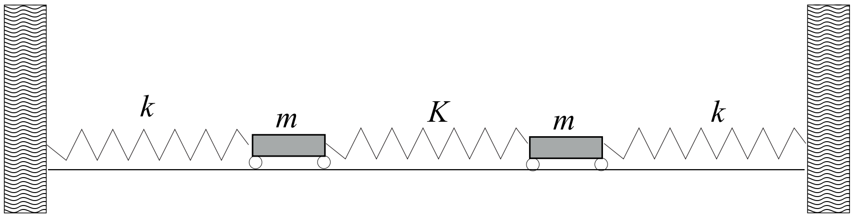
Paper 1: Mechanical Engineering

Examples Paper 4

Elementary exercises are marked †, problems of Tripos standard ∗.
Answers can be found at the back of the paper.

Normal Modes

1 † Two masses m and three springs, k, K, k, are arranged as shown below. The springs are unstretched when the system is in its equilibrium configuration.

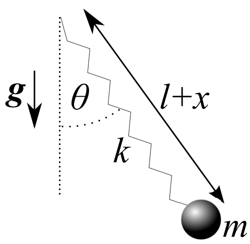


(a) Write down the Lagrangian for the system in terms of x_1 and x_2, the displacements of the masses from equilibrium.

(b) Identify the M and K matrices from the Lagrangian, and hence find the system's normal modes of oscillation .

(c) Sketch how the how the normal mode frequencies (in units of $\sqrt{k/m}$) vary as a function of K/k.

2 A *spring pendulum* is formed from a mass m hung on a light spring, with spring constant k and natural length l.



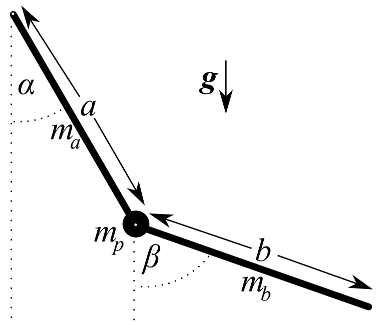
(a) Find the Lagrangian for the system in terms of the angle θ the extension of the spring x.

(b) Find the equations of motion for θ and x, and identify the (stable) equilibrium configuration, x_0, θ_0.

(c) Identify the M and K matrices, firstly by linearizing the equations of motion around x_0 and θ_0, and secondly by expanding the Lagrangian around x_0 and θ_0. Verify these two approaches give the same answer.

(d) Find the frequencies for the normal modes of oscillation, and sketch the corresponding motions.

3 *The double pendulum used in the double pendulum lab can be modeled as two uniform rigid rods, one of length a and mass m_a, and the other with length b and m_b, connected by a pivot with mass m_p, as shown below.



A generic double pendulum has a Lagrangian of the form:

$$\mathcal{L} = \frac{1}{2}\left(A\dot{\alpha}^2 + B\dot{\beta}^2 + 2C\dot{\alpha}\dot{\beta}\cos(\alpha - \beta)\right) + D\cos\alpha + E\cos\beta,$$

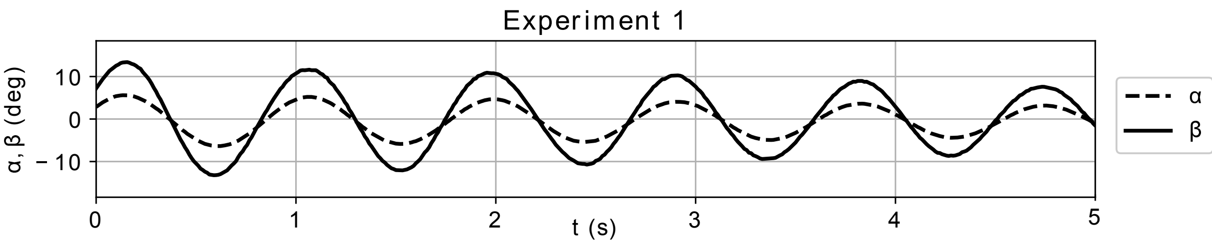
and you calculated $A....E$ for various examples, including the lab double pendulum, on the previous examples sheet.

