Filled In – As delivered Lent 2023

Part IB Paper 1 MECHANICS

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1 Introduction

This course picks where Part IA Mechanics left off. It begins with eight lectures on Newtonian Mechanics applied to rigid bodies and plane mechanisms. The remaining eight lectures is an introduction to Lagrangian mechanics and non-linear dynamics. Section 7 has a copy of the handout for the Experiment D1. I'll say more about doing this experiment in lectures.

involves displacements & velocities

2 Kinematics in Two Dimensions - Revision

planar motion

These concepts covered in Part IA Mechanics will be important in Part IB Mechanics, so it's as well to go over them.

2.1 Centre of Mass and Moment of Inetrtia

A lamina moving in a plane has a Centre of Mass (CoM) – usually denoted 'G' on diagrams – using the symbol ◒.

Difference between CoM & CoG ?

Finding the centre of mass: If G is located at $\overline{x}, \overline{y}$ and the total body mass is M then $M\overline{x} = \int x dm$ and $M\overline{y} = \int y dm$

$(\overline{x}, \overline{y})$ is the centroid of the body.

The weight force mg acts downwards through G.

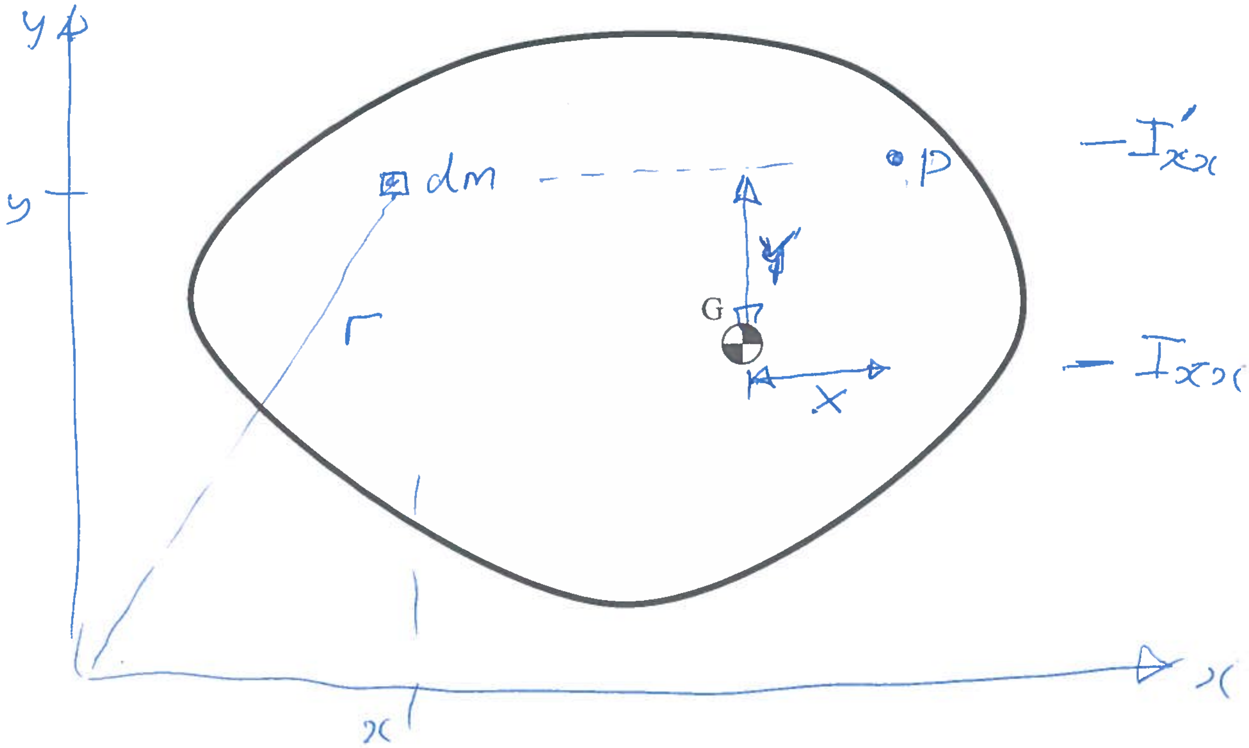
The **Moment of Inertia** of the body about the x axis is $I_{xx} = \int y^2 dm$ and about the y axis is $I_{xx} = \int x^2 dm$

All these formulae are in Mechanics Data Book

The **Perpendicular Axis Theorem** states that $I_{xx} + I_{yy} = I_{zz}$ which only works for a plane lamina.

$r^2 = x^2 + y^2$ $\qquad \int r^2 dm = \int x^2 dm + \int y^2 dm$

The **Parallel Axis Theorem** states that if I_{xx} and I_{yy} are moments of inertia through G then the moments of inertia I'_{xx} and I'_{yy} about parallel axes through P at X, Y are given by $I'_{xx} = I_{xx} + \cancel{MX^2}$ MY^2 and $I'_{yy} = I_{yy} + \cancel{MY^2}$ MX^2.



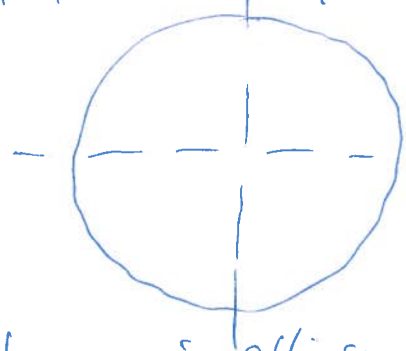
disc radius a

$\boxed{k^2 = \frac{1}{2}a^2}$

In most cases the **Mechanics Data Book** can be used to look up moments of inertia and the location of G. Note that the Data Book lists the **Radius of Gyration** k defined, for example, such that $I_{xx} = Mk_x^2$. The value of k is the distance at which all of the mass of a body can be concentrated to give the same moment of inertia. For a hoop, k_z is equal to the radius of the hoop.

Disc

Data book 5.4.3 p18



$I_{xx} = I_{yy} = m\frac{a^2}{4}$

$b = a$ in ellipse

$\therefore I_{zz} = I_{xx} + I_{yy} = \frac{ma^2}{2}$

Solid sphere radius a

from 5.5.2 Spheroid $a = b$

$\boxed{k^2 = \frac{2a^2}{5}}$

$k = \sqrt{\frac{2}{5}}\,a$ is radius of a hoop with same moment of inertia

Example

A square plate of side $2a$ and mass m is held at one corner A with all edges either horizontal or vertical. What is the initial angular acceleration of the plate when released from rest. ?

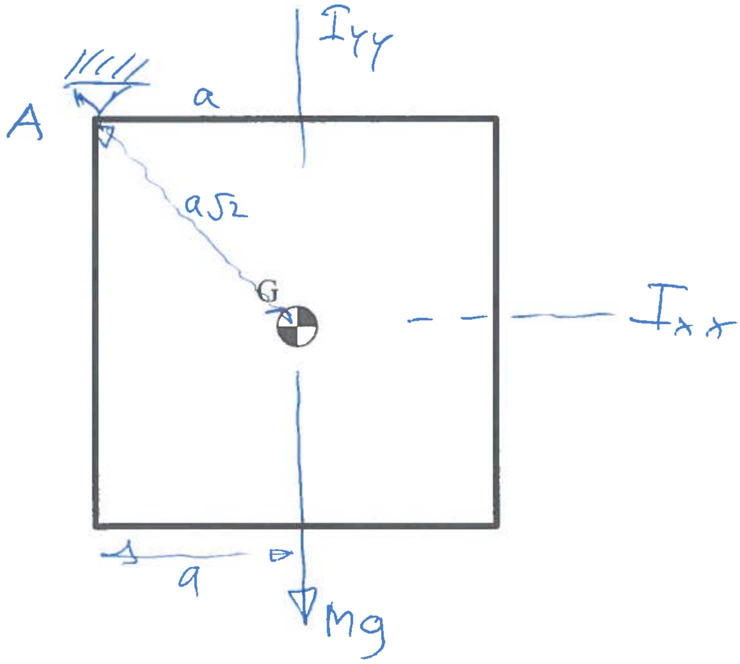
$I_{xx} = I_{yy} = \frac{1}{3}ma^2$

$I_{zz} = \frac{2}{3}ma^2$

$I_{zz}' = I_{zz} + m(a\sqrt{2})^2$

$= \frac{2}{3}ma^2 + 2ma^2$

$= \frac{8}{3}ma^2$



Moments about A

$Mga = I_{zz}'\ddot{\theta} = \frac{8}{3}ma^2\ddot{\theta} \quad \therefore \boxed{\ddot{\theta} = \frac{3}{8}\frac{g}{a}}$

check units: $\frac{m s^{-2}}{m} = s^{-2}$ ✓

2.2 Kinetic and Potential Energy

flat rigid body

The **Kinetic Energy** (KE or T) of a lamina of mass m and moment of inertia I at G is given by

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

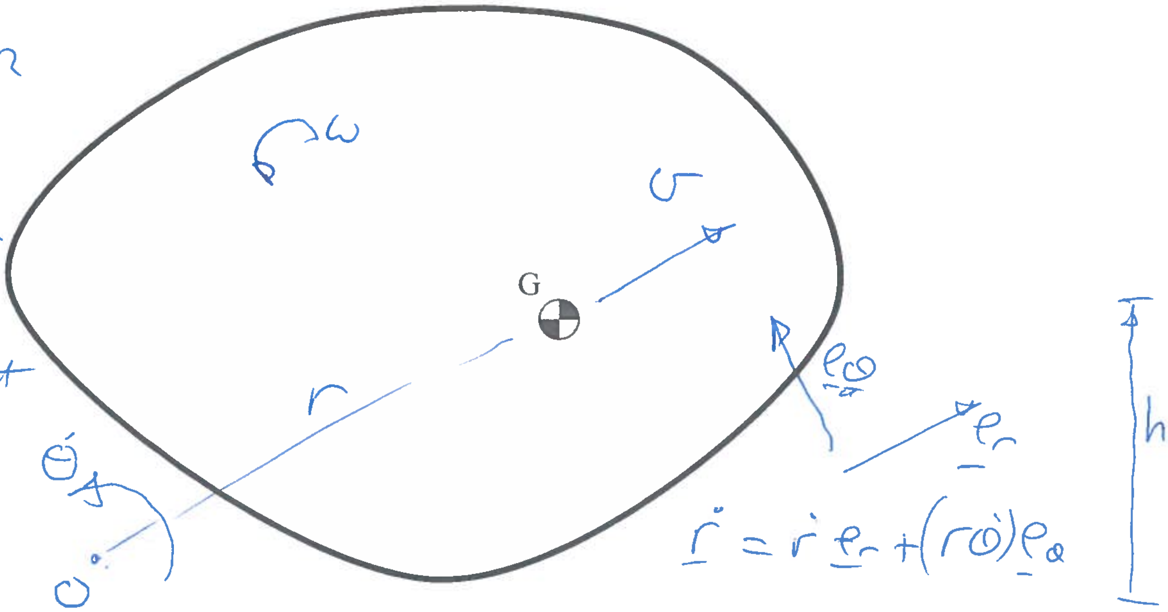
where v is the speed of the CoM and ω is the angular velocity of the lamina. Do not use the parallel axis theorem to find kinetic energy about any point other than G because it doesn't always work! The only exception is for motion about a fixed pivot, but it's safest to use G every time.

$v^2 = \dot{x}^2 + \dot{y}^2$

or in polars

$v^2 = \dot{r}^2 + (r\dot{\theta})^2$

Convenient that KE has "v^2" in it



$\dot{\underline{r}} = \dot{r}\,\underline{e}_r + (r\dot{\theta})\underline{e}_\theta$

A conservative force moving through a distance stores **Potential Energy** (PE or V) which can be recovered. The gravitational force, spring force, electrostatic force etc. are all conservative. Friction is *not* a conservative force. The most common PE formulae in mechanics are

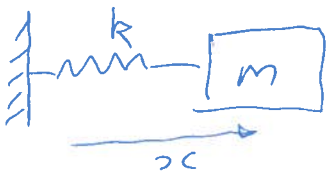
1. Gravitational PE (close to the Earth's surface): $V = mgh$

2. Gravitational PE (for satellite motion etc): $V = -\frac{GMm}{r}$ $= -\frac{GMm}{R+h} = -\frac{GMm}{R(1+\frac{h}{R})} = -\frac{GMm}{R}\left(1-\frac{h}{R}\right)$ $= \frac{GM}{R^2} m h$ (g !)

3. Spring PE: $V = \frac{1}{2}kx^2$

$F = kx$ $\quad V = \int F\,dx = \frac{1}{2}kx^2$

Many simple problems can be solved by **Conservation of Energy** where PE+KE is constant. Very very useful



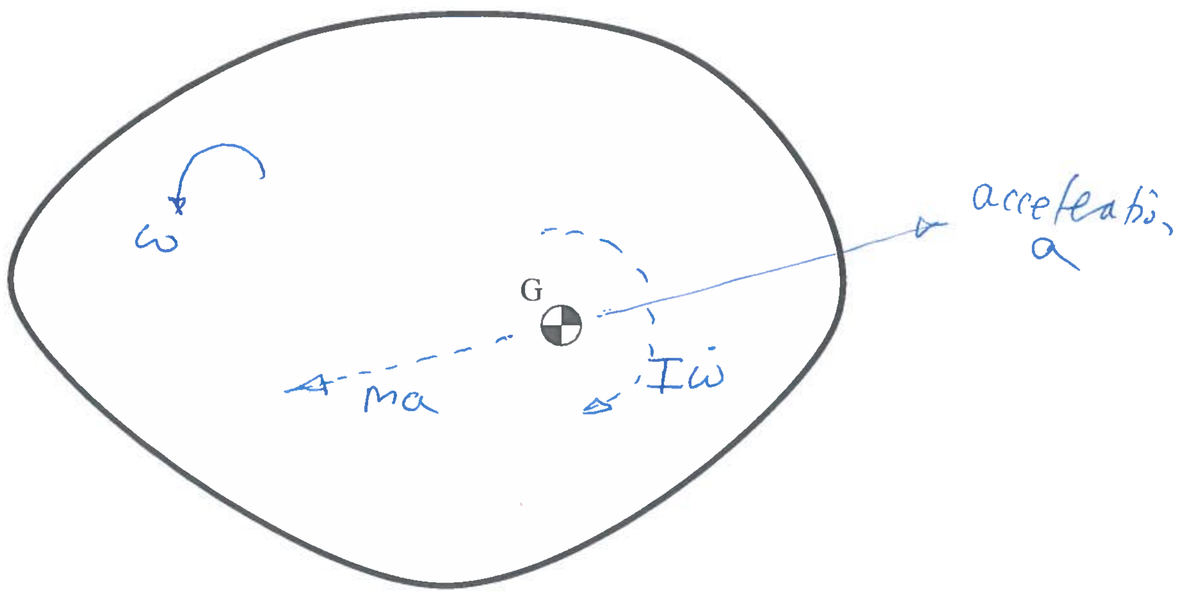
$\frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 = c$

Note: $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right)$!! check this

Take $x = 0$ at equilibrium $\therefore$ $\boxed{m\ddot{x} + kx = 0}$ ✓

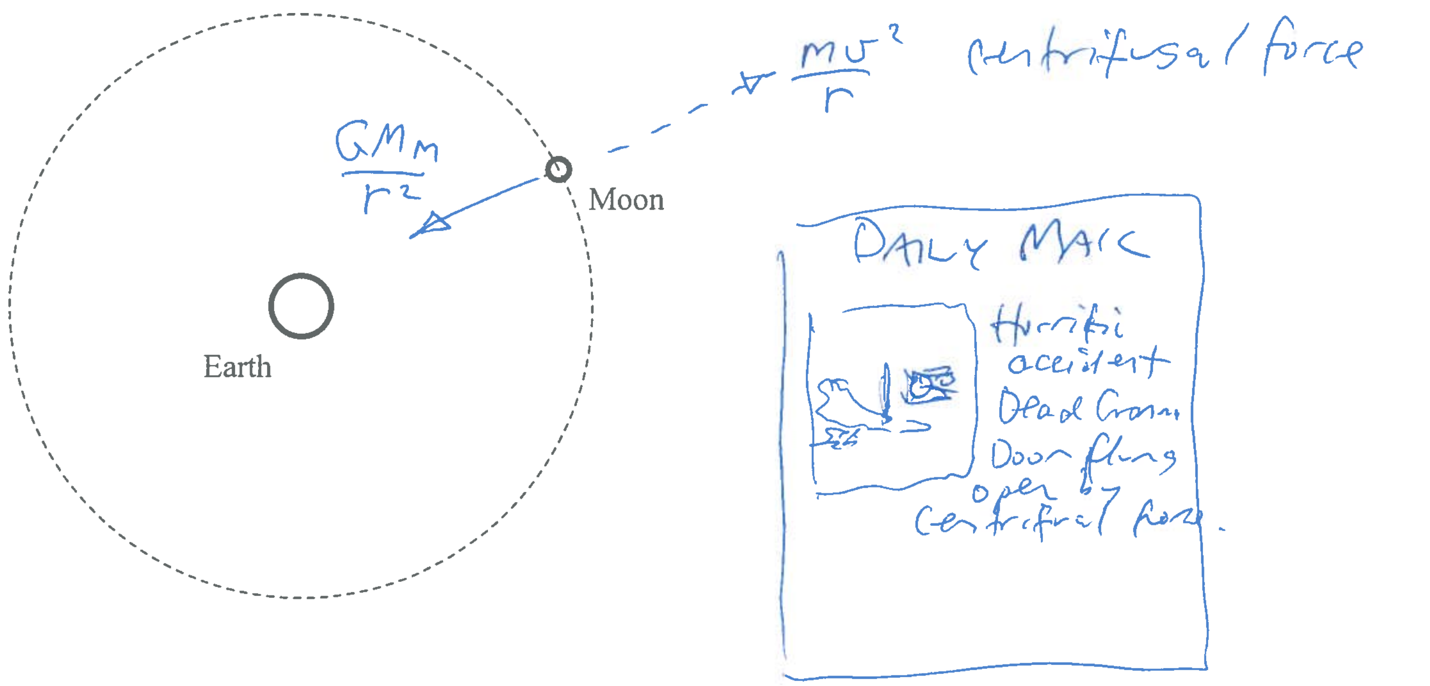
2.3 d'Alembert's Principle

Consider a body of mass m and moment of inertia I at G. It is in motion with linear acceleration $\mathbf{a}$ at G and angular acceleration $\dot{\omega}\mathbf{k}$. The **d'Alembert Inertia Force** is $-m\mathbf{a}$ acting at G and the **d'Alembert Inertia Couple** is $-I\dot{\omega}$. They can be treated as any other force or couple so that the methods of statics (equilibrium of forces and moments) can be used. To distinguish inertia forces and inertia couples from other forces they are conventionally shown dashed, just for clarity.



