Paper 1: Mechanical Engineering

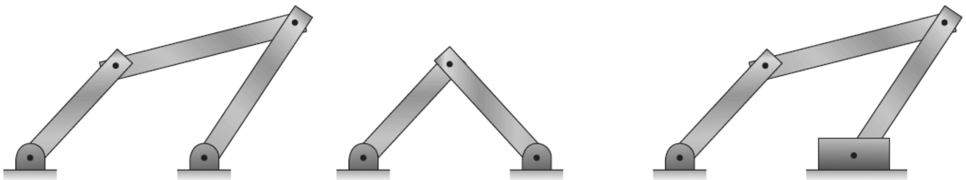
Examples Paper 3

Elementary exercises are marked †, problems of Tripos standard ∗.
Answers can be found at the back of the paper.

Degrees of freedom

1 † For each of the following systems, identify the number of degrees of freedom and propose a suitable set of generalized coordinates:

(a) seasaw in a playground
(b) cylinder rolling, without slip, down an inclined plane
(c) pair of spectacles floating in the space station
(d) office chair that can move across the floor and swivel
(e) the following three mechanisms.



Newtonian Mechanics with constraints

2 † A pendulum bob, of mass m, hangs from a pivot by a light inextensible string of length l and tension T.

(a) Using Cartesian coordinates, with the origin at the pivot, show that the equations of motion of the pendulum are

$$-T\frac{x}{l} = m\ddot{x}$$
$$-mg - T\frac{y}{l} = m\ddot{y},$$

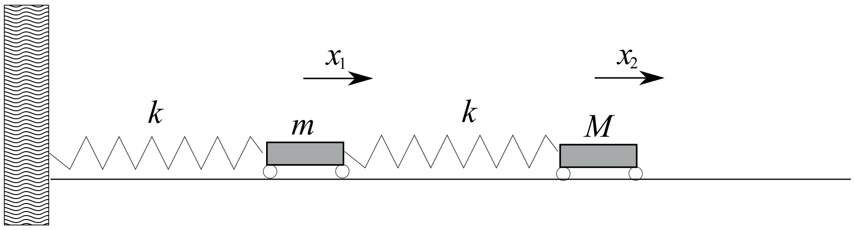
(b) By substituting $x = l\sin\theta$, and $y = -l\cos\theta$, find the equation of motion for θ, and an expression for T.

Lagrangian Mechanics

3 † A square lamina, with sides a, hangs from one of its corners under gravity. Use Lagrangian mechanics to find the square's equation of motion, and find its time period for small oscillations.

Would this time period increase or decrease if we use a square picture frame instead?

4 † A system with two masses (m and M) and two springs (both k) is shown below. The generalized coordinates x_1 and x_2 give the masses' displacement from their equilibrium positions.



(a) Write down the Lagrangian for the system, and find the Lagrangian equations of motion for x_1 and x_2. What are the physical interpretations of the generalized momenta and forces?

(b) Show than an exactly analogous Lagrangian arrises for a system of two disks (I_1 and I_2) and two matching torsional springs, leading to the same equations of motion for the disks' angular displacements θ_1 and θ_2. What now are the physical interpretations of the generalized momenta and forces?

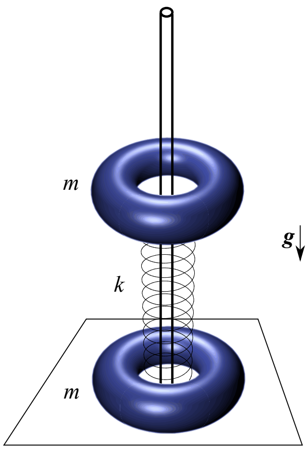
(c) Show that these equations of motion agree with those obtained by directly considering forces on the two masses or torques on the two disks.

5 A satellite has position r and θ in a plane-polar coordinate system centered on Earth.

(a) Write down the Lagrangian for the satellite, and find the Lagrangian equations of motion for r and θ.

(b) Show that these equations of motion agree with those obtained by directly applying Newton's second law in polar coordinates.

