# Part IB Paper 6: Information Engineering

# LINEAR SYSTEMS AND CONTROL

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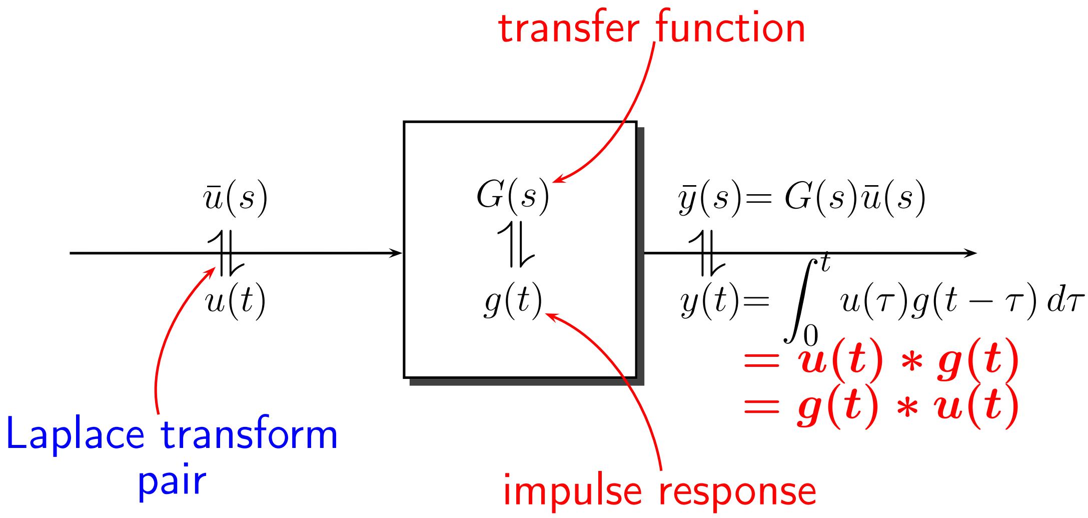
## HANDOUT 2

## "Impulse responses, step responses and transfer functions."



# Summary

The

- impulse response,
- step response and
- transfer function

of a Linear, Time Invariant and causal (LTI) system *each* completely characterize the input-output properties of that system.

Given the input to an LTI system, the output can be deterermined:

- In the time domain: as the *convolution* of the impulse response and the input.
- In the Laplace domain: as the *multiplication* of the transfer function and the Laplace transform of the input.

They are related as follows:

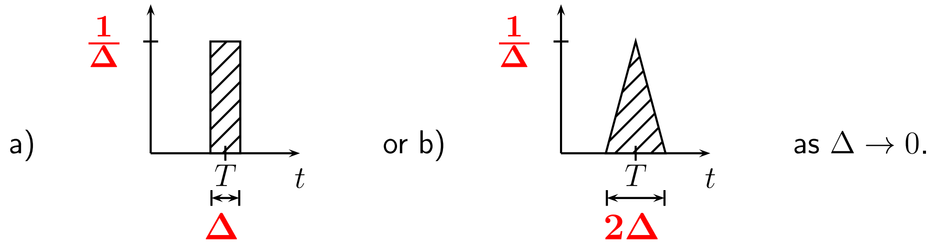
- The step response is the integral of the impulse response.
- The transfer function is the Laplace transform of the impulse response.

# Contents

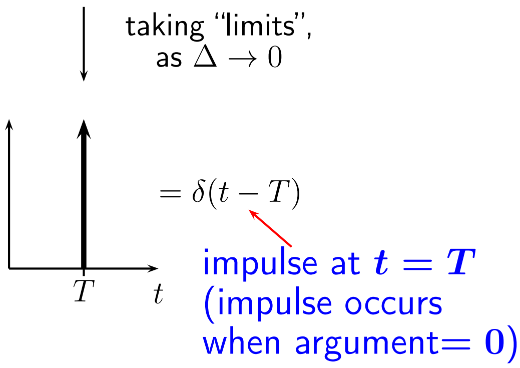
## 2.1 Preliminaries

### 2.1.1 Definition of the impulse "function"

The impulse can be defined in many different ways, for example:




(area$=$ **1** in each case.)




Note: The "convergence", as $\Delta \to 0$, occurs in the sense that both functions a) and b) have the same properties in the limit.

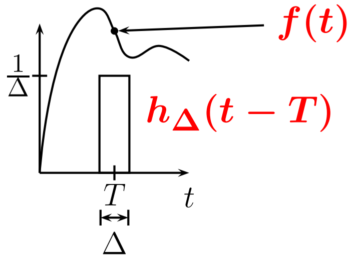
The *impulse* function is the limit of a square or triangular pulse that has area equal to 1 and is zero everywhere apart within an interval of width $\Delta$ that tends to 0.

As we will see later in this handout it is an important concept in the study of linear systems.

