Lecture 4

The Gradient of a Scalar Field

4.1 The vector operator 'Del'

You have met the *vector operator*, 'del', in Part IA. We represent 'del' by the 'upside down triangle' ∇ (which has nothing to do with big delta, Δ). In Cartesian coordinates, ∇ is defined by,

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \quad . \tag{4.1}$$

We call ∇ an operator because it acts on, or *operates* on, whatever comes immediately after it. For example, if ϕ is a scalar function $\phi = \phi(x,y,z)$ then,

$$\nabla\phi = \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right)\phi = \underline{i}\frac{\partial\phi}{\partial x} + \underline{j}\frac{\partial\phi}{\partial y} + \underline{k}\frac{\partial\phi}{\partial z}$$

$\nabla\phi$ is called the gradient of ϕ (or 'grad ϕ') and is a *vector*. ∇ can only operate directly on a scalar function, and the result is a vector.

Example

Find the gradient of the scalar field $\phi = x^2 y \sin z$.

$$\frac{\partial\phi}{\partial x} = 2x\, y \sin z$$

$$\frac{\partial\phi}{\partial y} = x^2 \sin z$$

$$\frac{\partial\phi}{\partial z} = x^2 y \cos z$$

$$\nabla\phi = (2xy\sin z)\,\underline{i} + (x^2 \sin z)\,\underline{j} + (x^2 y \cos z)\,\underline{k}$$

Two 'vector' identities involving the gradient that are useful are,

$$\nabla(f+g) = \nabla f + \nabla g \tag{4.2}$$

$$\nabla(fg) = f\nabla g + g\nabla f \tag{4.3}$$

where f and g are both scalar fields. The easiest way to prove these is by expanding terms in Cartesian coordinates.

Example

Prove, $\nabla(fg) = f\nabla g + g\nabla f$.

$$\nabla(fg) = \underline{i}\frac{\partial}{\partial x}(fg) + \underline{j}\frac{\partial}{\partial y}(fg) + \underline{k}\frac{\partial}{\partial z}(fg)$$

$$= f\left(\underline{i}\frac{\partial g}{\partial x} + \underline{j}\frac{\partial g}{\partial y} + \underline{k}\frac{\partial g}{\partial z}\right)$$

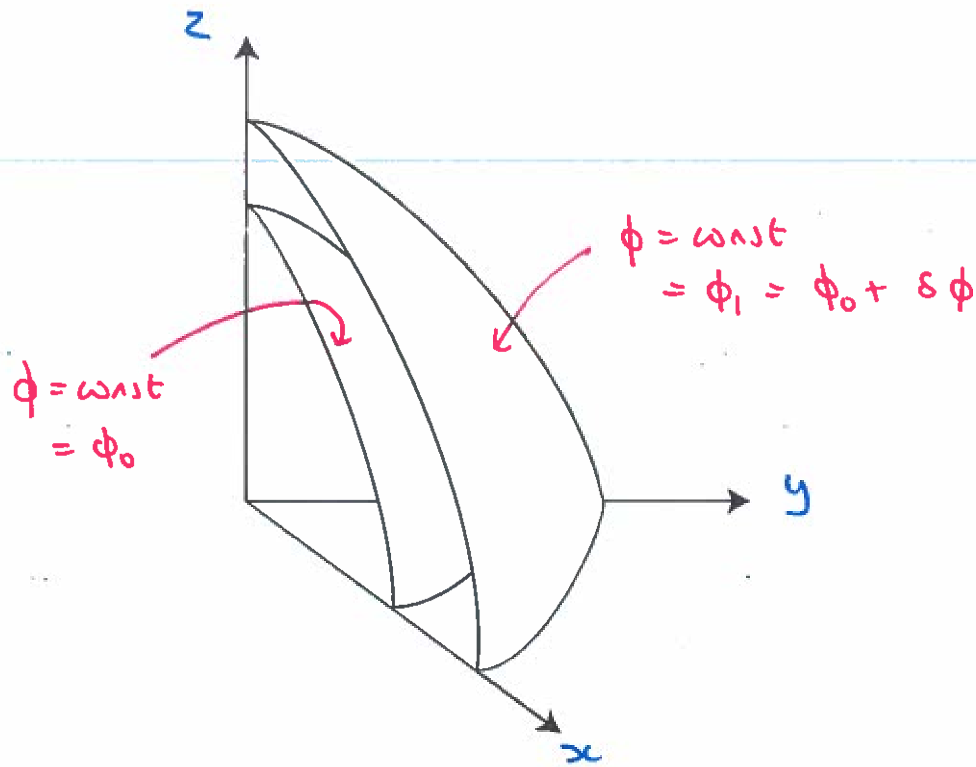
$$+ g\left(\underline{i}\frac{\partial f}{\partial x} + \underline{j}\frac{\partial f}{\partial y} + \underline{k}\frac{\partial f}{\partial z}\right)$$

