1B Paper 6: Communications

Handout 3: Digitisation of Analogue Signals

Ramji Venkataramanan

Signal Processing and Communications Lab
Department of Engineering
rv285@cam.ac.uk

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Two Types of Sources

1. **Analogue**: Continuous-time, continuous-amplitude sources, e.g., speech, music

2. **Digital**: Can be represented as bits, e.g., email, computer files, JPEG image files, mp3 music files

   In general, a digital source is a *discrete-time* sequence of symbols drawn from a *finite alphabet*

$$\text{E.g.,} \quad X_1, X_2, X_3, \ldots \qquad X_i \in \{a, b, c, d\} \text{ for all } i$$

In this handout, we will learn how to effectively convert analogue signals into digital

Digitisation of Analogue Signals

Digitisation:
The process by which an analogue signal is converted into digital format, i.e., from a *continuous* signal (in time and amplitude) to a *discrete* signal (in time and amplitude). It consists of

- Sampling (discretises the time axis)
- Quantisation (discretises the signal amplitude axis)

Digitisation is also called analogue-to-digital conversion (ADC).

Why do we want to do this?

Why Digital ?

There are many advantages of transmitting digital signals:

- **Robustness**: In analogue communication systems, the effect of channel noise, signal distortion etc. are *cumulative*.
  In contrast, regenerators can be used to recover and retransmit a digital signal *exactly* before it excessively degrades.
- **Performance**: Powerful *error-correcting codes* can correct *bit* errors that may occur in the transmission of digital signals
- **Encryption**: Digital communication systems can be made highly secure by exploiting powerful encryption algorithms

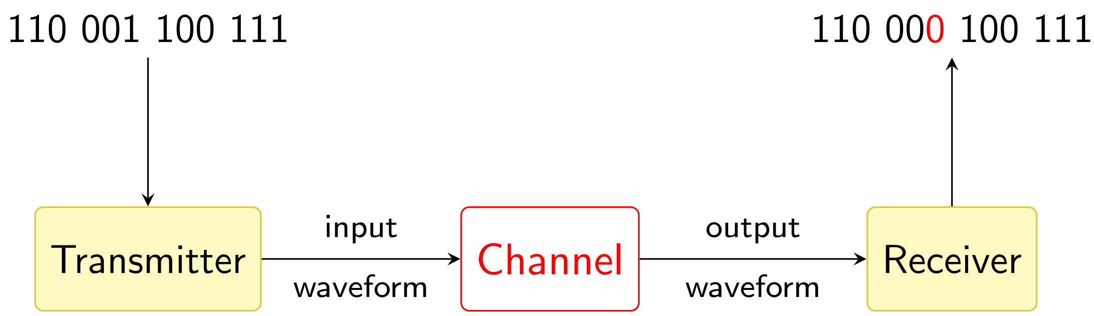
Digital communication does increase system complexity.
But dramatic improvements in hardware technology have made design and implementation very cost-effective.

End-to-end Digital Communication System

Step 1: Digitisation: Sampling + Quantisation

$$x(t) \quad \xrightarrow{\textit{sampling}} \quad x(nT) \quad \xrightarrow{\textit{quant.}} \quad \ldots 0101110111000 \ldots \text{ (bits)}$$

Step 2:



Channel noise can cause errors at the receiver. We'll later see how to deal with this using *coding*.

