IB Paper 7: Linear Algebra   Handout 1

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Aims

To introduce the ideas and techniques of Linear Algebra, and illustrate some applications to Engineering.

Syllabus

- Solution of the matrix equation $\mathbf{Ax} = \mathbf{b}$: Gaussian elimination, LU factorization.
- Four fundamental subspaces of a matrix.
- Least squares solution of $\mathbf{Ax} = \mathbf{b}$ for an $m \times n$ matrix with n independent columns: Solving $\mathbf{A}^\mathbf{T} \mathbf{A}\, \mathbf{x} = \mathbf{A}^\mathbf{T}\, \mathbf{b}$, Gram-Schmidt orthogonalization, **QR** decomposition.
- Solution of $\mathbf{Ax} = \lambda\mathbf{x}$, eigenvectors and eigenvalues.
- Singular Value Decomposition

Text book

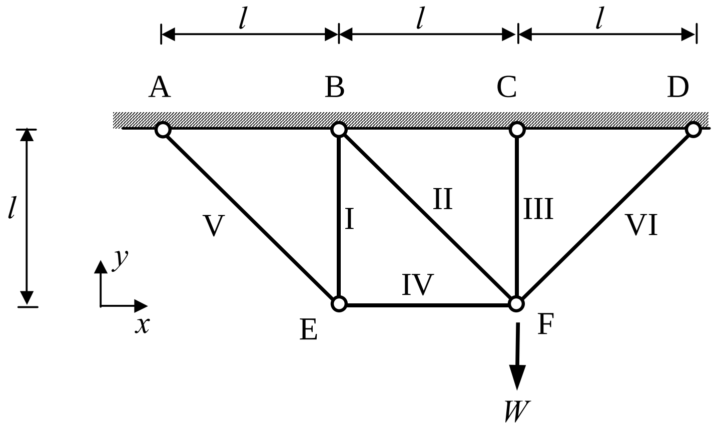
Gilbert Strang, Linear Algebra and its Applications, Harcourt Brace Jovanich 3rd edition,1988.
EC 62

Examples Papers

According to Rita

1 Example Application

Many problems in engineering involve linear equations. The most recent example you have met is probably in structures, involving a statically indeterminate truss.

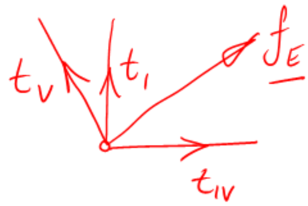


(a) The *equilibrium matrix* relates the forces in the members to the applied external forces.

$$A\underline{t} = \underline{f}$$

Cases like this were analysed in the Structures Course, where **A** was shown to be:

E.g. joint E



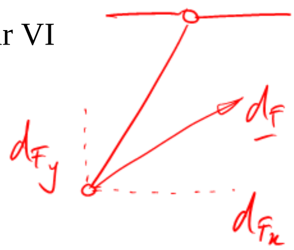
$$R\ (\longleftrightarrow)\quad f_{Ex} + t_{IV} - t_V \frac{1}{\sqrt{2}} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 1/\sqrt{2} & 0 \\ -1 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 1 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -1 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} t_{I} \\ t_{II} \\ t_{III} \\ t_{IV} \\ t_{V} \\ t_{VI} \end{bmatrix} = \begin{bmatrix} f_{Ex} \\ f_{Ey} \\ f_{Fx} \\ f_{Fy} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -W \end{bmatrix}$$

The *static indeterminacy* shows up in the fact that we have *4 equations for 6 unknowns*, indicating probably 2 redundant members.

