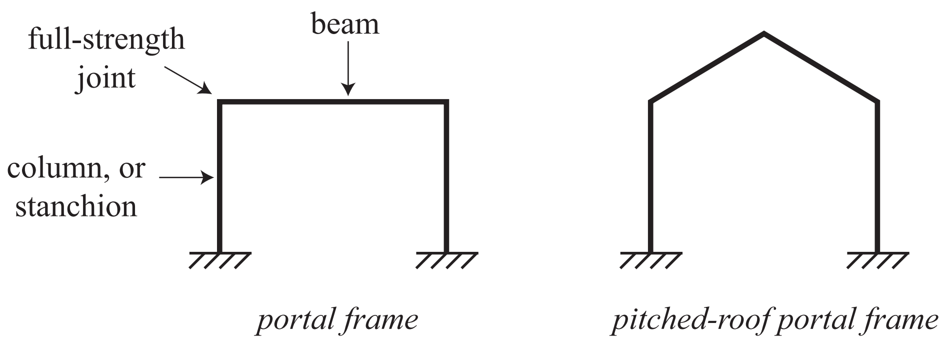
4.6 Plastic collapse of plane frames

Plastic theory can be applied to more complex structures than simple beams. In this section we will consider the collapse of *portal frames*, not only because they are important for many industrial structures, but they also provide a building block for more complex framed structures.



The loading on these structures is often simplified as two point loads, a horizontal load H to represent e.g. wind load, and a vertical load V to represent e.g. snow load.

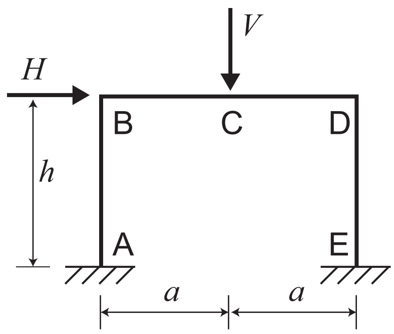
Assumptions

(1) Assume buckling does not occur. In practice, this is checked for separately.

(2) Neglect effect of axial load on plastic moment M_p. When calculating Z_p, we assumed there was no axial load in the member. In the columns of portal frames, however, there will be some axial load. In practice, this axial force would only cause a small area of the section to yield, and the effect on Z_p is minor.

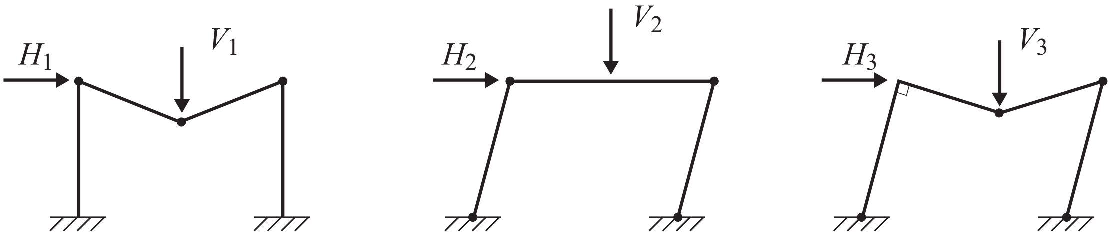
4.6.1 Example — Portal Frame

Calculate the possible failure modes of the portal frame shown, when subjected to positive loads V and H.



Feasible collapse mechanisms

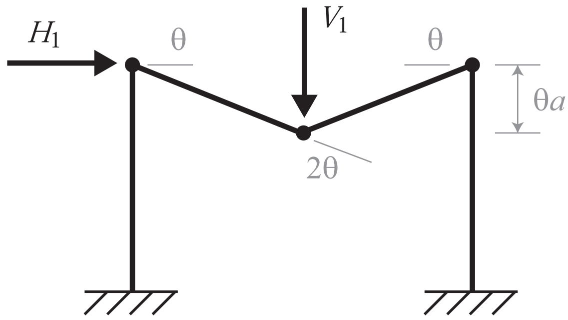
To find a good upper bound, it is only necessary to consider placing hinges where point loads are applied, or where the section changes. In this example, hinges will only form at A, B, C, D or E, which leaves the following as sensible mechanisms to be considered:



Beam mechanism *Sway mechanism* *Combined mechanism*

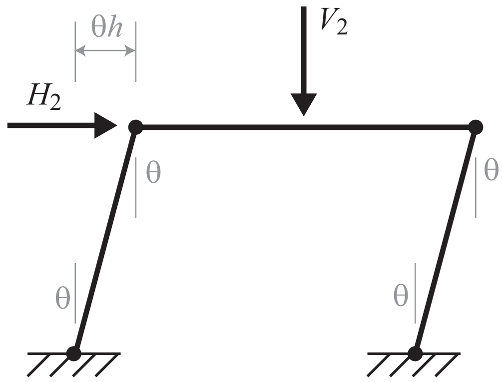
We will now examine the work equation for each of these mechanisms, each of which will give an upper bound on the collapse load

Beam mechanism



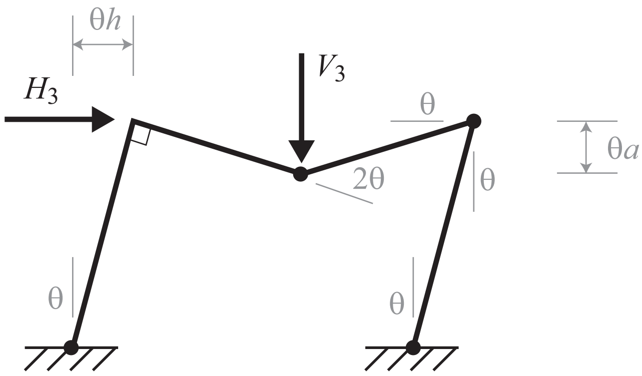
$$V_1\theta a = M_p(2\theta + \theta + \theta)$$
$$V_1 a = 4M_p \qquad (1)$$

Sway mechanism



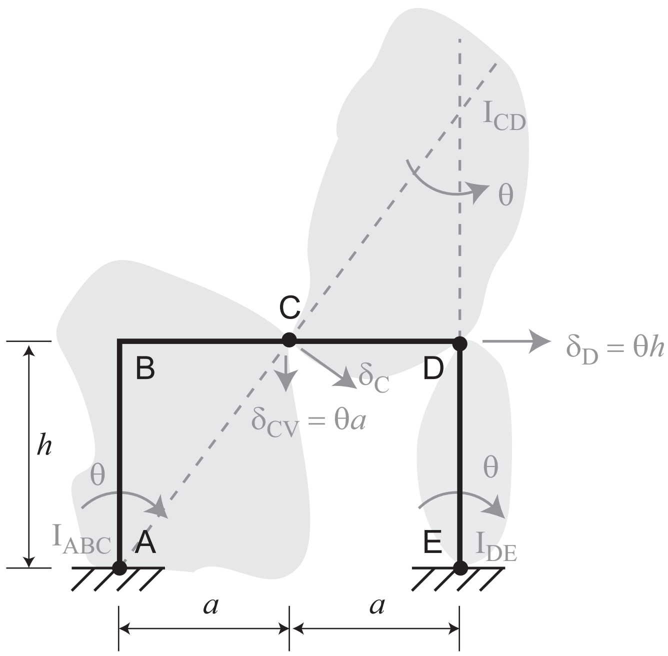
$$H_2\theta h = M_p 4\theta$$
$$H_2 h = 4M_p \qquad (2)$$

Combined mechanism



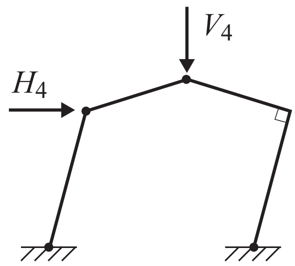
$$H_3\theta h + V_3\theta a = M_p(\theta + 2\theta + 2\theta + \theta)$$
$$H_3 h + V_3 a = 6M_p \qquad (3)$$

The geometry for this example is much harder. The best scheme is to consider using the Mechanics concept of Instantaneous Centres, I, for each of the rigid regions between plastic hinges — in this example ABC, CD and DE.



