1B Paper 6: Communications
Handout 2: Analogue Modulation

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Modulation

Modulation is the process by which some characteristic of a carrier wave is varied in accordance with an information bearing signal

A commonly used carrier is a sinusoidal wave, e.g., $\cos(2\pi f_c t)$.
f_c is called the *carrier frequency*.

- We are allotted a certain bandwidth centred around f_c for our information signal
- E.g. BBC Cambridgeshire: $f_c = 96$ MHz, information bandwidth ≈ 200 KHz
- Q: Why is f_c usually large?

  A: Antenna size $\propto \lambda_c \Rightarrow$ larger frequency, smaller antennas!

Analogue vs. Digital Modulation

Analogue Modulation: A *continuous information signal* $x(t)$ (e.g., speech, audio) is used to directly modulate the carrier wave.

We'll study two kinds of analogue modulation:

1. Amplitude Modulation (AM) : Information $x(t)$ modulates the *amplitude* of the carrier wave
2. Frequency Modulation (FM): Information $x(t)$ modulates the *frequency* of the carrier wave

We'll learn about:

- Power & bandwidth of AM & FM signals
- Tx & Rx design

In the last 4 lectures, we will study *digital* modulation:

- $x(t)$ is first digitised into bits
- Digital modulation then used to transport bits across the channel

Amplitude Modulation (AM)

- Information signal $x(t)$, carrier $\cos(2\pi f_c t)$
- The transmitted AM signal is

$$s_{\text{AM}}(t) = [a_0 + x(t)] \cos(2\pi f_c t)$$

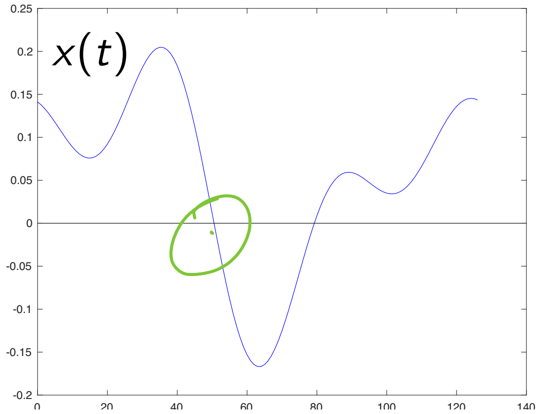
- a_0 is a positive constant chosen so that $\max_t |x(t)| < a_0$
- The **modulation index** of the AM signal is defined as

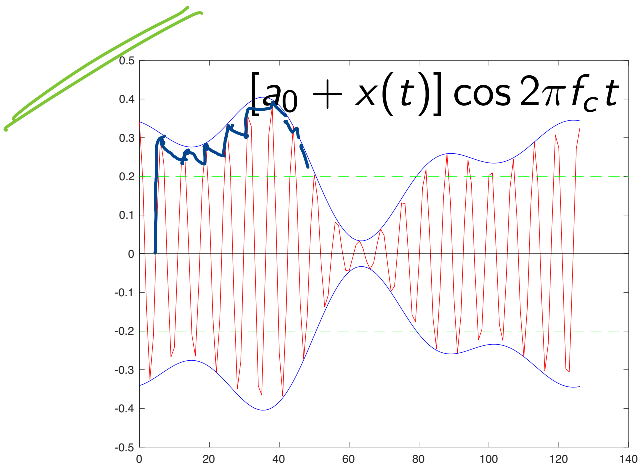
$$m_A = \frac{\max_t |x(t)|}{a_0}$$

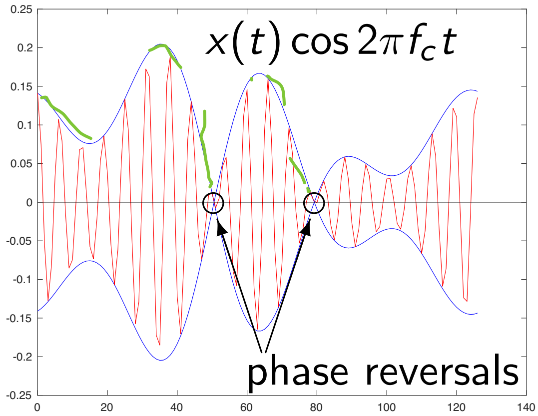
"The percentage that the carrier's amplitude varies above and below its unmodulated level"

Why is the modulation index important ?

$m_a < 1$ is desirable because we can extract the information signal $x(t)$ from the modulated signal by *envelope detection*.



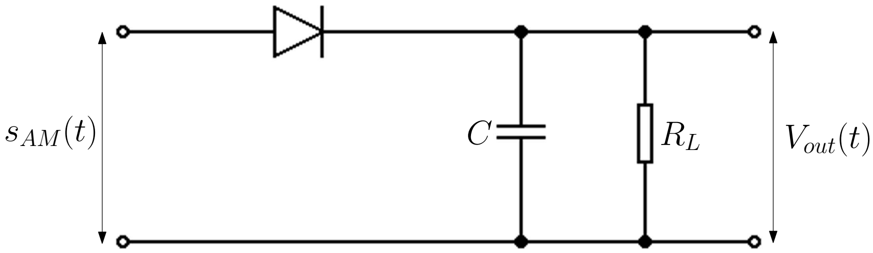




When modulation index > 1:

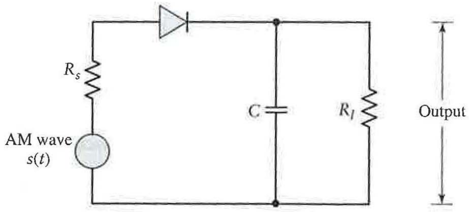
- Phase reversals occur
- $x(t)$ cannot be detected by tracing the $+$ve envelope

