Lecture 7

Line Integrals

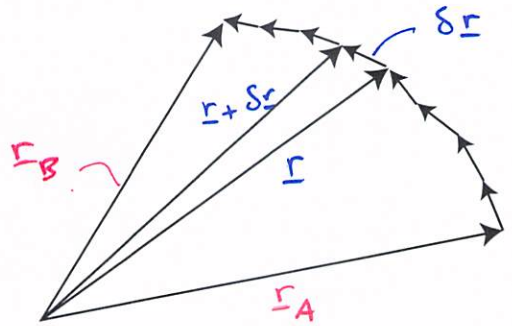
7.1 Line integration

In Lecture 2, we looked at integrals of scalar functions between limits specified by the independent variables, e.g $\int_{x_1}^{x_2} \phi(x)\, dx$. We now extend this to integration in 3-D along a *specified line in space*.

When we perform the familiar integral,

$$\int \phi(x)\, dx \quad , \tag{7.1}$$

we sum up the contributions of the product of δx elements (along the x-axis) multiplied by the 'weighting function' (the integrand), ϕ, evaluated at the same x.



In a line integral, the element is part of a specified line in 3-D space. The line element can be a scalar, δs measured a long the curve, or a vector $\delta \mathbf{r}$. The integrand can also be a scalar or a vector. For example,

$\int \rho(s)\, ds$	scalar	Total mass of a wire with mass per unit length, ρ
$\int q\, \underline{E}(s)\, ds$	vector	Total force on a wire, with net charge q per unit length, due to an electic field $\mathbf{E}$
$\int \underline{E} \cdot d\underline{r}$	scalar	Work done by a particle experiencing a force $\mathbf{F}$ as it moves along the line
$\int J\, (d\underline{r} \times \underline{B})$	vector	Total force experienced by a wire carrying a uniform current J in a magnetic field, $\mathbf{B}$

In general, the result of the line integral depends on the path taken between the start and end of the the line.

7.2 Line integrals of the form $\int \mathbf{F} \cdot d\mathbf{r}$

Integrals of the form $\int \mathbf{F} \cdot d\mathbf{r}$ are the most important type of line integral. For example, the work required to move a particle along a path from A to B in a force field $\mathbf{F}$ is,

$$W = \int_A^B \underline{F} \cdot d\underline{r}$$

In Cartesian coordinates,

$$\underline{F} = F_x \underline{i} + F_y \underline{j} + F_z \underline{k}$$

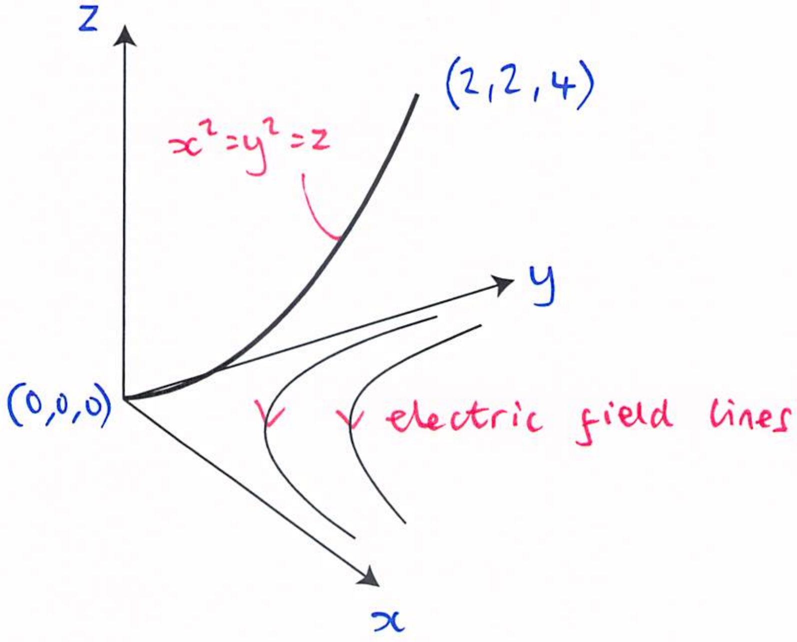
$$d\underline{r} = dx \, \underline{i} + dy \, \underline{j} + dz \, \underline{k}$$

$$W = \int_A^B F_x \, dx + F_y \, dy + F_z \, dz$$

It is often convenient to express the equation for the line, and the integrand, in parametric form. e.g. $x = x(s)$, $y = y(s)$, $F_x = F_x(s)$, etc.