(b) The *extensions* in all of the members must be compatible with the displacements. Written in matrix form, this introduces the *compatibility matrix* **C**, $\mathbf{C}\,\underline{d} = \underline{e}$

E.g. bar VI



$$e_{VI} = -d_{Fx}\frac{1}{\sqrt{2}} - d_{Fy}\frac{1}{\sqrt{2}}$$

which in this case is

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} d_{\mathrm{E}x} \\ d_{\mathrm{E}y} \\ d_{\mathrm{F}x} \\ d_{\mathrm{F}y} \end{bmatrix} = \begin{bmatrix} e_{\mathrm{I}} \\ e_{\mathrm{II}} \\ e_{\mathrm{III}} \\ e_{\mathrm{IV}} \\ e_{\mathrm{V}} \\ e_{\mathrm{VI}} \end{bmatrix}$$

This time we have 6 equations in 4 unknowns. This is an *over-specified problem* and at least 2 of these equations *must be simply linear combinations of 4 others*. There can only be 4 independent extensions, and 2 conditions on allowable $\underline{e}$'s to make this possible.

Coping with the redundancy and the compatibility, especially as the trusses get more complicated, involves considerable physical insight or the linear algebra methods we will develop in this course.

A close look at **A** and **C** shows that $C = A^T$. (This is no coincidence; it is always the case).

When a physical problem involves a matrix **A**, then $\mathbf{A}^{\mathbf{T}}$ is usually also involved.

The Big Questions

Q1 What set of forces can be held in equilibrium with this structure ?

i.e. How do we find/describe the set of $\underline{f}$, for which there is a solution to A $\underline{t} = \underline{f}$?

We shall call the set of $\underline{f}$ for which there is a solution, $\underline{t}_0$, the *Column Space* of **A** .

Q2 How can we tell a statically indeterminate case and, when we have one, what is the general solution ?

i.e. If we have a solution $\underline{t}_0$, is it unique ? If not, the general solution will be

$$\underline{t} = \underline{t}_0 + \underline{n} \qquad \text{so what is } \underline{n}\,?$$

$$\mathbf{A}\underline{t} = \mathbf{A}\underline{t}_0 + \mathbf{A}\underline{n} = \underline{f} + \mathbf{A}\underline{n} \qquad \text{so } \mathbf{A}\underline{n} = 0$$

We shall call the set of $\underline{n}$ for which $\mathbf{A}\underline{n} = 0$ the *Null Space* of **A** .

These correspond, for this application, to sets of bar forces for which there is no applied force.

i.e. states of self-stress

Q3 What set of bar extensions are compatible with keeping this structure fitting together ?

i.e. How do we find/describe the set of $\underline{e}$, for which there is a solution to **C** $\underline{d} = \underline{e}$?

For consistency with Q1, we should call this the *Column Space* of **C** , but since $C = A^T$, we prefer to call it the *Row Space* of **A**.

Q4 What set of nodal displacements produce zero extensions in the bars ?

i.e. How do we find/describe the set of $\underline{d}$, for which there is a solution to $\mathbf{C}\,\underline{d} = 0$?

Again for consistency with Q2, we should call this the *Null Space* of **C** , but since $C = A^T$, we prefer to call it the *Left Null Space* of **A**.

$$\mathbf{C}\underline{d} = 0 \;\Rightarrow\; \mathbf{A}^{\mathbf{T}}\underline{d} = 0 \;\Rightarrow\; \underline{d}^T A = 0 \text{ (taking the transpose)}$$

These correspond, for this application, to mechanisms

A similar set of questions tends to crop up in applications of matrix methods to other branches of engineering. The first part of this course, then, is devoted to the general solution of $\mathbf{A}\underline{x} = \underline{b}$, and methods to find the four sets of vectors which answer these four questions. i.e. to find the column space, the null space, the row space and the left-null space of the matrix **A**.

