UNIVERSITY OF CAMBRIDGE
Department of Engineering

IB Paper 5
Electromagnetic Fields & Waves

Lecture 5
Electromagnetic Waves at Interfaces

https://www.vle.cam.ac.uk/course/view.php?id=70081

Introduction to Waves at Interfaces

FLE143

- We have seen that the Maxwell Equations can be used to generate a wave equation
  - A solution to this is an electromagnetic wave which can propagate through free space
  - For a plane wave, the electric and magnetic field vectors are perpendicular to each other and perpendicular to the direction of wave propagation
  - The Poynting vector defines the direction of propagation

$$\mathbf{N} = \mathbf{E}\times\mathbf{H} \tag{4.23}$$

  - And the time-averaged power per unit area in the wave is

$$|\bar{\mathbf{N}}| = \frac{\mathbf{E}\times\mathbf{H}^*}{2} \tag{4.25}$$

  - $\mathbf{E}$ and $\mathbf{H}$ are related by the characteristic impedance

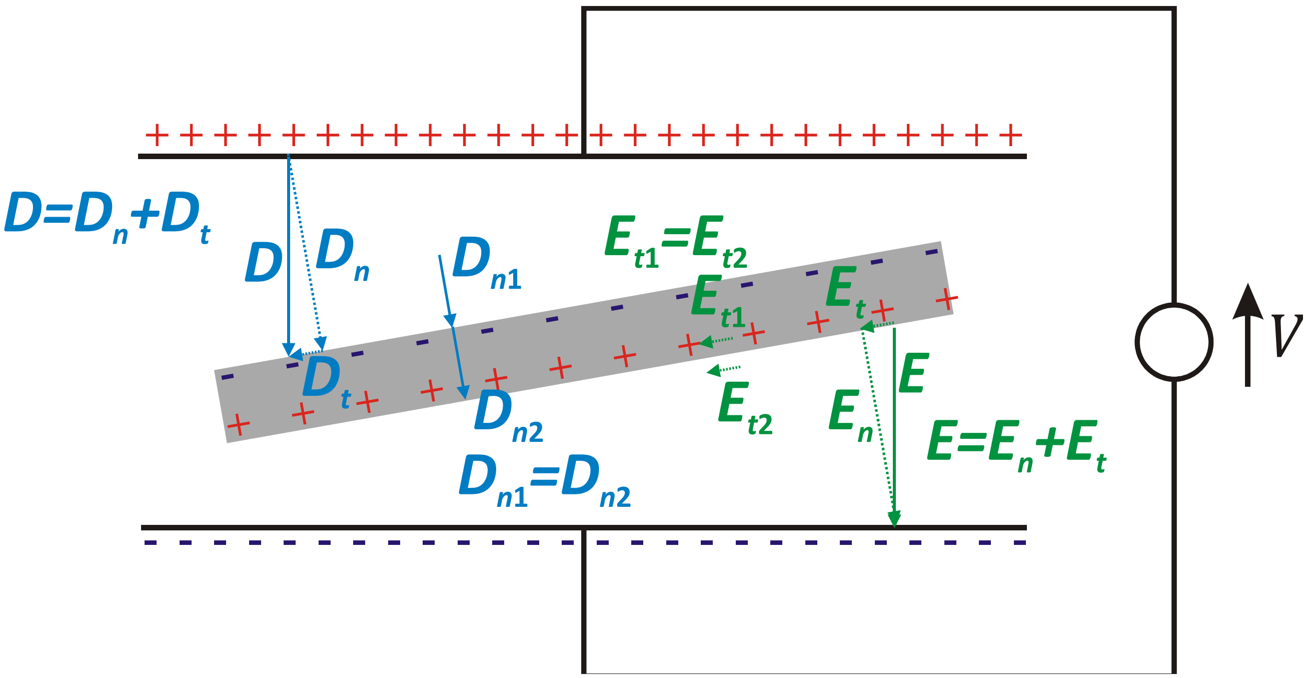
$$\eta = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} \tag{4.22}$$

- In this lecture, we are going to understand what happens when a wave meets an interface between media of different η
  - This is rather like the situation of joining transmission lines of different Z_0
  - However, to understand this we must first look at how fields are conserved at interfaces: ***boundary conditions***

Boundary Conditions

FLE144

- Let us consider a block of some material ‘2’ which is between the plates of a capacitor that is otherwise filled with air (material ‘1’)