AM Receiver - Envelope Detector

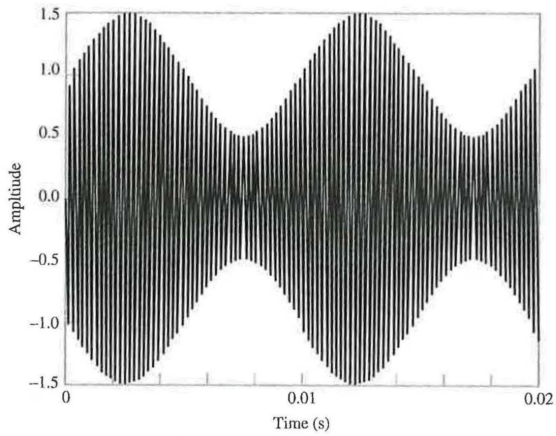


- On the positive half-cycle of the input signal, capacitor C charges *rapidly* up to the peak value of input $s_{\text{AM}}(t)$
- When input signal falls below this peak, diode becomes reverse-biased: capacitor discharges *slowly* through load resistor R_L
- In the next positive half-cycle, when input signal becomes greater than voltage across the capacitor, diode conducts again until next peak value
- Process repeats ...

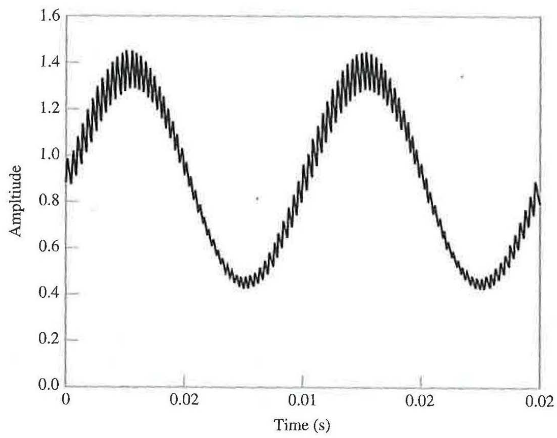
Very inexpensive receiver, but envelope detection needs $m_A < 1$.



Circuit Diagram



AM wave input



Envelope detector output

$x(t) \rightarrow X(f)$

$x(t)e^{j2\pi f_c t} \rightarrow X(f - f_c)$

Spectrum of AM

$a_0 \rightarrow a_0 \delta(f)$

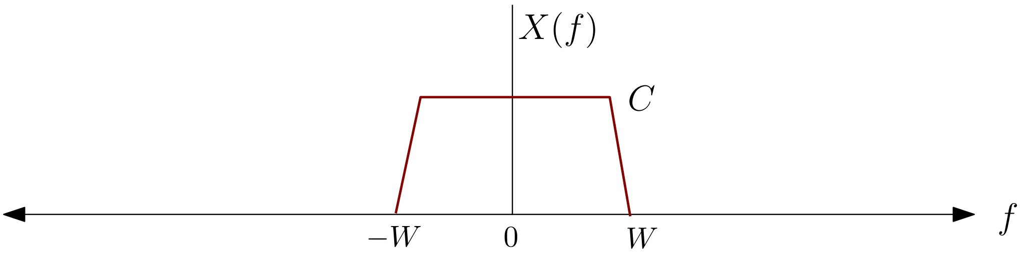
Next, let's look at the spectrum of $s_{\text{AM}}(t) = [a_0 + x(t)] \underbrace{\cos(2\pi f_c t)}_{\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}}$

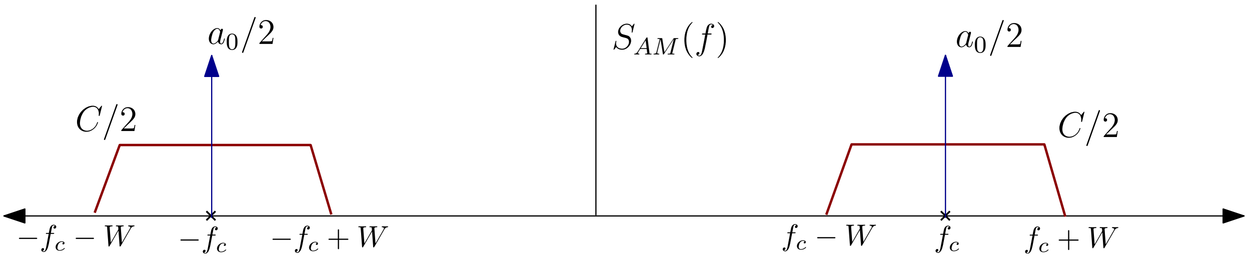
$$
\begin{aligned}
S_{\text{AM}}(f) &= \mathcal{F}[s_{\text{AM}}(t)] \\
&= \mathcal{F}\left[[a_0 + x(t)] \frac{(e^{j2\pi f_c t} + e^{-j2\pi f_c t})}{2} \right] \\
&= \underbrace{\frac{a_0}{2}[\delta(f - f_c) + \delta(f + f_c)]}_{\text{carrier}} + \underbrace{\frac{1}{2}[X(f - f_c) + X(f + f_c)]}_{\textit{information}}
\end{aligned}
$$

($\mathcal{F}[.]$ denotes the Fourier transform operation)

Example

$$S_{\mathrm{AM}}(f) = \frac{a_0}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right] \;+\; \frac{1}{2}\left[X(f - f_c) + X(f + f_c)\right]$$





Properties of AM

$$s_{\mathrm{AM}}(t) = [a_0 + x(t)]\cos(2\pi f_c t)$$

$$S_{\mathrm{AM}}(f) = \frac{a_0}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right] \;+\; \frac{1}{2}\left[X(f - f_c) + X(f + f_c)\right]$$

1. **Bandwidth**: From spectrum calculation, we see that if $x(t)$ is a baseband signal with (one-sided) bandwidth W, the AM signal $s_{\mathrm{AM}}(t)$ is passband with bandwidth

$$B_{\mathrm{AM}} = 2W$$

2. **Power**: We now prove that the power of the AM signal is

$$P_{\mathrm{AM}} = \frac{a_0^2}{2} + \frac{P_X}{2}$$

where P_X is the power of $x(t)$.