$$= f\nabla g + g\nabla f$$

4.2 Physical interpretation of the gradient

The vector gradient is the 3-D equivalent of the slope of a curve in 1-D.

$\phi = \phi(x,y,z)$ is a scalar field in Cartesian space (for example, the temperature at every point in space of a 3-D object). We can draw surfaces of constant ϕ:



The sketch shows two surfaces, one at ϕ_0 and one at $\phi_1 = \phi_0 + \delta\phi$. If we move from a point (x, y, z) on the ϕ_0 surface to a point $(x + \delta x, y + \delta y, z + \delta z)$ on the ϕ_1 surface, we can write,

$$\delta\phi = \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z$$

Now, using $\nabla\phi$, we can write this, more compactly, as,

$$\delta\phi = \left(\underline{i}\frac{\partial\phi}{\partial x} + \underline{j}\frac{\partial\phi}{\partial y} + \underline{k}\frac{\partial\phi}{\partial z}\right) \cdot \left(\delta x\,\underline{i} + \delta y\,\underline{j} + \delta z\,\underline{k}\right)$$

so that

$$\delta\phi = \nabla\phi \cdot \delta\mathbf{r} \quad , \tag{4.4}$$

and this holds for any coordinate system.

If we write $\delta\mathbf{r}$ as $\delta s\,\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit vector in the direction of $\delta\mathbf{r}$,

$$\delta\phi = \nabla\phi \cdot (\delta s\,\underline{\hat{n}})$$

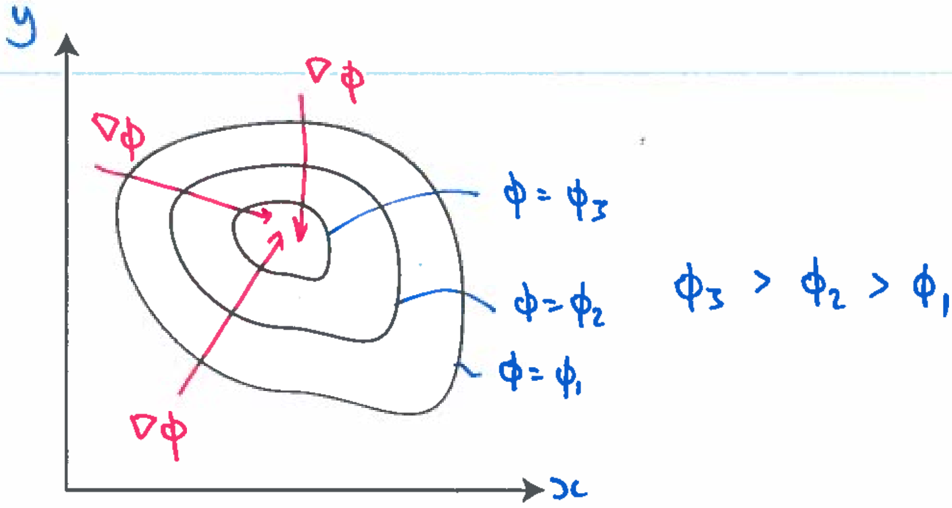
so that, as $\delta s \to 0$

$$\frac{d\phi}{ds} = \nabla\phi \cdot \hat{\mathbf{n}} \quad . \tag{4.5}$$

This is known as the 'directional derivative' and $d\phi/ds = \nabla\phi \cdot \hat{\mathbf{n}}$ is valid for any coordinate system. Notice that:

1. if $\hat{\mathbf{n}}$ lies on the surface of constant ϕ, $\nabla\phi \cdot \hat{\mathbf{n}} = d\phi/ds = 0$
2. the magnitude of $d\phi/ds$ is greatest when $\hat{\mathbf{n}}$ is parallel to $\nabla\phi$

3. the direction of $\nabla\phi$ is always in the direction of increasing ϕ ("$\nabla\phi$ always points up hill")



4.3 Flux-gradient empirical "laws"

For any scalar field ϕ it is always possible to obtain a vector field $\mathbf{V}$ using the relationship $\mathbf{V} = \nabla\phi$. However, if we have a particular vector field $\mathbf{V}_0$, it is not always possible to find a scalar field ϕ_0 such that $\mathbf{V}_0 = \nabla\phi_0$. For cases when we can find the required ϕ_0 field, ϕ_0 is known as the *scalar potential* and $\mathbf{V}_0$ is the *flux vector*. Considerable mathematical simplifications then follow: once we have obtained the scalar potential, we also know all three components of the vector field.

There are several engineering applications where the flux-gradient approach is used to model a physical process. Here, we will consider Fourier's law of heat conduction, Fick's law of diffusion and Ohm's law of current flow. Each of these is not actually a law, but rather a model that has been found to fit empirical data.

Heat conduction

Fourier's law of heat conduction tells us that heat flows down a temperature gradient. For example, in a straight metal bar aligned with the x-axis, the heat flow is given by $Q_x = -\lambda A \partial T/\partial x$ where T is the temperature, A is the cross-sectional area of the bar and λ is the thermal conductivity of the metal. The *heat flux* is the heat flow per unit area,

$$q_x = -\lambda \frac{\partial T}{\partial x}$$

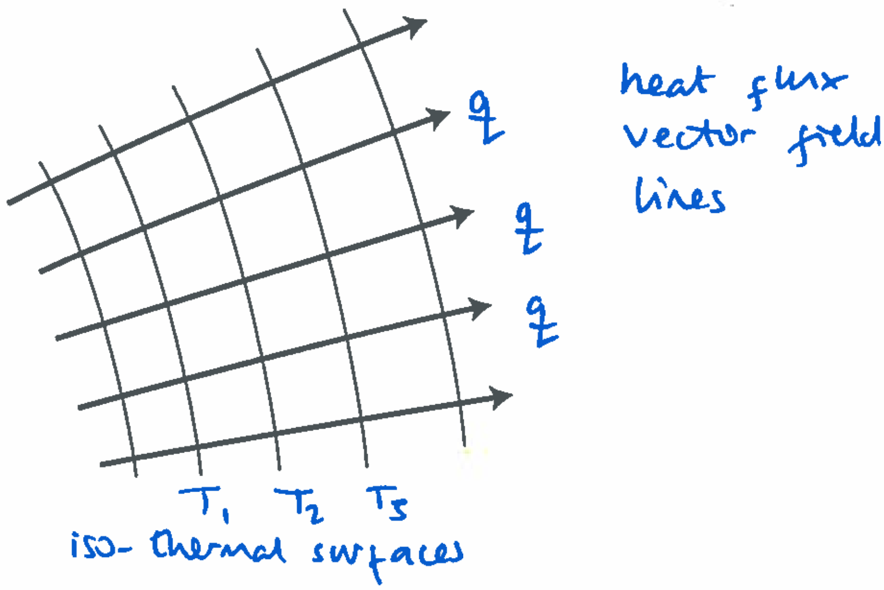
Similarly, in a three-dimensional problem, we would also have,

$$q_y = -\lambda \frac{\partial T}{\partial y} \qquad q_z = -\lambda \frac{\partial T}{\partial z}$$

We can express Fourier's law, concisely, as,

$$\underline{q} = -\lambda \nabla T$$

where **q** is the heat flux vector. We have derived this expression in Cartesian coordinates (where **q** has components q_x, q_y and q_z), but $\mathbf{q} = -\lambda \nabla T$ applies in any coordinate system. A common assumption is that λ is constant and so we may write $\mathbf{q} = \nabla(-\lambda T)$ and we see that $(-\lambda T)$ is the scalar potential.



