Substituting for the extensions $\mathbf{e} = \mathbf{F}\mathbf{t} = \mathbf{F}\mathbf{t}_0 + x\mathbf{F}\mathbf{s}$ gives

$$\begin{bmatrix} 1 \\ -1.4 \\ 1 \end{bmatrix} \cdot \left(\frac{l}{AE}\right) \left(\begin{bmatrix} 200\text{ N} \\ -100\text{ N} \\ 0 \end{bmatrix} + x \begin{bmatrix} 1.4 \\ -1.4 \\ 1.4 \end{bmatrix} \right) = 0$$

Which gives

$$340\text{ N} + x \times 4.8 = 0$$

$$x = -70\text{ N}$$

Now everything is known. We know the forces in the bars

$$\mathbf{t} = \mathbf{t}_0 + x\mathbf{s}$$

$$\mathbf{t} = \begin{bmatrix} 140\text{ N} \\ -100\text{ N} \\ 0 \end{bmatrix} - 70\text{ N} \times \begin{bmatrix} 1 \\ -1.4 \\ 1 \end{bmatrix} = \begin{bmatrix} 70\text{ N} \\ 0 \\ -70\text{ N} \end{bmatrix}$$

We also know the extensions of every bar

$$\mathbf{e} = \mathbf{F}\mathbf{t}$$

$$\mathbf{e} = \left(\frac{l}{AE}\right) \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 70\text{ N} \\ 0 \\ -70\text{ N} \end{bmatrix} = \left(\frac{l}{AE}\right) \begin{bmatrix} 100\text{ N} \\ 0 \\ -100\text{ N} \end{bmatrix}$$

3.2 Finding number of redundancies — Maxwell's equation

Maxwell's equation for the number of bars and joints required to make a structure rigid can be thought of as an expression for the size of the equilibrium matrix.

In the Structures Data Book, page 10, Maxwell's rule is described as

$$s - m = b + r - Dj$$

- b is the number of bars and, hence, describes the number of columns of the equilibrium matrix (the number of unknown tensions).

- $Dj - r$ is the number of equilibrium equations that can be written (in 2D, 2 for every joint, minus one for each restraint at a joint).

For a proper structure, the number of mechanisms, $m = 0$, and the number of redundancies/indeterminacies, i.e. the number of states of self stress s, is given by the number of unknowns b minus the number of equations $Dj - r$.

For some special structures, the equilibrium matrix is *rank-deficient*, and there can be mechanisms *and* states of self-stress. The equations of equilibrium turn out to be not independent. Maxwell's equation shows that in these cases, for every independent mechanism that exists, there must be an additional state of self-stress.

3.3 Principles of Structural Mechanics

There are three basic principles of structural mechanics that we will be using throughout the course:

Equilibrium Forces (stresses, moments etc.) must always be in equilibrium.

Compatibility Displacements (strains, curvatures etc.) must always be compatible.

Material Laws The constitutive laws (relating stress-strain, moment-curvature etc.) must follow a consistent material model (e.g. linear elastic, elasto-plastic).

The forces in statically-determinate problems can be solved by considering only equilibrium. The material law can then be used to find extensions, and finally compatibility can be used to find displacements. For statically-*indeterminate* problems, all three principles need to be applied in parallel.

Later in the course, we shall see other, *plastic* methods of analysis that use only Compatibility and a Material Law, or only Equilibrium and a Material Law, to find upper and lower bounds on the failure load of a structure.

3.4 Indeterminate Truss Structures — the Force Method

The analysis in Section 3.1.1 was an example of the *force method* of structural analysis, where member forces are taken as the primary unknowns. (Examples of an alternative scheme, the *displacement method* are covered in the third year). This section will outline formally how to analyse statically indeterminate pin-jointed truss structures, with the principle of structural mechanics invoked at each stage in **bold**:

---

1. Identify the number of redundancies, n.
   How many bars can be removed before the structure becomes statically determinate? This can be solved by a formal scheme such as Maxwell's rule, or by structural intuition — it is one less than the number of bars that can be removed before the structure becomes a mechanism.

---

2. Find a general solution for the bar forces in equilibrium with the applied load (**equilibrium**)

   (a) Find a particular set of bar forces, $\mathbf{t}_0$ in equilibrium with the applied loads.

   For a statically indeterminate structure, there will not be a unique answer for this: any equilibrium solution will be fine. A consistent scheme is to identify redundant bars (by ad-hoc methods, or a formal matrix scheme), and find the bar forces in equilibrium with the applied loads when the redundant bar forces are set to zero.

   (b) For each indeterminacy i, find a set of bar forces in equilibrium with zero applied load, $\mathbf{s}_i$.

   These are the states of self stress. A consistent scheme to find independent sets of self-stress is to set the force to 1 in one redundant bar at a time (while the other redundant bar forces are 0), and to then find the other bar forces in equilibrium with zero applied load.

(c) Write the general equilibrium solution, $\mathbf{t}$ as being the particular solution, $\mathbf{t}_0$, plus unknown amounts x_i of each of the states of self-stress, $\mathbf{s}_i$.

$$\mathbf{t} = \mathbf{t}_0 + x_1\mathbf{s}_1 + x_2\mathbf{s}_2 + \cdots + x_n\mathbf{s}_n$$

The problem is now reduced to finding the unknowns x_i.

---

3. Find the extensions of each bar (**material law**).

The flexibility of each bar, length/AE can be written along the diagonal of a flexibility matrix $\mathbf{F}$. Then, the extensions are the elastic extensions, plus any initial misfit $\mathbf{e}_0$ (e_0 is positive if the bar is initially too long, negative if too short).

$$\mathbf{e} = \mathbf{F}\mathbf{t} + \mathbf{e}_0 = \mathbf{F}\mathbf{t}_0 + x_1\mathbf{F}\mathbf{s}_1 + x_2\mathbf{F}\mathbf{s}_2 + \cdots + x_n\mathbf{F}\mathbf{s}_n + \mathbf{e}_0$$

---

4. Find the amount of each state of self-stress required to ensure that the structure fits together, i.e. the values of $x_1 \cdots x_n$ (**compatibility**).
The most convenient way to do this is to use Virtual Work.