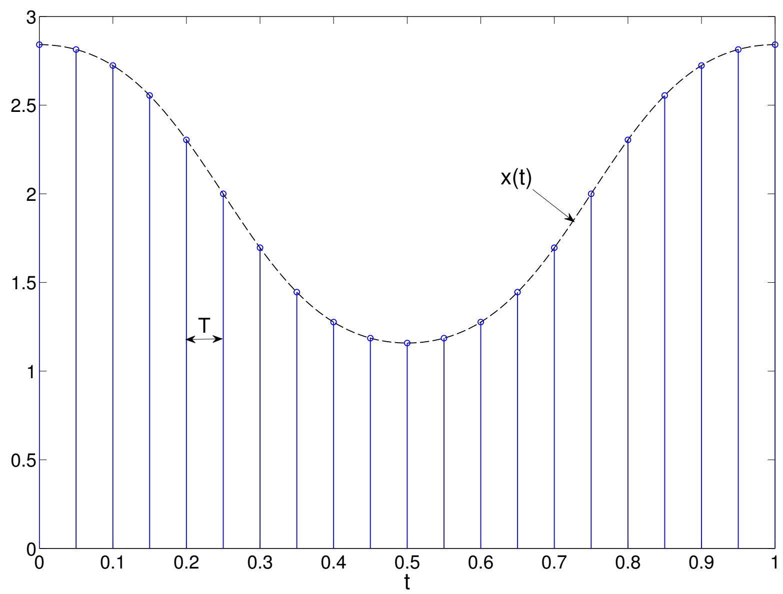
Let's start with Step 1

Sampling

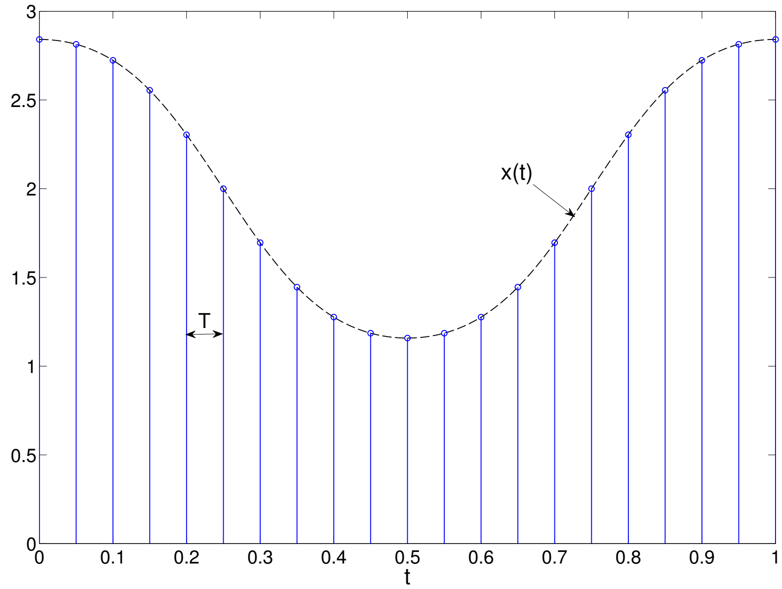
- Let $x(t)$ be a *continuous-time* signal for $-\infty < t < \infty$
- Choose a *sampling interval* T, and read off the values of $x(t)$ at times

$$\ldots, \ -3T, \ -2T, \ -T, \ 0, \ T, \ 2T, \ 3T, \ldots$$

- The obtained values $x(nT)$ are the *samples* of $x(t)$



Recovering $x(t)$ from its samples



- Can you recover $x(t)$ from just the samples $x(nT)$?

Yes, if the sampling rate $\frac{1}{T} > 2W$, where W is the bandwidth of $x(t)$ (in Hz)

Recovery easier to understand in frequency domain ...

Frequency domain interpretation of sampling

A continuous-time representation of the sampled signal is

$$x_s(t) = \sum_n x(nT)\delta(t - nT) = x(t) \sum_n \delta(t - nT)$$

$\sum_n \delta(t - nT)$ is periodic and can be expressed as a *Fourier series*

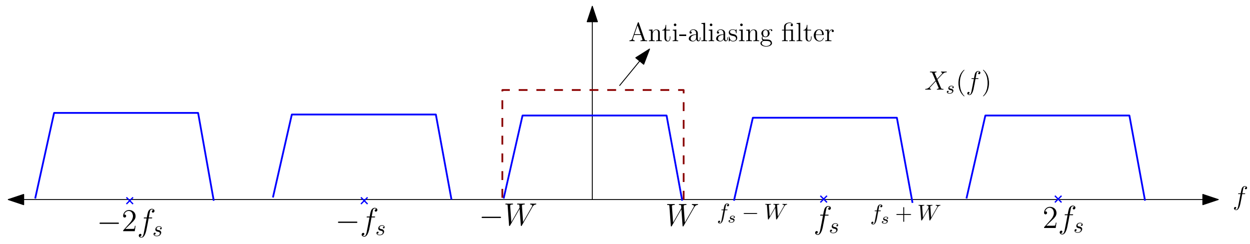
$$\sum_n c_n e^{jn\frac{2\pi}{T}t} \quad \text{with} \quad c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-j\frac{2\pi n}{T}t} dt = \frac{1}{T}$$

