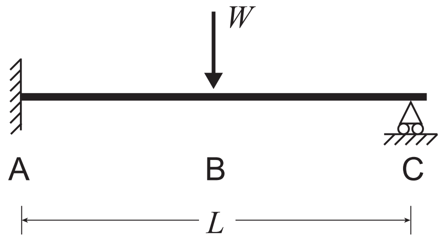
4.9 Lower-Bound Analysis

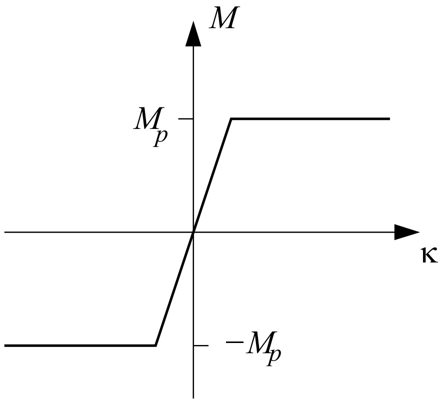
4.9.1 Intoduction

So far, our plasticity theory has only considered work equations for compatible collapse mechanisms (ignoring equilibrium), and this gives upper bounds on collapse loads. However, there is another side to plasticity theory. This Section will show that an alternative method of considering equilibrium *without considering compatibility* allows us to find a lower-bound on the collapse load.

4.9.2 Collapse of a propped cantilever

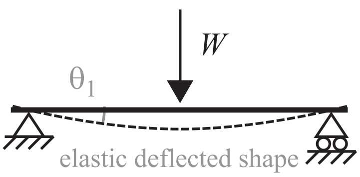


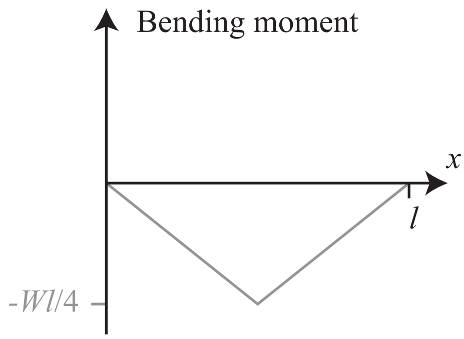
In Section 4.1 we looked in detail at the collapse of a statically determinate beam. Here we will look at the more complex series of events that occurs during the collapse of a statically indeterminate structure. We shall assume that the moment-curvature relationship is a simplified elastic-plastic relationship, for simplicity (see Section 4.1.4 for a more detailed analysis).



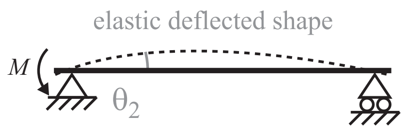
Initial Calculations

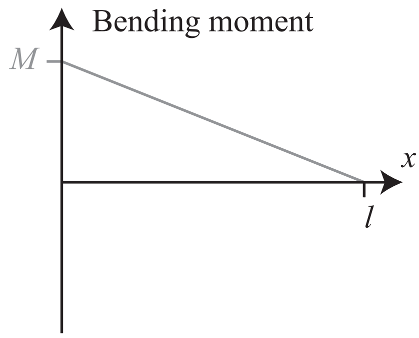
We shall make the structure statically determinate by adding a pin at the left hand support, and hence splitting the response in two.
Particular Equilibrium Solution in equilibrium with the applied load





State of Self-Stress in equilibrium with zero applied load





The Particular Equilibrium Solution must be present for equilibrium with the applied load. There is also an unknown amount of the State of Self Stress — but we cannot find the magnitude (M) by equilibrium alone.
Moments — For any value of M, the moment at the root is:

$$M_{\text{root}} = M \tag{4.1}$$

The moment at the centre is:

$$M_{\text{centre}} = -\frac{Wl}{4} + \frac{M}{2} \tag{4.2}$$

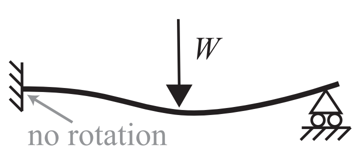
Data Book

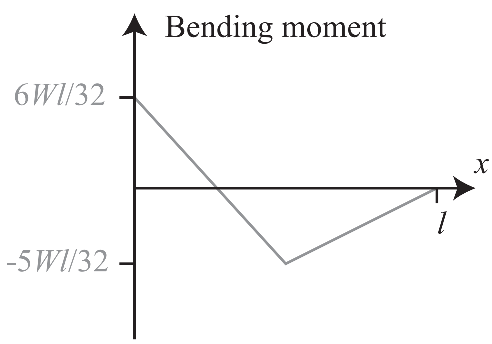
$$\theta_1 = \frac{Wl^2}{16EI}$$
$$\theta_2 = \frac{Ml}{3EI}$$

Before Yield

The structure is elastic, and we can use compatibility to find M, the magnitude of the state of self-stress
Compatibility $\theta_1 = \theta_2$

$$\Rightarrow M = \frac{3Wl}{16}$$





First Plastic Hinge Forms

The maximum moment is at the root of the cantilever. A plastic hinge will first form here when

$$\frac{6Wl}{32} = M_p$$
$$W = \frac{16M_p}{3l}$$

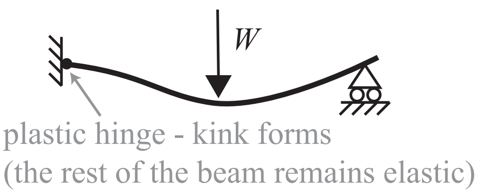
Although a plastic hinge has formed, *this does not lead to collapse*. As we know from the upper-bound method, a propped cantilever cannot collapse until two plastic hinges have formed.

