Paper 1: Mechanics

Examples Paper 2

Inertia forces in mechanisms

1 A turbine blade in a jet engine is modelled as a rigid uniform rod AB of mass m and length $2a$ as shown in Figure 1. The blade is freely pivoted at A to a rigid hub of radius R which is spinning about O at a constant angular velocity Ω as shown . The angle OAB is θ.

(a) In the context of this application why is reasonable to ignore the effects of gravity?

(b) Using the rotating unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ as shown find a vector expression for the acceleration of the centre of mass of the blade.

(c) Find a differential equation for θ and the frequency ω of small vibration of the blade.

(d) If the small-amplitude motion of the blade is $\theta = \theta_0 \sin \omega t$ find an expression for the reaction force at A using unit vectors $\mathbf{e}_2$ and $\mathbf{e}_2^*$.

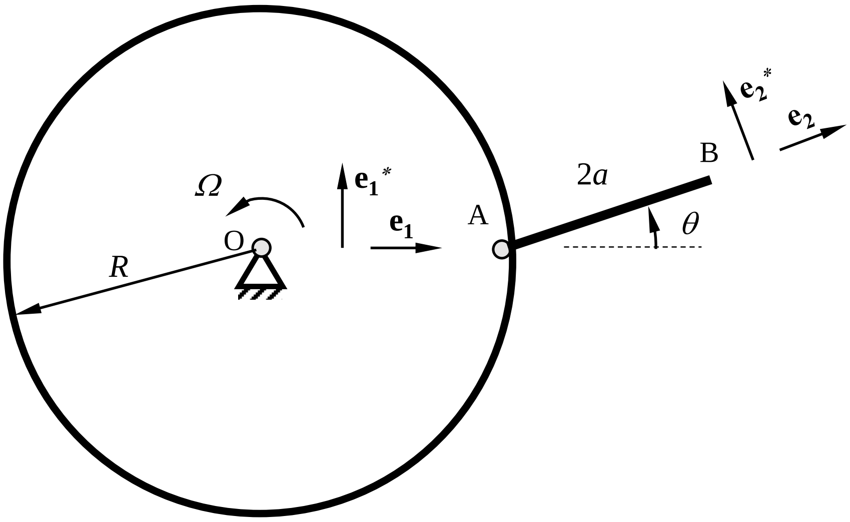

Figure 1

2 A mechanism is shown *in plan* in Figure 2. AB is a uniform rigid rod of length l and mass m hinged at a fixed pivot A. A similar uniform rigid rod CD of length l and mass m is hinged at its midpoint to the rod AB at B. The motion of CD is constrained by a light tie-rod CE, also of length l, fixed to a point E located so that ABE is 90°. The mechanism is initially at rest in the position shown with CD parallel to AB and the angle DEC equal to 60°. The distance between AB and CD is negligible. A torque T is applied to AB to give it an angular acceleration α. The effect of gravity can be ignored.

(a) Sketch an acceleration diagram and hence find the initial linear acceleration of point B and initial angular acceleration of CD.

(b) Determine the value of T and the corresponding force in CE.

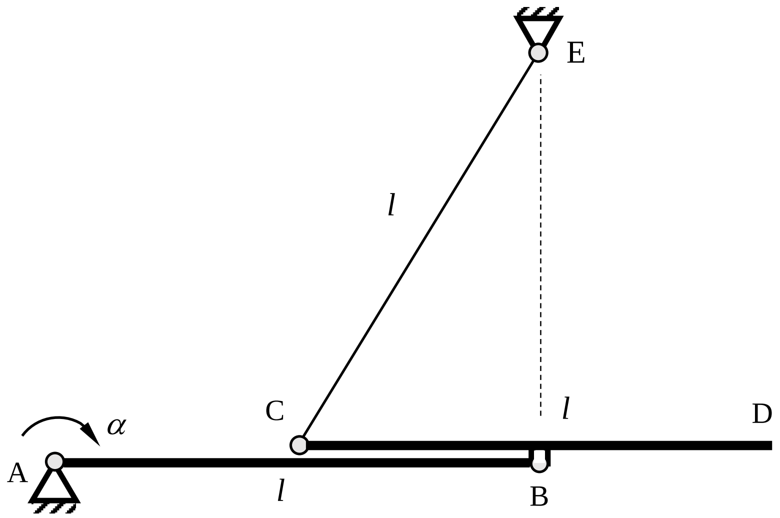

Figure 2

Bending moments and shear forces in mechanisms

3 A uniform bar of mass m per unit length and of length l is pivoted about a fixed vertical axis at one end P as shown in Figure 3. A torque T is applied to the bar at P. The effect of gravity can be ignored.