Therefore

$$x_s(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t) e^{j\frac{2\pi n}{T}t}$$

and

$$X_s(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{T}\right)$$



$X(f)$ can be recovered from $X_s(f)$ using an "ideal reconstruction" or "anti-aliasing" filter

Nyquist Rate

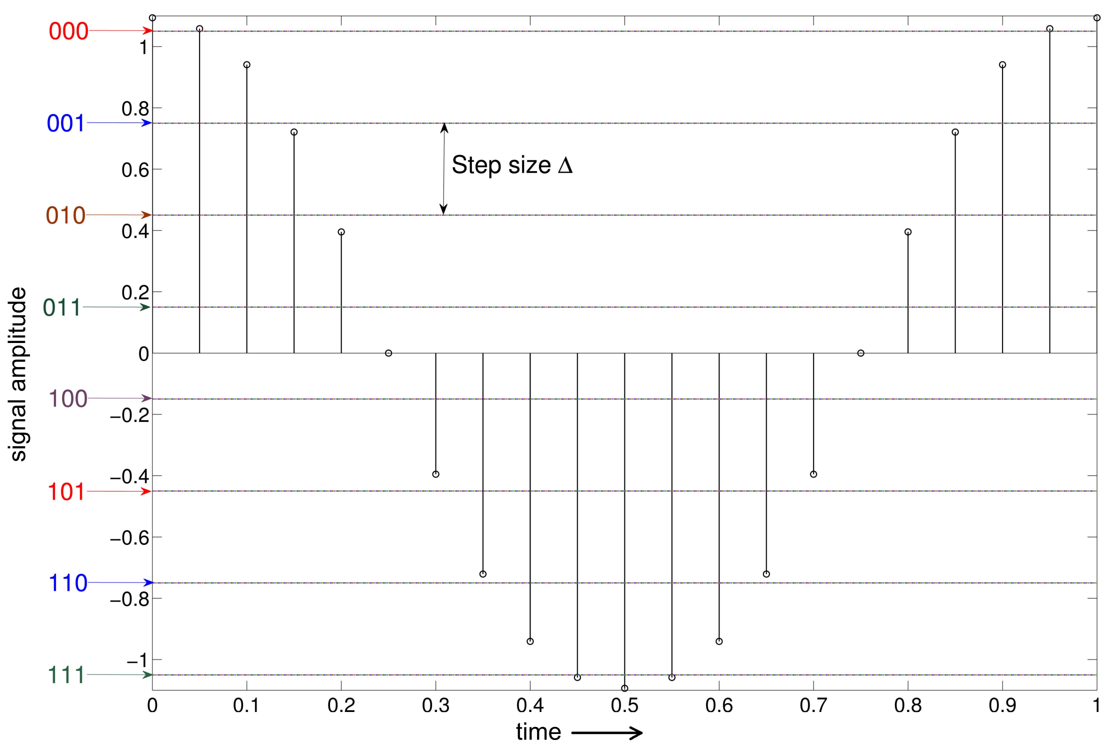
Consider a signal $x(t)$ with bandwidth W. Then, we can recover $x(t)$ from its samples $\{x(nT)\}$ provided that the sampling frequency $f_s = \frac{1}{T}$ satisfies $f_s > 2W$

The sampled version $\{x(nT)\}$, $n \in \mathbb{Z}$ is a *discrete-time* signal, but not yet digital!

