UNIVERSITY OF CAMBRIDGE
Department of Engineering

IB Paper 5
Electromagnetic Fields & Waves

Lecture 3
The Maxwell Equations of Electromagnetism

https://www.vle.cam.ac.uk/course/view.php?id=70081

Introduction

- Last year, we began to look at the origin and nature of electric and magnetic fields and saw that these could be calculated using the ***Gauss Law of Electrostatics*** and the ***Ampère Law of Magnetism***
- This year, we will see that these are two of a set of four equations that are collectively known as the ***Maxwell Equations of Electromagnetism***
  - Each of these equations tells us something about the fundamental physical origin or nature of electric and magnetic fields
  - They use surface and volume integrals and vector calculus to be able to express this physical nature in a rather elegant mathematical form
- In this lecture, we will aim to understand these four equations
- In the subsequent lectures, we will then see how we can use them to understand electric and magnetic fields as we experience them and as we employ them as engineers

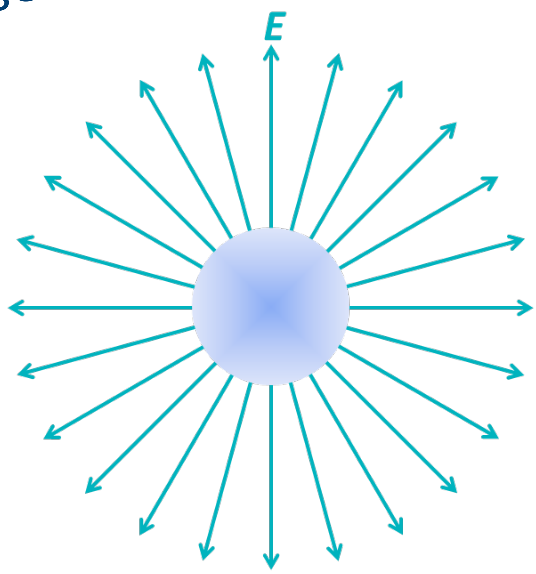
The Gauss Law of Electric Fields

FLE71, GER56

- We have previously seen that charge produces an electric field $\boldsymbol{E}$, rather as mass produces a gravitational field
  - A field is simply a region of space in which something with an appropriate property can experience a force
  - For an electric field, that property is charge (and it is mass for a gravitational field)
  - The field is a vector quantity, as the force $\mathbf{F}$ that it will produce on a charge q is also a vector

$$\boxed{\mathbf{F} = q\mathbf{E}} \tag{3.1}$$

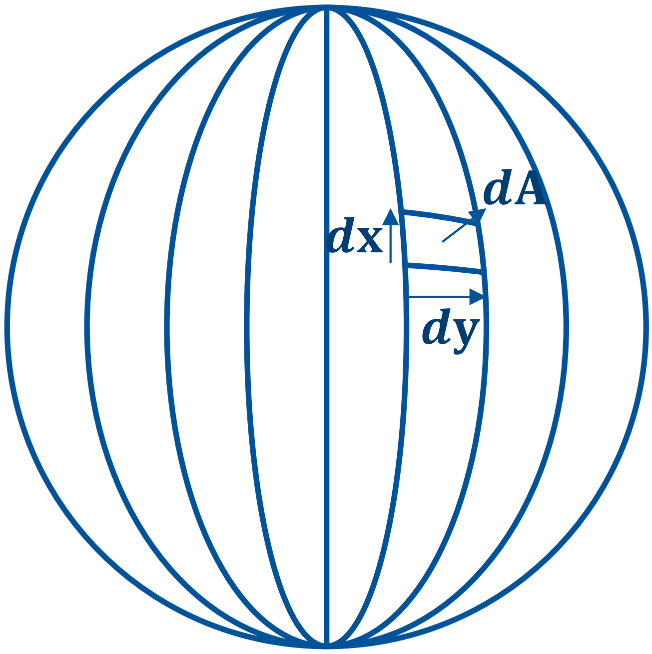
  - We therefore visualise the field as flux lines
  - The direction of the flux lines tells us the direction of the field and the density per unit area the magnitude
  - As lines of electric field begin on positive charges and end on negative charges, we can visualise the field around a sphere of charge



- The Gauss Law is simply a mathematical statement that lines of electric field begin and end on positive and negative charges respectively as

$$\boxed{\oint_S \mathbf{E}.\boldsymbol{d}\mathbf{A} = \frac{Q}{\varepsilon_0\varepsilon_r}} \tag{3.2}$$