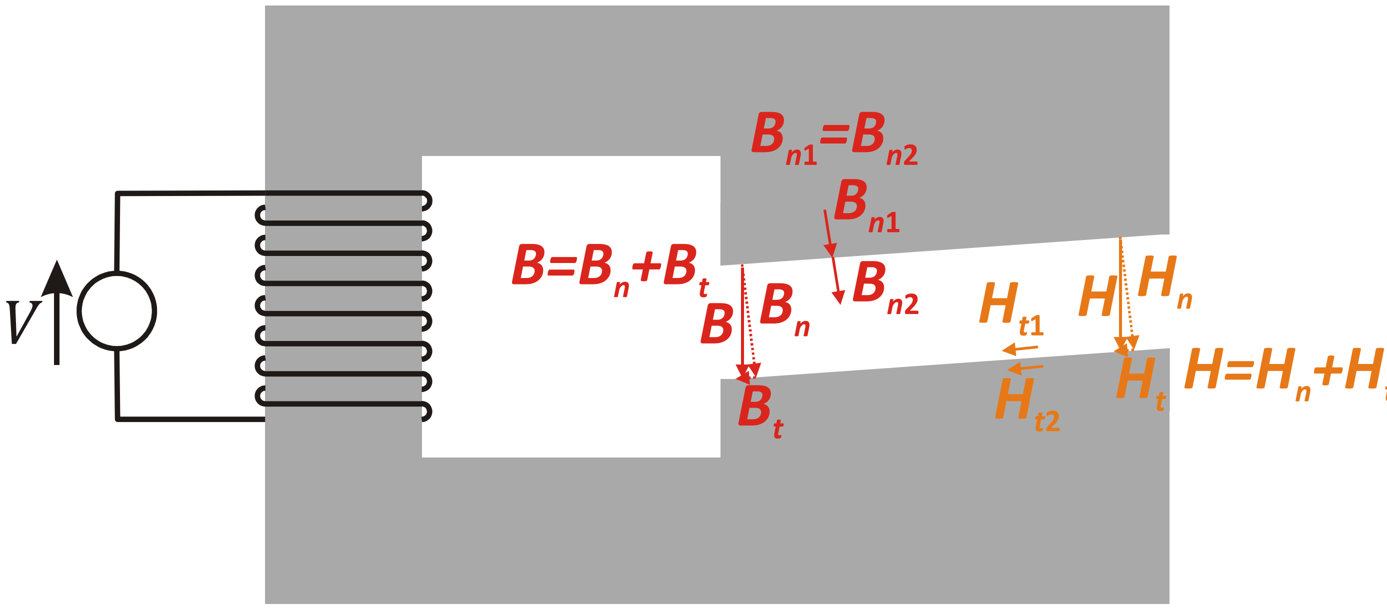
- Both an electric field $\mathbf{E}$ and electric flux density $\mathbf{D}$ will be set up
- Due to the build up of charge at the surface of the dielectric, $\mathbf{E}$ and $\mathbf{D}$ may be different in the two materials
- We can resolve both fields into components that are tangential to the interface ($\mathbf{E_t}$ and $\mathbf{D_t}$) and normal to the interface ($\mathbf{E_n}$ and $\mathbf{D_n}$)
- From the Maxwell Equations it can be shown that

$$\boxed{\begin{aligned}\mathbf{D_{n1}} &= \mathbf{D_{n2}} \\ \mathbf{E_{t1}} &= \mathbf{E_{t2}}\end{aligned}}$$

(5.1a)
(5.1b)

- This should 'feel' sensible as the polarisation charge over the surface will clearly change the normal component of $\mathbf{E}$ in the two materials, but if it is very thin, then it cannot change the tangential component

- We can also consider an iron (material 2) toroid with an air (material 1) gap



- Surface currents mean that $\mathbf{H}$ and $\mathbf{B}$ may be different in the two materials
- We can resolve both fields into components that are tangential to the interface ($\mathbf{H_t}$ and $\mathbf{B_t}$) and normal to the interface ($\mathbf{H_n}$ and $\mathbf{B_n}$)
- From the Maxwell Equations it can be shown that

$$\boxed{\begin{aligned} \mathrm{B_{n1}} &= \mathrm{B_{n2}} \\ \mathrm{H_{t1}} &= \mathrm{H_{t2}} \end{aligned}}$$

(5.1c)

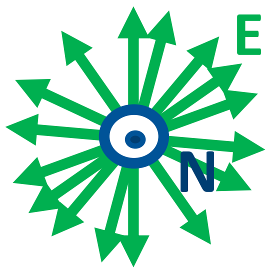
(5.1d)

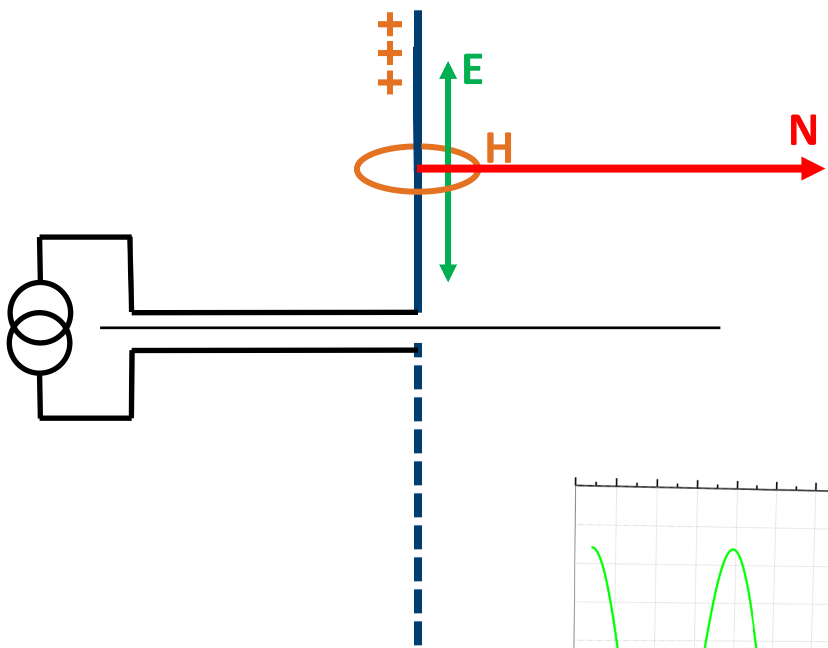
- ***The normal components of*** D ***and*** B ***and the tangential components of*** E ***and*** H ***are conserved at interfaces***