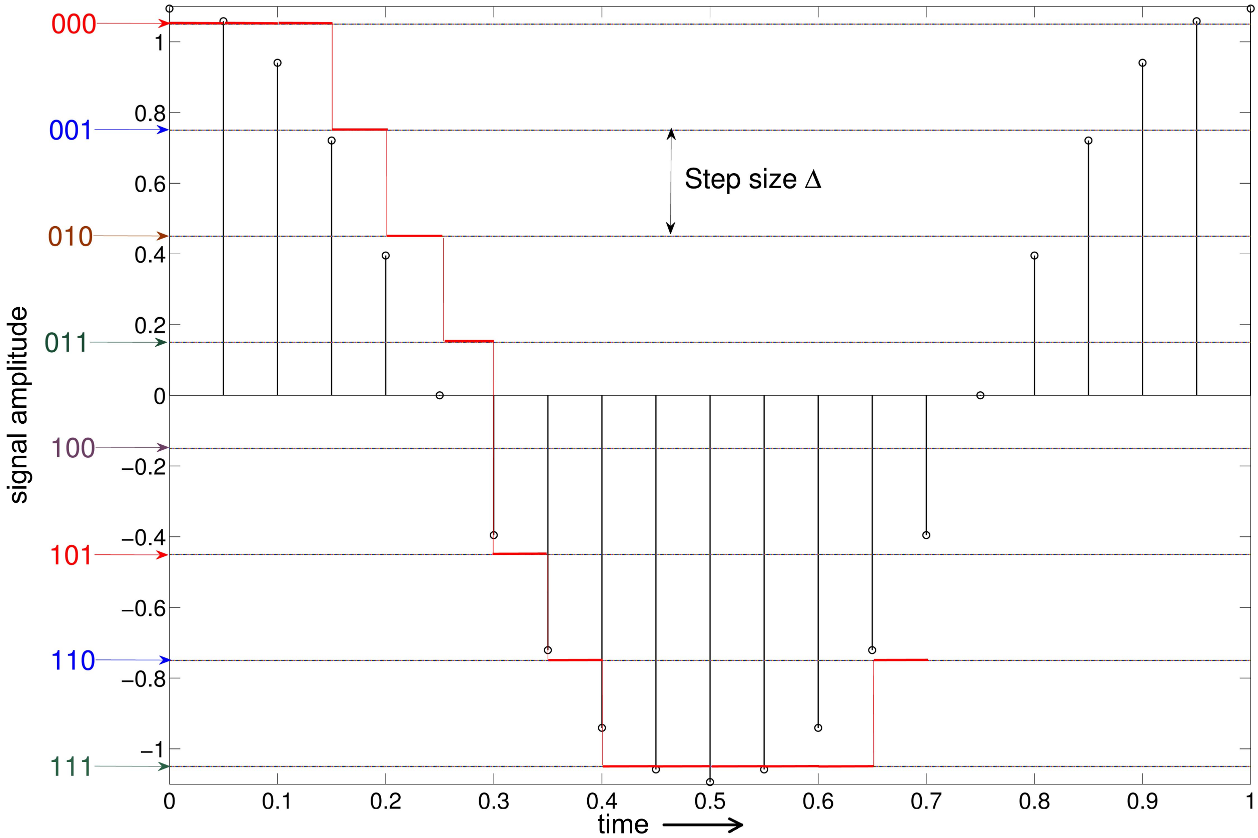
To represent the sampled signal using bits, we need to *quantise* it

Uniform Quantisation

The sampled signal can take continuous values. To convert it into digital, we: **1)** Assign a discrete amplitude from a *finite set* of levels (with step Δ), and **2)** Assign bits to those amplitudes



What the quantised signal looks like



Each sample $x(nT)$ is mapped to the nearest quantisation level

Sampling vs. Quantisation

Sampling is a *lossless* procedure as long as the sampling rate is greater than the Nyquist rate:

$\Rightarrow x(t)$ can be *perfectly* reconstructed from its samples $x(nT)$

Quantisation is *always lossy*: you cannot recover $x(nT)$ from its quantised value!