  - Let us remind ourselves what this physically means
  - Imagine a closed spherical surface



  - We can divide the surface up into small elements with side vectors $\boldsymbol{d}\mathbf{x}$ and $\boldsymbol{d}\mathbf{y}$ and then

$$\boldsymbol{d}\mathbf{A} = \boldsymbol{d}\mathbf{x} \times \boldsymbol{d}\mathbf{y} \tag{3.3}$$

  - $\boldsymbol{d}\mathbf{A}$ is then a vector normal to the surface with a magnitude equal to the area of the surface
  - $\mathbf{E}.\boldsymbol{d}\mathbf{A}$ is therefore the number of flux lines passing normally out of the surface element $\boldsymbol{d}\mathbf{A}$ and $\oint_S \mathbf{E}.\boldsymbol{d}\mathbf{A}$ is then the net number of flux lines leaving the closed Gaussian surface

- If a flux line merely passes in and out of the volume enclosed, then it has a net zero effect on this integration
- $\oint_S \mathbf{E}.\boldsymbol{dA}$ will only return a non-zero result if a flux line has either begun or ended within the volume, and as lines can only begin and end on charge, this means that a net charge must be inside
- In this way, **the Gauss Law is simply stating that charge produces electric fields**
- The permittivity is there to make sure that the magnitude of $\mathbf{E}$ is such that the relation between force and charge (Eqn. 3.1) is correct

- We should also remember that if we move a small charge q a distance $\boldsymbol{d}\mathbf{x}$ inside an electric field, then because the charge experiences a force, there much be a change in potential energy

$$\text{Change in potential energy} = -\mathbf{F}.\boldsymbol{d}\mathbf{x} \tag{3.4}$$

- However, the volt is defined as the potential energy per unit charge, so

$$\text{Change in potential energy} = q.dV \tag{3.5}$$

- Substituting Eqn 3.1 into Eqn. 3.4 and equating to 3.5 gives

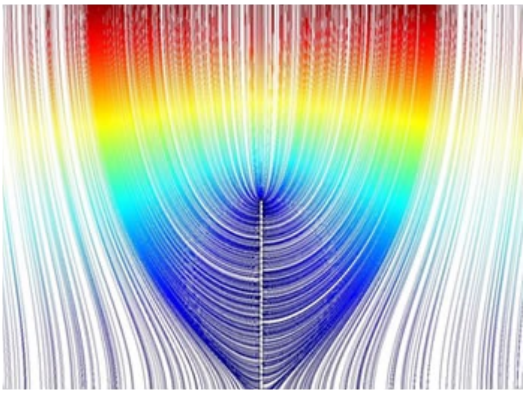
$$-q\mathbf{E}.\boldsymbol{d}\mathbf{x} = q.dV$$

$$|\mathbf{E}| = -dV/dx \tag{3.6}$$

- Or more generally, in a fully vector form

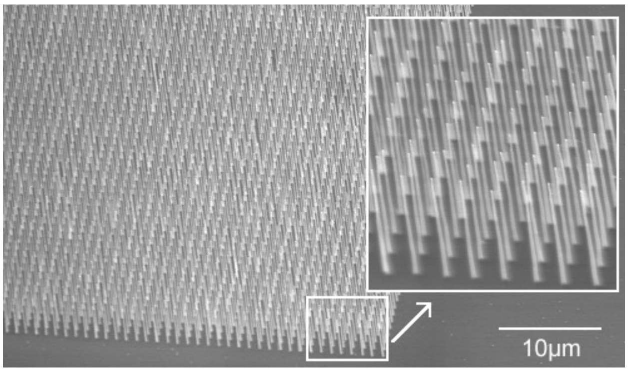
$$\boxed{\mathbf{E} = -\nabla V} \tag{3.7}$$