- Using a state of self-stress $\mathbf{s}_i$ as a *virtual equilibrium system*, in equilibrium with zero applied load, the equation of Virtual Work becomes

$$\sum s \cdot e = 0$$

Which, written as a dot-product between vectors is

$$\mathbf{s}_i \cdot \mathbf{e} = 0$$

- A different equation can be found for each state of self-stress
- There will be n simultaneous equations in the n unknowns x_i, and these equations can be solved to give the unknowns x_i.

---

5. Once the values of x_i have been found, all of the internal extensions and forces are known:

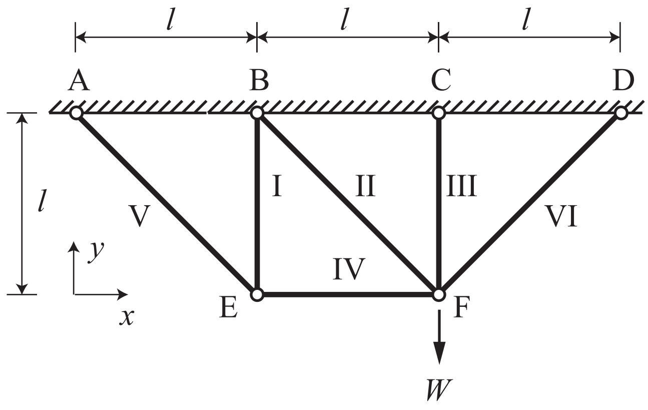
$$\mathbf{t} = \mathbf{t}_0 + x_1\mathbf{s}_1 + x_2\mathbf{s}_2 + \cdots + x_n\mathbf{s}_n$$

$$\mathbf{e} = \mathbf{F}\mathbf{t}_0 + x_1\mathbf{F}\mathbf{s}_1 + x_2\mathbf{F}\mathbf{s}_2 + \cdots + x_n\mathbf{F}\mathbf{s}_n + \mathbf{e}_0$$

If some external displacements are required, they can be found from Virtual Work with the now known internal extensions. There will be some freedom in the virtual equilibrium system that could be used, because of the indeterminacy, but any equilibrium system will give the same answer — an example later.

3.4.1 Example — another statically indeterminate truss

Find the forces in all the members of the truss shown, due to both the loading, and because bar EF is initially too long by αl.



redundancies/indeterminacies

There are six joints ($j = 6$), but 4 of these are restrained in 2 directions ($r = 8$) and there are six bars ($b = 6$), so Maxwell's rule gives:

$$s - m = b - 2j + r = 6 - 12 + 8 = 2$$

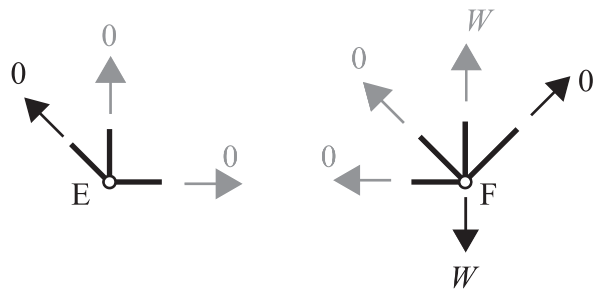
In this case, there are clearly no mechanisms (the equilibrium matrix is full rank, see below), $m = 0$, and hence $s = 2$.

We need to identify two redundant bars. A sensible choice would be V and VI; if they were removed, it would leave a determinate truss. A mistake to avoid is to identify bars that, if removed, would leave part of the structure statically indeterminate, and part a mechanism. Removing IV and V would be such a mistake.

The formal matrix scheme will identify suitable bars to be redundant, and will also deal well with some well known paradoxical cases.

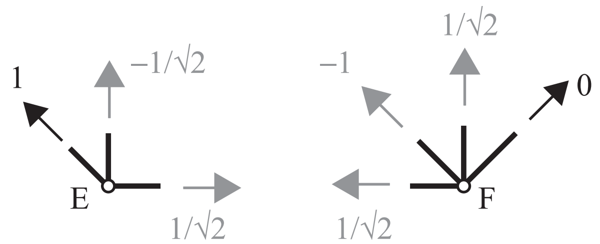
equilibrium

Initially we need to find a set of forces in equilibrium with the applied load. If we set $t_{\mathrm{V}} = t_{\mathrm{VI}} = 0$, we can find the other bar forces by joint equilibrium.



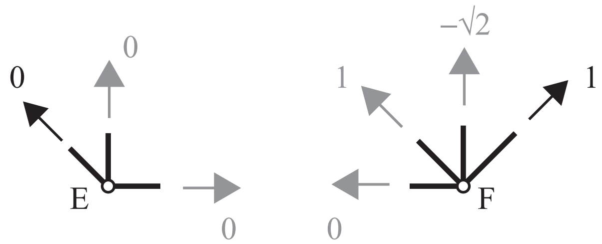
$$\mathbf{t}_0 = \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \\ t_{\mathrm{IV}} \\ t_{\mathrm{V}} \\ t_{\mathrm{VI}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ W \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We also need to find two independent sets of self stress. To find the first, $\mathbf{s}_1$, remove all external loads, set $t_{\mathrm{V}} = 1$ and $t_{\mathrm{VI}} = 0$, and find the other bar forces by equilibrium.



$$\mathbf{s}_1 = \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \\ t_{\mathrm{IV}} \\ t_{\mathrm{V}} \\ t_{\mathrm{VI}} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -1 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1 \\ 0 \end{bmatrix}$$

To find the second state of self-stress, $\mathbf{s}_2$, remove all external load, set $t_{\mathrm{V}} = 0$ and $t_{\mathrm{VI}} = 1$, and find the other bar forces by equilibrium.



$$\mathbf{s}_2 = \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \\ t_{\mathrm{IV}} \\ t_{\mathrm{V}} \\ t_{\mathrm{VI}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -\sqrt{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

All possible sets of bar forces in equilibrium with the applied loads are given by

$$\mathbf{t} = \mathbf{t}_0 + x_1\mathbf{s}_1 + x_2\mathbf{s}_2$$

Using the Equilibrium Matrix

An alternative scheme to find $\mathbf{t}_0$, $\mathbf{s}_1$ and $\mathbf{s}_2$ is to write the equilibrium matrix: two equations at each free node.