Power of AM signal

$\frac{1}{T}\int^{T} \frac{(a_0+x(t))^2}{2}dt = \frac{1}{T}\int_0^T \frac{a_0^2}{2}dt + \frac{1}{2T}\int_0^T (x(t))^2 dt + \frac{1}{T}\int_0^T a_0 x(t)\,dt$

$$P_{\text{AM}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T [a_0+x(t)]^2\cos^2(2\pi f_c t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T [a_0+x(t)]^2\,\frac{1+\cos(4\pi f_c t)}{2}\,dt$$

$= \frac{a_0^2}{2} + \frac{P_x}{2}$

$$= \frac{a_0^2}{2} + \frac{P_X}{2} + \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{[a_0+x(t)]^2}{2}\cos(4\pi f_c t)\,dt$$

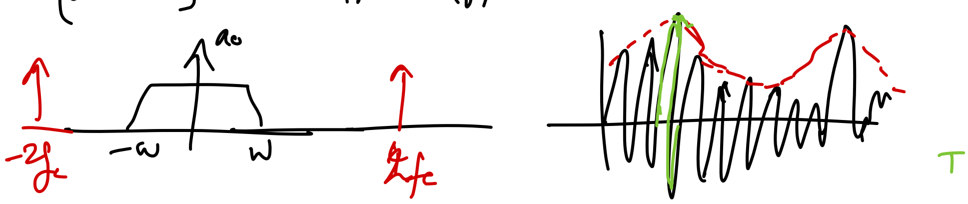
(We assume that $\frac{1}{T}\int_0^T x(t)dt = 0$ as a non-zero-mean can be absorbed into a_0.)

Now show that the last the last term is ≈ 0.

- $\cos(4\pi f_c t)$ is a high-frequency sinusoid with period $T_c = \frac{1}{2f_c}$.
- $g(t) = (a_0 + x(t))^2/2$ is a *baseband* signal which changes *much more slowly* than $\cos(4\pi f_c t)$.

$\frac{(a_0+x(t))^2}{2}\cos(4\pi f_c t)$

$[a_0 + x(t)] \to a_0\delta(f) + X(f)$



Hence, with $T = nT_c$, we have

$$\frac{1}{T}\int_0^T g(t)\cos(4\pi f_c t)\,dt \approx \frac{1}{nT_c}\Bigg(\int_0^{T_c} g(0)\cos(4\pi f_c t)\,dt +$$

$$+\int_{T_c}^{2T_c} g(T_c)\cos(4\pi f_c t)\,dt \ldots + \int_{(n-1)T_c}^{nT_c} g((n-1)T_c)\cos(4\pi f_c t)\,dt\Bigg)$$

$$= 0.$$

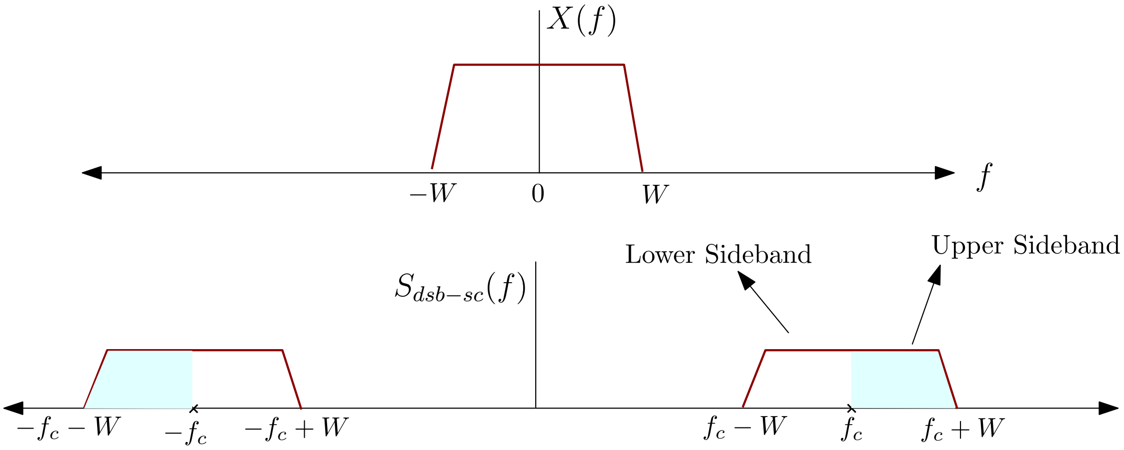
Hence $P_{AM} = \frac{a_0^2}{2} + \frac{P_X}{2}$. □

Double Sideband Suppressed Carrier (DSB-SC)

The power of AM signal is

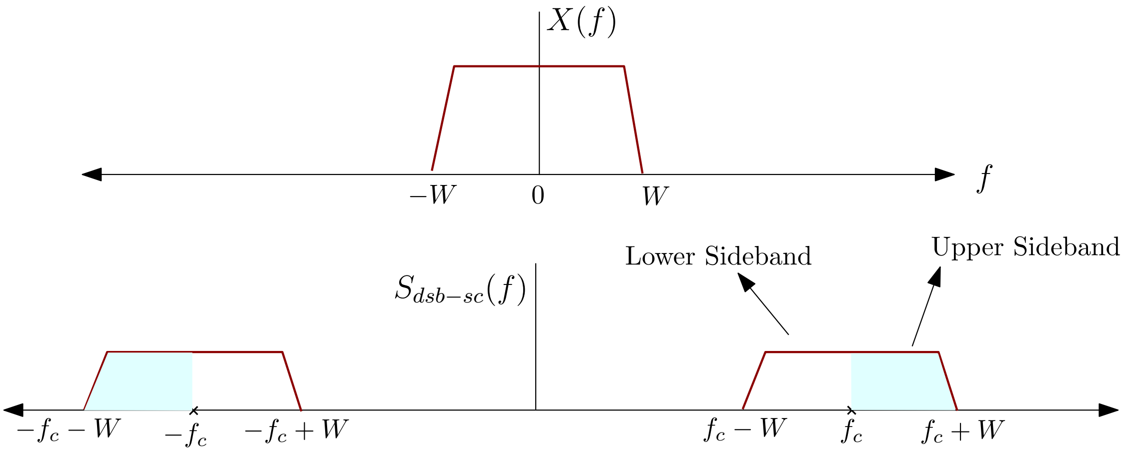
$$P_{\text{AM}} = \underbrace{\frac{a_0^2}{2}}_{\text{carrier}} + \frac{P_X}{2}$$

