2P7: Probability & Statistics

Continuous Random Variables

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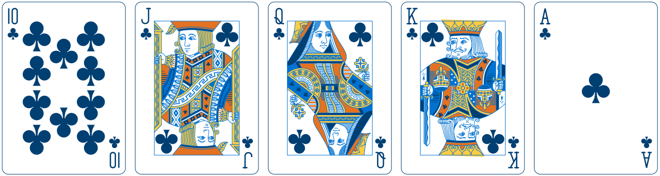
Lent 2024



the *royal flush*, the best possible hand in poker, has a probability 0.000154%

1. Probability Fundamentals
2. Discrete Probability Distributions
3. Continuous Random Variables
4. Manipulating and Combining Distributions
5. Decision, Estimation and Hypothesis Testing

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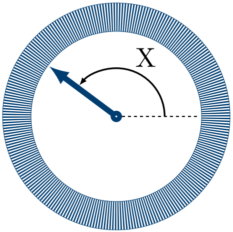
In the last lectures:

- We have seen how discrete random variables are defined and described by their probability mass function
- We have given important examples of probability mass functions:
  - Bernoulli
  - Geometric
  - Binomial
  - Poisson
- We have shown how to characterise probability mass functions via expectation, variance and other moments.

In this lecture, we will consider random variables with a continuous support, which are described by their probability density function, and give a few important examples.

- We have seen random variables assign a number to each outcome of the sample space.
- Discrete random variables have a discrete set of possible values.
- Continuous random variables will have a continuous set of values.
- The support can be finite (for example: $[0, 1]$, $[a, b]$) or infinite (for example: $[0, +\infty)$, $(-\infty, +\infty)$) in extent.

Example: spinner wheel



- The sample space is a continuous set of outcomes (orientations of the arrow)
- The angle with the horizontal is a continuous random variable X on a finite set $\mathbb{X} = [0, 2\pi)$.
- $\mathbb{P}[2.68 < \mathrm{X} \leq 2.69] = \frac{0.01}{2\pi}$
- $\mathbb{P}[\mathrm{X} = 2.68983285921430891716\ldots] = 0$

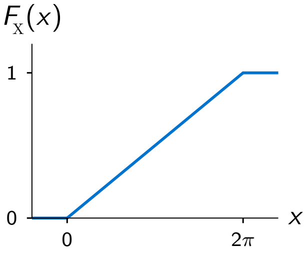
- In general, $\mathbb{P}[\mathrm{X} = a] = 0$ for continuous random variables.
- We can still consider events corresponding to intervals, $\mathbb{P}[a < \mathrm{X} \leq b]$, and we have seen

$$\mathbb{P}[a < \mathrm{X} \leq b] = F_{\mathrm{X}}(b) - F_{\mathrm{X}}(a)$$

  where $F_{\mathrm{X}}(x) = \mathbb{P}[\mathrm{X} \leq x]$ is the cumulative distribution function (CDF) of X.
- $F_{\mathrm{X}}(x)$ is an "informative" probability, even for a continuous random variable.

Example: spinner wheel

$$F_{\mathrm{X}}(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \dfrac{x}{2\pi} & \text{if } 0 \leq x < 2\pi, \\ 1 & \text{if } 2\pi \leq x. \end{cases}$$



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- Formally, we define the probability density function (PDF) as

$$f_{\mathrm{X}}(x) = \frac{\mathrm{d}F_{\mathrm{X}}(x)}{\mathrm{d}x}$$

- Interpretation:

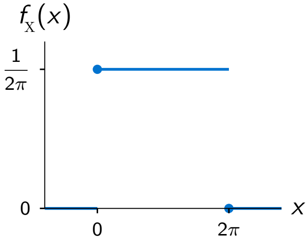
$$\begin{aligned} f_{\mathrm{X}}(x) &= \lim_{\mathrm{d}x \to 0} \frac{F_{\mathrm{X}}(x+\mathrm{d}x) - F_{\mathrm{X}}(x)}{\mathrm{d}x} \\ &= \lim_{\mathrm{d}x \to 0} \frac{\mathbb{P}[x < \mathrm{X} \le x + \mathrm{d}x]}{\mathrm{d}x} \quad \Leftrightarrow \quad f_{\mathrm{X}}(x)\mathrm{d}x \approx \mathbb{P}[x < \mathrm{X} \le x + \mathrm{d}x] \end{aligned}$$

  So $f_{\mathrm{X}}(x)\mathrm{d}x$ is the probability of X falling within the infinitesimal interval $(x, x + dx]$.

Example: spinner wheel

$$f_{\mathrm{X}}(x) = \begin{cases} \dfrac{1}{2\pi} & \text{if } x \in [0, 2\pi), \\ 0 & \text{otherwise} \end{cases}$$

gives a "good picture" of the uniform distribution of X.



Note: we can extend the support to $\mathbb{R}$ by setting $f_{\mathrm{X}}(x) = 0$ for $x \notin \mathbb{X}$.

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- Reminder on the properties of F_{X}:
  (1) F_{X} is non-decreasing: $F_{\mathrm{X}}(a) \leq F_{\mathrm{X}}(b)$ if $a \leq b$
  (2) $\lim_{x \to -\infty} F_{\mathrm{X}}(x) = 0$ and $\lim_{x \to \infty} F_{\mathrm{X}}(x) = 1$
- From (1), the probability density function is positive:

$$f_{\mathrm{X}}(x) \geq 0 \qquad \text{for all } x \in \mathbb{R}$$

- From $f_{\mathrm{X}}(x) = F'_{\mathrm{X}}(x)$:

$$\int_a^b f_{\mathrm{X}}(x)\mathrm{d}x = F_{\mathrm{X}}(b) - F_{\mathrm{X}}(a) = \mathbb{P}[a < \mathrm{X} \leq b]$$

- From (2), the probability density function is *normalised*:

$$\int_{-\infty}^{+\infty} f_{\mathrm{X}}(x)\mathrm{d}x = 1$$

- In general, the $\sum$ seen with discrete mass distributions become $\int$ with density functions.
- Note that f_{X} is not a probability. It has the dimension of X^{-1}.

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- For two continuous random variables X and Y, we defined the joint probability density function $f_{\mathrm{XY}}(x, y)$ from the joint CDF $F_{\mathrm{XY}}(x, y) = \mathbb{P}[\mathrm{X} \leq x \cap \mathrm{Y} \leq y]$:

$$f_{\mathrm{XY}}(x, y) = \frac{\partial^2 F_{\mathrm{XY}}(x, y)}{\partial x \partial y}$$

- The sum rule becomes an integral rule and marginalisation is stated as

$$\int_{-\infty}^{+\infty} f_{\mathrm{XY}}(x, y) \mathrm{d}y = f_{\mathrm{X}}(x)$$

- Conditional probability density function[1] and product rule:

$$f_{\mathrm{X|Y}}(x|y) = \frac{f_{\mathrm{XY}}(x, y)}{f_{\mathrm{Y}}(y)} \quad \Rightarrow \quad f_{\mathrm{XY}}(x, y) = f_{\mathrm{X|Y}}(x|y) f_{\mathrm{Y}}(y)$$

[1] We define the conditional PDF $f_{\mathrm{X|Y}}(x|y) = \frac{\partial}{\partial x} F_{\mathrm{X|Y}=y}(x|\mathrm{Y}=y)$ with:

$$F_{\mathrm{X|Y}=y}(x|\mathrm{Y}=y) = \lim_{\mathrm{d}y \to 0} \frac{\mathbb{P}[(\mathrm{X} \leq x) \cap (y < \mathrm{Y} \leq y + \mathrm{d}y)]}{\mathbb{P}[y < \mathrm{Y} \leq y + \mathrm{d}y]} = \lim_{\mathrm{d}y \to 0} \frac{F_{\mathrm{XY}}(x, y + \mathrm{d}y) - F_{\mathrm{XY}}(x, y)}{F_{\mathrm{Y}}(y + \mathrm{d}y) - F_{\mathrm{Y}}(y)} = \frac{1}{f_{\mathrm{Y}}(y)} \frac{\partial F_{\mathrm{XY}}(x, y)}{\partial y}$$

