UNIVERSITY OF CAMBRIDGE
Department of Engineering

IB Paper 5
Electromagnetic Fields & Waves

Lecture 2
Transmission Lines II

https://www.vle.cam.ac.uk/course/view.php?id=70081

Characteristic Impedance

FLE99, GER86

- A transmission line consists of two or more conductors that guide the flow of energy in the form of an electromagnetic wave
- We looked at an equivalent circuit for a short length of an ideal transmission line to give the ***Telegrapher's Equations***

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \qquad\qquad \frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} \tag{1.1}$$

- Combining these produced a ***wave equation*** for both current and voltage on the line

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} \qquad\qquad \frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} \tag{1.3}$$

- The solution to these are equations for voltage and current as a function of position and time

$$V = \overline{V_F}e^{j(\omega t-\beta x)} + \overline{V_B}e^{j(\omega t+\beta x)} \tag{1.11}$$

$$I = \overline{I_F}e^{j(\omega t-\beta x)} + \overline{I_B}e^{j(\omega t+\beta x)} \tag{1.13}$$

  - $\overline{V_F}$, $\overline{V_B}$, $\overline{I_F}$ and $\overline{I_B}$ are complex numbers representing both the amplitude and phase offset of the voltage and current waves travelling in the positive x-direction (forward) and negative x-direction (backwards)
- An attenuation term is added to Eqns. 1.11 and 1.13 if the transmission line is 'lossy'

- The voltage and current on the transmission line are clearly related to each other by the Telegraphers Equations (Eqn. 1.1)

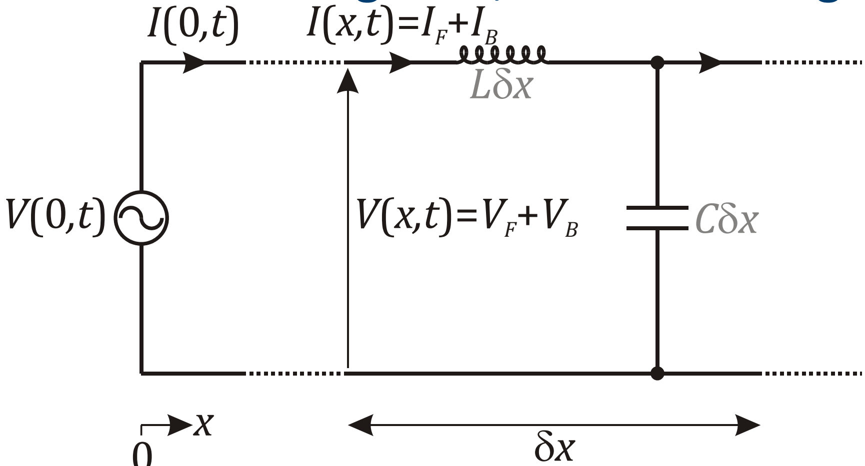
$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}$$

$$\left(-\beta\overline{I_F}e^{-j\beta x} + \beta\overline{I_B}e^{j\beta x}\right) = -C\left(\omega\overline{V_F}e^{-j\beta x} + \omega\overline{V_B}e^{j\beta x}\right)$$

  - We have cancelled the common factor of $je^{j\omega t}$
  - We can also separately equate the forward ($e^{-j\beta x}$) and backward ($e^{j\beta x}$) terms separately to give

$$\frac{\overline{V_F}}{\overline{I_F}} = \frac{\beta}{\omega C} \qquad\qquad \frac{\overline{V_B}}{-\overline{I_B}} = \frac{\beta}{\omega C} \qquad (2.1)$$

- Why are the equations for the relationship between V and I the same for both the forward and backward waves apart from the '–' on $\overline{I_B}$?
  - The voltages are defined as being the voltage on the 'upper' conductor with respect to the lower, and the currents are defined as being in the positive x-direction
  - When the wave is travelling in the negative x-direction, the current will be negative, but the voltage is not



- We can therefore *define* a ***Characteristic Impedance***, Z_0 as being *the ratio between the voltage and current of a unidirectional wave at any point on a transmission line*

$$Z_0 = \frac{\overline{V_F}}{\overline{I_F}} = \frac{\overline{V_B}}{-\overline{I_B}} = \frac{\beta}{\omega C} \tag{2.2}$$

  - However, $\beta = 2\pi/\lambda$ and $\omega = 2\pi f$, so from Eqn. 1.7

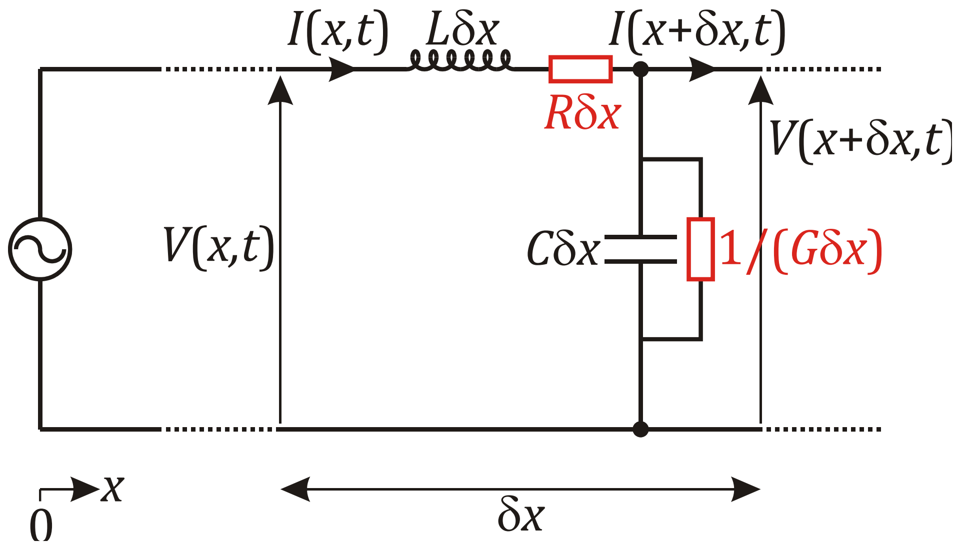
$$\frac{\beta}{\omega} = \frac{1}{c} = \sqrt{LC}$$