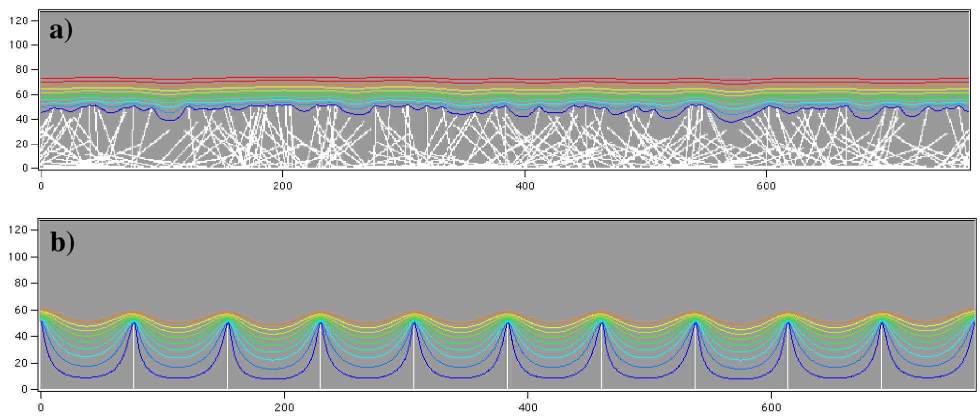
- Therefore, electric flux lines and surfaces of constant potential must always intersect perpendicularly
- It is for this reason that sharp conducting tips concentrate electric fields
  - A conductor has to be at a constant potential so electric flux lines have to intersect the surface perpendicularly
  - Hence the flux lines around a single carbon nanotube:



T. D. Wilkinson *et al.*, *Adv. Mater.*, **20**, 363 (2008)

  - We can engineer arrays of CNTs to emit electrons





- Vector calculus gives us the concept of the divergence of a vector quantity, where for some vector $\mathbf{Y}$

$$\mathrm{div}(\mathbf{Y}) = \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} = \nabla.\mathbf{Y} \qquad (3.7)$$

  - Its interpretation is the net flux of $\mathbf{Y}$ per unit volume out of a small volume element
  - In Section 8.1 of the IB Mathematics (2P7) course, you met the Gauss Theorem for divergence which relates the surface integral of flux to the volume integral of divergence as

$$\int_V (\nabla.\mathbf{Y})dV = \oint_S \mathbf{Y}.d\mathbf{A} \qquad (3.8)$$

  - Hence, the Gauss Law of Electric Fields (Eqn. 3.2) becomes

$$\oint_S \mathbf{E}.d\mathbf{A} = \int_V (\nabla.\mathbf{E})dV = \frac{Q}{\varepsilon_0\varepsilon_r}$$

  - The total charge Q inside the closed surface is just the integral of the charge density ρ over the enclosed volume

$$Q = \int_V \rho.dV \qquad (3.9)$$

$$\int_V (\nabla.\mathbf{E})dV = \frac{1}{\varepsilon_0\varepsilon_r}\int_V \rho.dV$$

$$\boxed{\nabla.\mathbf{E} = \frac{\rho}{\varepsilon_0\varepsilon_r}} \qquad (3.10)$$

- We have just re-expressed the Gauss Law of Electric Fields in differential form
  - This turns out to be a much more powerful form of the Gauss Law as it allows us to use vector calculus to analyse electromagnetic problems, rather as doing so in fluid mechanics allows better engineering in that field
- Finally, we know that an electric field is affected by the material that it is in
  - This is because positive charges experience a force in the direction of $\mathbf{E}$ and negative charges experience a force in the opposite direction
  - Although this has no net effect in the bulk of a homogenous material, it can lead to the build up of a surface charge which then reduces the electric field inside the material by a factor of ε_r
  - In Part IA we therefore defined the electric flux density