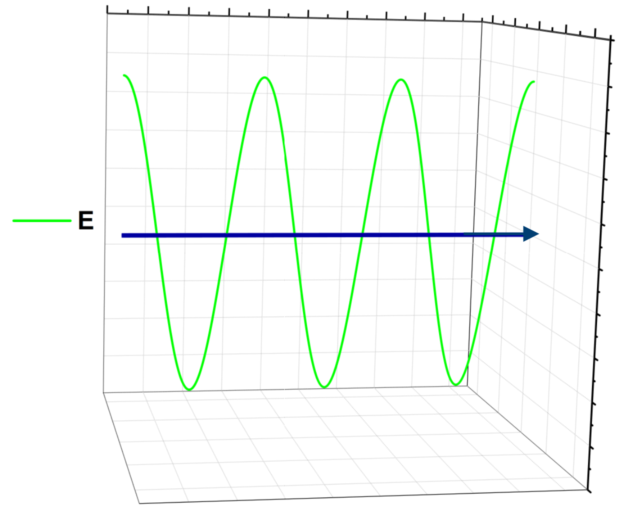
Polarised Plane Electromagnetic Waves

FLE121

- In an arbitrary plane electromagnetic wave, such as light from the sun, although the electric field vector is perpendicular to the direction of propagation, its direction will vary randomly within this perpendicular plane
  - However, in the case of the simple dipole antenna, the electric field in the electromagnetic wave was always pointing vertically up or down

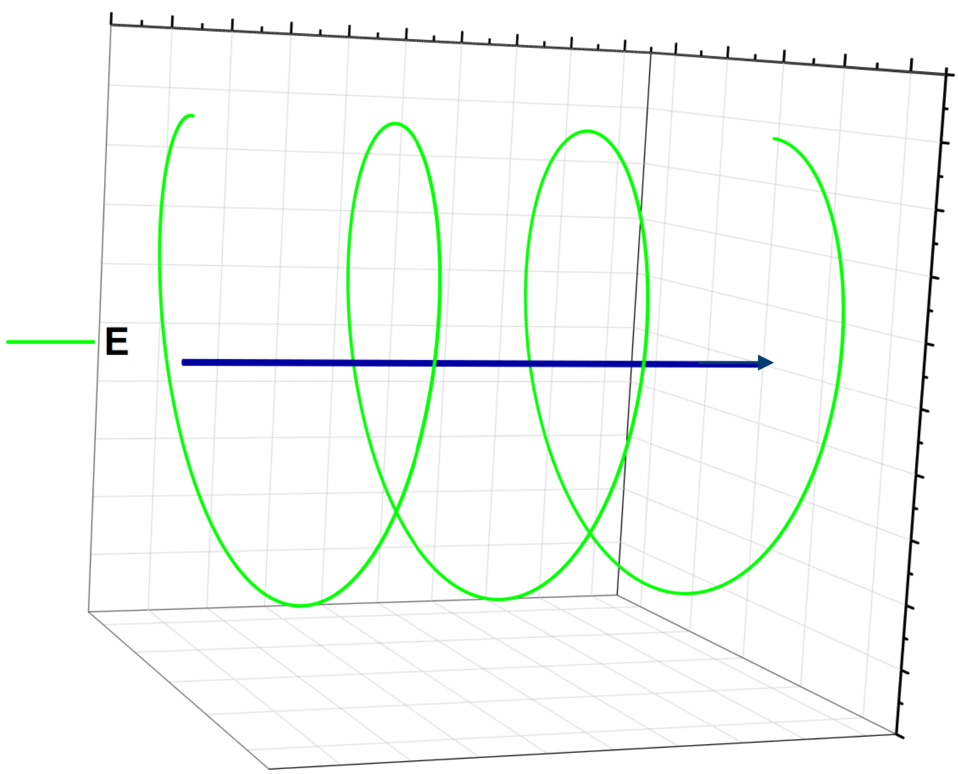






- We call this a linearly polarised wave

- Polarised waves are widely employed in their own right
  - Radio and television transmission
  - Sunglasses
  - Liquid crystal displays
- We can also create more complex polarisations, for example circular polarisation (frequently found in lasers), by superposing two linear polarisations at 90° to each other and with a 90° phase shift

