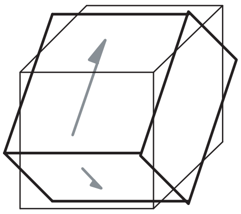
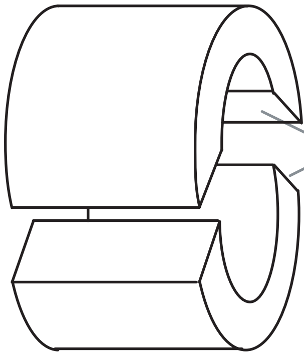
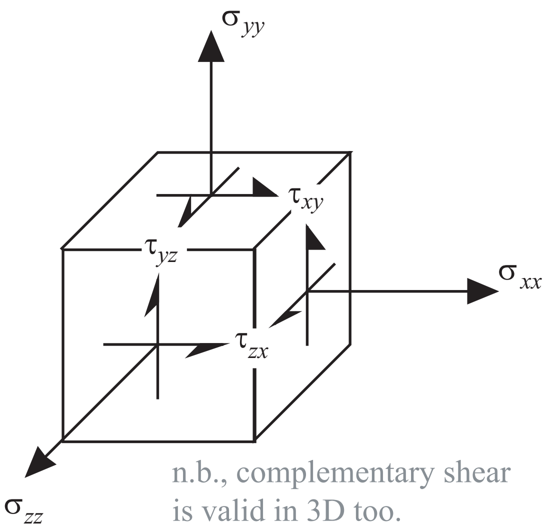




Warning: Only use Mohr's Circle to rotate around principal axes!

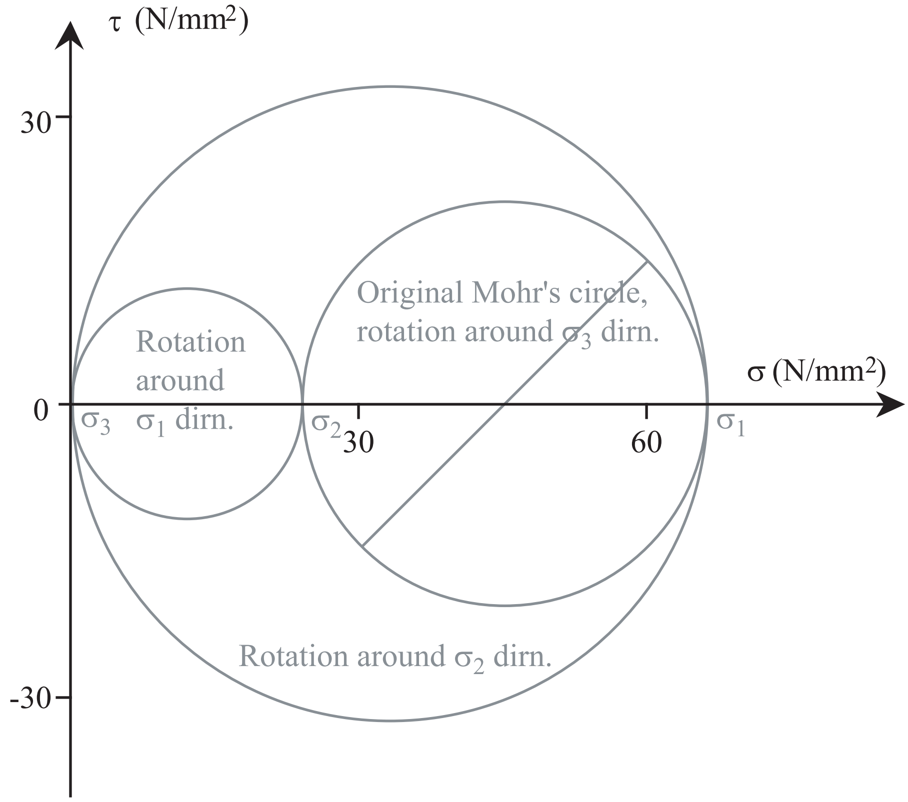
If you rotate about other axes, the stresses you plot will still form a circle, but you would lose track of information about shear on the face you are rotating around (along with the complementary shear on the other faces).



Rotating around the x-direction will correctly calculate the change in σ_{zz}, σ_{yy} and τ_{yz}, but will completely lose track of the changes in τ_{xy}, τ_{zx}. This will not occur if you only rotate about principal axes (because then τ_{xy}, τ_{zx} always remain at zero).

2.4.2 Example

Sketch the 3D Mohr's circles for the earlier example, where the principal stresses were found to be 0, 24 and 66 N/mm^2.



If you know one principal direction (in this case we knew the through-thickness direction was a principal direction), you can rotate around that direction to find the other principal stresses — and then complete the 3D Mohr's circles.

Try Questions 4 and 5, Examples Sheet 2/2

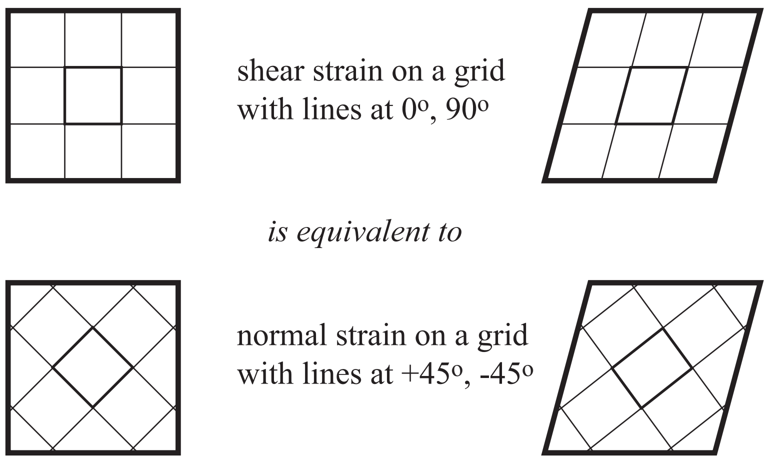
2.4.3 Stress Tensor

Because the basic Mohr's circle is only suitable for rotating around principal directions, it doesn't give complete freedom in calculating stresses in any arbitrary direction. An alternative approach is to go back to the basic equilibrium relationships, and derive a new representation of stress as a 3×3 matrix, called the stress tensor. This is covered in the third year.

$$\begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

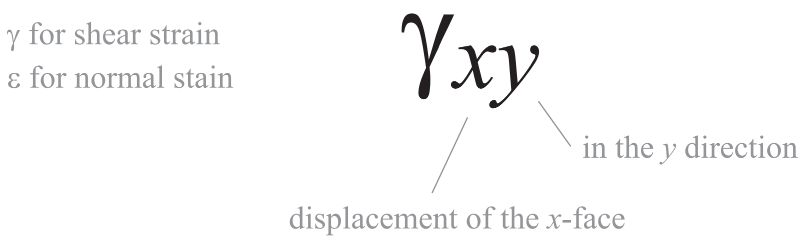
2.5 2-D strain state

We have seen that our measure of stress varies with the orientation of the axes used. The same is also true of strain; it is clearly shown by drawing different grids on a piece of material that is then deformed:

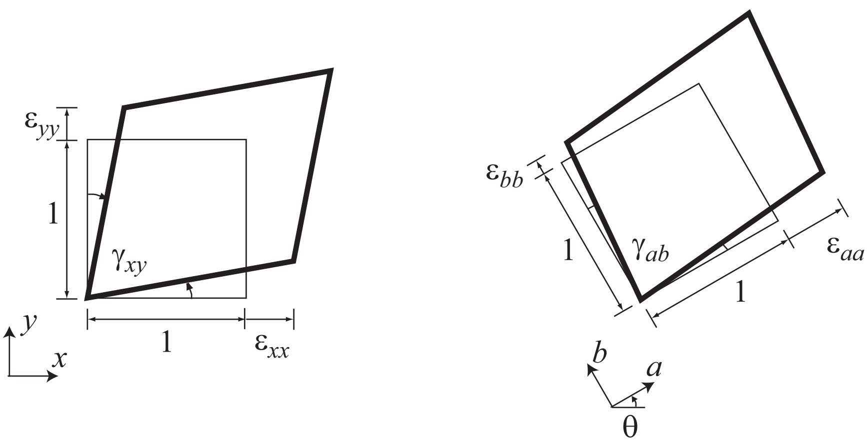


The relationship between strain and axis orientation is simply a matter of *geometry* (compare the relationship between stress and axis orientation, which was a matter of *equilibrium*). However, to derive the relationship can be rather involved, and we won't do it here — derivations may be found in Crandall, Dahl and Lardner pp. 236–238 of 2nd edn., or Case, Chilver and Ross, pp. 109–111 of 4th edn. We will simply quote the result (Question 7, on examples paper 2/2 will explore this a little).

Notation used for strains



2.5.1 Compatibility relationships



$$\varepsilon_{aa} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\cos\theta\sin\theta$$
$$\frac{\gamma_{ab}}{2} = -\varepsilon_{xx}\sin\theta\cos\theta + \varepsilon_{yy}\sin\theta\cos\theta + \frac{\gamma_{xy}}{2}(\cos^2\theta - \sin^2\theta)$$

These equations will be found in the Structures Data Book, page 3, and are proved in an examples paper question.

Try Question 6, Examples Sheet 2/2

For the introductory example shown above, there was pure shear strain about 0°, 90° axes,

$$\varepsilon_{xx} = 0 \qquad \varepsilon_{yy} = 0 \qquad \gamma_{xy} \neq 0$$

Using the equations above, we have, for $\theta = +45^\circ$,

$$\varepsilon_{45} = \gamma_{xy}/2 > 0 \qquad \gamma_{45} = 0$$

and for $\theta = -45^\circ$

$$\varepsilon_{-45} = -\gamma_{xy}/2 < 0 \qquad \gamma_{-45} = 0$$

These results agree with the deformation of the grid, shown clearly by the figure.

There is a remarkable correspondence between the *compatibility relationships* for strain given in Section 2.5.1, and the *equilibrium relationships* for stress given in Section 2.2.2. The two sets of equations are identical, if the following substitutions are made:

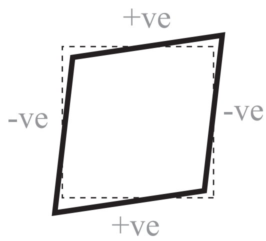
$$\begin{aligned}
\varepsilon_{xx} &\Leftrightarrow \sigma_{xx} \\
\varepsilon_{yy} &\Leftrightarrow \sigma_{yy} \\
\frac{\gamma_{xy}}{2} &\Leftrightarrow \tau_{xy} \\
\varepsilon_{aa} &\Leftrightarrow \sigma_{aa} \\
\varepsilon_{bb} &\Leftrightarrow \sigma_{bb} \\
\frac{\gamma_{ab}}{2} &\Leftrightarrow \tau_{ab}
\end{aligned}$$

This correspondence leads naturally to a Mohr's circle of strain.

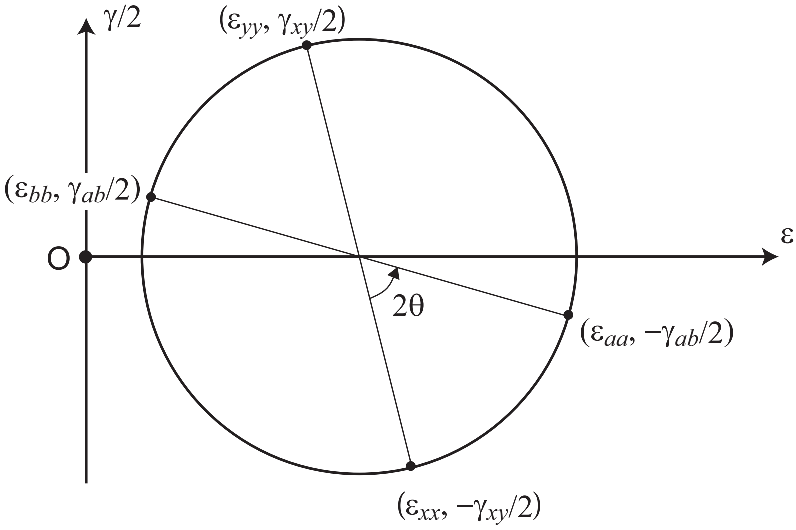
2.5.2 Mohr's circle of strain

Consider plotting *normal strain* against *half* the *shear strain* for various orientations of axes. For shear strain, use the special sign convention:

For Mohr's circle, shear strain is plotted positive when the centre of the face has moved in a clockwise direction relative to the centre of the square.



With this sign convention, the strains plot as a circle, *Mohr's circle of strain*.



2.5.3 Properties of Mohr's circle of strain

1. The strain state in two perpendicular directions (e.g. x and y, a and b) lie at opposite ends of a diameter of the circle.
2. The centre of the circle lies on the ε axis, at the mean value of the normal strains on perpendicular faces, $\frac{1}{2}(\varepsilon_{xx} + \varepsilon_{yy})$. This mean value doesn't depend on the choice of axes.
3. To find the strain in directions at an orientation θ to the original axes, rotate around Mohr's circle by 2θ in the same direction.
4. There are two directions in which there is no shear strain. These are called *principal directions of strain*.

2.5.4 Principal Strain Directions

It is possible to show that: (but not proved here)

> For any state of strain, there will be three perpendicular directions in which there is no shear strain. These directions are called the principal strain directions. In these directions, the only strain is a normal strain, and these three strains are called the principal strains.