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E} \tag{3.11}$$

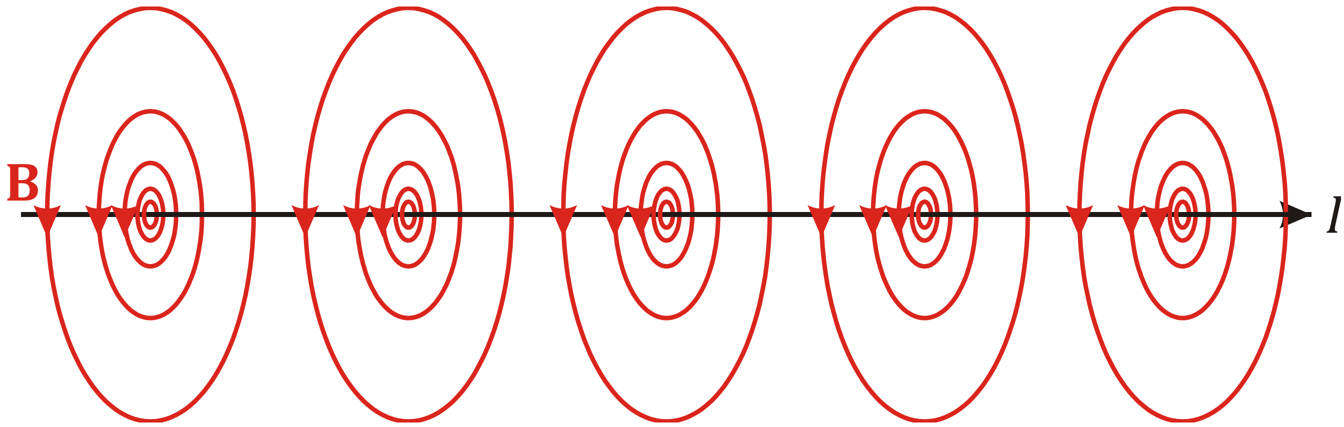
  - $\mathbf{D}$ is independent of medium (although we will refine this statement when we look at boundary conditions), and so it is usually easier to use this in calculations and find the electric field at the end
  - Substituting Eqn. 3.11 into 3.10 gives us a ***general form of the Gauss Law of Electric Fields which can be used in any medium***

$$\boxed{\nabla . \mathbf{D} = \rho} \tag{3.12}$$

The Gauss Law of Magnetic Fields

FLE73

- There is also a Gauss Law of Magnetic Fields which basically states that there are no magnetic monopoles – only dipoles – so lines of magnetic flux density $\mathbf{B}$ form closed loops
  - For example, the $\mathbf{B}$ around a current-carrying wire is:



  - If we imagine drawing a closed Gaussian surface anywhere, the net flux through the surface will always be zero as the flux lines have no beginning or ends; there are no monopoles for them to begin or end on, so

$$\boxed{\oint_S \mathbf{B}.d\mathbf{A} = 0} \qquad (3.13)$$

  - From the Gauss Theorem for divergence

$$\oint_S \mathbf{B}.d\mathbf{A} = \int_V (\nabla.\mathbf{B})dV = 0$$

$$\boxed{\nabla.\mathbf{B} = \mathbf{0}} \qquad (3.14)$$

  - Eqns. 3.13 and 3.14 are the ***Gauss Law for Magnetic Fields*** in integral and differential form respectively