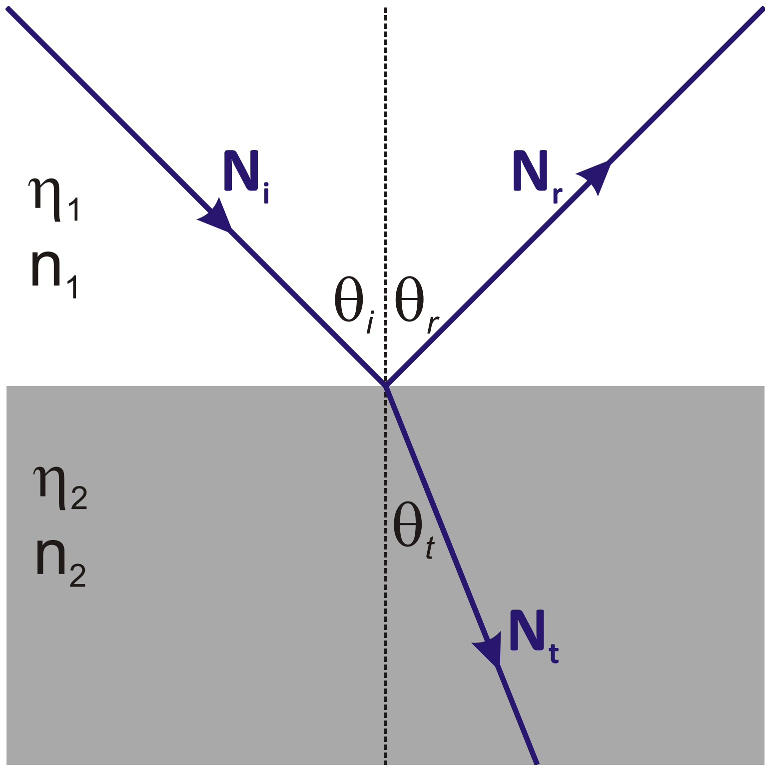


- We can also resolve any arbitrary electromagnetic wave into two components with their electric fields perpendicular to each other

Reflection and Refraction of Plane Waves

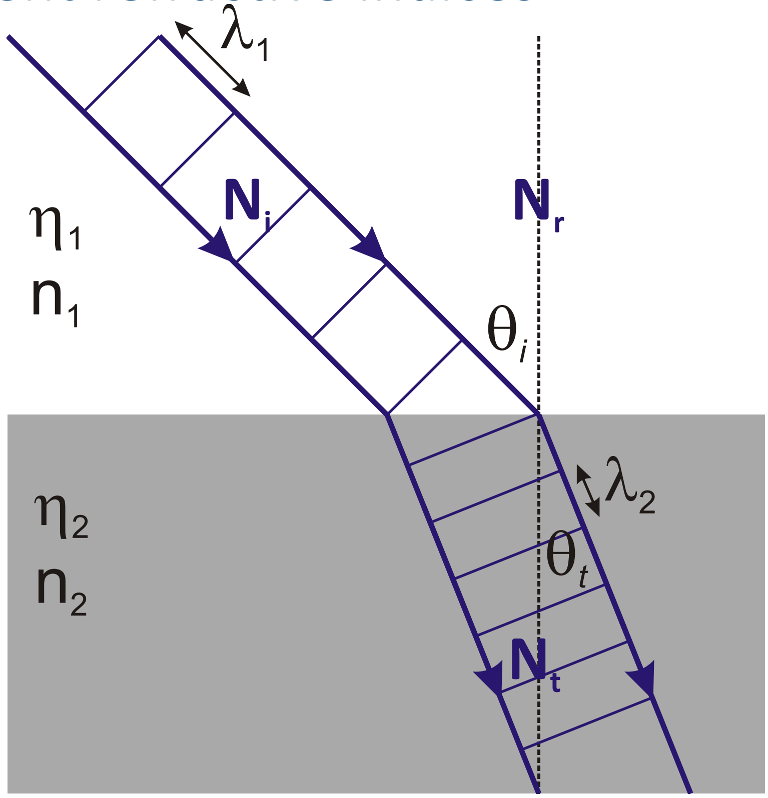
FLE147

- We are going to consider the general case of a plane electromagnetic wave arriving at a flat interface between two materials with some angle of incidence θ_i



- We will assume both materials 1 and 2 to be infinite in thickness
- The wave is travelling from medium 1 to medium 2
- At the interface, some of the wave is transmitted and some is reflected
- We know that for reflection, $\theta_i = \theta_r$, but for the transmitted wave, refraction means that $\theta_i \neq \theta_t$

- We can determine the relationship between θ_t and θ_i very straightforwardly using geometry and the fact that the speed of the wave changes due to the different refractive indices