Example

An electric field is determined by the equation $\mathbf{E} = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$. Find the work required to move a charge q from the point $(0,0,0)$ to the point $(2,2,4)$ along the curve $x^2 = y^2 = z$.



Force on charge $= q\underline{E}$

$$W = \int_A^B \underline{F} \cdot d\underline{r} = \int_A^B q\underline{E} \cdot d\underline{r}$$

Parametric eq^n for curve:

$$x = t \qquad y = t \qquad z = t^2$$

Here $\quad dx = dt \qquad dy = dt \qquad dz = 2t\,dt$

$$d\underline{r} = dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k}$$
$$= (\underline{i} + \underline{j} + 2t\,\underline{k})\,dt$$

$$W = q\int_{t=0}^{2} (t\,\underline{i} - t\,\underline{j} + t^2\underline{k}) \cdot (\underline{i} + \underline{j} + 2t\,\underline{k})\,dt$$

$$= q\int_0^2 t - t + 2t^3 = q\int_0^2 2t^3\,dt = 8q$$

Example

Although it is often simpler to use a parametric approach (requiring one integration), as in the previous example, it is also possible to retain the coordinate system independent variables e.g. (x,y,z), (requiring 3 integrations).

For example, evaluate,

$$I = \int_{(0,0,0)}^{(1,1,1)} \mathbf{F} \cdot d\mathbf{r} \qquad \text{where} \quad \mathbf{F} = (y-z)\,\mathbf{i} + (z-x)\,\mathbf{j} + (x-y)\mathbf{k} \tag{7.2}$$

and the integral is along the path $y = x^2$, $z = x^3$.

$$I = \int_{(0,0,0)}^{(1,1,1)} (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$$

$$I = \int_{x=0}^{1} (y-z)\,dx + \int_{y=0}^{1} (z-x)\,dy + \int_{z=0}^{1} (x-y)\,dz$$

Using the specified curve, we can make substitutions:

$$I = \int_{x=0}^{1} (x^2 - x^3)\,dx + \int_{y=0}^{1} (y^{3/2} - y^{1/2})\,dy + \int_{z=0}^{1} (z^{1/3} - z^{2/3})\,dz$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 + \left[\frac{2}{5}y^{5/2} - \frac{2}{3}y^{3/2}\right]_0^1 + \left[\frac{3}{4}z^{4/3} - \frac{3}{5}z^{5/3}\right]_0^1$$

$$= -\frac{1}{30}$$

7.3 Conservative fields

If a vector field $\mathbf{V}$ is '*conservative*' then the integral $\int \mathbf{V}.d\mathbf{r}$ between any two points A and B is *independent of the path of integration*. We now investigate the conditions which ensure that a field $\mathbf{V}$ is conservative.

In Cartesian coordinates,

$$\int_A^B \mathbf{V} \cdot d\mathbf{r} = \int_A^B V_x\,dx + V_y\,dy + V_z\,dz \tag{7.3}$$

If we can obtain $\mathbf{V}$ by taking the gradient of the scalar ϕ, $\mathbf{V} = \nabla\phi$, then,

$$\mathbf{V} \cdot d\mathbf{r} = \nabla\phi \cdot d\mathbf{r} = \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) \tag{7.4}$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