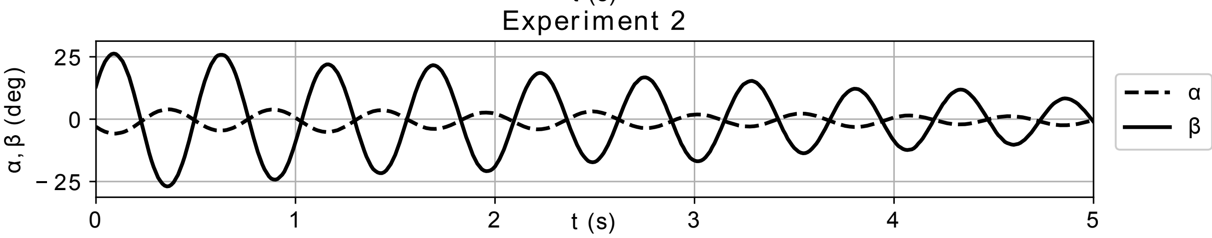
(a) Expand the generic Lagrangian to quadratic order, assuming α and β are small, and hence identify the M and K matrices, and the frequencies of the normal modes.

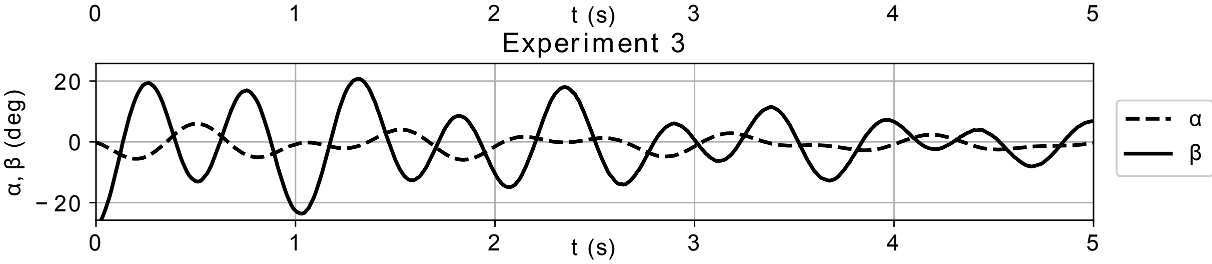
(b) The lab double pendulum has $a = 0.185$ m $b = 0.172$ m, $m_a = 0.044$ kg , $m_b = 0.019$ kg, $m_p = 0.022$ kg. What are the numerical values for the frequencies and mode shapes $\chi = \alpha/\beta$ for the lab double pendulum? Do these agree with your data?

(c) Find the frequencies/growth rates for the lab double pendulum if you expanding around the $\alpha = \pi$, $\beta = 0$ equilibrium state. Is this state stable or unstable?

(d) Three low amplitude motions from the double pendulum lab are shown below. Do they appear consistent with your normal modes calculation?

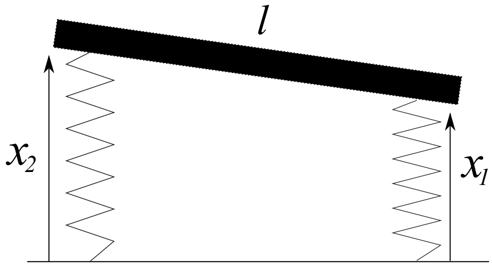






Use the python notebook double_pendulum_vibrations.ipynb to fit theoretical modes to this data. Bring graphs of the best fits to your supervision. Reminder: a plot showing your best fit between theory and your version of experiment 3 is part of the D1 submission/credit (see 2P1 moodle.)

4 A uniform rod, of length l and mass m, is supported by two springs, each with spring constant k and natural length l. The position of the rod is described using x_1 and x_2, the heights of the two ends, and gravity can be ignored.

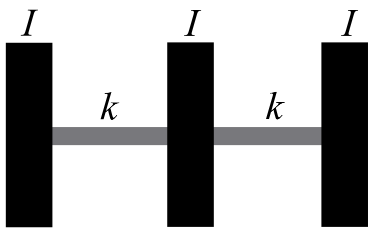


(a) Identify the equilibrium values of x_1 and x_2, and find the Lagrangian for the system assuming x_1 and x_2 are close to these equilibrium values.

(b) Identify the M and K matrices from the Lagrangian, and find the rod's normal modes of oscillation.

(c) (None examinable extension) Rewrite the Lagrangian in terms of the normal coordinates for the system, and comment on their physical interpretation.

5 * A simple model of a jet engine comprises three identical thin rigid discs mounted equidistant on a uniform light shaft. The disks have moment of inertia I, and the shaft between two masses behaves as torsional spring, storing potential energy $(1/2)k\theta^2$ when the disks have an angular separation θ.



