Part IB Paper 6: Information Engineering

LINEAR SYSTEMS AND CONTROL

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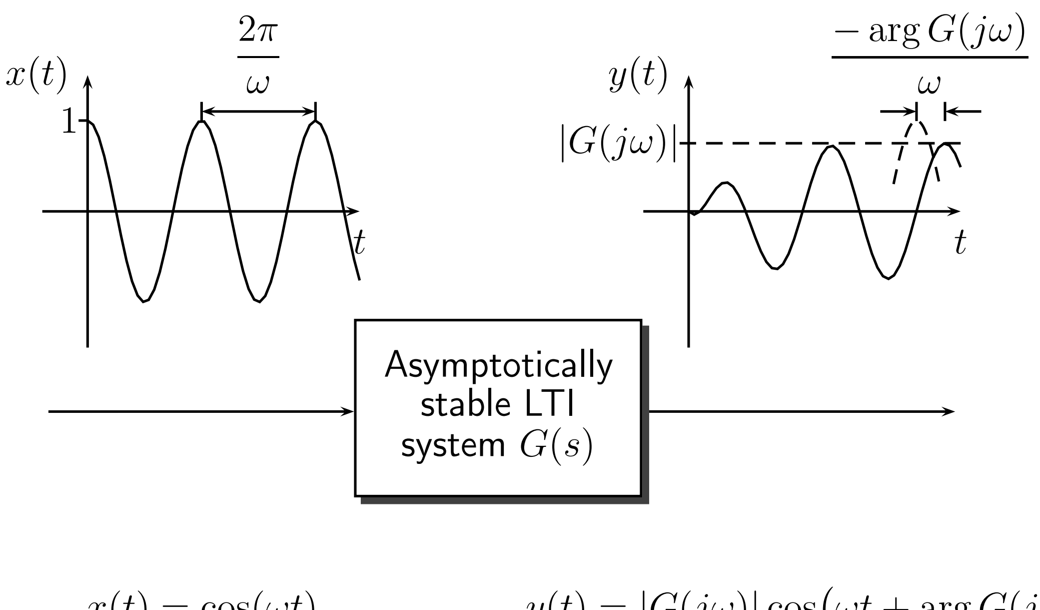
HANDOUT 4

“The Frequency Response $G(j\omega)$”



$x(t) = \cos(\omega t)$ $\qquad$ $y(t) = |G(j\omega)| \cos\big(\omega t + \arg G(j\omega)\big)$ +starting transient

Summary

If a pure sinusoid is input to an asymptotically stable LTI system, then the output will also settle down, eventually, to a pure sinusoid. This *steady-state* output will have the same frequency as the input but be at a different amplitude and phase. The dependence of this amplitude and phase on the frequency of the input is called the *frequency response* of the system.

In this handout we shall:

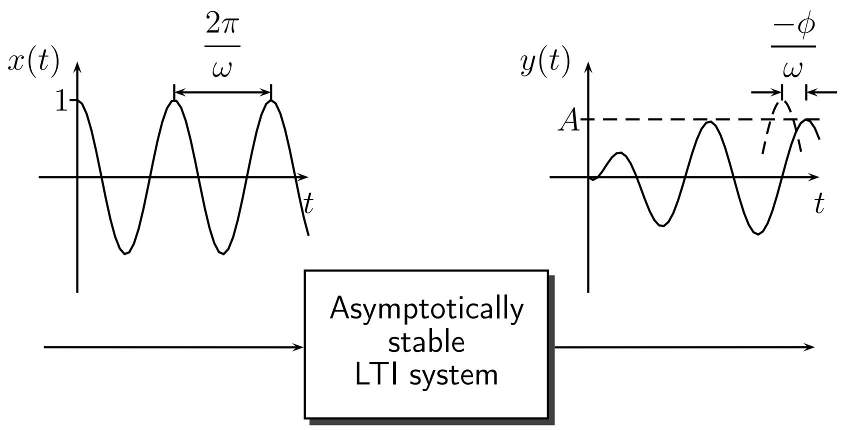
- Show how the frequency response can be derived from the transfer function.

  (by substituting $j\omega$ for s)

- Study how the frequency response can be represented graphically. (using the Bode diagram)

Contents

4.1 What is the Frequency Response?



$x(t) = \cos(\omega t)$

$y(t) = \quad A\cos\big(\omega t + \phi\big)$
+starting transient

If the input to an asymptotically stable LTI system is a pure sinusoid then the steady state output will also be a pure sinusoid, of the same frequency as the input but at a different amplitude and phase.

How do we find A, the gain, and ϕ, the phase shift ?

Our analysis in section 4.1 will be as follows:

(i) We will first use complex number approaches studied in part IA to illustrate that for a system with transfer function $G(s)$ that has as input a sinusoid, the output is also a sinusoid with the same frequency. Furthermore, the gain A, and phase shift ϕ are given by

$$A = |G(j\omega)|,$$
$$\phi = \angle G(j\omega)$$

(ii) We will then show a more general and practically relevant result. We will show that even when the system is initially at rest and then a sinusoidal input is applied, then as $t \to \infty$ the output will be a sinusoid, with gain and phase shift given by the same expressions.

(iii) This result will be initially shown for linear systems described by differential equations. It will then be generalised to any linear system.

How does this relate to Part IA? Consider again a system with input u and output y, If

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = a\frac{du}{dt} + bu.$$

then we can use the usual trick, letting

$$u = e^{j\omega t}$$

so that

$$\Re(u(t)) = \cos(\omega t).$$

We will find the response to the input $\cos(\omega t)$ by taking the real part of y, the response to $u = e^{j\omega t}$.

To find the solution y, we assume it takes the form

$$y(t) = Ye^{j\omega t} \tag{4.1}$$

for some complex number $Y = |Y|e^{j\arg Y}$, so that

$$\Re(y(t)) = \Re(|Y|e^{j\arg Y}e^{j\omega t}) = |Y|\cos(\omega t + \arg Y).$$

Substituting (4.1) into the differential equation, and noting that

$$\frac{dy}{dt} = [j\omega]Ye^{j\omega t}, \quad \frac{d^2y}{dt^2} = [j\omega]^2Ye^{j\omega t}, \quad \text{etc}$$

we obtain

$$Y[j\omega]^2e^{j\omega t} + \alpha Y[j\omega]e^{j\omega t} + \beta Ye^{j\omega t} = a[j\omega]e^{j\omega t} + be^{j\omega t}$$

or

$$Y = \frac{a[j\omega] + b}{[j\omega]^2 + \alpha[j\omega] + \beta}$$

Note that the transfer function from $\bar{u}(s)$ to $\bar{y}(s)$ is given by

$$G(s) = \frac{as + b}{s^2 + \alpha s + \beta}$$

So, it would appear that $Y = G(j\omega)$, suggesting:

ANSWER: If system has the transfer function $G(s)$, then

$$A = |G(j\omega)|, \qquad \phi = \arg G(j\omega)$$

You've done this in Linear Circuits, Maths and Mechanical Vibrations in Part IA, so I'm assuming that you are familiar with these arguments! The notation above is as used in Maths and Mechanical Vibrations. In Linear Circuits $\tilde{y}$ was used to represent a complex phasor, rather than the Y above.

Example: **The Capacitor** is described by the differential equation

$$i = C\frac{dv}{dt}$$

so $\bar{i}(s) = Cs\bar{v}(s)$ in the absence of initial conditions giving the transfer function

$$\frac{\bar{v}(s)}{\bar{i}(s)} = \frac{1}{sC}.$$

So, the frequency response of a capacitor (from current to voltage) is $\frac{1}{j\omega C}$, which equals its impedance, $\tilde{v}/\tilde{i}$. The advantage of transfer functions is that they can be used to deduce the response to all possible inputs, whereas the impedance can only be used for sinusoidal (ac) signals, providing the change in their amplitude and phase.