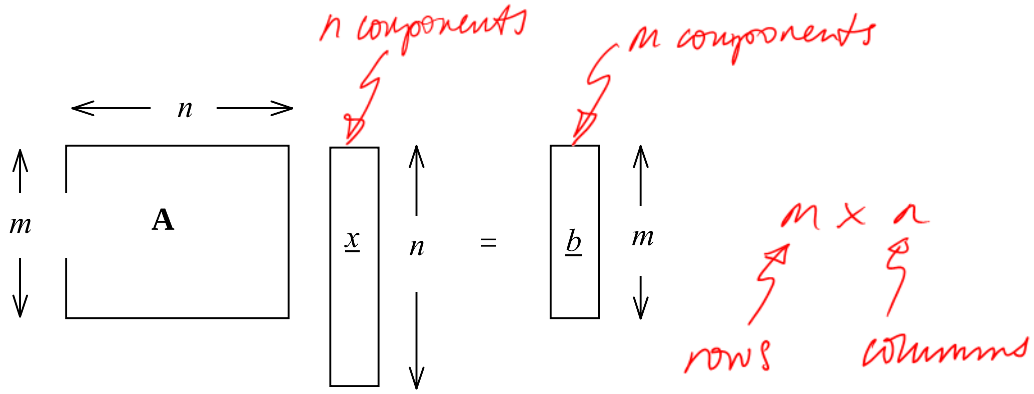
2 The Geometry of *n* dimensions

Geometrical interpretation is a great help when considering how to solve systems of equations, which in 3 dimensions are planes, lines, etc. In this section, we will try and extend the ideas of lines and planes to dimensions higher than 3.

Considered as a mapping, the 4 × 6 matrix **A** above maps a 6-dimensional vector into a 4-dimensional one

$$\begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \\ t_{\mathrm{IV}} \\ t_{\mathrm{V}} \\ t_{\mathrm{VI}} \end{bmatrix} \xrightarrow{A} \begin{bmatrix} f_{\mathrm{E}x} \\ f_{\mathrm{E}y} \\ f_{\mathrm{F}x} \\ f_{\mathrm{F}y} \end{bmatrix} \qquad\qquad \begin{bmatrix} d_{\mathrm{E}x} \\ d_{\mathrm{E}y} \\ d_{\mathrm{F}x} \\ d_{\mathrm{F}y} \end{bmatrix} \xrightarrow{A^T} \begin{bmatrix} e_{\mathrm{I}} \\ e_{\mathrm{II}} \\ e_{\mathrm{III}} \\ e_{\mathrm{IV}} \\ e_{\mathrm{V}} \\ e_{\mathrm{VI}} \end{bmatrix}$$

In general, an $m \times n$ matrix **A**, transforms an n-dimensional vector $\underline{x}$ into a corresponding m-dimensional vector $\underline{b}$. $\mathbf{A}\,\underline{x} = \underline{b}$



As we move to dimensions higher than 3, most of the familiar vector properties generalise, and only a few do not.

4 dimensions

(i) We still have 4 independent unit vectors along the "axis" directions

$$\underline{e}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad \underline{e}_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad \underline{e}_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad \text{and} \quad \underline{e}_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

(ii) Any vector has 4 components $\qquad \underline{x} \; = \; x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3 + x_4\underline{e}_4$

(iii) Length is

$$|\underline{x}| \; = \; \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

(iv) Dot product survives

$$\underline{x} \cdot \underline{y} \; = \; x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \; = \; \underline{x}^{\mathrm{T}}\,\underline{y} \; = \begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}$$

We can think of this as defining an angle between two vectors $\cos\theta = \dfrac{\underline{x}\cdot\underline{y}}{|\underline{x}||\underline{y}|}$, and if we do so

$$\underline{x} \perp \underline{y} \Leftrightarrow \quad \underline{x} \cdot \underline{y} = 0$$

As in 3-d $|\underline{x}|^2 \; = \underline{x} \cdot \underline{x} \; = \underline{x}^T\underline{x}$

(v) All of this is compatible with the corresponding definitions in 3-d, and we still have

$$\underline{x}\cdot\underline{e}_1 \; = \; x_1 \qquad \underline{e}_1 \cdot \underline{e}_1 = 1 \qquad \underline{e}_1 \cdot \underline{e}_2 = 0 \quad \text{, etc.}$$

with the angle between $\underline{e}_1$ and $\underline{e}_2$ being 90°, etc.

(vi) An example of something which *doesn't* generalise is cross product $\underline{x} \times \underline{y}$. We could try using $|\underline{x}||\underline{y}|\sin\theta\,\underline{\hat{n}}$, but we come unstuck with $\underline{\hat{n}}$ since, in four dimensions, there are *two* unit vectors perpendicular to $\underline{x}$ and $\underline{y}$.