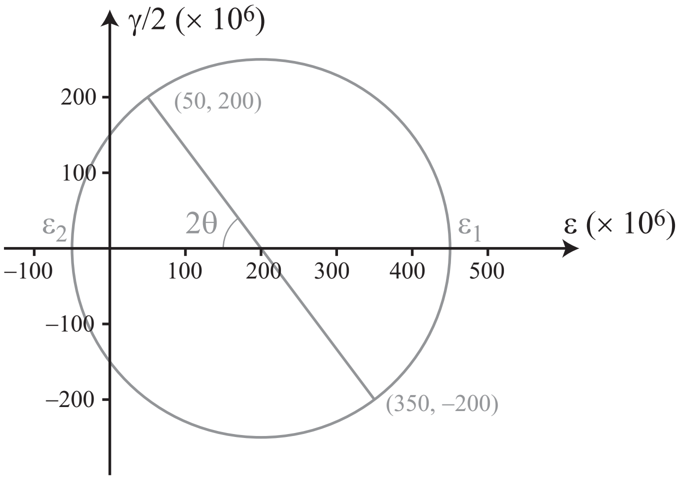
For an isotropic material, no shear *strain* implies no shear *stress*, and hence the principal strain directions, and the principal stress directions, coincide. This may not be true for e.g. a fibre-reinforced structure.

Example

The strain at a point on a thin-walled aluminium-alloy structure has been measured to be $\varepsilon_{xx} = 350 \times 10^{-6}$, $\varepsilon_{yy} = 50 \times 10^{-6}$, $\gamma_{xy} = 400 \times 10^{-6}$. Find the principal strains, the principal stresses, and the orientation of the principal directions to the x and y axes.

Because the structure is thin-walled, the structure is in a state of *plane stress*. Thus the through-thickness (z) axis is a principal stress direction, and this principal stress is zero. Because the structure is isotropic, the z-axis is also a principal strain direction — but we cannot tell directly what is the corresponding strain.

Plot Mohr's circle of strain:



Centre of Mohr's circle, $(50+350)/2 = 200$

Radius of Mohr's circle, $R^2 = 150^2 + 200^2$, $R = 250$

Orientation, $2\theta = \tan^{-1}(200/150) = 53.1°$, $\theta = 26.6°$

$$\varepsilon_1 = 200 + 250 = 450 \times 10^{-6}, \ 26.6° \text{ anticlockwise from } x\text{-axis}$$
$$\varepsilon_2 = 200 - 250 = -50 \times 10^{-6}, \ 26.6° \text{ anticlockwise from } y\text{-axis}$$

Find stresses

In Handout 1, we found strains in terms of stresses. Now we want to invert that relationship. This is straightforward for plane stress ($\sigma_{zz} = 0$) — the required expressions can be found on page 1 of the Structures Data Book.

$$\sigma_1 = \frac{E}{(1-\nu^2)}(\varepsilon_1 + \nu\varepsilon_2)$$
$$\sigma_2 = \frac{E}{(1-\nu^2)}(\varepsilon_2 + \nu\varepsilon_1)$$

For aluminium-alloy, $E = 70 \times 10^9 \ \mathrm{N/m^2}$, $\nu = 0.33$

$$\sigma_1 = 78.6 \times 10^9 (450 \times 10^{-6} - 0.33 \times 50 \times 10^{-6}) = 34 \times 10^6 \ \mathrm{N/m^2}$$

$$\sigma_2 = 78.6 \times 10^9 (-50 \times 10^{-6} + 0.33 \times 450 \times 10^{-6}) = 7.7 \times 10^6 \ \mathrm{N/m^2}$$

Also, $\sigma_3 = 0$

Find the third principal strain

We can now find ε_3

$$\varepsilon_3 = \frac{1}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2)$$

$$\varepsilon_3 = \frac{1}{70\times 10^9}(0 - 0.33\times 34\times 10^6 - 0.33\times 7.7\times 10^6 = -197\times 10^{-6}$$

Try Questions 7, 8 and 9, Examples Sheet 2/2

2.5.5 3-D strain state, and 3-D Mohr's circle of strain

A complete state of strain in 3D requires three shear strains and three normal strains to be defined. There is a 3D Mohr's circle of strain analogous to the 3D stress case, but we won't consider it in detail. One important point to note, however, is just as for the stress case, Mohr's circle leads to loss of information unless the rotation of the axes is about a principal axis.

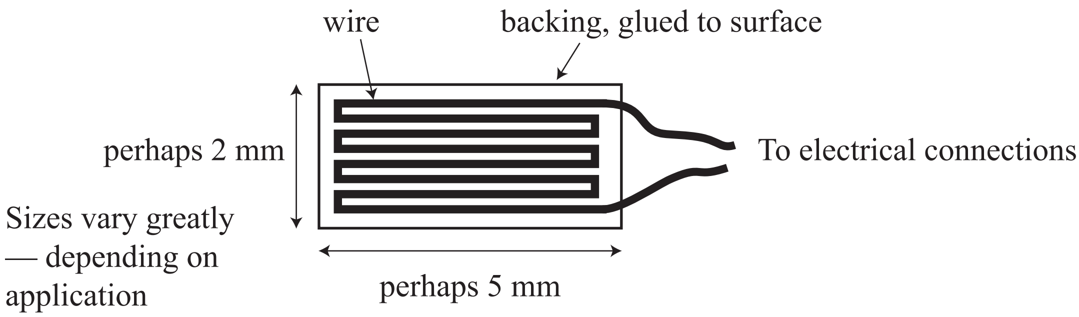
2.5.6 Strain Tensor

Just as there is a 3×3 matrix, called the stress tensor, there is also a 3×3 matrix, called the strain tensor. However, to ensure that this fits the general pattern for transformation of tensors requires a redefinition of shear strain. Thus the *mathematical shear strain* ε_{xy} is defined in terms of the *engineering shear strain* γ_{xy} as $\varepsilon_{xy} = \gamma_{xy}/2$ (note this gets rid of the factor of 1/2 for Mohr's circle of strain). The strain tensor is then defined as

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yx} & \varepsilon_{zx} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{zy} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}$$

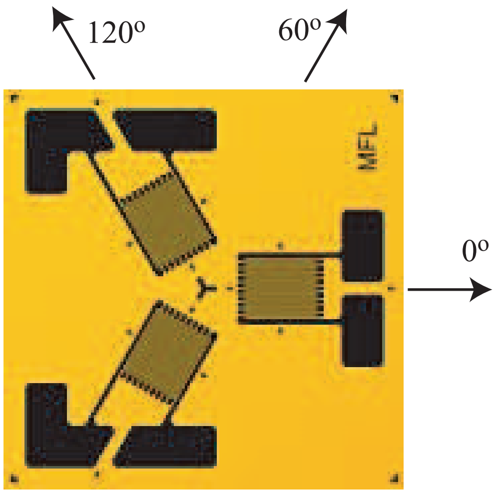
2.6 Strain Gauges

Resistance strain gauges are commonly used for measuring the surface strains on a body. They rely on the fact that the resistance of a wire, when stretched, will increase. To increase the sensitivity, the wire is run backwards and forwards, but in as compact an area as possible; the change in resistance is then measured by a Wheatstone Bridge.