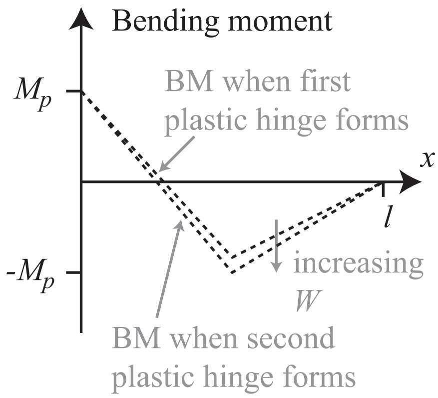
Single Plastic Hinge

We can no longer use compatibility to find M, the magnitude of the state of self-stress. Because a plastic hinge has formed, it is quite possible for a kink to appear here, and hence all we can say is that $\theta_1 \geq \theta_2$.

We can, however, use our knowledge of plasticity to find M. As a plastic hinge has formed, we know that

$$M = M_p$$

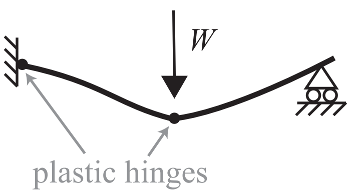


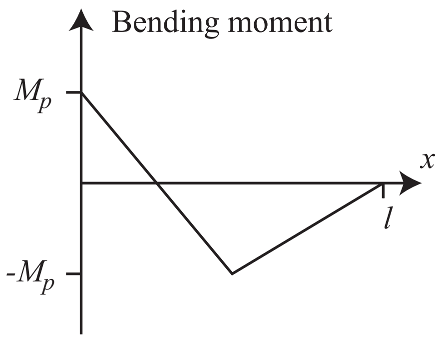


The plastic hinge prevents any increase in the magnitude of the state of self-stress. The load can keep increasing, however, but the load is taken by increasing the particular solution only.

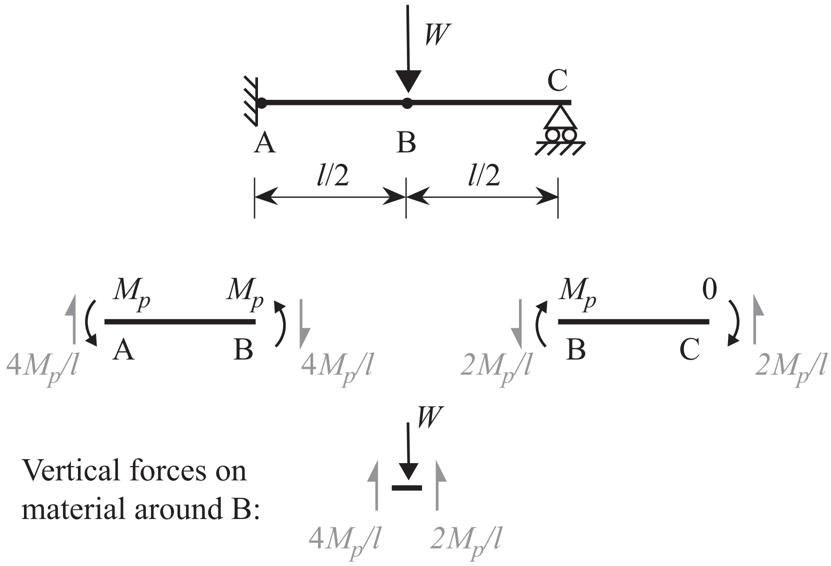
Final Collapse

Once the bending moment at the centre also reaches M_p, the structure can carry no more load, and the final collapse load is reached.





What load corresponds to this collapse load? There are two approaches: (i) consider the Particular Equilibrium Solution and State of Self-Stress (we'll do this later); (ii) Consider free-body diagrams of the sections of the cantilever as it collapses — we know the moment at the root is $+M_p$, and at the centre is $-M_p$.



$$W = \frac{2M_p}{l} + \frac{4M_p}{l} = \frac{6M_p}{l}$$

The exact sequence of events leading to collapse depends on the initial conditions — we have assumed that the structure was initially stress-free. If it wasn't, due to e.g. settlement of supports, the sequence would change, and for instance the first hinge to form may be the one in the centre. However, the final collapse state would be the same *whatever the initial conditions*.

4.9.3 Lower-bound theorem of plasticity

> *If a set of internal stresses can be found in the structure that are in equilibrium with an applied load W_{equil}, and nowhere violate the yield condition, then the applied load will be less than, or equal to, the actual collapse load W_c.*
>
> $$W_{\text{equil}} \leq W_c$$

A proof of this theorem will be given at the end of the handout.

We can use the lower-bound theorem to avoid having to consider the detailed sequence of events leading to collapse. For beam structures, all we need to do is to find a set of moments in equilibrium with the applied load that nowhere exceed the plastic moment M_p. We must have the Particular Equilibrium Solution, but we can add as much or as little of the State of Self-Stress to give the largest possible load.

We don't have to consider *compatibility* at all — kinks will form as necessary at plastic hinges.

Equilibrium systems for the propped cantilever

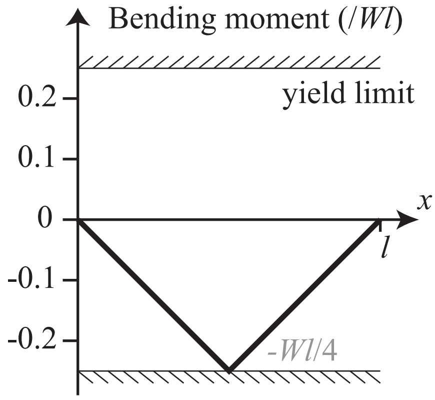
All the systems below are in equilibrium with the applied load, but only one is the optimum lower-bound solution. We will use the Particular Equilibrium Solution and State of Self-Stress shown in Section 4.9.2. Equations 4.1 and 4.2 then give the moments at the root and the centre in terms of the magnitude of the Self-Stress, M

$$M_{\text{root}} = M$$

The moment at the centre is:

$$M_{\text{centre}} = -\frac{Wl}{4} + \frac{M}{2}$$

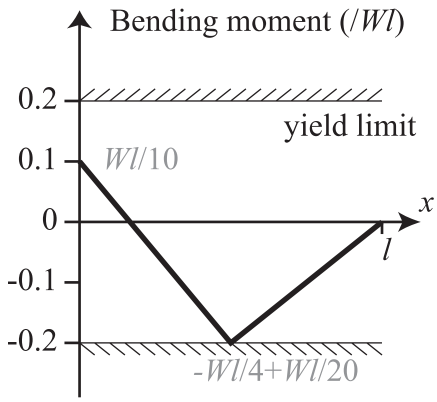
1. Choose $M = 0$.