(vii) We can replace 4 by m or n in the above with obvious generalisations.

$$\underline{x} \; = \; x_1\underline{e}_1 + x_2\underline{e}_2 + \ldots + x_n\underline{e}_n$$

The n-dimensional "world" is referred to as R^n. We live in R^3 . An $m \times n$ matrix $\mathbf{A}$ maps R^n to R^m .

n real coordinates

2.1 The Column Picture for Matrix Multiplication

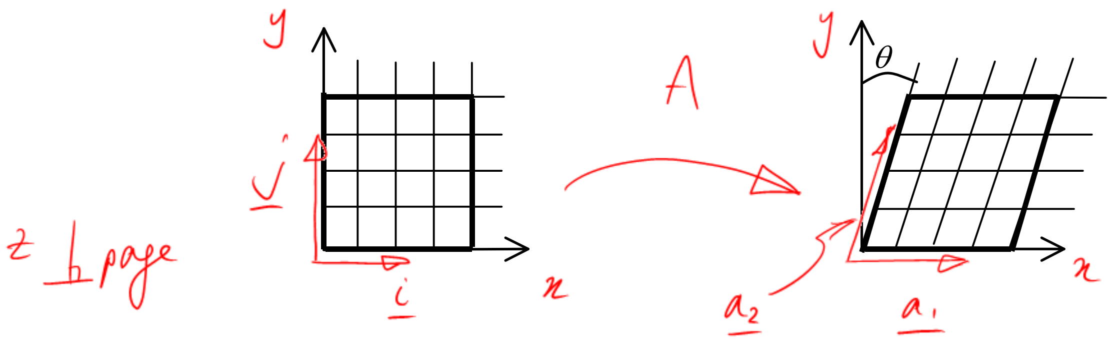
The simplest way of understanding the effect of multiplying by a matrix is to think of the effect on the co-ordinate base vectors. In 3-d, for example,

$$\mathbf{A}\underline{i} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \underline{a}_1 \text{ the first column of } \mathbf{A}$$

$$\mathbf{A}\underline{j} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \underline{a}_2 \text{ the second column of } \mathbf{A}$$

$$\mathbf{A}\underline{k} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \underline{a}_3 \text{ the third column of } \mathbf{A}$$

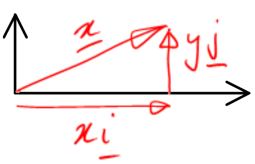
Knowing what happens to the base vectors when we operate on them with **A** enables us to tell immediately what happens to any vector $\underline{x}$. E.g. Simple Shear

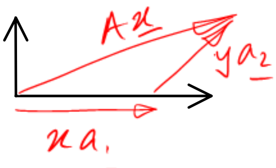


$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} \tan\theta \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \mathbf{A} = \begin{bmatrix} 1 & \tan\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then for *any* vector **x**,

$$\begin{aligned} \mathbf{A}\underline{x} &= \mathbf{A}\left(x\underline{i} + y\underline{j} + z\underline{k}\right) \\ &= x\mathbf{A}\underline{i} + y\mathbf{A}\underline{j} + z\mathbf{A}\underline{k} \\ &= x\underline{a}_1 + y\underline{a}_2 + z\underline{a}_3 \end{aligned}$$





When you multiply a vector by a matrix then, the original co-ordinates multiply the columns of **A** after the transformation.

$$x\underline{i} + y\underline{j} + z\underline{k} \xrightarrow{A} x\underline{a}_1 + y\underline{a}_2 + z\underline{a}_3$$

2.2 The Column Picture for Simultaneous Equations.

Let us stay in R^3 for the present and consider the problem of solving 3 equations in 3 unknowns.

$$\mathbf{A}\underline{x} = \underline{b} \qquad \begin{array}{rcrcrcr} x & + & 3y & - & z & = & 11 \\ 3x & - & 2y & - & z & = & 7 \\ -x & + & y & + & 4z & = & -9 \end{array}$$

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & -2 & -1 \\ -1 & 1 & 4 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 11 \\ 7 \\ -9 \end{bmatrix}$$

