Lecture 2

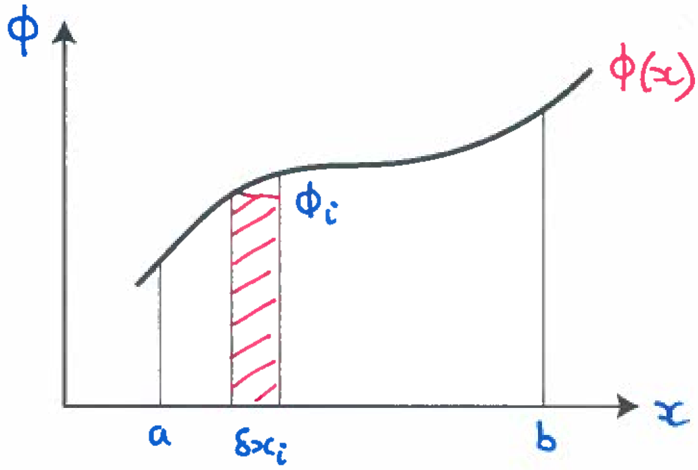
Scalar Fields - Integration

2.1 Integration of a function of one variable

If ϕ is a scalar function of one independent variable, $\phi = \phi(x)$, we define the integration of ϕ between the limits of $x = a$ and $x = b$ as,

$$\int_a^b \phi(x)\, dx = \lim_{\delta x_i \to 0} \sum_{i=1}^{N} \phi_i\, \delta x_i$$

The result of this integration is the "area under the curve": the area enclosed by the curve, the x-axis, and the limits of $x = a$ and $x = b$.

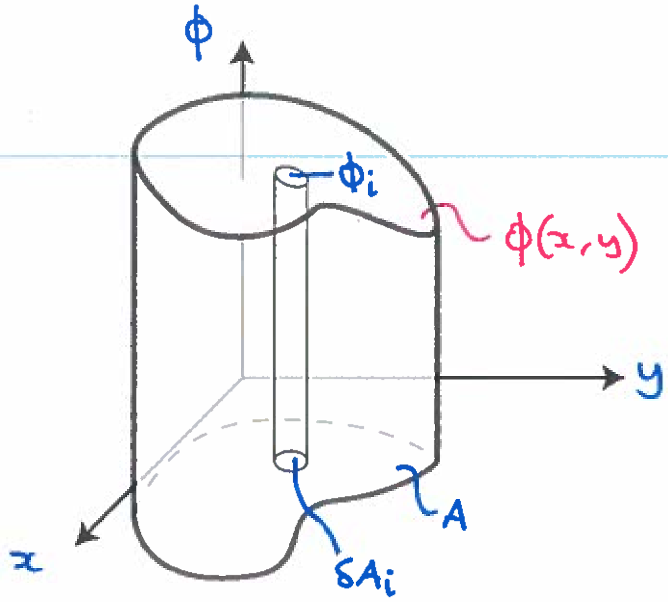


2.2 Integration of a function of two variables

If ϕ is a function of two independent variables, $\phi = \phi(x, y)$, we define the integration as,

$$\int_A \phi\, dA = \lim_{\delta A_i \to 0} \sum_{i=1}^{N} \phi_i\, \delta A_i$$

where A is an area on the $x-y$ plane. The result of this integration is the *volume* enclosed by the surface $\phi = \phi(x, y)$, the area A (on the $x-y$ plane), and the 'vertical curtain' connecting the boundary of A with the ϕ surface.



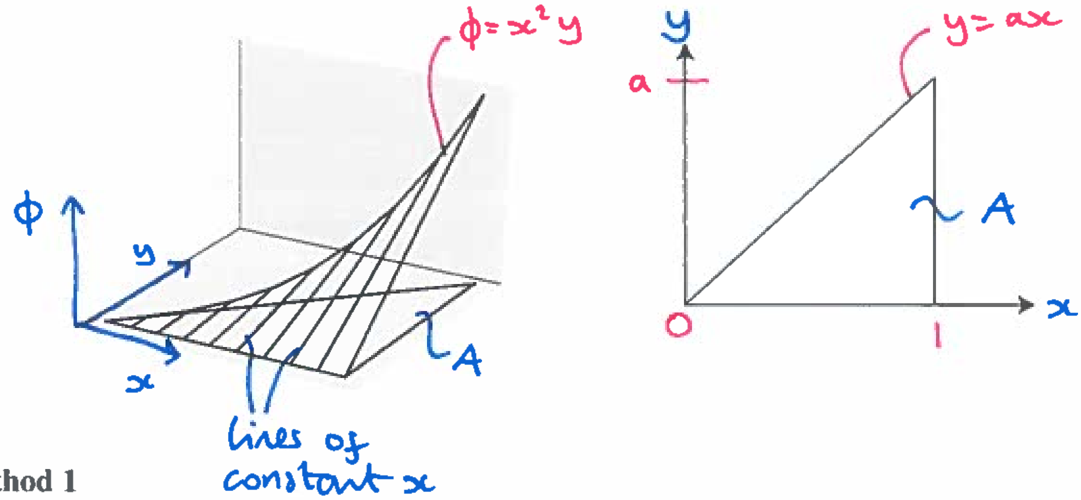
To illustrate that the integration is done over an area A in the $x-y$ plane, and therefore over two dimensions, x and y, we often use the double integral notation,