If this solution is about to violate the yield condition (at the centre):

$$-\frac{W_1 l}{4} = -M_p$$
$$W_1 = 4\frac{M_p}{l}$$

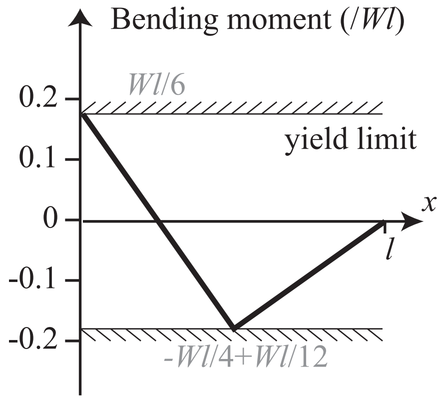
2. Choose $M = Wl/10$.



If this solution is about to violate the yield condition (at the centre):

$$-\frac{5W_2 l}{20} + \frac{W_2 l}{20} = -M_p$$
$$W_2 = 5\frac{M_p}{l}$$

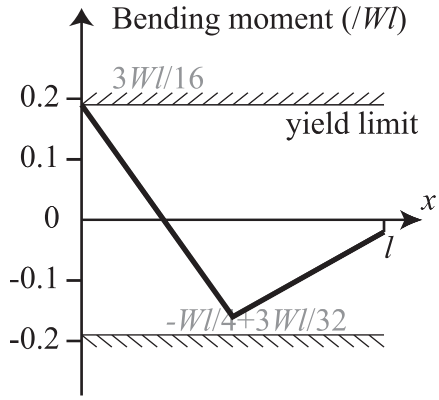
3. Choose $M = Wl/6$.



If this solution is about to violate the yield condition (at the centre and root simultaneously):

$$-\frac{3W_3 l}{12} + \frac{W_3 l}{12} = -M_p$$
$$W_3 = 6\frac{M_p}{l}$$

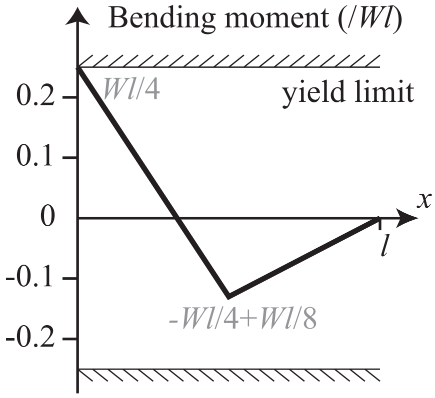
4. Choose $M = 3Wl/16$ (elastic solution).



If this solution is about to violate the yield condition (at root):

$$\frac{3W_4 l}{16} = M_p$$
$$W_4 = \frac{16}{3}\frac{M_p}{l} = 5.33\frac{M_p}{l}$$

5. Choose $M = Wl/4$.



If this solution is about to violate the yield condition:

$$\frac{W_5 l}{4} = +M_p$$
$$W_5 = \frac{4M_p}{l}$$

All of the solutions given are in equilibrium, and do not violate yield, and are therefore lower-bounds on the true collapse load W_c

$$W_1 = W_5 < W_2 < W_4 < W_3 = W_c = \frac{6M_p}{l}$$

W_3 is the largest (optimum) lower-bound solution, which equals the smallest (optimum) upper-bound solution from a collapse analysis (see Section 4.5.1), and is therefore the true collapse load.

Although the lower-bound method can be applied to many different systems, e.g. frames or slabs, equilibrium for these systems can be difficult. We will instead only concentrate on multi-span beams, where a Particular Equilibrium Solution, and States of Self-Stress, are very easy to calculate.

4.9.4 Useful equilibrium results

It is helpful to remember some equilibrium results for a simply-supported beam of length l (although these can, of course, easily be calculated from first principles). The moment sign convention is given in the Structures Data Book, and all forces are applied vertically down.

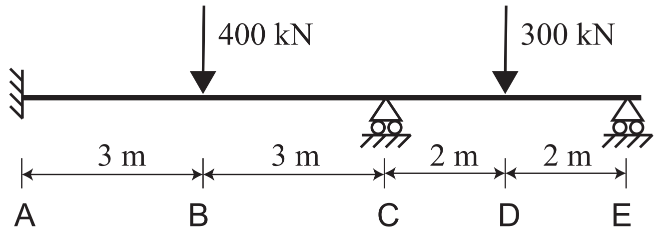
- When a central point load of magnitude W is applied, the largest magnitude bending moment is $M = -Wl/4$ at the centre of the beam, and the bending moment diagram is piecewise linear.
- When a point load of magnitude W is applied at a distance a from a support, the largest magnitude bending moment is $M = -Wa(l-a)/l$ at the point where the load is applied, and the bending moment diagram is piecewise linear.

- When a distributed load of magnitude w per unit length is applied to the entire beam, the bending moment at a distance x from a support is $M = -wx(l-x)/2$, and the largest magnitude bending moment is $M = -wl^2/8$.
- When a couple C is applied at a support, the bending moment at a distance x from the other support is $M = \pm Cx/l$.