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 1/\sqrt{2} & 0 \\ -1 & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1 & 0 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & -1 & 0 & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \\ t_{\mathrm{IV}} \\ t_{\mathrm{V}} \\ t_{\mathrm{VI}} \end{bmatrix} = \begin{bmatrix} f_{\mathrm{E}x} \\ f_{\mathrm{E}y} \\ f_{\mathrm{F}x} \\ f_{\mathrm{F}y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -W \end{bmatrix}$$

Gaussian elimination cannot proceed immediately, as there is a number zero in the top left corner of the equilibrium matrix. However, if the rows are swopped so that row 1 becomes row 4, and the other rows are moved up:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1 & 0 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & -1 & 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 & -1 & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \\ t_{\mathrm{IV}} \\ t_{\mathrm{V}} \\ t_{\mathrm{VI}} \end{bmatrix} = \begin{bmatrix} f_{\mathrm{E}y} \\ f_{\mathrm{F}x} \\ f_{\mathrm{F}y} \\ f_{\mathrm{E}x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -W \\ 0 \end{bmatrix}$$

Now, if the new row 3 is replaced by the addition of row 2 and row 3, Gaussian elimination is complete.

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1 & 0 & -1/\sqrt{2} \\ 0 & 0 & -1 & 1 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & -1 & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \\ t_{\mathrm{IV}} \\ t_{\mathrm{V}} \\ t_{\mathrm{VI}} \end{bmatrix} = \begin{bmatrix} f_{\mathrm{E}y} \\ f_{\mathrm{F}x} \\ f_{\mathrm{F}x} + f_{\mathrm{F}y} \\ f_{\mathrm{E}x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -W \\ 0 \end{bmatrix}$$

There are two columns where we have been unable to do any elimination, V and VI, and these are our choice for redundant bars. If we set the tension in them to zero, and solve the equations one at a time, starting from the final one (back-substitution), we find

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1 & 0 & -1/\sqrt{2} \\ 0 & 0 & -1 & 1 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & -1 & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ W \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -W \\ 0 \end{bmatrix}$$

and this gives $\mathbf{t}_0$.

To find states of self-stress, we must remove all load. If we now set the tension in bar V to 1, and bar VI to 0, and solve by back substitution we find

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1 & 0 & -1/\sqrt{2} \\ 0 & 0 & -1 & 1 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & -1 & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ -1 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and this gives $\mathbf{s}_1$. If we now set the tension in bar V to 0, and bar VI to 1, and solve by back substitution we find

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1 & 0 & -1/\sqrt{2} \\ 0 & 0 & -1 & 1 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & -1 & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -\sqrt{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and this gives $\mathbf{s}_2$.

An advantage of the formal matrix scheme is that is cannot incorrectly identify redundant bars. If elimination cannot be carried out on a column, the corresponding bar will be redundant, and elimination should continue on the next column, if this is possible. It will also identify if the matrix is rank-deficient — then elimination will leave one or more equations as all zeros.

Material Law

Now we know the tensions in each member (either by an ad-hoc scheme, or a formal matrix scheme), we can find the extensions. The extensions are a combination of the elastic extensions, and any initial misfit[1].

$$\mathbf{e} = \mathbf{F}\mathbf{t} + \mathbf{e}_0$$

[1] A good way to think about the sign of initial extensions is to consider that the members are initially the correct length to fit together perfectly, but that turnbuckles are added which then extend members that were actually longer than this length — and this will have to be accommodated through elastic deformation of the structure.

$$\mathbf{e} = \mathbf{F}(\mathbf{t}_0 + x_1\mathbf{s}_1 + x_2\mathbf{s}_2) + \mathbf{e}_0$$

$$\begin{bmatrix} e_{\mathrm{I}} \\ e_{\mathrm{II}} \\ e_{\mathrm{III}} \\ e_{\mathrm{IV}} \\ e_{\mathrm{V}} \\ e_{\mathrm{VI}} \end{bmatrix} = \left(\frac{l}{AE}\right) \begin{bmatrix} 1 & & & & & \\ & \sqrt{2} & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \sqrt{2} & \\ & & & & & \sqrt{2} \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ W \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} -1/\sqrt{2} \\ -1 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -\sqrt{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha l \\ 0 \\ 0 \end{bmatrix}$$

3.4.2 Compatibility

The extension given above will only fit together for a particular value of x_1 and x_2. There are two compatibility equations that we have to solve

$$\mathbf{s}_1 \cdot \mathbf{e} = 0$$
$$\mathbf{s}_2 \cdot \mathbf{e} = 0$$

The first equation gives

$$W\left(\frac{1}{\sqrt{2}}\right)\frac{l}{AE} + x_1\left(2\sqrt{2} + \frac{3}{2}\right)\frac{l}{AE} + x_2\left(-\sqrt{2} - 1\right)\frac{l}{AE} + \left(\frac{1}{\sqrt{2}}\right)\alpha l = 0$$