"Centrifugal Force" is an example of an inertia force.

Example
What holds the Moon up? (this is a question that vexed Isaac Newton)



L23
end L1

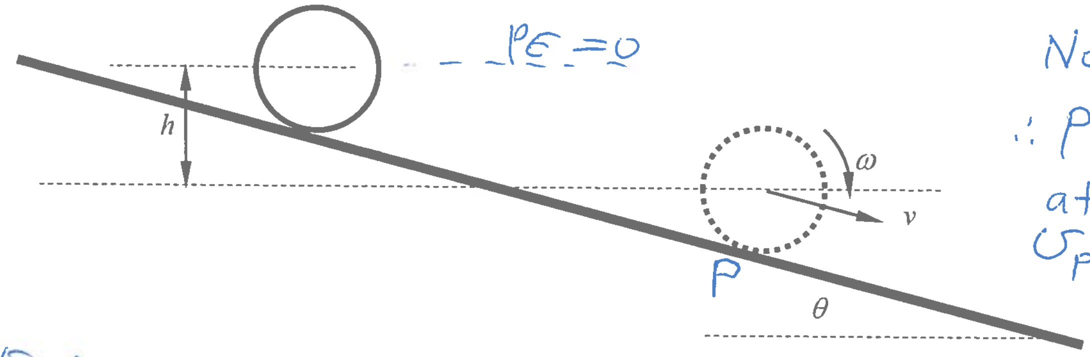
Example Data book $I_G = \frac{2}{5}ma^2$

A solid ball of radius a and mass m is released from rest and rolls without slip down a slope whose angle to the horizontal is θ.

(a) What is the angular velocity of the ball after it has descended a vertical height h?

(b) What is the acceleration of the ball down the slope?

(c) What is the minimum coefficient of friction μ required for no slip?



No slip at P

∴ P in ball is at rest

$v_P = 0$

$\underline{v = a\omega}$

(a) Energy $PE + KE = \text{const} = 0$

$$\therefore \; -mgh + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = 0$$

$$\therefore \; 2gh = \frac{7}{5}a^2\omega^2 \qquad \therefore \; \omega^2 = \frac{10}{7}\frac{gh}{a^2}$$

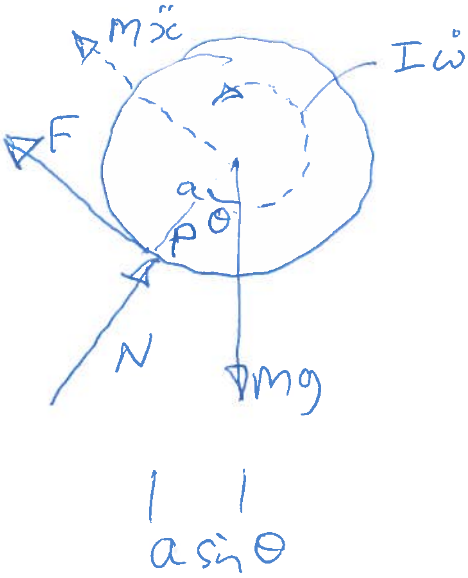
(units ok)

(b) d'Alembert, take moments about P

No slip also says $\dot{v} = a\dot{\omega}$

$\ddot{x} = a\dot{\omega}$

$$\sum M_P \; : \; mga\,\cancel{8}\sin\theta - I\dot{\omega} - m\ddot{x}a = 0$$



$$\therefore \; \ddot{x} = \frac{5g}{7}\sin\theta$$

(c) $$\mu = \frac{F}{N} = \frac{-m\ddot{x} + mg\sin\theta}{mg\cos\theta}$$

$$= \frac{2}{7}\tan\theta$$

(check this for yourself)

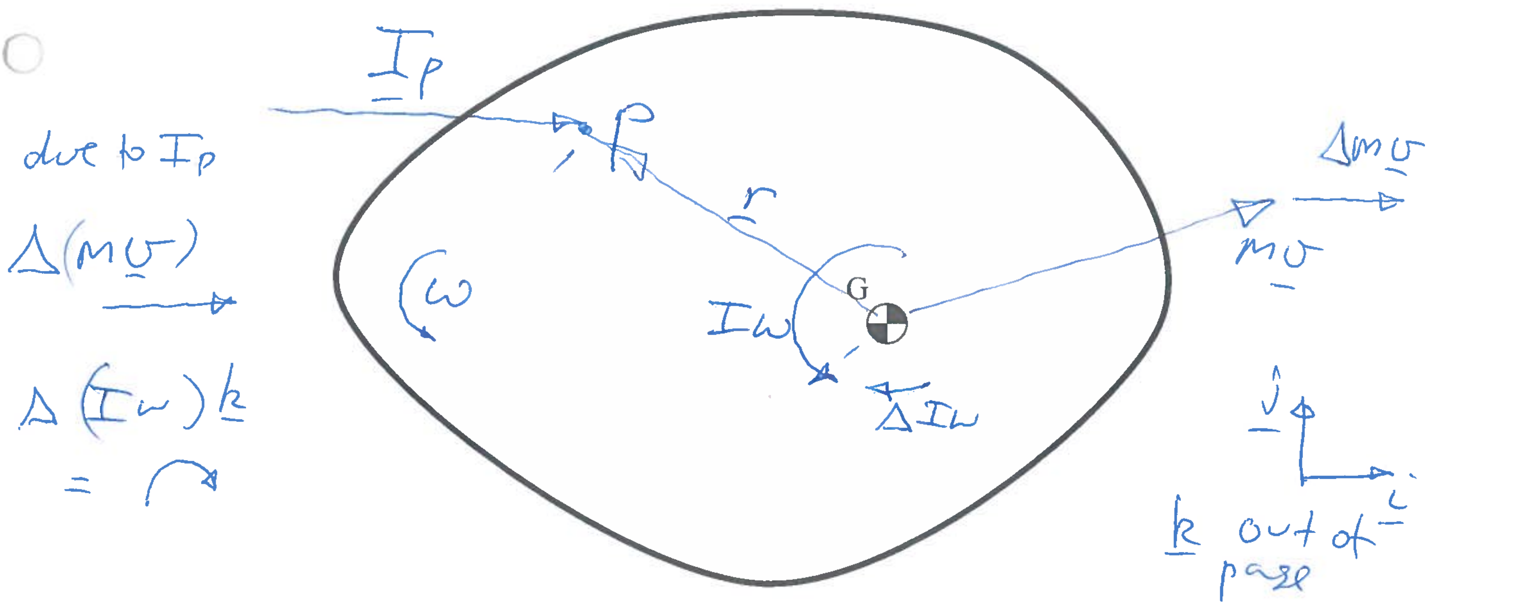
2.4 Linear and Angular Momentum

A body has mass m and moment of inertia I at G. It is in motion with linear velocity $\mathbf{v}$ at G and angular velocity $\omega\mathbf{k}$. The **Linear Momentum** is $\mathbf{p} = m\mathbf{v}$ through G and the **Angular Momentum** is $\mathbf{h} = I\omega\mathbf{k}$.

If a force $\mathbf{F}$ acts on the body then $\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$ and if $\mathbf{M}$ is the moment of all forces about G then $\mathbf{M} = \frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}$
If $\mathbf{F}$ is impulsive in nature so that the impulsive force $\mathbf{I}_P = \int \mathbf{F} dt$ then the impulse $\mathbf{I}_P = \Delta\mathbf{p}$ and the moment about G of the impulse $\mathbf{r} \times \mathbf{I}_P = \Delta\mathbf{h}$

$\underset{\sim}{I}_p \quad \Delta(m\underline{v})$ Impulse = change in momentum

$\underline{r} \times \underline{I}_p = \Delta(I\omega)\underline{k}$ moment of impulse = change in angular momentum



If there is a point P through which all impulsive forces pass them angular momentum is conserved about P.

Finite forces (like gravity and spring forces) can be ignored when compared with impulsive forces.

Example

A pencil of mass m and length $2a$ is dropped from height h so as just to clip the edge of a filing cabinet. Assume that no energy is lost in the collision.

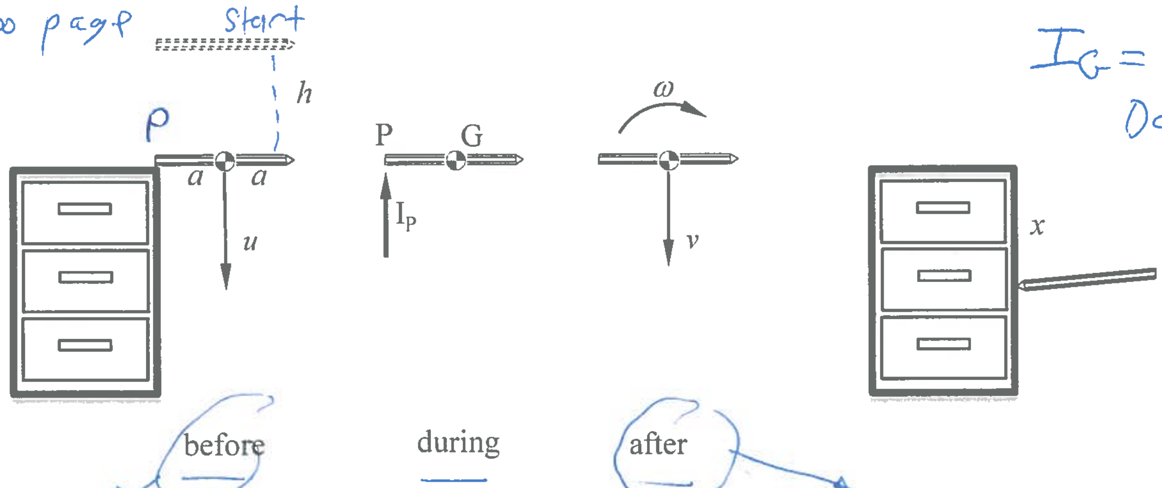
(a) What is the velocity of G and the angular velocity immediately after impact?

(b) How far down the cabinet does the tip of the pencil hit the cabinet?

[Video of this motion see http://www2.eng.cam.ac.uk/~hemh1/movies.htm#pencils]

$\underline{k}$ into page

$+$



$I_G = \frac{1}{3}ma^2$
Data book

rotated by angle π (approx)

Showing Impulse I_P (ignore mg because its small)

a/ $h_P = mua\,\underline{k}$ (Moment of Momentum!) $= (mva + \frac{1}{3}ma^2\omega)\underline{k}$ ①

$KE = \frac{1}{2}mu^2 = mgh = \frac{1}{2}mv^2 + \frac{1}{2}\frac{1}{3}ma^2\omega^2$ ②

① $\rightarrow$ $a\omega = 3(u-v)$

② $\rightarrow$ $u^2 - v^2 = \frac{1}{3}(a\omega)$

$v = \frac{u}{2}$

$a\omega = \frac{3u}{2}$

b/ $\omega t = \pi \quad \therefore \quad t = \frac{\pi}{\omega} = \frac{2\pi a}{3u}$

SUVAT : "$x = ut + \frac{1}{2}gt^2$"

v is here

(KE→h $v = \sqrt{2gh}$)

$\therefore \quad x = \frac{\pi a}{3}\left(1 + \frac{\pi a}{3h}\right)$

$\pi \approx 3 \quad \therefore \quad x \approx a + \frac{a^2}{h}$

say $h = \frac{a}{2}$

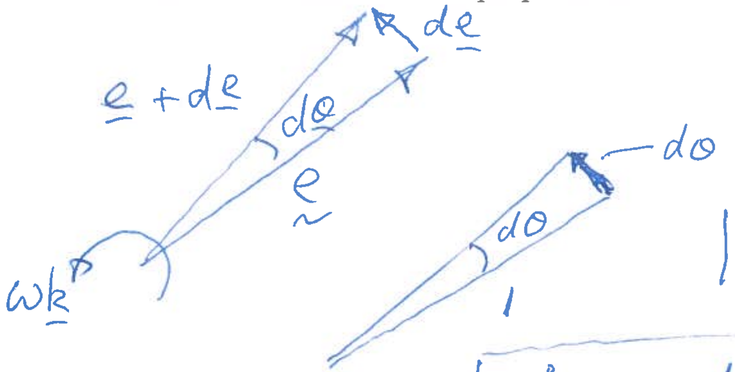
$\therefore x = 3a$

as observed

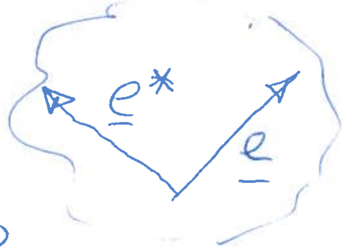
2.5 Differentiation of Rotating Vectors

Really important in Part IB Mech & 3C5 Dynamics

If a unit vector $\mathbf{e}$ is rotating at angular velocity $\omega = \dot{\theta}\mathbf{k}$ then $\frac{d}{dt}\mathbf{e} = \dot{\mathbf{e}} = \omega \times \mathbf{e} = \dot{\theta}\mathbf{e}^*$
The notation $\mathbf{e}^*$ denotes a vector perpendicular to $\mathbf{e}$ in the direction of rotation defined by ω



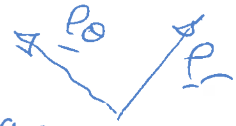
$|d\underline{e}| = d\theta$



$$\left|\frac{d\underline{e}}{dt}\right| = \frac{d\theta}{dt} = \omega$$

$$\boxed{\dot{\underline{e}} = \frac{d\underline{e}}{dt} = \omega\,\underline{e}^*}$$

just like $\dot{\underline{e}}_r = \omega\,\underline{e}_\theta$ last year



Example
In polar coordinates the position of P is

$$\mathbf{r} = r\mathbf{e}_r$$

The velocity of P is found by differentiation of $\mathbf{r}$, using the product rule.

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

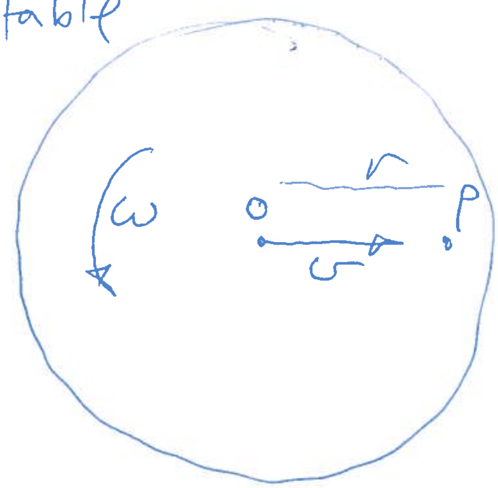
be sure you can do this

The acceleration of P is found by differentiation of $\dot{\mathbf{r}}$, using the product rule.

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\mathbf{e}_\theta$$

radial   centripetal   coriolis   circumferential

turn table



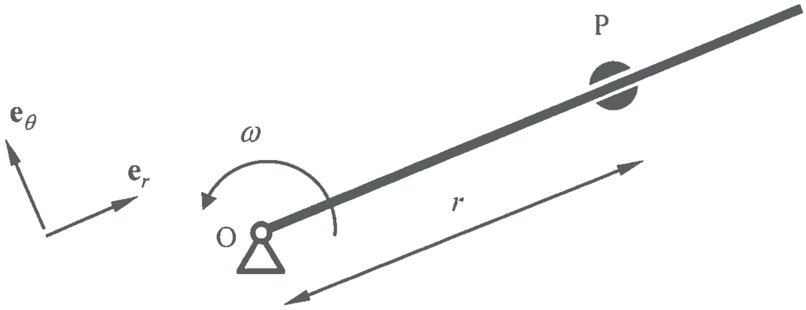
at O, my speed is zero
at P, my speed is rω ↑
So I need a force ↑ to be accelerated up to speed.
This is the coriolis force

2.6 Equations of Motion from F=ma

Equations of motion can found by application of Newton II: $\mathbf{F} = m\mathbf{a}$

This is a vector equation
∴ 2 equations in 2D

Example
Consider a particle sliding without friction on a rod.



$$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{e}_\theta$$

$\underline{F} = m\ddot{\underline{r}}$, no friction in $\underline{e}_r$ direction

$\therefore \quad \ddot{r} - r\dot{\theta}^2 = 0$

Case ① $\quad \dot{\theta} = \omega = \text{const} \qquad \ddot{r} - \omega^2 r = 0$

$\therefore \quad r = A\cosh\omega t + B\sinh\omega t$

Apply boundary conditions eg
starting from rest ...

Case ② No torque at O ∴ no force
on particle in $\underline{e}_\theta$ direction

$\therefore \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

try $\frac{d}{dt}(r^2\dot{\theta}) = 0 \quad = \quad 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0$ ✓

conservation of angular momentum

$\boxed{\text{end L2 L23}}$

2.7 Equations of Motion from Energy

Equations of motion from Energy and from Newton II are one and the same. For example think of a falling ball, for which $F = ma$ gives $-mg = m\ddot{y}$. Integrate this to get the "Conservation of Energy" equation.