4.1.1 Derivation of gain and phase shift

The previous argument is not the whole story though - we have assumed that the input $u = \cos(\omega t)$ has been present since the beginning of time, and have only shown that $y = A\cos(\omega t + \phi)$ is a *possible* response. What if the system is at rest until $t = 0$, and then a sinusoid is applied? Take an *asymptotically stable* system with input $\bar{u}(s)$ output $\bar{y}(s)$ and rational transfer function $G(s) = \frac{n(s)}{d(s)}$, so

$$\bar{y}(s) = G(s)\bar{u}(s).$$

Let $u(t) = e^{j\omega t}$, so $\bar{u}(s) = \frac{1}{s-j\omega}$ then, since $G(s)$ can't have a pole at $s = j\omega$,

$$\bar{y}(s) = G(s)\frac{1}{s-j\omega} = \frac{\lambda_1}{s-p_1} + \frac{\lambda_2}{s-p_2} + \cdots + \frac{\lambda_n}{s-p_n} + \frac{\lambda_0}{s-j\omega}$$

How can we find λ_0 without calculating all the other terms?

Ans: Cover-up rule.

Multiply both sides by $(s - j\omega)$ to obtain:

$$G(s) = \frac{\lambda_1(s-j\omega)}{s-p_1} + \frac{\lambda_2(s-j\omega)}{s-p_2} + \cdots + \frac{\lambda_n(s-j\omega)}{s-p_n} + \lambda_0$$

$$\implies \lambda_0 = G(j\omega)$$

$$\implies y(t) = \underbrace{\lambda_1 e^{p_1 t} + \lambda_1 e^{p_2 t} + \cdots + \lambda_n e^{p_n t}}_{\boldsymbol{y_{\mathrm{tr}}(=\mathrm{CF})}} + \underbrace{G(j\omega)e^{j\omega t}}_{\boldsymbol{y_{\mathrm{ss}}(=\mathrm{PI})}}$$

(with the obvious modifications if any poles are repeated).

Since all the p_k, the poles of $G(s)$, have a negative real part, then $y_{\mathrm{tr}}(t) \to 0$ as $t \to \infty$ – leaving the steady-state response y_{ss}. The steady-state response to the input $u(t) = \cos(\omega t)$ is given by the real part of this:

$$\begin{aligned}\Re(y_{\mathrm{ss}}(t)) &= \Re\left(e^{j\omega t}G(j\omega)\right) \\ &= \Re\left(|G(j\omega)|e^{j\left(\omega t + \arg G(j\omega)\right)}\right) \quad \text{as } \boldsymbol{z = |z|e^{j\arg z}} \\ &= \underbrace{|G(j\omega)|}_{\text{gain } \boldsymbol{A}} \cos\left(\omega t + \underbrace{\arg G(j\omega)}_{\text{phase shift } \boldsymbol{\phi}}\right)\end{aligned}$$

Example:

Consider a system with transfer function

$$G(s) = \frac{1}{s^2 + 0.1s + 2}$$

and an input

$$x(t) = \cos(0.5t)$$

(a sinusoid at 0.5rad/s, or $\mathbf{0.5/(2\pi) = 0.0796}$ **Hz**)

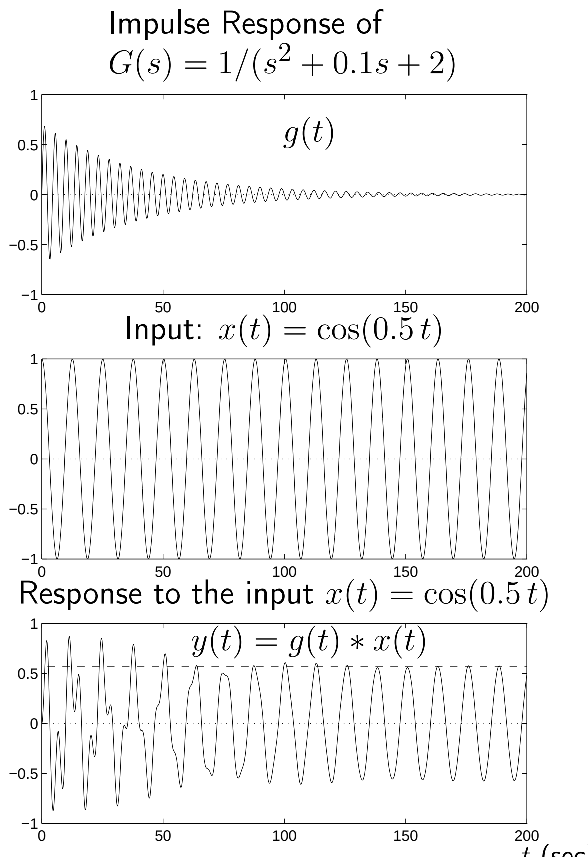
So, $\omega = 0.5$ and

$$G(j\omega) = \frac{1}{(0.5j)^2 + 0.1(0.5j) + 2} = \frac{1}{-0.25 + 0.05j + 2}$$
$$= \frac{1}{1.75 + 0.05j} = \frac{1}{1.7507e^{0.0286j}} = 0.571e^{-0.0286j}$$
$$= \mathbf{0.571 \angle -1.64^\circ}$$

The following figures show the impulse response of this system, the input $x(t) = \cos(0.5t)$, and the response to this input.

Note how the output initially contains a component near the resonant frequency of the system $\omega_n = \sqrt{2}$ (0.225 Hz) but that this quickly decays leaving only a sinusoid at the frequency of the input $\omega = 0.5$ (0.08 Hz) (and at an amplitude of 0.571).

(The phase lag of 1.64° is too small to be seen on this diagram)



4.1.2 Frequency response of an arbitrary linear system

It's not just systems described by ODEs which have a frequency response, *any* asymptotically stable system, with impulse response $g(t)$ and transfer function $G(s) = \mathcal{L}g(t)$, does. (See Q5 on Examples Paper 2 for an example). Consider the response of such a system to a sinusoidal input beginning suddenly at $t = 0$.

Let

$$x(t) = \Re(e^{j\omega t}) \ \big(= \cos\omega t\,\big)$$

then

$$y(t) = \int_0^t \underbrace{\Re\big(e^{j\omega(t-\tau)}\big)}_{x(t-\tau)} g(\tau)\,d\tau = \Re\left(\int_0^t e^{j\omega(t-\tau)} g(\tau)\,d\tau\right)$$

$$= \underbrace{\Re\left(\int_0^\infty e^{j\omega(t-\tau)} g(\tau)\,d\tau\right)}_{y_{\mathsf{ss}}(t)} - \underbrace{\Re\left(\int_t^\infty e^{j\omega(t-\tau)} g(\tau)\,d\tau\right)}_{y_{\mathsf{tr}}(t)}$$

Consider the second term:

$$|y_{\mathsf{tr}}(t)| \le \int_t^\infty \underbrace{|e^{j\omega(t-\tau)}|}_{1}\,|g(\tau)|\,d\tau = \int_t^\infty |g(\tau)|\,d\tau$$

but $\displaystyle\int_0^\infty |g(\tau)|\,d\tau$ is finite, and hence $\lim_{t\to\infty} |y_{\mathsf{tr}}(t)| = 0$.

$y_{\mathsf{tr}}(t)$ is called the *transient part of the response* , as it decays to zero, eventually (and so is transitory in nature).

Now consider the first term:

$$y_{\mathsf{ss}}(t) = \Re\left(e^{j\omega t} \underbrace{\int_0^\infty e^{-j\omega\tau} g(\tau)\,d\tau}_{G(s)\big|_{s=j\omega}}\right) = \Re\left(e^{j\omega t} G(j\omega)\right)$$

$$= \Re\left(|G(j\omega)| e^{j\left(\omega t + \angle G(j\omega)\right)}\right) = |G(j\omega)| \cos\big(\omega t + \angle G(j\omega)\big)$$

$y_{\mathsf{ss}}(t)$ is called the *steady-state* part of the response, as it is the part that remains when the transients have decayed.