- The presence of a_0 makes envelope detection possible, but requires extra power of $\frac{a_0^2}{2}$ corresponding to the carrier
- In DSB-SC, we eliminate the a_0:
  We transmit only the sidebands, and *suppress* the carrier



The transmitted DSB-SC waveform is

$$s_{\text{dsb-sc}}(t) = x(t)\cos(2\pi f_c t)$$

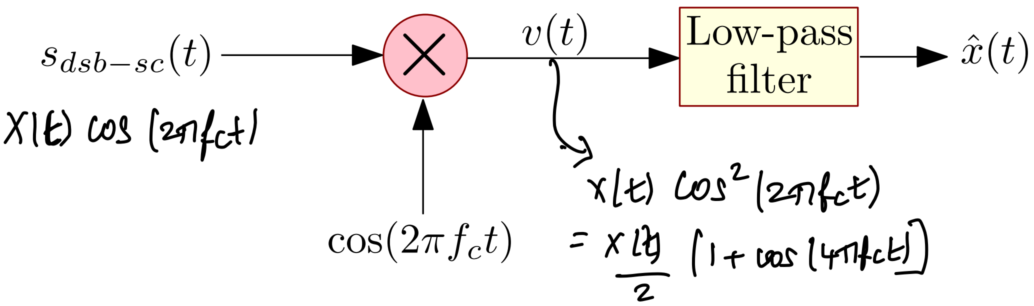


How to recover $x(t)$ at the receiver?

Phase reversals $\Rightarrow$ cannot use envelope detection

DSB-SC receiver

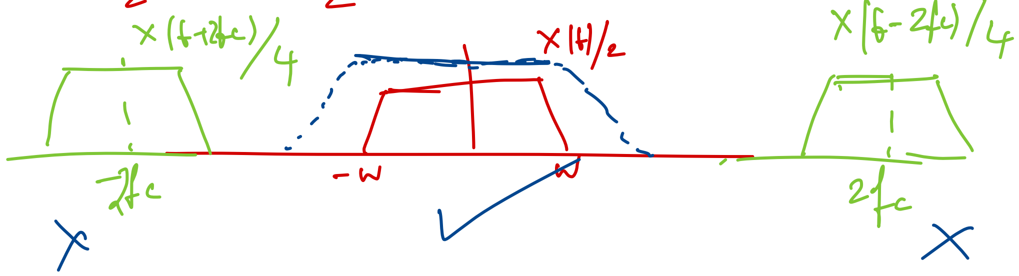
DSB-SC Receiver: Product Modulator + Low-pass filter



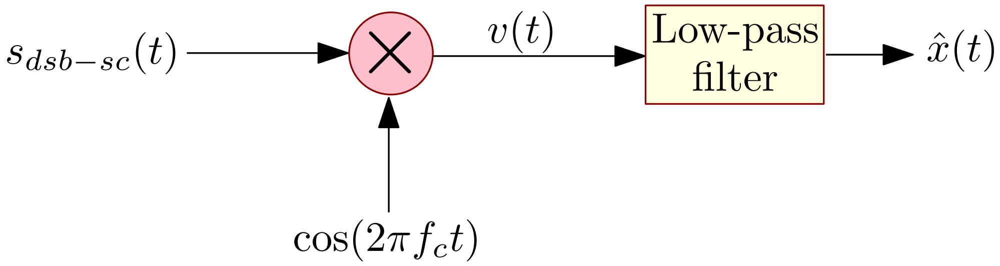
Step 1: Multiplying received signal by $\cos(2\pi f_c t)$ gives

$$v(t) = x(t)\,\cos^2(2\pi f_c t) = \underbrace{\frac{x(t)}{2}}_{\text{low freq.}} + \underbrace{\frac{x(t)\cos(4\pi f_c t)}{2}}_{\text{high freq.}}$$

$\frac{x(t)}{2} \rightarrow \frac{X(f)}{2}$



DSB-SC receiver



Step 2: Low-pass filter eliminates the high-frequency component.

Ideal low-pass filter has $H(f) =$ constant for $-W \leq f \leq W$, and zero otherwise

Properties of DSB-SC

$$s_{\text{dsb-sc}}(t) = x(t)\cos(2\pi f_c t)$$

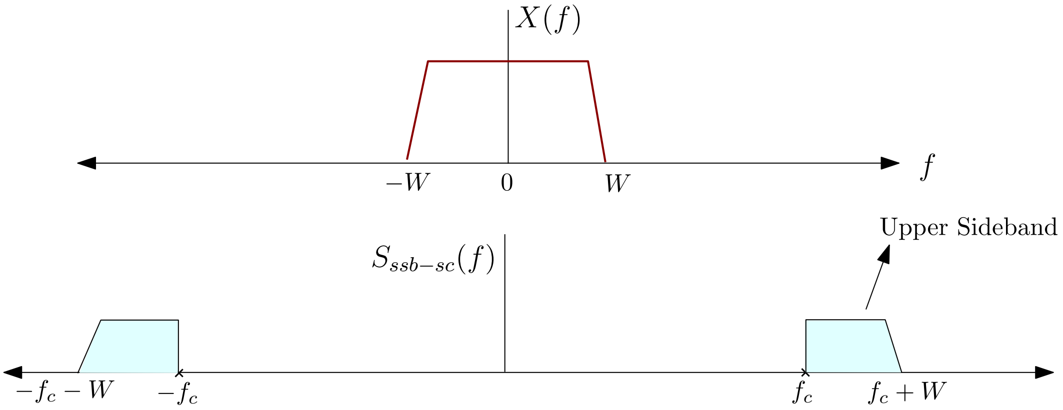
$$S_{\text{dsb-sc}}(f) = \frac{1}{2}(X(f+f_c) + X(f-f_c))$$