The second equation gives

$$-W\sqrt{2}\frac{l}{AE} + x_1\left(-\sqrt{2} - 1\right)\frac{l}{AE} + x_2\left(2\sqrt{2} + 2\right)\frac{l}{AE} = 0$$

and the solution to these equations is given by

$$x_1 = \quad 0W - 0.227\alpha AE$$
$$x_2 = 0.293W - 0.113\alpha AE$$

Hence, the tension in the members is given by

$$\mathbf{t} = \begin{bmatrix} 0 \\ 0.293 \\ 0.586 \\ 0 \\ 0 \\ 0.293 \end{bmatrix} W + \begin{bmatrix} 0.160 \\ 0.113 \\ 0 \\ -0.160 \\ -0.227 \\ -0.113 \end{bmatrix} \times \alpha AE$$

and the extension of the members is given by

$$\mathbf{e} = \begin{bmatrix} 0 \\ 0.414 \\ 0.586 \\ 0 \\ 0 \\ 0.414 \end{bmatrix} \left(\frac{Wl}{AE}\right) + \begin{bmatrix} 0.160 \\ 0.160 \\ 0 \\ 0.840 \\ -0.320 \\ -0.160 \end{bmatrix} \times \alpha l$$

3.5 Finding deflections in the structure

Once the equations have been solved to find the force, and hence the compatible extension of every member, everything is known about the structure. If, for instance, we wish to know about the deflection of a node, it is a simple matter of selecting the correct information. We could draw a displacement diagram, but we will use Virtual Work.

$$\sum t \cdot e = \sum f \cdot \delta$$

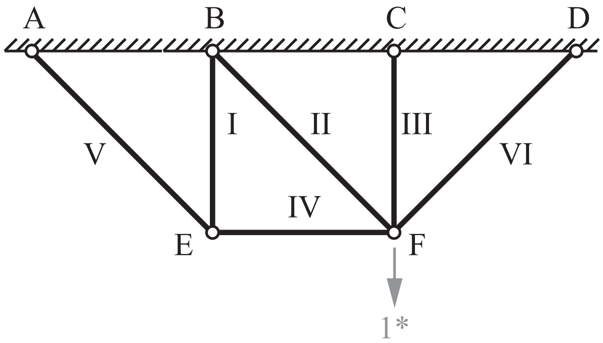
If a virtual force of (1) is applied where the displacement is required, the Virtual Work equation becomes

$$\delta_{\text{required}} = \mathbf{t}^* \cdot \mathbf{e}$$

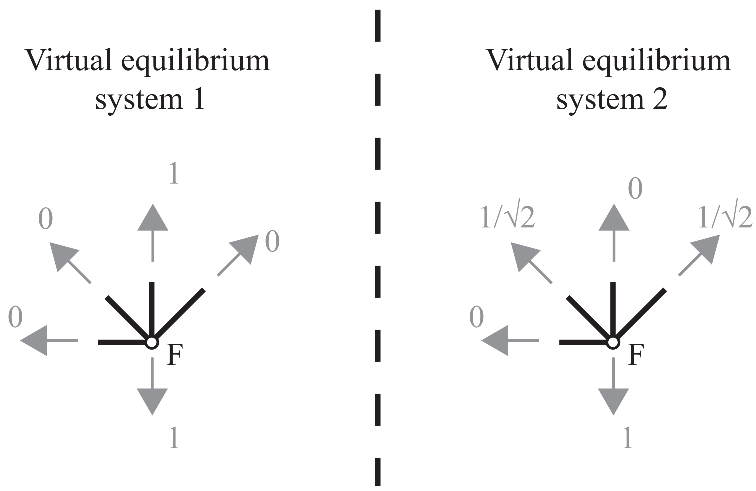
For an indeterminate structure, there will be a choice of equilibrium systems $\mathbf{t}^*$, but each will give the same δ_{required}.

Example

For the truss in Section 3.4.1, find the vertical deflection of node F.
Apply a virtual load at the point at which the displacement is required:



Find a virtual force system in equilibrium:



Apply Virtual Work:

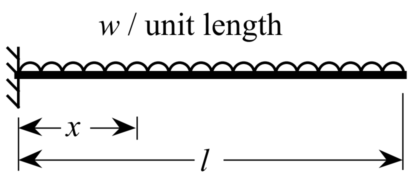
$$\mathbf{t}_1^* = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{t}_2^* = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\delta_{F_{\text{vertical}}} = \mathbf{t}_1^* \cdot \mathbf{e} = \mathbf{t}_2^* \cdot \mathbf{e} = 0.586 \left(\frac{Wl}{AE}\right)$$

Try Questions 2, 3 and 4, Examples Sheet 2/3

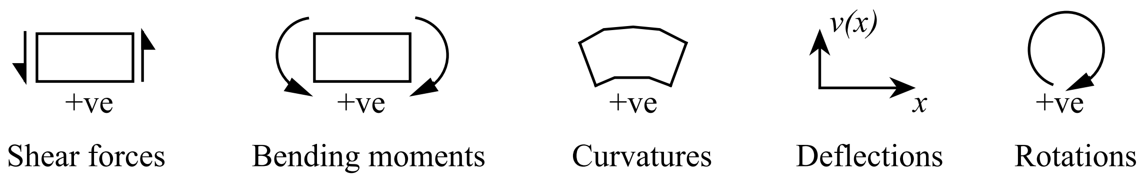
3.6 Three Approaches to Deflections in Beams

The previous section has shown that, to be able to analyse statically indeterminate structures, it is essential to be able to calculate the deformations of the structure. This section will start to consider beam structures. It will consider how deflections in statically determinate beams can be calculated. The first two methods are revision of first-year material, but the third, Virtual Work in beams, is new. We will examine the three methods by reference to a simple example, a cantilever carrying a uniformly distributed load.



In each case we will calculate the deflection and rotation of the free end of the cantilever.

Sign convention for beams

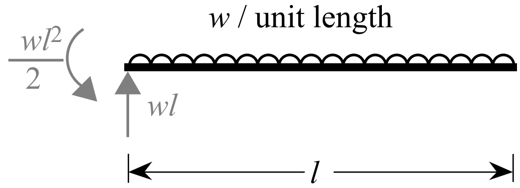


3.6.1 Differential Equation for Bending (IA revision)

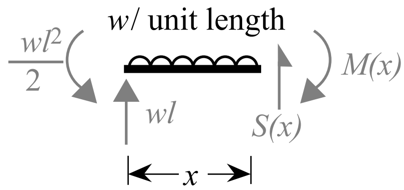
The first method we will look at is to set up a differential equation for bending.