$$\int_A \phi \, dA = \iint \phi \, dA = \int_{y_1}^{y_2} \int_{x_1(y)}^{x_2(y)} \phi \, dx \, dy \quad . \tag{2.1}$$

The order in which the integration is performed is: first, the inner integral (in this case, ϕ with respect to x - note that the limits are functions of y) then the outer.

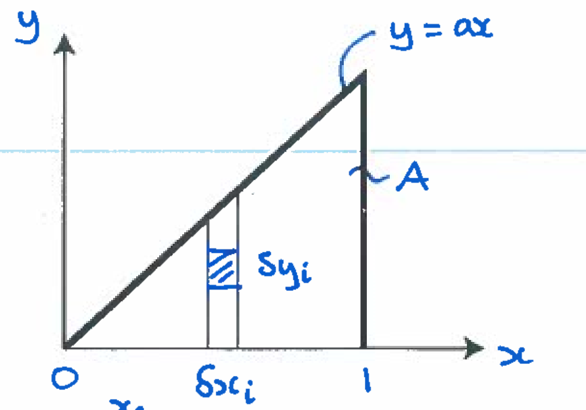
Example

Consider $\phi = x^2 y$. Find $\int_A \phi dA$ for the triangular region shown below.



Method 1

Divide the domain of integration into elements of area $\delta x_i \delta y_i$. First sum contributions to a vertical strip of width δx_i. Then add contributions from all strips.



$$I = \int_A \phi \, dA = \int_{x_1}^{x_2} \left[\int_{y_1(x)}^{y_2(x)} \phi \, dy \right] dx$$

where $x_1 = 0$, $x_2 = 1$, $y_1 = 0$, $y_2 = ax$.

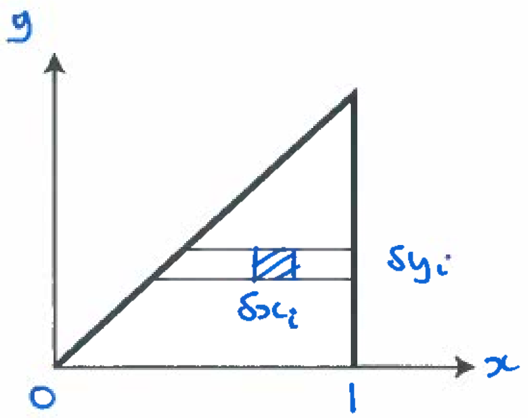
$$I = \int_0^1 \left[\int_0^{ax} x^2 y \, dy \right] dx \tag{2.2}$$

$$= \int_0^1 \left[x^2 \int_0^{ax} y \, dy \right] dx$$

$$= \int_0^1 x^2 \left[\frac{y^2}{2} \right]_0^{ax} dx$$

$$= \int_0^1 \frac{x^2 (ax)^2}{2} \, dx = \frac{a^2}{10}$$

Method 2



We now reverse the order so that we first sum up all elements in a horizontal strip of height δy_i, and then add up all the strips.

$$I = \int_A \phi dA = \int_{y_1}^{y_2} \left[\int_{x_1(y)}^{x_2(y)} \phi \, dx \right] dy \tag{2.3}$$

$$I = \int_0^a \left[\int_{y/a}^1 x^2 y \, dx \right] dy$$

$$= \int_0^a y \left[\frac{x^3}{3} \right]_{y/a}^1 dy$$

$$= \frac{1}{3} \int_0^a y - \frac{y^4}{a^3} \, dy = \frac{a^2}{10}$$

Both methods give the same answer since the order of the summations (the order of doing the integrations) does not change the total area.