Try Questions 7 and 8, Examples Sheet 2/5

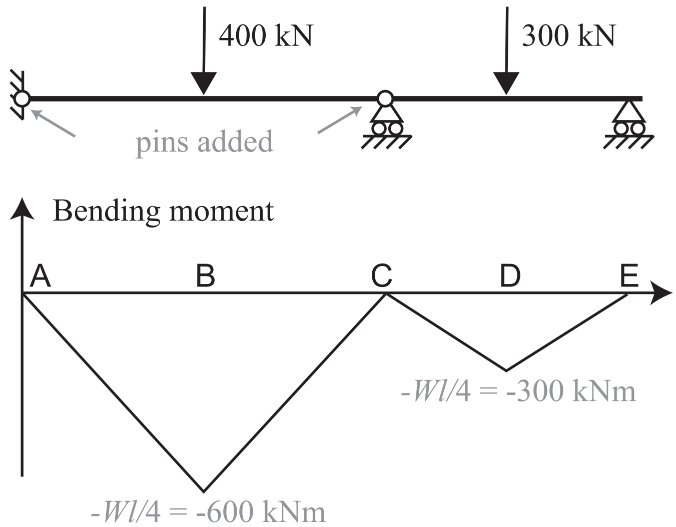
4.9.5 Example — two-span beam

For the structure shown below, use the lower-bound theorem to find a UB (Universal Beam) that would be suitable for the entire span AE. The loads shown include a safety factor. The beam is to be made from steel with yield stress $\sigma_y = 300\ \mathrm{N/mm^2}$.

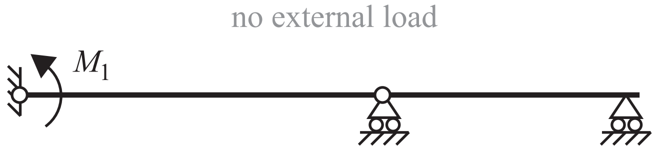


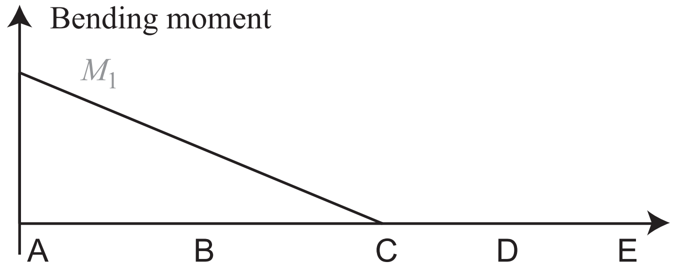
We will make this determinate by adding pins at the joints.

Particular Equilibrium Solution

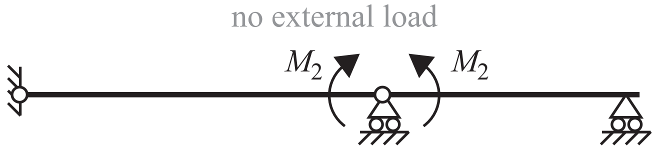


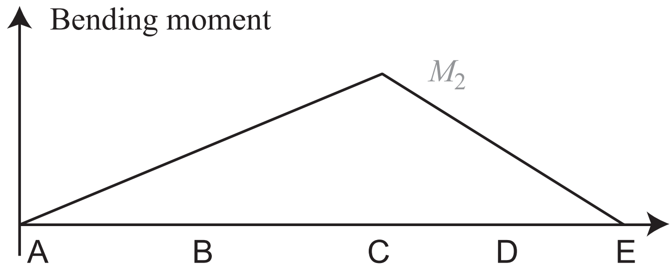
State of Self-Stress 1





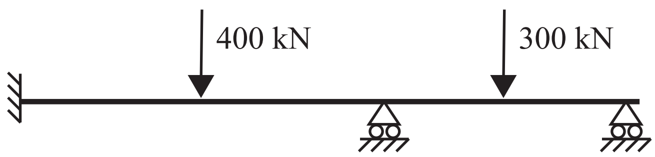
State of Self-Stress 2

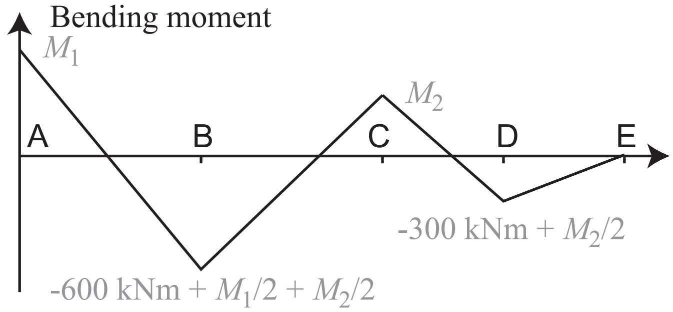




All possible equilibrium systems must contain the Particular Equilibrium Solution, but can include any amount of the two States of Self-Stress.

General Bending Moment Diagram





Optimum solution

We want to design a beam that is as small as possible, while still finding a set of moments where $|M| \leq M_p$ everywhere.

The maximum moments occur at A and C, $M_A = M_1$, and $M_C = M_2$. An optimum solution will have

$$M_2 = M_1$$

The minimum moments occur at B and D. Try optimizing at B first, and check D afterwards

$$M_B = -600 \text{ kNm} + \frac{M_1}{2} + \frac{M_2}{2}$$

Make the magnitude of the minimum moment (at B) equal to the magnitude of the maximum moment (at A and C)

Choose $M_B = -M_A = -M_C = -M_1$

$$\begin{aligned} -M_1 &= -600 \text{ kNm} + M_1 \\ -2M_1 &= -600 \text{ kNm} \\ M_1 &= 300 \text{ kNm} \quad (= M_2) \end{aligned}$$

We have ensured that $|M| \leq 300$ kNm everywhere in span AC, but we also need to check span CE. What is the minimum moment, at D?

$$\begin{aligned} M_D &= -300 \text{ kNm} + \frac{M_2}{2} \\ M_D &= -150 \text{ kNm} \end{aligned}$$

We have now ensured that $|M| \leq 300$ kNm everywhere in the entire structure. If we now pick a beam that is at least this strong, then the lower-bound theorem shows that this is safe. We require a beam where