- **Bandwidth** of DSB-SC is $B_{\text{dsb-sc}} = 2W$, same as AM
- **Power** of $DSB - SC$ is $P_{\text{dsb-sc}} = \frac{P_X}{2}$
  (follows from AM power calculation)
- DSB-SC requires *less power* than AM as the carrier is not transmitted
- But DSB-SC receiver is *more complex* than AM !
  We assumed that receiver can generate locally generate a frequency f_c sinusoid that is synchronised perfectly in phase and frequency with transmitter's carrier
- Effect of phase mismatch at Rx is explored in Examples paper

Single Sideband Suppressed Carrier (SSB-SC)

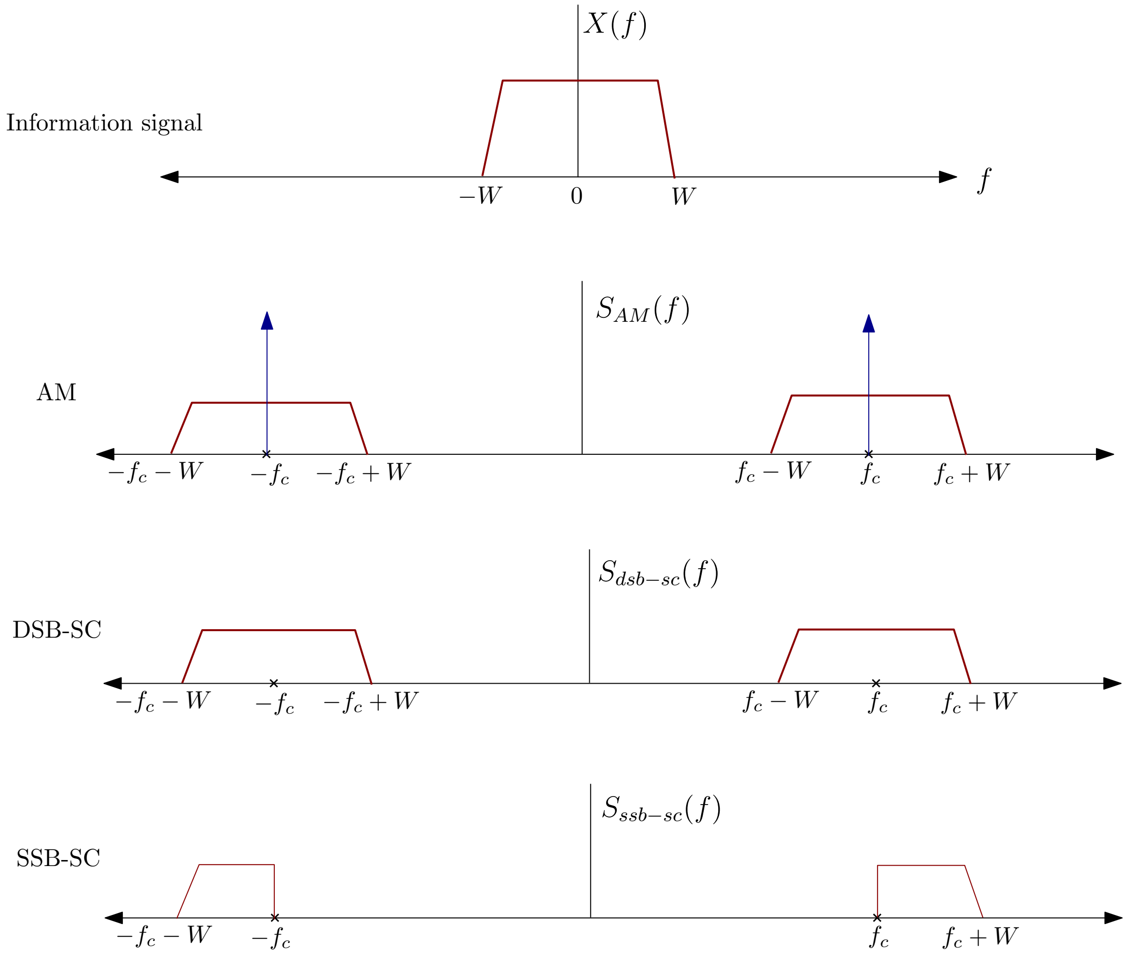
DSB-SC transmits less power than AM. Can we also save bandwidth?

- $x(t)$ is real $\Rightarrow X(-f) = X^*(f)$
  $\Rightarrow$ Need to specify $X(f)$ only for $f > 0$
- In other words, transmission of both sidebands is not strictly necessary: we could obtain one sideband from the other!



- **Bandwidth** is $B_{\text{ssb-sc}} = W$, half of that of AM or DSB-SC!
- **Power** is is $P_{\text{ssb-sc}} = \frac{P_X}{4}$, half of DSB-SC

Summary: Amplitude Modulation



You can now do Questions 1–5 on Examples Paper 8.

Frequency Modulation (FM)

In FM, the information signal $x(t)$ modulates the *instantaneous frequency* of the carrier wave.