Diffusion

Fick's law governs the diffusion in solids, liquids and gases. In one dimension, the mass transfer rate of the diffusing species across a plane of area A is given by $M_x = -DA\partial c/\partial x$ where c is the concentration of the species (mass per unit volume) and D is the diffusion coefficient. The diffusive mass flux is $m_x = M_x/A$ and,

$$m_x = -D\frac{\partial c}{\partial x}$$

governs our 1-D diffusion. In 3-D, Fick's law is captured by the vector equation,

$$\underline{m} = -D\nabla c$$

where **m** is the diffusive mass flux vector. Just as **q** is perpendicular to lines of constant T, we see that **m** must be perpendicular to lines of constant c.

Current flow

The current flowing in a conductor aligned with the x-axis obeys Ohm's Law, $I_x = \sigma A\partial V/\partial x$ where σ is the electrical conductivity, A is the cross-sectional area and V is the electric potential. The current per unit cross-sectional area is $j_x = I_x/A$ and this is called the current density (the terminology would be more consistent if j_x was known as the current flux). In 1-D, we have

$$j_x = -\sigma\frac{\partial V}{\partial x}$$

and the general, 3-D, expression is,

$$\underline{j} = -\sigma\nabla V$$

If σ is constant, we see that the scalar potential in this case is $(-\sigma V)$ and the current density vector is everywhere normal to surfaces of constant electric potential.

Example

The concentration of a species is axi-symmetric, $c = c(r)$, and is given by $c = c_0 - a\ln(r/r_0)$. Given that the diffusion coefficient is D (constant), find an expression for the diffusive mass flux in the radial direction and for the total diffusive mass flow rate crossing radius $r = R$.

$$\underline{m} = -D\nabla c$$

$$m_r = -D\frac{\partial c}{\partial r} = aD\frac{1}{r}$$

At $r = R$ $\quad M_r = 2\pi R a D\frac{1}{R} = 2\pi a D$ (per unit in z dir^n)

4.4 A return to the substantive derivative

In Lecture 1 we found the rate of change of temperature, as measured by a probe moving through a time-varying temperature field $T = T(x,y,z,t)$, was given by the total derivative,

$$\frac{dT}{dt} = \left(V_x\frac{\partial T}{\partial x} + V_y\frac{\partial T}{\partial y} + V_z\frac{\partial T}{\partial z}\right) + \frac{\partial T}{\partial t} \tag{4.6}$$

where V_x, V_y and V_z are the components of the velocity of the probe. We can use vector notation to write the first term on the right hand side as a scalar product,

$$\frac{dT}{dt} = (V_x\underline{i} + V_y\underline{j} + V_z\underline{k})\cdot\left(\underline{i}\frac{\partial T}{\partial x} + \underline{j}\frac{\partial T}{\partial y} + \underline{k}\frac{\partial T}{\partial z}\right) + \frac{\partial T}{\partial t}$$

which we now recognise as, $\quad \dfrac{dT}{dt} = \underline{V}\cdot\nabla T + \dfrac{\partial T}{\partial t}$

where $\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$ is the velocity of the probe. We can see that the combination $(\mathbf{V}\cdot\nabla)$ acts on the temperature field, T,

$$\mathbf{V}\cdot\nabla = V_x\frac{\partial}{\partial x} + V_y\frac{\partial}{\partial y} + V_z\frac{\partial}{\partial z} \quad , \tag{4.7}$$

and is a 'scalar operator' (due to the dot product) that can act on either a scalar field (as in the above example) or a vector field (as in $(\mathbf{V}\cdot\nabla)\mathbf{V}$ used in 1B Paper 4).