### 2.1.2 Properties of the impulse "function"

Consider a <u>continuous</u> function $f(t)$, and let $h_\Delta(t)$ denote the pulse approximation to the impulse ( (a) on previous page):

$$h_\Delta(t) = \begin{cases} 1/\Delta & \text{if } -\Delta/2 < t < \Delta/2 \\ 0 & \text{otherwise} \end{cases}$$

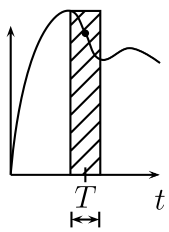



Let

$$I_\Delta = \int_{-\infty}^{\infty} f(t) h_\Delta(t-T)\, dt = \int_{T-\frac{\Delta}{2}}^{T+\frac{\Delta}{2}} f(t) \frac{1}{\Delta}\, dt \quad = \frac{1}{\Delta} \int_{T-\frac{\Delta}{2}}^{T+\frac{\Delta}{2}} f(t)\, dt$$

Now,

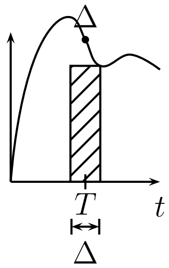
$I_\Delta \leq$ $\frac{1}{\Delta}\times$




$$= \frac{1}{\cancel{\Delta}} \times \cancel{\Delta} \times \max_{T-\Delta/2 \leq t \leq T+\Delta/2} f(t)$$

and also

$I_\Delta \geq$ $\frac{1}{\Delta}\times$




$$= \frac{1}{\cancel{\Delta}} \times \cancel{\Delta} \times \min_{T-\Delta/2 \leq t \leq T+\Delta/2} f(t)$$

So, it must be the case that $\lim_{\Delta\to 0} I_\Delta = \boldsymbol{f(T)}$, that is

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} f(t) h_\Delta(t-T)\, dt = \boldsymbol{f(T)}$$

The impulse function $\delta(t)$ satisfies the property

$$\int_{-\infty}^{\infty} f(t)\delta(t-T)\, dt = f(T)$$

This is shown on the left by considering the integral above when $\delta(t-T)$ is replaced with a pulse $h_\Delta(t-T)$ of width $\Delta$ and area 1. Upper and lower bounds to this integral are obtained and it is then shown that when $\Delta \to 0$ the integral tends to $f(T)$.

A similar argument leads to the same result for the triangular approximation to the impulse ( defined above as (b) ).

Formally, the *unit impulse* is defined as **any** "function" $\delta(t)$ which has the property

$$\int_{-\infty}^{\infty} f(t)\delta(t-T)\,dt = f(T)$$

(*Strictly speaking,* $\delta(t)$ *is a* distribution *rather than a function.*)

**Laplace transform of** $\delta(t)$:

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt \boldsymbol{= e^{-s\times 0} = 1}$$

### 2.1.3 The Impulse and Step Responses

> DEFINITION: The *impulse response* of a system is the output of the system when the input is an impulse, $\delta(t)$, and all initial conditions are zero.

> DEFINITION: The *step response* of a system is the output of the system when the input is a step, $H(t)$, and all initial conditions are zero.

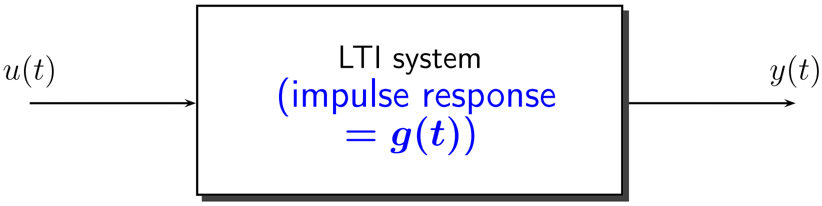
where $H(t)$ is the unit step function

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

If you know the impulse response of a system, then the response of that system to *any* input can be determined using convolution, as we shall now show:

## 2.2 The convolution integral

### 2.2.1 Direct derivation of the convolution integral




| INPUT | OUTPUT |
|---|---|
| $\delta(t)$ | $g(t)$ ← impulse response |
| $\delta(t-\tau)$ | $g(t-\tau)$ |
| $u(\tau)\delta(t-\tau)$ | $u(\tau)g(t-\tau)$ |
| $\underbrace{\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\,d\tau}_{u(t)}$ | $\int_{-\infty}^{\infty} u(\tau)g(t-\tau)\,d\tau$ |

Hence, the response to the input $u(t)$ is given by:

$$\boxed{y(t) = \int_{-\infty}^{\infty} u(\tau)g(t-\tau)d\tau}$$

The slide on the left shows that the output $y(t)$ of a linear system for a given input $u(t)$, is equal to the convolution of the impulse response $g(t)$ of the system with $u(t)$.

This is a very important result as it shows that the impulse response of a linear system fully characterizes the system. This is in the sense that it allows to deduce the output $y(t)$ for any input $u(t)$.