$\Rightarrow f_{\mathrm{X|Y}}(x|y) = \frac{f_{\mathrm{XY}}(x,y)}{f_{\mathrm{Y}}(y)}$. This is conditional to "$\mathrm{Y} = y$" exactly.

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- Law of total probability

$$f_{\mathrm{X}}(x) = \int_{-\infty}^{+\infty} f_{\mathrm{X|Y}}(x|y) f_{\mathrm{Y}}(y)\mathrm{d}y$$

- Bayes' rule

$$f_{\mathrm{Y|X}}(y|x) = \frac{f_{\mathrm{X|Y}}(x|y) f_{\mathrm{Y}}(y)}{\int_{-\infty}^{+\infty} f_{\mathrm{X|Y}}(x|y) f_{\mathrm{Y}}(y)\mathrm{d}y}$$

- Independence

$$\begin{aligned} \mathrm{X} \text{ and } \mathrm{Y} \text{ independent} &\Leftrightarrow f_{\mathrm{XY}}(x,y) = f_{\mathrm{X}}(x) f_{\mathrm{Y}}(y) \\ &\Leftrightarrow f_{\mathrm{X|Y}}(x|y) = f_{\mathrm{X}}(x) \qquad \text{for all } x, y \in \mathbb{R}\times\mathbb{R} \\ &\Leftrightarrow f_{\mathrm{Y|X}}(y|x) = f_{\mathrm{Y}}(y) \end{aligned}$$

- The probability density function can be used to compute expectations:

$$\mathbb{E}[g(\mathrm{X})] = \int_{-\infty}^{+\infty} g(x) f_{\mathrm{X}}(x) \mathrm{d}x \quad \left(\mathbb{E}[g(\mathrm{X},\mathrm{Y})] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y) f_{\mathrm{XY}}(x,y) \mathrm{d}x \mathrm{d}y \right)$$

- In particular, we call

$$\begin{aligned} \mathbb{E}[\mathrm{X}^n] &\quad \text{the } n^{\text{th}} \text{ moment} \\ \mathbb{E}[(\mathrm{X} - \mathbb{E}[\mathrm{X}])^n] &\quad \text{the } n^{\text{th}} \text{ central moment} \end{aligned}$$

  The following moments are important:
  - The mean (or *first moment*)

$$\mathbb{E}[\mathrm{X}] = \int_{-\infty}^{+\infty} x \, f_{\mathrm{X}}(x) \mathrm{d}x$$

  - The variance (or *second central moment*)

$$\mathrm{Var}[\mathrm{X}] = \mathbb{E}[(\mathrm{X} - \mathbb{E}[\mathrm{X}])^2] = \mathbb{E}[\mathrm{X}^2] - \mathbb{E}[\mathrm{X}]^2$$

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There are many ways to characterise the distribution of a random variable X. For example:

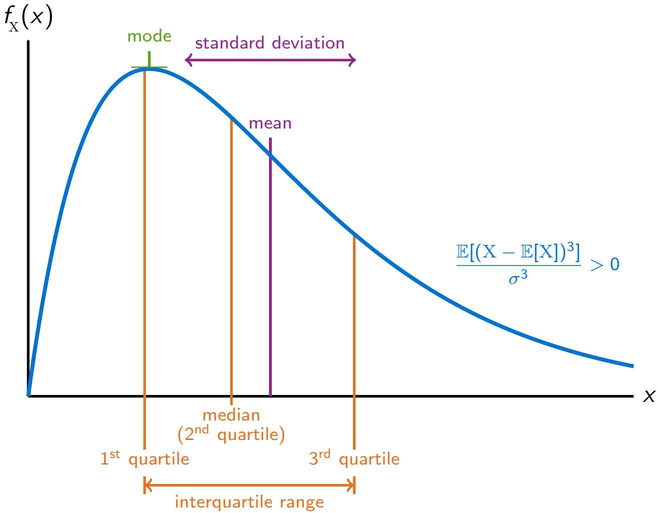
- The standard deviation is $\sigma = \sqrt{\mathrm{Var}[\mathrm{X}]}$
- The mode is the value of x at which $f_{\mathrm{X}}(x)$ is maximum
- The median is the value $Q_{1/2}$ of x at which $F_{\mathrm{X}}(x) = \frac{1}{2}$ (split area under the PDF in two equal parts):

$$\int_{-\infty}^{\text{median}} f_{\mathrm{X}}(x)\mathrm{d}x = \frac{1}{2} = \int_{\text{median}}^{+\infty} f_{\mathrm{X}}(x)\mathrm{d}x$$

- The 1^{st} and 3^{rd} quartiles are the values $Q_{1/4}$ and $Q_{3/4}$ of x at which $F_{\mathrm{X}}(x) = \frac{1}{4}$ and $\frac{3}{4}$, respectively
- The interquartile range: $Q_{3/4} - Q_{1/4}$
- The skewness $\mathbb{E}[(\mathrm{X} - \mathbb{E}[\mathrm{X}])^3]/\sigma^3$. If the skewness is positive, the distribution is *skewed to the right* (the "tail" of the distribution is longer to the right)

Probability Density Function

Characteristics of a PDF



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What is the time/distance between two successive successes?

- Consider $\mathrm{X}_t \sim \mathrm{Pois}(\lambda t)$ the number of successes (or arrivals) over a time interval t with an average rate of arrivals λ.
- We wish to derive the density $f_{\mathrm{T}}(t)$ of the time intervals T between arrivals.
- The probability $f_{\mathrm{T}}(t)\mathrm{d}t = \mathbb{P}[t < \mathrm{T} \leq t + \mathrm{d}t]$.
- The event $\{t < \mathrm{T} \leq t + \mathrm{d}t\}$ means both:
  - No arrivals happen between $[0, t]$: $\{\mathrm{X}_t = 0\}$
  - Exactly one arrival happens between $[t, t + \mathrm{d}t]$: $\{\mathrm{X}_{\mathrm{d}t} = 1\}$
- So

$$
\begin{aligned}
f_{\mathrm{T}}(t)\mathrm{d}t &= \mathbb{P}[\mathrm{X}_t = 0 \cap \mathrm{X}_{\mathrm{d}t} = 1] = P_{\mathrm{X}_t}(0) \times P_{\mathrm{X}_{\mathrm{d}t}}(1) \\
&= \frac{(\lambda t)^0 e^{-\lambda t}}{0!} \times \frac{(\lambda \mathrm{d}t)^1 e^{-\lambda \mathrm{d}t}}{1!} = \lambda e^{-\lambda t} e^{-\lambda \mathrm{d}t} \mathrm{d}t
\end{aligned}
$$

after simplification and taking $\mathrm{d}t \to 0$, $f_{\mathrm{T}}(t) = \lambda e^{-\lambda t}$

