Engineering Tripos Part IB **SECOND YEAR**

Part IB Paper 7 : Mathematical Methods

VECTOR CALCULUS AND PDEs

Examples Paper 4

Straightforward questions are marked †

Partial Differential Equations

1.† The function $\phi(r, \theta)$ satisfies Laplace's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

in the cylindrical region $1 \le r \le 2$ for $0 \le \theta \le 2\pi$, with boundary conditions $\phi(1, \theta) = \sin\theta$ and $\phi(2, \theta) = 2\sin\theta$. If $\phi(r, \theta) = R(r)\,T(\theta)$, show that

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{T}\frac{d^2T}{d\theta^2} = const$$

Explain why $T(\theta)$ can be taken as $\sin\theta$ and find the boundary conditions for R.

By looking for solutions of the form $R = Ar^{\alpha}$, show that the general solution of the differential equation for R is $R = B\,r + C\,r^{-1}$. Find $\phi(r, \theta)$.

2. Consider the Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\cos\left(\frac{\pi x}{2L}\right)$$

subject to the boundary conditions: $x = \pm L,\ \phi = 0$; $y = \pm d,\ \phi = 0$.

(i) Show that the 'particular integral' $\phi = A\cos(\pi x/2L)$ satisfies the differential equation and find the value of A.

(ii) Assume that $\phi = \phi_0 + A\cos(\pi x/2L)$ and find the differential equation and boundary conditions which must be satisfied by the 'complementary function' $\phi_0 = \phi_0(x, y)$.

(iii) Find the complete solution for ϕ.

3. The function $f(x,t)$ satisfies the equation

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + 2\omega\frac{\partial f}{\partial x}$$

in $x \geq 0$ where ω is a real positive constant. Using the method of separation of variables, find $f(x,t)$ subject to the boundary conditions

$$f = \sin(\omega t) \quad \text{and} \quad \frac{\partial f}{\partial x} = 0 \quad \text{at } x = 0 \text{ for all } t.$$

4. A triangular metal plate of uniform thickness and thermal conductivity lies in the x-y plane with corners A at (0,0), B at (1,0) and C at (1,1). For steady-state heat conduction in the plate, the temperature distribution $T = T(x,y)$ satisfies Laplace's equation $\nabla^2 T = 0$. The temperatures at points A, B and C are 300 K, 400 K and 800 K, respectively, and along the boundaries AB and BC the temperature varies linearly with distance. The boundary AC is well insulated and there is no heat flux across it. (The heat flux vector is $\underline{q} = -\lambda\nabla T$, where λ is the thermal conductivity.)

(i) Use the method of separation of variables: what value of separation constant do you need in order to be consistent with the form of variation along the boundaries AB and BC? Hence show that the temperature distribution in the plate is of the form

$$T = (\alpha x + \beta)(\gamma y + \delta) = a + bx + cy + dxy\,.$$

(ii) Show that the boundary condition on AC is $\partial T/\partial x = \partial T/\partial y$. Hence find the values of the constants a, b, c and d.

5.† A nuclear reactor takes the form of a plane slab of material with faces at $x = \pm d/2$ and is essentially of infinite extent in the other directions. The neutron density n varies throughout the slab but leakage at the faces maintains $n = 0$ at $x = \pm d/2$. For $t < 0$, the reactor is operating steadily and the neutron density distribution is given by $n = n_0 \cos(\pi x/d)$. At $t = 0$, the nuclear reaction ceases and the neutron density decays as the neutrons diffuse out of the slab. The neutron density is then governed by the unsteady diffusion equation

$$\frac{\partial n}{\partial t} = \alpha\frac{\partial^2 n}{\partial x^2}$$

where α is the diffusivity of the neutrons. Find an expression, valid for $t \geq 0$, for the neutron density distribution in the slab and find the time for the neutron density at the centre of the slab to fall to half its initial value.

6. One-dimensional sound wave propagation in air is governed by the wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2\frac{\partial^2 p}{\partial x^2}$$

where $p = p(x,t)$ is the excess pressure (above atmospheric) induced by the passage of the wave and c is the speed of the wave, assumed constant. In air at 20°C, $c = 344$ m/s.

Calculate the length of an organ pipe which resonates at 'middle C' (261.6 Hz) as its fundamental frequency. The boundary conditions are $\partial p/\partial x = 0$ for all t at the closed end (which corresponds to zero flow velocity) and $p = 0$ for all t at the open end. Find the second natural frequency of the pipe. Sketch the corresponding mode shapes (i.e. the excess pressure as a function of distance along the pipe).

7. (a) The equation of motion for the local displacement y of a beam can be written as:

$$\rho A \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = 0.$$

By separation of variables show that this can be written as two ordinary differential equations:

$$\frac{d^4 X}{dx^4} - k^4 X = 0 \qquad\qquad \frac{d^2 T}{dt^2} + \omega^2 T = 0$$

where $X = X(x)$, $T = T(t)$ and k and ω are constants. What is the relationship between k and ω?

(b) The travelling wave response of an infinite beam to an input chain of unit impulses at $x = 0$ can be shown to be:

$$y(x,t) = \frac{1}{T} \sum_{n=0}^{\infty} \cos(\omega_n t - k_n x)$$

where T is the time between impulses, $\omega_n = 2\pi n/T$ and the relationship between k_n and ω_n was found in part (a).