### 2.2.2 Alternative statements of the convolution integral

The convolution integral

$$y(t) = \int_{-\infty}^{\infty} u(\tau)g(t-\tau)d\tau$$

is abbreviated as

$$\boldsymbol{y(t) = u(t) * g(t)}$$

Let $T = t - \tau$, so $\tau = t - T$ and $d\tau = -dT$

It follows that

$$\boxed{\boldsymbol{y(t) = \int_{-\infty}^{\infty} u(t-T)g(T)\,dT}}$$

$$= g(t) * u(t)$$

NOTE: in either form of the convolution integral,

the arguments of $\boldsymbol{u(\cdot)}$ and $\boldsymbol{g(\cdot)}$ add up to $\boldsymbol{t}$

If, in addition,

- $g(t) = 0$ for $t < 0$ (CAUSALITY)
- $u(t) = 0$ for $t < 0$ (standing assumption), then

as $\boldsymbol{g(t-\tau) = 0}$ for $\boldsymbol{t - \tau < 0}$ (i.e. $\boldsymbol{\tau > t}$)

$$y(t) = \int_{0}^{t} u(\tau)g(t-\tau)\,d\tau$$

as $\boldsymbol{u(\tau) = 0}$ for $\boldsymbol{\tau < 0}$

and also

as $\boldsymbol{u(t-\tau) = 0}$ for $\boldsymbol{t - \tau < 0}$ (i.e. $\boldsymbol{\tau > t}$)

$$y(t) = \int_{0}^{t} u(t-\tau)g(\tau)\,d\tau$$

as $\boldsymbol{g(\tau) = 0}$ for $\boldsymbol{\tau < 0}$

It is shown on the left how the convolution integral can be simplified. In particular, we exploit the fact that the impulse response $g(t)$ is 0 for $t < 0$ (due to causality). Furthermore, we will use throughout the course the convention that $u(t) = 0$ for $t < 0$. These two properties imply that the convolution integral only needs to be evaluated from 0 to $t$ (rather than from $-\infty$ to $\infty$).

## 2.3 The Transfer Function (for ODE systems)

As an alternative to convolution, the response of a linear system to arbitrary inputs can be determined using Laplace transforms. This is clear when the system is described as a linear ODE:

For example, if a linear system has input $u$ and output $y$ satisfying the ODE

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = a\frac{du}{dt} + bu$$

and if *all initial conditions are zero*, i.e. $\left.\frac{dy}{dt}\right|_{t=0} = y(0) = u(0) = 0$, then taking Laplace transforms gives

$$s^2\bar{y}(s) + \alpha s\bar{y}(s) + \beta\bar{y}(s) = a\,s\bar{u}(s) + b\bar{u}(s)$$

or

$$\Big(s^2 + \alpha s + \beta\Big)\,\bar{y}(s) = \Big(a\,s + b\Big)\bar{u}(s)$$

and so

$$\bar{y}(s) = \underbrace{\frac{a\,s + b}{s^2 + \alpha s + \beta}}_{\text{Transfer function}}\bar{u}(s)$$

The function $\frac{a\,s+b}{s^2+\alpha s+\beta}$ is called the *transfer function* from $\bar{u}(s)$ (the input) to $\bar{y}(s)$ (the output).

Clearly the same technique will work for higher order linear ordinary differential equations (with constant coefficients). For such systems, the transfer function can be regarded as a placeholder for the coefficients of the differential equation.

### 2.3.1 Laplace transform of the convolution integral

We have seen that both convolution with the impulse reponse (in the time domain) and multiplication by the transfer function (in the Laplace domain) can both be used to determine the output of a linear system. What is the relationship between these techniques?

Assume that $g(t) = u(t) = 0$ for $t < 0$,

$$
\begin{aligned}
\boxed{\mathcal{L}\Big(g(t) * u(t)\Big)} &= \mathcal{L}\left(\int_{-\infty}^{\infty} g(\tau)u(t-\tau)\,d\tau\right) \\
&= \int_{0}^{\infty} e^{-st} \int_{-\infty}^{\infty} g(\tau)u(t-\tau)\,d\tau\,dt \\
&= \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-st} g(\tau)u(t-\tau)\,dt\,d\tau
\end{aligned}
$$

Note that $\tau$ is constant for the inner integration, and let
$T = t - \tau$, which implies $t = T + \tau$ and $dt = dT$,
giving (so $e^{-st} = e^{-sT}e^{-s\tau}$)

$$
\begin{aligned}
\mathcal{L}\Big(g(t) * u(t)\Big) &= \int_{\cancel{-\infty}\;0}^{\infty} \int_{\cancel{-\tau}\;0}^{\infty} e^{-s\tau}e^{-sT} g(\tau)u(T)\,dT\,d\tau \\
&= \int_{0}^{\infty} e^{-s\tau}g(\tau)d\tau \times \int_{0}^{\infty} e^{-sT}u(T)dT \\
&= \boxed{\mathcal{L}\Big(g(t)\Big) \times \mathcal{L}\Big(u(t)\Big)} = \bar{g}(s)\,\bar{u}(s)
\end{aligned}
$$

In words, a Laplace transform (easy when you get used to them) turns a convolution (always hard!) into a multiplication (very easy).