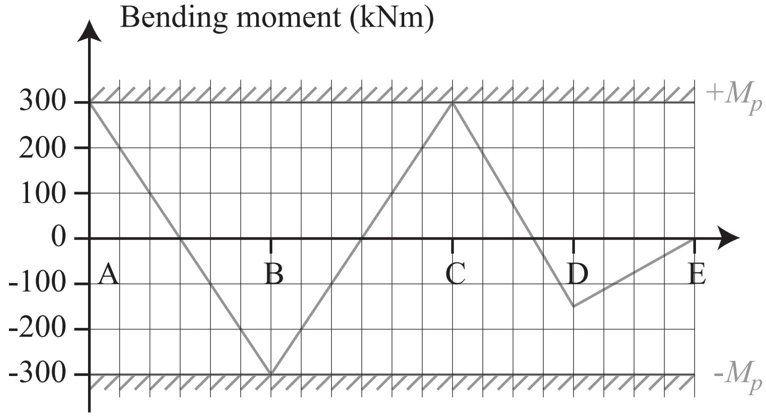
$$Z_p = \frac{M_p}{\sigma_y} \geq \frac{300 \times 10^3 \text{ Nm}}{300 \times 10^6 \text{ Nm}^{-2}}$$
$$Z_p \geq 1000 \times 10^{-6} \text{ m}^3 = 1000 \text{ cm}^3$$

From the structures data book, a suitable section would be

a UB $356 \times 171 \times 57$

$$\begin{aligned} Z_p &= 1010 \text{ cm}^3 \\ M_p &= 303 \text{ kNm} \end{aligned}$$

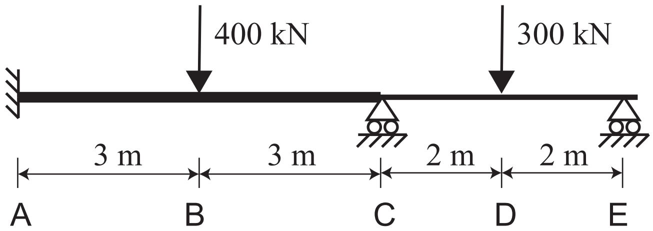
The final bending moment diagram that we chose is shown below:

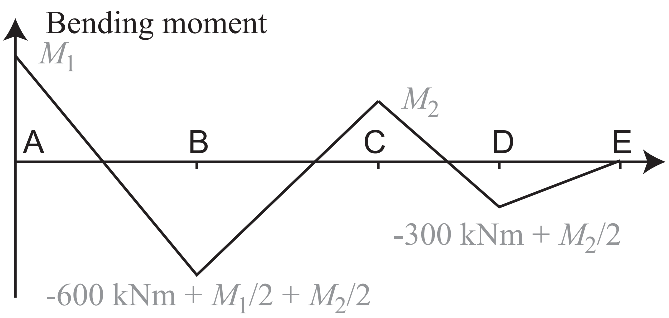


Because this BMD: (i) is in equilibrium with the applied load; (ii) nowhere exceeds the yield limit, then this is a safe design, by the lower-bound theorem.

4.9.6 Example — two-span beam 2

The previous structure was optimized for span AC, but span CE was stronger than it needed to be. An alternative would be to use different beams for each span, welded together at C.





At C, we have to ensure that the moment does not exceed the plastic moment for the weaker beam. Because of this, it is easier to design the weaker beam CE first.

$$M_C = M_2$$
$$M_D = -300\text{ kNm} + \frac{M_2}{2}$$

Make the magnitude of the maximum moment (at C) equal to the magnitude of the minimum moment (at D)

$$
\begin{aligned}
M_C &= -M_D \\
M_2 &= 300\text{ kNm} - \frac{M_2}{2} \\
\frac{3M_2}{2} &= 300\text{ kNm} \\
M_2 &= 200\text{ kNm}
\end{aligned}
$$

Now we know $M_C = M_2$ we can design AC

$$
\begin{aligned}
M_A &= M_1 \\
M_B &= -600\text{ kNm} + \frac{M_1}{2} + \frac{M_2}{2} \\
&= -500\text{ kNm} + \frac{M_1}{2}
\end{aligned}
$$

Make the magnitude of the maximum moment (at A) equal to the magnitude of the minimum moment (at B).

$$
\begin{aligned}
-M_A &= M_B \\
-M_1 &= -500\text{ kNm} + \frac{M_1}{2} \\
M_1 &= 333\text{ kNm}
\end{aligned}
$$

We have now ensured that $|M| \leq 333$ kNm in AC, and $|M| \leq 200$ kNm in CE. For AC we require

$$
\begin{aligned}
Z_p = \frac{M_p}{\sigma_y} &\geq \frac{333 \times 10^3\text{ Nm}}{300 \times 10^6\text{ Nm}^{-2}} \\
Z_p &\geq 1111 \times 10^{-6}\text{ m}^3 = 1111\text{ cm}^3
\end{aligned}
$$

From the structures data book, a suitable section would be a UB $356 \times 171 \times 67$

$$
\begin{aligned}
Z_p &= 1211\text{ cm}^3 \\
M_p &= 363\text{ kNm}
\end{aligned}
$$

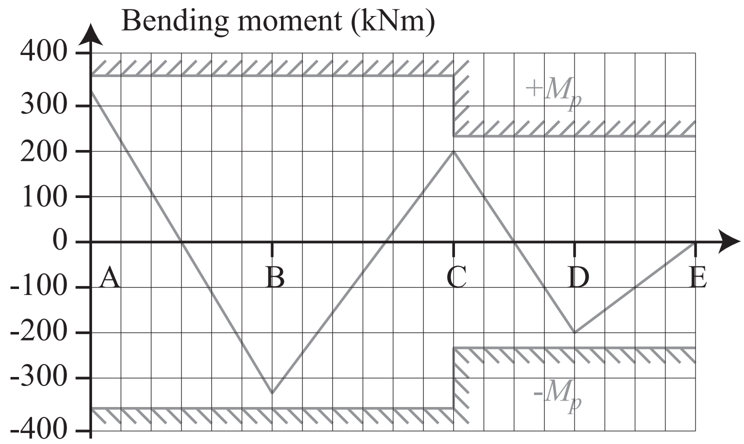
For CE we require

$$
\begin{aligned}
Z_p = \frac{M_p}{\sigma_y} &\geq \frac{200 \times 10^3\text{ Nm}}{300 \times 10^6\text{ Nm}^{-2}} \\
Z_p &\geq 667 \times 10^{-6}\text{ m}^3 = 667\text{ cm}^3
\end{aligned}
$$