  - Substituting this into Eqn. 2.2 gives

$$\boxed{Z_0 = \sqrt{L/C}} \tag{2.3}$$

- Z_0 is always a real number for an ideal, lossless line
- Although Z_0 has units of Ω, it ***does not dissipate power*** as V and I are not between the same point
- It is the apparent impedance that is 'seen' if looking into an infinitely long line at $x = 0$ (i.e. $V(0,t)/I(0,t)$)
- Substitution of L and C for a real transmission line results in an equation of the form

$$Z_0 = \text{Geometry Factor} \times \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}} \tag{2.4}$$



- If the line is 'lossy' then the characteristic impedance gains a frequency dependence

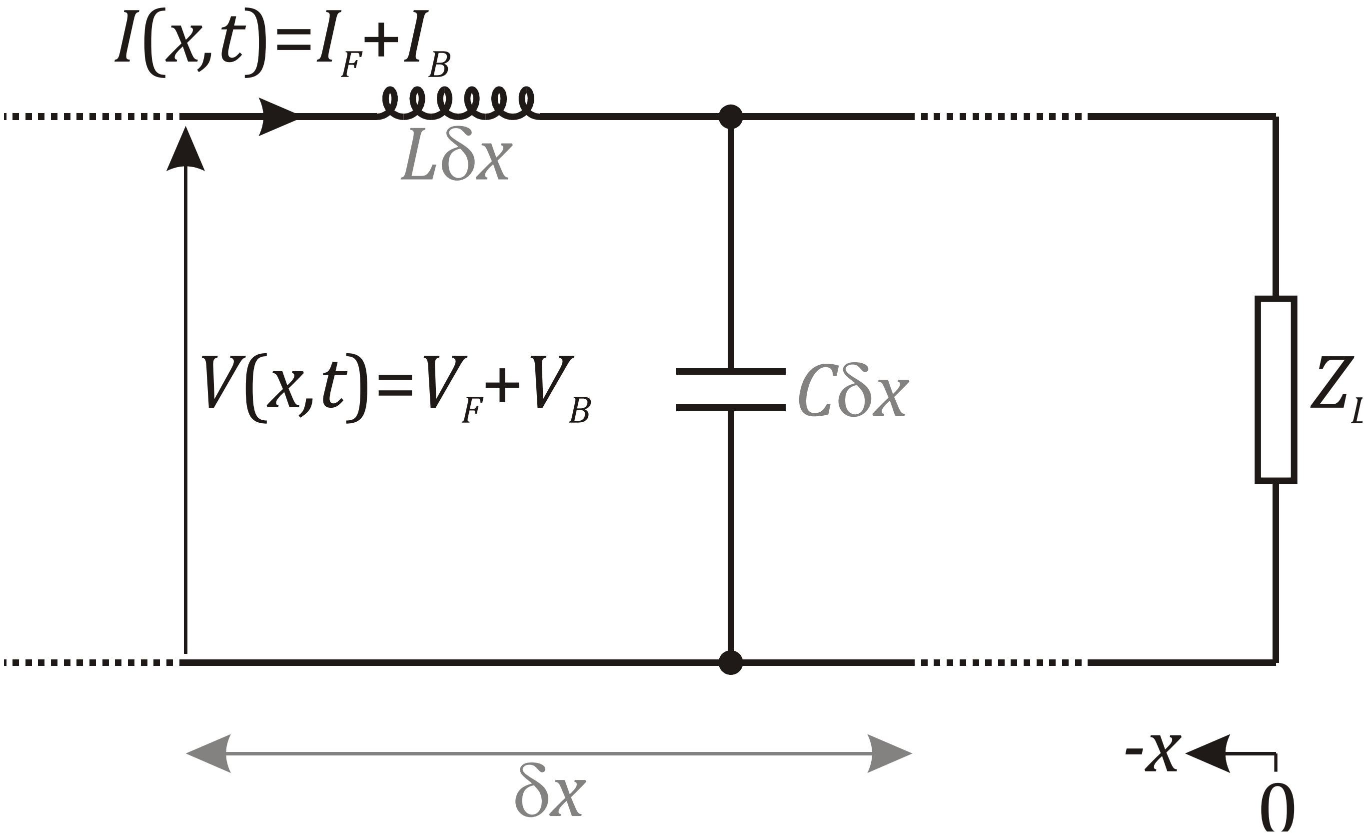
$$\boxed{Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}} \qquad (2.5)$$