$mg \downarrow \quad \downarrow y$

$$m\ddot{y} + mg = 0$$

$$\therefore m\frac{d}{dy}\left(\frac{1}{2}\dot{y}^2\right) + mg = 0$$

$$\therefore \frac{1}{2}m\dot{y}^2 = -mgy$$

$\therefore$ KE PE

Note the useful result

$$\ddot{y} = \frac{d}{dy}\left(\frac{1}{2}\dot{y}^2\right)$$

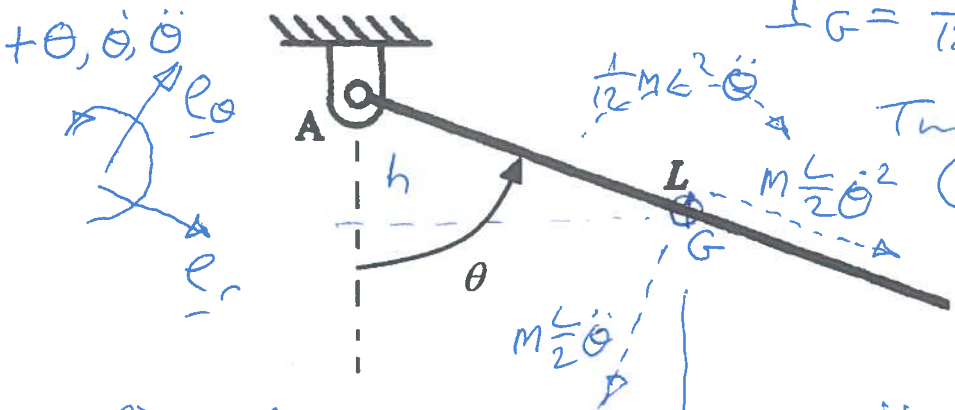
proof: $\frac{d}{dy}\left(\frac{1}{2}\dot{y}^2\right) = \frac{1}{2}\frac{d}{dt}\left(\dot{y}^2\right)\frac{dt}{dy} = \frac{1}{2}2\dot{y}\ddot{y}\frac{1}{\dot{y}} = \ddot{y}$ ✓

chain rule

Example

A rod of mass m and length $2a$ is swinging freely. It is released from rest when horizontal, ie $\theta = \pi/2$. Find the angular acceleration of the rod when $\theta = \pi/4$ and find the reaction at A.

$I_G = \frac{1}{12}mL^2$ (data book)



Two methods: ① Newton II & d'Alembert ② Energy

① d'Alembert $\quad mg\downarrow$

$$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (\cancel{2\dot{r}\dot{\theta}} + r\ddot{\theta})\underline{e}_\theta$$

$$\Sigma M_A: \quad mg\frac{L}{2}\sin\theta + m\frac{L}{2}\ddot{\theta}\frac{L}{2} + \frac{1}{12}mL^2\ddot{\theta} = 0$$

$$\therefore \frac{1}{2}g\sin\theta = -\frac{1}{3}L\ddot{\theta}$$

$$\therefore \ddot{\theta} = -\frac{3}{2}\frac{g}{L}\sin\theta$$

At $\theta = \frac{\pi}{4}$, $\ddot{\theta} = -\frac{3\sqrt{2}g}{4L}$ //ans

You can now do questions 1-4 on Examples Paper 1

② Energy method

$$mgh = \frac{1}{2}m\left(\frac{L}{2}\dot{\theta}\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2$$

$$\equiv \quad \text{"}\tfrac{1}{2}mv^2\text{"} \qquad \text{"}\tfrac{1}{2}I\omega^2\text{"}$$

$$\Rightarrow mg\frac{L}{2}\cos\theta = \frac{1}{2}\frac{1}{3}mL^2\dot{\theta}^2$$

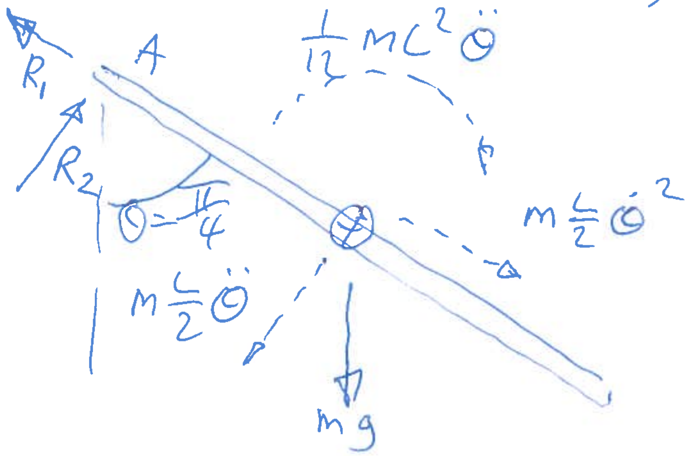
$$\therefore \dot{\theta}^2 = 3\frac{g}{L}\cos\theta = \frac{3\sqrt{2}}{2}\frac{g}{L} \quad \text{at } \theta = \frac{\pi}{4}$$

check use $\frac{1}{2}\frac{d}{d\theta}(\dot{\theta}^2) = \ddot{\theta}$

$$\therefore \ddot{\theta} = \frac{3}{2}\frac{g}{L}\int\cos\theta\, d\theta$$

$$= -\frac{3}{2}\frac{g}{L}\sin\theta \qquad \text{as before !}$$

Find Reactions at A, with $\theta = \frac{\pi}{4}$



$$R_2 = +m\frac{L}{2}\ddot{\theta} + mg\cos\theta = +\frac{mL}{2}\left(-\frac{3\sqrt{2}}{4}\frac{g}{L}\right) + mg\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{8}mg$$

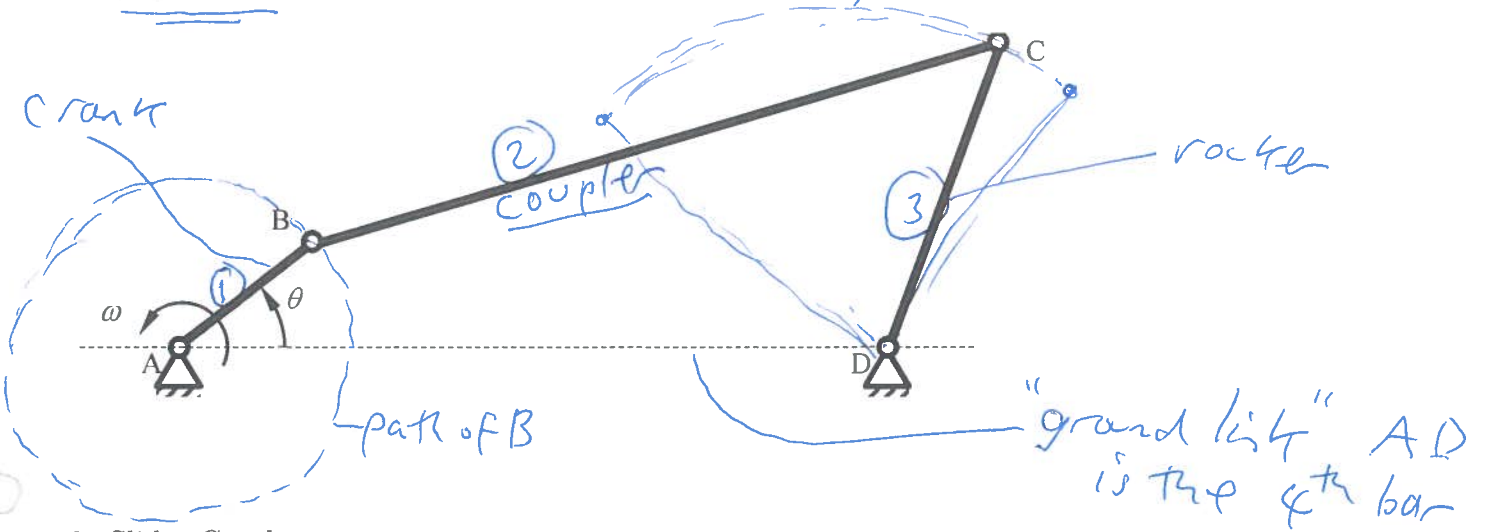
$$R_1 = m\frac{L}{2}\dot{\theta}^2 + mg\sin\theta$$

$$= m\frac{L}{2}\left(\frac{3\sqrt{2}}{2}\frac{g}{L}\right) + mg\frac{\sqrt{2}}{2} \qquad \therefore R_1 = \frac{5}{4}\sqrt{2}\,mg$$

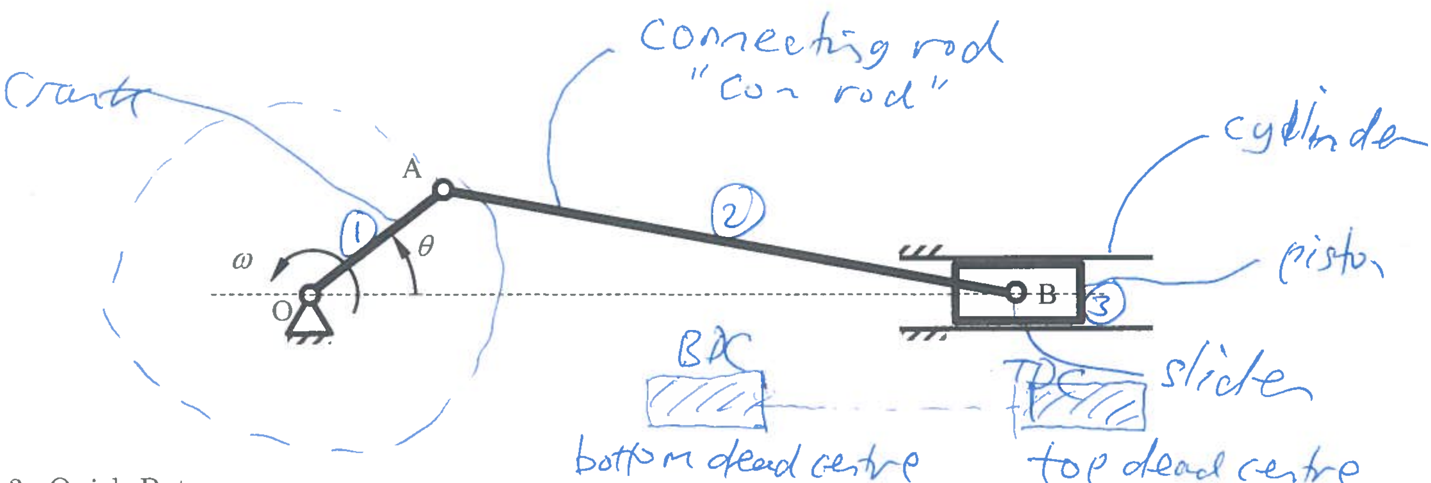
3 Kinematics of plane mechanisms

There are many types of mechanisms. In Part IB Mechanics we will look mainly at "four bar mechanisms". The following are examples:

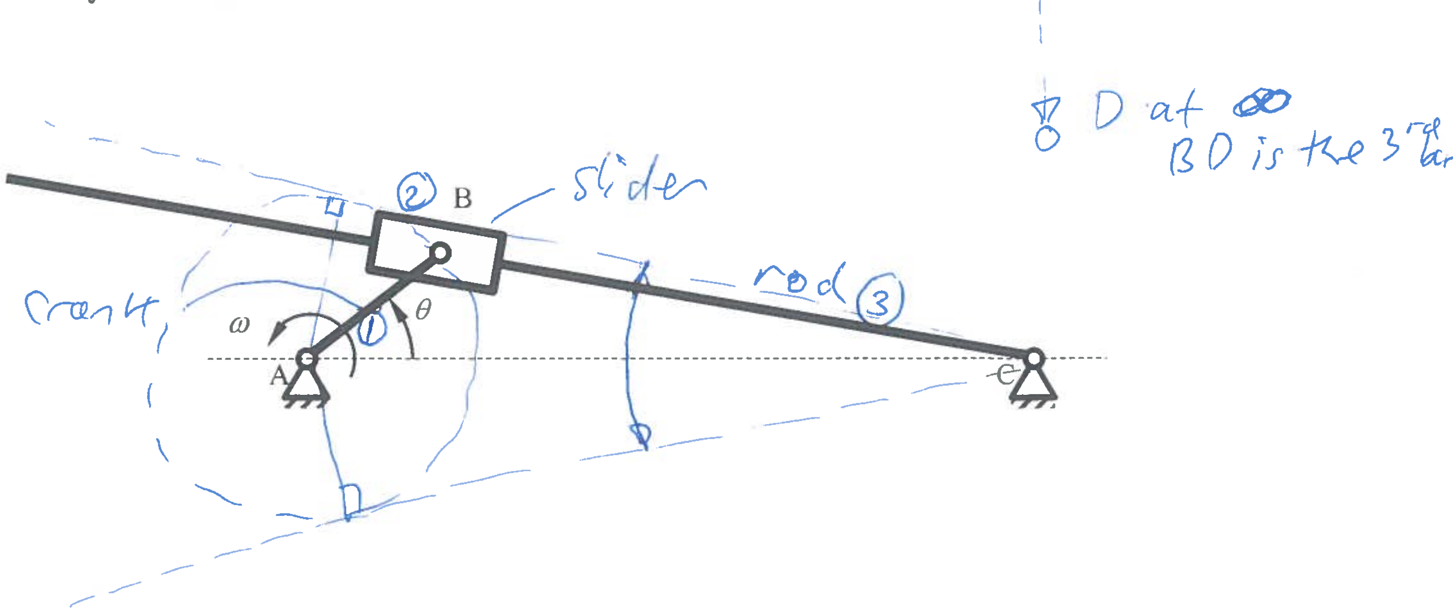
1. Crank Rocker



2. Slider Crank



3. Quick Return



3.1 Instantaneous Centres and Velocity Diagrams

The easiest way of determining the velocities and angular velocities of the components of a mechanism is by the method of **Instantaneous Centres**. If the direction of the velocities of two points in a rigid body are known at a particular instant then the instantaneous centre I about which the body is instantaneously rotating is found at the intersection of the perpendiculars to these velocities.
Once velocities have been determined then a **Velocity Diagram** can be drawn.
The velocity diagram is a graphical representation of the vectors showing the relative velocities between components of the mechanism, usually the pivots which are labelled with Capital Letters.

I_2 Instantaneous Centre of body (2) at intersection of lines $\perp$ v_B & v_C

Example
Crank Rocker

ω_2

The directions of velocities of points B and C are known

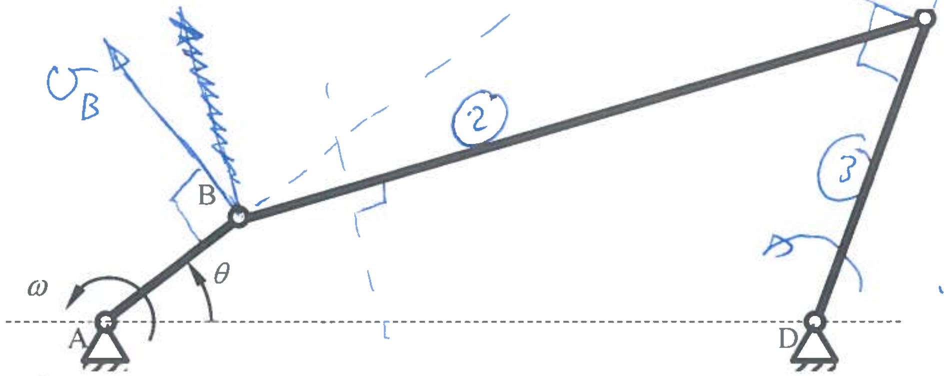
Given ω

$\therefore v_B = \overline{AB}\,\omega$

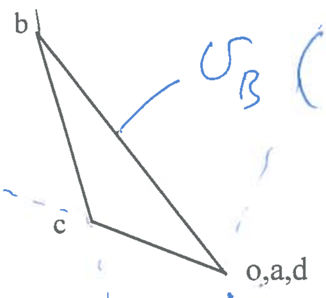
$v_B = \overline{AB}\,\omega = \overline{I_2B}\,\omega_2$

$\therefore \boxed{\omega_2 = \frac{\overline{AB}}{\overline{I_2B}}\,\omega}$

Space Diagram



$\therefore \omega_3 = \frac{v_C}{CD} = \frac{I_2C\,\omega_2}{CD}$



$v_{B/C}$ is $\perp$ BC

v_B (given, $\overline{AB}\omega$ ↖)

origin

v_C is $\perp$ DC

Velocity Diagram

It is conventional to label the
velocity diagram with lower-case letters
and not to put arrowheads on vectors.

end L3 L23

Example
Slider Crank

The directions of velocities of points A and B are known

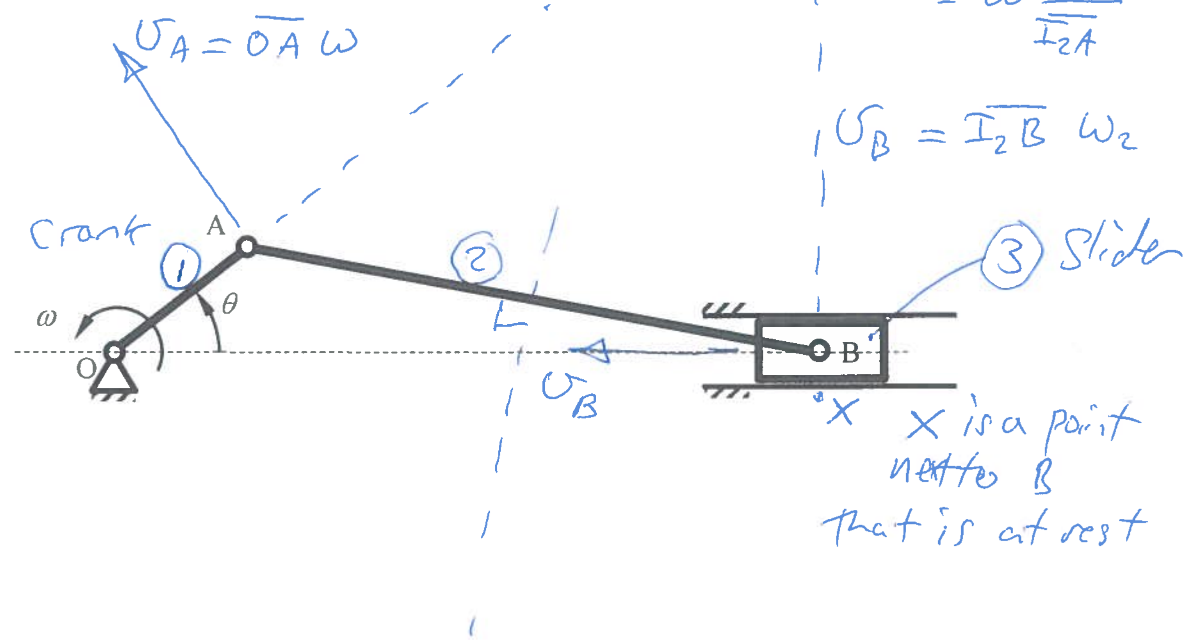
I_2 found at X

$\omega_2 = \frac{v_A}{\overline{I_2 A}}$

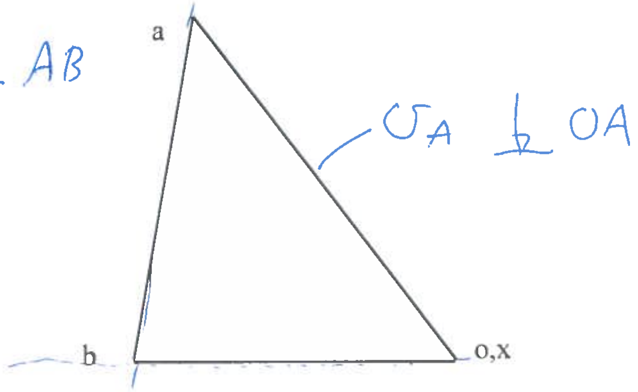
$= \omega \frac{\overline{OA}}{\overline{I_2 A}}$

$v_B = \overline{I_2 B}\, \omega_2$

Space Diagram



$v_{B/A}$ is $\perp$ AB



v_B is in this direction

Velocity Diagram

The sliding velocity of the slider is represented by labelling a point X on the cylinder instantaneously adjacent to B

Example

Quick Return (A tricky one)

The directions of the velocity of point B on the slider is known – but we need another point to locate the instantaneous centre of the slider. Imagine extending the slider so that part of it is over point C. The direction of the velocity of the point on the slider instantaneously over point C is known.