The instantaneous frequency $f(t)$ is varied linearly with $x(t)$:

$$f(t) = f_c + k_f x(t)$$

This translates to an instantaneous phase $\theta(t)$ given by

$$\theta(t) = 2\pi \int_0^t f(u)du = 2\pi f_c t + 2\pi k_f \int_0^t x(u)du$$

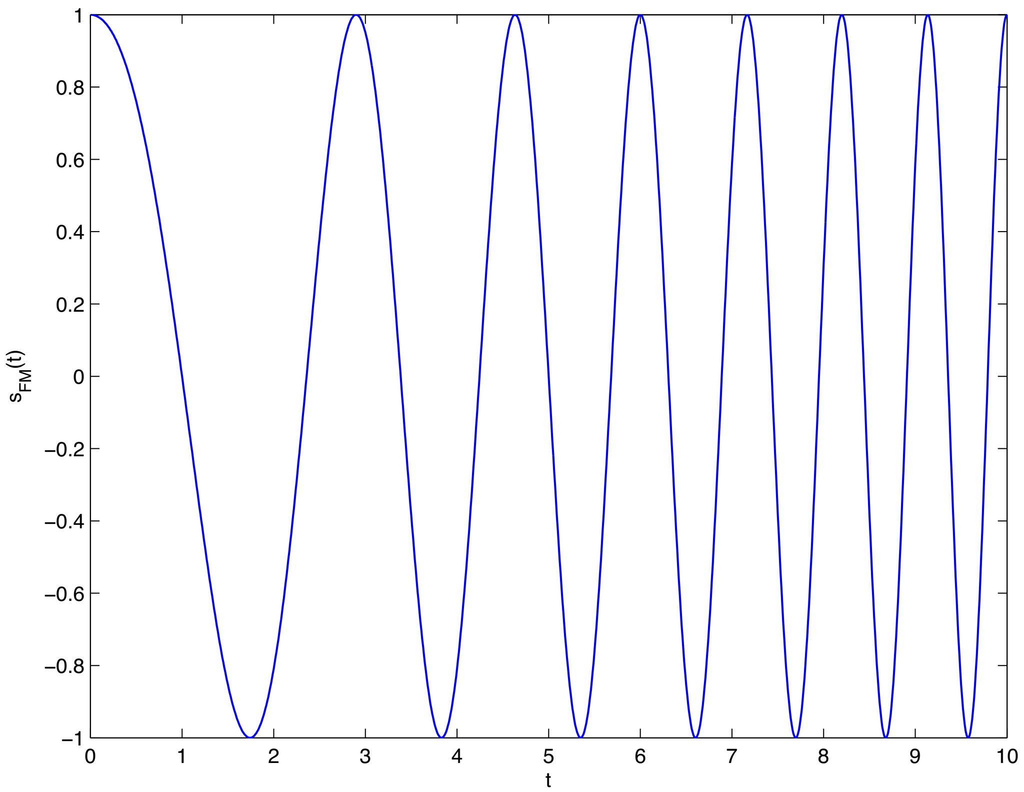
The modulated FM signal

$$s_{\mathrm{FM}}(t) = A_c \cos(\theta(t)) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t x(u)du\right)$$

- A_c is the carrier amplitude
- k_f is called the *frequency-sensitivity factor*

Example

What information signal does this FM wave correspond to?



(a) a constant, (b) a ramp, (c) a sinusoid, (d) no clue

FM Demodulation

At the receiver, how do we recover $x(t)$ from the FM wave? (ignoring effects of noise)

$$s_{\text{FM}}(t) = A_c \cos\left(2\pi f_c t \,+\, 2\pi k_f \int_0^t x(u)du\right)$$

The derivative is

$$\frac{ds_{\text{FM}}(t)}{dt} = -2\pi A_c[f_c + k_f x(t)] \sin\left(2\pi f_c t \,+\, 2\pi k_f \int_0^t x(u)du\right)$$

- The derivative is a passband signal with amplitude modulation by $[f_c + k_f x(t)]$
- If f_c large enough, we can recover $x(t)$ by *envelope detection* of $\frac{ds_{\text{FM}}(t)}{dt}$!
- Hence FM demodulator is a *differentiator* + *envelope detector*
- Differentiator: $\frac{d}{dt} \xrightarrow{\mathcal{F}} j2\pi f$ (frequency response). See Haykin-Moher book for details on how to build a differentiator

Properties of FM

$$s_{\text{FM}}(t) = A_c \cos\left(2\pi f_c t \,+\, 2\pi k_f \int_0^t x(u)du\right)$$

- **Power** of FM signal $= \frac{A_c^2}{2}$, *regardless* of $x(t)$
- Non-linearity: $FM(x_1(t) + x_2(t)) \neq FM(x_1(t)) + FM(x_2(t))$
- FM is more robust to additive noise than AM.

  Intuitively, this is because the message is "hidden" in the frequency of the signal rather than the amplitude.
- But this robustness comes at the cost of increased transmission bandwidth
- What is the bandwidth of the FM signal $s_{\text{FM}}(t)$?