## 2.4 The transfer function for *any* linear system

It follows from the result of the previous section that, if

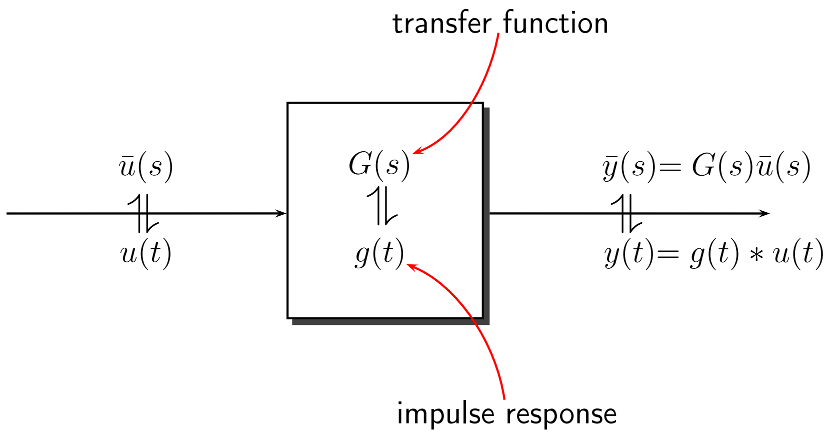
$$y(t) = g(t) * u(t)$$

is the response of an LTI system with impulse response $g(t)$ to the input $u(t)$, then we can also write

$$\boldsymbol{\bar{y}(s) = G(s)\,\bar{u}(s)}$$

where $G(s) = \mathcal{L}g(t)$ is called the *transfer function* of the system.

It follows that *all* LTI systems have transfer functions (given by the Laplace transform of their impulse response).




*We shall use the notation $\bar{x}(s)$ to represent the Laplace transform of a signal $x(t)$, and uppercase characters to represent transfer functions (e.g. $G(s)$).*

In general a system may have more than one input. In this case, the transfer function from a particular input to a particular output is defined as the Laplace transform of that output when an impulse is applied to the given input, all other inputs are zero and all initial conditions are zero.

This is most easily seen in the Laplace domain: If an LTI system has an input $u$ and an output $y$ then we can always write

$$\bar{y}(s) = G(s)\bar{u}(s) + \text{other terms independent of } u.$$

$G(s)$ is then called the transfer function from $\bar{u}(s)$ to $\bar{y}(s)$. (Or, the transfer function relating $\bar{y}(s)$ and $\bar{u}(s)$)

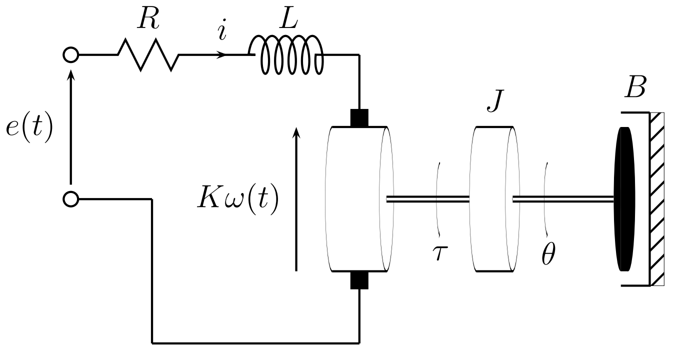
Here the "other terms" could be a result of non-zero initial conditions or of other non-zero inputs (disturbances, for example).

**NOTE: Remember that although the transfer function is defined in terms of the impulse response, it is usually most easily calculated directly from the system's differential equations.**

## 2.5 Example: DC motor

The following example will illustrate the impulse and step responses and the transfer function. We take the input to the motor to be the applied voltage $e(t)$, and the output to be the shaft *angular velocity*

$$\omega := \dot{\theta}(t).$$




1) $\boldsymbol{\tau(t) = Ki(t)}$ Linearized motor equation

2) $\boldsymbol{\tau(t) - B\omega(t) = J\dot{\omega}(t)}$ Newton

3) $\boldsymbol{e(t) = Ri(t) + L\frac{di(t)}{dt} + K\omega(t)}$ Kirchoff

**Find the effect of $e(t)$ on $\omega(t)$:**

Take Laplace Transforms, assuming that $\omega(0) = i(0) = 0$:

1) $\bar{\tau}(s) = \boldsymbol{K\bar{i}(s)}$

2) $\bar{\tau}(s) - B\boldsymbol{\bar{\omega}(s)} = J\left(\boldsymbol{s\bar{\omega}(s) - \omega(0)}\right)$

3) $\bar{e}(s) = R\boldsymbol{\bar{i}(s)} + L\left(\boldsymbol{s\bar{i}(s) - i(0)}\right) + K\bar{\omega}(s)$

We can now eliminate $\bar{i}(s)$ and $\bar{\tau}(s)$ to leave one equation relating $\bar{e}(s)$ and $\bar{\omega}(s)$:

First 1) and 2) give:

$$K\bar{i}(s) = (Js + B)\bar{\omega}(s)$$

and rearranging 3) gives:

$$\bar{e}(s) = (Ls + R)\bar{i}(s) + K\bar{\omega}(s)$$

putting these together leads to

$$\bar{e}(s) = \left( (Ls + R)\frac{(Js + B)}{K} + K \right) \bar{\omega}(s)$$

or, in the standard form (outputs on the left)

$$\begin{aligned} \bar{\omega}(s) &= \frac{K}{(Ls + R)(Js + B) + K^2}\bar{e}(s) \\ &= \frac{k}{(sT_1 + 1)(sT_2 + 1)}\bar{e}(s) \end{aligned}$$

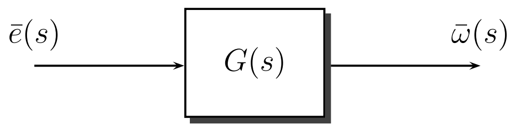
for suitable definitions of $k$, $T_1$ and $T_2$ (assuming real roots).

That is,

$$\overset{\text{output}}{\bar{\omega}(s)} \;=\; \underbrace{\frac{\boldsymbol{k}}{\boldsymbol{(T_1 s + 1)(T_2 s + 1)}}}_{\text{Transfer Function}} \overset{\text{input}}{\bar{e}(s)}$$

We call $G(s) = \frac{k}{(sT_1+1)(sT_2+1)}$ the transfer function *from* $\bar{e}(s)$ *to* $\bar{\omega}(s)$ (or, relating $\bar{\omega}(s)$ and $\bar{e}(s)$).

The diagram




represents this relationship between the input and the output:

By using Laplace transforms, all LTI blocks can be treated as multiplication by a transfer function.

*For the motor used in the lego demonstrations we have* $k \approx 2.2$, $T_1 \approx 54$*ms and* $T_2 \approx 1$*ms.*

As illustrated in the example it is more convenient to find the output of a system when a particular input is applied by working in the Laplace domain. In particular we find first the *transfer function* relating the input and output of interest.

Then the Laplace transform of the output can be evaluated by multiplying the transfer function with the Laplace transform of the input.

### 2.5.1 Impulse Response of the DC motor

To find the impulse response, we let

$$e(t) = \delta(t) \longleftarrow \text{unit impulse}$$

which implies

$$\bar{e}(s) = \mathbf{1}$$

and put

$$\omega(0) = i(0) = 0.$$

So,

$$\bar{\omega}(s) = \frac{k}{(T_2 s + 1)(T_1 s + 1)} \times \mathbf{1}$$

Split into partial fractions:

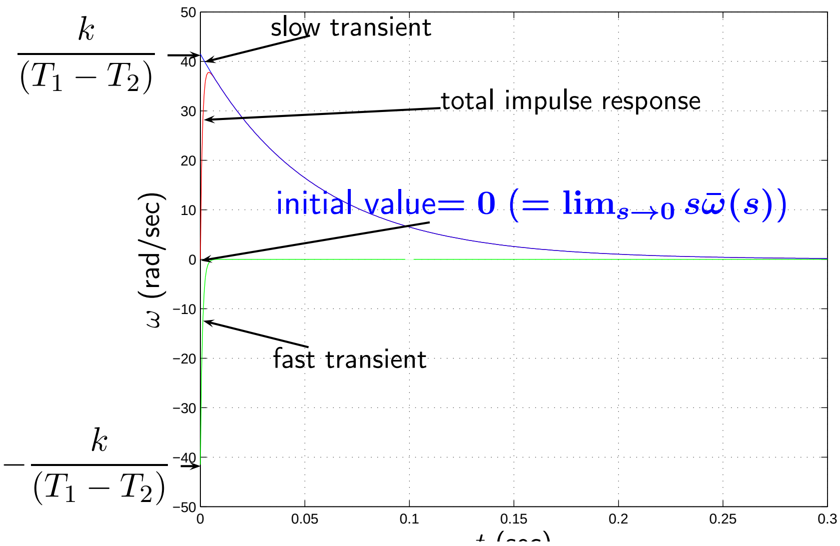
$$\bar{\omega}(s) = k\left[\frac{1}{1-\dfrac{T_1}{T_2}} \times \frac{1}{sT_2+1} + \frac{1}{1-\dfrac{T_2}{T_1}} \times \frac{1}{sT_1+1}\right]$$

$$= \frac{k}{T_1 - T_2}\left[-\frac{1}{s+1/T_2} + \frac{1}{s+1/T_1}\right]$$

Hence,

$$\omega(t) = \frac{k}{T_1 - T_2}\left[\underbrace{-e^{-t/T_2}}_{\text{Fast Transient}} + \underbrace{e^{-t/T_1}}_{\text{Slow Transient}}\right]$$

$$\text{e.g.} \quad T_2 \approx 1\text{ms} \qquad T_1 \approx 54 \text{ ms}$$




It should be noted that the response of the system is dominated by the slow transient, i.e. the term with the largest time constant $T_i$.

Note also that the coefficients multiplying the time $t$ in the exponential terms are equal to the poles of the transfer function.