Equilibrium

First consider the overall equilibrium of the cantilever



Now examine equilibrium by cutting at a general distance x



$$\begin{aligned} S(x) &= -w(l-x) \\ M(x) &= \frac{wl^2}{2} + wx\frac{x}{2} - wlx \\ &= \frac{w}{2}(x^2 - 2lx + l^2) \end{aligned}$$

Elastic Law

Bending moments cause curvature

$$\kappa(x) = \frac{M(x)}{EI}$$

For our example

$$\kappa(x) = \frac{w}{2EI}(x^2 - 2lx + l^2)$$

(N.B. We assume that shear forces cause no deformation, a good assumption for long, slender beams. The IB Model Structures lab. shows that this is not always true!)

Compatibility

The curvature is related to the rotation by $\kappa = d\theta/ds$
For small rotations,

$$\theta = -\frac{dv}{dx} \quad \text{(clockwise +ve)} \quad , \qquad ds = dx$$

$$\kappa = -\frac{d^2v}{dx^2}$$

For our example

$$\frac{d^2v}{dx^2} = -\frac{w}{2EI}(x^2 - 2lx + l^2)$$

We can solve this equation by integrating twice, and then applying two boundary conditions

$$\frac{dv}{dx} = -\frac{w}{2EI}\left(\frac{x^3}{3} - lx^2 + l^2x\right) + C$$
$$v = -\frac{w}{2EI}\left(\frac{x^4}{12} - \frac{lx^3}{3} + \frac{l^2x^2}{2}\right) + Cx + D$$

Boundary Conditions

at $x = 0$, $v = 0$, $\quad\Rightarrow\quad D = 0$

at $x = 0$, $\frac{dv}{dx} = 0$, $\quad\Rightarrow\quad C = 0$

$$v = -\frac{wx^2}{24EI}(x^2 - 4lx + 6l^2)$$
$$\frac{dv}{dx} = -\frac{wx}{6EI}(x^2 - 3lx + 3l^2)$$

At $x = l$,

Tip deflection $\quad v(l) = -\frac{wl^4}{8EI}$

Tip rotation $\quad \theta(l) = -\left.\frac{dv}{dx}\right|_{x=l} = \frac{wl^3}{6EI}$

By solving the differential equation, we have ensured that these deflections are *compatible* with the curvatures derived from *equilibrium* and the *elastic law*.

3.6.2 Virtual Work

Section 3.1 contained a reminder about Virtual Work applied to pin-jointed trusses. In this section, we will show how Virtual Work can also be applied to beams, and can be used instead of differential equations to find external displacements that are *compatible* with internal curvatures. To apply Virtual Work to beams, we need to find the external work done, and the internal work done, by a compatible displacement on a beam loaded by concentrated and distributed forces, and by couples.

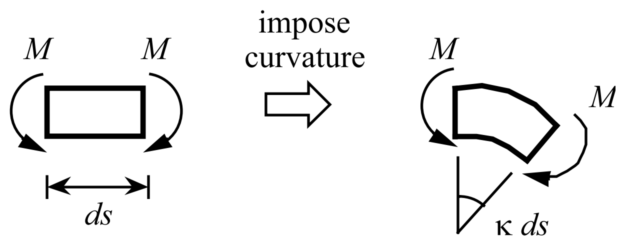
External Work

If a beam is loaded by concentrated forces F_i, by distributed forces $w(s)$, and by couples C_i, the work done by a compatible displacement will be:

$$\text{external work done} = \sum F_i \times \delta_i + \int w(s) \times \delta(s)\, ds + \sum C_i \times \theta_i$$

Internal Work

It is usually assumed that beams undergo no axial deformation, no shear deformation, and hence all work is done in bending. How *much* work is done in bending?
Take a small length of beam, and impose a curvature κ.



The moment has done work $M \quad \kappa\, ds$

For the complete beam, the internal work done will be $\int M(s)\, \kappa(s)\, ds$

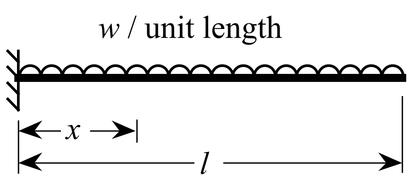
Thus the equation of Virtual Work for a beam will be

$$\sum F_i \delta_i + \int w(s) \delta(s)\, ds + \sum C_i \theta_i = \int M(s)\, \kappa(s)\, ds$$

where $s =$ distance along beam
Points to note

1. The forces, couples and bending moments must be in equilibrium.
2. The displacements, rotations and curvatures must be compatible.
3. The equilibrium set and the compatible set don't have to be related. Any compatible set can be imposed on any equilibrium system. Thus we can use a *virtual* set to discover something about the real equilibrium or compatible set.

We will now use Virtual Work to find the deflection and rotation at the end of our example cantilever



As before, we first need to calculate the actual curvatures of the beam.

Equilibrium

$$M(x) = \frac{w(l-x)^2}{2}$$

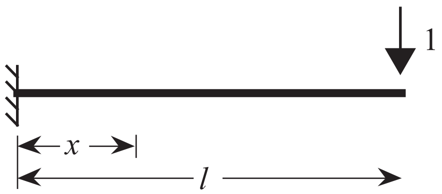
Elastic Law

$$\kappa(x) = \frac{1}{EI}\frac{w(l-x)^2}{2}$$

This was derived in Section 3.6.1.