(a) What is the angular acceleration of the bar?

(b) What is the reaction at the pivot perpendicular to the bar?

(c) Find expressions for the shear force and bending moment at a section A-A distance x along the bar. Sketch the shear force and bending moment diagrams.

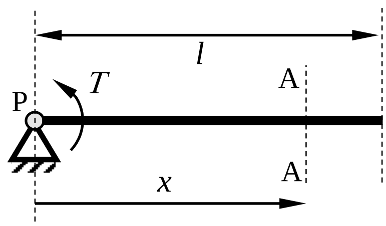

Figure 3

4 A uniform bar of length $(a+b)$ is pivoted about a horizontal axis at a distance a from one end as shown in Figure 4. The bar is held in a horizontal position and is then released.

(a) If $a < b$ find the initial acceleration of the bar.

(b) At the instant of release find the distance from P of the maximum sagging bending moment in the longer part of the bar, for the case $a < b/2$. What happens when $a > b/2$?

5 A mass M is attached to a fixed pivot P by a light rod, as shown in plan view in Figure 5. The centre of mass G is located a distance l from P and the moment of inertia about G is $I = Mk^2$. The rod is driven by a steady torque T applied at P. The effect of gravity can be ignored.

(a) What is the angular acceleration of the arm as a result of the torque T?

(b) At a given instant after the application of T the angular velocity of the body is ω. What are the components of the the reaction at P (i) along and (ii) perpendicular to the rod?

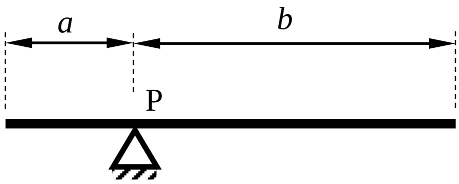

Figure 4

(c) At section A-A in the rod a distance a from G what are the values of (i) the shear force and (ii) the bending moment? Sketch the Shear Force and Bending Moment diagrams along PG.

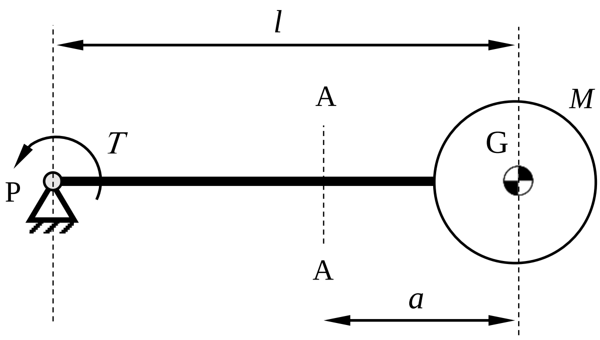

Figure 5

6 A uniform rod pendulum is shown in Figure 6. It has length $2a$ and mass m and it is free to pivot about A. It is swinging in a vertical plane and its motion is described by the angle θ measured from the horizontal. The rod is released from rest when $\theta = 0$.

(a) What is the angular acceleration of the rod, expressed as a function of θ?

(b) For $\theta = 45°$ what distance from the pivot does the maximum bending moment occur?

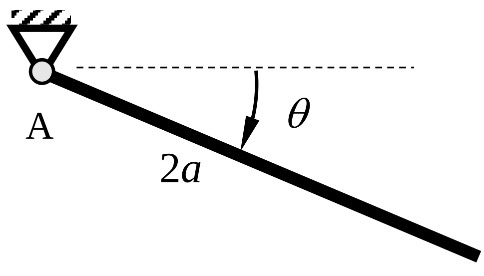

Figure 6

7 A car is turning a corner at a constant speed. Its wheels have radius r. The centre of one of its wheels traces a path of radius R centred on O moving with a uniform speed v as shown in Figure 7. The centre of the wheel is labelled Q and the point on the tyre instantaneously in contact with the ground is labelled P. A unit vector $\mathbf{e}_1$ is aligned with OQ and a unit vector $\mathbf{e}_2$ is aligned with QP. The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are a *fixed* reference frame where $\mathbf{k}$ is the upward vertical and $\mathbf{e}_1$ is instantaneously aligned with $\mathbf{i}$.