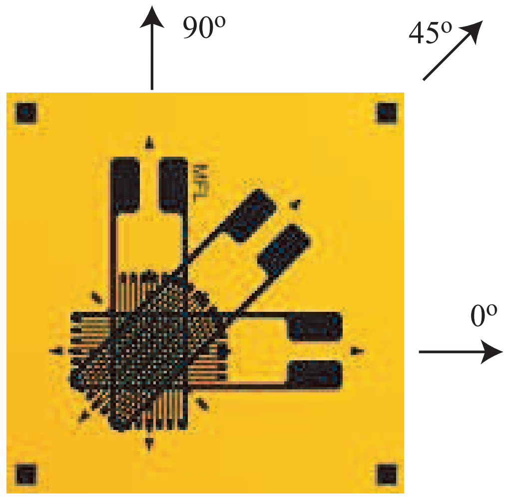


Strain gauges can only measure *normal* strain along the length of the gauge (the only strain that causes the wire to increase in length). Strain gauges *cannot* measure shear strain directly.

To measure the complete 2D state of strain at a point requires three measurement from a strain gauge *rosette*, as shown below. Two types of rosette are common, a 45° rosette that measures the normal strain at 0°, 45°, and 90°, and a 60° rosette that measures the normal strain at 0°, 60°, and 120°. With some processing, both of these provide enough data to determine the complete 2D state of strain at a point.



single-plane 60º rosette



stacked 45º rosette

2.6.1 45° rosette

The rosette measures strains ε_0, ε_{45} and ε_{90}. If the x and y axes are aligned with the 0° and 90° directions, then

$$\varepsilon_{xx} = \varepsilon_0$$

$$\varepsilon_{yy} = \varepsilon_{90}$$

The strain γ_{xy} is more difficult, but can straightforwardly be found using the compatibility equation:

$$\varepsilon_{aa} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\cos\theta\sin\theta$$

Substituting ε_{xx} and ε_{yy} from above, with $\theta = 45°$, gives

$$\varepsilon_{45} = \frac{\varepsilon_0 + \varepsilon_{90} + \gamma_{xy}}{2}$$

and thus

$$\gamma_{xy} = 2\varepsilon_{45} - \varepsilon_0 - \varepsilon_{90}$$

2.6.2 60° rosette

The rosette measures strains ε_0, ε_{60} and ε_{120}. If the x axis is aligned with the 0° direction, then

$$\varepsilon_{xx} = \varepsilon_0$$

The strains ε_{yy} and γ_{xy} are more difficult, but again can be found using the compatibility equation:

$$\varepsilon_{aa} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\cos\theta\sin\theta$$

Substituting ε_{xx} from above, with $\theta = 60^\circ$, and $\theta = 120^\circ$, gives

$$\varepsilon_{60} = \frac{\varepsilon_0 + 3\varepsilon_{yy} + \sqrt{3}\gamma_{xy}}{4} \tag{2.1}$$

$$\varepsilon_{120} = \frac{\varepsilon_0 + 3\varepsilon_{yy} - \sqrt{3}\gamma_{xy}}{4} \tag{2.2}$$

Adding 2.1 and 2.2 gives

$$\varepsilon_{60} + \varepsilon_{120} = \frac{\varepsilon_0 + 3\varepsilon_{yy}}{2}$$

and so

$$\varepsilon_{yy} = \frac{-\varepsilon_0 + 2\varepsilon_{60} + 2\varepsilon_{120}}{3}$$

Subtracting 2.2 from 2.1 gives

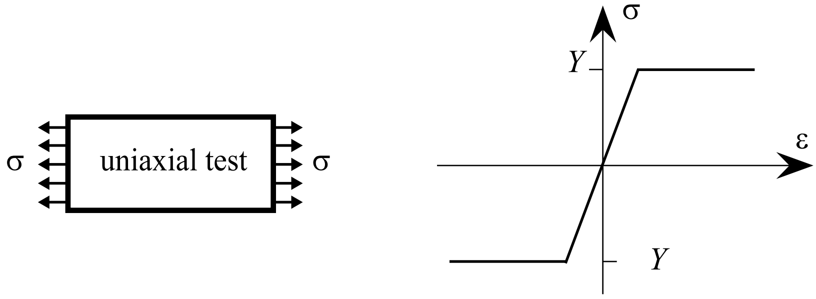
$$\varepsilon_{60} - \varepsilon_{120} = \frac{\sqrt{3}\gamma_{xy}}{2}$$

and so

$$\gamma_{xy} = \frac{2\varepsilon_{60} - 2\varepsilon_{120}}{\sqrt{3}}$$

2.7 Yield Criteria

A common assumption is that a material has a distinct yield stress. For instance, an elastic-perfectly plastic material in a uniaxial test would give a stress-strain curve:



Y is called the uniaxial yield stress of the material.

If instead there were a complex 3D stress state in the material, how can we can determine when it will yield? Experiments on annealed metals have shown that:

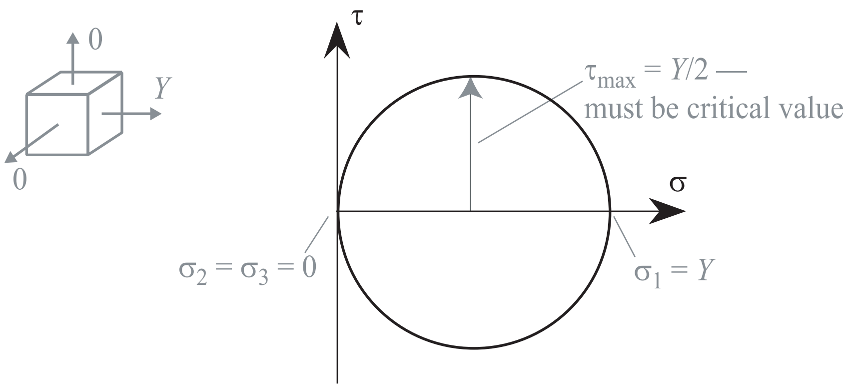
- Plastic straining is due to the movement of dislocations driven by shear stresses.
- Hydrostatic stress never causes yield.

Two alternative yield criteria have been suggested that fit these results, the Tresca Yield Criterion, and the von Mises Yield Criterion. To dissociate these criteria from any particular set of axes, they will be defined in terms of the three principal stresses in the material, σ_1, σ_2 and σ_3.

2.7.1 Tresca Yield Criterion

A body will yield when the maximum shear stress reaches a critical value.