- Imagine an incoming plane wave
- In order to preserve the planes of peaks and troughs, it must take the same time for the wave to travel a distance λ_1 in medium 1 as it does to travel λ_2 in medium 1, so

$$\frac{\lambda_1}{\sin\theta_i} = \frac{\lambda_2}{\sin\theta_t} \tag{5.2}$$

- Frequency is the same in both media, so

$$c_1 = c/n_1 = f\lambda_1$$
$$\lambda_1 = \frac{c}{fn_1} \tag{5.3}$$

- Similarly,

$$\lambda_2 = \frac{c}{f n_2} \tag{5.4}$$

- Hence, substitution of Eqns. 5.3 and 5.4 into 5.2 gives

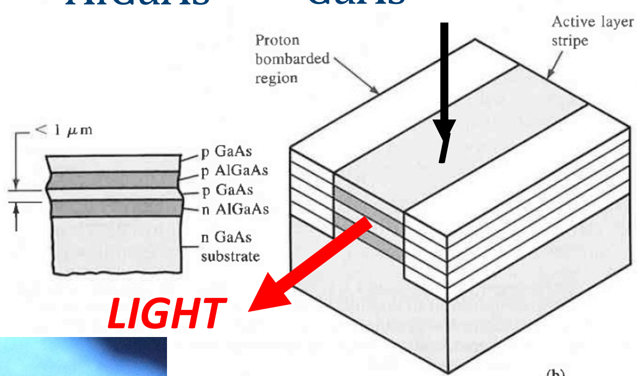
$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t} \tag{5.5}$$

- This is the ***Snell Law of Refraction***: if a wave is passing from one material to another of higher refractive index ($n_2 > n_1$) then the transmitted wave is refracted so that its path is closer to the normal ($\theta_i > \theta_t$)
- Also, if the wave is going from one material into one of a lower refractive index, then if the angle of incidence is greater than a critical value θ_{ic} then there can be no transmission and the wave is totally reflected, where

$$\theta_{ic} = \sin^{-1}(n_2/n_1)$$

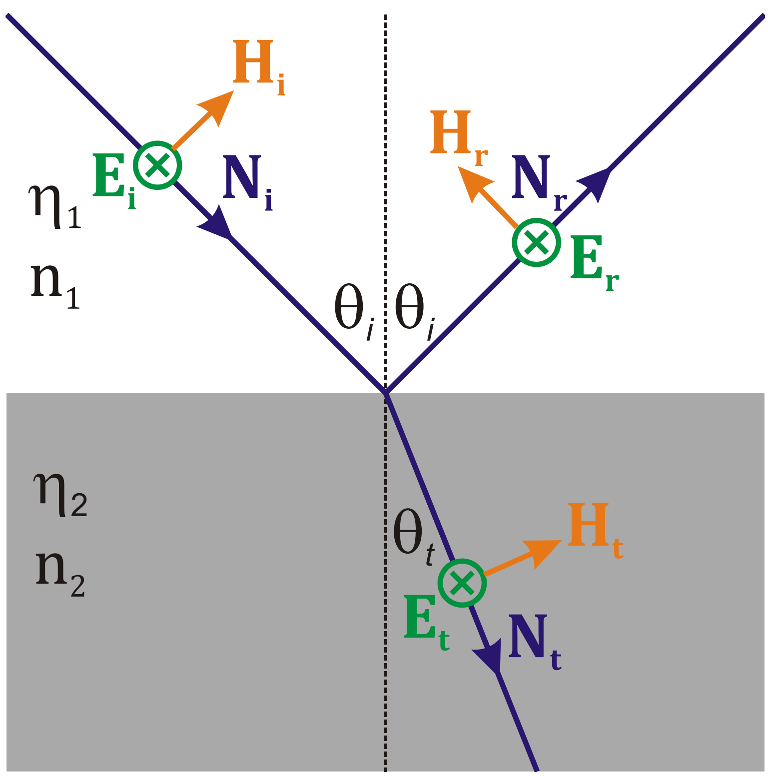
- This is the basis behind the semiconductor laser diode and the optical fibre where light is confined in a high refractive index material by a low index surround

$$n_{\mathrm{AlGaAs}} < n_{\mathrm{GaAs}}$$



From Streetman, *'Solid State Electronics'*, 3rd Ed.

- Let us now consider how much of the wave is reflected and refracted
  - For this, we will need to consider two scenarios
    1. the electric field is perpendicular to the plane of incidence, and
    2. the electric field is parallel to the plane of incidence
  - We have to do this because different boundary conditions apply in each case
  - We can resolve any arbitrary wave into these two components
- Consider the situation where the electric field is perpendicular to the plane of incidence first