In general, we always have a choice of the order in which we perform the integrations because:

$$I = \int_{x_1}^{x_2} \left[\int_{y_1(x)}^{y_2(x)} \phi(x,y) \, dy \right] dx = \int_{y_1}^{y_2} \left[\int_{x_1(y)}^{x_2(y)} \phi(x,y) \, dx \right] dy \tag{2.4}$$

A final comment on notation, multiple integrals are sometimes written using 'left-to-right' notation,

$$\int_{y_1}^{y_2} \left[\int_{x_1(y)}^{x_2(y)} \phi(x,y) \, dx \right] dy = \int_{x_1(y)}^{x_2(y)} dx \int_{y_1}^{y_2} dy \, \phi(x,y) \quad . \tag{2.5}$$

2.3 Integration of a function of three variables

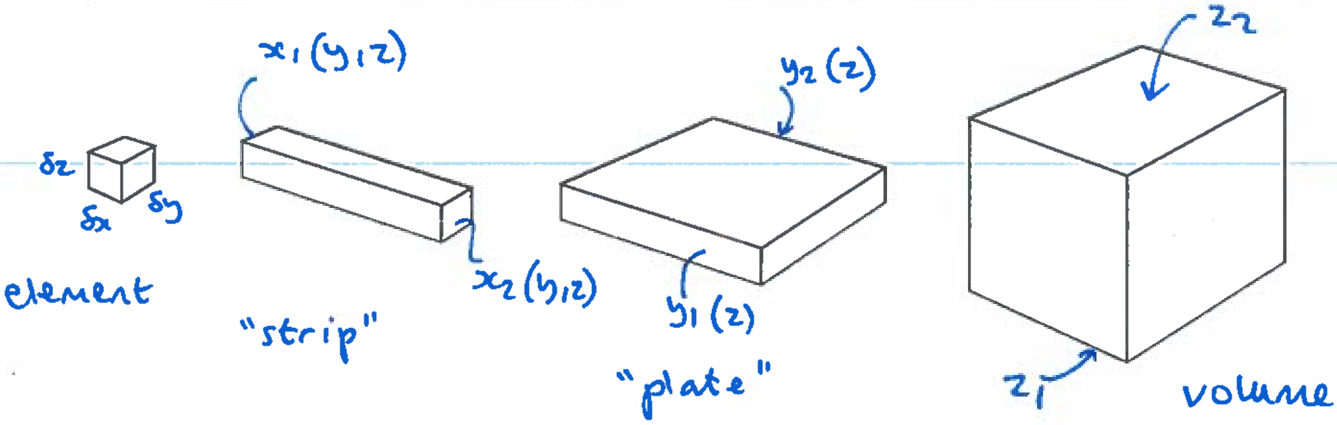
We can also evaluate integrals of three (or more!) variables. For example, if ρ is the density of a body and varies over the volume, $\rho = \rho(x,y,z)$, then we may find the mass of the body by summing up the elements of volume δv (with mass $\rho \delta v$),

$$m = \int_V \rho dv = \lim_{\delta V_i \to 0} \sum_{i=1}^{N} \rho_i \delta v_i \quad . \tag{2.6}$$

In Cartesian coordinates, $\delta v = \delta x \, \delta y \, \delta z$ and so,

$$m = \int_{z_1}^{z_2} \left[\int_{y_1(z)}^{y_2(z)} \left[\int_{x_1(y,z)}^{x_2(y,z)} \rho(x,y,z) \, dx \right] dy \right] dz \tag{2.7}$$

The order of the integration is, again, inner-to-outer. The process is illustrated, for a cuboid body, in the diagram below. First, the elements of volume are added in the x direction to form a 'strip' at constant y and z. All the strips are then added in the y direction, at constant z, to form a 'plane'. Finally, in the outer-most integral, all the planes are added in the z direction.



Application of integration to find average values

A useful application of area and volume integrals is in finding averages. For example, we could define the average density of an object $\bar{\rho}$ such that,

$$m = \bar{\rho}\, V_{tot}$$

where m is the mass of the object, and V_{tot} is its volume.

We can use integrations over the volume to evaluate m and V_{tot} so that the average density is given by,

$$\bar{\rho} = \frac{\iiint_V \rho(x,y,z)\, dx\, dy\, dz}{\iiint_V dx\, dy\, dz}$$

Similarly, in two-dimensions, we could evaluate the average height of an area (the average height of a mountain range, say) using,

$$\bar{h} = \frac{\iint_A h(x,y)\, dx\, dy}{\iint_A dx\, dy}$$

2.4 Change of variable and the Jacobian

Functions of one variable

If $\phi = \phi(x)$ and we would like to evaluate the integral, I,

$$I = \int_{x_1}^{x_2} \phi(x) dx \quad , \tag{2.8}$$

we may find it more convenient to use a new independent variable u where $x = x(u)$. Since,

$$dx = \frac{dx}{du} du \quad ,$$

we may write,

$$I = \int_{u_1}^{u_2} \phi(x(u))\frac{dx}{du}du \quad ,$$

where $x(u_1) = x_1$ and $x(u_2) = x_2$.