The solution of which can be shown to be $x = 3 \quad y = 2 \quad z = -2$

This problem can be written in vector form as

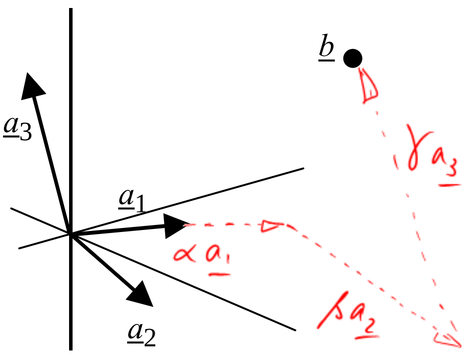
$$x\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + y\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + z\begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \\ -9 \end{bmatrix}$$

and the vectors on the left hand side are, as expected, the *columns* of the matrix **A**.

The problem then is to find which linear combination of the columns on the LHS will give the vector on the RHS. We will refer to this as *column visualization* or as the *column picture*.

In this case

$$3\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \\ -9 \end{bmatrix}$$



If the vectors which are the columns of **A** are *independent* (or rather *linearly independent*), i.e. you can not express one as a linear combination of the other two, then *any* vector can be written as a linear combination of them (in a unique way). So the equations are guaranteed to have a (unique) solution for any RHS $\underline{b}$.

using $\alpha \underline{a}_1 + \beta \underline{a}_2$ can get anywhere on a plane
using $\gamma \underline{a}_3$ can always get onto that plane

$$\underline{b} - \gamma \underline{a}_3 = \alpha \underline{a}_1 + \beta \underline{a}_2$$

When might the equations *not* have a solution?

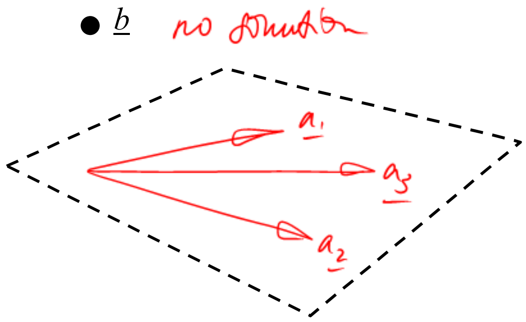
If the matrix **A** is *singular*, then the columns of **A** are *not independent* as is the case for the following set

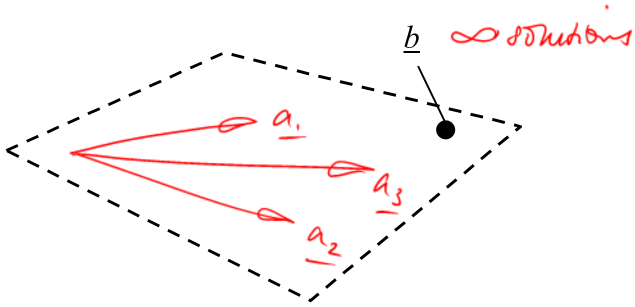
$$x\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + y\begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix} + z\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \\ -9 \end{bmatrix}$$

This has the third column lying in the plane spanned by the first two (see below).
i.e. $\underline{a}_3 \;=\; \alpha\underline{a}_1 + \beta\underline{a}_2$. It follows that **A** *anything* must also lie in the plane of $\underline{a}_1$ and $\underline{a}_2$:

$$\mathbf{A}\underline{x} \;=\; x\underline{a}_1 + y\underline{a}_2 + z\underline{a}_3 \;=\; x\underline{a}_1 + y\underline{a}_2 + z\left(\alpha\underline{a}_1 + \beta\underline{a}_2\right) = \left(x + z\alpha\right)\underline{a}_1 + \left(y + z\beta\right)\underline{a}_2$$

If $\underline{b}$ does not also lie in this plane, then there is no solution.





$\underline{b} = \lambda\underline{a}_1 + \mu\underline{a}_2$
$= \lambda'\underline{a}_2 + \mu'\underline{a}_3$
or any linear combination

If $\underline{b}$ *does* lie in this plane, then there is an infinite number of solutions.