What will the critical value be? The simplest approach is to refer back to the uniaxial test, for which the Mohr's Circles at yield are given by:



In a general 3-D stress state, the maximum shear stress is the radius of the largest Mohr's circle. Thus the Tresca Yield Criterion may be written, in terms of the diameters of the 3D Mohr's circles, as:

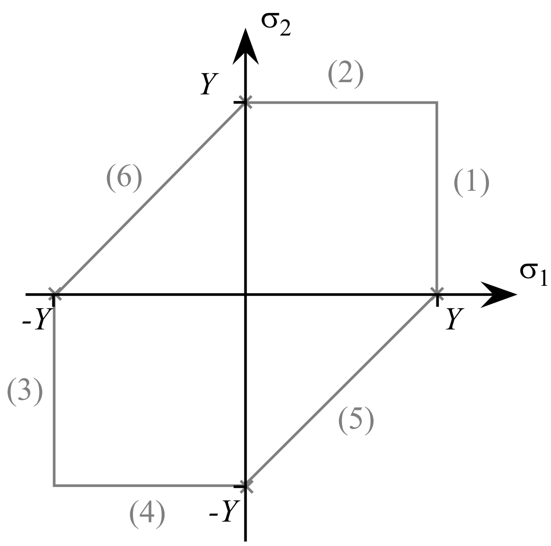
According to the Tresca Criterion, yield will occur when

$$\max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) = Y$$

Structures Data Book, page 4.

Visualising Tresca for plane stress

Often, one of the principal directions is unstressed, giving $\sigma_3 = 0$ (e.g. thin-walled structures). What combinations of σ_1 and σ_2 will then cause yield according to Tresca? We will plot the *Yield Surface* on an *interaction diagram*:



Four points are known from uniaxial tests: marked by $\times$

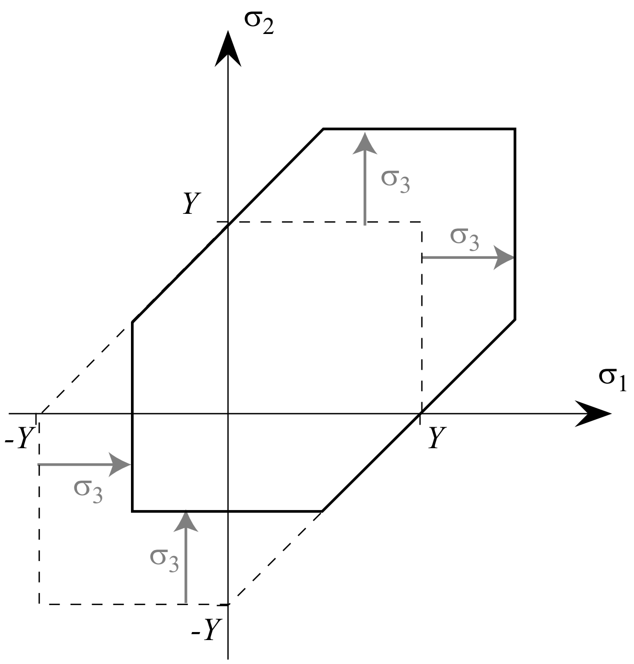
- When $\sigma_1 = 0$, yield will occur at $\sigma_2 = \pm Y$.
- When $\sigma_2 = 0$, yield will occur at $\sigma_1 = \pm Y$

The rest can be filled in by considering the six following possibilities:

1. $\sigma_1 \geq \sigma_2 \geq \sigma_3 (= 0) \quad \Rightarrow \quad \sigma_1 - \sigma_3 = Y, \quad \sigma_1 = Y$
2. $\sigma_2 \geq \sigma_1 \geq \sigma_3 (= 0) \quad \Rightarrow \quad \sigma_2 - \sigma_3 = Y, \quad \sigma_2 = Y$
3. $\sigma_3 (= 0) \geq \sigma_2 \geq \sigma_1 \quad \Rightarrow \quad \sigma_3 - \sigma_1 = Y, \quad \sigma_1 = -Y$
4. $\sigma_3 (= 0) \geq \sigma_1 \geq \sigma_2 \quad \Rightarrow \quad \sigma_3 - \sigma_2 = Y, \quad \sigma_2 = -Y$
5. $\sigma_1 \geq \sigma_3 (= 0) \geq \sigma_2 \quad \Rightarrow \quad \sigma_1 - \sigma_2 = Y$
6. $\sigma_2 \geq \sigma_3 (= 0) \geq \sigma_1 \quad \Rightarrow \quad \sigma_2 - \sigma_1 = Y$

Visualising Tresca for $\sigma_3 \neq 0$

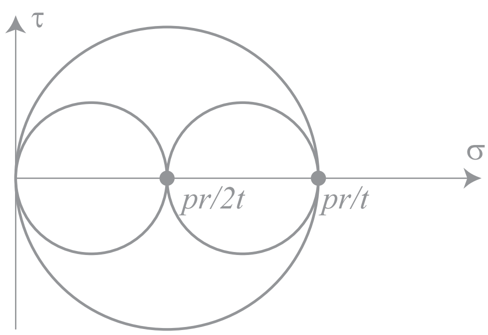
If $\sigma_3 \neq 0$, the basic shape of the yield criterion will be unaffected, but the lines defined by cases (1) – (4), above, will all be offset by σ_3, giving:



Example

A thin-walled cylinder with closed ends has radius 50 mm, thickness 1 mm, and is made of material with a uniaxial yield stress of 100 N/mm^2. What internal pressure can the cylinder withstand before it yields, according to the Tresca Yield Criterion?

$$\sigma_1 = \frac{pr}{t}, \qquad \sigma_2 = \frac{pr}{2t}, \qquad \sigma_3 = 0$$



$$\max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) = Y$$

$$\frac{pr}{t} = Y$$

$$\text{At yield, } p = \frac{100 \times 1}{50} = 2\ \text{N/mm}^2$$

2.7.2 Von Mises Yield Criterion

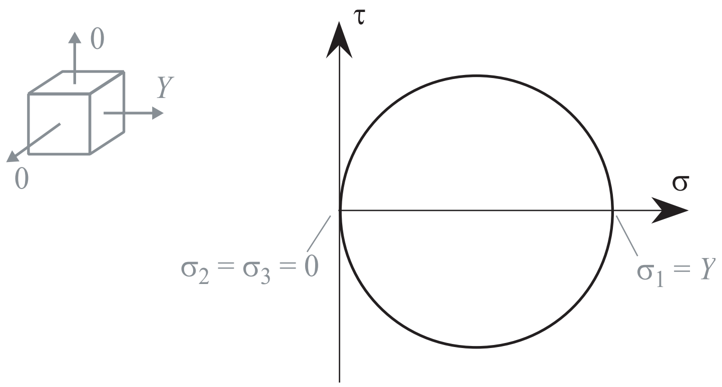
A body will yield when the strain energy of distortion reaches a critical value.

