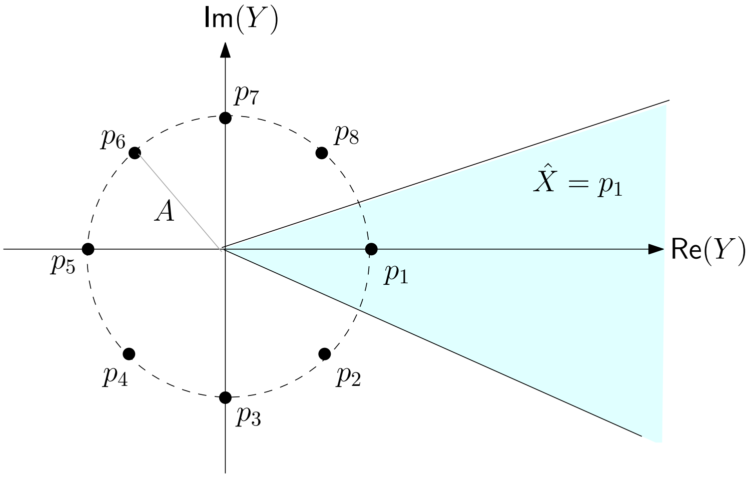
(Same detection principle as PAM, but the symbols are complex in QAM)

Example 1: The decision regions for QPSK

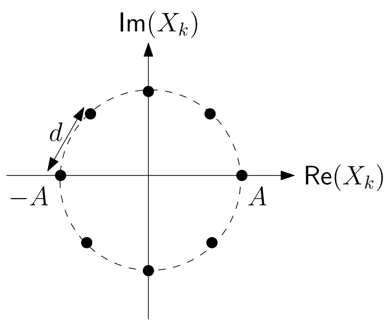


$$\hat{X} = \arg\min_{x \in \mathcal{C}} |Y - x|^2$$

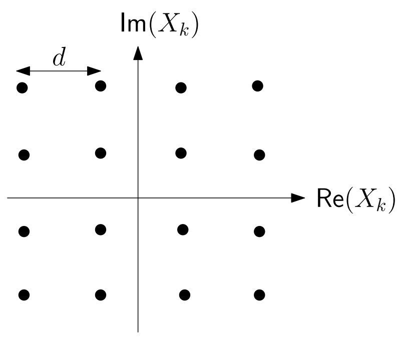
Example 2: The decision regions for 8-PSK



- Can similarly sketch the decision regions for other constellations
- We can also calculate the probability of detection error $P_e = P(\hat{X} \neq X)$ for various constellations



8-PSK



16-QAM

We won't do an exact calculation of QAM probability of error P_e, but can be shown that:

- P_e is a $\mathcal{Q}-$function depending on $\frac{d}{\sigma}$, where d is the separation between adjacent constellation points and σ^2 is the variance of the Gaussian noise
- P_e decays exponentially with d^2/σ^2

Suppose we increase the number of constellation points, e.g., 16 QAM $\rightarrow$ 64 QAM $\rightarrow$ 256 QAM, while keeping E_s constant (E_s = average energy per symbol) :

- Transmission rate increases: 4 bits/symbol $\rightarrow$ 6 bits/sym. $\rightarrow$ 8 bits/sym. (good!)
- To keep E_s constant, d has to decrease $\Rightarrow$ P_e increases (bad!)

Note that both up-converted PAM and QAM are essentially amplitude modulation for *digital* information:

- The (baseband) information signal $x_b(t)$ is generated from bits via a constellation as

$$x_b(t) = \sum_k X_k p(t - kT).$$

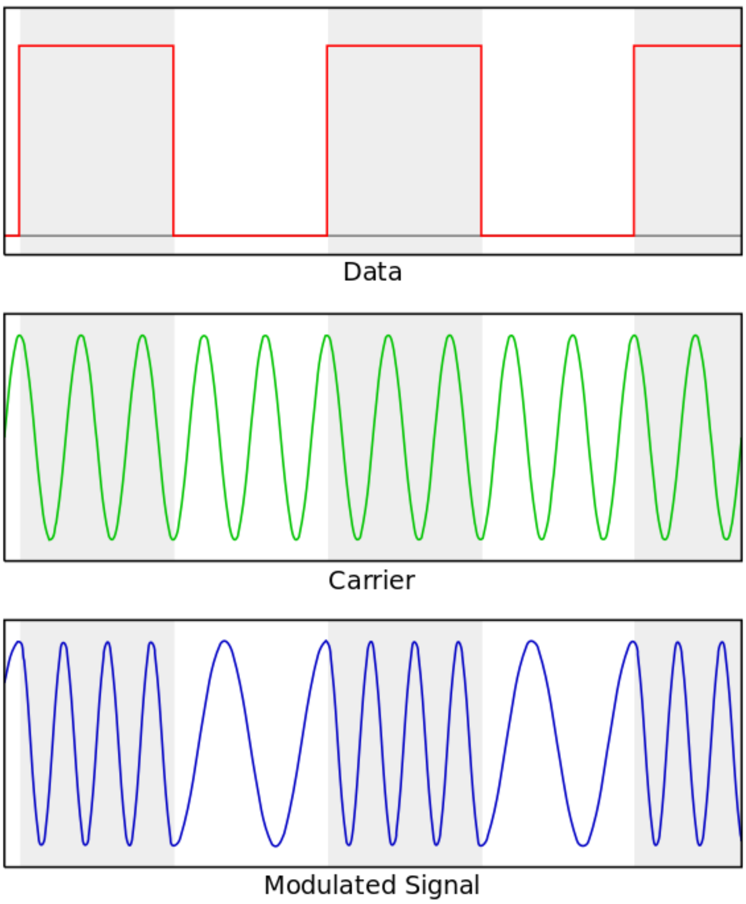
- The real and imaginary parts of $x_b(t)$ modulate the amplitude of the carriers $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$, respectively.

