IB Paper 6: Signal and Data Analysis

Handout 5: Sampling Theory

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Sampling and Aliasing

- All of our work so far with Fourier series and Fourier transforms has worked with signals and functions in continuous time.
- Calculation of Fourier coefficients requires integrals over continuous time.
- This is fine when you consider special functions whose formula can be written down (sine, cosine, δ-function, etc.)

- However, in the real world, signals don't have a pre-specified formula - we just have to measure them.
- Nowadays, signals are measured in digital form on computers, which means discrete time sampling, or analogue to digital conversion.
- Can we still do signal analysis when continuous time signals have been sampled and stored in digital format?
- The theory of sampling and aliasing shows how to do this in a proper fashion.

Digital Sampling

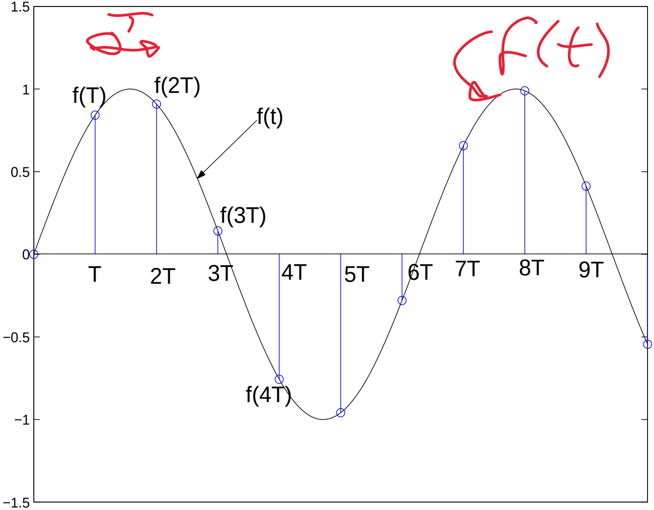


Figure 1: Digital sampling of a continuous waveform

Firstly define what is meant by Digital sampling:

- Suppose a continuous time signal is given by $f(t)$, $-\infty < t < +\infty$
- Choose a sampling interval T and read off the value of $f(t)$ at times:

$$-\infty, \ldots -2T, -T, 0, T, 2T, \ldots, +\infty$$

  i.e. at times nT, $n = -\infty, \ldots, -1, 0, 1, 2, \ldots, \infty$.

- The obtained values $f(nT)$ are the sampled version of $f(t)$.
- The practical procedure, known as analogue to digital conversion, is discussed further in the Communications course (P6 2nd half of Lent term).

The big question is: how should you choose T, the sampling interval?

- Too large a value of T will mean loss of detail from $f(t)$:

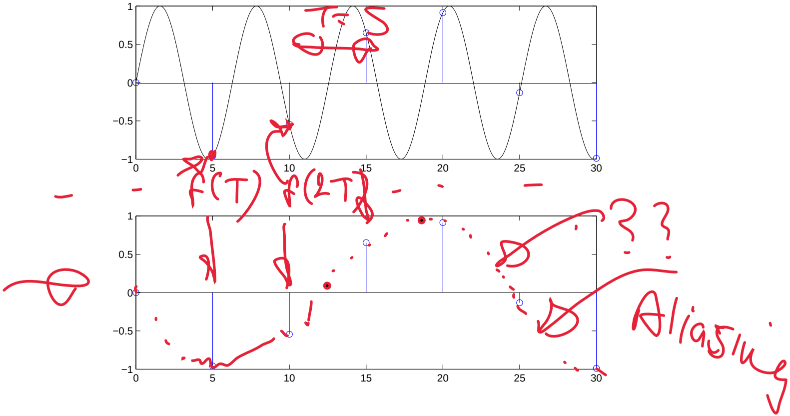


Figure 2: Sparse sampling of a continuous waveform

- Too small a value means unnecessary storage of over-detailed (redundant) data:

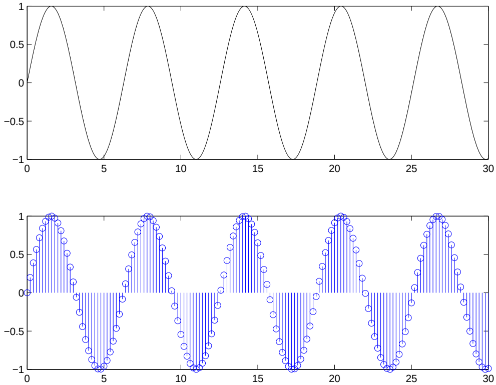


Figure 3: Dense sampling of a continuous waveform

T small

- We need the 'Goldilocks' principle!