The Faraday Law of Magnetic Fields

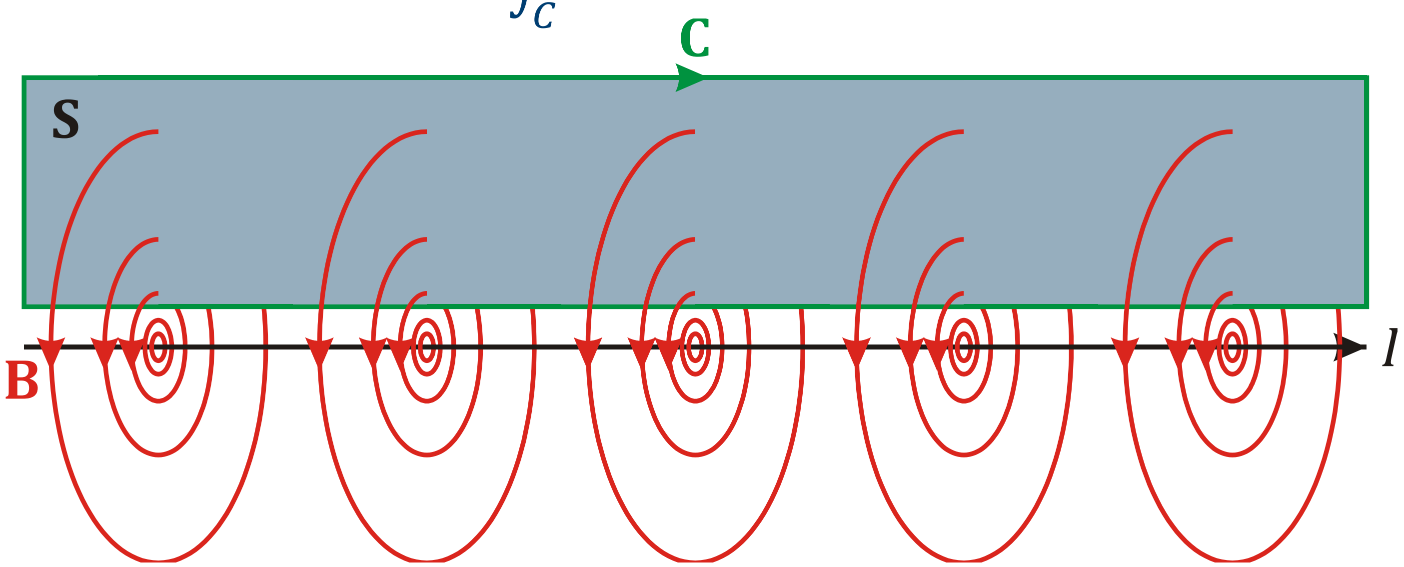
FLE74

- We met the Faraday Law last year in the form

$$V = -\frac{d\Phi}{dt} \tag{3.15}$$

  - where Φ is the magnetic flux
  - It basically states that a changing flux through a coil of wire produces an electromotive force (potential difference) that acts to oppose the changing flux
  - Technically, the minus sign is the Lenz Law
  - If there is a potential difference along the wire, then there must be an electric field in the direction of the wire, from Eqn. 3.6
  - If we imagine a loop of wire, then we can integrate this field around the wire to give the total potential difference around the closed loop

$$\oint_C \mathbf{E}.d\mathbf{r} = V \tag{3.16}$$



- Hence, by equating Eqns. 3.15 and 3.16 we get

$$\oint_C \mathbf{E}.\,d\mathbf{r} = -\frac{d\Phi}{dt} \tag{3.17}$$

- We also know that the total flux through the surface bounded by the closed loop is just the integral of $\mathbf{B}$ over that surface

$$\Phi = \int_S \mathbf{B}.\,d\mathbf{A} \tag{3.18}$$

- Hence, Eqn. 3.17 becomes

$$\boxed{\oint_C \mathbf{E}.\,d\mathbf{r} = -\int_S \frac{d\mathbf{B}}{dt}.\,d\mathbf{A}} \tag{3.19}$$

- This is the ***Faraday Law in integral form***

- Vector calculus also gives us the concept of the curl of a vector quantity, where for some vector field $\mathbf{Y}$

$$\text{curl}\ (\mathbf{Y}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ Y_x & Y_y & Y_z \end{vmatrix} = \nabla\times\mathbf{Y} \tag{3.20}$$

- Its interpretation is the circulation of $\mathbf{Y}$ in a small volume element

- In Section 11.1 of the IB Mathematics (2P7) course, you met the Stokes Theorem which relates the line integral of a vector around a closed loop to the surface integral through the loop as

$$\oint_C \mathbf{Y}.d\mathbf{r} = \int_S (\nabla\times\mathbf{Y}).d\mathbf{A} \tag{3.21}$$

- Hence, we can transform the line integral in the Faraday Law (Eqn. 3.19) to give

$$\int_S (\nabla\times\mathbf{E}).d\mathbf{A} = -\int_S \frac{d\mathbf{B}}{dt}.d\mathbf{A} \tag{3.22}$$

- This yields the ***differential form of the Faraday Law***

$$\boxed{\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}} \tag{3.23}$$

- Although this might feel rather abstract in this form, we should not forget that all it is saying is that a changing magnetic field will induce an electric field
- Note that this is not a divergent electric field, but a circulating electric field, and no charge is required for its production
- This is important for us to understand how electromagnetic waves can propagate through free space in Lecture 4