From the structures data book, a suitable section would be a UB $356 \times 171 \times 45$

$$
\begin{aligned}
Z_p &= 775\text{ cm}^3 \\
M_p &= 233\text{ kNm}
\end{aligned}
$$

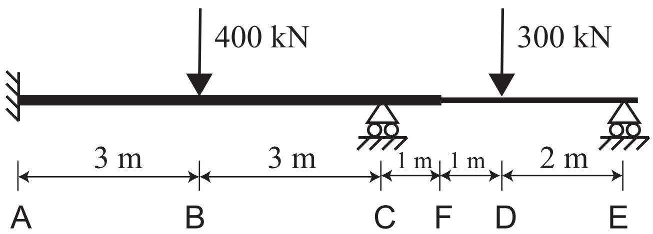
The final bending moment diagram is:

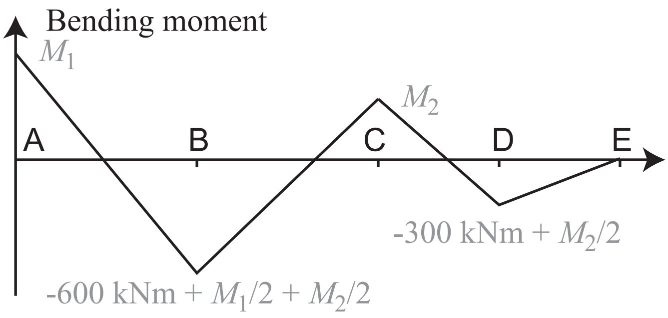


Again, because this BMD: (i) is in equilibrium with the applied load; (ii) nowhere exceeds the yield limit, then this is a safe design, by the lower-bound theorem.

4.9.7 Example — two-span beam 3

In some ways, the previous structure was a foolish design, as the strength was reduced at a point where there was a peak in the bending moment (at C). A better design would reduce the strength a little further along, e.g. at F, below.





Because C is now full-strength, we can design the left-hand part of the beam (AF) as for the first design. Hence we choose $M_1 = M_2 = 300$ kNm, giving $|M|_{\max}$ as 300 kNm.

We can now investigate FE. The minimum moment will occur at D

$$M_D = -300\text{ kNm} + \frac{M_2}{2} = -150\text{ kNm}$$

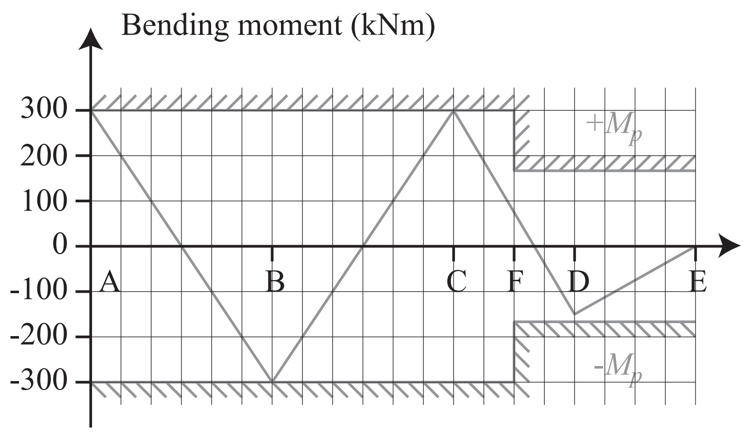
M_p for FE must be at least this — we also need to check the moment at F.

$$M_F = \frac{-300\text{ kNm}}{2} + \frac{3M_2}{4} = 75\text{ kNm}$$

We have now ensured that $|M| \leq 300$ kNm in AF, and $|M| \leq 150$ kNm in FE. For AF we require $Z_p \geq 1000\text{ cm}^3$, and can choose UB $356 \times 171 \times 57$ as before. For FE we require $Z_p \geq 500\text{ cm}^3$, and from the structures data book, we can choose
a UB $356 \times 127 \times 33$

$$Z_p = 543\text{ cm}^3$$
$$M_p = 163\text{ kNm}$$

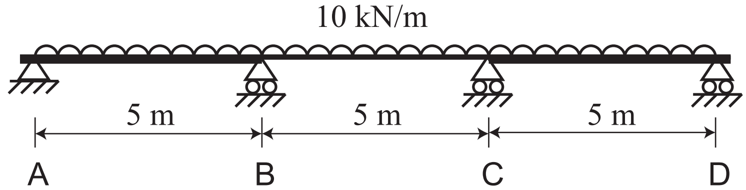
The final bending moment diagram is shown below:



Despite all of this optimization, it is likely that the initial design would be the cheapest, because of the simplicity in construction.

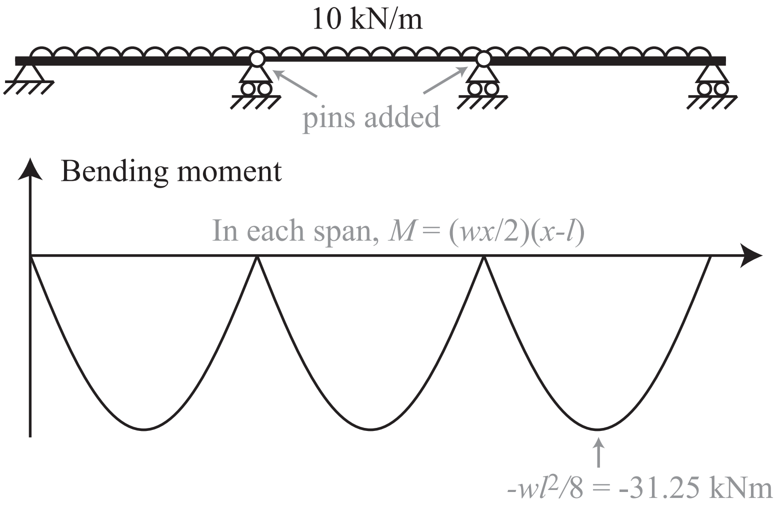
4.9.8 Example — A three-span beam

It has been suggested that a suitable design for the three-span beam shown below it to have $M_p = \{32;16;32\}$ kNm in the three spans $\{AB;BC;CD\}$. Use the Lower-Bound Theorem to show that this is a safe design