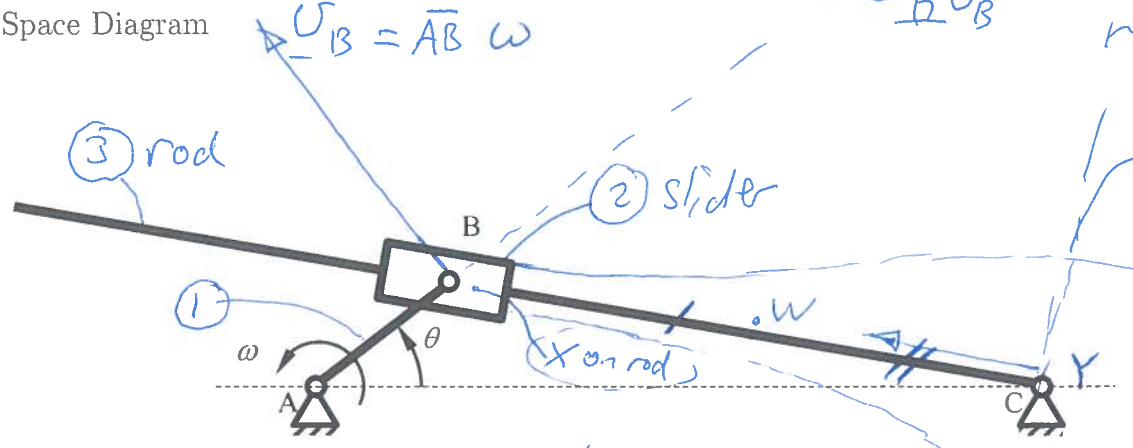
I_2 found at Y

$\omega_2 = \frac{v_B}{\overline{I_2 B}}$

$\perp v_B$

$\perp v_Y$

Space Diagram

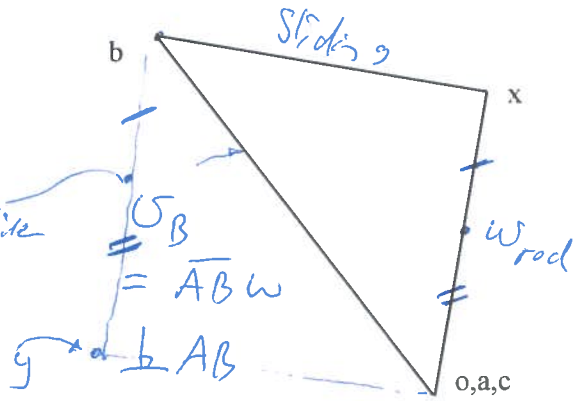


extension of slide 2 to include point Y which is immediately adjacent to C where there is pure sliding so v_Y ←

$\boxed{\omega_{rod} = \omega_2}$

angular velocity of rod = angular velocity of slider

velocity of point W on slide



point X is on rod next to B

Image theorem (see later) is used to find velocity of point W_{rod} & W_{slide}

Velocity Diagram

The sliding velocity of the slider is represented by labelling a point X on the rod instantaneously adjacent to B

Note that the angular velocity of the rod is *the same* as the angular velocity of the slider.

3.2 The Velocity Image

$r_{B/A}$ means position of B relative to A

Once you have a velocity diagram then intermediate points on a rigid lamina that has angular velocity $\omega\mathbf{k}$ can be obtained by inspection using the **Velocity Image**.
Suppose the velocities of two points A and B on the lamina are known and they are placed on a velocity diagram. You can locate the velocity of point C on the diagram by noting that triangles ABC and abc are similar, with abc rotated 90° from ABC in the direction of ϖ.
Proof:

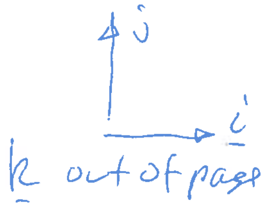
$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \quad \dot{\mathbf{r}}_B = \dot{\mathbf{r}}_A + \dot{\mathbf{r}}_{B/A} = \dot{\mathbf{r}}_A + \omega\mathbf{k} \times \mathbf{r}_{B/A}$

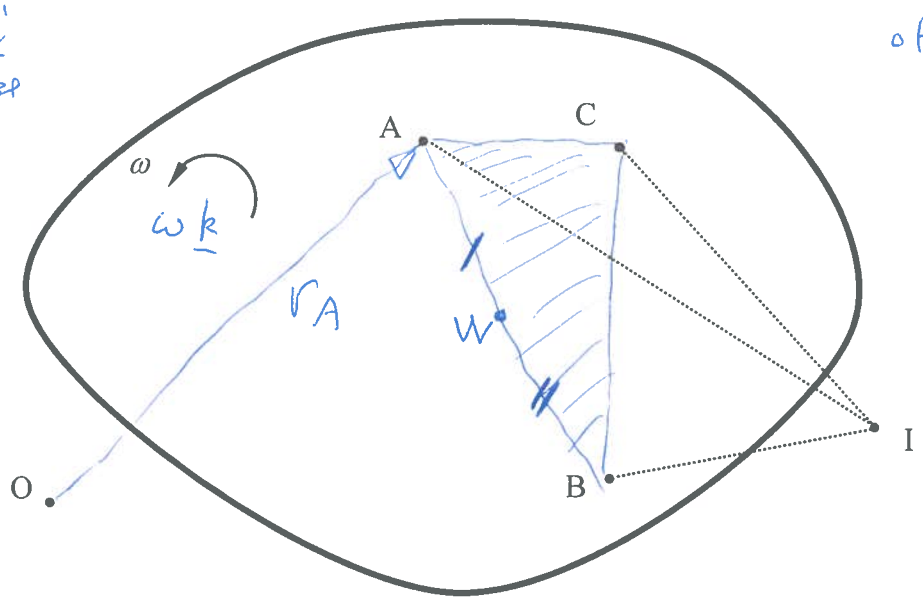
$\mathbf{r}_C = \mathbf{r}_A + \mathbf{r}_{C/A} \quad \dot{\mathbf{r}}_C = \dot{\mathbf{r}}_A + \dot{r}_{C/A} = \dot{\mathbf{r}}_A + \omega\mathbf{k} \times \mathbf{r}_{C/A}$

$\boxed{\underset{\sim}{v} = \underline{\omega} \times \underline{r}}$

$= \omega \underline{k} \times \underline{r}$

∴ "$\underline{\omega} \times$" is a rotation of 90° ↺



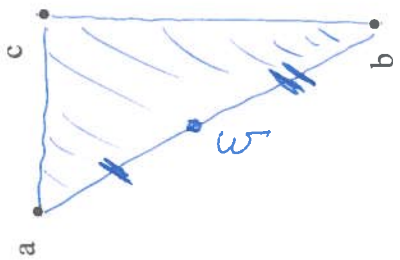


instantaneous centre of the body is at o

i
o

On a velocity diagram

Triangle abc is similar to triangle ABC rotated by 90° ↺ in direction of ω

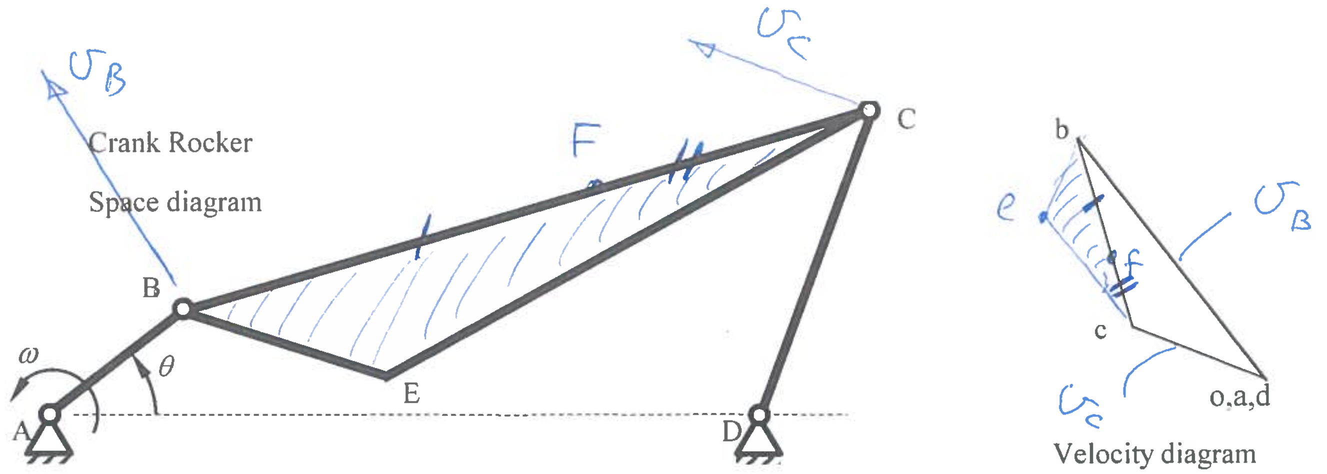


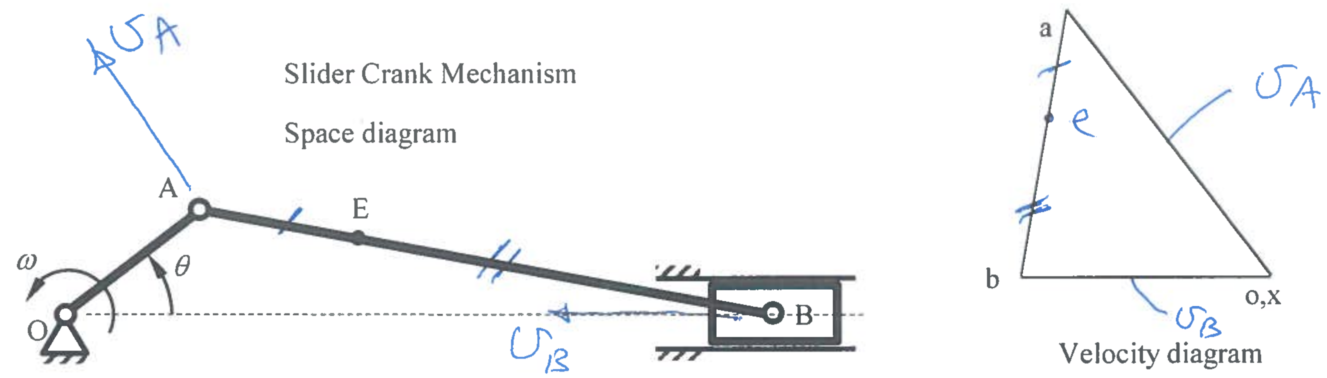
Use —/—•—//— to show that you are using image theorem

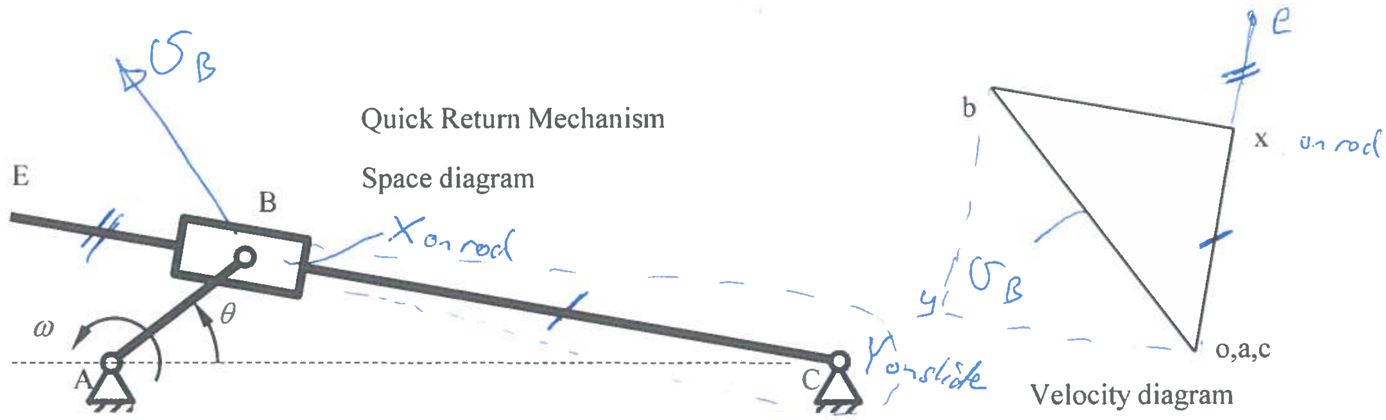
Velocity Diagram, note that abc is ABC rotated by 90° in the direction of $\omega\mathbf{k}$ and that the instantaneous centre I is instantaneously at rest, so on the velocity diagram i is at o.

Examples

Use "velocity image" to locate the point E on the velocity diagram for each of the following:







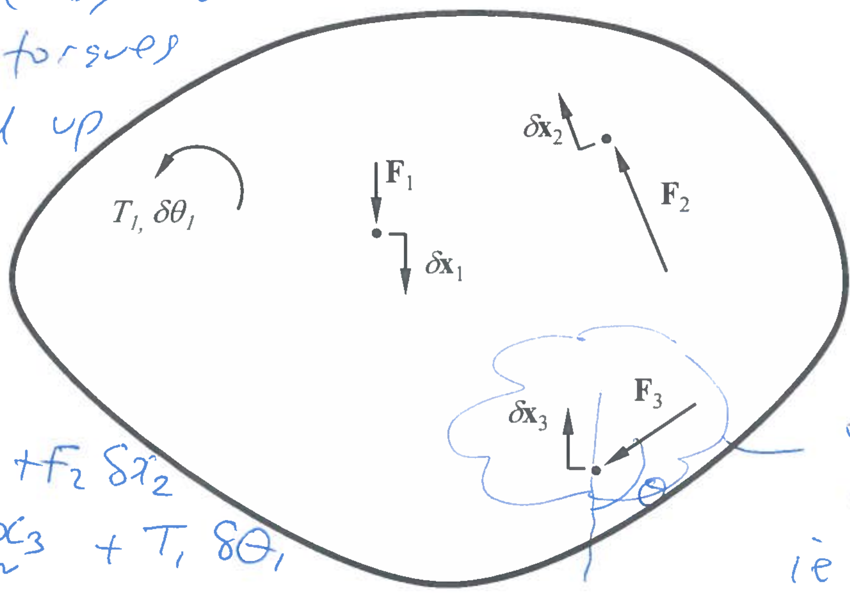
Take care with rotating sliders
(identify points on rod & on slider

3.3 Virtual Power

Mechanisms are used to transmit force and power. It is often necessary to calculate the torque required to drive a mechanism, or to evaluate the effect of friction at sliders and joints. The method of **Virtual Power** is useful.

First consider a system of parts, moving slowly without friction. It does not absorb any energy and kinetic energy can be ignored. The net work done on the mechanism as it moves by a small amount is zero. Consider all forces $\mathbf{F}_i$ acting on the mechanism. In a small time interval δt the points of application of these forces displace an amount $\delta \mathbf{x}_i$. Likewise any torques T_i turn through a small angle $\delta\theta_i$.

work done by all forces & torques must add up to zero when the body moves



use $F_3 \, \delta x_3 \cos\theta$ ie dot product

$\delta W = F_1 \delta x_1 + F_2 \delta x_2 + \underset{\sim}{F_3} \cdot \underset{\sim}{\delta x_3} + T_1 \delta\theta_1 = 0$

dot product

During the time interval δt the total work done is

$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{x}_i + \sum_i T_i \delta\theta_i = 0$$

(Note that for a planar mechanism all torques and rotations are in the $\mathbf{k}$ direction.)
Now divide by δt and note that $\delta \mathbf{x}_i / \delta t = \mathbf{v}_i$ and $\delta\theta_i/\delta t = \omega_i$ giving

$$\sum_i \mathbf{F}_i \cdot \mathbf{v}_i + \sum_i T_i \omega_i = 0$$

Power equation

Use the method of instantaneous centres and/or velocity diagrams to find velocities $\mathbf{v}_i$ and angular velocities ω_i at the point of application of all forces and torques.