- We know from our boundary condition Eqn. 5.1b that the component of $\mathbf{E}$ tangential to the interface is conserved across the boundary

$$\mathbf{E_{t1}} = \mathbf{E_{t2}} \tag{5.1b}$$

- So from the geometry of the system

$$E_i + E_r = E_t \tag{5.7}$$

- The tangential component is $\mathbf{H}$ is also conserved (Eqn. 5.1d), so

$$\mathbf{H_{t1}} = \mathbf{H_{t2}} \tag{5.1d}$$

$$H_i \cos\theta_i - H_r \cos\theta_i = H_t \cos\theta_t \tag{5.8}$$

- We can convert expressions in terms of $\mathbf{H}$ into $\mathbf{E}$ using the characteristic impedance

$$H_i = E_i/\eta_1 \qquad H_r = E_r/\eta_1 \qquad H_t = E_t/\eta_2 \tag{5.9}$$

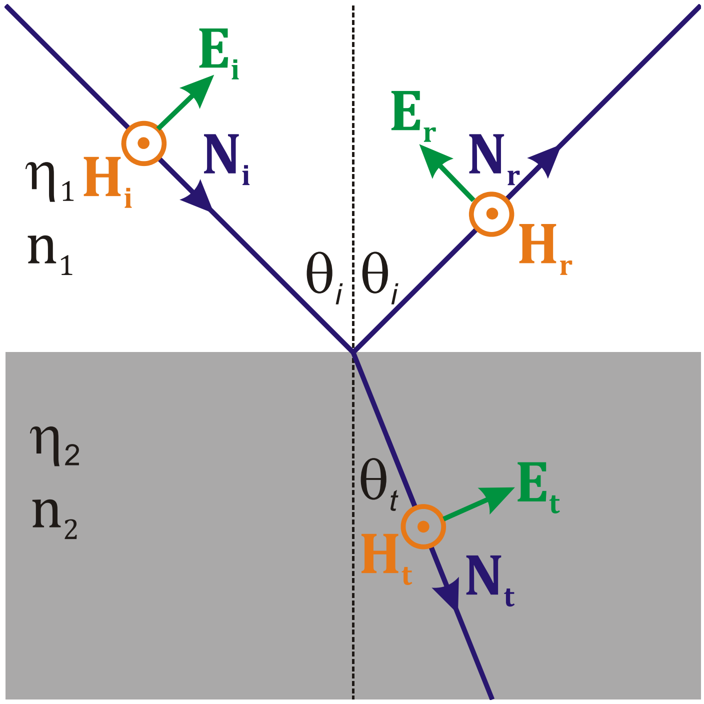
- Therefore, substituting Eqn. 5.9 into Eqn. 5.8 gives

$$\frac{E_i}{\eta_1}\cos\theta_i - \frac{E_r}{\eta_1}\cos\theta_i = \frac{E_t}{\eta_2}\cos\theta_t \tag{5.10}$$

- By substituting for either E_r or E_t from Eqn. 5.7 in Eqn. 5.10 gives expressions for the transmitted and reflected electric fields

$$\boxed{\begin{aligned} \left(\frac{E_t}{E_i}\right)_\perp &= \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} && (5.11a) \\ \left(\frac{E_r}{E_i}\right)_\perp &= \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} && (5.11b) \end{aligned}}$$

- Now consider the situation where the electric field is parallel to the plane of incidence



- Applying the conservation of tangential electric field gives

$$E_i \cos\theta_i - E_r \cos\theta_i = E_t \cos\theta_t \tag{5.12}$$

- Conservation of tangential magnetic field gives

$$H_i + H_r = H_t \tag{5.13}$$

- Using the same characteristic impedance relations for $\mathbf{H}$ and $\mathbf{E}$ in Eqn. 5.9 allows Eqn. 5.13 to be rewritten as