For the remainder of this question, use the Jupyter Notebook for this examples paper at: https://colab.research.google.com/github/CambridgeEngineering/PartIB-Paper7-Mathematics-Vector-Calculus

The notebook calculates the response at 100m using the first N harmonics. A time-vector is stored in `t` and the result in `y`. Edit the notebook to plot the response as a function of time. Zoom in on one of the impulses. What do you notice about the frequency as a function of time?

(c) The notebook writes out an audio file called `response.wav` that you can download and play on your computer. The relationship between frequency and time should sound very clear: note this is also similar to the sound you can sometimes hear at the side of a railway as a train approaches, or from rocks landing on a frozen lake (see https://www.youtube.com/watch?v=UiCOiWRxI0I).

8. A large, hot steel ingot is initially at a uniform temperature, T_0. At time $t = 0$ cooling water is applied to the bottom face of the ingot (at $y = 0$) which drops the surface temperature there to $T = T_0 - \Delta T$. The ingot may be treated as semi-infinite, occupying the region $y > 0$. Show that a characteristic length-scale of the problem is $\sqrt{\alpha t}$ where α is the thermal diffusivity. Hence show that,

$$\frac{T_0 - T(y,t)}{\Delta T} = 1 - \frac{1}{\sqrt{\pi}} \int_0^{\eta} \exp\left(-\frac{\eta^2}{4}\right) d\eta$$

where $\eta = y/\sqrt{\alpha t}$. Find an expression for the surface temperature gradient, $\left(\frac{\partial T}{\partial y}\right)_{y=0}$, as a function of time.

Linear Algebra revision

9. If $\underline{x} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^t$, find (i) $\underline{x}^t \underline{x}$ (ii) $\underline{x}\,\underline{x}^t$

10. (a) **A** is a 3 × 3 matrix. Show that

$$\mathbf{A}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \text{the first column of } \mathbf{A}.$$

(b) If the columns of **A** are represented by the vectors $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$, show that

$$\mathbf{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\underline{a}_1 + y\underline{a}_2 + z\underline{a}_3$$

(c) The 3 × 3 matrix **Q** represents a rotation of vectors about the y-axis. If the vector $\begin{bmatrix} \frac{1}{\sqrt{2}} & 2 & \frac{1}{\sqrt{2}} \end{bmatrix}^t$ is mapped to $\begin{bmatrix} -\frac{1}{2} & 2 & \frac{\sqrt{3}}{2} \end{bmatrix}^t$ by this rotation, find **Q**.

11. If the 3 × 3 matrix **A** has *rows* represented by the vectors $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$ and the 3 × 3 matrix **X** has *columns* represented by the vectors $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$

$$\mathbf{A} = \begin{bmatrix} \leftarrow & \underline{a}_1 & \rightarrow \\ \leftarrow & \underline{a}_2 & \rightarrow \\ \leftarrow & \underline{a}_3 & \rightarrow \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \underline{x}_1 & \underline{x}_2 & \underline{x}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

What is b_{21}, the 2-1 element of the matrix **B** = **AX** ?

What is c_{21}, the 2-1 element of the matrix **C** = $\mathbf{X}^t \mathbf{A}^t$?

12. $\underline{x}_1$ and $\underline{x}_2$ are 2-dimensional vectors related to the vectors $\underline{u}_1$ and $\underline{u}_2$ by

$$\underline{x}_1 = \alpha\underline{u}_1 + \beta\underline{u}_2$$
$$\underline{x}_2 = \gamma\underline{u}_1 + \delta\underline{u}_2$$

where α, β, γ and δ are scalars.

(a) Find the matrix **A**, such that **X** = **UA** where the columns of the matrix **X** are the vectors $\underline{x}_1$ and $\mathbf{x}_2$ and the columns of matrix **U** are $\underline{u}_1$ and $\underline{u}_2$.

(b) Find the matrix **P** which swaps the columns of **U**.
(c) How would you represent the matrix **Y** which has *rows* $\underline{x}_1$ and $\underline{x}_2$?

Answers

4.1 $r \sin \theta$

4.2 $\phi = (2L/\pi)^2 \cos(\pi x/2L)\left[1 - \dfrac{\cosh(\pi y/2L)}{\cosh(\pi d/2L)}\right]$

4.3 $f(x,t) = (1+\omega x)\exp(-\omega x)\sin(\omega t)$

4.4 (i) Zero (ii) $a = 300$, $b = 100$, $c = 100$, $d = 300$.

4.5 $n(x,t) = n_0 \exp(-\alpha\pi^2 t/d^2)\cos(\pi x/d)$, $\quad t = \dfrac{d^2 \ln 2}{\alpha\pi^2}$

4.6 0.329 m, 784.8 Hz .

4.7 (a) $\omega = k^2\sqrt{\dfrac{EI}{\rho A}}$

4.8 $\Delta T/\sqrt{\pi\alpha t}$

4.9 (i) 14 (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

4.10 (c) $\begin{bmatrix} .2588 & 0 & -.9659 \\ 0 & 1 & 0 \\ .9659 & 0 & .2588 \end{bmatrix}$

4.11 $b_{21} = \underline{a}_2^{\mathrm{t}}\,\underline{x}_1 = \underline{a}_2 . \underline{x}_1$, $\quad c_{21} = \underline{a}_1^{\mathrm{t}}\,\underline{x}_2 = \underline{a}_1 . \underline{x}_2$

4.12 (a) $\mathbf{A} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$ (b) $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (c) $\mathbf{Y} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \times$ matrix with $\underline{u}_1$ and $\underline{u}_2$ as rows

GP Michaelmas 2023