1. Reaction forces at fixed pivots do no work as they do not move through any distance.

2. Friction at sliders can be considers as an external force - use the relative sliding velocity.

3. Friction at pivots can be considered as an external torque - use the relative angular velocity.

4. Friction always acts to oppose the motion.

5. Be very careful with signs - use a consistent sign convention.

6. Be aware of the dot product in $\mathbf{F}_i \cdot \mathbf{v}_i$ and be sure to evaluate the displacement in the direction of the force (for instance look at forces $\mathbf{F}_3$ in the diagram above).

end L4 / L23

Example

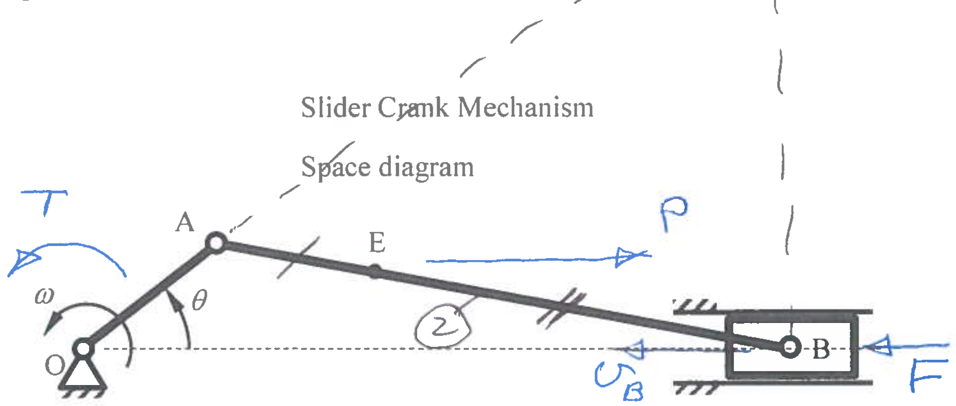
(a) what torque T at O is required to resist a gas pressure force F acting to the left at B?

(b) what horizontal force at E will resist the gas pressure force F at B?

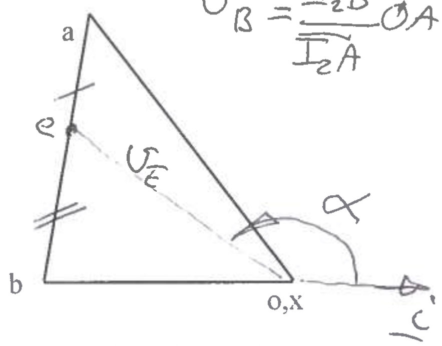
(c) what gas pressure force at B is required to overcome a friction torque Q acting at each of the pivots O, A and B?

Slider Crank Mechanism

Space diagram



$$v_B = \frac{\overline{I_2B}}{\overline{I_2A}} \overline{OA}\,\omega$$



Velocity diagram

a/ $T\omega + F v_B = 0$ Zero net power

$\therefore\ T = -\dfrac{F v_B}{\omega} = -\dfrac{F}{\cancel{\omega}} \dfrac{\overline{I_2B}}{\overline{I_2A}} \overline{OA}\,\cancel{\omega}$

negative as expected units ok!

b/ First find v_E, use image theorem

then $\underset{\sim}{P} \cdot \underset{\sim}{v_E} + F v_B = 0$ (net power $= 0$)

$\therefore\ P v_E \cos\alpha = -F v_B$. Say $\cos\alpha = -0.75$

$\therefore\ P = \frac{4}{3} F \dfrac{v_B}{v_E}$

c/ look at joint A

sign convention for angular velocity

Power ~~Work done~~ at joint A $= -Q\,|\omega_1 - \omega_2|$

You can now do questions 5 and 6 on Examples Paper 1 tabulate all joints

	$\lvert\omega\rvert$ at joint			
joint at O	$\lvert\omega\rvert$	$= \omega$	$=$	ω
joint at A	$\lvert\omega_2 - \omega_1\rvert$	$= \left\lvert -\omega\frac{OA}{I_2A} - \omega\right\rvert$	$=$	$\omega\left(\frac{OA}{I_2A} + 1\right)$
joint at B	$\lvert\omega_3 - \omega_2\rvert$	$= \left\lvert 0 - -\omega\frac{OA}{I_2A}\right\rvert$	$=$	$\omega\,\frac{OA}{I_2A}$

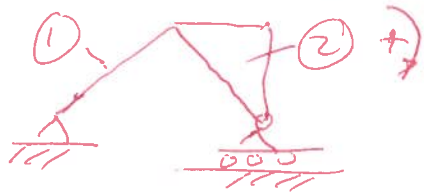
$$+ \overline{\left(2 + 2\frac{OA}{I_2A}\right)\omega}$$

$$F\,v_B = 2Q\left(1 + \frac{OA}{I_2A}\right)\omega$$

$$\therefore\ F =$$

Hint for EP1 Q5

tabulate all joints



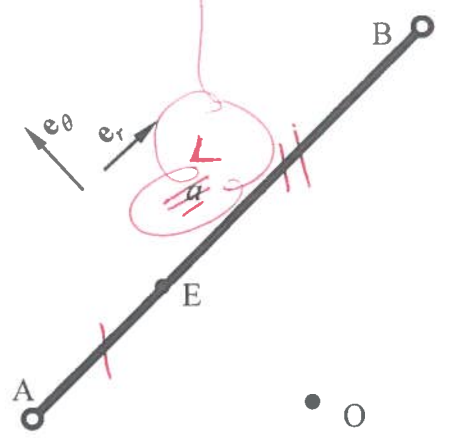
EP1 Q6, note $\omega_{slider} = \omega_{rod}$

fix this for next year!

3.4 Acceleration Diagrams and Acceleration Image

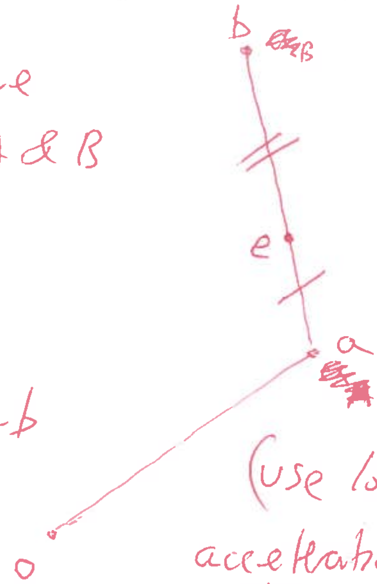
In mechanisms the relative motion between two points is well defined. There are two common cases:

1. a rigid link AB



lets say we have accelerations of A & B
$\underline{a}_A$ & $\underline{a}_B$

Use image theorem to get e on ab



(use lowercase) acceleration diagram

We can draw the acceleration diagram for the rigid link AB of length a by using the vector expression for the acceleration of B using unit vectors aligned with AB:

$$\mathbf{r}_B = \mathbf{r}_A + a\mathbf{e}_r$$

which can be differentiated to get velocities (see Section 2.5)

$$\dot{\mathbf{r}}_B = \dot{\mathbf{r}}_A + a\dot{\theta}\mathbf{e}_\theta$$

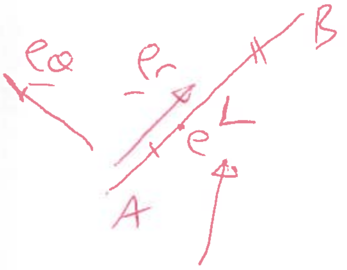
and then again to get the acceleration

$$\ddot{\mathbf{r}}_B = \ddot{\mathbf{r}}_A - a\dot{\theta}^2\mathbf{e}_r + a\ddot{\theta}\mathbf{e}_\theta$$

proof of image theorem for accelerations

It can readily be seen that a point E on the line AB can be placed on the acceleration diagram in proportion to its distance along AB - this is the **Acceleration Image** and it is just the same as the Velocity Image: any rigid body in the form of a polygon ABC will have a similar polygon abc in the acceleration diagram. Unlike the velocity image, the acceleration image abc is rotated from ABC through some angle that is not necessarily 90°.

Example suppose $L = 2m$, $\dot{\theta} = 3$ rad/s, $\ddot{\theta} = -6$ rad/s

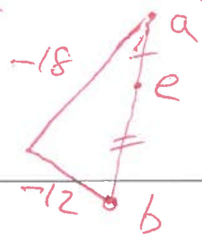


$\underline{r}_{B/A} = 2\,\underline{e}_r$ $\quad (r\,\underline{e}_r)$

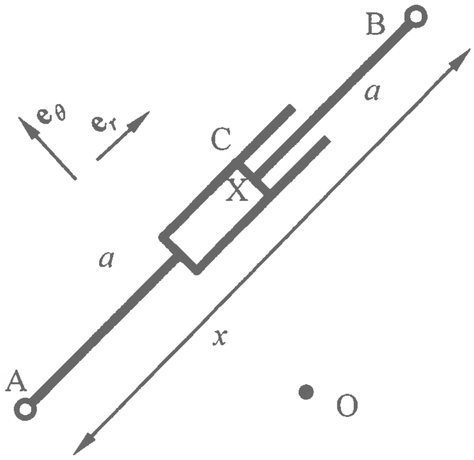
$\dot{\underline{r}}_{B/A} = (0\,\underline{e}_r + 2\times 3\,\underline{e}_\theta)$ $\quad (\dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta)$

$\ddot{\underline{r}}_{B/A} = (0 - 2\times 3^2)\,\underline{e}_r + (0 + 2\times(-6))\,\underline{e}_\theta$ $\quad (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{e}_\theta$

L is fixed is length $\therefore \dot{r}, \ddot{r} = 0$



2. a link AB with a slider (eg a piston)



Acceleration diagram for sliders — take care!

We need two points side by side, C on slider (cylinder), X on rod (piston)

$\underline{r}_{XC} = 0$ (instantaneous)

$\dot{\underline{r}}_{XC} \neq 0$ sliding velocity

$\ddot{\underline{r}}_{XC} \neq 0$

We can draw the acceleration diagram for the link AB as a slider of length x by using the vector expression for the acceleration of B using unit vectors aligned with AB:

$$\mathbf{r}_B = \mathbf{r}_A + x\mathbf{e}_r$$

which can be differentiated to get velocities (see Section 2.5), noting that x is not constant

$$\dot{\mathbf{r}}_B = \dot{\mathbf{r}}_A + \dot{x}\mathbf{e}_r + x\dot{\theta}\mathbf{e}_\theta$$

and then differentiate again to get the acceleration

$$\ddot{\mathbf{r}}_B = \ddot{\mathbf{r}}_A + \left(\ddot{x} - x\dot{\theta}^2\right)\mathbf{e}_r + \left(2\dot{x}\dot{\theta} + x\ddot{\theta}\right)\mathbf{e}_\theta$$

this is familiar

It can readily be seen that a point C is a fixed distance from A and can be placed on the acceleration diagram as for the rigid link and X (instantaneously adjacent to C) can likewise be placed relative to C. The acceleration of X relative to C is notable because the distance from X to C is zero:

$$\ddot{\mathbf{r}}_{X/C} = \ddot{x}\mathbf{e}_r + 2\dot{x}\dot{\theta}\mathbf{e}_\theta$$

Do the same for $\underline{r}_{X/C}$, noting that $\underline{r}_{XC} = 0$

then $\ddot{\underline{r}}_{X/C} = (\ddot{x} - \cancel{x\dot{\theta}^2})\,\underline{e}_r + (2\dot{x}\dot{\theta} + \cancel{x\ddot{\theta}})\,\underline{e}_\theta$

where here x is distance between C & X

Acceleration diagram:

Suppose at this instant

$AC = 2m \quad (=a)$

$CB = 2m \quad (=a)$

$x = 4m$

$\dot{x} = -1 m/s$ (contracting)

$\ddot{x} = 9 m/s^2$ (at unsteady rate)

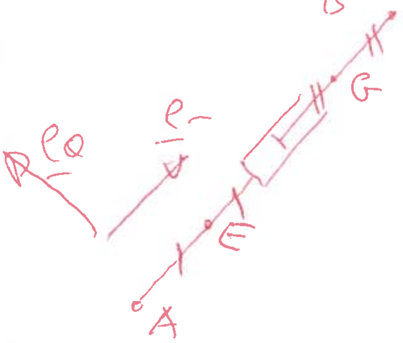
$\dot{\theta} = 3$ rad/s

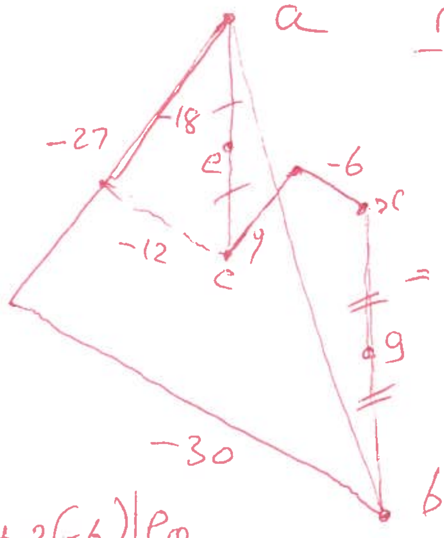
$\ddot{\theta} = -6$ rad/s

From page 23

$\ddot{r}_B = \ddot{r}_A + (9 - 4\times 3^2)\underline{e}_r + (2(-1)3 + 4(-6))\underline{e}_\theta$

$\quad = \ddot{r}_A - 27\,\underline{e}_r - 30\,\underline{e}_\theta$





$\underline{\ddot{r}}_{x/c} = \ddot{x}\,\underline{e}_r + 2\dot{x}\dot{\theta}\,\underline{e}_\theta$ (page 23)

$= 9\,\underline{e}_r - 6\,\underline{e}_\theta$

$\underline{\ddot{r}}_{c/A} =$ "$(\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{e}_\theta$"

$= (0 - 2\, 3^2)\underline{e}_r + (0 + 2(-6))\underline{e}_\theta$

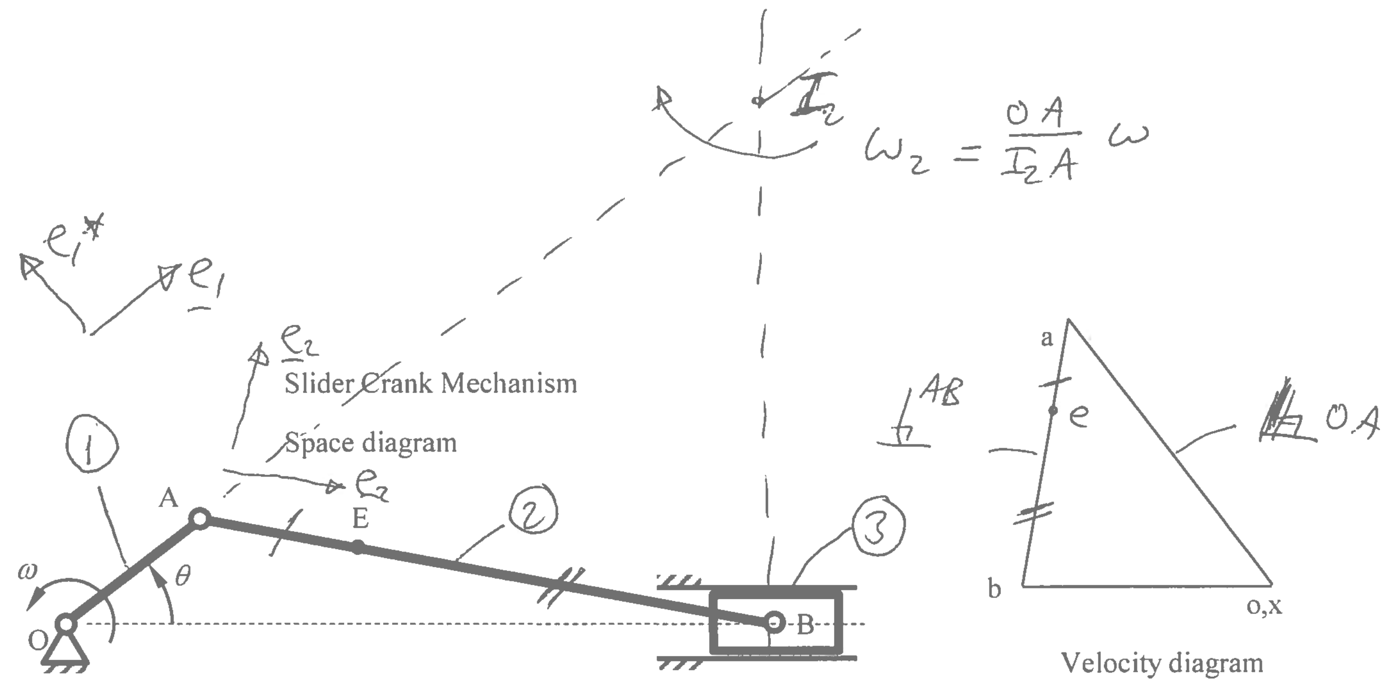
$= -18\,\underline{e}_r - 12\,\underline{e}_\theta$

acceleration diagram

end L5 / L23

Example

Sketch an acceleration diagram for the slider crank mechanism for the case where input angular velocity ω is constant. Note that velocities and angular velocities for the mechanism have already been obtained in Section 3.1 above. Use the Acceleration Image to locate the acceleration of point E on your diagram