The analysis in section 4.1.1 was for linear systems described by differential equations. This is reflected in the fact that $G(s)$ was assumed to be rational in s (polynomial divided by polynomial).

In this section it is proved that the same result holds (for the response when the input is a sinusoid), for any linear system. In particular, no assumption on the transfer function $G(s)$ is made and only use the fact that the output $y(t)$ is the convolution of the input $u(t)$ with the impulse response $g(t)$.

It should be noted that we are considering asymptotically stable systems, which implies that $\int_0^\infty |g(\tau)| d\tau$ is finite (from the definition of asymptotic stability given in the previous handout).

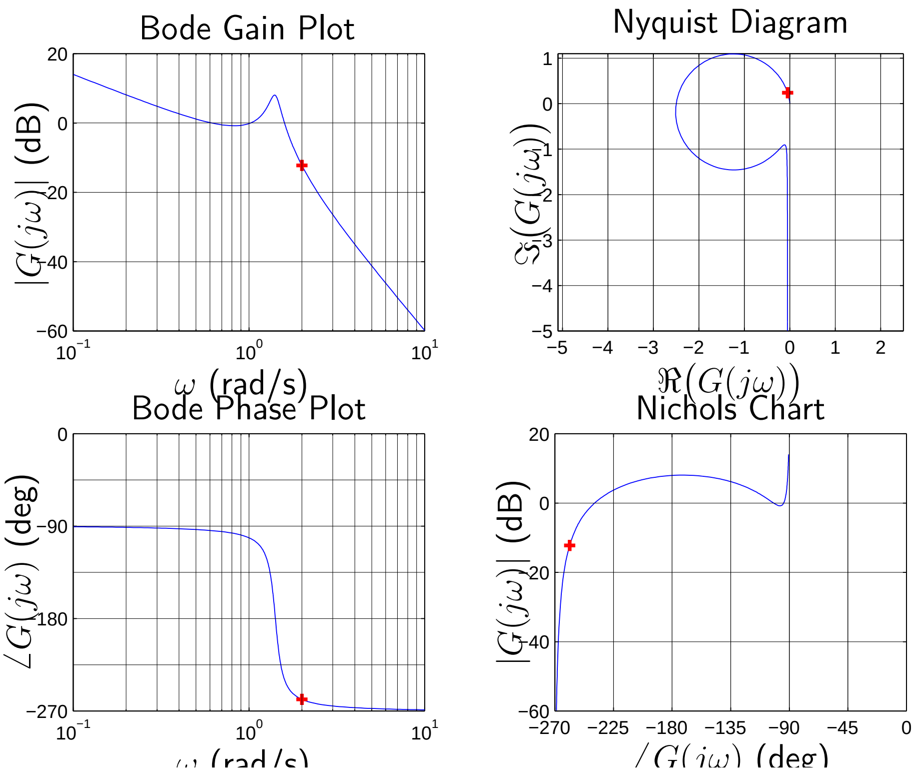
4.2 Plotting the frequency response

The frequency response $G(j\omega)$ is a *complex-valued* function of frequency ω. At each frequency ω, the complex number $G(j\omega)$ can be represented either in terms of its real and imaginary parts, or in terms of its gain (magnitude) and phase (argument).

There are a number of common ways of representing this information graphically:

- The **Bode** Diagram: Two separate graphs, one of $|G(j\omega)|$ vs ω (on log-log axes), and one of $\angle G(j\omega)$ (lin axis) vs ω (log axis).
- The **Nyquist** Diagram: One single parametric plot, of $\Re(G(j\omega))$ against $\Im(G(j\omega))$ (on linear axes) as ω varies.
- The **Nichols** Diagram: One single parametric plot, of $|G(j\omega)|$ (log axis) against $\angle(G(j\omega))$ (lin axis) as ω varies.

Example: $G(s) = \frac{1}{s(s^2+0.2s+2)}$



Each has its use:

The **Bode** diagram is relatively straightforward to sketch to a high degree of accuracy, is compact and gives an indication of the frequency ranges in which different levels of performance are achieved.

The **Nyquist** diagram provides a rigorous way of determining the stability of a feedback system.

The **Nichols** diagram combines some of the advantages of both of these (although is not quite as good in either specific application) and is widely used in industry. *We shall not study the Nichols diagram in this course(!), but the ideas behind its construction will be readily grasped once the two fundamental diagrams that we do study are understood.*

Consider $G(j\omega)$ for $\omega = 2$

$$G(j\omega)\Big|_{\omega=2} = \frac{1}{2j((2j)^2 + 0.2 \times (2j) + 2)} = \frac{1}{2j(-2 + 0.4j)}$$

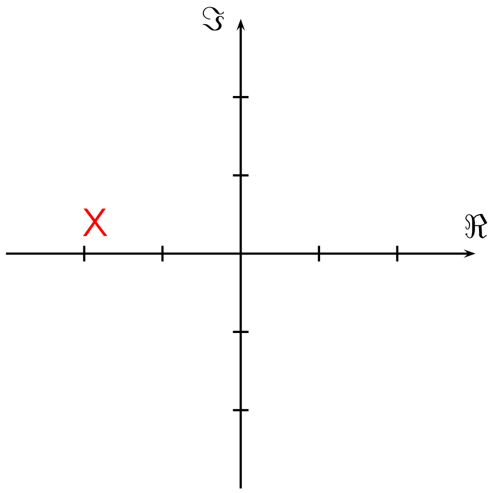
Hence

$$|G(j2)| = \frac{1}{2\sqrt{2^2 + 0.4^2}} = 0.2451$$

and

$$\begin{aligned} \arg G(j2) &= \arg 1 - \arg 2j - \arg(-2 + 0.4j) \\ &= 0 - \pi/2 - \underbrace{2.9442}_{\boldsymbol{\pi - \mathrm{atan}(.4/2)}} = -4.5150 \end{aligned}$$

because



So,

$$G(j2) = 0.2451e^{-4.5150j} = -0.0481 + 0.2404j$$

Also,

$$20\log_{10}|G(j2)| = -12.2dB, \ \angle G(j\omega) = -258.7^\circ$$

The corresponding point has been marked with a cross on each of the previous plots.

4.3 Sketching Bode Diagrams

The bode diagram of $G(s)$ consists of two curves

1. Gain Plot: Gain $|G(j\omega)|$ (log) vs freq ω (log)
2. Phase Plot: Phase $\angle G(j\omega)$ (lin) vs ω (log)

It is straightforward to sketch, and gives a lot of insight.

Basic idea: Consider a transfer function written as a ratio of factorized polynomials e.g.

$$G(s) = \frac{a_1(s)a_2(s)}{b_1(s)b_2(s)}.$$

Clearly

$$\log_{10}|G(j\omega)| = \log_{10}|a_1(j\omega)| + \log_{10}|a_2(j\omega)| - \log_{10}|b_1(j\omega)| - \log_{10}|b_2(j\omega)|,$$

so we can compute the gain curve by simply adding and subtracting gains corresponding to terms in the numerator and denominator. Similarly

$$\angle G(j\omega) = \angle a_1(j\omega) + \angle a_2(j\omega) - \angle b_1(j\omega) - \angle b_2(j\omega)$$

and so the phase curve can be determined in an analogous fashion.

Since a polynomial can always be written as a product of terms of the type

$$K, \quad sT, \quad 1+sT, \quad 1+2\zeta Ts+s^2T^2 \text{ (for } |\zeta|<1\text{, i.e. complex roots)}$$

it is suffices to be able to sketch Bode diagrams for these terms. The Bode plot of a complex system is then obtained by adding the gains and phases of the individual terms.

Note: *Always* rewrite the transfer function in terms of these building blocks before starting to sketch a Bode diagram – you will find it much easier. For example, if the transfer function has a term $(s+a)$ first rewrite this as $a \times (1+s/a)$ and then collect together all the constants that have been pulled out. The transfer functions in Question 6 on Examples Paper 2 are already given in the right form, but you will need to rewrite the transfer function in Question 8.