Compatibility

We can now use Virtual Work to carry out the compatibility calculation, in order to find the displacement at the tip of the cantilever. We need to choose a virtual force system that isolates the answer that we want. In this case, apply a virtual force of 1 where we want to find the deflection.
Virtual equilibrium system:



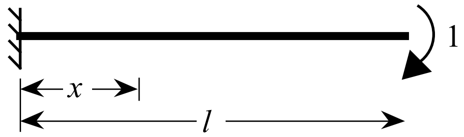
$$M^*(x) = (l-x)$$

The Virtual Work equation then becomes

$$\begin{array}{cccccc}
\sum F^* & \delta & = & \int M^*(s) & \kappa(s) & ds \\
\text{virtual force} & \text{actual displacement} & & \text{virtual moment} & \text{actual curvature} & \\
1 & -v(l) & = & \int_0^l (l-x) & \frac{1}{EI}\frac{w(l-x)^2}{2} & dx \\
 & -v(l) & = & -\frac{w}{EI}\left[\frac{(l-x)^4}{8}\right]_0^l & & \\
 & v(l) & = & -\frac{wl^4}{8EI} & &
\end{array}$$

Thus we have used Virtual Work to ensure that the displacement we have calculated is *compatible* with the actual curvatures in the beams.

By using a different virtual system, we can similarly calculate the rotation at the end of the beam:

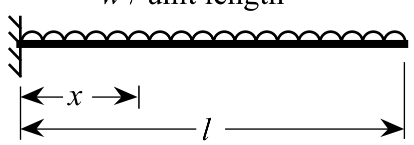


$$M^*(x) = 1$$

$$
\begin{array}{cccccc}
\sum C^* & \theta & = & \int M^*(s) & \kappa(s) & ds \\
\text{virtual couple} & \text{actual rotation} & & \text{virtual moment} & \text{actual curvature} & \\
1 & \theta(l) & = & \int_0^l 1 & \frac{1}{EI}\frac{w(l-x)^2}{2} & dx \\
 & \theta(l) & = & -\frac{w}{EI}\left[\frac{(l-x)^3}{6}\right]_0^l & & \\
 & \theta(l) & = & \frac{wl^3}{6EI} & &
\end{array}
$$

Again we have used Virtual Work to enforce a *compatibility* condition — that the rotation at the end is compatible with the actual curvatures calculated using equilibrium and the elastic law.

3.6.3 Data Book Coefficients (IA revision)



Looking in the data book page 6:

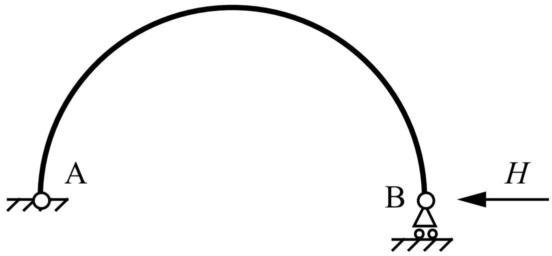
$$v(l) = -\frac{(wl)l^3}{8EI} = -\frac{wl^4}{8EI}$$

$$\theta(l) = \frac{(wl)l^2}{6EI} = \frac{wl^3}{6EI}$$

3.6.4 Comments on the Methods

If the problem can be split into data book cases, this will almost always be the simplest way to solve the problem. For other cases, such as curved beams, or beams with varying cross-sections, either the differential equation or the Virtual Work approach can be used. However, Virtual Work is usually simpler, because it allows us to use our knowledge of equilibrium to avoid integrating differential equations.

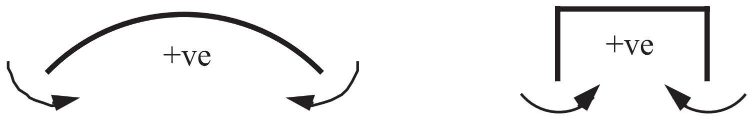
3.6.5 Example — Deflection of a curved beam

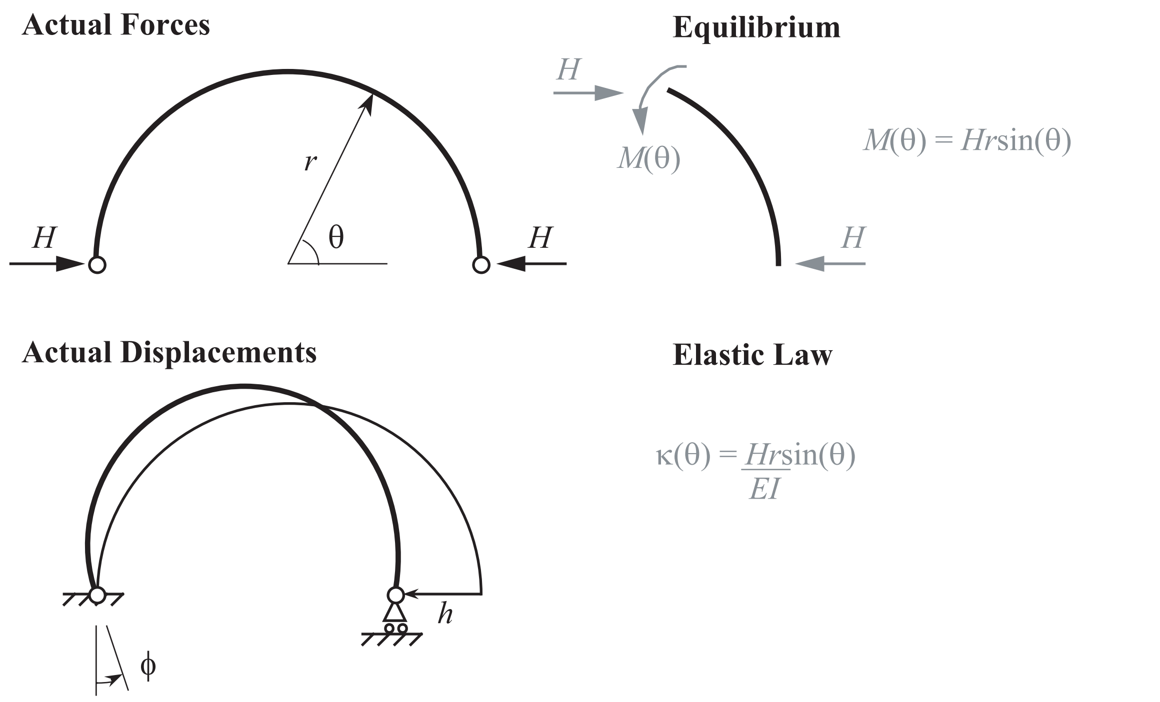


Find the deflection of B and the rotation of A due to the applied load.