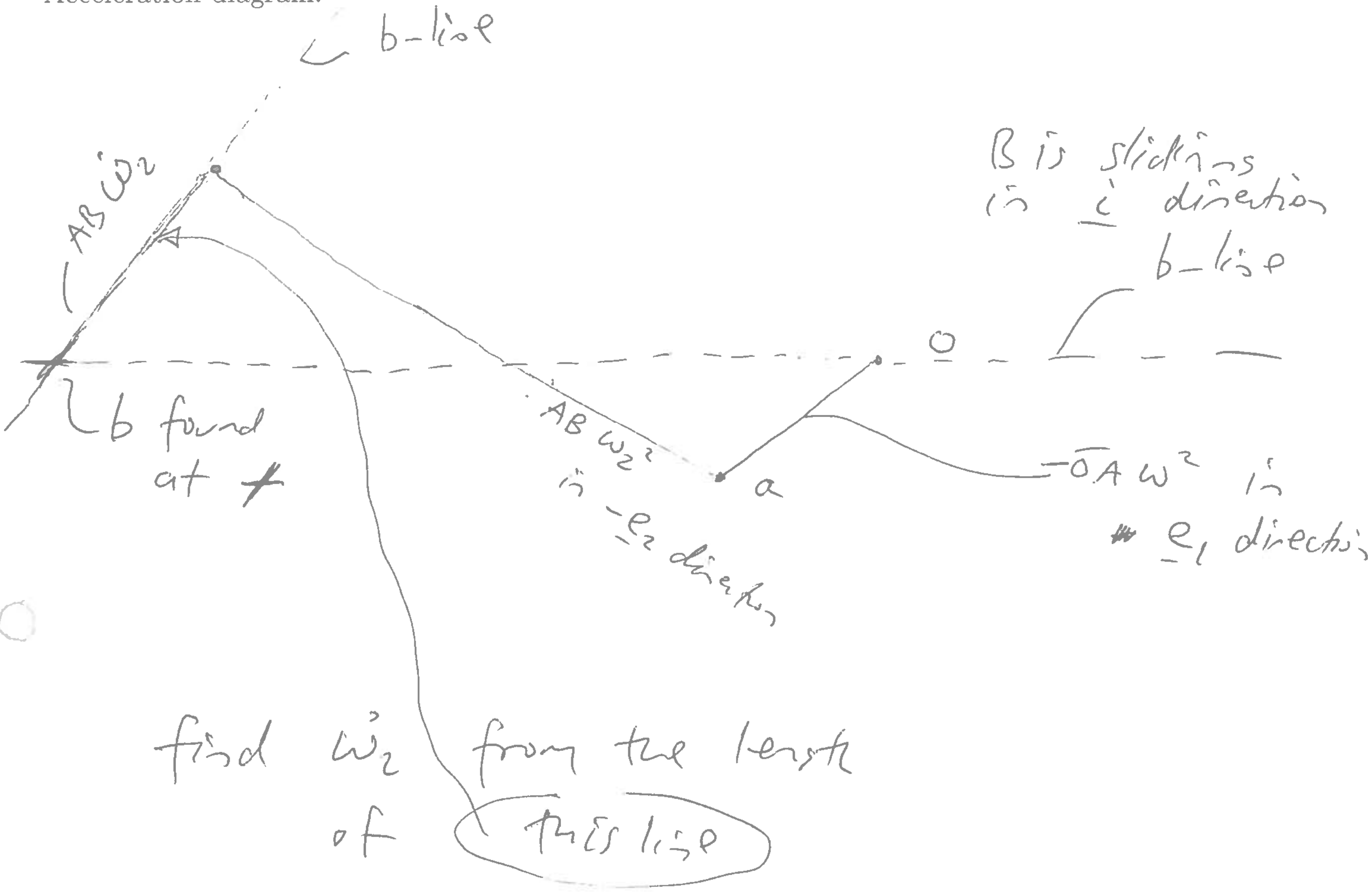
Velocity diagram

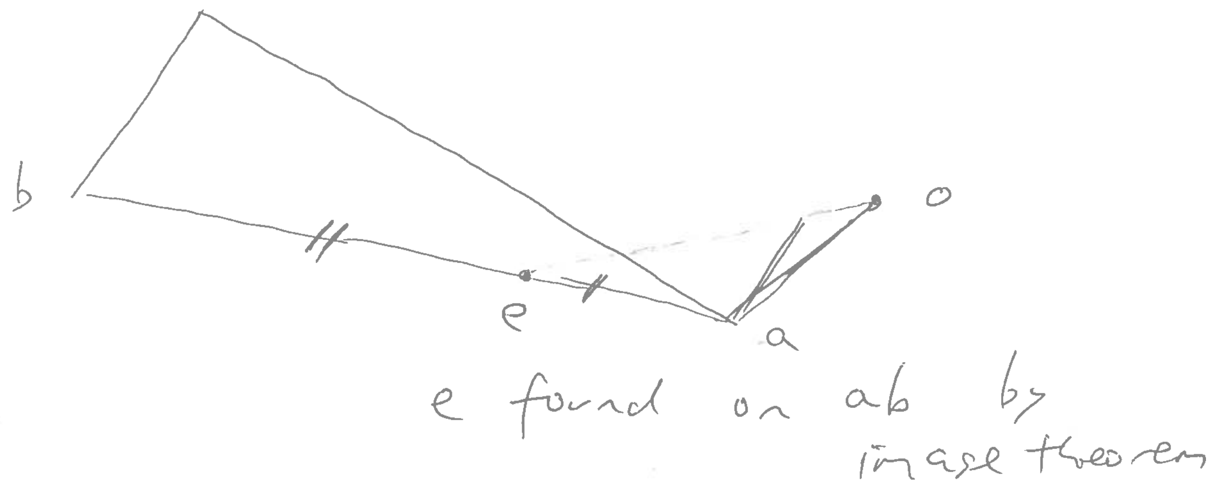
tabulate :

member	$\underset{\sim}{e}$ $\quad e^*$	$\underset{\sim}{e}$: $\ddot{r} - r\dot{\theta}^2$	e^*: $2\dot{r}\dot{\theta} + r\ddot{\theta}$	notes
OA ①	e_1, e_1^*, ω	$0 - \overline{OA}\,\omega^2$ $= -\overline{OA}\,\omega^2$	$0 + \cancel{\overline{OA}}\,0$ $= 0$	$r = \overline{OA}$ $\dot{r} = \ddot{r} = 0$ rigid link $\dot{\theta} = \omega$ $\quad \ddot{\theta} = 0$
AB ②	e_2^*, e_2, ω_2	$0 - \overline{AB}\,\omega_2^2$ $= -\overline{AB}\,\omega_2^2$	$0 + \overline{AB}\,\ddot{\theta}$ $\overline{AB}\,\dot{\omega}_2$	$r = \overline{AB}$ $\dot{r} = \ddot{r} = 0$ rigid link $\dot{\theta}$, $\ddot{\theta}$?

$\hookrightarrow \dot{\theta} = \omega_2 = \frac{OA}{I_2A}\omega$

Acceleration diagram:

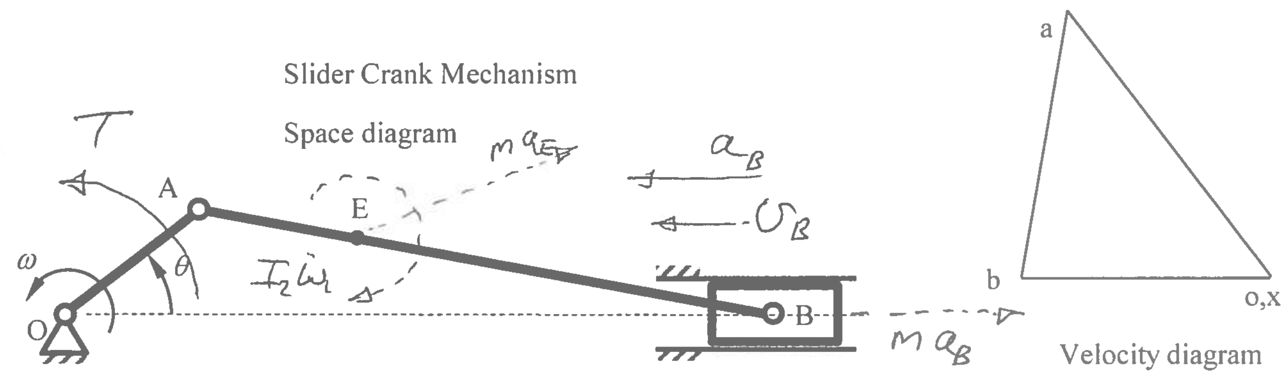




3.5 Inertia Forces in Mechanisms

As a mechanism moves the various links and sliders are accelerating. We have a means now of calculating the accelerations of all components. The d'Alembert Inertia forces and couples can be applied as if they were external forces and couples (see Section 2.3) acting at the centres of mass of the links and sliders. The method of Virtual Power (Section 3.3) can be used to determine, for example, the driving torque needed to resist the inertia forces of the accelerating mechanism.

Example
For the slider crank mechanism with constant input angular velocity ω , find the torque T necessary at O to drive the mechanism considering only the inertia force of the slider at B which has mass m. All other links are considered massless and there is no friction anywhere.



get acceleration a_B from acceleration diagram page 26 $\overleftarrow{a_B}$

get velocity v_B from instant centres or velocity diagram $\overleftarrow{v_B}$

Power $T\omega + (-m a_B) v_B$

$$\therefore \boxed{T = \frac{m a_B v_B}{\omega}}$$

Suppose AB is <u>not</u> massless and CofM is at E and I_G is moment of inertia

Note $\dot{\omega}_2$ is ↻ from accel diag

Power: extra term for AB

$$(m \underset{\sim}{a_E}) \cdot \underset{\sim}{v_E} + (I_G \dot{\omega}_2)(-\omega_2)$$

You can now do questions 7-10 on Examples Paper 1

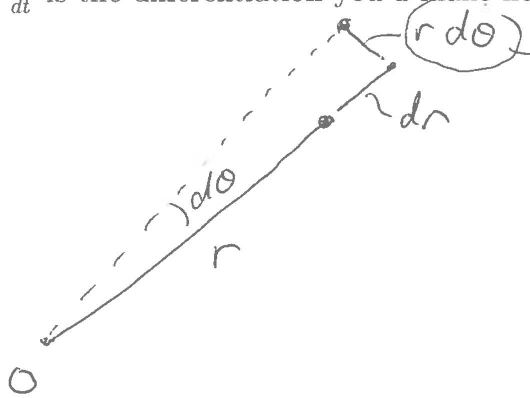
4 Planar vectorial dynamics - inertia forces in mechanisms

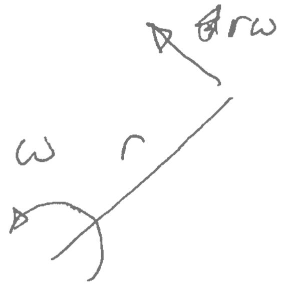
4.1 Differentiation of vectors - summary of important results

The general result for differentiation of a vector $\mathbf{r}$ which is rotating at an angular velocity $\boldsymbol{\omega}$ is

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}$$

where $\frac{d\mathbf{r}}{dt}$ is the differentiation you'd make not take into account the rotation due to $\boldsymbol{\omega}$. This works in 3D.





4.2 Differentiation of vectors - special case for planar motion

The position of B relative to A is

$$\mathbf{r} = r\mathbf{e}$$

where $\mathbf{e}$ is a unit vector along AB. If the angular velocity of AB is $\dot{\theta}\mathbf{k}$ then the velocity of B relative to A is

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e} + r\dot{\theta}\mathbf{e}^*$$

using $\underline{e}$, $\underline{e}^*$ notation

and $\mathbf{e}^* = \mathbf{k} \times \mathbf{e}$ is a unit vector perpendicular to $\mathbf{e}$. The acceleration of B relative to A is

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\mathbf{e}^*$$

just like $\underline{e}_r$ & $\underline{e}_\theta$ but use new * notation

Example

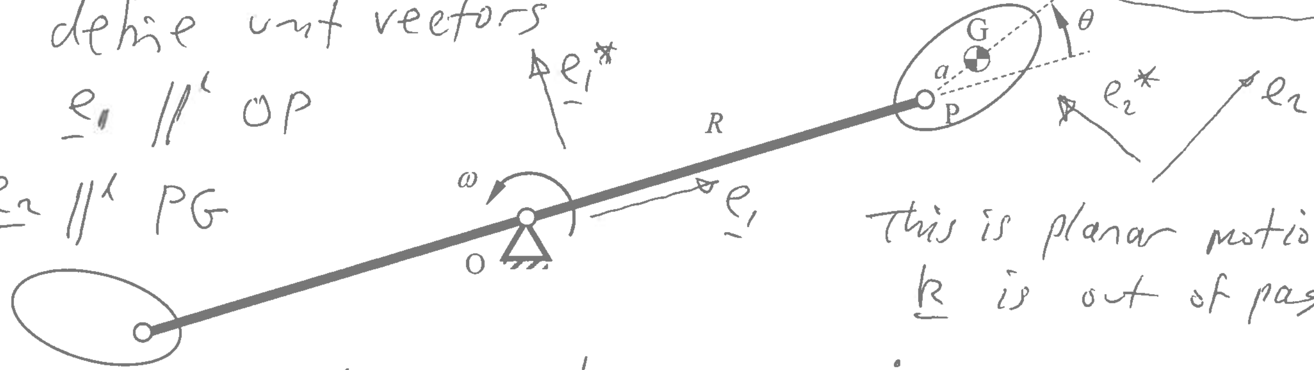
The "Booster" ride, found in most fairgrounds, comprises a long beam rotating in a vertical plane with a pair of pods located at each end. Thrillseekers sit in the pods which ~~move in an apparently-random way~~. Suppose the beam OP is of length R, rotating about O at a steady rate ω. The pod, freely pivoted at P, has its centre of mass at G with PG=a. The mass of the pod is m and the moment of inertia of the pod about G is $I_G = mk^2$ where k is the radius of gyration. Find an equation of motion for the pod in terms of the angle θ shown.

Note θ measured relative to OP

define unit vectors

$\underline{e}_1 \parallel^l$ OP

$\underline{e}_2 \parallel^l$ PG



This is planar motion

$\underline{k}$ is out of page

$$\underset{\sim}{\omega}_1 = \omega \underline{k} \quad , \quad \dot{\omega}_1 = 0$$

$$\underline{\omega}_2 = (\omega + \dot{\theta})\underline{k} \quad , \quad \dot{\omega}_2 = \ddot{\theta}\,\underline{k}$$

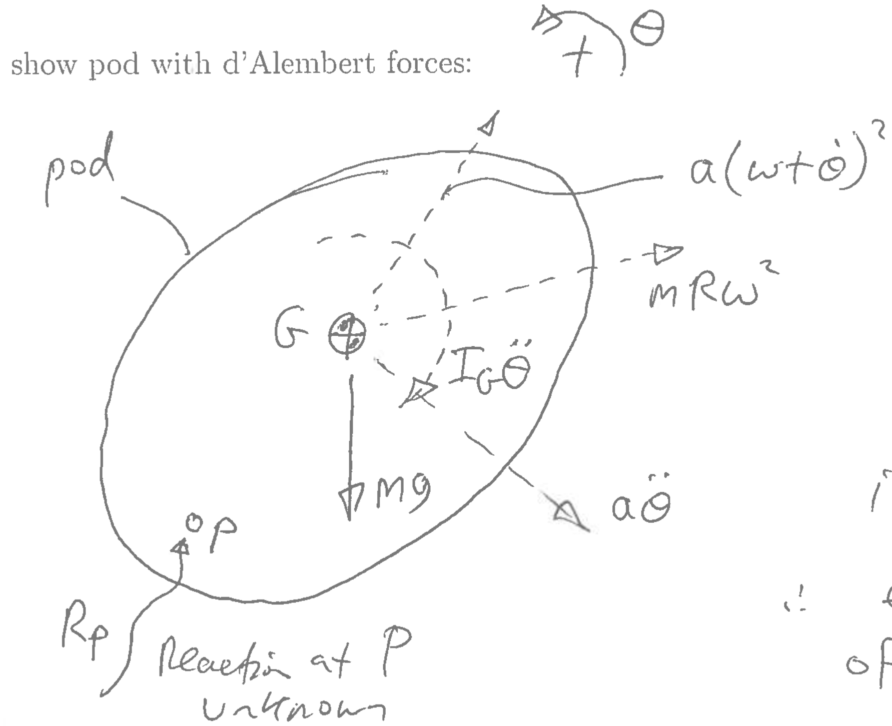
we need acceleration of G, start from $\underline{r}_G$

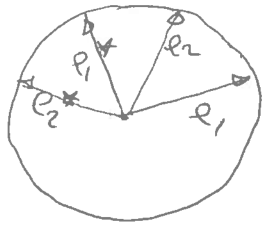
$$\underline{r}_G = R\underline{e}_1 + a\underline{e}_2 \quad , \quad \dot{\underline{r}}_G = R\omega \underline{e}_1^* + a(\omega + \dot{\theta})\underline{e}_2^*$$

$$\ddot{\underline{r}}_G = -R\omega^2 \underline{e}_1 + a\ddot{\theta}\,\underline{e}_2^* - a(\omega + \dot{\theta})^2 \underline{e}_2$$

d'Alembert at G

show pod with d'Alembert forces:





Now take moments about P just as in Part IA Structures

∴ equation in terms of $\ddot{\theta}$, $\dot{\theta}$, θ which you can solve

EP2 Q1 looks at solving this for small θ

4.3 Special case - "release from rest"

The acceleration of B relative to A is

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\mathbf{e}^*$$

, $\dot{r}=0$, $\ddot{r}=0$ for rigid link

but if AB is a rigid link instantaneously at rest then $\dot{r} = 0$, $\ddot{r} = 0$ and $\dot{\theta} = 0$ so that

release from rest ∴ $\dot{\theta}=0$

$$\ddot{\mathbf{r}} = r\ddot{\theta}\mathbf{e}^*$$

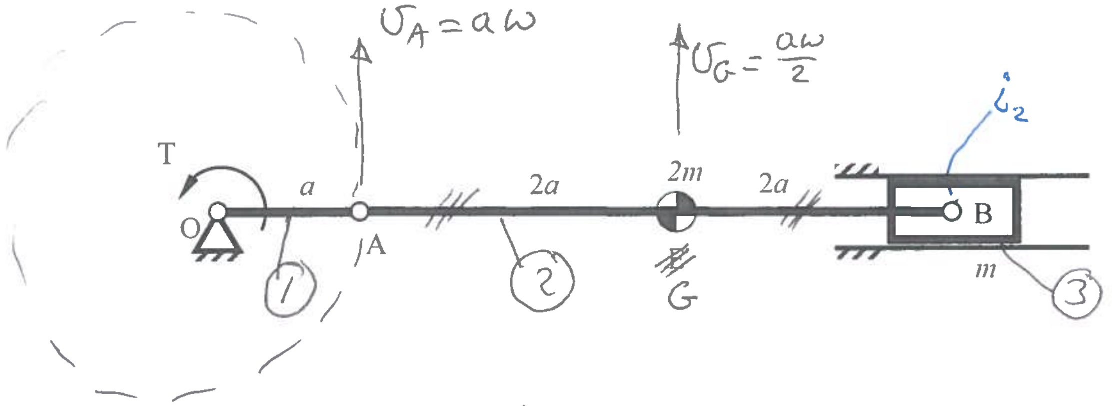
This dramatically simplifies the construction of an acceleration diagram.

