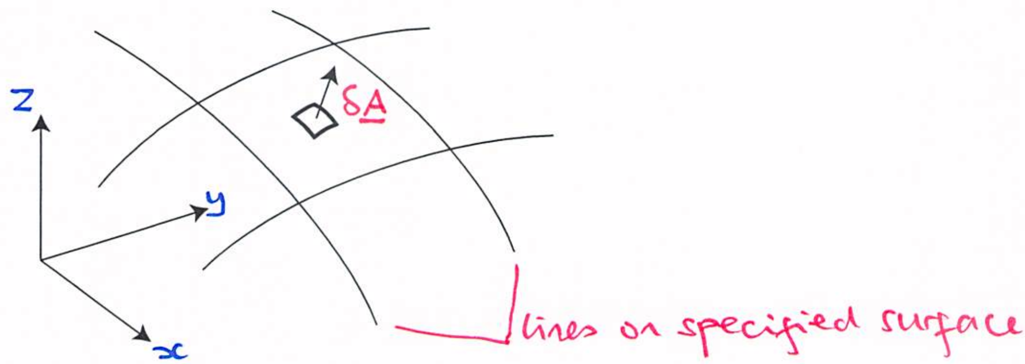
Lecture 8

Surface and Volume Integrals

8.1 Surface integrals

Surface integrals are the extension to 3-D of the scalar double integral $\iint \phi(x,y)dA$. Instead of integrating over an area in the (x,y) plane, a surface integral is the integration of a function over a *specified surface* in 3-D space. To do this, we must define an element of area on the surface in vector form.

The vector quantity $\delta\mathbf{A}$ has magnitude equal to the elemental area, and direction perpendicular to the surface. For closed 3-D surfaces, the convention is that $\delta\mathbf{A}$ points *out* of the enclosed volume.



The result of a surface integral can be a vector or a scalar. Some engineering examples are:

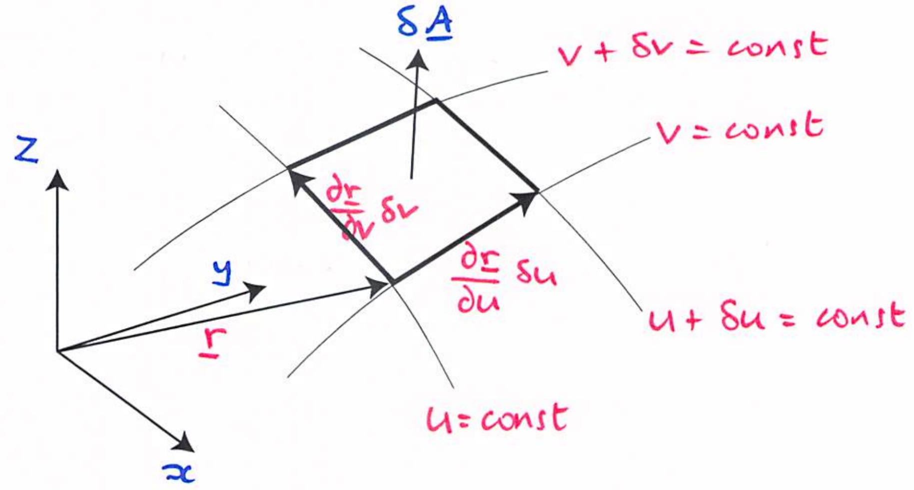
$\iint_S -p\, d\underline{A}$	a vector	Total force on surface S due to pressure (acts normal to surface)
$\iint_S -\underline{f}\, dA$	a vector	Total force on surface S due to some distributed force per unit area $\mathbf{f}$
$\iint_S \rho \underline{V}\cdot d\underline{A}$	a scalar	Total mass flowrate of fluid passing through surface S
$\iint_S -p(\underline{r}\times d\underline{A})$	a vector	Total moment about the origin caused by the pressure acting on surface S

Often the main difficulty in performing a surface integral comes in evaluating $d\mathbf{A}$. Any surface can be defined by two parameters (u,v),

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k} \tag{8.1}$$

where $\mathbf{r}$ is the position vector (in Cartesian coordinates) of a point on the surface.

To find $d\mathbf{A}$ we consider the element of area bounded by the lines of constant u, $u+\delta u$, v and $v+\delta v$ (which are all on the surface).