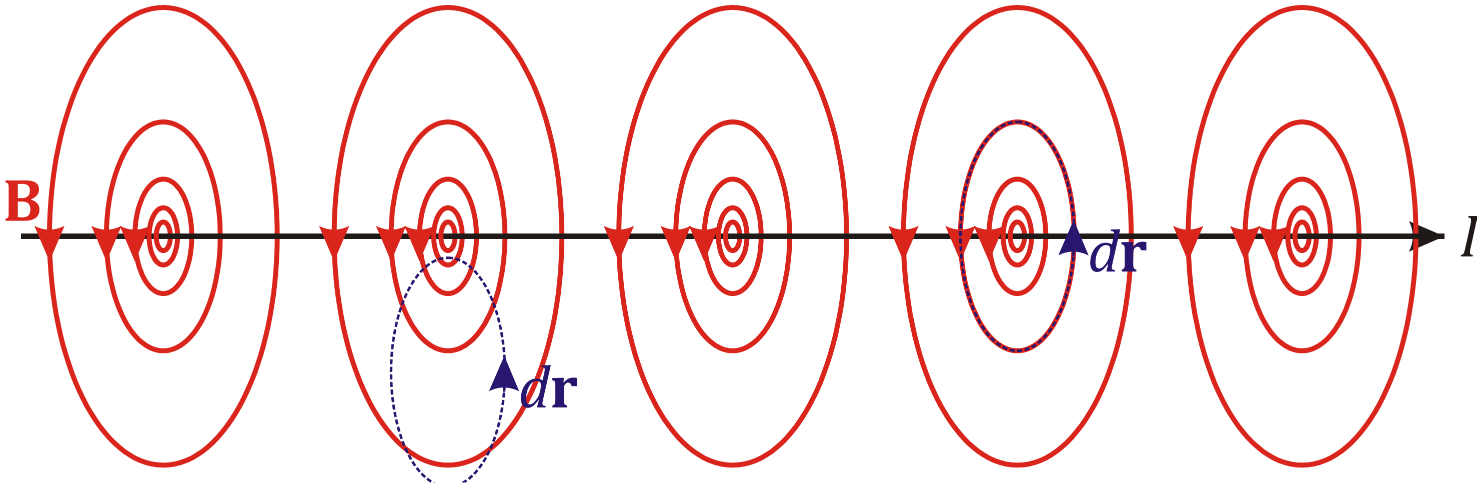
The Ampère-Maxwell Law

FLE76

- We met the ***Ampère Law in integral form*** last year

$$\boxed{\oint_C \mathbf{H}.\,d\mathbf{r} = I} \qquad (3.24)$$

  - It is a statement that a moving charge (i.e. a current I) produces a magnetic field $\mathbf{H}$ that circulates round the current
  - Let us use the example of the current-carrying wire again



  - If we imagine a closed Ampèrian loop which does not enclose a current then the integral of $\mathbf{H}.\,d\mathbf{r}$ around the loop will be zero
  - If the loop does surround the current, and we follow a field line at some radius R, then $\mathbf{H}.\,d\mathbf{r}$ becomes an ordinary product as the two vectors are parallel and

$$\oint_C \mathbf{H}.\,d\mathbf{r} = |\mathbf{H}|2\pi R = I$$

$$|\mathbf{H}| = \frac{I}{2\pi R} \qquad (3.25)$$

- We can again use the Stokes Theorem to convert the line integral of Eqn. 3.24 into a surface integral

$$\oint_C \mathbf{H}.d\mathbf{r} = \int_S (\nabla \times \mathbf{H}).d\mathbf{A} \tag{3.26}$$

  - However, the current I through the Ampèrian loop is just the integral of the current density through the loop

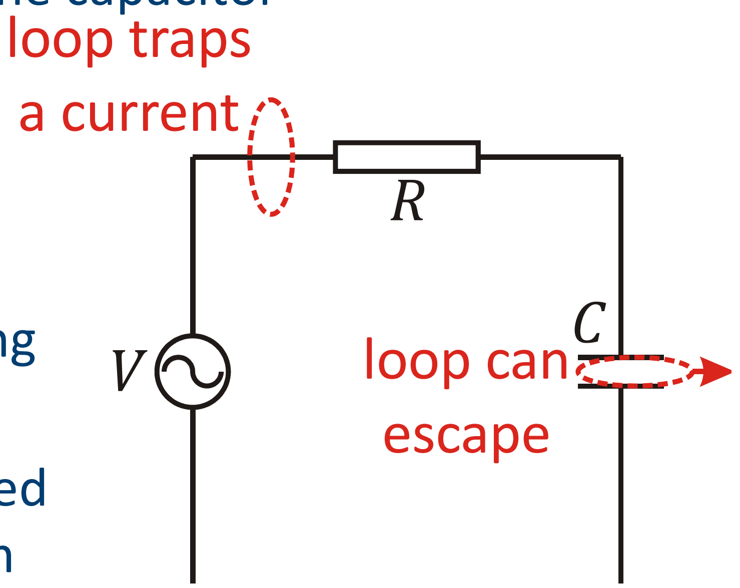
$$I = \int_S \mathbf{J}.d\mathbf{A} \tag{3.27}$$

  - So the Ampère Law becomes

$$\int_S (\nabla \times \mathbf{H}).d\mathbf{A} = \int_S \mathbf{J}.d\mathbf{A}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{3.29}$$