$$E_i/\eta_1 + E_r/\eta_1 = E_t/\eta_2 \tag{5.14}$$

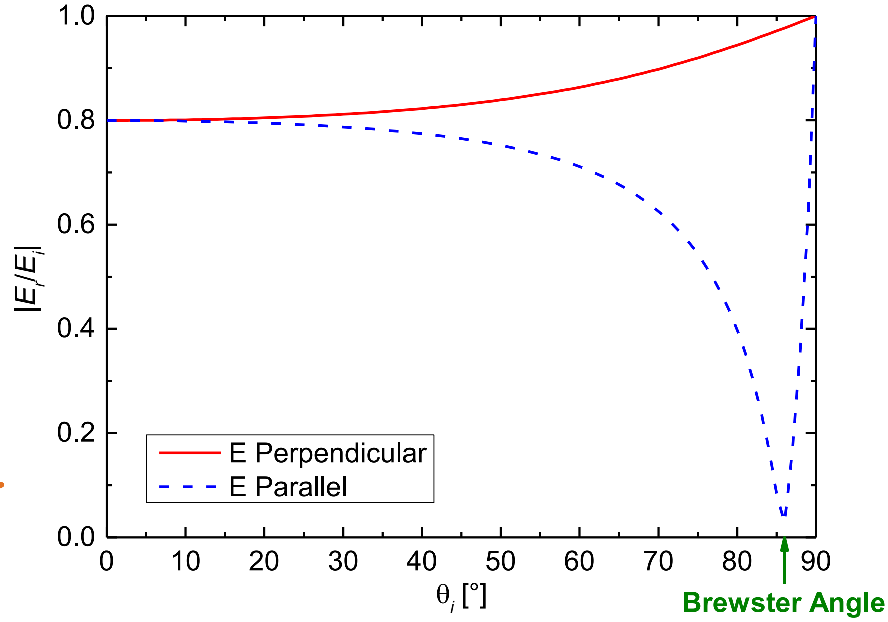
- So substitution now gives

$$\boxed{\begin{aligned} \left(\frac{E_t}{E_i}\right)_{\parallel} &= \frac{2\eta_2\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t} \quad (5.15a) \\ \left(\frac{E_r}{E_i}\right)_{\parallel} &= \frac{\eta_1\cos\theta_i - \eta_2\cos\theta_t}{\eta_1\cos\theta_i + \eta_2\cos\theta_t} \quad (5.15b) \end{aligned}}$$

Polarisation by Reflection: the Brewster Angle

FLE153, GER116

- A key result from this analysis is that the parallel and perpendicular electric field polarisations of an electromagnetic wave are reflected differently
  - Let us take the case of light reflecting off water where the characteristic impedance of water is 42 Ω



5/7 Q6

  - At the ***Brewster Angle*** θ_B, the component of an electromagnetic wave whose electric field is parallel to the plane of incidence is not reflected
  - The reflected wave is now plane polarised with the electric field perpendicular to the plane of incidence
  - This occurs when Eqn. 5.15b is zero
  - A common use for this is in sunglasses, where the polaroid is oriented to block the perpendicular electric field component, thereby significantly reducing reflected glare
  - This assumes you are upright!

Anti-Reflection Coatings

FLE155

- Let us consider a special case where the incident electromagnetic wave has $\theta_i = 0°$
  - In this case, the equation for the transmitted wave for both polarisations (Eqns. 5.11a and 5.15a) reduce to

$$\left(\frac{E_t}{E_i}\right)_{\perp,\parallel,\theta_i=0} = \frac{2\eta_2}{\eta_1 + \eta_2} \tag{5.16}$$

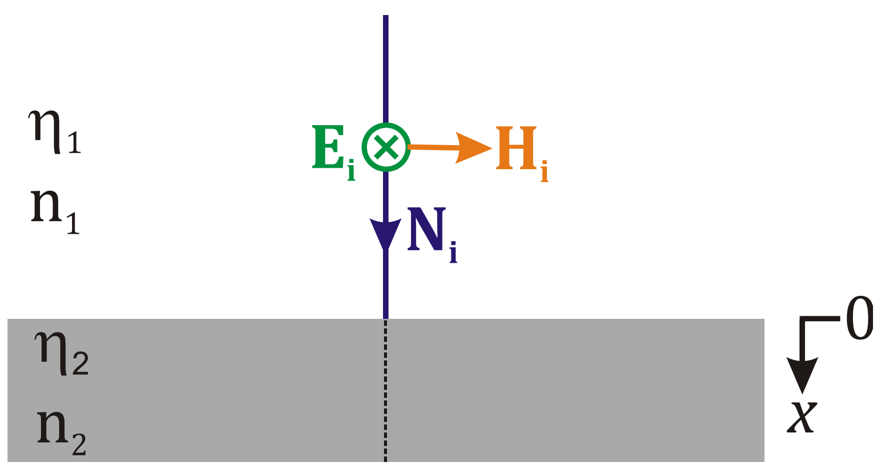
  - This is identical to the equation for transmitted voltage at a junction in a transmission line
  - For the reflected wave the equation for the two polarisations are slightly different

$$\left(\frac{E_r}{E_i}\right)_{\perp,\theta_i=0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \tag{5.17a}$$

$$\left(\frac{E_r}{E_i}\right)_{\parallel,\theta_i=0} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \tag{5.17b}$$

  - The perpendicular case is identical to the voltage reflection on the transmission line (Eqn. 2.10)
  - Eqn. 5.17 shows us that, just as for the transmission line, an incident wave at 0° angle of incidence will be reflected whenever there is a change in characteristic impedance
  - Therefore, if we want electromagnetic waves to propagate without reflections then we need to avoid step changes in the characteristic impedance

- This is a major issue in optical and optoelectronic systems, and examples include
  - coupling of light into and out of optical fibres
  - coupling sunlight into a solar cell
  - minimising reflections on camera lenses and glasses
  - eyes
- In many cases, we are coupling light that has been travelling through the air, and so we cannot simply match impedances
- Rather as for the transmission line, we can define an effective characteristic impedance some distance x from an interface
  - We will use the case for the electric field perpendicular to the plane of incidence



- The effective impedance at any distance x away from the interface is

$$\eta(x) = \frac{\overline{E_i} e^{j(\omega t - \beta x)} + \overline{E_r} e^{j(\omega t + \beta x)}}{\overline{H_i} e^{j(\omega t - \beta x)} - \overline{H_r} e^{j(\omega t + \beta x)}} \tag{5.18}$$