The strain energy stored in a body can be split into the *hydrostatic* strain energy (the energy associated with uniformly enlarging or contracting the body), and the *distortional* strain energy. von Mises assumes only the latter is important to yielding.

It can be shown (see e.g. Case, Chilver and Ross, pp. 130–132 of 4th edn.) that the strain energy of distortion, U_D, for a linear-elastic isotropic solid with shear modulus G is related to the principal stresses σ_1, σ_2, σ_3 by

$$U_D = \frac{1}{12G}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]$$

What critical value must U_D reach before yield occurs? Again the simplest approach is to refer back to the uniaxial test, for which the Mohr's Circles at yield are given by:



$$\begin{aligned}\text{critical } U_D &= \frac{1}{12G}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]\\ &= \frac{1}{12G}[(Y-0)^2+(0-0)^2+(0-Y)^2] = \frac{Y^2}{6G}\end{aligned}$$

Substituting the critical value of U_D back into the general expression, and cancelling G, gives:

According to the von Mises Criterion, yield will occur when $$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 = 2Y^2$$

Structures Data Book, page 4.

Visualising von Mises for plane stress

If one of the principal directions is unstressed, giving $\sigma_3 = 0$, what combinations of σ_1 and σ_2 will cause yield?

The same four points are known from uniaxial tests:

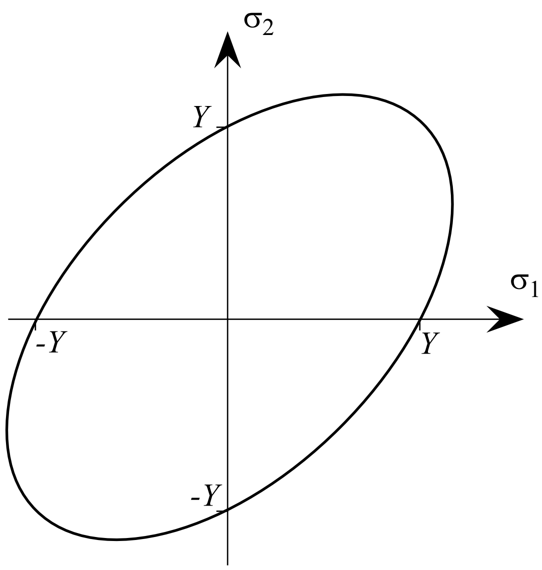
- When $\sigma_1 = 0$, yield will occur at $\sigma_2 = \pm Y$.

- When $\sigma_2 = 0$, yield will occur at $\sigma_1 = \pm Y$.

The rest can be filled in by rewriting the yield criterion for $\sigma_3 = 0$ as:

$$(\sigma_1 - \sigma_2)^2 + \sigma_1^2 + \sigma_2^2 = 2Y^2$$
$$\text{or } \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = Y^2$$

This yield surface plots as an ellipse.

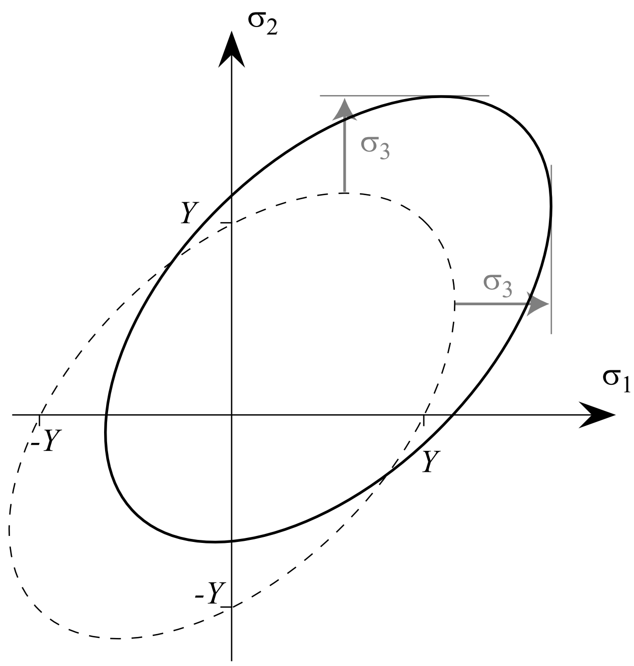


Visualising von Mises for $\sigma_3 \neq 0$

The von Mises Yield Criterion can be re-written as:

$$(\sigma_1 - \sigma_3)^2 - (\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3) + (\sigma_2 - \sigma_3)^2 = Y^2$$

The basic form of this equation is the same as the one for the ellipse plotted above, but now the ellipse has been offset by σ_3 in both the σ_1 and σ_2 directions.



Example

A thin-walled cylinder with closed ends has radius 50 mm, thickness 1 mm, and is made of material with a uniaxial yield stress of 100 N/mm^2. What internal pressure can the cylinder withstand before it yields, according to the von Mises Yield Criterion?

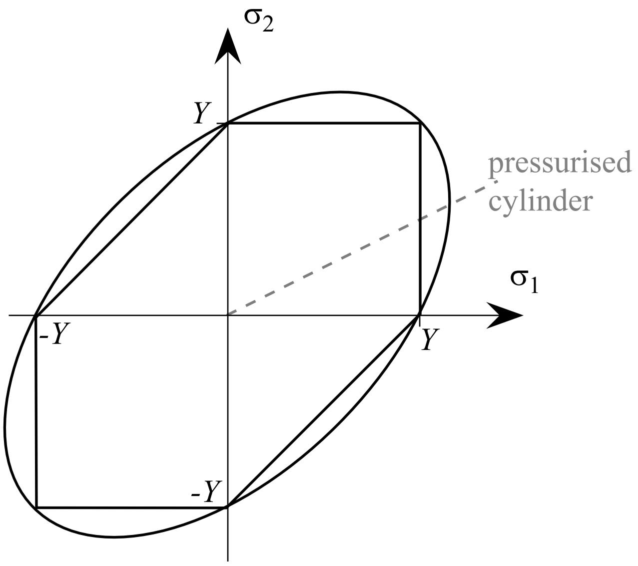
$$\sigma_1 = \frac{pr}{t}; \quad \sigma_2 = \frac{pr}{2t}; \quad \sigma_3 = 0$$

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2$$
$$\left(\frac{pr}{2t}\right)^2 + \left(\frac{pr}{2t}\right)^2 + \left(\frac{pr}{t}\right)^2 = 2Y^2$$
$$\frac{6}{4}\left(\frac{pr}{t}\right)^2 = 2Y^2$$
$$p = 2.3\ \mathrm{N/mm^2}$$

15 % greater than the Tresca prediction.