6 A jumping toy consists of two toroidal masses, m, connected by a spring with spring-constant k and natural length l. The masses are constrained to move vertically by a central pole.

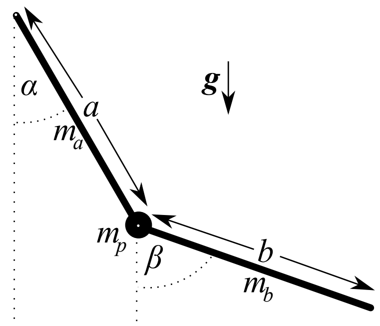


(a) Using the mass positions, z_1 z_2, as generalized coordinates, find the Lagrangian for the system, firstly in the case that the bottom mass is in contact with the floor, and secondly when the toy is in the air.

(b) In the case where the toy is in the air, find the Lagrangian using the center, $z_c = \frac{z_2+z_1}{2}$ and separation $z_s = z_1 - z_2$ of the masses as generalized coordinates. Physically, what is the generalized momenta for z_c?

(c) Find and solve the Lagrangian equations of motion for z_c and z_s, and hence describe the toy's motion whilst it jumps.

7 *The double pendulum used in the double pendulum lab can be modeled as two uniform rigid rods, one of length a and mass m_a, and the other with length b and m_b, connected by a pivot with mass m_p, as shown below.



A generic double pendulum has a Lagrangian of the form:

$$\mathcal{L} = \frac{1}{2}\left(A\dot{\alpha}^2 + B\dot{\beta}^2 + 2C\dot{\alpha}\dot{\beta}\cos(\alpha-\beta)\right) + D\cos\alpha + E\cos\beta$$

(a) Find the values of A, B... E for

(i) A double pendulum consisting of point masses, m, connected by light rods of length l.

(ii) A double pendulum consisting of two uniform rigid rods of length l and mass m, without further point masses.

(iii) The double pendulum from the double-pendulum lab.

(b) Find the equations of motion for α and β for the generic double pendulum.

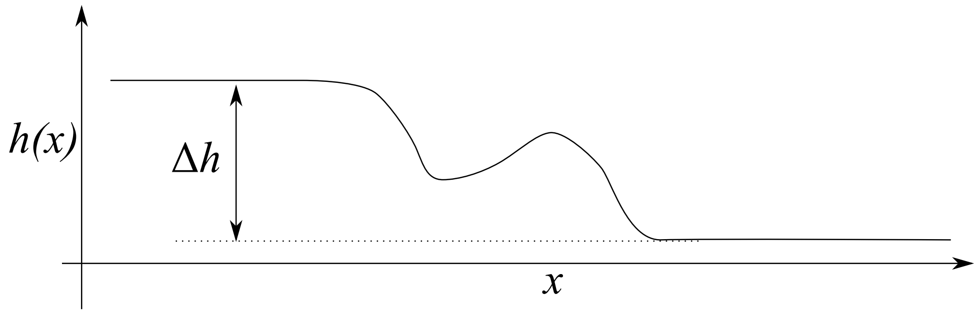
(c) Find the equilibrium states of the double pendulum.

(d) A double pendulum is released from rest at α_0 β_0. Show the lower rod will never flip if

$$\frac{D}{E}\cos\alpha_0 + \cos\beta_0 > \frac{D}{E} - 1.$$

1D dynamics and Effective potentials

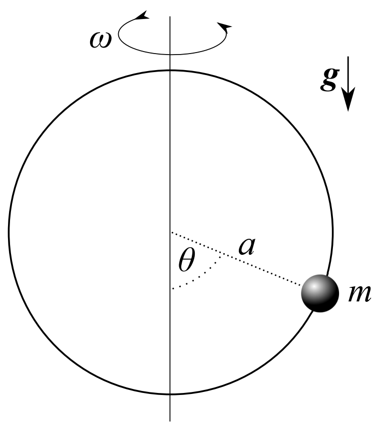
8 An ice hockey puck slides under gravity on an uneven pitch with height $h(x, y) = h(x)$ as plotted below. Friction is negligible, the puck never leaves the surface, and you may assume height variations are small compared to the length of the pitch, so you can ignore the kinetic energy associated with vertical motion.