Moving from a point on the surface (u,v) to $(u+\delta u, v)$ requires the following change of position,

$$\delta \underline{r}_v \approx \frac{\partial \underline{r}}{\partial u}\delta u$$

where $\delta \mathbf{r}_v$ denotes the change in $\mathbf{r}$ along a constant v line.

Similarly,

$$\delta \underline{r}_u \approx \frac{\partial \underline{r}}{\partial v}\delta v$$

The vector of area $\delta\mathbf{A}$ is given by the cross product,

$$\delta \underline{A} = \delta \underline{r}_v \times \delta \underline{r}_u \approx \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right)\delta u\,\delta v$$

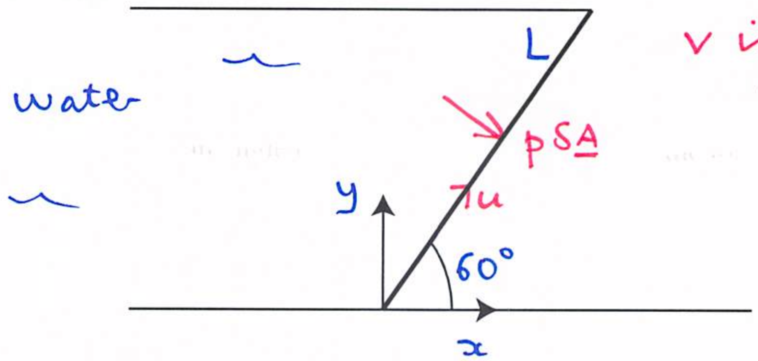
As δu and $\delta v \to 0$,

$$d\mathbf{A} = \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) du\,dv \quad . \tag{8.2}$$

The order of the cross product must be chosen so that $d\mathbf{A}$ points in the direction of the convention being used.

Example

A water retaining dam consists of a rectangular flat plate of length L and breadth B (normal to the paper). The plate is at an angle of 60° to the horizontal. Find the total force on the plate due to the water.



v is normal to page

$$x = u\cos 60^\circ = u/2$$

$$y = u\sin 60^\circ = \sqrt{3}u/2$$

$$z = v$$

$\underline{r}$ is position vector of a point on the plate:

$$\underline{r} = \frac{u}{2}\underline{i} + \frac{\sqrt{3}u}{2}\underline{j} + v\underline{k}$$

$$d\underline{A} = \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) du\,dv = \left[\left(\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}\right) \times \underline{k}\right] du\,dv$$

$$= \left(\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}\right) du\,dv$$

$$p = \rho g (L-u)\sin 60^\circ = \frac{\sqrt{3}}{2}\rho g (L-u)$$

$$\underline{F} = \iint p\, d\underline{A} = \int_{v=0}^{B}\int_{u=0}^{L} \frac{\sqrt{3}}{2}\rho g (L-u)\left(\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}\right) du\,dv$$

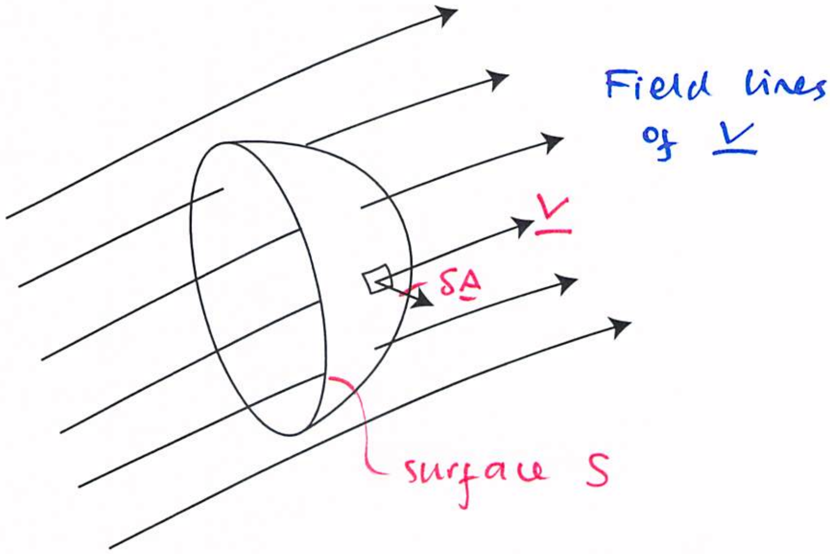
$$= \frac{3}{8}\rho g B L^2 \underline{i} - \frac{\sqrt{3}}{8}\rho g B L^2 \underline{j}$$

8.2 Flux-type surface integrals

Flux-type surface integrals are perhaps the most important of all surface integrals. They have the form,

$$\iint_S \underline{V} \cdot d\underline{A}$$

and the result is a *scalar* quantity.