- Why is the Characteristic Impedance important?
  - Let's imagine that we want to transmit an a.c. signal down a coaxial cable from a signal generator with an output impedance Z_s
  - The maximum power transfer theorem shows that the impedance of a load has to match the output impedance of a source for maximum power transfer into the load
  - Therefore, we need $Z_s = Z_0$ to maximise the signal in the coaxial cable
  - It is for this reason that almost all coaxial cables are either 50 Ω or 75 Ω and equipment with BNC inputs or outputs are also 50 Ω or 75 Ω (e.g. the output from a satellite TV receiver box)

Reflections from a Load Impedance

FLE102, GER87

- We will always want to connect our transmission line to something
  - An analogue electronic circuit for processing the signal
  - A digital logic circuit for processing data
  - A load for dissipating power
- Let us consider a situation where we have a wave from some source that is travelling in the forward direction down a transmission line which is terminated in a load impedance Z_L at $x = 0$
- What happens when the wave reaches the load?



- We clearly have a forward travelling wave from the source, but we should allow for the fact that some of the wave may be reflected by the load to give a backwards travelling wave
  - As we have defined the load as being at $x = 0$, Eqns. 1.11 and 1.13 at the load become

$$V(0,t) = \overline{V_F} e^{j(\omega t - \beta 0)} + \overline{V_B} e^{j(\omega t + \beta 0)} = \left(\overline{V_F} + \overline{V_B}\right) e^{j\omega t} \quad (2.6)$$

$$I(0,t) = \overline{I_F} e^{j(\omega t - \beta 0)} + \overline{I_B} e^{j(\omega t + \beta 0)} = \left(\overline{I_F} + \overline{I_B}\right) e^{j\omega t} \quad (2.7)$$

  - As these are the voltage and current across the load Z_L

$$Z_L = \frac{V(0,t)}{I(0,t)} = \frac{\left(\overline{V_F} + \overline{V_B}\right)}{\left(\overline{I_F} + \overline{I_B}\right)} \quad (2.8)$$

  - From Eqn. 2.2 we also know that $\overline{V_F}$ and $\overline{I_F}$, and $\overline{V_B}$ and $\overline{I_B}$ are related by Z_0, so we can substitute the currents for voltages

$$Z_L = \frac{\left(\overline{V_F} + \overline{V_B}\right)}{\left(\overline{V_F}/Z_0 - \overline{V_B}/Z_0\right)} = Z_0 \frac{\left(\overline{V_F} + \overline{V_B}\right)}{\left(\overline{V_F} - \overline{V_B}\right)} \quad (2.9)$$

  - This can be rearranged to make $\overline{V_B}$ the subject so we can see how big the reflected wave is

$$\overline{V_B} = \overline{V_F} \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) \quad (2.9)$$

