Lecture 14

Self-similar solutions of the diffusion equation

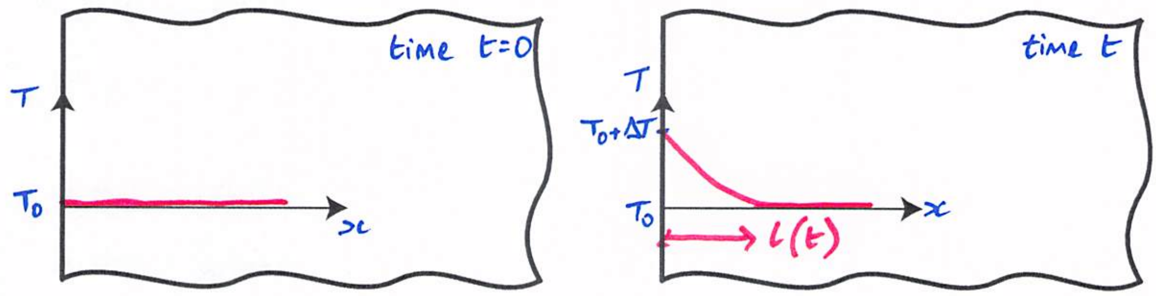
14.1 Introduction

We have used the separation of variables method to find solutions to Laplaces's equation, the diffusion equation and the wave equation. We now introduce another technique that can be applied to the diffusion equation: self-similar solutions. Self-similar solutions apply when the problem domain has no inherent length-scale but can, instead, be considered infinite, or semi-infinite, during the period of interest.

14.2 Diffusion length

Self-similar solutions are based on the idea of *diffusion length*. The diffusion length is the characteristic distance a substance (or heat, or momentum) will diffuse in a time t.

The diagram shows the edge of a block, i.e. a semi-infinite domain.



The block is at a uniform temperature T_0 for $t < 0$. Then, at $t = 0$ the temperature of the edge at $x = 0$ is raised instantaneously to $T = T_0 + \Delta T$. At a time t later, the zone of increased temperature has reached a distance $l = l(t)$ where l is the diffusion length.

We can use dimensional analysis to obtain an estimate for $l(t)$. We expect $l(t)$ to depend on diffusivity, α ($\alpha = \lambda/(\rho c)$ for our heat conduction problem) because a higher diffusivity means faster diffusion. The diffusion length must also depend on t, but we then run out of variables. We have,

$$l = f(\alpha, t)$$

for some function f. α has units of m^2s^{-1}.

We can use Buckingham's Pi theorem to determine how many non-dimensional groups are needed in the problem. We have 3 parameters (P), and 2 dimensions (D), so the Pi theorem tells us that the number of groups $G = P - D = 1$. A suitable non-dimensional group is,

$$\Pi = \frac{l}{\sqrt{\alpha t}}$$

Since we only have one non-dimensional group in our problem, we conclude that Π must be a constant, c, and,

$$l = c\sqrt{\alpha t}$$

where c varies from problem to problem, but will always be of order unity.

Note that this estimate for $l(t)$ works only because there is no imposed geometric length-scale in our problem. Suppose that the heated block had a finite depth, H. Then, after a certain time, this finite depth would influence the diffusion process and we should write $l = F(\alpha, t, H)$; we would have two dimensionless groups and could no longer conclude that $l = c\sqrt{\alpha t}$.

14.3 Self-similar solutions

Returning to our semi-infinite heat conduction problem (where we have no imposed geometric length-scale), it is natural to seek a solution of the diffusion equation that has the form,

$$\frac{T - T_0}{\Delta T} = f\left(\frac{x}{l}\right) \quad \text{where} \quad l = \sqrt{\alpha t}$$

This is called a *self-similar solution* because, as long as x is normalised by l, the solution looks the same from one moment to the next. Time does not *explicitly* appear in the solution, only *implicitly* through the variable $l(t)$.