### 2.5.2 Step Response of the DC motor

To find the step response, we let

$$e(t) = H(t)$$

which implies that

$$\bar{e}(s) = \frac{\mathbf{1}}{\boldsymbol{s}}$$

and put

$$\omega(0) = i(0) = 0$$

(as for the impulse response.) So,

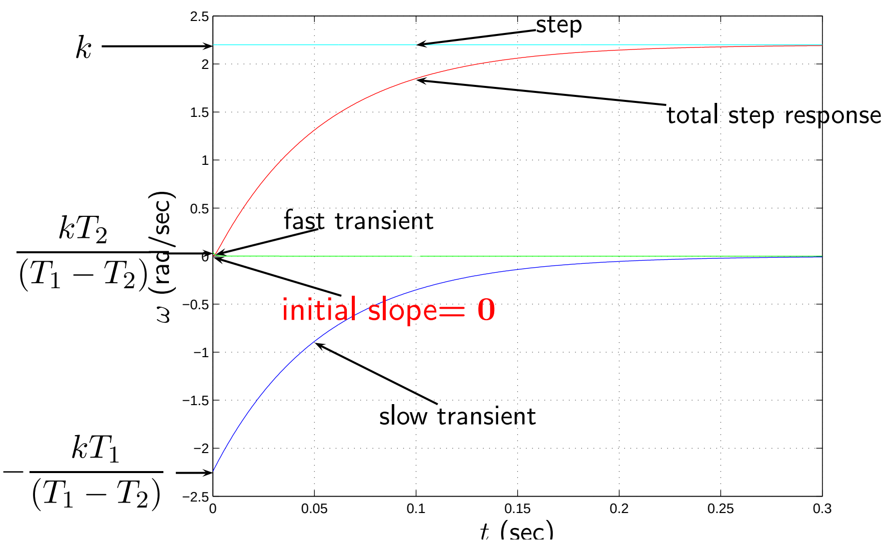
$$\bar{\omega}(s) = \frac{k}{(T_2 s + 1)(T_1 s + 1)} \times \frac{\mathbf{1}}{\boldsymbol{s}}$$

Split into partial fractions:

$$\bar{\omega}(s) = k\left[\frac{1}{s} + \frac{T_2}{T_1 - T_2} \times \frac{1}{s + \frac{1}{T_2}} - \frac{T_1}{T_1 - T_2} \times \frac{1}{s + \frac{1}{T_1}}\right]$$

Hence,

$$\omega(t) = k\left[H(t) + \underbrace{\frac{T_2}{T_1 - T_2} \times \boldsymbol{e^{-t/T_2}}}_{\text{Fast Transient}} - \underbrace{\frac{T_1}{T_1 - T_2} \times \boldsymbol{e^{-t/T_1}}}_{\text{Slow Transient}}\right]$$




As in the case of the impulse response we see that the slow transient, i.e. the exponential term with the largest time constant $T_i$, determines how quickly the output converges to its equilibrium value.

### 2.5.3 Deriving the step response from the impulse response

For a system with impulse response $g(t)$, the step response is given by

$$y(t) = \int_{0^-}^{t} g(\tau)H(t-\tau)\,d\tau = \int_{0^-}^{t} g(\tau)\,d\tau$$

i.e.

$$\boxed{\text{step response} = \text{integral of impulse response}}$$

*Check this on the example. Make sure you understand where the term $H(t)$ in the step response comes from.*

## 2.6 Transforms of signals vs Transfer functions of systems

*Mathematically there is no distinction, but in practice they have different interpretations*

| $G(s)$ | Signals<br>$\mathcal{L}^{-1}G(s)$ | Systems<br>$\bar{y}(s) = G(s)\cdot\bar{u}(s)$<br>$\Updownarrow$ |
|---|---|---|
| $1$ | $\delta(t)$ | $y(t) = u(t)$ |
| $\frac{1}{s}$ | $H(t)$ | $y(t) = \int_0^t u(\tau)d\tau$ (integrator) |
| $\frac{1}{s+a}$ | $e^{-at}$ | $\dot{y}(t) + ay(t) = u(t)$ (lag) |
| $\frac{\omega}{s^2+\omega^2}$ | $\sin(\omega t)$ | $\ddot{y}(t) + \omega^2 y(t) = \omega u(t)$ |
| $e^{-sT}$ | $\delta(t-T)$ | $y(t) = u(t-T)$ (delay) |

In each case, the "signal" is the impulse response of the "system".

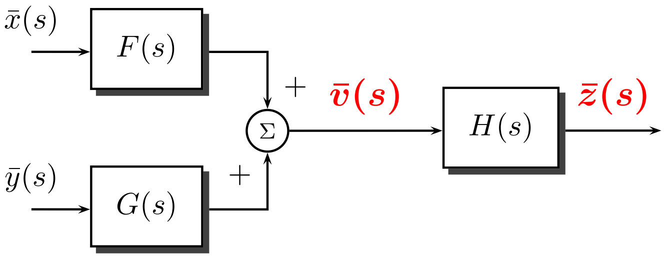
It should be noted that a function of $s$ could be either the Laplace transform of a *signal* or a *transfer function*.