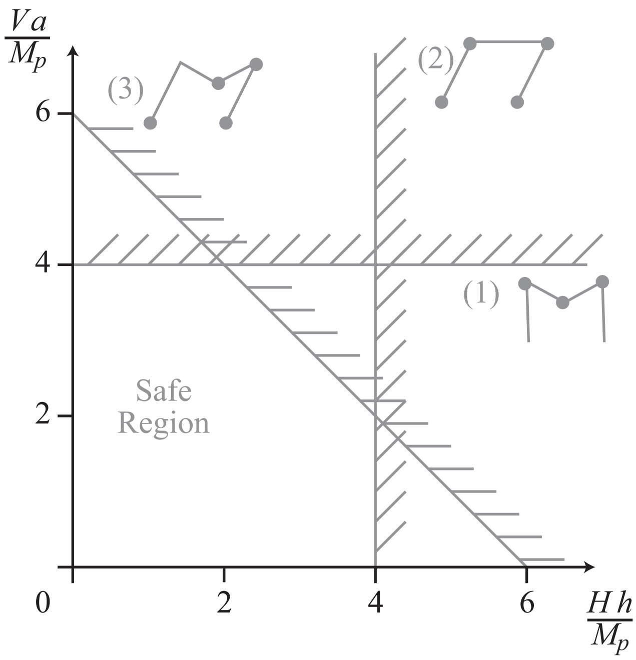
Other feasible mechanisms

There are other feasible mechanisms, for instance this one shown, or the frame swaying in the other direction, but they will not prove critical for positive V and H.

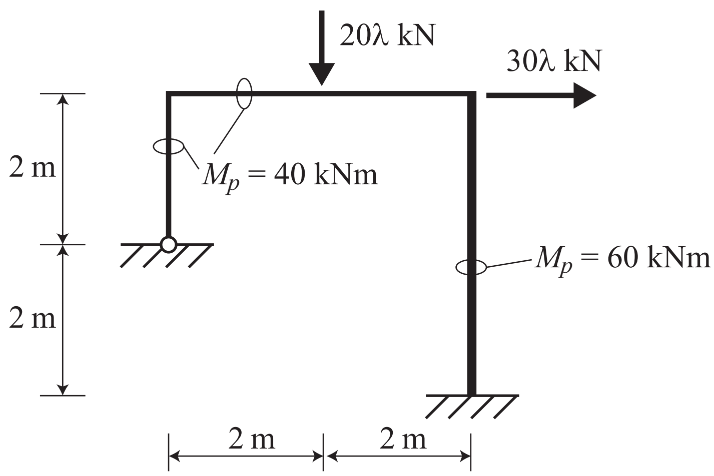


Interaction diagram

All the equations we have calculated have been *upper* bounds. Failure will occur at or before the loads calculated. A good way of understanding these results is to plot them on an interaction diagram for different values of V and H.

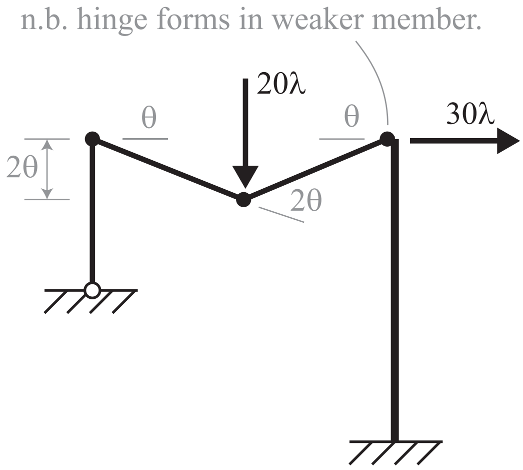


4.6.2 Example — Non-symmetric portal frame



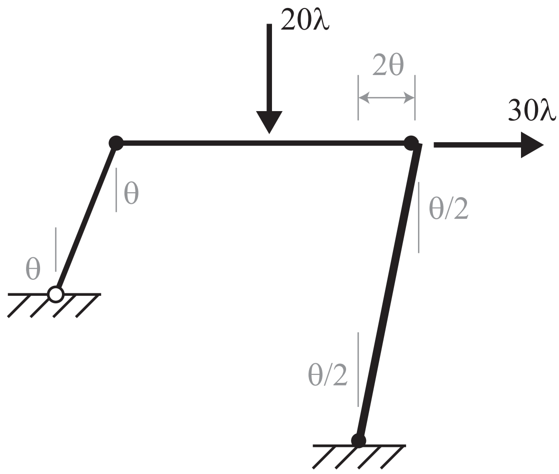
Calculate the load factor λ that would cause the structure to collapse.