  - We now define the voltage reflection coefficient ρ_L as

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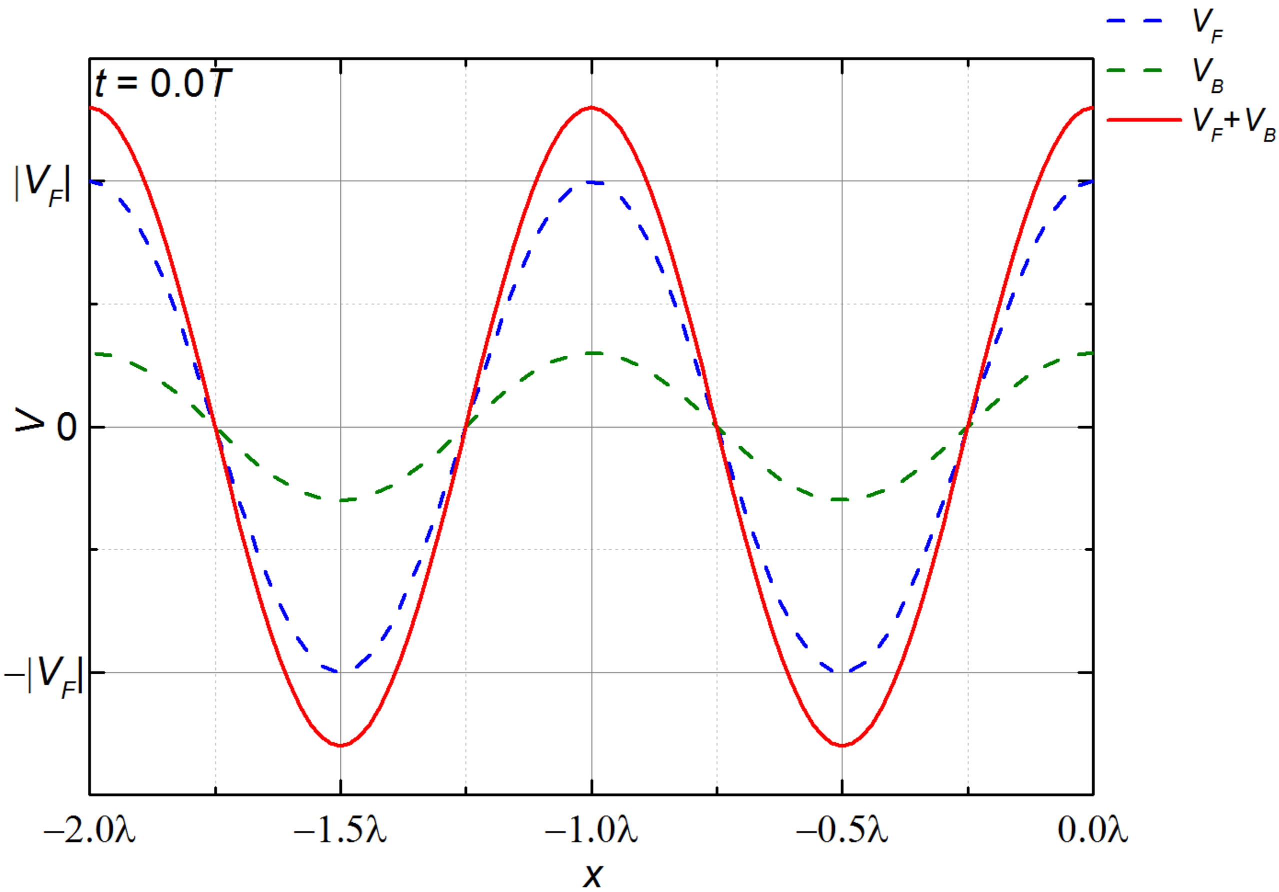
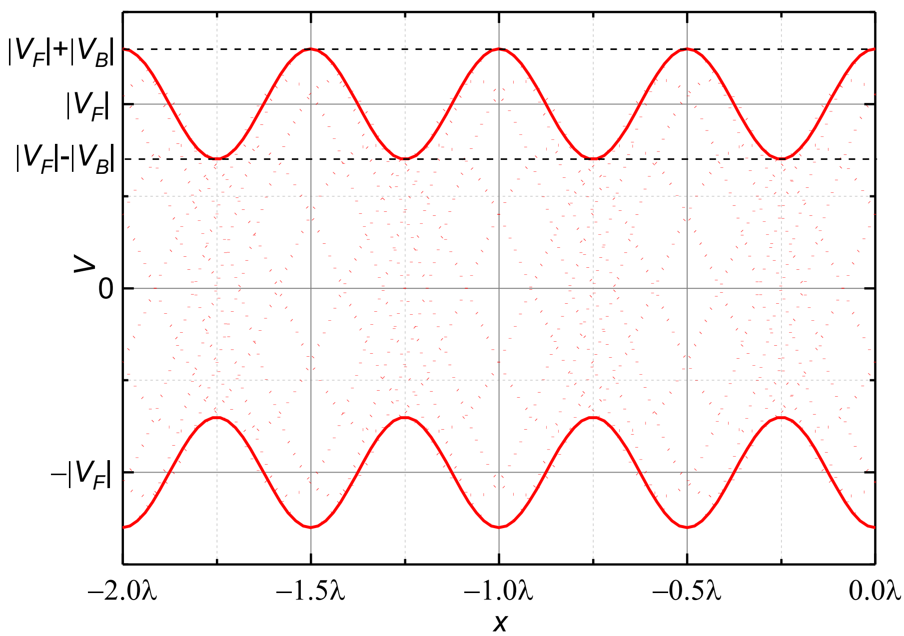
$$\boxed{\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0}} \quad (2.10)$$



- A proportion ρ_L of the forward wave V_F is being reflected to give a backward wave V_B and these two sum together to produce a ***standing wave***

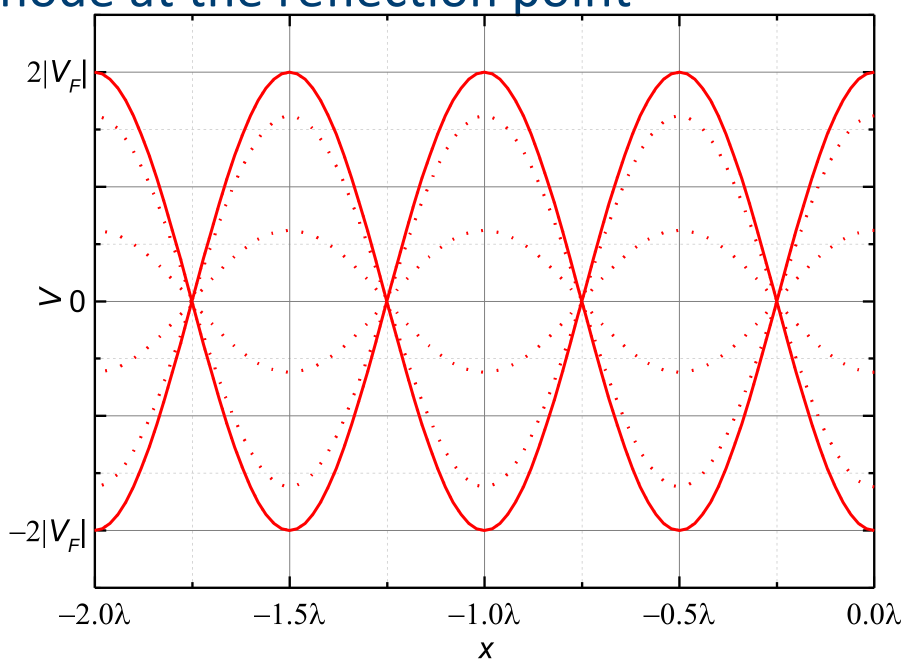
$$V(x,t) = \overline{V_F}\left(e^{-j\beta x} + \rho_L e^{j\beta x}\right)e^{j\omega t}$$