When we are dealing with 3 × 3 matrices, we know how to determine whether the columns of a matrix are independent and, if so, whether a given vector can be written in terms of them, although the methods we know are laborious. For the matrix referred to above:

(i) If we write the equations in the form **A** $\underline{x}$ **=** $\underline{b}$, then

$$\text{Determinant } \mathbf{A} \;=\; \begin{vmatrix} 1 & 3 & -1 \\ 3 & -2 & -1 \\ -1 & 8 & -1 \end{vmatrix} \;=\; 1(2+8) - 3(-3-1) - 1(24-2) \;=\; 10 + 12 - 22 \;=\; 0$$

This means that **A** is singular (has no inverse).

(ii) The columns of **A** can not, therefore, be independent. To prove the columns are not independent, we write the third one as a linear combination of the first two.

$$\text{Put} \quad \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \;=\; \alpha\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \beta\begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix}.$$

The first two equations give $\alpha = -\dfrac{5}{11}$ $\quad \beta = -\dfrac{2}{11}$ (and this also satisfies the third).

(iii) Since $\mathbf{A}\,\underline{x} \;=\; x_1\underline{a}_1 + x_2\underline{a}_2 + x_3\underline{a}_3 \;=\; \left(x_1 + \alpha x_3\right)\underline{a}_1 + \left(x_2 + \beta x_3\right)\underline{a}_2$, for a solution, $\underline{b}$ must also be a combination of the first two columns of **A**:-

This must be changed to 15 for a solution to exist.

$$\begin{bmatrix} 11 \\ 7 \\ -9 \end{bmatrix} = \gamma \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \delta \begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix}$$

Use the first two components to find γ and δ $\Rightarrow \gamma = \dfrac{43}{11} \quad \delta = \dfrac{26}{11}$ $\Rightarrow$ third component of $\underline{b}$ must be 15 for a solution to exist.

2.3 Vector Spaces and Subspaces of R^n.

We would like to hang onto these pictures, even though we will move into n and m dimensional space. For a general non-square $m \times n$ matrix

N.B. m components

$$\mathbf{A}\,\underline{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \underbrace{x_1 \underline{a}_1 + x_2 \underline{a}_2 + x_3 \underline{a}_3 + \ldots + x_n \underline{a}_n}_{n \text{ terms}}$$

where the $\underline{a}_i$ are the column vectors of $\mathbf{A}$. As the x_i vary, they sweep out the part of R^m that we can get to by multiplying a vector in R^n by $\mathbf{A}$.

For our equilibrium matrix in the problem described in section 1

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 1/\sqrt{2} & 0 \\ -1 & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1 & 0 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & -1 & 0 & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \\ t_{\mathrm{IV}} \\ t_{\mathrm{V}} \\ t_{\mathrm{VI}} \end{bmatrix} =$$

$$t_{\mathrm{I}} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t_{\mathrm{II}} \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} + t_{\mathrm{III}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + t_{\mathrm{IV}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_{\mathrm{V}} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} + t_{\mathrm{VI}} \begin{bmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

The region mapped out in R^m as the x_i vary (e.g. that in R^4 as the t_i vary) is called a *vector space* and it is a straightforward generalisation to arbitrary dimensions of the concept of a line or a plane.

The formal definition is

> A vector space in R^m is the set of $\underline{x}$ of the form
>
> $\underline{x} = \lambda \underline{u} + \mu \underline{v} + \nu \underline{w} + \ldots$

where $\underline{u}$, $\underline{v}$, **...** are fixed m-vectors and λ, μ, **...** are variable scalars taking all real values.

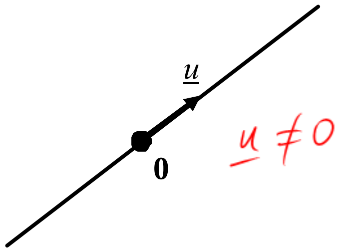
The vectors $\underline{u}$, $\underline{v}$, **...** are said to *span* the space.