If $Q(z)$ denotes the quantised valued of a sample $x(nT) = z$, the *quantisation noise* is defined as

$$e_Q(z) = z - Q(z)$$

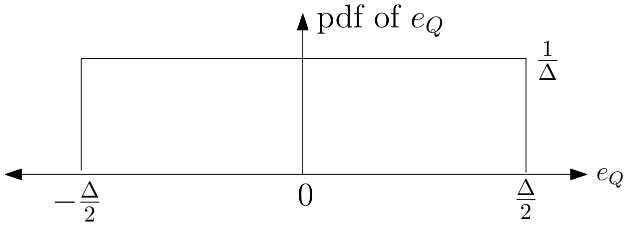
If the quantiser step size is Δ, then $e_Q(z)$ lies in the interval $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$

Quantisation Noise as a Random variable

$$e_Q(z) = z - Q(z)$$

We model e_Q as a random variable *uniformly distributed* in $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$. Why?

- If we quantise lots of samples and the step size Δ is small, the set of samples quantised to a level Q will be approximately uniformly distributed in a length-Δ interval centred around Q.



We can now easily compute the noise power

$$N_Q = \mathbb{E}[e_Q^2] = \int_{-\Delta/2}^{\Delta/2} u^2 \frac{1}{\Delta} du = \frac{1}{\Delta}\frac{u^3}{3}\Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{12}$$

and its corresponding RMS is $\frac{\Delta}{\sqrt{12}}$

Signal to Quantisation Noise Ratio

- Assume that the signal to be quantised is a sinusoid taking values between $-V$ and $+V$ (in Volts)
- The signal power is then $\frac{V^2}{2}$ and the RMS signal is $\frac{V}{\sqrt{2}}$
- The *signal-to-noise* ratio is

$$\text{SNR} = \frac{\text{signal power}}{\text{noise power}} = \frac{(\text{RMS signal})^2}{(\text{RMS noise})^2} = \frac{V^2/2}{\Delta^2/12}$$

- An n-bit uniform quantiser has 2^n levels, and step size $\Delta = \frac{2V}{2^n}$ (uses n bits to represent each sample of the signal)
- Hence the SNR can be written as

$$\text{SNR} = \frac{V^2/2}{\Delta^2/12} = 3 \times 2^{2n-1} = \mathbf{1.76 + 6.02n \ \ dB}$$

For fixed signal amplitude $\pm V$:

- Larger $n \Rightarrow$ smaller step size, better quality quantiser
- But larger n also means more bits to transmit !

Data Rate of the Digitised Source

Assuming we sample a signal $x(t)$ having bandwidth W at Nyquist rate, and we use an n-bit uniform quantiser, the *digitised* source will have a rate of

$$R = n2W \text{ bits per second}$$

(Because we have $2W$ samples/sec., each represented with n bits)

Assume we want to digitise a speech signal whose bandwidth $W = 3.2$kHz using Nyquist sampling and a 10-bit uniform quantiser

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$$\text{Bit Rate } R = 10 \times 2 \times 3200 = 64000 \text{ bits per second} = 64 \text{ kbps}$$

Non-uniform quantisation

Mobile phones use clever quantisers which reduce the bit-rate by a factor of 5, from 64kbps (our uniform quantiser) to 13 kbps!

- Idea: Smaller step sizes in the vicinity of smaller (or more frequently) occuring signal values, larger step sizes for larger (or rarer) signal values
- This is called *non-uniform* quantisation or sometimes, *companding* (Examples Paper 9, Problem 1.b)

Summary

- We learned how to digitise band-limited, continuous-valued sources: *Sample at Nyquist rate, then quantise*
- Sampling is a lossless operation (when sampling rate $>$ Nyquist rate), but quantisation is lossy
- Trade-off in Quantisation: Bits/sample (n) vs SNR:

  Larger n, lower quantisation noise, but more bits to transmit
- We will next learn how to associate bits with signals in order to transmit them across communication channels
- To transport bits across a channel, it doesn't matter where they came from!

  You should now be able to do all of Examples Paper 8