2.7.3 Comparison of von Mises and Tresca

If $\sigma_3 = 0$, the two criteria can be visualised on the σ_1, σ_2 plane as:



Obviously the criteria are in fairly good agreement. The maximum discrepancy is for pure shear ($\sigma_1 = -\sigma_2$) where the difference is about 15%. Some interesting experiments to determine which criterion most closely represents the yielding of annealed metals were carried out in the Engineering Department in c. 1930, and are detailed in Examples Sheet 2 question 12. However, often the decision about which criteria to use will depend more on which is *easiest* to use for a particular calculation.

Try Questions 10–13, Examples Sheet 2/2

Handout 3

Elastic Structural Analysis

Filled Version

Text and pictures in grey are omitted from the version in lectures

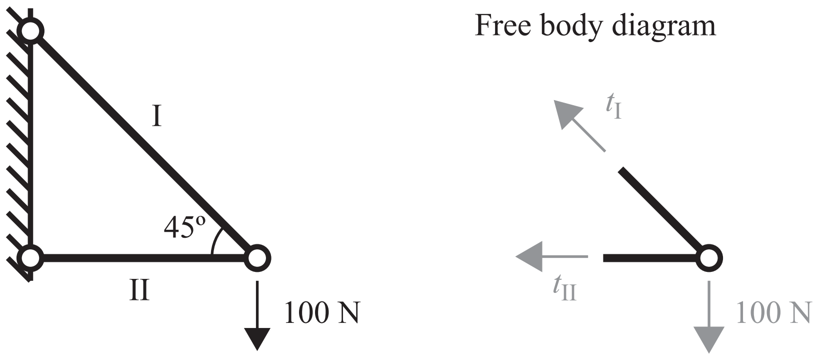
Cambridge University Engineering Department

3.1 Introduction

In the first-year course almost every structure that you analysed was deliberately chosen to be *statically determinate*. This means that, by *equilibrium* alone, it was possible to determine the forces carried by every member in those structures.

3.1.1 Example — a statically determinate truss

Consider the simple pin-jointed truss structure below.



The forces in each of the members can be simply calculated,

$$\begin{aligned} t_{\mathrm{I}} \sin 45^\circ &= 100\text{ N} \\ t_{\mathrm{I}} \cos 45^\circ + t_{\mathrm{II}} &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} t_{\mathrm{I}} &= 141\text{N} \\ t_{\mathrm{II}} &= -100\text{ N} \end{aligned}$$

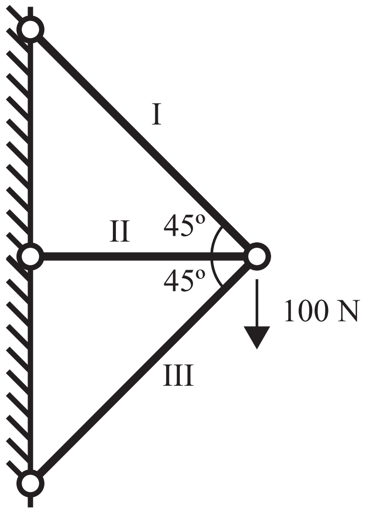
The two equations above can be written in a matrix form

$$\begin{bmatrix} \sqrt{1/2} & 0 \\ \sqrt{1/2} & 1 \end{bmatrix} \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} 100\text{ N} \\ 0 \end{bmatrix}$$

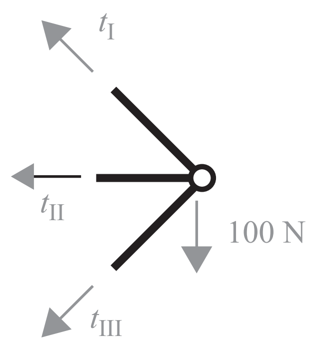
This *equilibrium matrix* is *square* and *non-singular*: there will be a unique solution for any possible loading.

3.1.2 Example — a statically indeterminate truss

Now consider adding an additional member to the structure, to give the new structure shown below.



Free body diagram



Again it is possible to write two equations of equilibrium at the free joint.

$$\begin{bmatrix} \sin 45^\circ & 0 & -\sin 45^\circ \\ \cos 45^\circ & 1 & \cos 45^\circ \end{bmatrix} \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \end{bmatrix} = \begin{bmatrix} 100\ \mathrm{N} \\ 0 \end{bmatrix}$$

These equations do not have a unique solution — the state of the structure cannot be found by equilibrium alone. To proceed, we need to also consider the *load-deflection* characteristics of each member, and the overall *compatibility* of the structure. Although there are infinitely many sets of tensions in equilibrium with the applied load, only one of them will lead to a structure that still fits together.

We can still make progress in solving the equilibrium equations by applying Gaussian Elimination to the equilibrium matrix. Re-writing the equations numerically (with $0.7 \approx \sqrt{1/2}$)

$$\begin{bmatrix} 0.7 & 0 & -0.7 \\ 0.7 & 1 & 0.7 \end{bmatrix} \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \end{bmatrix} = \begin{bmatrix} 100\ \mathrm{N} \\ 0 \end{bmatrix}$$

Replacing the second equation with the second equation minus the first gives

$$\begin{bmatrix} 0.7 & 0 & -0.7 \\ 0 & 1 & 1.4 \end{bmatrix} \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \end{bmatrix} = \begin{bmatrix} 100\ \mathrm{N} \\ -100\ \mathrm{N} \end{bmatrix}$$

Gaussian elimination can now proceed no further: eliminating in the third column would undo any progress. (Because we can't complete the elimination for this column, we can specify bar III as a *redundant* bar.) We can now look for two solutions:

1. A *particular solution* that is in equilibrium with the applied load;

2. A *homogeneous solution* that is in equilibrium with *zero* applied load (the null-space of the equilibrium matrix). In structures jargon, this is called a *state of self-stress*.

Of course, any of (2) can be added to (1), and it won't affect equilibrium with the external loads—we can find an infinite number of equilibrium solutions.