Example: We would rewrite $G(s) = \dfrac{500(s+3s+2)}{s(s^2+5s+100)} = \dfrac{500(s+1)(s+2)}{s(s^2+5s+100)}$ as

$$G(s) = \left(\frac{10}{s}\right) \times \frac{(1+s)(1+s/2)}{(1+0.05s+s^2/100)}$$

and begin by considering each term individually.

4.3.1 Powers of s: $(sT)^k$

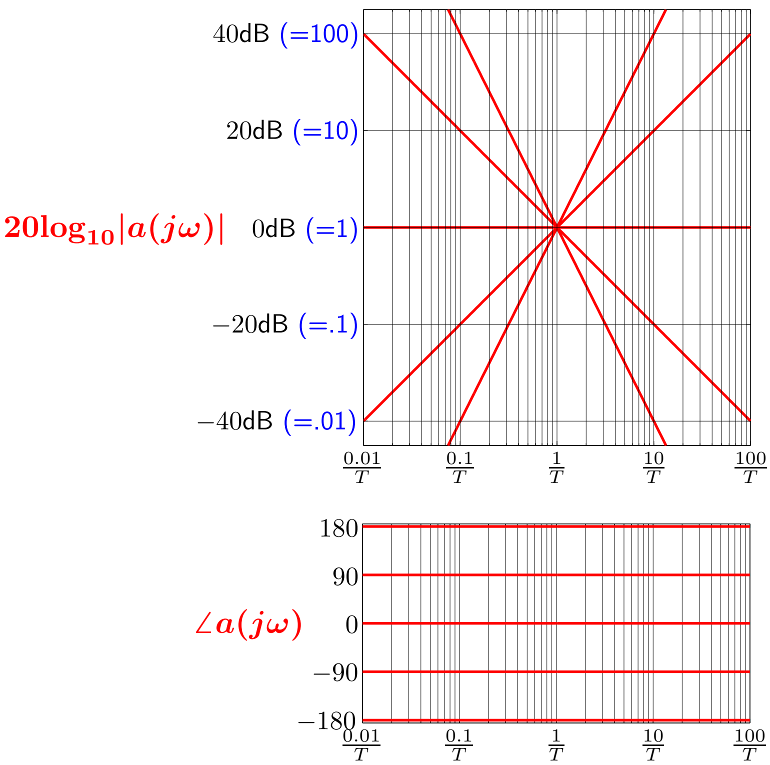
The simplest term in a transfer function is a power of s, which it is convenient to write in the form

$$a(s) = (sT)^k$$

where $k > 0$ if the term appears in the numerator and $k < 0$ if the term is in the denominator. The magnitude and phase of the term are given by

$$\log_{10}|a(j\omega)| = \log_{10}(\omega T)^k = k\log_{10}(\omega T), \ \angle G(j\omega) = 90k^\circ$$

The gain curve is thus a straight line with slope k decades/decade, or $20k$ dB/decade, intersecting the 0dB line at $\omega T = 1$. The phase curve is a constant at $90^\circ \times k$. For $T = 1$, the case when $k = 1$ corresponds to a differentiator, and has a slope 20dB/decade and phase 90°. The case when $k = -1$ corresponds to an integrator and has a slope -20dB/decade and phase -90°.



The first figure on the left illustrates the *Gain plot* of $a(s) = (sT)^k$ for $k = 0, 1, 2, -2, 1$. Note that the slope is $20k$ dB/decade.

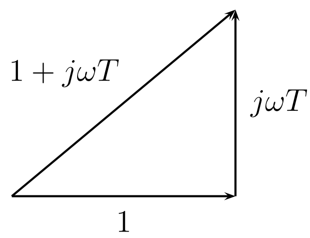
The second figure illustrates the corresponding *Phase plot*. Note that $\angle a(j\omega) = 90k^\circ$

4.3.2 First Order Terms: $(1 + sT)$

Bode plot of $\boxed{G(s) = (1 + sT)}$ (for $T > 0$)

... replace s by $j\omega$ to get $\boxed{G(j\omega) = (1 + j\omega T)}$

$$
\begin{aligned}
\Longrightarrow\ |G(j\omega)| &= \underbrace{|1 + j\omega T|}_{\sqrt{1+\omega^2T^2}} \\
\angle G(j\omega)| &= \underbrace{\angle(1 + j\omega T)}_{\text{atan}(\omega T)}
\end{aligned}
$$



Asymptotes:

$\underline{\omega \to 0}$: (i.e. $\omega \ll 1/T$)

$$
\begin{aligned}
20\log_{10}|G(j\omega)| &\to 20\log_{10}1 = 0 \\
\angle G(j\omega) &\to \angle 1 = 0
\end{aligned}
$$

$\underline{\omega \to \infty}$: (i.e. $\omega \gg 1/T$)

$$
\begin{aligned}
20\log_{10}|G(j\omega)| &\to 20\log_{10}|j\omega T| \\
&= 20\log_{10}\omega - 20\log_{10}1/T
\end{aligned}
$$

(which is a straight line with slope=20dB/decade and x-axis (i.e. 0db) intercept at $\omega = 1/T$)

$$\angle G(j\omega) \to \angle j\omega T = 90^\circ$$

At $\omega = 1/T$, we get

$$
\begin{aligned}
20\log_{10}|G(j\omega)| &= 20\log_{10}|1 + j| \\
&= 20\log_{10}\sqrt{2} \quad \text{(3dB)} \\
\angle G(j\omega) &= \angle(1 + j) = 45^\circ
\end{aligned}
$$

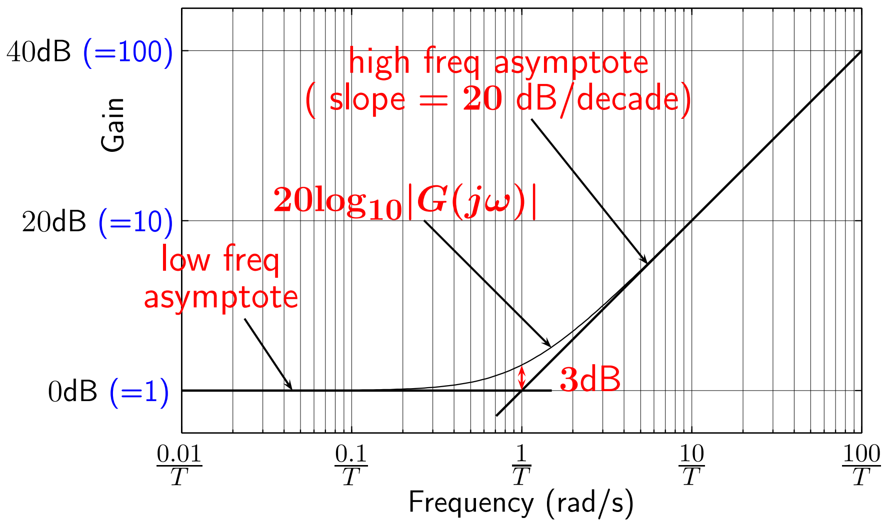
For the Gain plot the main features illustrated are the two asymptotes when $\omega \to 0$ (i.e. $\omega \ll 1/T$) and $\omega \to \infty$ (i.e. $\omega \gg 1/T$). These asymptotes intersect at the $\omega = 1/T$.