R^m is itself a vector space and "smaller" ones within it are said to be *sub-spaces* of R^m.

i.e. fewer than m spanning vectors

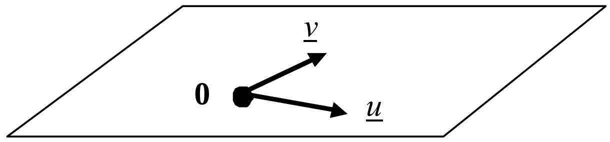
For example: The non-trivial sub-spaces of R^3 are

(a) $\underline{x} = \lambda \underline{u}$ lines through the origin

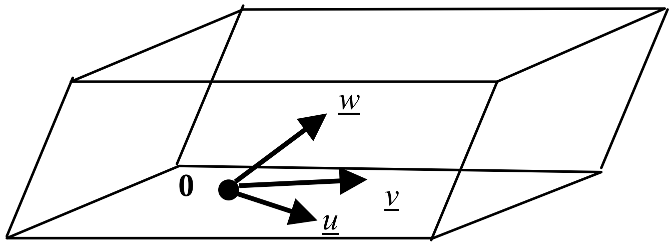


$\underline{u} \neq 0$

(b) $\underline{x} = \lambda \underline{u} + \mu \underline{v}$ planes through the origin.



(c) $\underline{x} = \lambda \underline{u} + \mu \underline{v} + \nu \underline{w}$ = whole of R^3



2.4 How many dimensions ?

We would naturally describe a line as a one-dimensional sub-space of R^3 and a plane as a two-dimensional subspace since within these spaces we have one and two "degrees of freedom". The term "dimension" is thus used in *two senses* – number of degrees of freedom *and* number of co-ordinates. Every point on a line has 3 co-ordinates and is thus a *three-dimensional vector* but the line is a *one-dimensional object*. This ambiguity does not, in general, cause confusion.

In drawing the diagrams above, we have assumed that the vectors $\underline{u}$, $\underline{v}$ and $\underline{w}$ are *independent*. For case (d), if $\underline{u}$, $\underline{v}$, $\underline{w}$ are *not* independent (i.e. one of them can be written as a sum of the other two), we can drop it from the list of vectors necessary to span the space. If

$$\underline{w} = \alpha \underline{u} + \beta \underline{v} \Rightarrow \underline{x} = (\lambda - \nu\alpha)\underline{u} + (\mu + \nu\beta)\underline{v} \Rightarrow \underline{x} = \lambda' \underline{u} + \mu' \underline{v}$$

and we only have a plane. Similarly considerations apply for (b) and (c).

A set of vectors $\underline{u}$, $\underline{v}$, **...** which span the space and are also *independent* are said to form a *basis* and the number of these is the *dimension* of the sub-space. The basis of a vector sub-space is not unique – any full set of independent vectors will do, but it will always contain the same *number* of vectors.

e.g. The sub-spaces of R^3 which are given by

$$\underline{x} = \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \underline{x} = \lambda' \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mu' \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \underline{x} = \lambda'' \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mu'' \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + \nu'' \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

turn out to be the same and, moreover, these are not the only way of representing the plane. Whatever vectors are used, however, it needs *two and only two* independent vectors to describe it.

Sets of Independent Vectors (typical properties)

1) Complete the following, for vectors in R^3 :

Two vectors are linearly dependent if they lie on the same line (through the origin)

Three vectors are linearly dependent if they lie on the same plane (— " —)

Four vectors are certain to be linearly dependent

2) What is the maximum number of vectors in an independent set in R^6 ? 6

3) The mathematical test for linear independence is:

The vectors $\underline{x}_i$, $i = 1, ..., n$ are linearly independent if, whenever

$$\lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \ldots + \lambda_n \underline{x}_n = 0 \quad \text{for any scalars } \lambda_i , i = 1, ..., n ,$$

then we must have $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$

(If one of the λ's is non-zero, then we can solve this equation for the vector it multiplies in terms of the others).