  The spectral analysis is a bit complicated, but we will do it for a simple case ... where $x(t)$ is a sinusoid (a pure tone)

FM modulation of a tone

Consider FM modulation of a tone $x(t) = a_x \cos(2\pi f_x t)$. We have

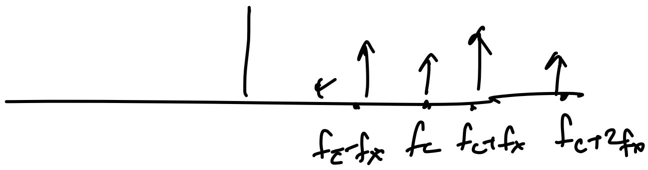
$$f(t) = f_c + k_f a_x \cos(2\pi f_x t)$$
$$\theta(t) = 2\pi f_c t + \frac{k_f a_x}{f_x} \sin(2\pi f_x t)$$

- $\Delta f = k_f a_x$ is called the *frequency deviation*
  Δf is the max. deviation of the carrier frequency $f(t)$ from f_c
- $\beta = \frac{k_f a_x}{f_x} = \frac{\Delta f}{f_x}$ is called the *modulation index*
  β is the max. deviation of the carrier phase $\theta(t)$ from $2\pi f_c t$

Then the FM signal becomes

$$s_{\text{FM}}(t) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_x t)\right)$$

The spectrum of the FM signal



We want to study the frequency spectrum of

$$s_{\text{FM}}(t) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_x t)\right)$$

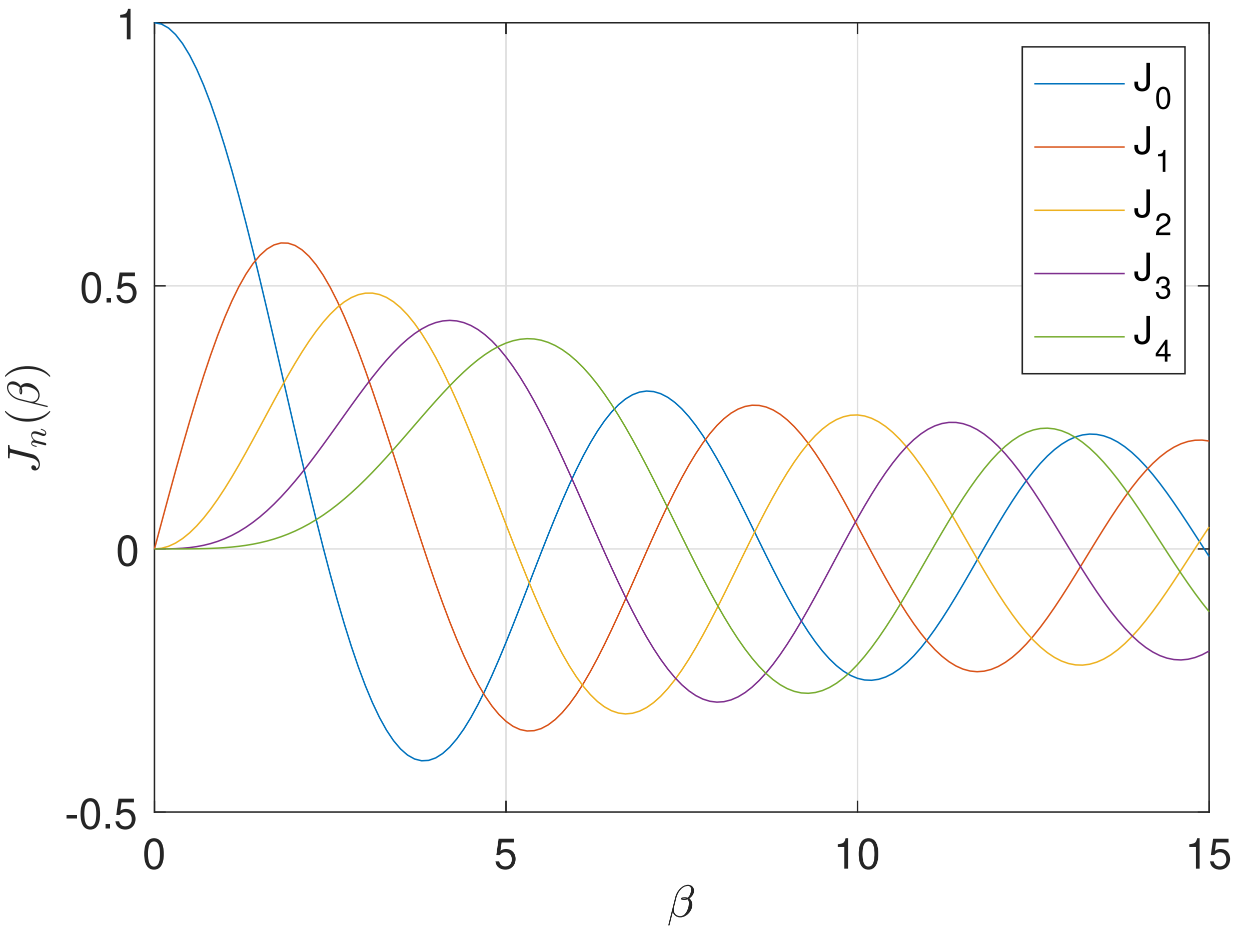
You will show in the Examples Paper that

$$S_{\text{FM}}(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[\delta(f - f_c - nf_x) + \delta(f + f_c + nf_x)\right]$$

where $J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin u - nu)} \, du$

$J_n(.)$ is called the *nth order Bessel function* of the first kind.