One can also transmit digital information by modulating the phase/frequency of a carrier.

- *Frequency shift keying* (FSK) is one such method
- Binary FSK: In each symbol time $[kT, (k+1)T)$, transmit one bit X_k via

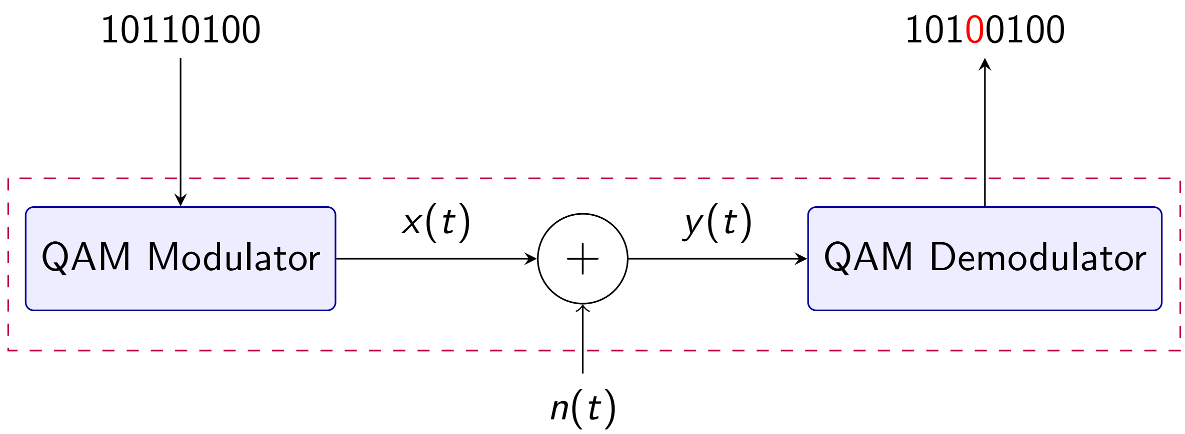
$$x(t) = \begin{cases} \cos(2\pi(f_c - \Delta_f)t) & \text{if } X_k = 0, \\ \cos(2\pi(f_c + \Delta_f)t) & \text{if } X_k = 1. \end{cases}$$

Example of binary FSK waveform for information bits 1 0 1 0 1 . . .



(Image source: Wikipedia)

Quadrature Amplitude Modulation - The Key Points



QAM is a technique to convert bits to a *passband* waveform:

1. QAM constellations are complex in general
2. Thus the baseband waveform is also complex
   $x_b(t) = \sum_k X_k p(t - kT)$
3. We then up-convert and transmit a *real passband* waveform:

$$x(t) = \text{Re}\left[x_b(t)\, e^{j2\pi f_c t}\right]$$
$$= \sum_k p(t - kT)\, [\mathbf{Re}(\mathbf{X_k}) \cos(2\pi f_c t) - \mathbf{Im}(\mathbf{X_k}) \sin(2\pi f_c t)]$$

At the receiver:

- Down-convert using product modulator + low-pass filter (separately for the sine, cosine carriers)
- Demodulate the baseband waveforms using matched filter
- Detection rule: Pick the constellation symbol closest to the (complex) output symbol

Properties of the QAM signal:

- Rate $= 1/T$ QAM symbols/s or $\frac{\log_2 \mathbf{M}}{\mathbf{T}}$ bits/s, where M is the constellation size
- Passband bandwidth of QAM waveform $= 2W$, where W is the bandwidth of the (baseband) pulse $p(t)$
- Note that up-converted PAM also has the same bandwidth $2W$.

QAM is a *very widely* used modulation scheme. E.g., 4G LTE uses QPSK/16-QAM/64-QAM, also used in optical fibre communication

You can now do Questions 1–5 in Examples Paper 9