4) Show that if $\underline{x}_i$, $i = 1, ..., n$ are a basis for the vector space S, then every vector in S has a unique representation in terms of them.

$\underline{x} \in S \Rightarrow \underline{s} = \lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \ldots + \lambda_n \underline{x}_n$

if also: $\underline{s} = \mu_1 \underline{x}_1 + \mu_2 \underline{x}_2 + \ldots + \mu_n \underline{x}_n$ then subtracting gives:

$0 = (\lambda_1 - \mu_1)\underline{x}_1 + (\lambda_2 - \mu_2)\underline{x}_2 + \ldots + (\lambda_n - \mu_n)\underline{x}_n \Rightarrow 0 = \lambda_1 - \mu_1$
$= \lambda_2 - \mu_2$
etc.

2.5 Column Space

The vector space spanned by the columns of a general $m \times n$ matrix **A** is called the *column space* of **A**. The dimension (in the degrees of freedom sense) of column space is called the *rank* of **A**. So, for example, if **A** is a 3×3 matrix

If rank(**A**) = 3 column space = whole of R^3

If rank(**A**) = 2 column space = a plane (we lose 1 dimension in the mapping)

If rank(**A**) = 1 column space = a line (we lose 2 dimensions in the mapping)

If rank(**A**) = 0 column space = 0 (we lose 3 dimensions in the mapping)

If we lose dimensions, then we can not *reverse* a mapping. If **A** is $n \times n$, then if it loses dimensions, rank(**A**) < n, it is singular. If it doesn't, rank(**A**) = n, then its inverse will exist.

Now for a general $m \times n$ matrix because

$$\mathbf{A}\,\underline{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + x_3 \underline{a}_3 + \ldots + x_n \underline{a}_n$$

the following should be fairly clear.

(i) Column space is part of R^m and so the number of independent columns of **A** can not exceed m, i.e. rank(**A**) $\le m$. In addition, there are only n columns, so rank(**A**) $\le n$. The 4×6 equilibrium matrix studied earlier can not, therefore, have more than 4 independent columns.

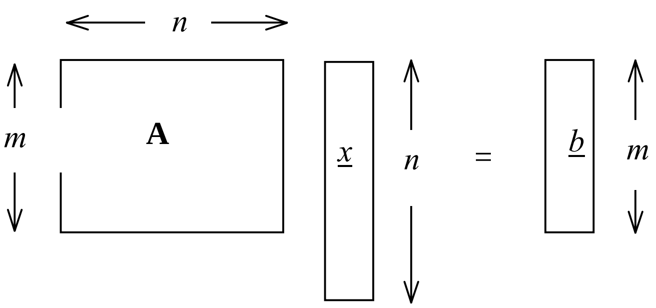
(ii) If $\underline{b}$ lies in column space, then $\mathbf{A}\underline{x} = \underline{b}$ has at least one solution.

(iii) If $\underline{b}$ is not in column space, then $\mathbf{A}\underline{x} = \underline{b}$ has no solution.

(iv) If rank(**A**) = m, so that column space = whole of R^m, then $\underline{b}$ must lie in column space.

Key Points from Lecture

- In general, an $m \times n$ matrix **A**, transforms an n-dimensional vector $\underline{x}$ into a corresponding m-dimensional vector $\underline{b}$. $\mathbf{A}\,\underline{x} = \underline{b}$



- The n-dimensional "world" is R^n. An $m \times n$ matrix **A** maps R^n to R^m.

$$\mathbf{A}\,\underline{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + x_3 \underline{a}_3 + \ldots + x_n \underline{a}_n$$

where the $\underline{a}_i$ are the column vectors of **A** .

- As the x_i vary, they sweep out the part of R^m that we can get to by multiplying a vector in R^n by **A** . This is Column Space which is a vector sub-space of R^m (i.e a generalisation of a line or a plane)

- The dimension of Column Space = number of independent columns of **A** is called the rank(**A**) $\le$ smaller of m and n. A (far from unique) set of independent vectors which span the column space of **A**, i.e. a basis, will contain rank(**A**) vectors.

You can now do Examples Paper 1 Questions 1 - 3