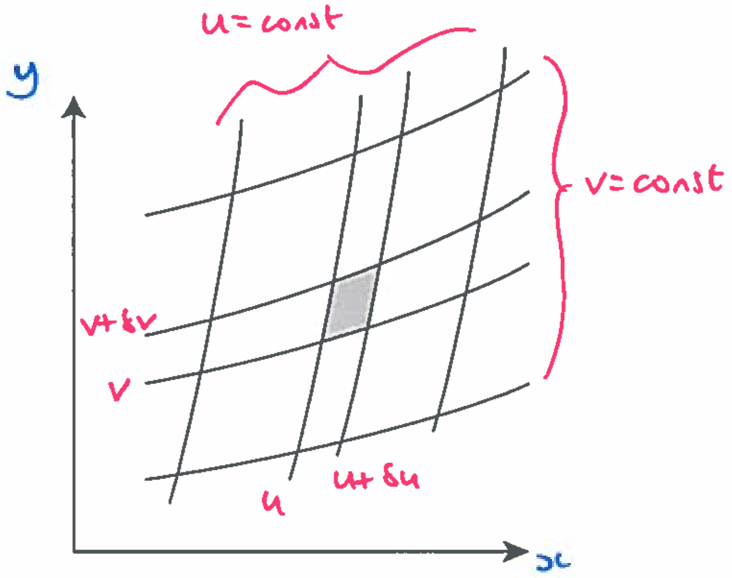
Functions of two variables

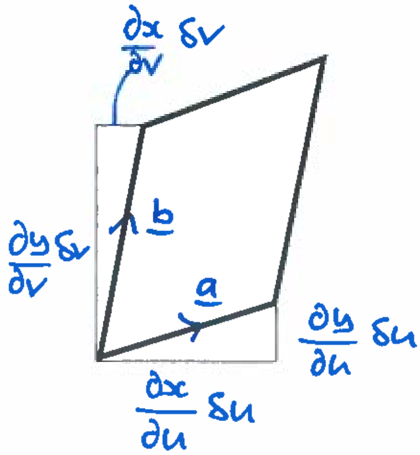
Similarly, if ϕ is a function of two independent variables, $\phi = \phi(x,y)$, and we wish to evaluate the integral,

$$I = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \phi(x,y)\,dx\,dy \quad , \tag{2.9}$$

it may be easier to switch to new independent variables u and v such that $x = x(u,v)$ and $y = y(u,v)$. However, just as in one dimension we found that $\iint \phi(x)dx \neq \iint \phi(u)du$ (we needed to multiply du by a 'scale factor' of dx/du), it is also true, in two dimensions, that $\iint \phi(x,y)dxdy \neq \iint \phi(u,v)dudv$. We now seek the correct 'scale factor' in this case.

The diagram below shows lines of constant u and v in the $x-y$ plane. The shaded area (bounded by constant u lines that are δu apart and constant v lines that are δv apart) is not $\delta u\delta v$ because the u and v lines are not perpendicular to each other.





The general expression for the area of a parallelogram is $|\mathbf{a} \times \mathbf{b}|$. In this case, $\mathbf{a}$ is given by moving δu in the direction of constant v, and $\mathbf{b}$ is given by moving δv in the direction of constant u. $\mathbf{a}$ and $\mathbf{b}$ are defined by,

$$\underline{a} = \frac{\partial x}{\partial u}\delta u\,\underline{i} + \frac{\partial y}{\partial u}\delta u\,\underline{j}$$

$$\underline{b} = \frac{\partial x}{\partial v}\delta v\,\underline{i} + \frac{\partial y}{\partial v}\delta v\,\underline{j}$$

The area of the parallelogram of interest is then

$$|\underline{a} \times \underline{b}| = \text{magnitude of} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial x/\partial u & \partial y/\partial u & 0 \\ \partial x/\partial v & \partial y/\partial v & 0 \end{vmatrix} \delta u\, \delta v$$