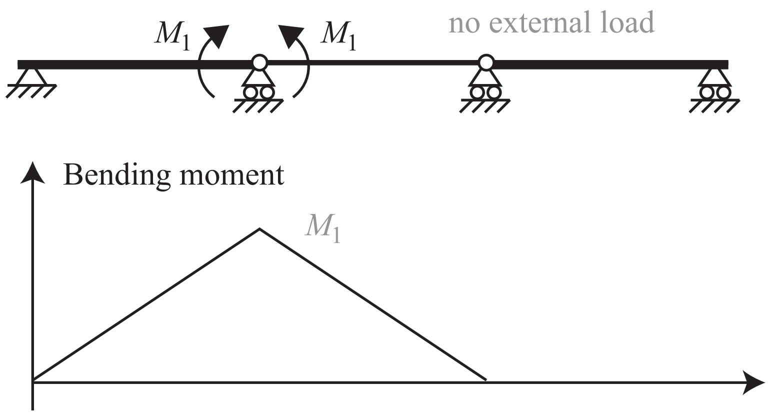


Make the structure statically determinate by adding pins over the supports, and calculate the Particular Equilibrium Solution and the States of Self-Stress.

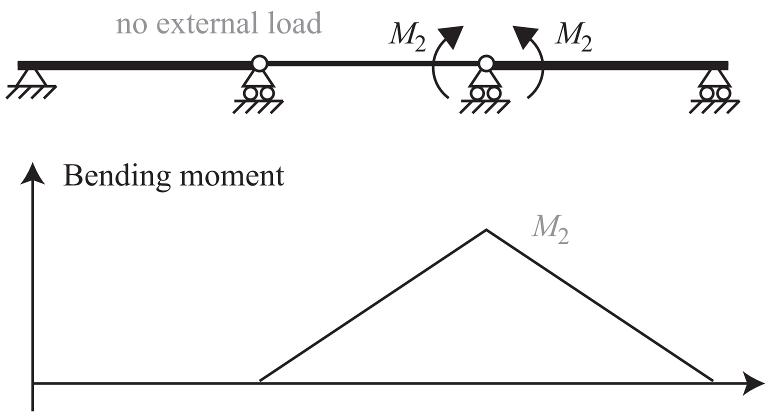
Particular Equilibrium Solution



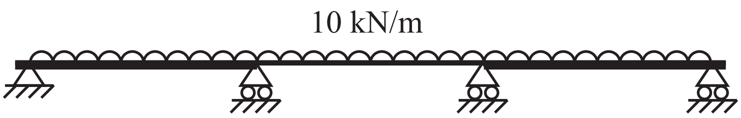
State of Self-Stress 1

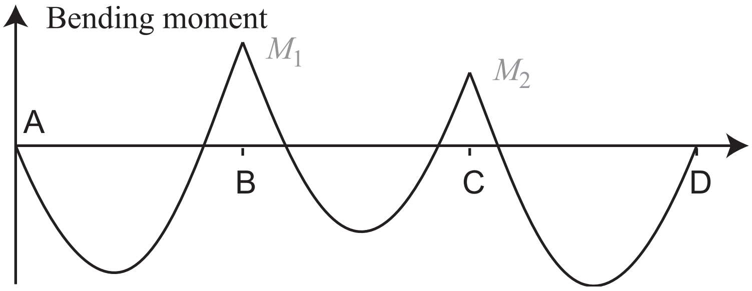


State of Self-Stress 2



General Bending Moment Diagram





All possible equilibrium systems must contain the Particular Equilibrium Solution, but can include any amount of the two States of Self-Stress.

We are not trying to find an optimum design, simply to show that the one we are given is suitable. Hence, try making M_1 and M_2 as large as possible without violating the yield criterion, and check that the resulting moments elsewhere in the beam are suitable.

$$
\begin{aligned}
M_1 &= 16 \text{ kNm} \\
M_2 &= 16 \text{ kNm (moments in } BC \text{ govern)}
\end{aligned}
$$

The central span is easy to check. The minimum moment will occur in the centre of the beam.

$$
\begin{aligned}
M_{\text{BCmin}} &= -31.25 \text{ kNm} + \frac{M_1}{2} + \frac{M_2}{2} \\
&= -15.25 \text{ kNm} \geq -16 \text{ kNm}
\end{aligned}
$$

The outer spans are more difficult. We can no longer say that the minimum moment will occur in the centre of the beam. However, it is straightforward to find an expression for the moment.

$$M_{\text{AB}} = \frac{10x}{2}(x-5) + \frac{x}{5} \times 16 \text{ — all units kN,m}$$

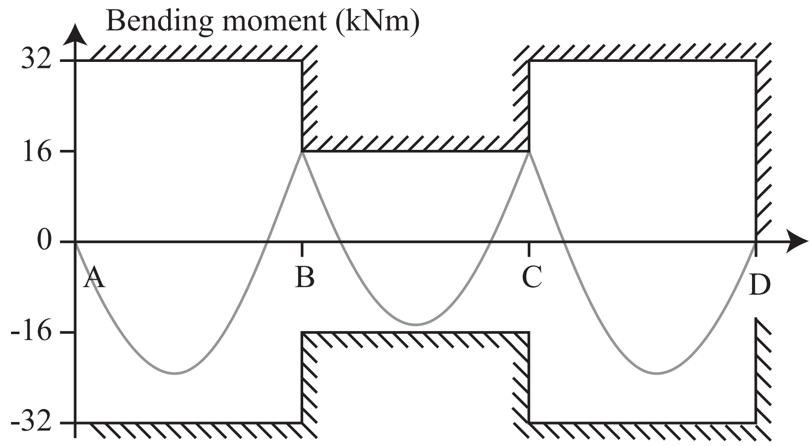
The minimum will occur where $dM/dx = 0$.

$$\frac{dM_{\text{AB}}}{dx} = 5(x-5) + 5x + \frac{16}{5} = 10x - \frac{109}{5} = 0$$

$$\text{Minimum at } x = 2.18 \text{ m}$$

$$M_{\text{ABmin}} = 5 \times 2.18 \times (2.18 - 5) + 2.18 \times \frac{16}{5} = -23.76 \text{ kNm} \geq -32 \text{ kNm}$$

Thus the moments everywhere are less than the plastic moments, and the design is safe. The chosen bending moment diagram, which shows this clearly, is below.