(a) Categorize the different types of motion the puck can undergo, and sketch (in x–y 2-D coordinates) the different types of motion as seen from above.

(b) The puck is launched in the top flat region towards the bottom flat region. If it is launched at speed v_i and making an angle θ_i with the x axis, what angle, θ_f, will it be moving at when it is in the lower flat region.

9 * A wire hoop spins around a vertical diameter at fixed angular velocity ω. A bead, of mass m, is free to slide on the hoop without friction, and there is a gravitational field g.



(a) Find the Lagrangian equation of motion for θ, and show it corresponds to motion in an effective potential,

$$ma^2\ddot{\theta} = -V'_{eff}(\theta), \quad \text{where} \quad V_{eff}(\theta) = ma^2\left(-\frac{g}{a}\cos\theta - \frac{1}{2}\omega^2\sin^2\theta\right).$$

(b) Find the equilibrium points, and show that the $\theta = 0$ equilibrium is unstable if $\omega^2 > g/a$.

(c) Sketch the effective potential for the $a\omega^2/g \sim 1/2$ and $a\omega^2/g \sim 2$, and hence draw phase portraits with axes θ and $\dot{\theta}$ for the two cases.

Suitable past Tripos questions

Degrees of freedom: This is just warm up material.
Newtonian Mechanics with constraints: This is just warm up material.
Lagrangian Mechanics: 3C5 2004 Q4, 3C5 2005 Q5 (the force f is associated with a potential energy fx), 3C5 2005 Q5a, 3C5 2006 Q4a-b, IB sample paper 2019 Q4, Revision sheet Q1.
1D dynamics and Effective potentials: 3C5 2014 Q4,3C5 2018 Q3, IB sample paper 2019 Q6, Revision sheet Q2, 2P1 2019 Q1

Answers

3. $T = 2\pi\sqrt{2\sqrt{2}a/(3g)}$

4(a). $\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}kx_1^2$.

5(a). $\mathcal{L}(r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}$, $\ddot{r} = -GM/r^2 + r\dot{\theta}^2$, $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$

6(a). $\mathcal{L} = \frac{1}{2}m\dot{z}_1{}^2 - mgz_1 - \frac{1}{2}k(z_1 - l)^2$, $\quad \mathcal{L} = \frac{1}{2}m\dot{z}_1{}^2 + \frac{1}{2}m\dot{z}_2{}^2 - mgz_1 - mgz_2 - \frac{1}{2}k(z_1 - z_2 - l)^2$.

6(b). $\mathcal{L} = m\dot{z}_c^2 + \frac{1}{4}m\dot{z}_s^2 - 2mgz_c - \frac{1}{2}k(z_s - l)^2$

7(a). (i) $A = 2ml^2,\ B = ml^2,\ C = ml^2$, $D = 2mgl$ and $E = mgl$.

7(a). (ii) $A = 4ml^2/3,\ B = ml^2/3,\ C = ml^2/2$, $D = 3mgl/2$ and $E = mgl/2$.

7(a). (iii) $A = m_pa^2 + \frac{1}{3}m_aa^2 + m_ba^2, B = m_b\frac{1}{3}b^2, C = \frac{1}{2}m_bab, D = g\left(m_pa + m_a\frac{a}{2} + m_ba\right), E = gm_b\frac{b}{2}$.

7(b). $A\ddot{\alpha} + C\ddot{\beta}\cos(\alpha - \beta) + C\dot{\beta}^2\sin(\alpha - \beta) = -D\sin\alpha$,
$B\ddot{\beta} + C\ddot{\alpha}\cos(\alpha - \beta) - C\dot{\alpha}^2\sin(\alpha - \beta) = -E\sin\beta$.

7(c). $(\theta_1, \theta_2) = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi)$

8(b). $\frac{\sin\theta_i}{\sin\theta_f} = \sqrt{1 + \frac{2gh}{v^2}}$