  'Not too big; and not too small; but just right'

  But, seriously:

- The Sampling Theorem tells us the maximum value of T we can take and still perfectly reconstruct $f(t)$ from $f(nT)$ - a remarkable and perhaps not obvious result.

The Sampling Theorem

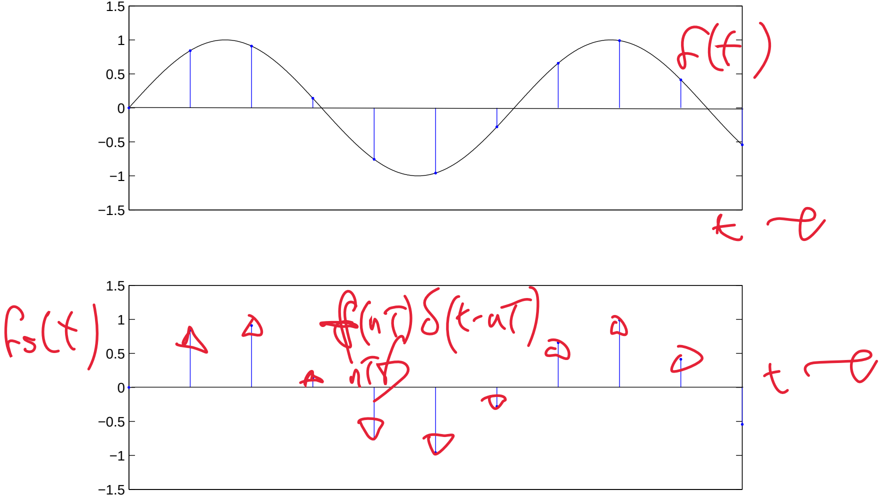


Figure 4: Representing the sampled data as a train of δ-functions

Firstly define a mathematical representation of the sampled signal using a train of δ-functions:
Do this by taking each sample $f(nT)$ and multiplying it by a δ-function centered at $t = nT$:

$$f(nT)\delta(t - nT)$$

But this equals

$$f(t)\delta(t - nT) \quad [\text{since } \delta(t - nT) \text{ is zero except at } t = nT]$$

Then sum all such samples to give the whole sampled signal as:

$$\begin{aligned} f_s(t) &= \sum_{n=-\infty}^{\infty} f(t)\delta(t - nT) \\ &= f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= f(t)\, \delta_p(t) \end{aligned}$$

$f_s(t)$ is then a continuous time signal which contains only the sampled data information $f(nT)$ - it is zero elsewhere.

Note: think of this as an conceptual version of the sampled signal – in no way are we implying that there are infinities in a real sampled signal.

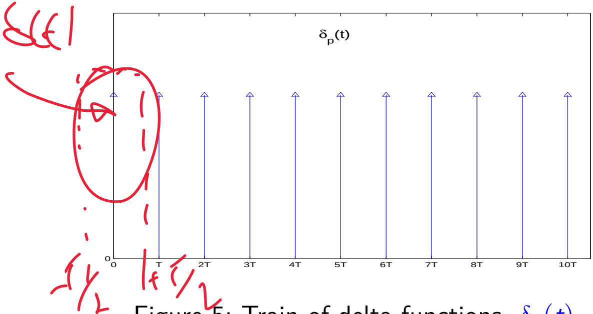


Figure 5: Train of delta functions, $\delta_p(t)$

$\delta_p(t)$ is a periodic function and can therefore be represented as a Fourier series:

$$\delta_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{1}$$

where $\omega_0 = 2\pi/T$ and the c_n are given by

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta_p(t) e^{-jn\omega_0 t} dt = \frac{1}{T} \quad \text{for all } n \tag{2}$$

$$> \frac{1}{T}\int \delta(t) e^{-jn\omega_0 t} dt$$

(This is also a question on example sheet 6/6). We therefore have an alternative formula for the sampled signal:

$$f_s(t) = f(t)\,\delta_p(t) = f(t)\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \tag{3}$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} f(t)\, e^{jn\omega_0 t} \tag{4}$$

It turns out that this formula is much easier to understand in the frequency domain. We will therefore determine the Fourier Transform of $f_s(t)$.