A particular solution can be found by setting the value of t_{III} to zero (or any other value!). The second equation now becomes:

$$0t_{\mathrm{I}} + t_{\mathrm{II}} + 1.4 \times (t_{\mathrm{III}} = 0) = -100\ \mathrm{N}$$
$$t_{\mathrm{II}} = -100$$

And the first equation can be solved by *back-substitution* as

$$0.7t_{\mathrm{I}} + 0 \times (t_{\mathrm{II}} = -100\ \mathrm{N}) - 0.7 \times (t_{\mathrm{III}} = 0) = 100\ \mathrm{N}$$
$$t_{\mathrm{I}} = 140\ \mathrm{N}$$

Thus a particular solution for the equilibrium equations is given by

$$\mathbf{t}_0 = \begin{bmatrix} 140\ \mathrm{N} \\ -100\ \mathrm{N} \\ 0 \end{bmatrix}$$

A state of self stress can be found by solving the same equations with zero on the right hand side. If we set t_{III} to zero this time, however, we end up with everything being zero — which is not very interesting. Instead, set t_{III} to one (or anything else *except* zero!), and solve again by back-substitution. The second equation now becomes:

$$0t_{\mathrm{I}} + t_{\mathrm{II}} + 1.4 \times (t_{\mathrm{III}} = 1) = 0$$
$$t_{\mathrm{II}} = -1.4$$

And the first equation can be solved by *back-substitution* as

$$0.7t_{\mathrm{I}} + 0 \times (t_{\mathrm{II}} = -1.4) - 0.7 \times (t_{\mathrm{III}} = 1) = 0$$
$$t_{\mathrm{I}} = 1$$

Thus a state of self-stress for the equilibrium equations is given by

$$\mathbf{s} = \begin{bmatrix} 1 \\ -1.4 \\ 1 \end{bmatrix}$$

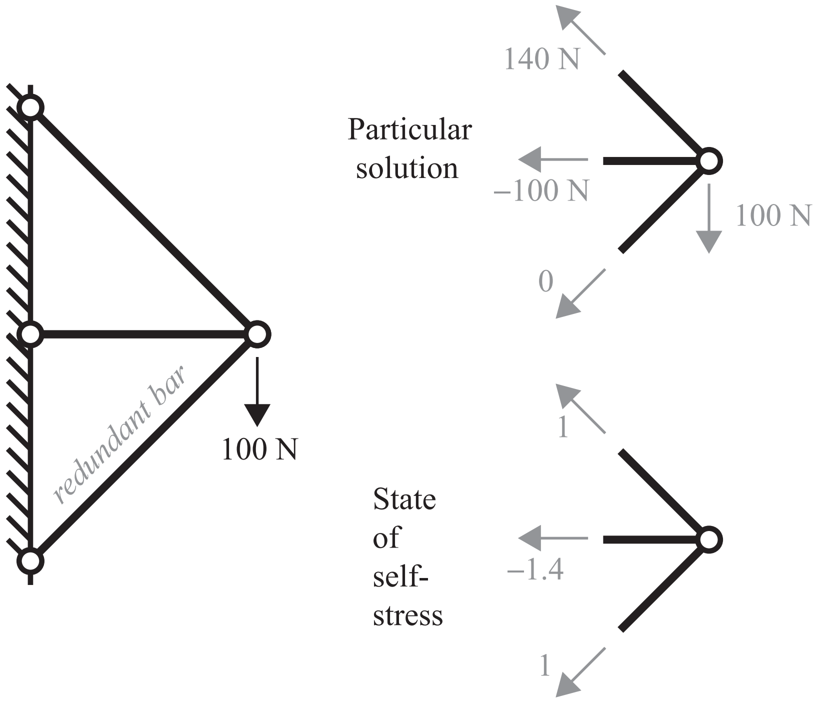
The complete solution is thus the particular solution, plus some (as yet unknown) proportion of the state of self-stress, which we could write as

$$\mathbf{t} = \mathbf{t}_0 + x\mathbf{s}$$

We have reduced the problem to finding x — how much self-stress should be included? Before solving that, we shall consider a less formal way to find the equilibrium solutions.

An alternative scheme

Gaussian elimination of the equilibrium matrix enabled us to identify a *redundant* bar. Setting the stress in this bar equal to zero allowed us to find a particular solution. Setting it equal to one (with no external load) allowed us to find a state of self stress. As long as we are able to identify a suitable redundant bar, we could have found these solutions without resorting to a formal matrix scheme:



Finding the elastic solution

Whatever method is used for finding the equilibrium solutions, to find a single solution for a statically indeterminate structure requires us to consider the deformations of the structure. For our structure, let us assume each bar is elastic with Young's Modulus E, and cross sectional area A. The extension of each bar is now axial force $\times$ length$/AE$, and this can be written using the *flexibility* matrix $\mathbf{F}$ as $\mathbf{e} = \mathbf{F}\mathbf{t}$.

$$\begin{bmatrix} e_{\mathrm{I}} \\ e_{\mathrm{II}} \\ e_{\mathrm{III}} \end{bmatrix} = \begin{bmatrix} \sqrt{2}l/AE & 0 & 0 \\ 0 & l/AE & 0 \\ 0 & 0 & \sqrt{2}l/AE \end{bmatrix} \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \end{bmatrix}$$

Substituting $\mathbf{t} = \mathbf{t}_0 + x\mathbf{s}$ gives

$$\mathbf{e} = \left(\frac{l}{AE}\right) \begin{bmatrix} \sqrt{2} & & \\ & 1 & \\ & & \sqrt{2} \end{bmatrix} \left(\begin{bmatrix} 140\text{ N} \\ -100\text{ N} \\ 0 \end{bmatrix} + x \begin{bmatrix} 1 \\ -1.4 \\ 1 \end{bmatrix} \right)$$

$$\mathbf{e} = \left(\frac{l}{AE}\right)\left(\begin{bmatrix} 200\ \text{N} \\ -100\ \text{N} \\ 0 \end{bmatrix} + x\begin{bmatrix} 1.4 \\ -1.4 \\ 1.4 \end{bmatrix}\right)$$

For a general set of extensions, the structure will not fit together — the lengths of two bars fix the position of the joint: there is then no freedom in the length of the third bar. How can we find the value of x that allows the structure to fit together? One possibility would be to draw some sort of displacement diagram, but as x changes every length, this will in general be very tricky (although Question 1 on Examples Paper 3 uses this method) . A much better scheme is to use Virtual Work.

Try Question 1, Examples Sheet 2/3

A reminder about Virtual Work
Virtual Work for a pin-jointed truss:

$$\sum t \cdot e = \sum f \cdot \delta$$

or, written as a dot-product between vectors,

$$\mathbf{t} \cdot \mathbf{e} = \mathbf{f} \cdot \boldsymbol{\delta}$$

This is a fundamental statement of static equilibrium.
If a set of loads and internal forces are in *equilibrium*, then there will be no work done for *any* possible (small) movement of the structure. The Virtual Work equation just adds up all the work components to make sure that any extra internal work is balanced by some extra external work — the system as a whole does no work. If the external loads did more work, the structure would be accelerating in that direction; if they did less, it would accelerate in the opposite direction.
Another way of saying that a movement of the structure is possible is to say that the displacements and extensions are *compatible*: the structure will still fit together after the deformation.
Note that the *compatible* deformations can be any deformations: they do not have to be connected with any particular *equilibrium* system.

We have a set of extensions, and we want to make sure that they are compatible — however, we know nothing about the external displacements of the joint. If we use as a *virtual equilibrium system* a state of self-stress, then this problem disappears, as every external displacement is multiplied by a zero load.

$$\mathbf{s} \cdot \mathbf{e} = \mathbf{f} \cdot \boldsymbol{\delta} = 0$$