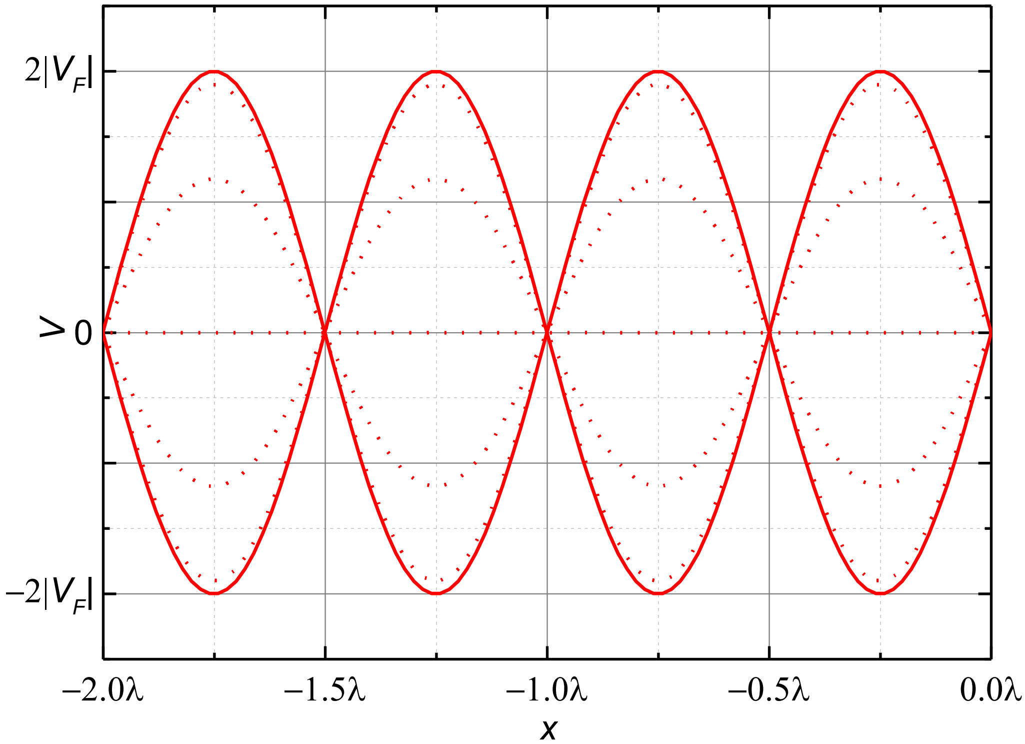
- We can define a ***voltage standing wave ratio*** (VSWR) as being the ratio of the maximum voltage amplitude and minimum voltage amplitude

$$\boxed{\mathrm{VSWR} = \frac{|\overline{V_F}| + |\overline{V_B}|}{|\overline{V_F}| - |\overline{V_B}|} = \frac{1 + |\rho_L|}{1 - |\rho_L|}} \quad (2.11)$$

- If some of the wave is being reflected, then this implies that some power is being reflected too
  - As power is proportional to V^2 then the proportion of the incident power that is reflected is $|\rho_L|^2$
- Three special cases for Z_L now emerge
  1. If the transmission line is terminated with an **open circuit**, then $Z_L = \infty$ and $\rho_L = 1$
     - The incident wave is reflected with no change in phase for voltage at the reflection point
     - A perfect standing wave ($\mathrm{VSWR} = \infty$) is formed with an antinode at the reflection point



2. If the transmission line is terminated with a **closed circuit**, then $Z_L = 0$ and $\rho_L = -1$
   - The incident wave is reflected with a π phase change for voltage at the reflection point
   - A perfect standing wave ($VSWR = \infty$) is formed with a node at the reflection point