The PDE that governs our heat conduction (diffusion) problem is,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad . \tag{14.1}$$

The non-dimensional temperature field f must also satisfy this PDE,

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{\alpha}\frac{\partial f}{\partial t}$$

Writing $\eta = x/l = x/\sqrt{\alpha t}$,

$$\frac{\partial \eta}{\partial t} = \frac{x}{\sqrt{\alpha}} \cdot \frac{-1}{2} t^{-3/2} = \frac{-\eta}{2t}$$

$$\frac{\partial f}{\partial t} = \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{\eta}{2t}\frac{df}{d\eta}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\underbrace{\left(\frac{1}{\sqrt{\alpha t}}\frac{df}{d\eta}\right)}_{\partial f/\partial x} = \frac{1}{\alpha t}\frac{d^2 f}{d\eta^2}$$

Substituting these expressions into the PDE, we obtain the ODE:

$$\frac{d^2 f}{d\eta^2} + \frac{\eta}{2}\frac{df}{d\eta} = 0 \tag{14.2}$$

By writing $g = df/d\eta$, we have

$$\frac{dg}{d\eta} + \frac{\eta}{2} g = 0 \quad , \tag{14.3}$$

which we can integrate to give,

$$g = \frac{df}{d\eta} = A\exp(-\eta^2/4) \quad . \tag{14.4}$$

Integrating again to obtain f,

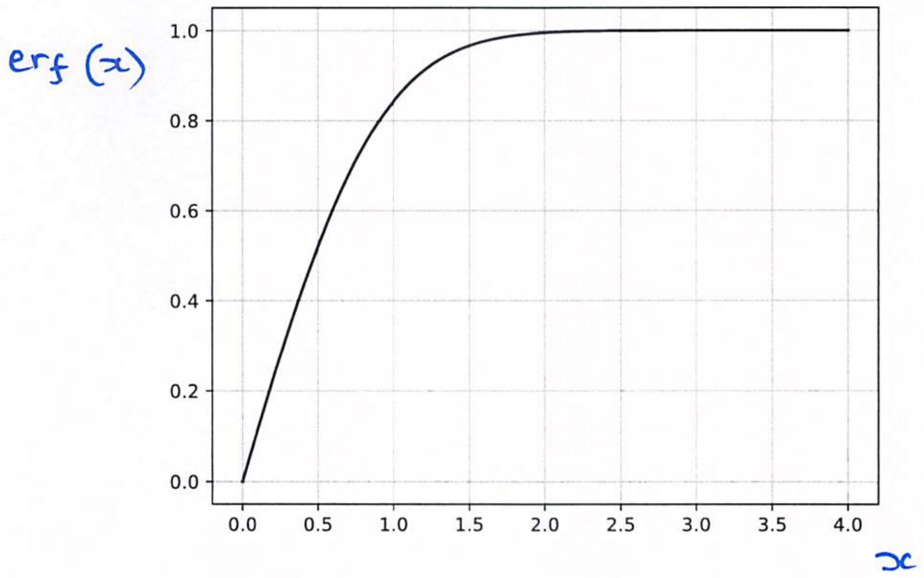
$$f(\eta) = A\int_0^\eta \exp(-\eta^2/4)\, d\eta + B \tag{14.5}$$

where A and B are constant that are set by the boundary conditions of the problem.

We can write out result for f more neatly by using a modified diffusion length, $l' = 2l = 2\sqrt{\alpha t}$ so that $\eta^2/4 = (x/l')^2$. We can also use the *error function*,

$$\mathrm{erf}(a) = \frac{2}{\sqrt{\pi}}\int_0^a \exp(-z^2)\, dz \tag{14.6}$$

which has the following shape:



In terms of l' and the error function, our solution becomes:

$$f(x/l') = C\text{erf}(x/l') + B \quad . \tag{14.7}$$

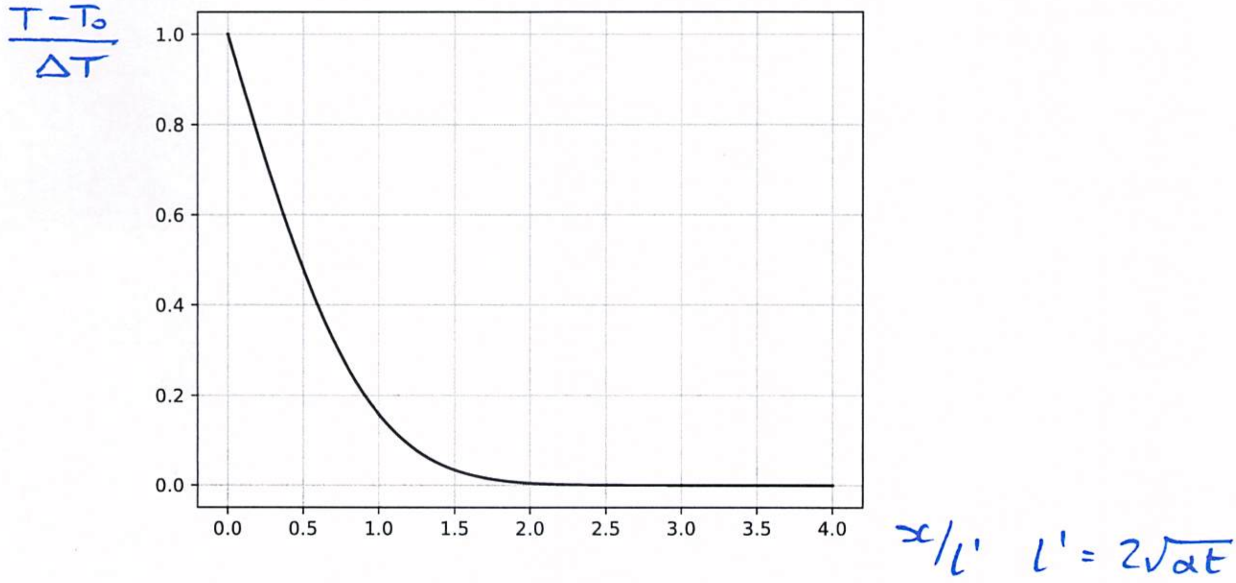
Remembering that $f = (T - T_0)/\Delta T$,

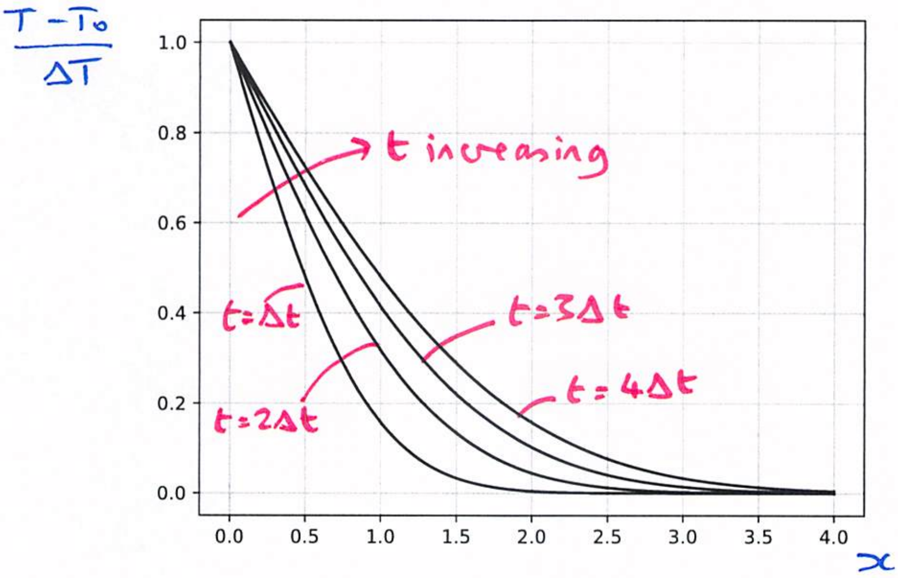
$$T - T_0 = \Delta T \left(C \,\text{erf}\left(\frac{x}{l'}\right) + B \right)$$