Because the beam is curved, we will have to use Virtual Work to find compatible deflections. However, the first part of the calculation remains identical whatever the approach — find moments in the beam in *equilibrium* with the applied force, and use the *elastic law* to find the curvatures.

Sign Convention
For frames and ring structures we will use the sign convention that positive moments and curvatures cause *tension* on the *outside* of the frame.

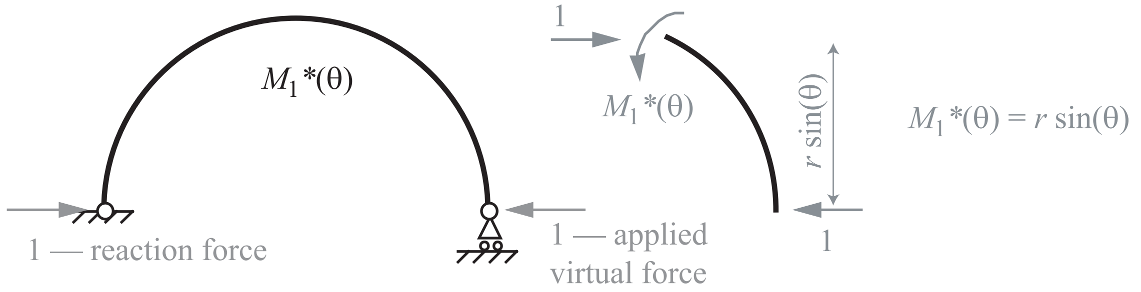




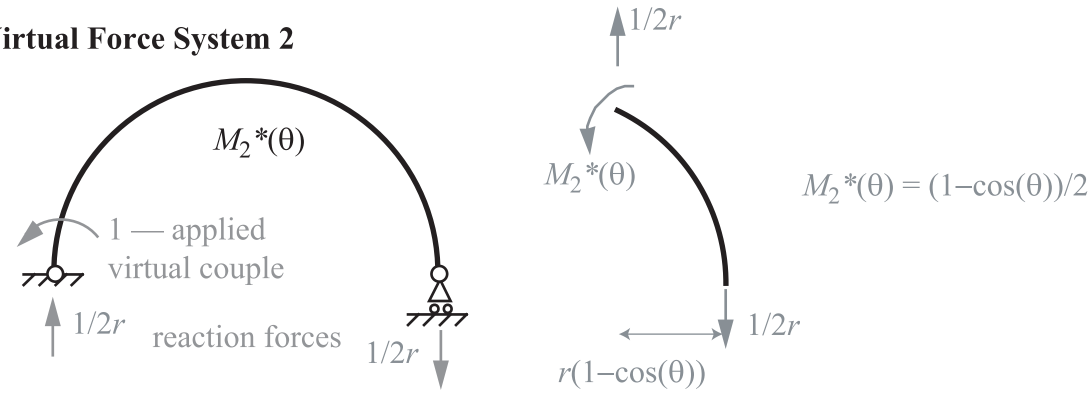
Compatibility by Virtual Work

We will use Virtual Work to find the rotation ϕ and the displacement h which are compatible with the curvatures we have calculated. We will use two virtual equilibrium sets, one to find each of the displacements we require.

Virtual Force System 1



Virtual Force System 2



For displacement

$$1 \times h = \int \kappa(\theta) \times M_1^*(\theta)\, ds$$

$$\begin{aligned} h &= \int_0^{\pi} \frac{Hr\sin\theta}{EI}(r\sin\theta)(r\,d\theta) \\ &= \frac{Hr^3}{EI}\int_0^{\pi} \sin^2\theta\, d\theta) \\ &= \frac{H\pi r^3}{2EI} \end{aligned}$$

For rotation

$$1 \times \phi = \int \kappa(\theta) \times M_2^*(\theta)\, ds$$

$$\begin{aligned} \phi &= \int_0^{\pi} \frac{Hr\sin\theta}{EI}\left(\frac{1-\cos\theta}{2}\right) r\,d\theta \\ &= \frac{Hr^2}{2EI}\int_0^{\pi}(\sin\theta - \sin\theta\cos\theta)\,d\theta \\ &= \frac{Hr^2}{EI} \end{aligned}$$