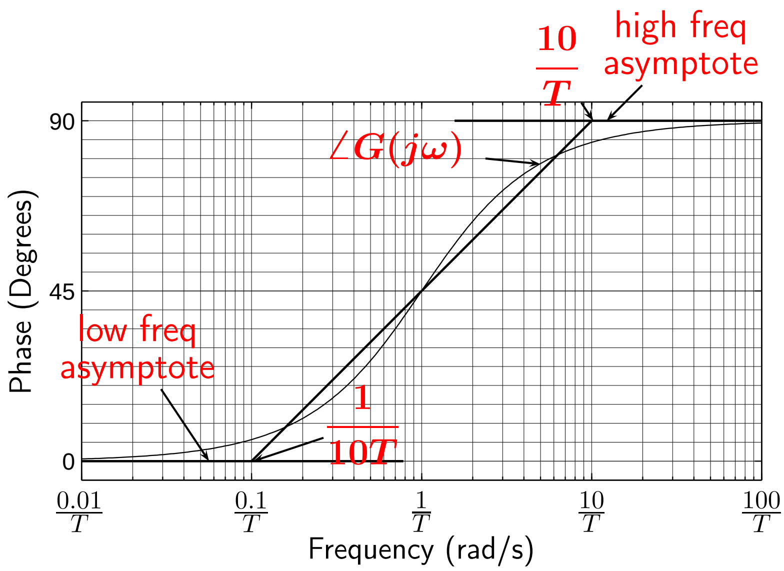
Note also that the low frequency asymptote is horizontal and the high frequency asymptote has slope 20dB/decade.

The phase $\angle G(j\omega)$ increases from 0 to 90° as ω increases from 0 to ∞ with the phase being 45° at $\omega = 1/T$.

See also the graphical illustration in the next page.

Bode diagram of $G(s) = (1 + sT)$:





In both the Gain and Phase plots we illustrate the two asymptotes when $\omega \to 0$ and $\omega \to \infty$, respectively.

In the gain plot the asymptotes intersect at $\omega = 1/T$ (follows from the expressions derived for the asymptotes in the previous page). In the phase plot the phase is $45°$ at this frequency.

To facilitate the drawing of the phase plot a straight line that approximates the plot can be drawn that passes through the point with $\omega = 1/T$, $\angle G(j\omega) = 45°$, it intersects the high frequency asymptote at $\omega = 10/T$, and the low frequency asymptote at $\omega = 1/T$ (see the phase diagram on the left).

4.3.3 Second order terms: $(1 + 2\zeta sT + s^2T^2)$

Bode plot of $\boxed{G(s) = \dfrac{1}{1 + 2\zeta sT + s^2T^2}}$ (for $T > 0, 0 \le \zeta \le 1$)

... replace s by $j\omega$ to get $G(j\omega) = \dfrac{1}{1 + 2\zeta j\omega T - \omega^2T^2}$

$$\implies 20\log_{10}|G(j\omega)| = -20\log_{10}\left|1 - \omega^2T^2 + 2\zeta j\omega T\right|$$
$$\angle G(j\omega) = -\angle\left(1 - \omega^2T^2 + 2\zeta j\omega T\right)$$

Asymptotes:

$\underline{\omega \to 0}$: (i.e. $\omega \ll 1/T$)

$$20\log_{10}|G(j\omega)| \to -20\log_{10}1 = 0$$
$$\angle G(j\omega) \to -\angle 1 = 0$$

$\underline{\omega \to \infty}$: (i.e. $\omega \gg 1/T$)

$$\begin{aligned} 20\log_{10}|G(j\omega)| &\to -20\log_{10}|-\omega^2T^2| \\ &= -40\log_{10}\omega T \\ &= 40\log_{10}1/T - 40\log_{10}\omega \end{aligned}$$

(which is a straight line with slope$=-40$dB/decade and x-axis (i.e. 0db) intercept at $\omega = 1/T$)

$$\angle G(j\omega) \to -\angle - \omega^2T^2 = -180^\circ$$

At $\omega = 1/T$, we get

$$\begin{aligned} 20\log_{10}|G(j\omega)| &= -20\log_{10}|2\zeta j| \\ &= 20\log_{10}\frac{1}{2\zeta} \\ \angle G(j\omega) &= -\angle(2\zeta j) = -90^\circ \end{aligned}$$

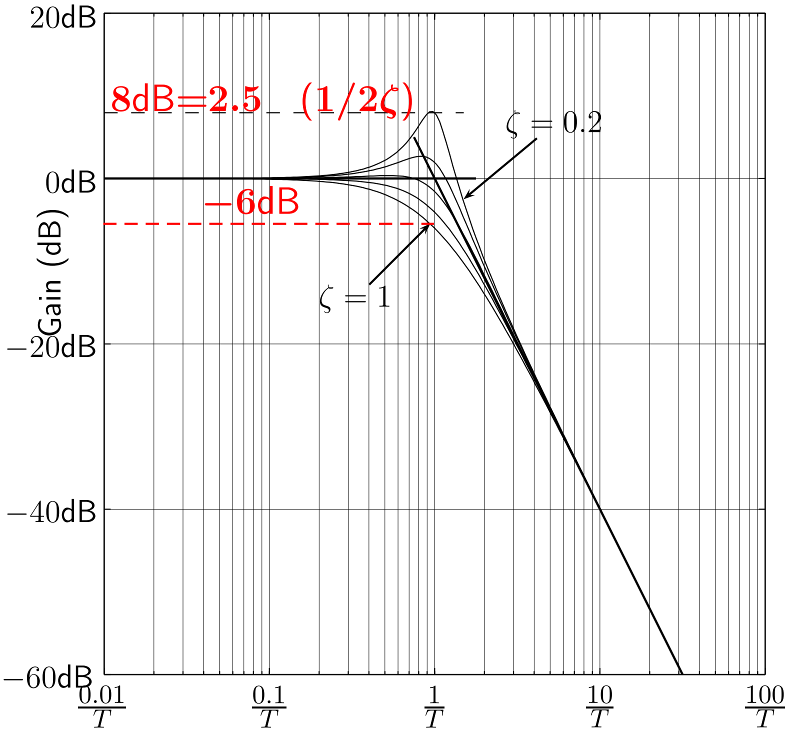
As in the case of first order terms a main feature we illustrate in the Bode plot of second order terms, in both the Gain plot and the Phase plot, are the asymptotes when $\omega \to 0$ and $\omega \to \infty$, respectively.

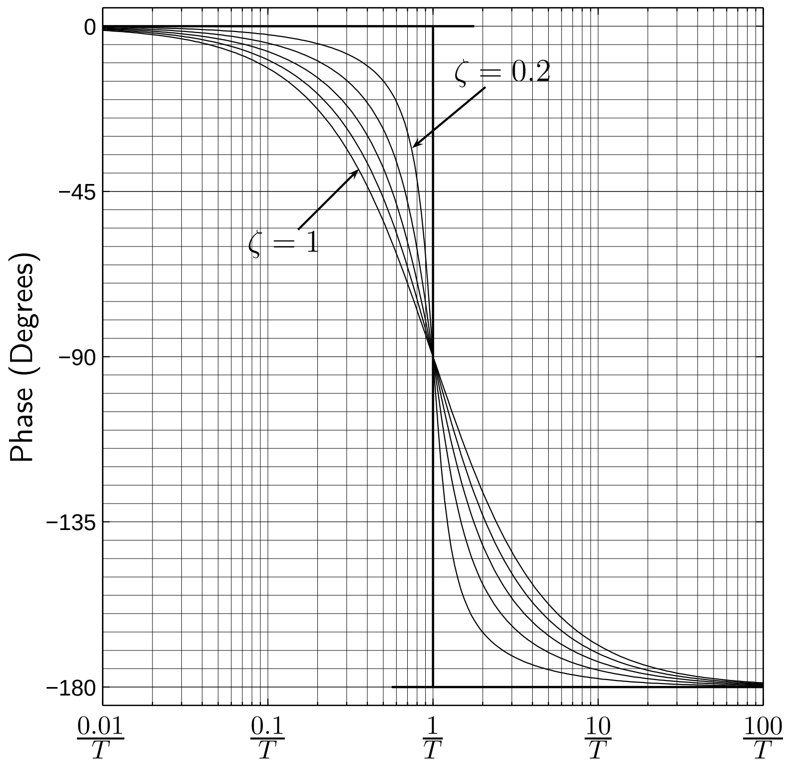
As shown on the left the high frequency asymptote has slope -40dB/decade (in contrast to 20dB/decade in the first order term previously analysed).