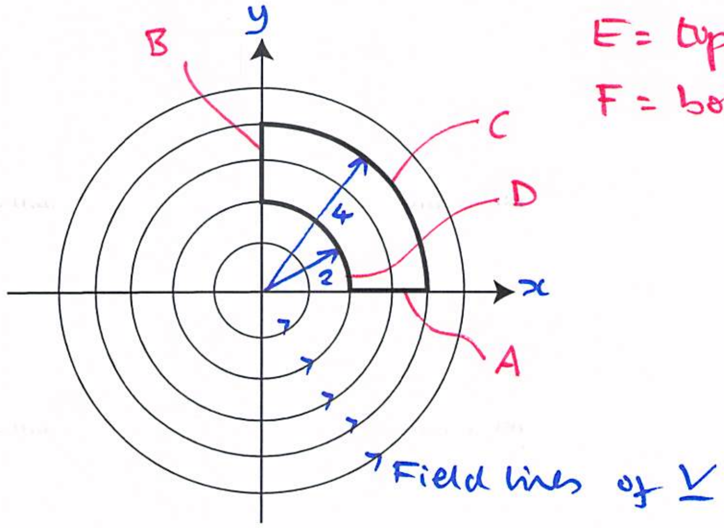


Examples of flux integrals include,

1. If $\mathbf{V}$ is the velocity field, and ρ is the density field, then $\iint_S \rho\mathbf{V} \cdot d\mathbf{A}$ is the total mass flowrate crossing surface S.
2. If $\mathbf{B}$ is the magnetic field, then $\iint_S \mathbf{B} \cdot d\mathbf{A}$ is the total magnetic flux crossing surface S.
3. If $\mathbf{q}$ is the heat flux vector field, then $\iint_S \mathbf{q} \cdot d\mathbf{A}$ is the total heat transfer rate crossing surface S.

Example

Calculate $\iint \mathbf{V} \cdot d\mathbf{A}$ for the vector field $\mathbf{V} = -y\,\mathbf{i} + x\,\mathbf{j}$ for each of the surfaces A, B, C, D, E and F in the diagram. Assume that surfaces A, B, C, D, extend unit distance in the z direction.



E = top (z = 1)
F = bottom (z = 0)

In this problem, we can save time by first investigating the form of the vector field $\mathbf{V}$. The filed lines are given by,

$$\frac{dy}{dx} = \frac{V_y}{V_x} = \frac{-x}{y}$$

and this integrates to,

$$x^2 + y^2 = \text{const}$$

so the field lines are concentric circles, in the (x, y) plane, centred on the origin.

Surfaces C, D, E and F: $\mathbf{V}$ and $d\mathbf{A}$ are orthogonal so $\iint \mathbf{V} \cdot d\mathbf{A} = 0$

Surface A: the element of area is $-(\delta x\, \delta z)\,\mathbf{j}$ (outward pointing normal)

$$\iint_A \underline{V} \cdot d\underline{A} = \int_0^1 \int_2^4 (-y\,\underline{i} + x\,\underline{j}) \cdot (-\underline{j})\, dx\, dz$$

$$= \int_0^1 \int_2^4 -x\, dx\, dz = -6$$

Surface B: the element of area is $-(\delta y\, \delta z)\,\mathbf{i}$

$$\iint_B \underline{V} \cdot d\underline{A} = \int_0^1 \int_2^4 (-y\,\underline{i} + x\,\underline{j}) \cdot (-\underline{i})\, dy\, dz$$