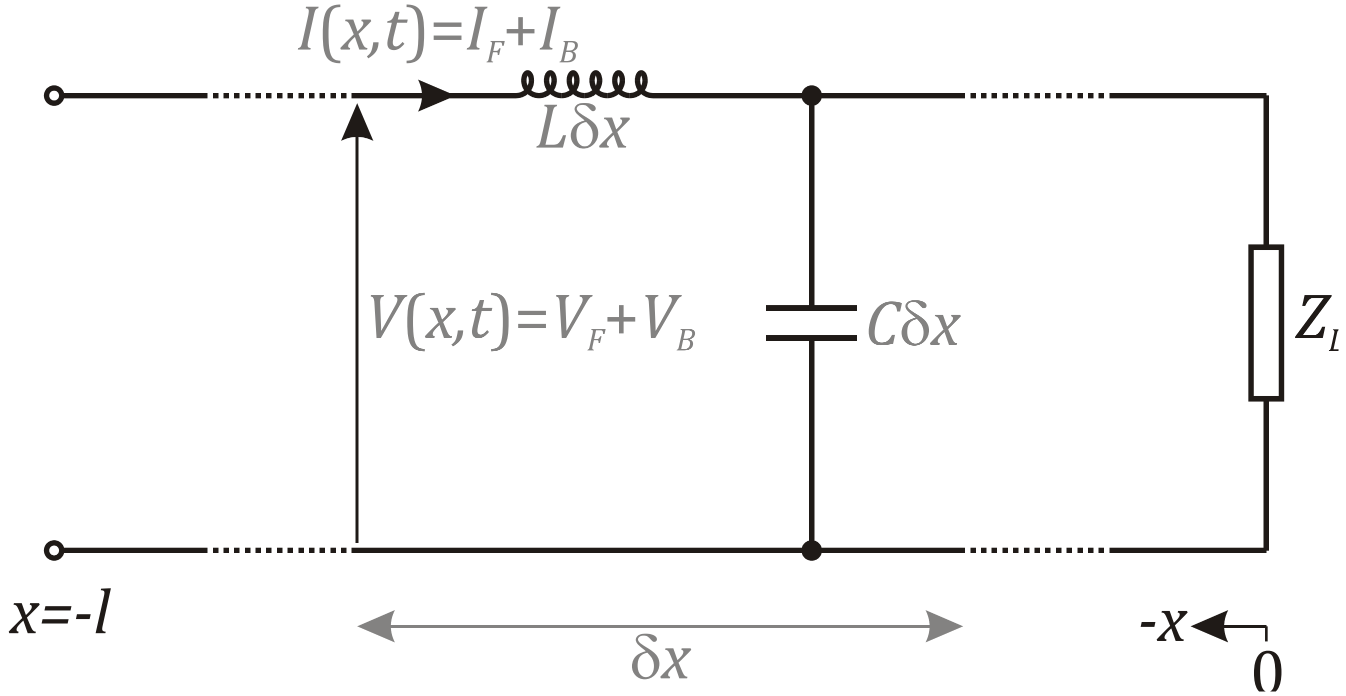


3. If the transmission line is terminated with a load whose impedance is the same as the characteristic impedance of the transmission line ($Z_L = Z_0$) then $\rho_L = 0$
   - No wave is reflected and all of the power is dissipated in the load
   - This is the ideal scenario for most situations, such as the BNC signal input to a television, which is made to present a load of 50 Ω to match the characteristic impedance of the coaxial cable supplying the signal

Input Impedance of a Terminated Line

FLE109

- Very often, we have to connect our line to a load that is not a matched impedance
  - An example might be a WiFi box where we are trying to broadcast a 2.45 GHz signal from an antenna
    - The antenna is designed to maximise the broadcast power, and so its impedance might not be equal to the cable supplying the signal
  - We will want to attach a signal source to the input of the cable, and so we will need to know what impedance the source will 'see'
- Let us take a scenario where the cable has a characteristic impedance Z_0
  - It is terminated with an impedance of Z_L at $x = 0$
  - The cable has a length l, and so the input is at $x = -l$



- We can calculate an apparent impedance at any point x in the line as being simply the ratio of the voltage to current at that point using Eqns. 1.11 and 1.13

$$Z(x) = \frac{\overline{V_F} e^{j(\omega t - \beta x)} + \overline{V_B} e^{j(\omega t + \beta x)}}{\overline{I_F} e^{j(\omega t - \beta x)} + \overline{I_B} e^{j(\omega t + \beta x)}} \tag{2.12}$$

  - We can lose a factor of $e^{j\omega t}$ and use the characteristic impedance to turn currents into voltages (Eqn. 2.2)

$$Z(x) = \frac{\overline{V_F} e^{-j\beta x} + \overline{V_B} e^{j\beta x}}{\frac{\overline{V_F}}{Z_0} e^{-j\beta x} - \frac{\overline{V_B}}{Z_0} e^{j\beta x}} \tag{2.13}$$

  - We can also divide all through by $\overline{V_F}$

$$Z(x) = Z_0 \frac{e^{-j\beta x} + \frac{\overline{V_B}}{\overline{V_F}} e^{j\beta x}}{e^{-j\beta x} - \frac{\overline{V_B}}{\overline{V_F}} e^{j\beta x}} \tag{2.14}$$

  - This allows us to re-express the apparent impedance in terms of the voltage reflection coefficient using Eqn. 2.9

$$Z(x) = Z_0 \frac{e^{-j\beta x} + \rho_L e^{j\beta x}}{e^{-j\beta x} - \rho_L e^{j\beta x}} \tag{2.15}$$

  - We can also explicitly show the dependence on Z_L using Eqn. 2.10

$$Z(x) = Z_0 \frac{(Z_L + Z_0) e^{-j\beta x} + (Z_L - Z_0) e^{j\beta x}}{(Z_L + Z_0) e^{-j\beta x} - (Z_L - Z_0) e^{j\beta x}} \tag{2.16}$$

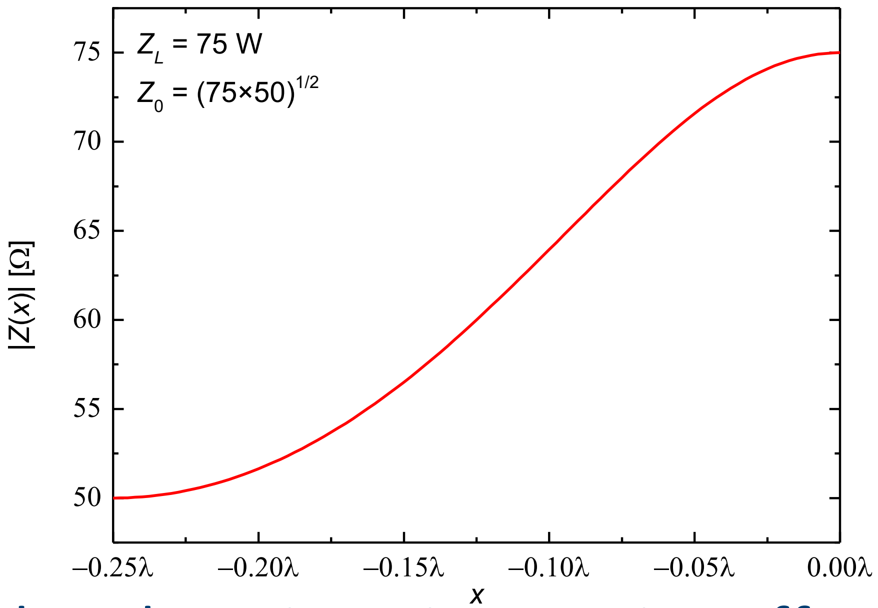
- In this situation, its turns out to be easier to visualise what is going on if we use the de Moivre Theorem to turn our complex exponentials into trigonometric functions