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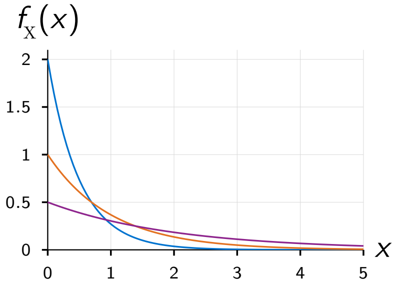
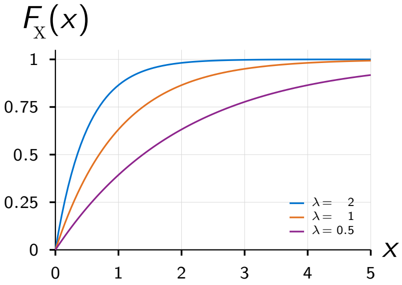
A random variable X is said to have an Exponential distribution with parameter $\lambda > 0$ if:

$$\mathrm{X} \sim \mathrm{Exp}(\lambda) \quad \Leftrightarrow \quad f_{\mathrm{X}}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The support of X, $\mathbb{X} = [0, \infty)$, is continuous infinite.





Verify that $\int_{-\infty}^{+\infty} f_{\mathrm{X}}(x)\mathrm{d}x = 1$.

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- Expectation $\mathbb{E}[\mathrm{X}] = \frac{1}{\lambda}$ [DB]

  $\mathbb{E}[\mathrm{X}] = \int_0^\infty x\lambda e^{-\lambda x}\mathrm{d}x = \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\mathrm{d}x = \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^\infty$ □

- Variance $\mathrm{Var}[\mathrm{X}] = \frac{1}{\lambda^2}$ [DB]

  $\mathbb{E}[\mathrm{X}^2] = \int_0^\infty x^2\lambda e^{-\lambda x}\mathrm{d}x = \left[-x^2 e^{-\lambda x}\right]_0^\infty + \int_0^\infty 2xe^{-\lambda x}\mathrm{d}x = \frac{2}{\lambda^2}$ □

- Mode $x_{\mathrm{max}} = 0$

  Obvious from the curve... □

- Median $Q_{1/2} = \frac{\ln 2}{\lambda}$

  See next □

- Quartile $Q_p = -\frac{\ln(1-p)}{\lambda}$, $Q_{1/4} = \frac{\ln \frac{4}{3}}{\lambda}$, $Q_{3/4} = \frac{\ln 4}{\lambda}$

  $F_{\mathrm{X}}(x) = \int_0^x f_{\mathrm{X}}(\xi)\mathrm{d}\xi = 1 - e^{-\lambda x}$ so $F_{\mathrm{X}}(Q_p) = p \Leftrightarrow 1 - e^{-\lambda Q_p} = p$ □

- Skewness $2 > 0$ (strongly right-tailed)

  Tedious but not difficult □

The Gaussian Density

Definition

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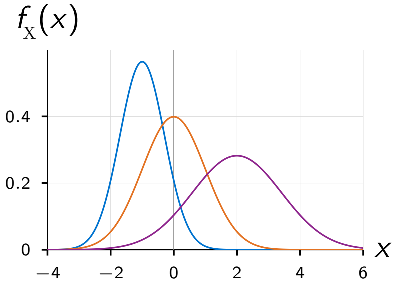
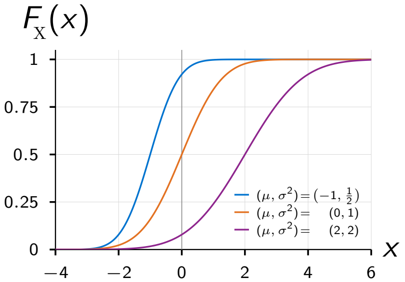
A random variable X is said to have a Gaussian (or Normal) distribution with mean μ and variance σ^2 if:

$$\mathrm{X} \sim \mathcal{N}(\mu, \sigma^2) \;\Leftrightarrow\; f_{\mathrm{X}}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for all } x \in \mathbb{R}$$

The support of X, $\mathbb{X} = \mathbb{R}$, is continuous infinite.





Verify that $\int_{-\infty}^{+\infty} f_{\mathrm{X}}(x)\mathrm{d}x = 1$.