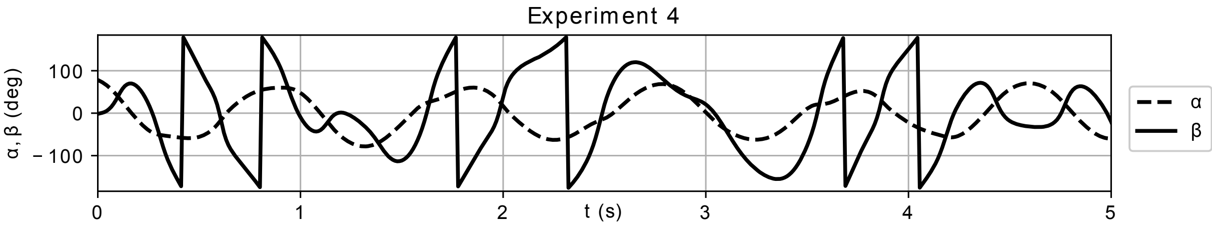
(a) Find the Lagrangian for the system, and identify the M and K matrices.

(b) Find the normal modes of the system, and comment on why they are orthogonal.

(c) The entire engine is rotating at angular velocity ω when a small object enters the engine produces an abrupt change $\Delta\Omega$ in the angular velocity of the first disk. Calculate the maximum resultant angle of twist of the shaft between the discs.

Numerical integration and Chaos

6 An example of high-amplitude motion from the lab double pendulum is shown below.



The python notebook double_pendulum_simulation.ipynb provides a numerical integrator for the double pendulum and several functions to help plot the output data. Complete the notebook, making sure you fulfill the following tasks:

(a) Make plots showing your best fit between simulation and Experiments 1 and 4.

(b) Conduct a simulation with initial condition $(\alpha, \beta, \dot{\alpha}, \dot{\beta}) = (\alpha_0, 0, 0, 0)$ with $\alpha_0 = 10°$. Compare this motion with motion starting from $\alpha_0 + \Delta\alpha$ for $\Delta\alpha = 10, 1, 0.1, 0.01°$....

(c) Repeat the above exercise but with $\alpha_0 = 90°$, and make a plot of divergence time vs $\Delta\alpha$. Can we ever make long term predictions for this system?

(d) Create Poincare sections for simulations with a range of initial conditions, including examples which are periodic, quasiperiodic and chaotic.

Bring plots summarizing your results for each task to your supervision. Reminder: a plot showing your best fit between simulation and your version of Experiment 4 is part of the D1 submission/credit (see 2P1 moodle.)

Time varying Lagrangians

7 † A pendulum consists of a mass m on a light rod l. The pendulum is mounted in an elevator, which is accelerating upwards at a.

(a) Find the time period for small oscillations of the pendulum.

(b) What acceleration will cause the downward configuration to loose stability?

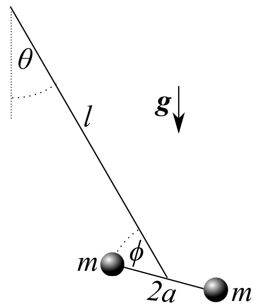
8 * The system from question one is prepared with three identical springs, $K = k$. An additional horizontal force $F = F_0 \cos(\omega t)$ is applied to the left hand mass.

(a) Write down the Lagrangian for the system, including the driving force.

(b) Find the equations of motion of the system, and write them in matrix form.

(c) Find the steady-state response of the system (the particular integral) and sketch the amplitude of the responses of x_1 and x_2 as a function of ω.

9 * A simple model of a child on a swing is sketched below. The swing is modeled as a light rod, of length l, which makes an angle θ with the vertical. The child is modeled as two point masses m on either end of a light rod of length $2a$. The rod is attached at its center to the end of the swing, and makes an angle ϕ with the swing rod. The angle $\phi(t)$ is not a dynamical degree of freedom, but rather controlled by the child in order to keep the swing moving.



(a) Find the system's Lagrangian and show that the equation of motion for θ is:

$$(l^2 + a^2)\ddot{\theta} + a^2\ddot{\phi} = -gl\sin\theta.$$

(b) The child pumps the swing by alternating between a laying flat and sitting upright, giving $\phi = \phi_0 + b\sin(\omega t)$. What value of ω should the child select to swing the most effectively?