We could have chosen M_1 and M_2 to be slightly smaller, and still have found a bending moment diagram that didn't violate yield. However, there is no need — a single valid equilibrium state shows that the design is safe.

Try Questions 9, 10 and 11, Examples Sheet 2/5

4.9.9 Justification for Elastic Analysis

We have seen for a number of examples that an elastic analysis is not suitable for finding the collapse load of a structure. Indeed, the basic initial assumptions that a structure in unstressed in its initial state are probably not valid. Despite this, elastic analysis is still commonly used — largely because it is straightforward to write a computer program to do the analysis.

The Lower-Bound theorem provides a justification for the elastic designer. The elastic solution is certainly an equilibrium solution (that also satisfies various compatibility constraints). And an equilibrium solution that does not violate yield is safe — hence the elastic design is safe, even though it is unlikely to be an optimum design.

4.10 Summary of Plasticity

4.10.1 Assumptions

For both lower and upper-bound methods:

Ductility It is important that the complete failure mechanism is able to form before the material loses load-bearing capacity at any point. We have often assumed that the material is rigid-perfectly plastic.

Small deformations Although plastic deformation may be large compared with elastic deformation, we still assume that the deformations are small compared with the overall dimensions of the structure.

Ignore other failure modes We have assumed that failure will not occur due to e.g. buckling.

4.10.2 Bound theorems

We will only prove the bound theorems for beam structures with point loads applied, for brevity. Our proof will also take a few (intuitively reasonable) shortcuts, such as assuming that there is only one collapse load for any chosen load distribution. A more general proof will be found in Calladine's 'Plasticity for Engineers'.

Lower-bound theorem

If a set of internal stresses can be found in the structure that are in equilibrium with an applied load W_L, and nowhere violate the yield condition, then the applied load will be less than, or equal to, the actual collapse load W_C.

Consider a complete collapse solution, where loads W_i^C are displacing by δ_i at positions i, and moments M_j^C have developed at critical points j in the beam, where kinks θ_j occur. The loads W_i^C are in equilibrium with the moments M_j^C, and the displacements δ_i are compatible with the rotations θ_j. We can thus write a statement of Virtual Work:

$$\sum W_i^C \delta_i = \sum M_j^C \theta_j \tag{4.3}$$

Now consider another set of loads where $W_i^L = \beta W_i^C$. Also consider a set of moments M_j^L that are everywhere less than or equal to the plastic moment, and are in equilibrium with the new loads W_i^L (note that M_j^L are not necessarily βM_j^C — we are happy with any set of equilibrium moments, not just a set that are scaled from the collapse solution). We can also write, by Virtual Work, using the new equilibrium system, and the original compatible system:

$$\sum W_i^L \delta_i = \sum M_j^L \theta_j \tag{4.4}$$

Hence

$$\beta \sum W_i^C \delta_i = \sum M_j^L \theta_j \tag{4.5}$$

4.3 and 4.5 give

$$\sum M_j^L \theta_j = \beta \sum M_j^C \theta_j \tag{4.6}$$

We now explore every possible critical point in the beam j. There are three possibilities for the complete collapse solution:

1. A 'positive' plastic hinge has formed, $\theta_j > 0$, $M^C = M_p$, and because $M^L \leq M_p$, $M_j^C \theta_j \geq M_j^L \theta_j$.

2. A 'negative' plastic hinge has formed, $\theta_j < 0$, $M^C = -M_p$, and because $M^L \geq -M_p$, $M_j^C \theta_j \geq M_j^L \theta_j$.

3. No plastic hinge has formed, there is no kink, $\theta_j = 0$, and hence $M_j^C \theta_j = M_j^L \theta_j$.

Overall, for every possible critical point, $M_j^C \theta_j \geq M_j^L \theta_j$, and so

$$\sum M_j^C \theta_j \geq \sum M_j^L \theta_j \tag{4.7}$$

Thus, substituting into 4.6, $\beta \leq 1$. Thus, if a set of moments can be found in equilibrium with some applied load that nowhere violate yield, it would be necessary to multiply those loads by $1/\beta \geq 1$ to cause collapse, proving the lower-bound theorem.

Upper-bound theorem

> *An estimate of the plastic collapse load, W_{mech}, calculated for any arbitrary compatible mechanism by equating the work done by the applied load, and the plastic energy dissipated, will either be greater than, or equal to, the actual collapse load W_C.*

Consider that we have a complete collapse solution, where there are loads W_i^C at positions i on the beam, and moments M_j^C have developed at critical points j. The loads W_i^C are in equilibrium with the moments M_j^C.

Consider a postulated collapse mechanism — a compatible set of displacements, δ_i^U and hinge rotations θ_j^U that may or may not be the correct collapse mechanism. By Virtual Work we can write:

$$\sum W_i^C \delta_i^U = \sum M_j^C \theta_j^U \tag{4.8}$$

Now consider doing a work calculation to find a load factor γ for this assumed mechanism. Each assumed hinge does work $M_p|\theta_j^U|$, giving:

$$\gamma \sum W_i^C \delta_i^U = \sum M_p |\theta_j^U| \tag{4.9}$$

At every section, $-M_p \leq M_j^C \leq M_p$, and hence $M_j^C \theta_j^U \leq M_p|\theta_j^U|$. Thus

$$\sum M_j^C \theta_j^U \leq \sum M_p |\theta_j^U| \tag{4.10}$$

Substituting 4.8 and 4.9 into 4.10 gives $\gamma \geq 1$. Thus, the collapse load calculated from a work calculation with a compatible collapse mechanism will either be high, or correct, proving the upper-bound theorem.