4.5 ∇ in non-Cartesian coordinate systems

So far, we have made use of the definition of the ∇ operator in Cartesian coordinates,

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \quad . \tag{4.8}$$

But we can use the result,

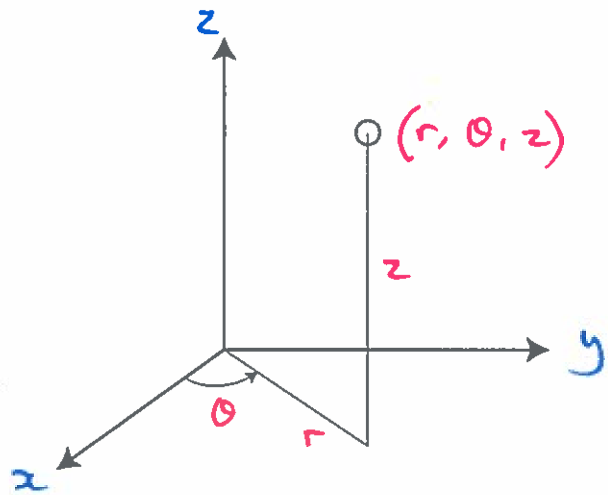
$$\delta f = \nabla f \cdot \delta \mathbf{r} \quad , \tag{4.9}$$

to define ∇ in other coordinate systems.

Cylindrical polar coordinates (r, θ, z)

A cylindrical polar coordinate system (r, θ, z) has base vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$. The small change in position vector as we move from (r, θ, z) to $(r + \delta r, \theta + \delta\theta, z + \delta z)$ is given by,

$$\delta \underline{r} = \delta r \, \underline{e}_r + r \, \delta\theta \, \underline{e}_\theta + \delta z \, \underline{e}_z$$



In order to satisfy $\delta f = \nabla f \cdot \delta \mathbf{r}$, we can see that,

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \quad . \tag{4.10}$$

As a check,

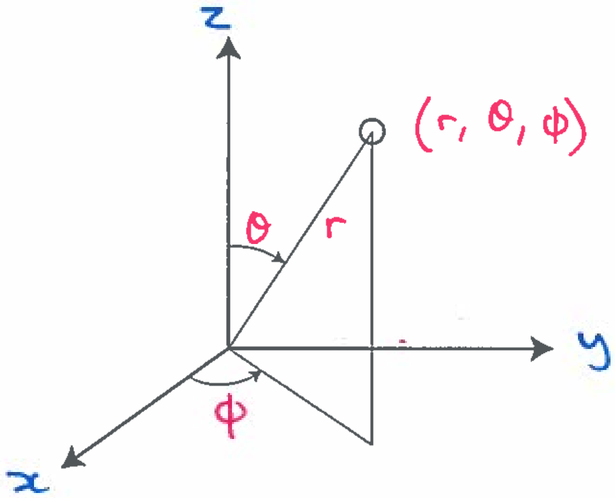
$$\nabla f \cdot \delta \mathbf{r} = \left(\mathbf{e}_r \frac{\partial f}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{e}_z \frac{\partial f}{\partial z}\right) \cdot (\delta r \, \mathbf{e}_r + r\delta\theta \, \mathbf{e}_\theta + \delta z \, \mathbf{e}_z) \tag{4.11}$$

$$= \frac{\partial f}{\partial r}\delta r + \frac{\partial f}{\partial \theta}\delta\theta + \frac{\partial f}{\partial z}\delta z = \delta f \tag{4.12}$$

Spherical polar coordinates (r, θ, ϕ)

We can follow the same procedure in spherical polar coordinates (r, θ, ϕ). The base vectors are now $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ and the small change in position vector as we move from (r, θ, ϕ) to $(r+\delta r, \theta+\delta\theta, \phi+\delta\phi)$ is given by,

$$\delta\mathbf{r} = \delta r \mathbf{e}_r + r\delta\theta\mathbf{e}_\theta + r\sin\theta\,\delta\phi\,\mathbf{e}_\phi \tag{4.13}$$



In order to satisfy $\delta f = \nabla f \cdot \delta\mathbf{r}$, we must have,

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \quad . \tag{4.14}$$

You can now do Examples Paper 1: Q8, 9, 10, 11 and 12