(a) Express the angular velocity vectors $\boldsymbol{\omega_1}$ and $\boldsymbol{\omega_2}$ of the car and wheel respectively in terms of v, R and r using the unit vectors given.

(b) Write down an expression for the position vector of P using $\mathbf{e}_1$ and $\mathbf{e}_2$ and hence find vector expressions for (i) the velocity and (ii) the acceleration of point P (do not expand cross products).

(c) Evaluate the instantaneous velocity and acceleration of P in the fixed $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ reference frame.

(d) Sketch the paths traced out by Q and by P as the wheel rolls along its curved path. Hence account on geometric grounds for the *outward* direction of the acceleration of point P.

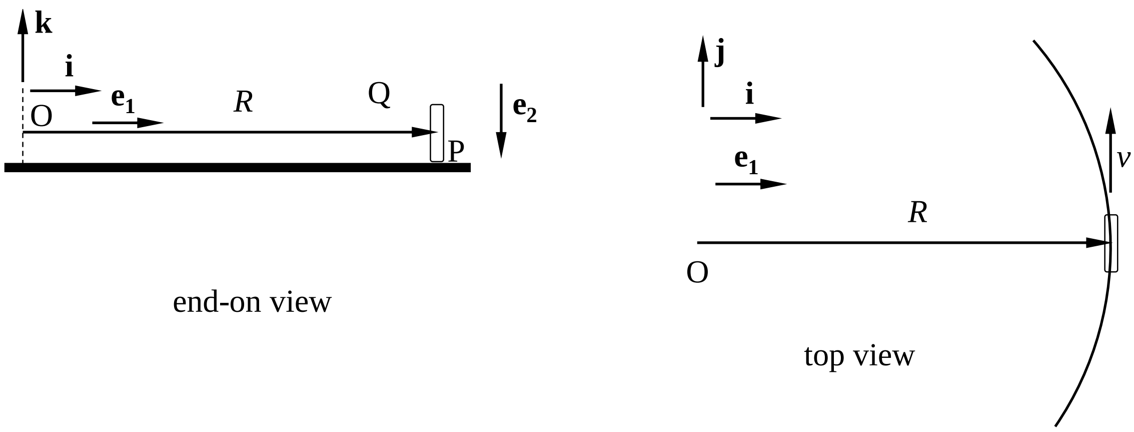

Figure 7

8 The turbine in a ship's power plant has a has a mass of 2500kg with its centre of mass at G as shown in Figure 8. The turbine has a radius of gyration of 0.3m and is mounted in bearings at A and B with its axis aligned with that of the ship. It is spinning at a speed of 8000rpm as viewed from the stern of the ship. The ship is moving with a speed of 20 knots (1 knot = 0.514 m/s).

(a) Find the vertical reactions at the bearings A and B when the ship is moving in a straight line.

(b) The ship is making a turn to the starboard of 500m radius. Determine the vertical components of the bearing reactions.

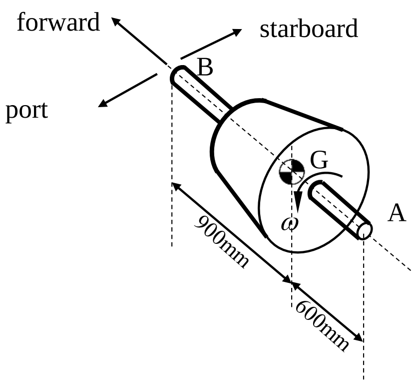

Figure 8

9 A toy gyroscope consists of a thin uniform disc of mass m and radius a in a set of light gimbals which is supported by means of a frictionless pivot P at the top of a small, light conical tower as shown in Figure 9. The supporting tower stands on a rough table. The disc is spinning with angular velocity ω, clockwise as viewed from the tower. The centre of mass of the disc is at G. Distance PG is d.
The gyroscope is in steady precession with PG horizontal.