$$Z(x) = Z_0 \frac{Z_L \cos(\beta x) - jZ_0 \sin(\beta x)}{-jZ_L \sin(\beta x) + Z_0 \cos(\beta x)} \tag{2.16}$$

- Therefore, the impedance looking into the line of length l is

$$\boxed{Z(x=-l) = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}} \tag{2.17}$$

- The input impedance depends on the length of the line



- In particular, there is an interesting effect when then length of the line $l = \lambda/4$, as at this point $\beta l = \pi/2$ and $\tan(\beta l) = \infty$

$$Z(x=-\lambda/4) = Z_0 \frac{Z_L + jZ_0 \infty}{Z_0 + jZ_L \infty}$$

$$Z(x=-\lambda/4) = \frac{Z_0^2}{Z_L} \tag{2.18}$$

- We can use this phenomenon to connect mismatched loads to a transmission line

5/6 Q8

  - The technique is called ***quarter wave matching***
  - For example, imagine that we wish to connect a transmission line with a characteristic impedance of 50 Ω to a load impedance of 75 Ω without any reflections
  - We connect a short length of another transmission line between the 50 Ω line and the load
  - This additional line should have a length of $\lambda/4$ and a characteristic impedance, Z_0, equal to the geometric mean of the 50 Ω transmission line and the load, so

$$Z_0 = \sqrt{Z_L Z_{in}}$$

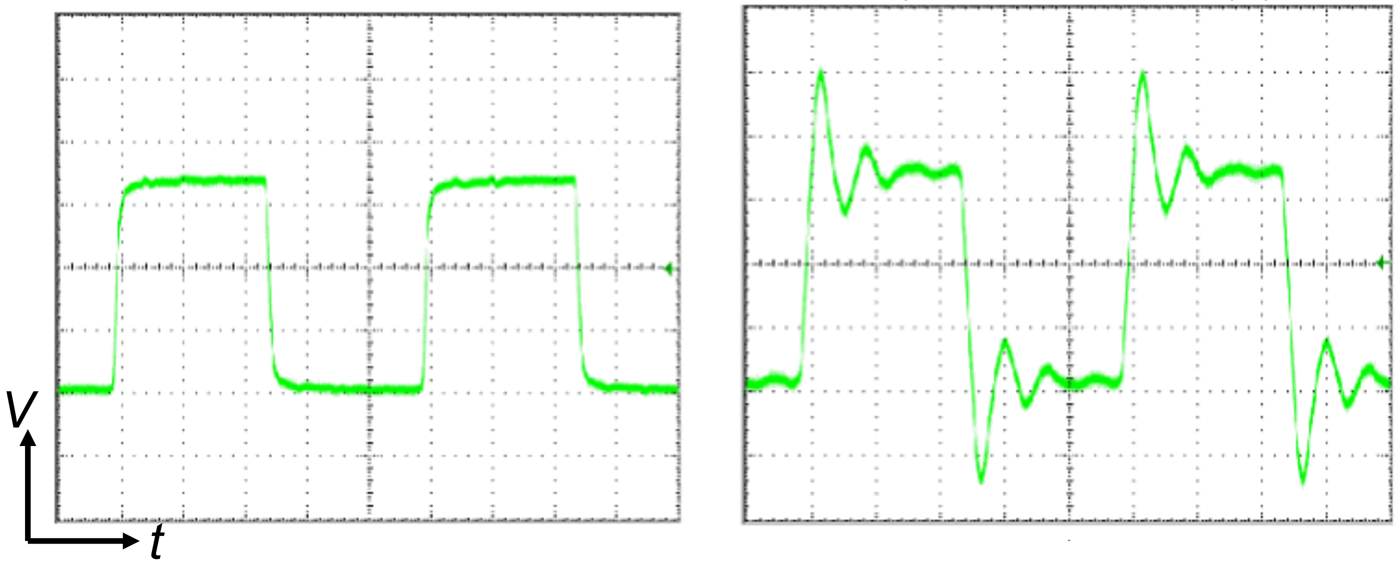
  - This 'matching line' will appear to have an input impedance of 50 Ω, and so it will not cause any reflections at the junction with the 50 Ω transmission line as their impedances are the same
  - The apparent impedance then appears to vary smoothly up to 75 Ω at the load
  - As the load is also 75 Ω, then there is no reflection here either
  - All the input power is therefore dissipated in the load with no reflections, as required