Hint: calculate $\left(\int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x\right)^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y$ in cylindrical coordinates

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- $\mathcal{N}(0,1)$ is called the standard Gaussian distribution. We will show in the next lecture that:

$$\mathrm{Y} \sim \mathcal{N}(0,1) \quad \Leftrightarrow \quad \mathrm{X} = \mu + \sigma \mathrm{Y} \sim \mathcal{N}(\mu, \sigma^2)$$

- The cumulative distribution function of $\mathrm{Y} \sim \mathcal{N}(0,1)$ is:

$$F_{\mathrm{Y}}(y) = \Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{\xi^2}{2}} \mathrm{d}\xi$$

  $\Phi(y)$ is tabulated p.29 of the Maths Databook. By symmetry, you can verify $\Phi(-y) = 1 - \Phi(y)$ and $\Phi(0) = \frac{1}{2}$.

- The CDF of $\mathrm{X} \sim \mathcal{N}(\mu, \sigma^2)$ is $\Phi\left(\frac{x-\mu}{\sigma}\right)$.

- Most computing environments (Python, MATLAB...) have an "error function" called `erf`. Be *cautious* that

$$\Phi(y) = \frac{1}{2}\left[1 + \mathtt{erf}\left(\frac{y}{\sqrt{2}}\right)\right]$$

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In the following, we write $\mathrm{X} = \mu + \sigma \mathrm{Y}$ with $\mathrm{Y} \sim \mathcal{N}(0, 1)$

- Expectation $\mathbb{E}[\mathrm{X}] = \mu$ [DB]

  $\mathbb{E}[\mathrm{Y}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y\, e^{-y^2} \mathrm{d}y = 0$ (integrand is odd), $\mathbb{E}[\mathrm{X}] = \sigma \mathbb{E}[\mathrm{Y}] + \mu$ □

- Variance $\mathrm{Var}[\mathrm{X}] = \sigma^2$ [DB]

  $\mathbb{E}[\mathrm{Y}^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y^2\, e^{-y^2} \mathrm{d}y = \frac{1}{\sqrt{2\pi}} \left(\left[-\frac{y}{2}\, e^{-y^2} \right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{-y^2} \mathrm{d}y \right) = 1,$

  $\mathbb{E}[\mathrm{X}^2] = \sigma^2 \mathbb{E}[\mathrm{Y}^2] + 2\sigma\mu \mathbb{E}[\mathrm{Y}] + \mu^2$ □

- Mode $x_{\mathrm{max}} = \mu$

  Obvious from the curve... □

- Median $Q_{1/2} = \mu$

  See next □

- Quartile $Q_p = \mu + \sigma \Phi^{-1}(p)$

  Quartile of Y is $\Phi^{-1}(p)$ □

  $\Phi^{-1}(1/2) = 0$ and $\Phi^{-1}(3/4) = -\Phi^{-1}(1/4) \approx 0.6745$

- Skewness 0

  By symmetry □

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- **Confidence interval**: $\mathbb{P}[|\mathrm{X} - \mu| \leq m\sigma] = 2\Phi(m) - 1$

$$\begin{aligned} \mathbb{P}[|\mathrm{X} - \mu| \leq m\sigma] &= \mathbb{P}[|\mathrm{Y}| \leq m] \\ &= \mathbb{P}[-m \leq \mathrm{Y} \leq m] \\ &= \Phi(m) - \Phi(-m) \\ &= 2\Phi(m) - 1 \end{aligned} \qquad \square$$

How likely is X within m standard deviations of the mean?