useful in EP2 Q2
Q4

Example

A car engine, modelled as a Slider-Crank mechanism, has one piston at top-dead-centre (TDC) and the engine is being started from at rest. The crank OA is length a and is considered light. The uniform Connecting Rod AB is length $4a$ and mass $2m$ and the piston at B has mass m. A torque T is applied at O. What is the initial acceleration of B and what are the angular accelerations of OA and OB?

we'll use virtual power, keep ω in the picture even though we're starting from rest



$I_G = \frac{1}{12}2m(4a)^2$

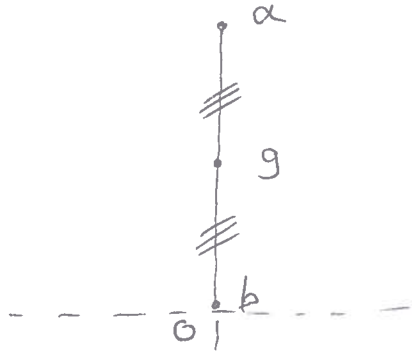
$= \frac{8}{3}ma^2$

$\ddot{r}_A = a\dot{\omega}\uparrow$, $\ddot{r}_B$ must be purely horizontal

and $\ddot{\underline{r}}_B = \ddot{\underline{r}}_A + \ddot{r}_{B/A}$

$\ddot{\underline{r}}_{B/A} = \overline{AB}\,\dot{\omega}_2 \downarrow$

$= 4a\dot{\omega}_2$



acceleration diagram

d'Alembert:

$\frac{8}{3}ma^2\frac{\dot{\omega}}{4}$

$= \frac{2}{3}ma^2\dot{\omega}$

$2m\left(2a\frac{\dot{\omega}}{4}\right) = ma\dot{\omega}$

Virtual power

$T\omega = ma\dot{\omega}\left(\frac{a\omega}{2}\right)$

$+ \frac{2}{3}ma^2\dot{\omega}\left(\frac{\omega}{4}\right)$

Note: inertia forces are in opposite direction to velocities – take care with signs

You can now do questions 1 and 2 on Examples Paper 2

end L6, Lent 23

5 Bending moments and shear forces in mechanisms

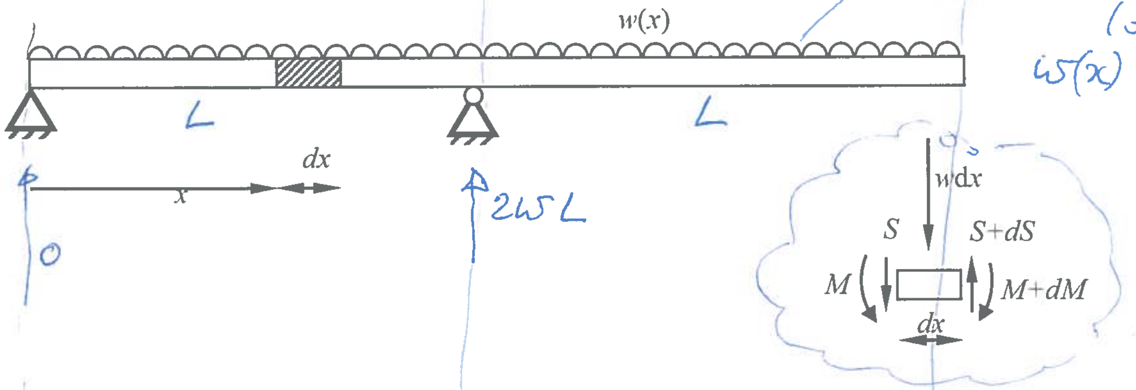
One of the most important applications of d'Alembert forces is for the calculation of bending moments and shear forces in the components of a mechanism in motion.

5.1 Revision - Bending Moments and Shear Forces

Consider a beam subject to a distributed load $w(x)$

"UDL" Uniformly distributed load

$w(x) = w$
$= \text{constant}$ (say)



Equilibrium of the element dx gives

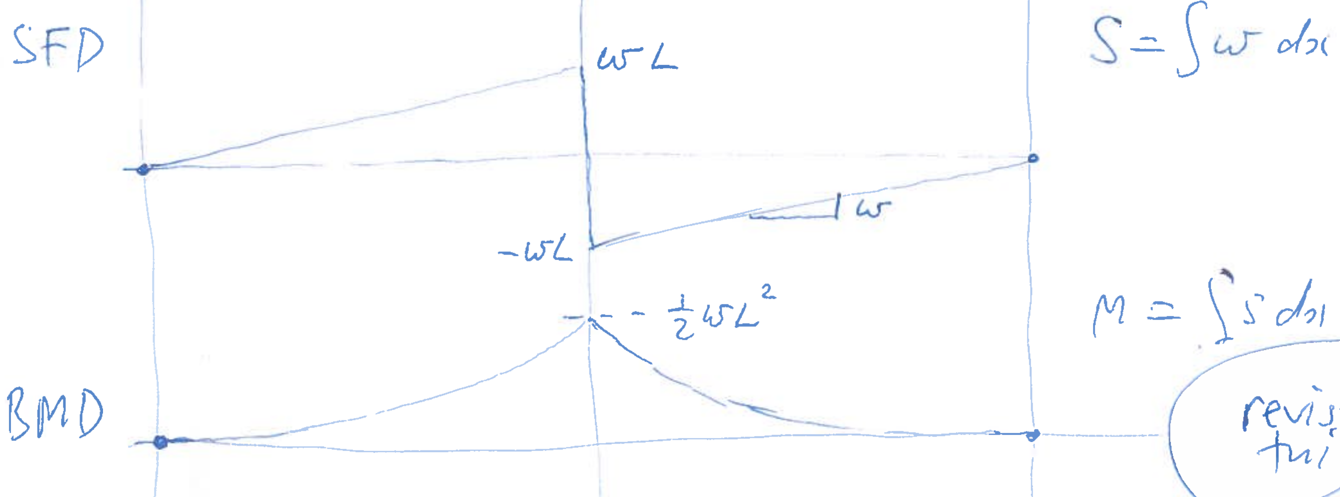
$$w = \frac{dS}{dx}$$

$$S = \frac{dM}{dx}$$

Note that the maximum bending moment occurs where $S = 0$.

Example
Sketch Shear Force and Bending Moment diagrams for the case above with $w(x) = w$ (constant) and where the distance between the supports is L and the beam is of length $2L$.

$S = \int w \, dx$

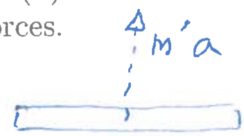


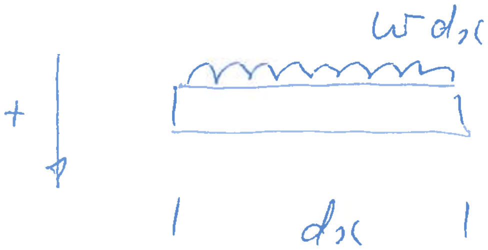
$M = \int S \, dx$

revise this

5.2 Inertia forces in beams

An element of beam dx with mass-per-unit-length m' is accelerating with acceleration a then the d'Alembert inertia force is $-m'adx$. This can be considered as a distributed load $w(x)$ on a beam and the usual methods of statics can be used to find bending moments and shear forces.





d'Alembert

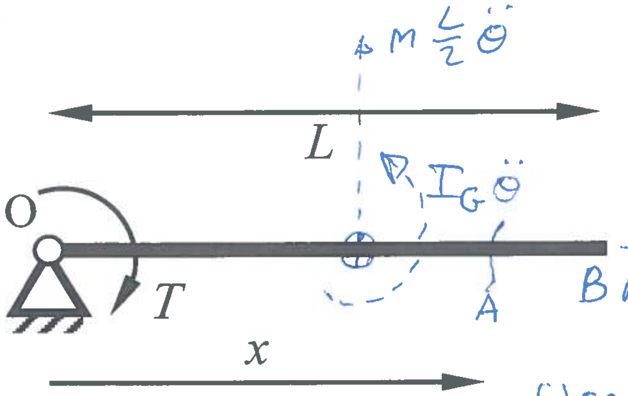
put $w = -m'a\,dx$

kg/m m/s² m $\equiv$ N/m for w

Example

A beam of length L and mass m is pivoted at its left end. It is accelerated by a torque T at the pivot. Find the shear force and bending moment at a distance x from the pivot. Where is the maximum bending moment? ** NB $S = \frac{dM}{dx}$ $\therefore$ Max BM is where $S = 0$

$I_G = \frac{1}{12} mL^2$



$m\frac{L}{2}\dot{\theta}^2$ (not needed for BM & SF calculations)

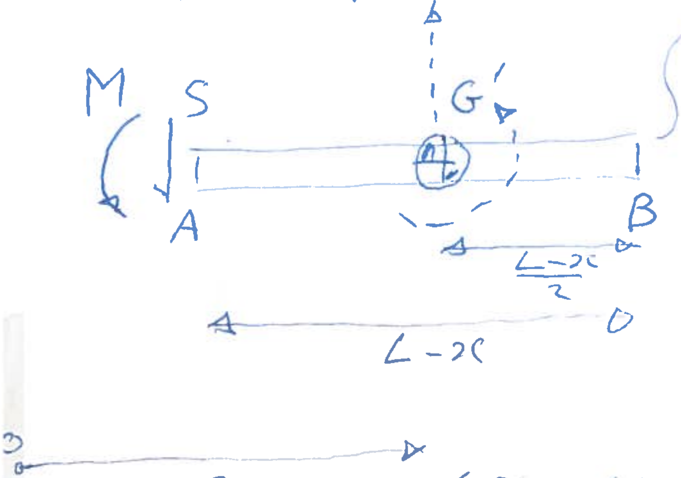
Use d'Alembert & equilibrium

$T = m\frac{L}{2}\ddot{\theta}\frac{L}{2} + \frac{1}{12}mL^2\ddot{\theta}$

$\therefore \boxed{\ddot{\theta} = \frac{3T}{mL^2}}$

example continued ...

Cut the beam at A (free end)



G' is at C of M of the short bit AB

M_{AB} = Mass of AB $= \frac{m}{L}(L-x)$

$I_{G'}$ of AB $= \frac{1}{12} M_{AB} (L-x)^2$

$r_{G'} = x + \frac{L-x}{2} = \frac{L+x}{2}$

Equilibrium of AB

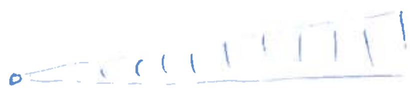
$$S = \underbrace{\frac{m}{L}(L-x)}_{M_{AB}} \underbrace{\frac{L+x}{2}}_{r_{G'}} \underbrace{\frac{3T}{mL^2}}_{\ddot{\theta}}$$

d'Alembert force

$$= \frac{3T}{2L^3}\left(L^2 - x^2\right)$$

$S = 0$ gives location of Mass M $\therefore x = L$

<u>sketch BMD</u> , $w(x)$



$S = \int w(x)\,dx$

$M = \int S\,dx$

Example

A chimney of length $2a$ is being demolished. After an explosive charge goes off at its base the chimney begins to rotate under the action of gravity. The chimney is observed to break at a distance about one-third the way up. Show that this is consistent with the location of the maximum bending moment.

$$KE + PE = \text{const}$$

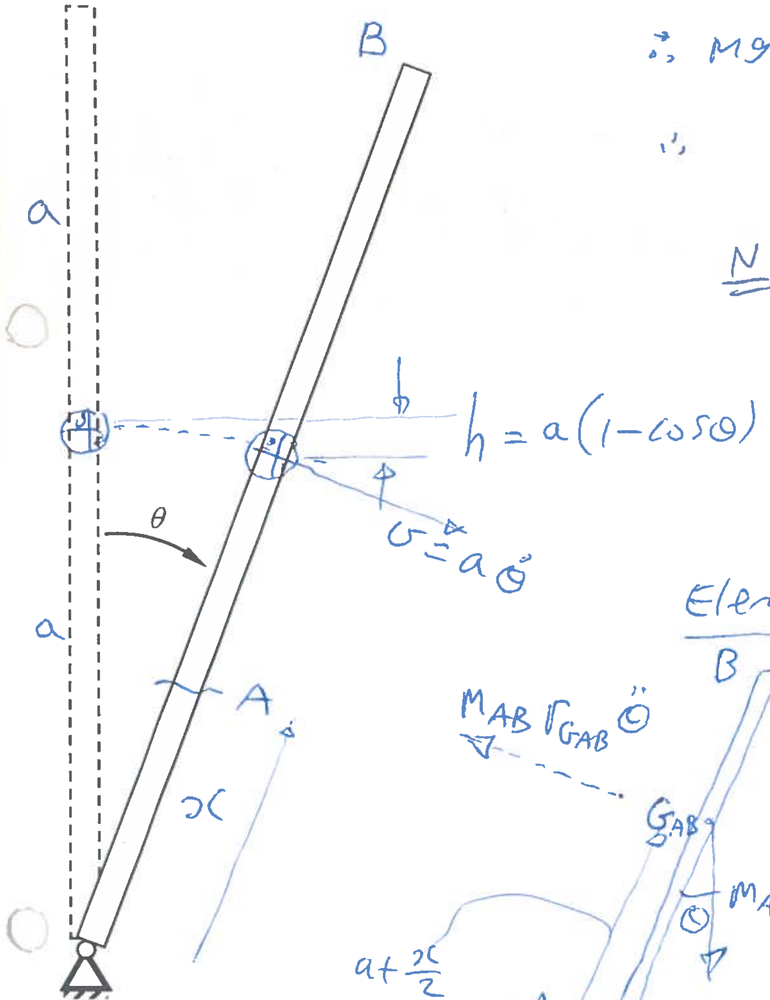
$$\therefore \; mgh = \tfrac{1}{2} m v^2 + \tfrac{1}{2} I_G \dot{\theta}^2$$

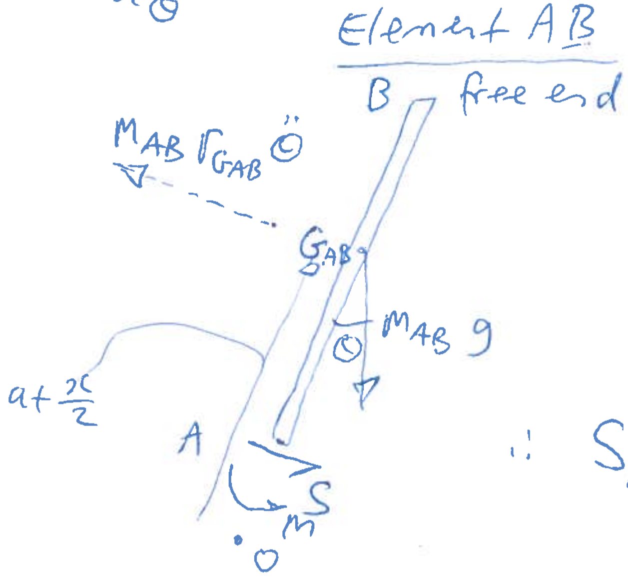
$$\therefore \; mga(1-\cos\theta) = \tfrac{1}{2} m (a\dot{\theta})^2 + \tfrac{1}{2}\tfrac{1}{3} m a^2 \dot{\theta}^2$$

$$\therefore \; \tfrac{1}{2}\dot{\theta}^2 = \frac{3g}{4a}(1-\cos\theta)$$

$$\underline{\underline{NB}} \quad \frac{d}{d\theta}\left(\tfrac{1}{2}\dot{\theta}^2\right) = \ddot{\theta}$$

$$\therefore \; \boxed{\ddot{\theta} = \frac{3g}{4a}\sin\theta}$$





$$L_{AB} = 2a - x$$

$$m_{AB} = \frac{M}{2a}(2a - x)$$

$$r_{G_{AB}} = a + \frac{x}{2}$$

$$\therefore \; S_A = m_{AB} r_{G_{AB}} \ddot{\theta} - m_{AB} g \sin\theta$$

$$= 0 \quad \text{for max BM}$$

$$\therefore \; \left(a + \frac{x}{2}\right) \underbrace{\frac{3}{4}\frac{g}{a}\sin\theta}_{\ddot{\theta}} = g\sin\theta$$

$$\therefore \; a + \frac{x}{2} = \frac{4}{3}a \qquad \therefore \; x = \frac{2}{3}a \left(= \frac{1}{3}L\right)$$

You can now do questions 3-6 on Examples Paper 2

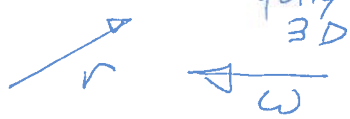
6 3D kinematics and gyroscopic mechanics

This part of the course is a brief snapshot of where Mechanics goes in three dimensions. The Part IIA course 3C5 Dynamics takes this all a lot further. It's partner in Part IIA is 3C6 Vibration and typically students take both of these papers.