This provides our rule for changing the element of area in the (x,y) coordinate system to the (u,v) coordinate system: we replace $dx\,dy$ with,

$$\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| du\,dv \quad .$$

The expression inside the $|\ldots|$ is called the *Jacobian* (after the mathematician, Jacobi). It is sometimes also written,

$$J = \frac{\partial(x,y)}{\partial(u,v)} \quad ,$$

so that our rule is,

$$dx\,dy = \frac{\partial(x,y)}{\partial(u,v)} du\,dv \quad .$$

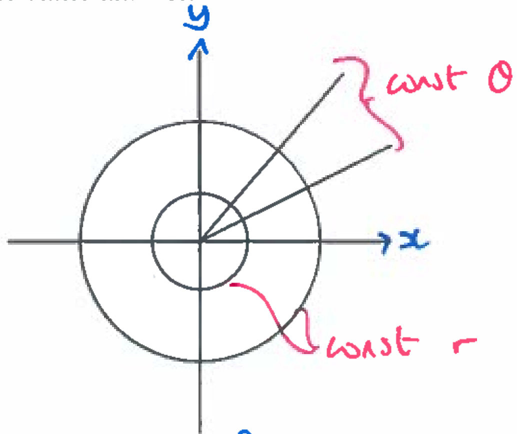
We have seen that the Jacobian, J, is really just the ratio of elemental areas in one coordinate system (x,y) to another (u,v). It follows from this that the ratio of areas in (u,v) coordinates to the equivalent in (x,y) coordinates (i.e. making the reverse change in variables) is given by the reciprocal of the Jacobian,

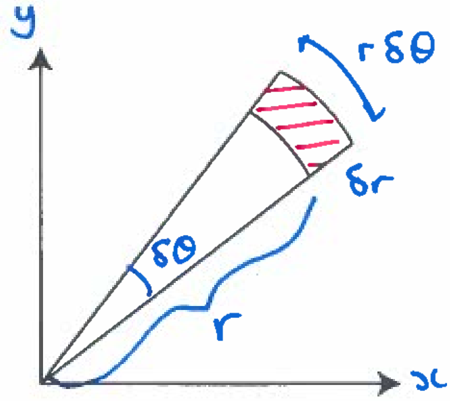
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} \quad .$$

This is a useful property because, depending on how the relationship between the old and new set of independent variables is expressed, it may be easier to evaluate $\partial(u,v)/\partial(x,y)$ than $\partial(x,y)/\partial(u,v)$.

Example

A common change of independent variable is from Cartesian to polar coordinates. We can use this example as a way to confirm that the algebraic and geometric interpretations of the Jacobian yield the same answer.





$x = r\cos\theta$
$y = r\sin\theta$

$\text{Area} = r\,\delta\theta\,\delta r$

To evaluate $J = \partial(x,y)/\partial(r,\theta)$ we need the following partial derivatives,

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

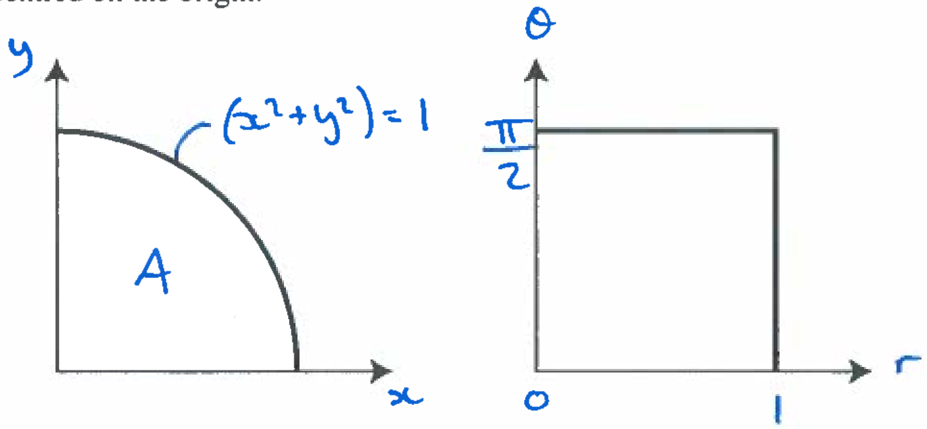
So the Jacobian is given by,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

We can use this result to evaluate the area integral,

$$I = \iint_A \left(x^2 + y^2\right) dx\, dy$$

where the region A is in the first quadrant, bounded by the x-axis, the y-axis and the circle of unit radius centred on the origin.



If we transform to polar coordinates,

$$x^2 + y^2 = r^2$$

$$dx\, dy \longrightarrow r\, d\theta\, dr$$

We can now evaluate I as follows:

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^2 \; r \, dr \, d\theta$$

$$= \int_{0}^{\pi/2} \left[\frac{r^4}{4}\right]_0^1 d\theta = \frac{\pi}{8}$$

You can now do Examples Paper 1: Q2, 3 and 4