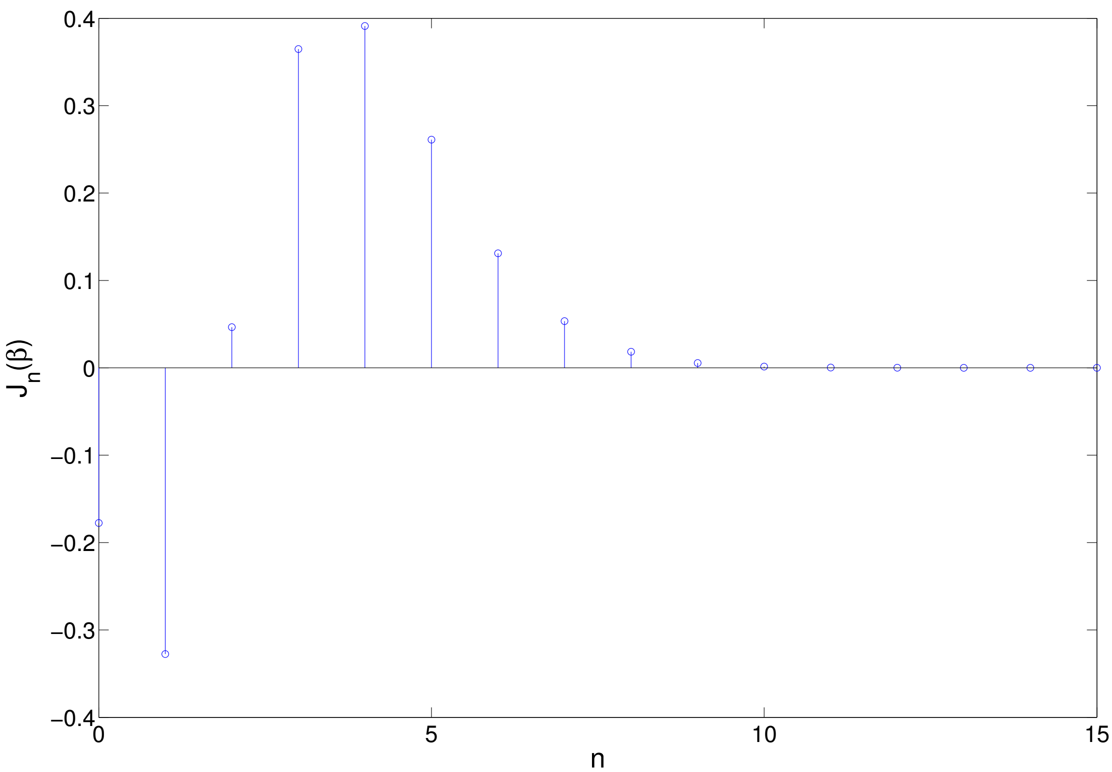
Plots of $J_n(\beta)$ vs β



Example

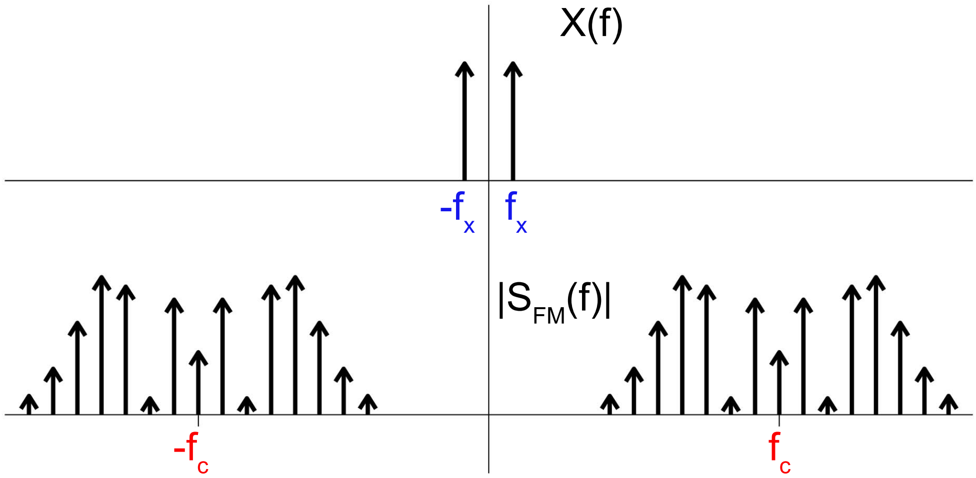
What is the spectrum of the FM signal when $x(t)$ is a pure tone and the modulation index $\beta = 5$?

$J_n(\beta)$ vs n for $\beta = 5$



The spectrum is

$$S_{\mathrm{FM}}(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(5) \left[\delta(f - f_c - nf_x) + \delta(f + f_c + nf_x) \right]$$



Bandwidth of FM signals

To summarise, even for the case where $x(t)$ has a single frequency f_x, the spectrum of the FM wave is rather complicated:

- There is a carrier component at f_c, and components located symmetrically on either side of f_c at $f_c \pm f_x$, $f_c \pm 2f_x, \ldots$
- The absolute bandwidth is infinite, but ... the side components at $f_c \pm nf_x$ become negligible for large enough n

Carson's rule for the *effective* bandwidth of FM signals:

1. The bandwidth of an FM signal generated by modulating a single tone is
$$B_{\mathrm{FM}} \approx 2\Delta f + 2f_x = 2\Delta f \left(1 + \tfrac{1}{\beta}\right)$$
2. For an FM signal generated by modulating a general signal $x(t)$ with bandwidth W, the bandwidth $B_{\mathrm{FM}} \approx 2\Delta f + 2W$

(Recall: for any FM wave, Δf is the frequency deviation around f_c)

Example

BBC Radio Cambridgeshire: $f_c = 96$ MHz and $\Delta f = 75$ kHz.
Assuming that the voice/music signals have $W = 15$ kHz, we have

$$\beta = \frac{\Delta f}{W} = \frac{75}{15} = 5$$

and the bandwidth

$$B_{\text{FM}} = 2(\Delta f + W) = 2(75 + 15) = 180\,\text{kHz},$$

while

$$B_{\text{AM}} = 2W = 30 \text{ kHz}$$

FM signals have larger bandwidth than AM, but have better robustness against noise.

Summary: Analogue Modulation

Amplitude Modulation with information signal of bandwidth W

- **AM** modulated signal: Bandwidth $2W$, high power, simple Rx using envelope detection
- **DSB-SC**: Bandwidth $2W$, lower power, more complex Rx
- **SSB-SC**: Bandwidth W, even lower power, Rx similar to DSB-SC

Frequency Modulation with information signal of bandwidth W:

- FM signal has constant carrier amplitude $\Rightarrow$ constant power
- Bandwidth of FM signal depends on both β and W

  Can be significantly greater than $2W$
- Better robustness to noise than AM as the information is "hidden" in the phase