  - It might look as though we have arrived at the differential form of the Ampère Law, but Maxwell noticed a flaw
  - If we have a capacitor in a circuit, then although a current flows around the circuit, no current actually flows between the two plates of the capacitor
  - Therefore, we could put an Ampèrian loop between the plates of the capacitor and appear to get no magnetic field from the current flowing in the rest of the circuit
  - We need to add what is called a ***displacement current*** term



- Let us imagine a small volume of space with some charge density $\rho(\mathbf{r})$ inside it, so that it contains a total charge

$$Q = \int_V \rho(\mathbf{r}) dV \tag{3.30}$$

- If this charge is moving, then the total amount of charge within the volume will be changing, and we can express the rate of change of total charge as either the integral of current density through the surface surrounding the volume (noting that a positive current density reduces the charge enclosed)

$$-\frac{dQ}{dt} = \int_S \mathbf{J}.\, d\mathbf{A} \tag{3.31}$$

- or in terms of the change in the enclosed charge density

$$\frac{dQ}{dt} = \int_V \frac{\partial \rho(\mathbf{r})}{\partial t} dV \tag{3.32}$$

- Applying the Divergence Theorem to Eqn. 3.31 and equating to 3.32 gives

$$-\int_S \mathbf{J}.\, d\mathbf{A} = -\int_V \boldsymbol{\nabla}.\mathbf{J} dV = \int_V \frac{\partial \rho(\mathbf{r})}{\partial t} dV$$

$$\boxed{\nabla.\mathbf{J} = -\frac{\partial \rho(\mathbf{r})}{\partial t}} \tag{3.33}$$

- This should feel intuitively right, as it says that a divergent current density means that the local charge density must be changing, and is a ***continuity equation for charge***

- From the Gauss Law of Electric Fields (Eqn. 3.10), we know

$$\rho = \varepsilon_0 \varepsilon_r \nabla . \mathbf{E}$$

- Therefore, remembering that $\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$

$$\nabla . \mathrm{J} = -\varepsilon_0 \varepsilon_r \nabla . \left(\frac{\partial \mathbf{E}}{\partial t} \right) = -\nabla . \left(\frac{\partial \mathrm{D}}{\partial t} \right) \qquad (3.34)$$

- From vector calculus $\nabla . \nabla \times \mathbf{u} = 0$ (Maths Data Book p16), so $\nabla . \nabla \times \mathbf{H} = 0$
- If the simple differential form of the Ampère Law (Eqn. 3.29) was correct, then this would mean that $\nabla . \mathrm{J} = 0$, which is inconsistent with Eqn. 3.34
- We have to add the term inside the divergence on the right hand side of Eqn. 3.34 to the right hand side of Eqn. 3.29 to make everything consistent, giving the Ampère-Maxwell Law

$$\boxed{\nabla \times \mathbf{H} = \mathrm{J} + \frac{\partial \mathrm{D}}{\partial t}} \qquad (3.35)$$

- We can now verify consistency of our equations as

$$\nabla . \nabla \times \mathbf{H} = \nabla . \left(\mathrm{J} + \frac{\partial \mathrm{D}}{\partial t} \right)$$

$$0 = -\frac{\partial \rho(\mathbf{r})}{\partial t} + \frac{\partial}{\partial t} (\nabla . \mathbf{D})$$

- So from the Gauss Law of Electric Fields (Eqn. 3.12)

$$0 = -\frac{\partial \rho(\mathbf{r})}{\partial t} + \frac{\partial \rho(\mathbf{r})}{\partial t}$$

$$0 = 0$$

Summary of the Maxwell Equations

FLE81

5/7 Q1

- The four Maxwell Equations of Electromagnetism each describe the origin and nature of electric and magnetic fields
- The Gauss Law of Electric Fields

$$\nabla.\mathbf{D} = \rho \tag{3.12}$$

  - Charge produces an electric field
- The Gauss Law of Magnetic Fields

$$\nabla.\mathbf{B} = 0 \tag{3.14}$$

  - There are only magnetic dipoles (no monopoles) so lines of magnetic flux form closed loops
- The Faraday Law of Magnetic Fields

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{3.23}$$

  - A changing magnetic flux density induces an electric field
- The Ampère-Maxwell Law

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \tag{3.35}$$

  - Both a moving charge and a changing electric flux density induce a magnetic field
- We can now use the Maxwell Equations to understand the electromagnetic world in which we live and how to engineer it