Looking at each term of the summation, we have from the frequency shift theorem: $f(t) \xrightarrow{FT} F(\omega)$

$$f(t)e^{jn\omega_0 t} \stackrel{FT}{\longleftrightarrow} F(\omega - n\omega_0) \tag{5}$$

Hence the Fourier transform of the sum is (by linearity):

$$\boxed{F_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_0)} \tag{6}$$

i.e. **The Fourier transform of the sampled signal is simply 1/T times the Fourier transform of the continuous signal repeated every integer multiple of the sampling frequency and summed together.**

$F(\omega)$

$F(\omega - n\omega_0)$

(sinc)

$f(t)$:

$F(\omega)$

ω

Example:

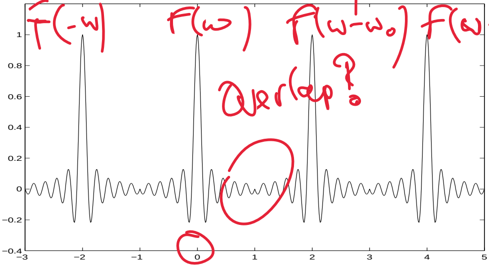


Figure 6: Spectrum repeated every integer multiple of the sampling frequency

We will see shortly that this fundamental result is all we need to answer the original question:

> what is the optimal sampling frequency $1/T$ for perfect reconstruction of the original signal $f(t)$ from its samples $f(nT)$?

Discrete-time Fourier Transform

As well as the above analysis, note that the Fourier transform of the sampled signal can also be written as follows:

$$F_s(\omega) = \int_{-\infty}^{\infty} f_s(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} f(t)\delta(t - nT) \right\} e^{-j\omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \delta(t - nT) dt \right\}$$

$$= \sum_{n=-\infty}^{\infty} f(nT) e^{-jn\omega T}$$

[Using sifting property of δ]

This alternative formula is known as the **Discrete-Time Fourier Transform** or DTFT. The DTFT shows how to calculate the frequency content of the ideal sampled signal directly from its sampled values $f(nT)$.

Nyquist Frequency and Reconstruction

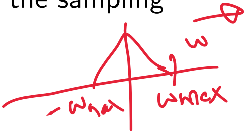
We have seen that the spectrum (=Fourier Transform) of a sampled signal consists of many repetitions of the spectrum of the original signal $f(t)$:

$$F_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_0) \tag{7}$$

Now, see Figure 7, which shows the sampled spectrum for a signal with bandwidth ω_{max} and for 3 possible values of the sampling frequency $\omega_0 = 2\pi/T$.

Say, $f(\omega)$)

real-valued



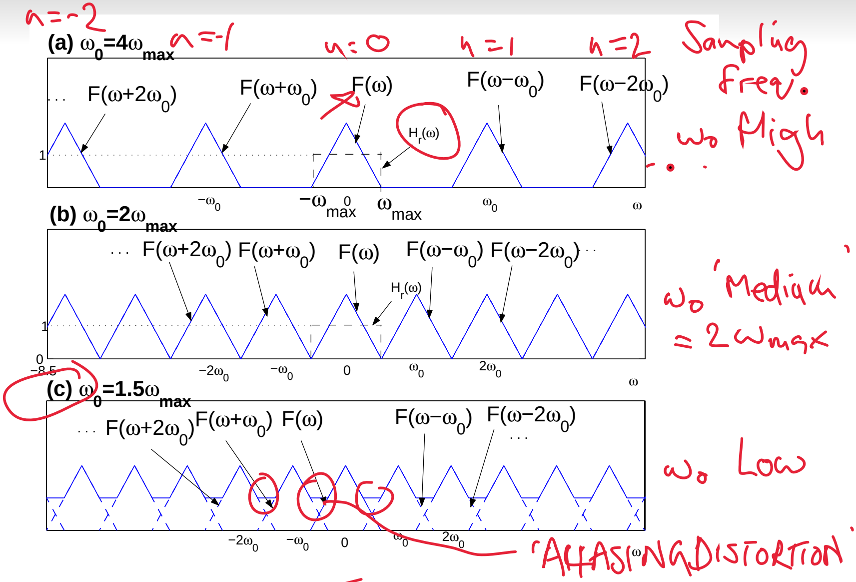


Figure 7: The sampled spectrum $|F_s(\omega)|$ for various values of sampling frequency

Now, attempt to extract just $F(\omega)$ from the sampled spectrum $F_s(\omega)$

- Apply a filter with frequency response $H_r(\omega)$ (shown dotted) to the sampled signal.
- Since $Y(\omega) = H_r(\omega)F_s(\omega)$ we can see from the diagram that for cases (a) and (b), $Y(\omega) = F(\omega)$ and we have reconstructed the original signal spectrum perfectly. In the third case (c), $F(\omega)$ is not properly reconstructed.
- In general it is possible to reconstruct the original spectrum only when there is no overlap between the periodic repetitions of $F(\omega)$.