The boundary conditions we must satisfy are: $T = T_0 + \Delta T$ at $x = 0$, and $T = T_0$ as $x \to \infty$. This leads to,

$$T - T_0 = \Delta T \left(1 - \text{erf}\left(\frac{x}{l'}\right)\right) \qquad l' = 2\sqrt{\alpha t}$$

This self-similar solution shows that, plotted as a function of $x/(2\sqrt{\alpha t})$, the temperature field always has the same shape.





Self-similar solutions are usually found if, as in the above case, the boundary conditions are steady in time and the problem has no inherent length-scale.

14.4 Summary

We have reached the end of the Part 1B Vector Calculus and PDEs course. Here is a very brief summary of what has been covered.

Vector fields (vectors which vary over space and, in general, time, $\mathbf{V} = \mathbf{V}(\mathbf{r}, t)$) are commonplace in engineering problems.

Vector calculus provides a compact notation and methodology for deriving and manipulating equations governing vector fields.

We have made much use of the 'del' operator, ∇:

$\nabla\phi$ grad ϕ - a vector field that indicates the magnitude and direction of the steepest rate of change of ϕ (a scalar field) at every point in space.

$\nabla \cdot \mathbf{V}$ div $\mathbf{V}$ - the net efflux of $\mathbf{V}$ from an elemental volume δv is $(\nabla \cdot \mathbf{V})\delta v$. If $\nabla \cdot \mathbf{V} = 0$, then $\mathbf{V}$ is a solenoidal field.

$\nabla \times \mathbf{V}$ curl $\mathbf{V}$ - a vector field representing twice the local angular velocity of a fluid particle. If $\nabla \times \mathbf{V} = 0$ then: (i) the field $\mathbf{V}$ is *irrotational* and *conservative*; (ii) a scalar field ϕ exists such that $\mathbf{V} = \nabla\phi$.

Gauss's theorem:

$$\iiint_{\text{vol}} (\nabla \cdot \mathbf{V})\, dv = \oiint_S \mathbf{V} \cdot d\mathbf{A} \quad . \tag{14.8}$$

The volume integral over the finite volume vol of $\nabla \cdot \mathbf{V}$ is equal to the net flux of $\mathbf{V}$ across the surface S that encloses vol. If $\mathbf{V}$ is solenoidal, the net flux across S will be zero (no sources or sinks of $\mathbf{V}$).

Stokes's theorem:

$$\oint_L \mathbf{V} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{V}) \cdot d\mathbf{A} \quad . \tag{14.9}$$

The line integral of $\mathbf{V} \cdot d\mathbf{r}$ around a closed loop L is equal to the flux of $\nabla \times \mathbf{V}$ across any surface S that spans L. If $\mathbf{V}$ is irrotational/conservative, the line integral of $\mathbf{V} \cdot d\mathbf{r}$ around any closed loop L is zero.

In a subset of cases, usually when the geometry is particularly simple, an analytical solution can be found for a partial differential equation (PDE).

We have looked at 3 linear, second order, PDEs, each with two independent variables:

$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ Laplace's equation (elliptic) in 2-D

$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\phi}{\partial t}$ Heat conduction (or diffusion) equation (parabolic) in 1-D

$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$ Wave equation (hyperbolic) in 1-D

Our principal technique for solving these PDEs has been to seek a separable solution, $\phi(x,y) = X(x)Y(y)$ or $\phi(x,t) = X(x)T(t)$. The solution must not only satisfy the PDE but also match the prescribed boundary conditions. A different approach, the self-similar solution, can be used to solve the diffusion equation when the problem has no inherent length-scale. In such cases, the diffusion length provides the characteristic length scale.

GP, Nov 2023