In particular, a transfer function corresponds to a system, and in the time domain it is represented by a differential equation relating the input and output. On the other hand, a signal in the time domain is a function of time. These are therefore two different interpretations. Examples of functions of $s$ and the corresponding signals and systems in the time domain are presented in the table on the left.

## 2.7 Interconnections of LTI systems

One of the main advantages of the use of transfer functions is the fact that they simplify significantly the analysis when we have interconnections of systems. In particular, finding the transfer function of a system that is an interconnection of other systems, becomes a relatively simple problem involving algebraic operations. Such examples are illustrated on the left and in the next page.

**a)**




represents the equations:

$$\boldsymbol{\bar{v}(s) = F(s)\bar{x}(s) + G(s)\bar{y}(s)}$$

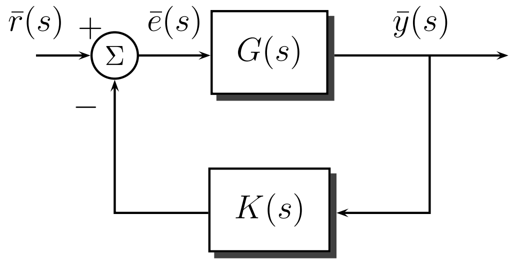
and

$$\boldsymbol{\bar{z}(s) = H(s)\bar{v}(s)}$$

$$\Longrightarrow \ \bar{z}(s) = H(s)\Big(F(s)\bar{x}(s) + G(s)\bar{y}(s)\Big)$$

$$= \underbrace{H(s)F(s)}_{\substack{\text{transfer function} \\ \text{from } \bar{x}(s) \text{ to } \bar{z}(s)}} \bar{x}(s) + \underbrace{H(s)G(s)}_{\substack{\text{transfer function} \\ \text{from } \bar{y}(s) \text{ to } \bar{z}(s)}} \bar{y}(s)$$

**b)**




represents the simultaneous equations:

$$\bar{e}(s) = \bar{r}(s) - K(s)\bar{y}(s)$$

and

$$\bar{y}(s) = G(s)\bar{e}(s)$$

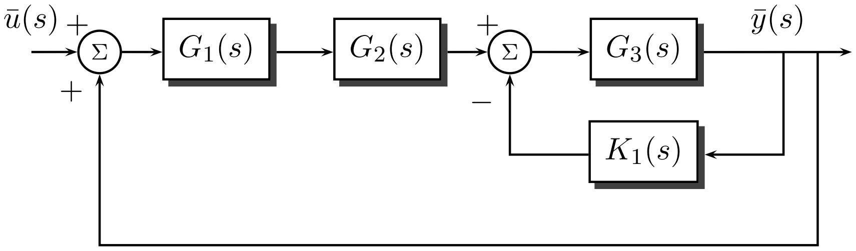
$$\boldsymbol{\Longrightarrow \ \bar{y}(s) = G(s)\bar{r}(s) - G(s)K(s)\bar{y}(s)}$$

$$\boldsymbol{\Longrightarrow \ (1 + G(s)K(s))\bar{y}(s) = G(s)\bar{r}(s)}$$

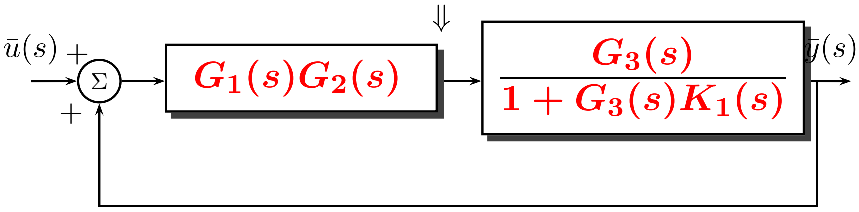
$$\boxed{\boldsymbol{\Longrightarrow \ \bar{y}(s) = \frac{G(s)}{1 + G(s)K(s)}\bar{r}(s)}}$$

### 2.7.1 "Simplification" of block diagrams

Recognizing that blocks represent multiplications, and using the above formulae, it is often easier to rearrange block diagrams to determine overall transfer functions, e.g. from $\bar{u}(s)$ to $\bar{y}(s)$ below.

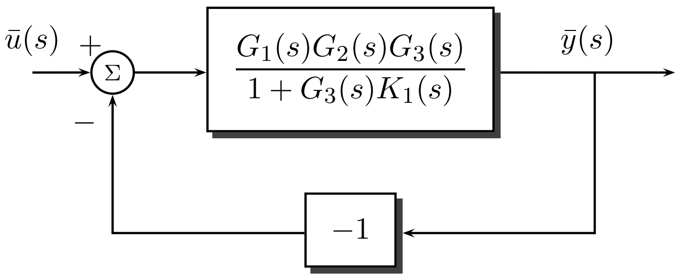



$$\Downarrow$$




$$\Downarrow$$

$$\Downarrow$$




$$\Downarrow$$

$$\bar{y}(s) = \frac{\dfrac{G_1(s)G_2(s)G_3(s)}{1+G_3(s)K_1(s)}}{1 - \dfrac{G_1(s)G_2(s)G_3(s)}{1+G_3(s)K_1(s)}}\bar{u}(s)$$

or

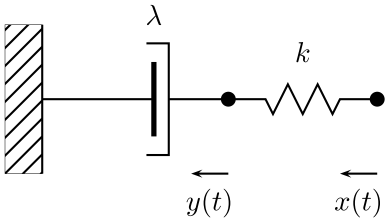
$$y(s) = \frac{\mathbf{G_1(s)G_2(s)G_3(s)}}{\mathbf{1 + G_3(s)K_1(s) - G_1(s)G_2(s)G_3(s)}}\bar{u}(s)$$