(a) What is the moment of inertia of the disc and what is the vertical reaction at P?

(b) What is the rate of precession Ω of the gyroscope, and what is the direction of precession as viewed from above?

(c) What is the horizontal reaction at P?

(d) The height of the tower is h and the base of the tower is of radius r. What is the minimum spin speed ω required to ensure that the tower does not topple over?

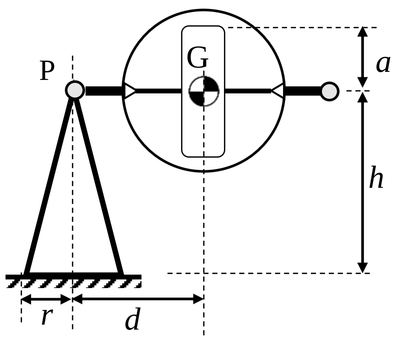

Figure 9

Suitable past Tripos questions

Inertia forces in mechanisms: 2015 Q1, 2016 Q1 Q3, 2018 Q2
Bending moments and shear forces in mechanisms: 2018 Q3, 2019 Q1, 2021 Q3
3D kinematics and gyroscopic mechanics: 2017 Q3, 2019 Q3

Answers

1(b). $-R\Omega^2\mathbf{e}_1 + a\ddot{\theta}\mathbf{e}_2^* - a(\dot{\theta}+\Omega)^2\mathbf{e}_2$

1(c). $\ddot{\theta} + \frac{3R\Omega^2}{4a}\sin\theta = 0, \quad \sqrt{\frac{3R\Omega^2}{4a}}$

1(d). $-m(R+a)\Omega^2\mathbf{e}_2 + \frac{1}{4}mR\Omega^2\theta_0\sin\omega t\mathbf{e}_2^*$

2(a). $l\alpha$ perpendicular to the rod, away from E, 2α clockwise,

2(b). $\frac{5}{3}ml^2\alpha$, $\frac{2}{3\sqrt{3}}ml\alpha$

3(a). $3T/ml^2$

3(b). $3T/2l$

3(c). $-\frac{3T}{2l}\left\{1-(\frac{x}{l})^2\right\}$ and $T\left\{1-\frac{3}{2}\frac{x}{l}+\frac{1}{2}\left(\frac{x}{l}\right)^3\right\}$

4(a). $\frac{3g(b-a)}{2(b^2-ab+a^2)}$

4(b). $\frac{4a^2-ab+b^2}{3(b-a)}$

5(a). $T/\left\{M(k^2+l^2)\right\}$

5(b). (i) $-Ml\omega^2\mathbf{i}$ (ii) $Tl/(k^2+l^2)\mathbf{j}$

5(c). (i) $-Tl/(k^2+l^2)$ (ii) $T(k^2+al)/(k^2+l^2)$

6(a). $\frac{-3g}{4a}\cos\theta$

6(b). $\frac{2a}{3}(\frac{2\sqrt{2}}{3}+1)$

7(a). $\frac{v}{R}\mathbf{k}$, $\frac{v}{R}\mathbf{k} - \frac{v}{r}\mathbf{e}_1$

7(b). $R\mathbf{e}_1 + r\mathbf{e}_2$, (i) $R(\boldsymbol{\omega}_1 \times \mathbf{e}_1) + r(\boldsymbol{\omega}_2 \times \mathbf{e}_2)$,
(ii) $R\left[(\dot{\boldsymbol{\omega}}_1 \times \mathbf{e}_1) + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{e}_1)\right] + r\left[(\dot{\boldsymbol{\omega}}_2 \times \mathbf{e}_2) + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{e}_2)\right]$

7(c). 0, $\frac{v^2}{R}\mathbf{i} + \frac{v^2}{r}\mathbf{k}$

8(a). 14.7kN, 9.8kN

8(b). 17.3kN, 7.2kN

9(a). $\frac{1}{2}ma^2$, mg

9(b). $\frac{2gd}{\omega a^2}$, anticlockwise.

9(c). $md\Omega^2$

9(d). $\frac{2d}{a^2}\sqrt{\frac{ghd}{r}}$