- We have $2\Phi(1) - 1 \approx 68\%$ confidence that $|\mathrm{X} - \mu| \leq \sigma$
- We have $2\Phi(2) - 1 \approx 95\%$ confidence that $|\mathrm{X} - \mu| \leq 2\sigma$

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- Suppose we observe that k out of n Bernoulli trials are successes. What is the probability density of the Bernoulli parameter p given this observation?
- From the Binomial distribution, $P_{\mathrm{k|p}}(k|p) = {}^n\mathrm{C}_k\, p^k(1-p)^{n-k}$
- Using Bayes rule:

$$f_{\mathrm{p|k}}(p|k) = \frac{P_{\mathrm{k|p}}(k|p) f_{\mathrm{p}}(p)}{\int_0^1 P_{\mathrm{k|p}}(k|p) f_{\mathrm{p}}(p) \mathrm{d}p}$$

- We assume that prior to any observation, all values of $\mathrm{p} \in [0,1]$ are believed to be equally likely, $f_{\mathrm{p}}(p) = 1$
- After some calculations we find

$$f_{\mathrm{p|k}}(p|k) = \frac{(n+1)!}{k!(n-k)!} p^k(1-p)^{n-k}$$

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A random variable X is said to have an Beta distribution with shape parameter $\alpha > 0$ and $\beta > 0$ if:

$$\mathrm{X} \sim \mathrm{Beta}(\alpha, \beta) \;\Leftrightarrow\; f_{\mathrm{X}}(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

where the Gamma function is defined $\Gamma(a) = \displaystyle\int_0^\infty \xi^{a-1} e^{-\xi} \mathrm{d}\xi$.

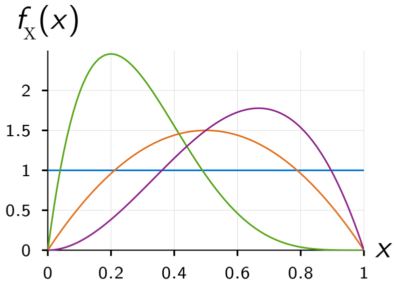
The support of X, $\mathbb{X} = [0, 1]$, is continuous finite.

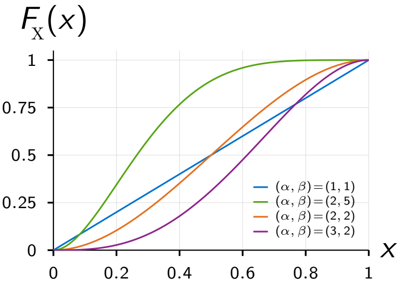
- The Gamma function is a generalisation of the factorial to non-integers
- It has the property $\Gamma(a) = (a-1)!$ when a is an integer.[2]

[2] From the previous slide, verify $f_{\mathrm{p|k}} = \mathrm{Beta}(\alpha, \beta)$ the probability density of p after the observation of $k = \alpha - 1$ successes and $n - k = \beta - 1$ fails.

The Beta Density

Properties of $X \sim \mathrm{Beta}(\alpha, \beta)$





- Expectation $\mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$ [DB]
- Variance $\mathrm{Var}[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ [DB]

No need to know this (but here for completeness)

- Mode $x_{\max} = \frac{\alpha-1}{\alpha+\beta-2}$ for $\alpha, \beta > 1$
- Median no closed-form expression...
- Quartile no closed-form expression...
- Skewness $\frac{2(\beta-\alpha)\sqrt{\alpha+\beta+1}}{(\alpha+\beta+2)\sqrt{\alpha\beta}}$ (tail's side depends on the sign of $\beta - \alpha$)

Additional Remarks

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Two additional remarks:

- It is possible to define a probability density function for a discrete random variable using the delta function!
  Consider a discrete random variable X with *probability mass function* P_{X} and support $\mathbb{X}$, then:

  $$f_{\mathrm{X}}(x) = \sum_{k \in \mathbb{X}} P_{\mathrm{X}}(k)\delta(x-k)$$

  is the *probability density function* of X.

- It is possible to define conditional expectations:

  $$\mathbb{E}[\mathrm{X}|\mathrm{Y}=y] = \int_{-\infty}^{+\infty} x\, f_{\mathrm{X}|\mathrm{Y}}(x|y)\mathrm{d}x \quad \text{(it is a function of } y\text{)}$$

You can attempt all problems in Examples Paper 5