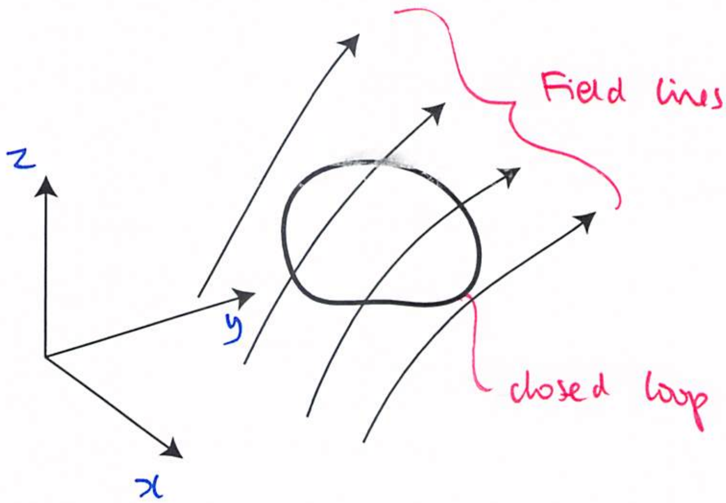
$$= d\phi$$

Hence, if $\mathbf{V} = \nabla\phi$, then $\mathbf{V} \cdot d\mathbf{r}$ is a *perfect differential* and the integral becomes,

$$\int_A^B \mathbf{V} \cdot d\mathbf{r} = \int_A^B d\phi = \phi_B - \phi_A \quad , \tag{7.5}$$

and *does not depend on the integration path*; $\mathbf{V}$, therefore, is a *conservative* field.

We have previously found that, if $\mathbf{V} = \nabla\phi$, then $\nabla \times \mathbf{V} = 0$ and the field is irrotational. The condition $\nabla \times \mathbf{V} = 0$, therefore, means that the field $\mathbf{V}$ is irrotational and also conservative - the two terms mean the same thing and are retained by convention. Traditionally, "irrotational" is used to describe a fluid velocity field $\mathbf{V}$ where $\nabla \times \mathbf{V} = 0$, and "conservative" is used to describe a force field $\mathbf{F}$ (such as the gravitational or electrostatic field) where $\nabla \times \mathbf{F} = 0$.



For a conservative field, the line integral around a closed loop in space,

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \quad , \tag{7.6}$$

because the final and initial values of ϕ are the same. The symbol $\oint$ is used to indicate integration around a closed loop because it does not matter at which point the integration starts.

For non-conservative fields,

$$\oint \mathbf{F} \cdot d\mathbf{r} = \Gamma \quad , \tag{7.7}$$

and Γ is generally non-zero. Γ is referred to as the *circulation*.

If $\nabla \times \mathbf{V} = 0$ everywhere, then the field is conservative or irrotational, and the circulation around any closed loop in the field is zero.

Example

The electrostatic field $\mathbf{E}$ at position $\mathbf{r}$ due to a point charge q at the origin is given by,

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^3}\mathbf{r} \quad . \tag{7.8}$$

Show that $\mathbf{E}$ is conservative and find its scalar potential.

Central force field $\Rightarrow$ spherical polar coordinates

$$\underline{E} = \frac{q}{4\pi\varepsilon_0 r^2}\,\underline{e}_r$$

components in $\underline{e}_\theta$ and $\underline{e}_\phi$ are zero

$$\nabla \times \underline{E} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \underline{e}_r & r\,\underline{e}_\theta & r\sin\theta\,\underline{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ \frac{q}{4\pi\varepsilon_0 r^2} & 0 & 0 \end{vmatrix}$$

$$= 0\,\underline{e}_r + 0\,\underline{e}_\theta + 0\,\underline{e}_\phi$$ $\therefore$ $\underline{E}$ is conservative

Scalar potential such that $\underline{E} = \nabla\psi$

$$\frac{\partial\psi}{\partial r} = \frac{q}{4\pi\varepsilon_0 r^2} \qquad \frac{\partial\psi}{\partial\theta} = 0 \qquad \frac{\partial\psi}{\partial\phi} = 0$$

$$\therefore \psi = \frac{-q}{4\pi\varepsilon_0 r} + \text{const.}$$

i.e. equipotential surfaces (ψ = const) are spheres centred at the origin, everywhere normal to field lines of $\underline{E}$

You can now do Examples Paper 2: Q8, 9, 10, 11 and 12