Also the change in phase as ω increases from 0 to ∞ is $-180°$, with the phase being $-90°$ at $\omega = 1/T$.

Furthermore, the gain at $\omega = 1/T$ depends on the damping ratio ζ, and tends to infinity as $\zeta \to 0$.

See also the graphical illustration in the next page.





The figures on the left illustrate the Bode plot of the second order term

$$G(s) = \frac{1}{1 + 2\zeta sT + s^2T^2}$$

for different values of ζ.

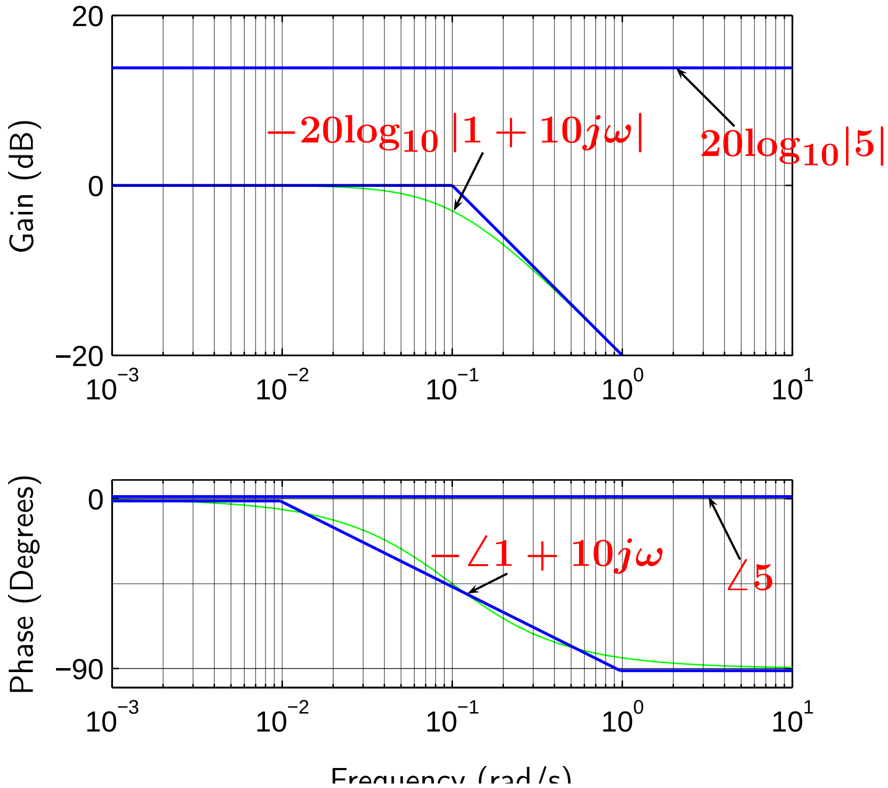
It should be noted that an overshoot is observed in the Gain plot for $\zeta < 1$.

4.3.4 Examples

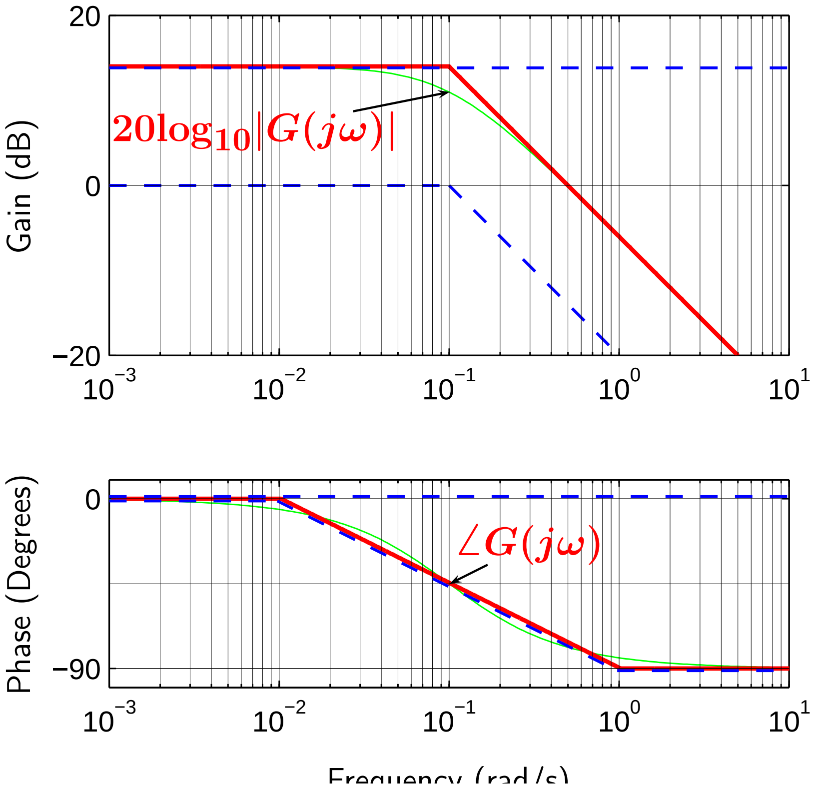
Example 1: $G(s) = \dfrac{5}{1+10s} \quad (K = 5,\ 1/T = 0.1)$

$$20\log_{10}|G(j\omega)| = 20\log_{10}5 - 20\log_{10}|1 + j\omega/0.1|$$
$$\angle G(j\omega)| = \angle 5 - \angle(1 + j\omega/0.1)$$

We now produce a sketch of the Bode diagram in two stages (this is for clarity – all these constructions would normally appear on one pair of graphs). First we plot the asymptotes and approximations to the true curves for the individual terms (The exact values are used for the plots here.)



In the next pair of diagrams, the contributions from the individual terms (now shown as dashed lines) have been added to give the Bode diagram of $G(s)$ (this has been done for both the asymptotes and the true gain and phase).



The next example has both a pole and a zero.
It is an example of what is known as a "phase-lead compensator" (will be studied again later in the course).

Example 2: $G(s) = 0.05\,\dfrac{1+10s}{1+s}$

So, $G(j\omega) = 0.05\,\dfrac{1+10j\omega}{1+j\omega}$,

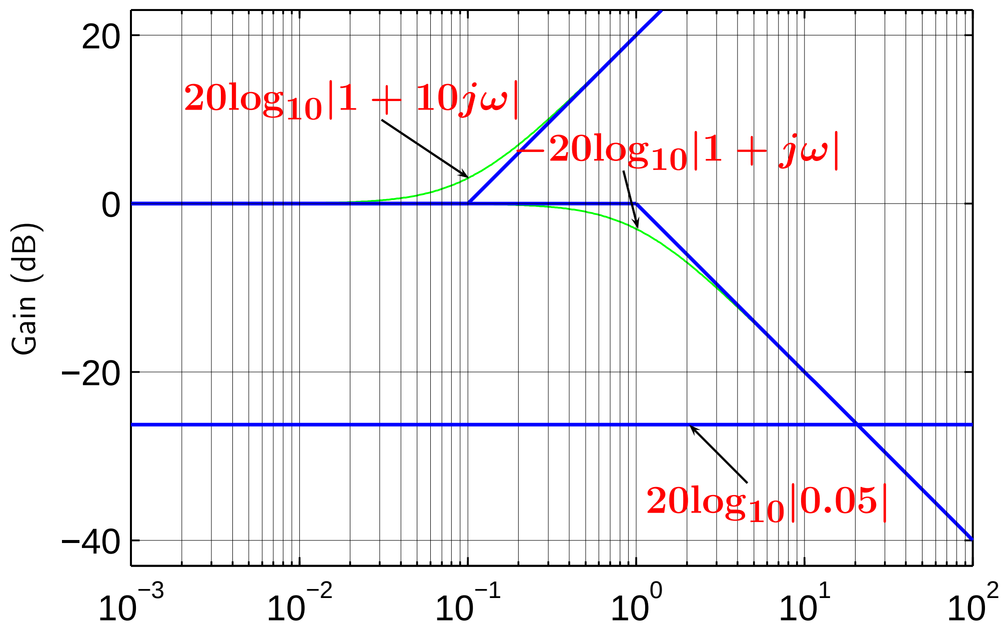
$$\begin{aligned}20\log_{10}|G(j\omega)| = 20\log_{10}0.05 \\ + 20\log_{10}|1+10j\omega| - 20\log_{10}|1+j\omega|\end{aligned}$$