$$= \int_0^1 \int_2^4 y\, dy\, dz = +6$$

If we sum the contributions from all the surfaces we find,

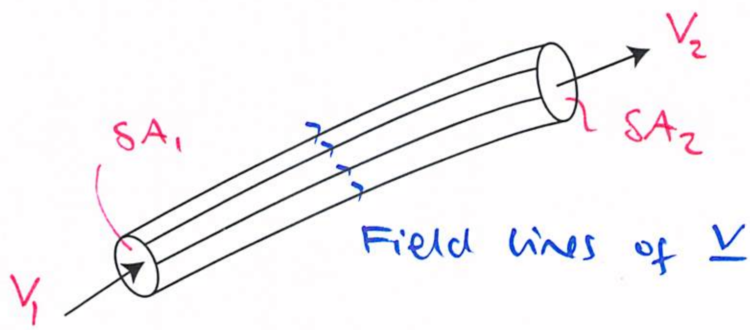
$$\oiint \mathbf{V} \cdot d\mathbf{A} = 0 \quad , \tag{8.3}$$

where $\oiint$ means that the integration is performed over a closed surface. Therfore, *in this particular case*, the net flux of $\mathbf{V}$ through our total closed surface is zero: the total flux entering (through surface A, in our case) is equal to the total flux leaving (through surface B).

The result $\oiint \mathbf{V} \cdot d\mathbf{A} = 0$ is not general but it is true for all vector fields $\mathbf{V}$ that are solenoidal. The net efflux of $\mathbf{V}$ from an elemental volume δv is given by $(\nabla \cdot \mathbf{V})\delta v$. If $\mathbf{V}$ is solenoidal, $\nabla \cdot \mathbf{V} = 0$ and there is zero net efflux of $\mathbf{V}$ from δv (there are no sources or sinks of $\mathbf{V}$). We can consider the volume enclosed by a surface as being made up of lots of elements δv and so, if $\mathbf{V}$ is solenoidal throughout, there can be no sources or sinks of $\mathbf{V}$ in the entire volume: the flux entering must equal the flux leaving.

The vector field in the above example, $\mathbf{V} = -y\,\mathbf{i} + x\,\mathbf{j}$, is solenoidal because,

$$\nabla \cdot \underline{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0$$



A "field tube" is a volume in 3-D space formed by a 'bundle' of field lines. By definition, there is no flux across the walls of the field tube. If the vector field is solenoidal, the total flux into the stream tube must be the same as flux leaving the field tube,

$$V_1 \, \delta A_1 = V_2 \, \delta A_2$$

In fact, the flux is constant all along the field tube (there are no sources or sinks of $\mathbf{V}$),

$$V \delta A = \text{const}$$

so, if δA reduces, the magnitude of $\mathbf{V}$ must increase, and vice-versa.

Some examples of field tubes are:

1. Incompressible fluid flow, $\nabla \cdot \mathbf{V} = 0$, the volume flowrate $V\delta A$ along a streamtube is constant.

2. Steady, variable density fluid flow, $\nabla \cdot (\rho \mathbf{V}) = 0$, the mass flowrate $V\rho\delta A$ along a streamtube is constant.

3. The magnetic field $\mathbf{B}$ is solenoidal, $\nabla \cdot \mathbf{B} = 0$, the magnetic flux $B\delta A$ is constant along a magnetic field tube.

8.3 Volume integrals

There are two kinds of volume integral, one yields a scalar and the other a vector. Examples are, where δv is an element of volume,

1. $\iiint_{\text{vol}} \rho \, dv$ = the mass (a scalar quantity) of fluid with density ρ (which could be variable) occupying volume 'vol'.
2. $\iiint_{\text{vol}} \rho \mathbf{V} \, dv$ = the total momentum (a vector quantity) of fluid of density ρ and velocity $\mathbf{V}$ occupying volume 'vol'.

Example

A solid cylindrical body of radius R, height H and density ρ is rotating about its axis with angular velocity Ω. Calculate the total kinetic energy of the cylinder.

consider a volume element δv at radius r:

$$\delta(KE) = \rho \frac{V^2}{2} \delta v = \rho \frac{r^2 \Omega^2}{2} \delta v$$

$$KE = \iiint_{\text{Vol}} \rho \frac{r^2 \Omega^2}{2} dv$$

$$dv = 2\pi r H \, dr$$

$$KE = \int_0^R \rho \frac{r^3 \Omega^2}{2} 2\pi H \, dr$$

$$= \frac{\pi \rho \Omega^2 H R^4}{4}$$

You can now do Examples Paper 3: Q1, 2, and 3