The Nyquist sampling theorem can now be stated as:

If a signal $f(t)$ has a maximum frequency content (or bandwidth) ω_{max}, then it is possible to reconstruct $f(t)$ perfectly from its sampled version $f_s(t)$ provided the sampling frequency is at least

$$\omega_0 = 2\omega_{max}, \text{ the 'Nyquist frequency'}$$

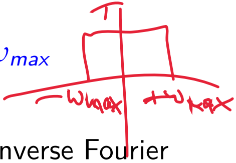
- The minimum sampling frequency of $2 \times \omega_{max}$ is known as the Nyquist Frequency, ω_{Nyq}.
- The repetitions of $F(\omega)$ in the sampled spectrum are known as aliasing
- When a signal is sampled at a rate less than ω_{Nyq} the distortion due to the overlapping spectra is called aliasing distortion

Case c)

Ideal Reconstruction filter

- The ideal filter frequency response for perfect reconstruction is the rectangle pulse function:

$$H_r(\omega) = \begin{cases} T, & -\omega_{max} < \omega < +\omega_{max} \\ 0 & \text{otherwise} \end{cases}$$



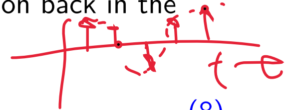
- The impulse response of the filter is then the inverse Fourier transform of $H_r(\omega)$. We know that the inverse Fourier transform of a rectangular pulse is a sinc function, i.e.

$$h_r(t) = \frac{\omega_{max} T}{\pi} \text{sinc}(\omega_{max} t)$$

Or, if we are sampling exactly at the Nyquist frequency, $\omega_{max} = \omega_0/2$, and the above becomes

$$h_r(t) = \frac{\omega_0 T}{2\pi}\mathrm{sinc}(\omega_0 t/2) = \mathrm{sinc}(\omega_0 t/2)$$

Since multiplication in the frequency domain implies convolution in the time domain, the equivalent recovery operation back in the time domain becomes:



$$f(t) = \mathrm{sinc}(\omega_0 t/2) * f_s(t) \tag{8}$$

By substituting $f_s(t) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT)$ and performing the convolution, equation (8) becomes

$$f(t) = \int_{-\infty}^{\infty} f_s(\tau)\mathrm{sinc}(\omega_0(t-\tau)/2)d\tau$$

$$= \int_{-\infty}^{\infty}\left\{\sum_{n=-\infty}^{\infty} f(nT)\delta(\tau-nT)\right\}\mathrm{sinc}(\omega_0(t-\tau)/2)d\tau$$

Swapping order of integral and summation gives

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \left\{ \int_{-\infty}^{\infty} \delta(\tau - nT) \text{sinc}(\omega_0 (t - \tau)/2) d\tau \right\} \quad (9)$$

which can be evaluated using the sifting property as

$$f(t) = \sum_{-\infty}^{\infty} f(nT) \text{sinc} \left[\frac{\pi}{T} (t - nT) \right] \quad (10)$$

This can be viewed as an *exact* interpolation formula for determining $f(t)$ from its samples $f(nT)$.

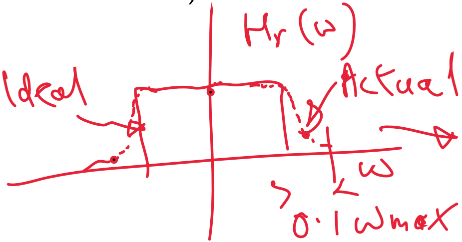
Practical considerations

This is all very idealised. How would this be modified in practice?

- Must determine the maximum frequency component ω_{max} present in a signal before sampling.
- In order to eliminate the aliasing effects of high frequency noise or unwanted high signal frequencies, first filter the data with a low-pass filter having frequency response $H_r(\omega)\,/\,T$, i.e. just a unity gain lowpass filter - call this filtered signal $f(t)$ and proceed with sampling.
- Then perform sampling at the Nyquist rate $\omega_0 = \omega_{Nyq} = 2 \times \omega_{max}$ to give digital samples $f(nT)$.
- Reconstruct the signal by passing the sampled signal through the same filter $H_r(\omega)$.

So $\omega_0 = 2 \times 1.1\, \omega_{max}$

- In practice though we cannot exactly implement the ideal filter $H_r(\omega)$. Must therefore allow say 10% extra signal bandwidth for the transition band of the filters (see Paper 6 Communications course)



Example: A music signal has bandwidth 20kHz.

a) Determine the sampling period for this signal, assuming ideal filter responses.

b) Determine a suitable sampling rate assuming a realistically achievable filter response.

$$\omega_0 = 2 \times 2\pi . 20 \times 10^3$$

$$T = \frac{2\pi}{\omega_0} = \frac{1}{40{,}000} \, s$$

$$\omega_0 = 1.1 \times 2\pi . 20 \times 10^3 \rightarrow 44 \, kHz$$

S. Godsill (2023),