## 2.8 More transfer function examples

*To obtain the transfer function, in each case, we take all initial conditions to be zero.*

The following three systems all have the same transfer function (a *1st order lag*)

1) Spring/damper system




$$\lambda \dot{y} = k(x - y)$$

$$\implies \quad \frac{\lambda}{k}\dot{y} + y = x$$

$$\implies \quad \frac{\lambda}{k} s\bar{y}(s) + \bar{y}(s) = \bar{x}(s)$$

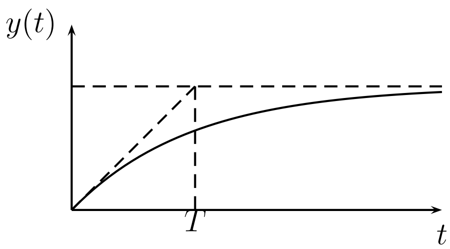
$$\implies \quad \left(\frac{\lambda}{k}s + 1\right)\bar{y}(s) = \bar{x}(s)$$

$$\implies \quad \boxed{\bar{y}(s) = \frac{1}{Ts+1}\bar{x}(s)} \qquad \boldsymbol{T = \lambda / k}$$

Step Response:

$$\bar{x}(s) = \frac{1}{s} \implies \bar{y}(s) = \frac{1}{s(sT+1)} = \frac{1}{s} - \frac{T}{sT+1}$$

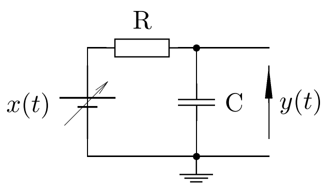
$$\implies \boldsymbol{y(t) = H(t) - e^{-t/T}}$$




The first example is a spring/damper system. The system has input $x$ and output $y$, which are the displacements at the two terminals of the spring, respectively.

The system equation relating $x$ and $y$ is derived by considering the differential equation satisfied by the damper, and Hooke's law applied to the spring.

2) RC network




$$i = C\frac{dy}{dt} = \frac{x-y}{R}$$

$$\Longrightarrow \qquad RC\dot{y} + y = x$$

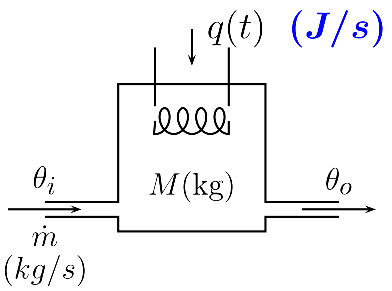
$$\Longrightarrow \qquad (RCs+1)\bar{y}(s) = \bar{x}(s)$$

$$\Longrightarrow \qquad \boxed{\bar{y}(s) = \frac{1}{Ts+1}\bar{x}(s)} \qquad T = RC$$

The second example is an RC electrical circuit where the input $x$ is the supply voltage and the output $y$ is the voltage across the capacitor, as shown in the diagram.

The differential equation relating $x$ and $y$ follows from the differential equation relating the voltage across a capacitor with the current through it (part IA).

3) Water Heater




(assuming perfect mixing – i.e. all the water in the tank is at the same temperature, $\boldsymbol{\theta_o}$)

specific heat capacity

$$q = Mc\dot{\theta}_0 + \dot{m}c(\theta_o - \theta_i)$$

$$\Longrightarrow \qquad (Mcs + \dot{m}c)\bar{\theta}_o(s) = \bar{q}(s) + \dot{m}c\bar{\theta}_i(s)$$

$$\Longrightarrow \qquad \boxed{\bar{\theta}_o(s) = \frac{1}{\dot{m}c}\frac{1}{Ts+1}\bar{q}(s) \; + \frac{1}{Ts+1}\bar{\theta}_i(s)} \qquad \boldsymbol{T = M/\dot{m}}$$

input disturbance

(water flow rate $\dot{m}$ is assumed to be constant)

The third example is a water heater system where the input $q$ is the rate with which energy is supplied to the system and the output $\theta_0$ is the temperature of the water in the tank.

The differential equation relating these variables is a statement of conservation of energy. It quantifies the fact that the energy supplied to the system is used to raise the temperature of the water already within the water tank, and of the water flowing into the tank with constant rate $\dot{m}$.

## 2.9 Key Points

- The step response is the integral (w.r.t time) of the impulse response.
- The transfer function is the Laplace transform of the impulse response.
- By using transfer functions, block diagrams represent simple algebraic relationships (all blocks become multiplications).