- Note the change in sign for $\overline{H_r}$ as the direction is changed on reflection (see the figure on Slide 10)
- If we were working with the other polarisation then $\overline{E_r}$ would change sign
- We can lose a factor of $e^{j\omega t}$ and use the characteristic impedance to turn magnetic fields into electric fields (Eqn. 4.22)

$$\eta(x) = \frac{\overline{E_i} e^{-j\beta x} + \overline{E_r} e^{j\beta x}}{\frac{\overline{E_i}}{\eta_1} e^{-j\beta x} - \frac{\overline{E_r}}{\eta_1} e^{j\beta x}} \tag{5.19}$$

- Then dividing all through by $\overline{E_i}$ gives

$$\eta(x) = \eta_1 \frac{e^{-j\beta x} + \frac{\overline{E_r}}{\overline{E_i}} e^{j\beta x}}{e^{-j\beta x} - \frac{\overline{E_r}}{\overline{E_i}} e^{j\beta x}} \tag{5.20}$$

- So if we define an electric field reflection coefficient as $\rho = \overline{E_r}/\overline{E_i}$, which is then given by Eqn. 5.11b, we have

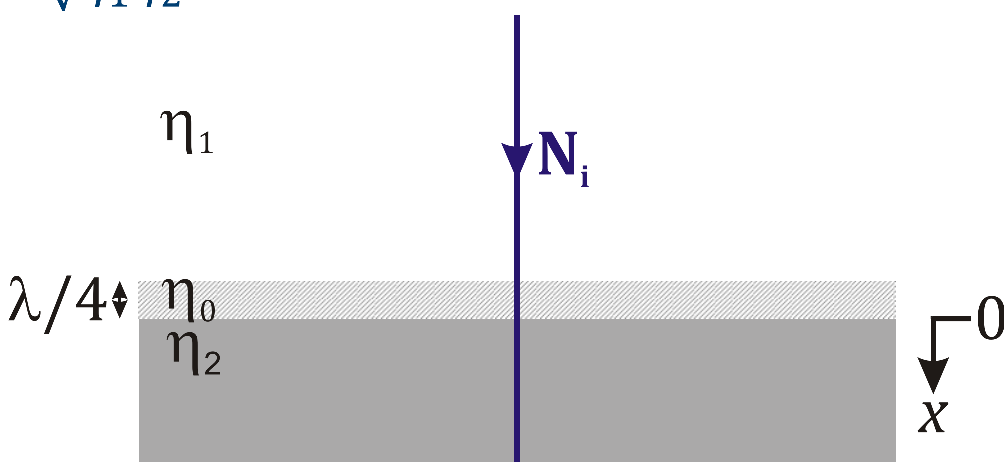
$$\eta(x) = \eta_1 \frac{e^{-j\beta x} + \rho e^{j\beta x}}{e^{-j\beta x} - \rho e^{j\beta x}} \tag{5.21}$$

5/7 Q7

- This is again analogous to the transmission line, and we find that we can again perform quarter wave matching
  - If we set $x = -\lambda/4$, then $\beta x = -\pi/2$, and Eqn. 5.20 becomes

$$\eta(x = -\lambda/4) = \eta_1 \frac{1-\rho}{1+\rho} = \frac{\eta_1^2}{\eta_2} \tag{5.21}$$

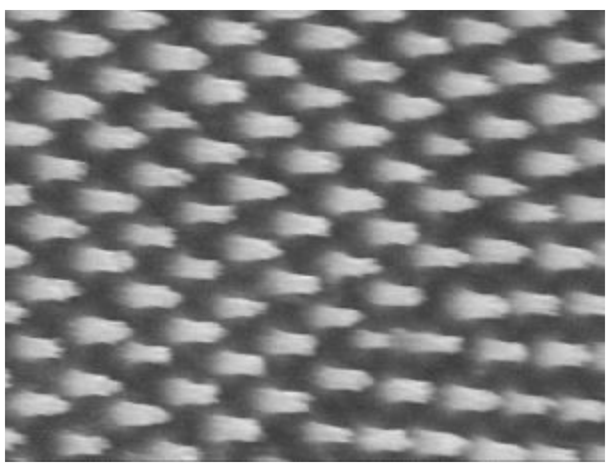
  - We can therefore make an anti-reflection coating whose thickness is $\lambda/4$ and whose characteristic impedance is $\eta_0 = \sqrt{\eta_1 \eta_2}$



  - A disadvantage of this approach is that it is tuned to one frequency, but this is OK if we are using laser light
- An alternative approach is to put a nanostructured material on the surface where the physical size of the nanostructures is much less than λ
  - In these circumstances, the electromagnetic wave does not interact with individual structures but sees a spatial average of the structure

- The result is that the nanostructure can appear to modulate the characteristic impedance smoothly as a function of depth into the layer
- As a result, reflections are minimised across a broad range of wavelengths

http://physicsworld.com/cws/article/news/2009/feb/09/moth-eyes-inspire-more-efficient-solar-cell



http://phys.org/news122899685.html