Try Questions 5, 6, 7 and 8, Examples Sheet 2/3

3.7 Indeterminate Beam and Frame Structures

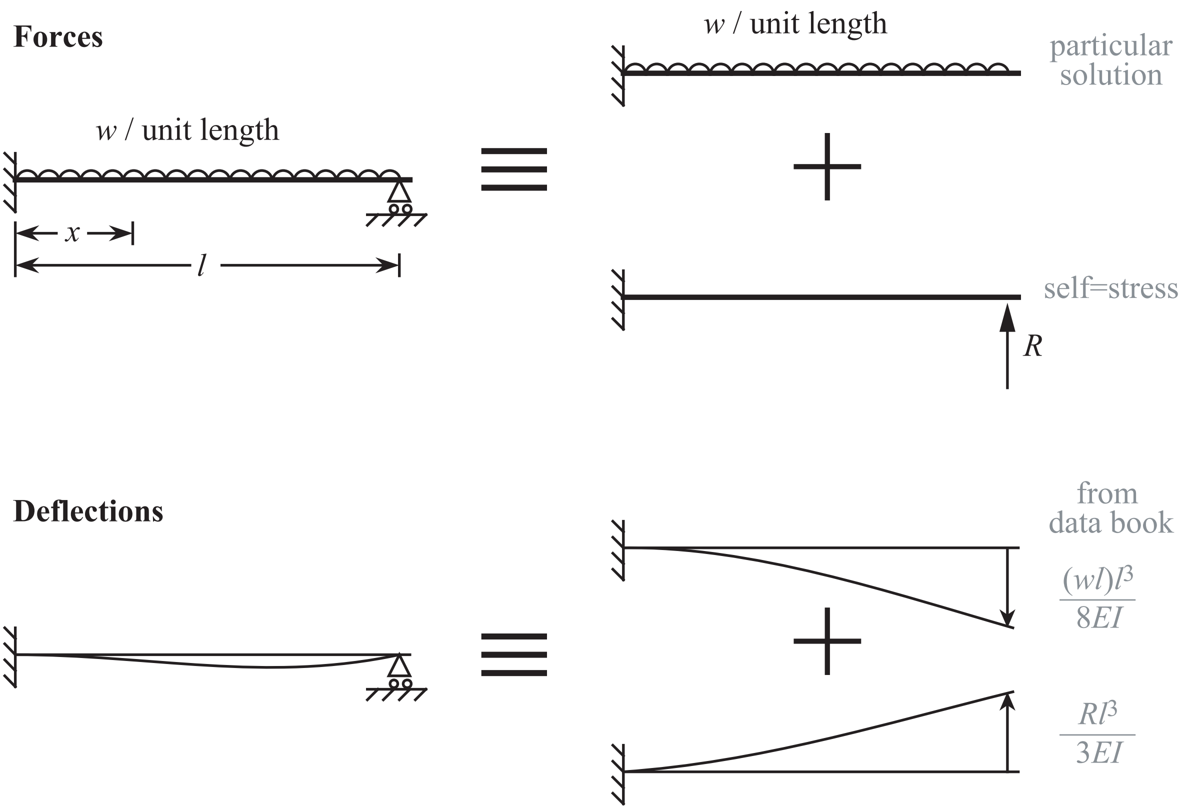
This section will extend the methods described for indeterminate pin-jointed truss structures to beam and frame structures.

3.7.1 Example — A Propped Cantilever

A propped cantilever has one support above the number needed to make it statically determinate.

Statically indeterminate pin-jointed structures can be made determinate by removing bars. We cannot do this here; instead, we can make it determinate by removing the end support, leaving the statically determinate system analysed in Section 3.6. Then, we reimpose the force we have removed, and make it large enough that the displacement at the end is zero — a compatibility condition. Because we only have to take away one support, the structure has only one redundancy or indeterminacy to solve for.

We consider the loading as the superposition of two loading cases:



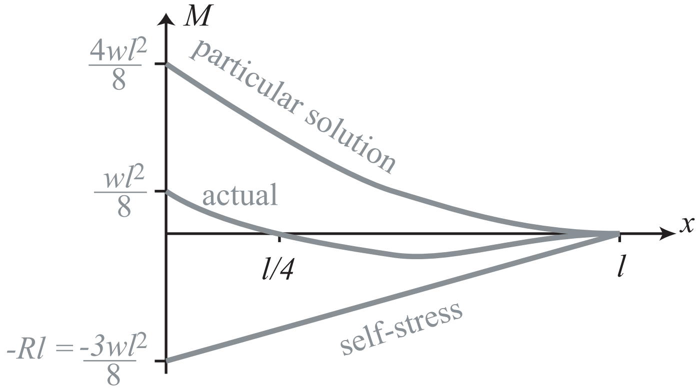
To impose compatibility at the support

$$\frac{Rl^3}{3EI} = \frac{wl^4}{8EI}$$

$$R = \frac{3wl}{8}$$

Having solved for the redundancy, the rest of the solution can proceed as normal. We can find the rest of the internal forces by equilibrium, all the curvatures from the elastic law, and any displacements from compatibility. We have already guaranteed that the displacement at the end of the cantilever is zero by our choice of the value of R.

Bending Moments



3.8 Identifying Redundancies in Beam and Frame Structures

To analyse indeterminate structures, we need to initially reduce them to determinate structures. For pin-jointed structures, we could always do this by removing bars. For beam and frame structures the problem is more difficult. Now, there are various ways of making the structure determinate — cutting the structure, adding pins, removing supports. Before we look at some examples of finding redundancies, we need to be able to recognise a statically determinate structure.

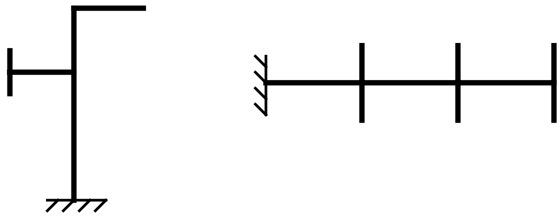
An alternative with many advantages would be to follow a formal equilibrium matrix scheme. However, we will not cover this for beam and frame structures because of lack of time.

3.8.1 Examples of Determinate Structures

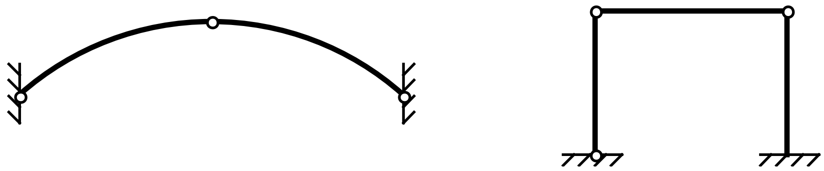
1. A Cantilever

2. A Simply-Supported Beam

3. A Tree Structure



4. A Three-Pinned Arch



One way of telling that a structure is statically determinate is that the removal of one more support, or the addition of one more pin, will cause the structure to become a mechanism. Note that the number of redundancies has nothing to do with the loading that is on the structure. Whatever loading was applied, you could calculate the bending moment, shear force and tension anywhere in the structure simply from equilibrium.