6.1 Differentiation of rotating vectors in 3D

It has been convenient to use the notation $\mathbf{e}$ and $\mathbf{e}^*$ for the case of planar motion, but in general, when $\boldsymbol{\omega}$ is not perpendicular to $\mathbf{r}$ the cross products for velocity and acceleration need to be handled in full.

fully 3D



The velocity is obtained by differentiation:

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}$$

The acceleration is obtained by differentiation, taking care to include all terms:

$$\ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} + \boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

and collecting terms

$$\ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

This formula is in the Mechanics Data Book.

centripetal acceleration

radial acceleration

coriolis acceleration

circumferential acceleration

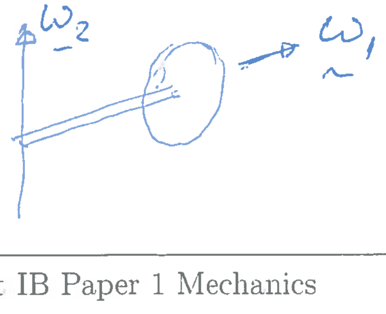
It is often simpler to consider to find accelerations from first principles.

page 2 Mechanics Data Book

You can use the formula there

or derive from first principles

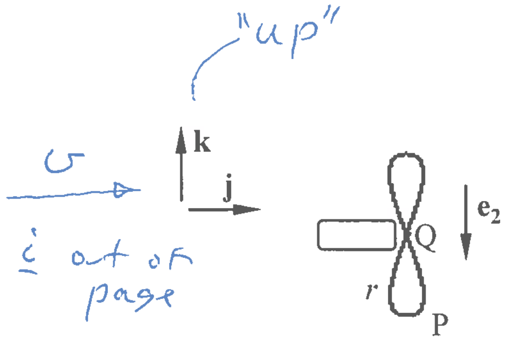
KEY is to keep track of angular velocities



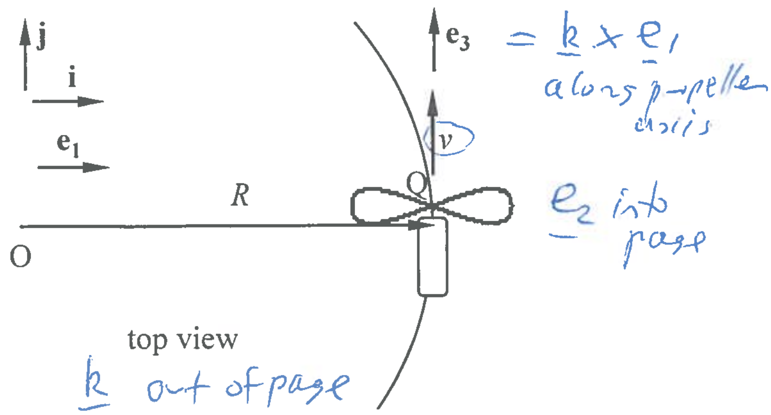
Know which unit vectors have which angular velocities

Example

A single-engined propellor aircraft is in steady level flight at speed v on a circular path of radius R. It has a propellor of radius r that is spinning at angular velocity ω clockwise as viewed by the pilot. At an instant the tip P of a propellor blade is at its lowest point. Find the instaneous acceleration of P.



side view of plane and propellor



top view

Note that vector $\mathbf{e}_1$ rotates with the plane and vector $\mathbf{e}_2$ rotates with the propellor. The vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ shown are fixed, instantaneously aligned with $\mathbf{e}_1$ and $\mathbf{e}_2$. Note that a vector aligned with the axis of the propellor is $\mathbf{e}_3 = \mathbf{k} \times \mathbf{e}_1$.

Carefully write down the angular velocity vectors for e_1, e_2 & e_3

$$\underbrace{\omega_1 = \frac{v}{R}\mathbf{k}}_{\text{out of plane due to turning radius } R}, \quad \underbrace{\omega_2 = \frac{v}{R}\mathbf{k} + \omega \mathbf{e}_3}_{\text{angular velocity of propeller}}$$

spin

$$\dot{\omega}_1 = 0$$

$$\dot{\omega}_2 = \omega \mathring{e}_3 = \omega\left(\frac{v}{R}k \times e_3\right) = -\frac{\omega v}{R} e_1$$

Position, velocity and acceleration of P:

$$\mathbf{r} = R\mathbf{e}_1 + r\mathbf{e}_2$$

$$\mathring{r} = R(\omega_1 \times \mathbf{e}_1) + r(\omega_2 \times \mathbf{e}_2)$$

$$\ddot{r} = R\left[(\dot{\omega}_1 \times \mathbf{e}_1) + \omega_1 \times (\omega_1 \times \mathbf{e}_1)\right] + r\left[(\dot{\omega}_2 \times \mathbf{e}_2) + \omega_2 \times (\omega_2 \times \mathbf{e}_2)\right]$$

Up to now assume everything is non-zero

Now evaluate these at the instant shown noting that $\dot{\omega}_1 = 0$, $\dot{\omega}_2 = -\frac{\omega v}{R}\mathbf{e}_1$, and that instantaneously $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = -\mathbf{k}$ and $\mathbf{e}_3 = \mathbf{j}$

$$\ddot{\underline{r}} = R\left[0 + \left(\frac{v}{R}\right)^2 \underline{k} \times (\underline{k} \times \underline{e}_1)\right] + r\left[-\frac{\omega v}{R}\underline{e}_1 \times \underline{e}_2 + \left(\frac{v}{R}\underline{k} + \omega \underline{e}_3\right) \times \left(\frac{v}{R}\underline{k} + \omega \underline{e}_3\right) \times \underline{e}_2\right]$$

$$= -\frac{v^2}{R}\underline{e}_1 + r\left[-\frac{\omega v}{R}\underline{e}_3 + \left[\frac{v}{R}\underline{k} + \omega \underline{e}_3\right]\left[0 + \omega(-\underline{e}_1)\right]\right]$$

$$= -\frac{v^2}{R}\underline{e}_1 \cancel{+ r[} - \frac{\omega v r}{R}\underline{e}_3 - \frac{\omega v r}{R}\underline{e}_3 - r\omega^2 \underline{e}_2$$

$$= -\frac{v^2}{R}\underline{e}_1 - 2\frac{\omega v r}{R}\underline{e}_3 - r\omega^2 \underline{e}_2$$

$$= -\frac{v^2}{R}\underline{i} - 2\frac{\omega v r}{R}\underline{j} + r\omega^2 \underline{k}$$

at this instant

centripetal due to cornering

coriolis acceleration

centripetal due to spin

6.2 Gyroscopic Mechanics

For motion of a particle we know that $\mathbf{F} = m\mathbf{a}$ and if we define the linear momentum as $\mathbf{p} = m\mathbf{v}$ then

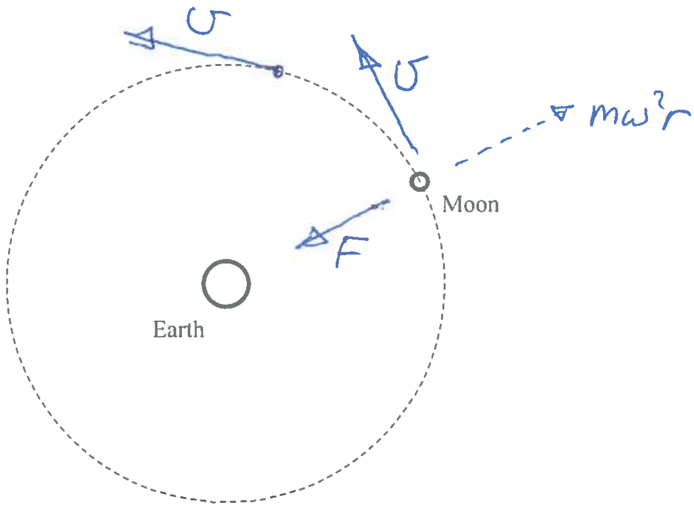
$$\mathbf{F} = \dot{\mathbf{p}}$$

If the speed of the particle is constant but it is moving on a circular path then $\mathbf{p}$ is rotating at an angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$ for motion in a plane. This means that

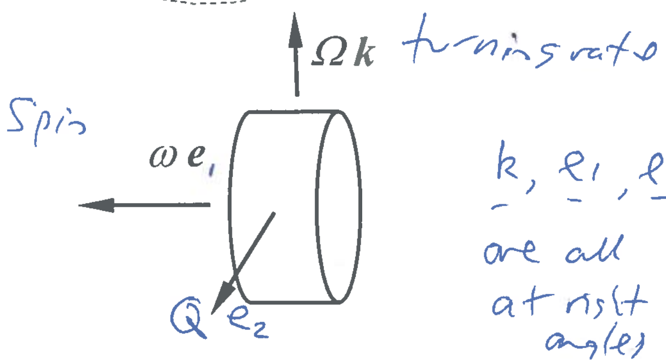
$$\dot{\mathbf{p}} = \boldsymbol{\omega} \times \mathbf{p}$$

With $v = r\omega$ this leads to the familiar formula for circular motion

$$F = m\omega^2 r$$



Linear Momentum changes in direction of applied force



$\underline{k}, \underline{e_1}, \underline{e_2}$ are all at right angles

Angular momentum $J\omega\, \underline{e}_1$ changes in direction of applied couple

Now consider the analogous case for angular momentum. Just as $\mathbf{F} = \dot{\mathbf{p}}$ then $\mathbf{Q} = \dot{\mathbf{h}}$ where $\mathbf{Q}$ is the applied couple (or "torque") and $\mathbf{h}$ is the angular momentum.
Consider a rotor whose angular momentum is $\mathbf{h}$. If the rotor is spinning with constant angular velocity $\omega\mathbf{e}$ where $\mathbf{e}$ is aligned with the axis of spin then $\mathbf{h} = J\omega\mathbf{e}$ where J is the polar moment of inertia of the rotor. If the axis of the rotor is turning at a rate $\Omega\mathbf{k}$ about an axis at right angles to $\mathbf{e}$ then

$$\dot{\mathbf{h}} = \Omega\mathbf{k} \times \mathbf{h} \quad = \quad J\omega\Omega\mathbf{k} \times \mathbf{e}_1$$

$\underline{k} \times \underline{e}_1 = \underline{e}_2$

$\underline{h} = J\omega\, \underline{e}_1$

$= J\omega\Omega\, \underline{e}_2$

$Q = \frac{dh}{dt}$

$h = J\omega$

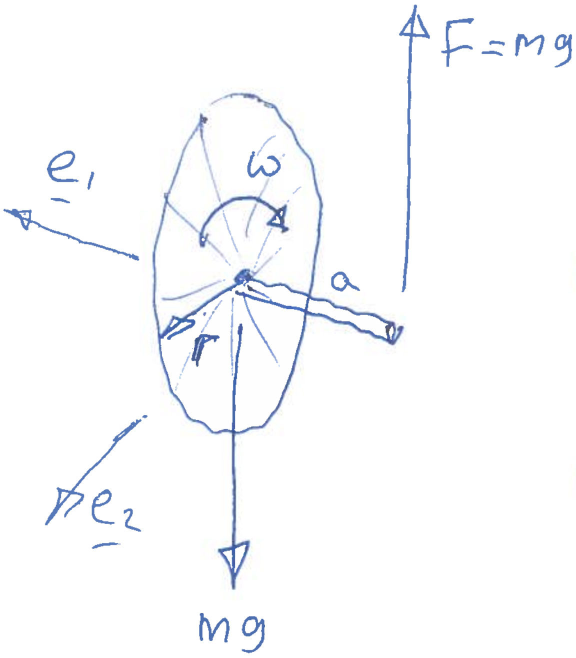
dh $d\theta$

which gives the formula for the gyroscopic couple

$$\boxed{Q = J\omega\Omega}$$

The rate of turning Ω is called *gyroscopic precession*

(not to be confused with nutation see 3C5 Dynamics)

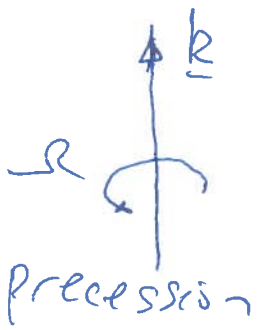


$$\underline{Q} = mga\,\underline{e}_2$$

$$Q = J\omega\Omega$$

$$\therefore \quad \Omega = \frac{Q}{J\omega}$$

wheel, thin rim, $J = mr^2$



$$\Omega = \frac{\cancel{m}ga}{\cancel{m}r^2\omega} = \frac{ga}{r^2\omega}$$

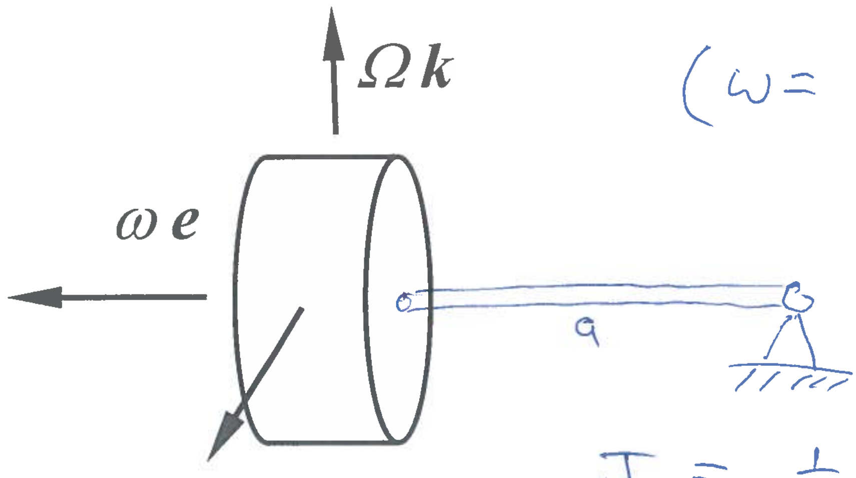
$$(\text{rad/s}) \equiv \left(\frac{m/s^2}{m \quad 1/s}\right) \checkmark$$

Example

A rotor of mass 2kg and radius 0.1m is supported at one end on a ball joint. The distance from the centre of mass to the support is 0.15m. The rotor is spinning at 5000rpm. What is the rate of precession of the rotor?

$$\omega = \frac{rpm}{10} = 500 \text{ rad/s}$$

$$\left(\omega = \frac{rpm}{60} \times 2\pi\right)$$

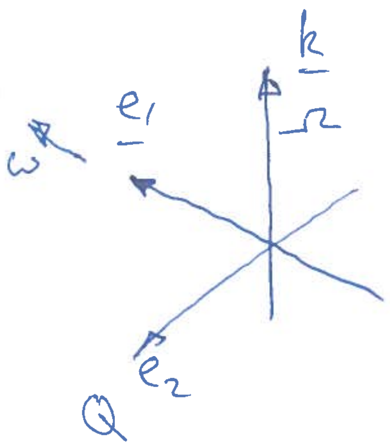


$$J = \frac{1}{2} m r^2 = \frac{1}{2} \times 2 \times (0.1)^2 = 0.01 \text{ kgm}^2$$

$$Q = mga = 2 \times 10 \times 0.15 = 3 \text{ Nm}$$

$$Q = J\omega\Omega \quad \therefore \quad \Omega = \frac{3}{0.01 \times 500} = 0.6 \text{ rad/s}$$

(0.573 exact)



You can now do questions 7 and 8 on Examples Paper 2

7 Experiment D1 - Double Pendulum

PART IB EXPERIMENTAL ENGINEERING

SUBJECT:	MECHANICS	EXPT D1
LOCATION:	SOUTH WING MECHANICS LAB	(SHORT)

DOUBLE PENDULUM

This lab is designed to be completed in 50 minutes. Please arrive at your designated time
D1E = "Early", starts at 9am or 11am D1L = "Late", starts at 10am or 12pm

OBJECTIVE

To observe and understand the motion of a double pendulum which is a non-linear system exhibiting chaos for large amplitudes and predictable periodic behaviour for small amplitudes. This experiment forms part of Examples Papers 3 and 4 in Part IB Mechanics.

INTRODUCTION

This is a short lab, designed to be completed in 50 minutes or less. Please don't waste time getting started.

EQUIPMENT

A double pendulum is supported on frictionless bearings. Its motion is monitored by a Raspberry Pi (RPi) fitted with an infrared camera. You should find the RPi powered up when you arrive. If not then lift up the brass knob and switch power on at the mains.

On powerup, the RPi will automatically begin the double pendulum lab software. If this does not happen, click on the icon on the desktop. You will be greeted by the intro screen showing a live stream from the camera pointed at the pendulum.

RUNNING THE SOFTWARE

Enter your crsids into the box one at a time and click "Enter". Check they are correct, as the program will email you the results at the end of the experiment. Once you have done this, click "continue" to proceed.

Figure 2: Intro screen

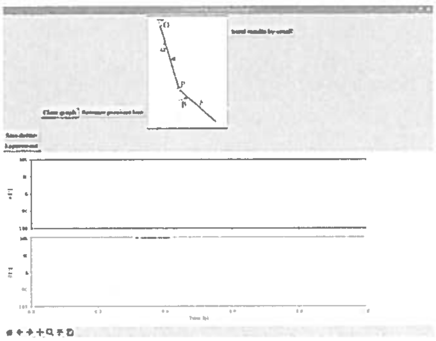


Figure 2: Main screen

You will be presented with the main screen. From here, you can either do an 'experiment' or a 'simulation'. To begin taking data from the pendulum, click "Experiment" in the main window. This will open the EXPERIMENT WINDOW. The procedure to take data is as follows:

1. Hold the pendulum in its initial position (don't obstruct the camera's view of the white reflective dots!).
2. Click "Start Countdown".

3. The program will countdown from 5. Release the pendulum from rest when it shows "GO!".
4. The program will automatically record 6 seconds of motion, analyse it, and then plot the results.

The program can also run a simulation of the experiment. To run a simulation, click "simulation" in the main window. This opens the SIMULATION WINDOW. Input the starting angles and length of simulation and click "Simulate". The program will show the simulation and plot the results.

If the angles remain small, then the system can be approximated as a linear 2 d.o.f with two modes of vibration. If you run either an experiment or a simulation where the angles are small throughout, then the program will do extra analysis on the results. It will add a row to the small angles table showing the frequency that is most prominent in the motion and the modeshape.

PROCEDURE

To start with, run an experiment three times with starting angles of is $\alpha = 90°$ and $\beta = 0°$ (see the diagram on the main screen for definitions of α and β). Compare the plots. Do they always diverge after the same amount of time?

Next, run a simulation with the same starting conditions and compare it to the previous plots. How does it compare? If you run more simulations with very slightly different starting conditions, how does it affect the graph?

Next, try and experimentally find the two modes of vibration of the pendulum with small angles. Run experiments for each mode, and the program will estimate the modeshape and natural frequency (if you wish for the experiment to run for a longer or shorter period, then click "More Options").

Click "calculate mode shapes" on the main screen. This will take you to a Jupyter notebook which will guide you through the calculations required to model the pendulum's motion for small angles.

When you have suitable starting conditions to isolate the modes, run simulations for each of them. Compare the natural frequencies and mode shapes with those collected experimentally. If they differ drastically, try the experimental readings again.

At the end of the experiment you should exit the programme and all your data will be automatically emailed to you. Keep it in a safe place – you'll need it in Examples Papers.

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