Ringing

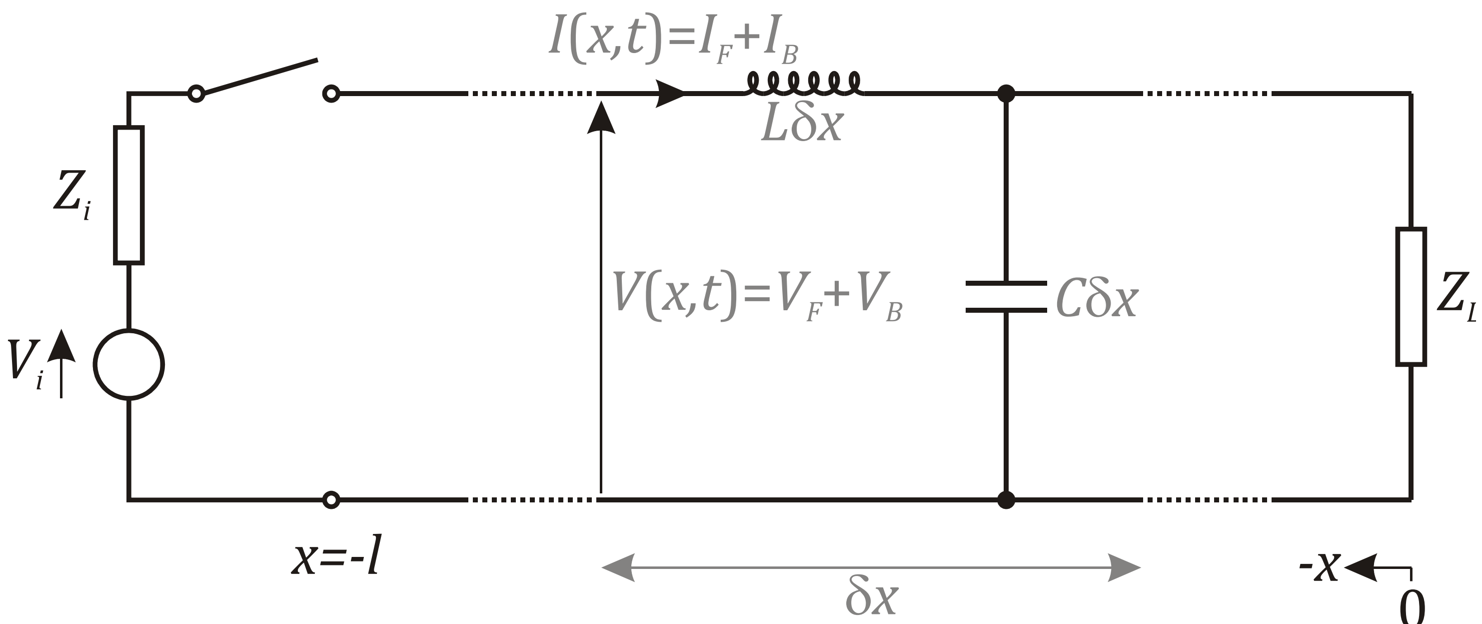
FLE112

- A common scenario is that we wish to send a digital data signal down a transmission line
  - Ideally this should be a square wave, but in practice it takes some time for the voltage to settle after each change

From http://www.ni.com/white-paper/3854/en/



  - Let us consider applying our input signal using a simple voltage source with an impedance Z_i and a switch connected to the transmission line of length l



- Let the switch close at time $t = 0$
- At that moment, the load impedance can have no effect on the input to the line, so, treating this as a simple potential divider, the voltage transmitted into the line is

$$V(x = -l, t = 0) = V_i\left(\frac{Z_0}{Z_i + Z_0}\right)$$

- This voltage will then be transmitted along the line at the wave velocity $c = 1/\sqrt{LC}$ (from Eqn. 1.7), arriving at the load some time T later, where

$$T = l/c = l\sqrt{LC}$$

- Upon reaching the load, the reflection coefficient will determine how much of the wave is reflected (Eqn. 2.10),

$$\rho_L = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)$$

- This will sum with the input voltage to give

$$V(x = 0, t = T) = V_i\left(\frac{Z_0}{Z_i + Z_0}\right)[1 + \rho_L] \tag{2.19}$$

- This will then take a further time T to travel back to the input end of the transmission line, where the reflected component will itself be reflected again with Z_i now looking like the load, so

$$\rho_i = \left(\frac{Z_i - Z_0}{Z_i + Z_0}\right)$$

- Hence

$$V(x=-l, t=2T) = V_i\left(\frac{Z_0}{Z_i+Z_0}\right)[1+\rho_L+\rho_L\rho_i] \quad (2.20)$$

- If there is another complete round-trip of reflections, then the voltage at the input will be

$$V(x=-l, t=4T) = V_i\left(\frac{Z_0}{Z_i+Z_0}\right)[1+\rho_L+\rho_L\rho_i+\rho_L^2\rho_i+\rho_L^2\rho_i^2] \quad (2.21)$$

- We can now generalise this to a series expression for n round-trips

$$V(x=-l, t=2nT) = V_i\left(\frac{Z_0}{Z_i+Z_0}\right)\left[1+\sum_{n=1}^{n}\left(\rho_L^n\rho_i^{n-1}+\rho_L^n\rho_i^n\right)\right] \quad (2.22)$$

- This geometric series then converges with time