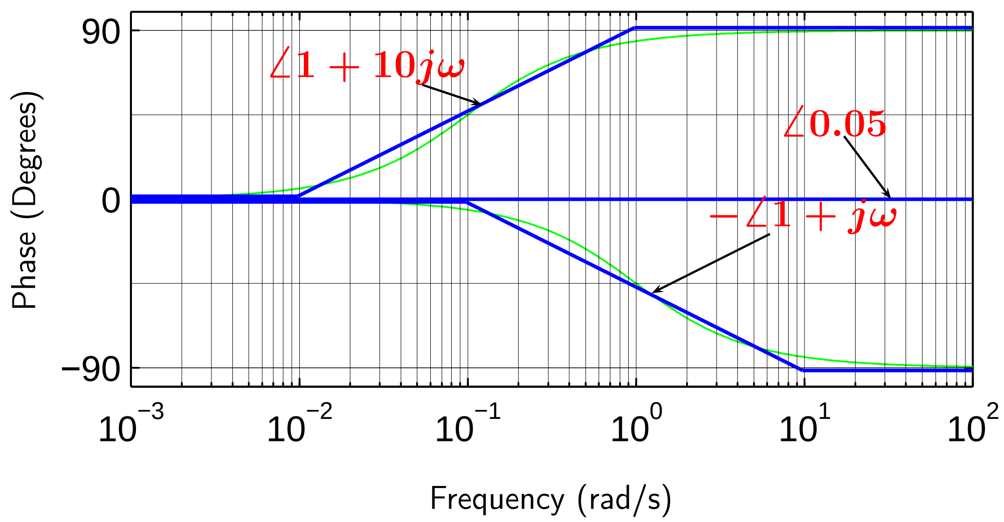
Note: $\mathbf{0.05 = -26}\boldsymbol{dB}$

and

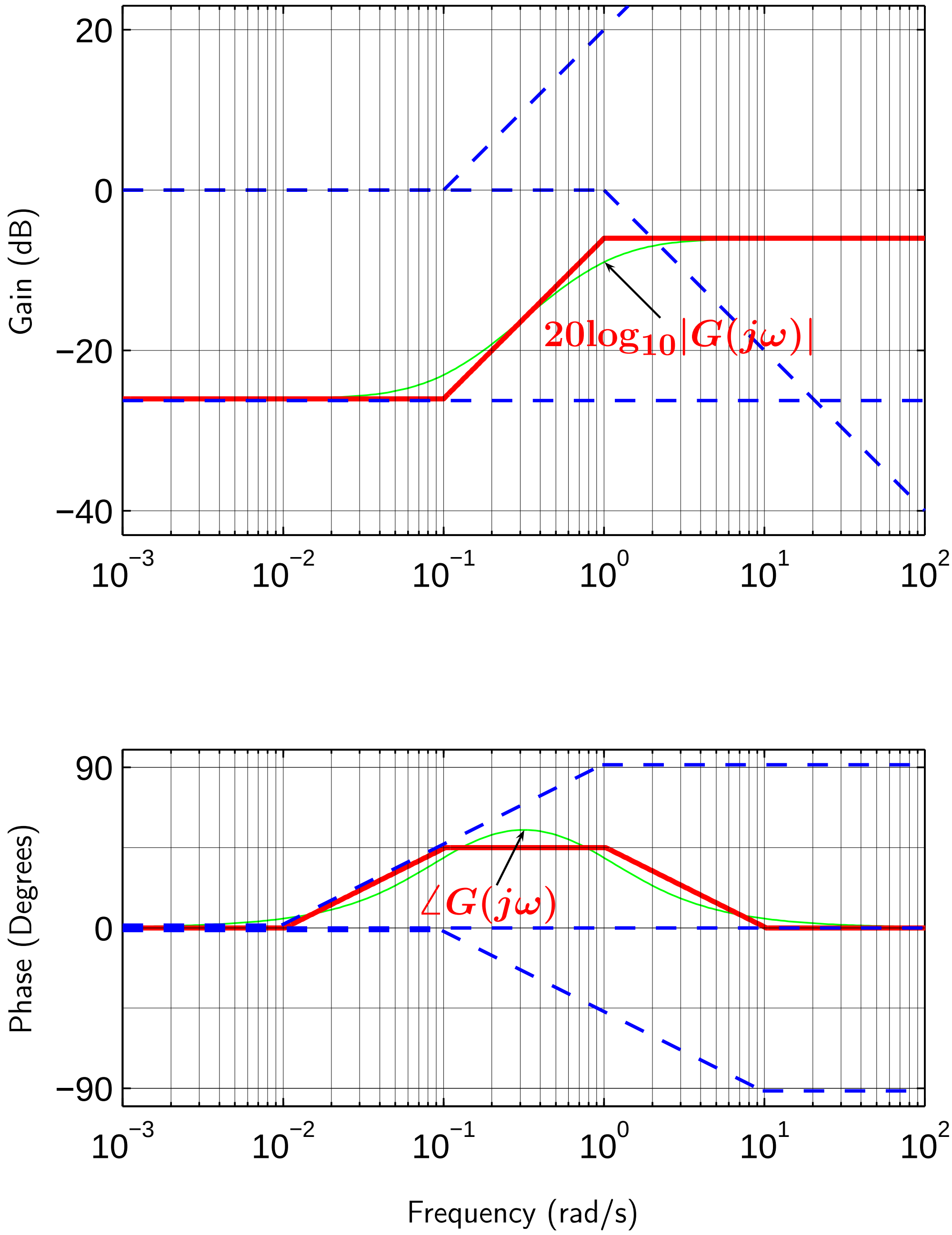
$$\angle G(j\omega) = \underbrace{\angle 0.05}_{\mathbf{0}} + \angle(1+10j\omega) - \angle(1+j\omega)$$

First we draw the individual terms:

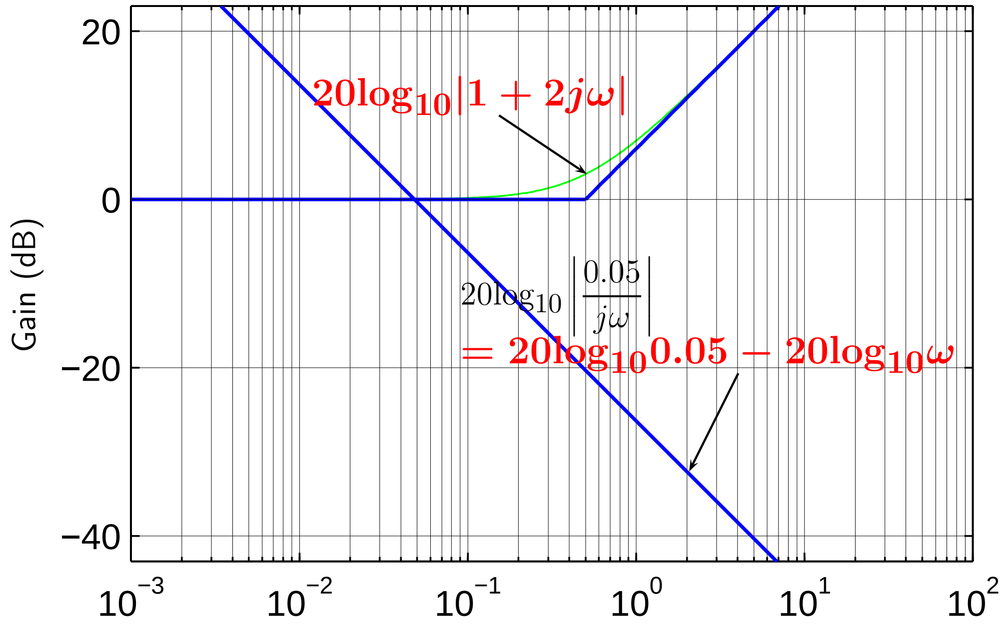


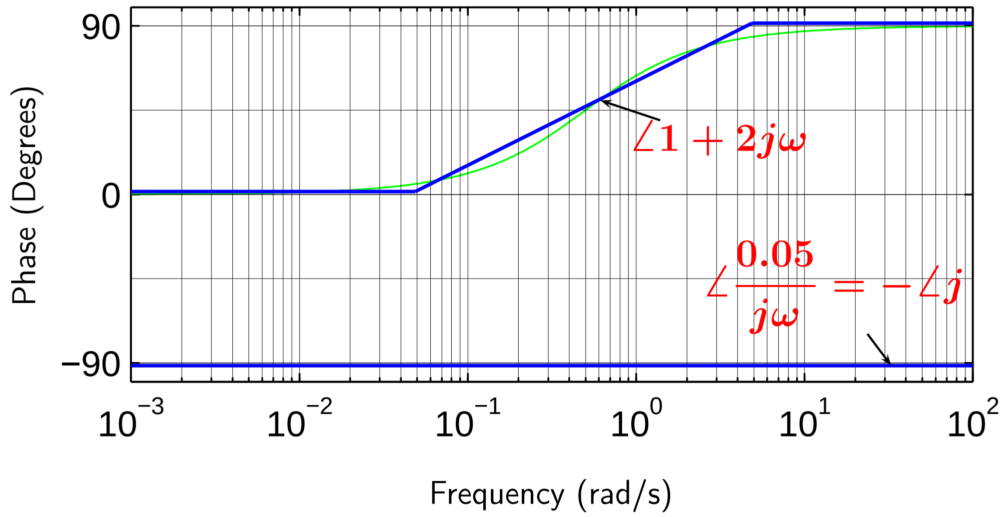


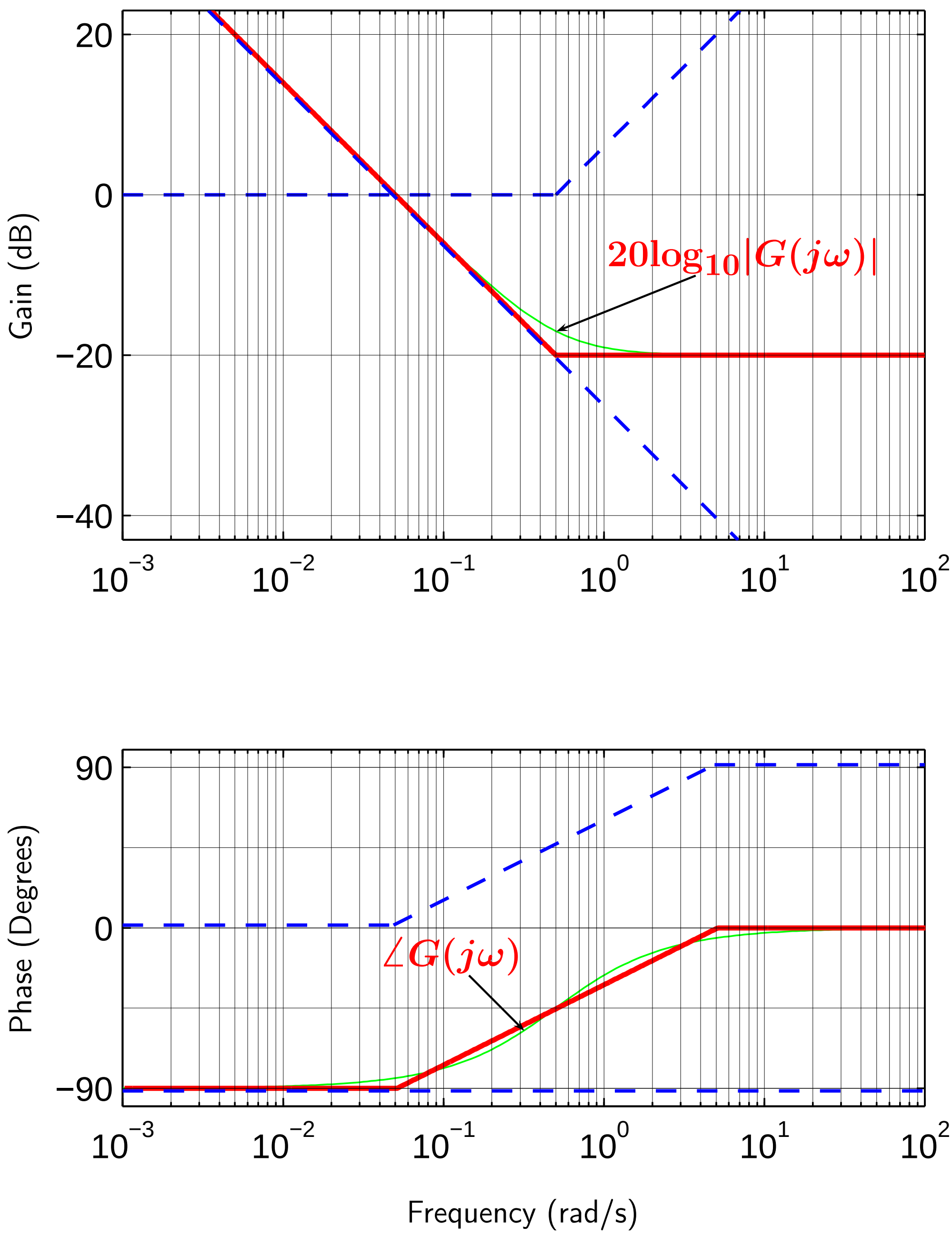
and then we sum them (Note how the phase terms sum to produce a maximum phase advance of only about $55°$.)



Example 3: $G(s) = 0.05\,\frac{1+2s}{s} = \frac{\mathbf{0.05}}{\boldsymbol{s}} \times (\mathbf{1} + \mathbf{2}\boldsymbol{s})$






20
0
−20
−40
Gain (dB)
$20\log_{10}|G(j\omega)|$
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
90
0
−90
Phase (Degrees)
$\angle G(j\omega)$
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)

RHP poles and zeros

$$\angle(1 - j\omega T) = \; - \; \angle(1 + j\omega T)$$

$$20\log_{10} |(1 - j\omega T)| = \; + \; 20\log_{10} |(1 + j\omega T)|$$

That is, if we have a term $(1 - sT)$ instead of $(1 + sT)$ then the gain plot is unchanged but the term's contribution to the phase plot is reversed in sign (so the contribution of a RHP zero to the overall phase diagram is the same as that of a LHP pole at the same location).

Comments on Bode sketching

A alternative technique for Bode diagrams would be to ignore the asymptotes, and just calculate and plot the true gain and phase over a grid of frequencies. This is *not* recommended for a number of reasons. Firstly, a lot more points are required to get the same accuracy (particularly when the diagram is to be used for control system analysis and design, as only a small region is required accurately in this case). Secondly, the structure of the diagram is then lost. A practising control engineer will often prefer a good sketch, with the asymptotes shown, to an accurate computer generated diagram – since this gives a better idea of how things can be changed to improve the behaviour of the controlled system.

When a question asks you to "draw" a Bode diagram (as in the questions on Examples Paper 2) it's really asking you to produce a drawing showing the straight line asymptotes and approximations and a rough approximation to the true gain and phase by rounding the corners appropriately.

4.4 Key Points

- The frequency response is obtained from the transfer function by replacing s with $j\omega$.
- At each frequency ω, $G(j\omega)$ is a complex number whose magnitude gives the gain of the system at that frequency and whose argument gives the phase shift of the system at that frequency.
- The gain and the phase shift are conveniently shown on the Bode diagram.