(c) A second strategy for swinging involves standing on the swing and alternating between crouching down and standing upright. In this case we have a parametric oscillator, with the length of the pendulum being changed as a function of time. Given this is a parametric oscillator, what frequency of crouching and standing should the child select to swing the most effectively?

Computing Help

The Python examples are very easy to run online without any installation:

1. Go to https://colab.research.google.com/github/CambridgeEngineering/PartIB-Mechanics (or short link https://bit.ly/3dEbTJk if you're typing it);
2. Click on the relevant Jupyter Notebook template file (e.g. doublependulumvibrations.ipynb);
3. This opens a copy of the Jupyter Notebook which you can edit and run;
4. You might be asked to login to a Google account. You can use your university email address (see https://help.uis.cam.ac.uk/service/collaboration/g-suite/g-suite-registration);
5. Click on Copy to Drive near the top so that you can save changes.

You can also run the files locally by installing Python. The notebooks can be downloaded directly from https://github.com/CambridgeEngineering/PartIB-Mechanics. The most straightforward way to instal Python is to download 'Anaconda' from: https://www.anaconda.com/download/. Once installed, then open the 'Jupyter Notebook' app from the start menu found inside the Anaconda folder. You can navigate to the folder where you are keeping your `*.ipynb` files and open the templates.

Suitable past Tripos questions

Normal Modes: 3C5 2017 Q4, 3C5 2012 Q4, 3C5 2011 Q4, 3C5 2010 Q4, 3C5 2007 Q4, 3C5 2003 Q5, IB sample paper 2019 Q5, Revision sheet Q3 & Q4, 2P1 2019 Q6.
Numerical integration and Chaos: This content is part of the double pendulum lab, not for examination
Time varying Lagrangians: 3C5 2009 Q5, 3C5 2003 Q4, 2P1 2019 Q5.

Answers

1(a). $\mathcal{L} = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 - \frac{1}{2}k(x_1 - x_2)^2$

1(b). $\omega^2 = k/m,\ (x_1, x_2) = (1, 1), \omega^2 = (2K + k)/m,\ (x_1, x_2) = (1, -1)$

2(a). $\mathcal{L} = \frac{1}{2}m\left((l + x)^2\dot{\theta}^2 + \dot{x}^2\right) + mg(l + x)\cos\theta - \frac{1}{2}kx^2.$

2(b). $m\ddot{x} = -kx + mg\cos\theta + m(l + x)\dot{\theta}^2$
$m(l + x)^2\ddot{\theta} + 2m(l + x)\dot{x}\dot{\theta} = -mg(l + x)\sin\theta$
$(x_0, \theta_0) = (mg/k, 0)$

2(d). $(x, \theta) = (1, 0)\ \omega^2 = k/m$; $(x, \theta) = (0, 1), \omega^2 = g/(l + mg/k)$

3(a). $\omega^2 = \frac{AE+DB\pm\sqrt{(AE+DB)^2-4(AB-C^2)DE}}{2(AB-C^2)}.$

3(b). $\omega = 6.80,\ 12.2\ \mathrm{rads}^{-1}$, $\chi = 0.529, -0.265$

3(c). $\omega = 10.1\mathrm{rads}^{-1}$ and $\omega = i/\tau$ with $\tau = 0.123$ s.

4(a). $\mathcal{L} = \frac{1}{6}m(\delta\dot{x}_1^2 + \delta\dot{x}_1\delta\dot{x}_2 + \delta\dot{x}_2^2) - \frac{1}{2}k(\delta x_1^2 + \delta x_2^2).$

4(b). $(\delta x_1, \delta x_2) = (1, 1), \omega^2 = 2k/m$ and $(\delta x_1, \delta x_2) = (1, -1), \omega^2 = 6k/m.$

5(a). $\mathcal{L} = \frac{1}{2}I\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right) - \frac{1}{2}k\left((\theta_1 - \theta_2)^2 + (\theta_3 - \theta_2)^2\right)$

5(b). $\omega_0^2 = 0, \omega_1^2 = k/I, \omega_2^2 = 3k/I.$

5(c). $\theta_1 - \theta_2 = \frac{\Delta}{2\omega_1}\left(1 + \frac{1}{\sqrt{3}}\right)$

7(a). $T = s\pi\sqrt{l/(g + a)}.$

7(b). $a < -g$

9(b). The natural frequency, $\omega_0^2 = \frac{gl}{l^2+a^2} \approx \frac{g}{l}$

9(c). Drive at $2\omega_0$