Beam mechanism



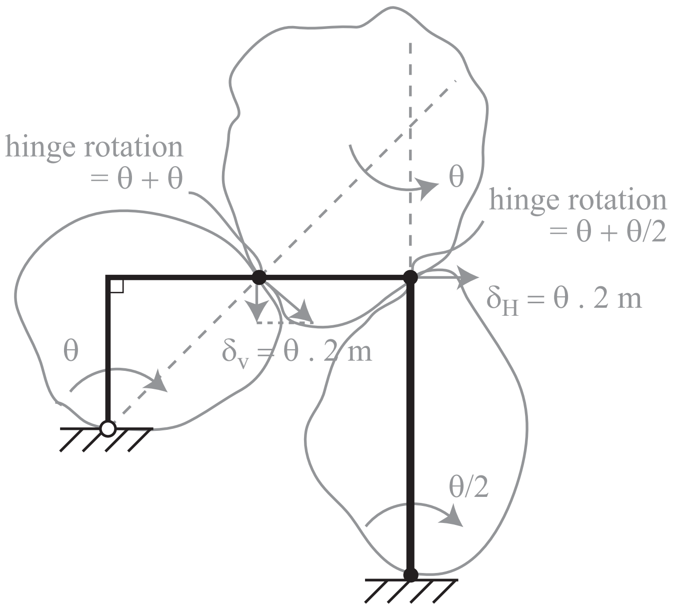
$$20\lambda \times 2\theta = 40(\theta + \theta + 2\theta)$$
$$\lambda = 4$$

Sway mechanism



$$30\lambda \times 2\theta = 40(\theta + \theta/2) + 60(\theta/2)$$
$$\lambda = 1.5$$

Combined mechanism

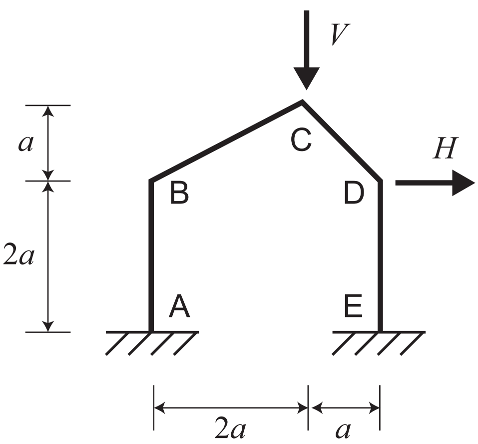


$$30\lambda \times 2\theta + 20\lambda \times 2\theta = 40(2\theta + 3\theta/2) + 60(\theta/2)$$
$$\lambda = 1.7$$

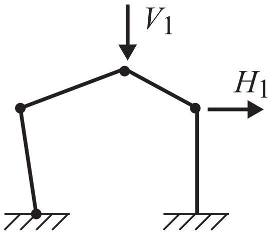
Sway mechanism is most critical, and the load factor at collapse is 1.5.

Example - pitched roof portal frame

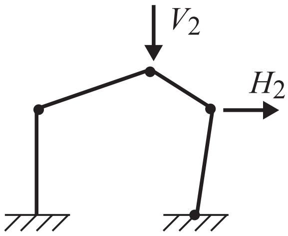
Draw an interaction diagram for the following structure, which has uniform plastic moment.



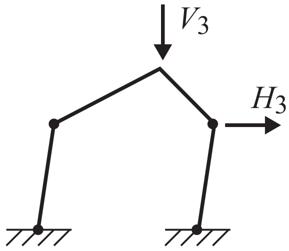
The same mechanisms are possible for this structure as the normal portal frame, but now the beam mechanism has to push out one of the legs:



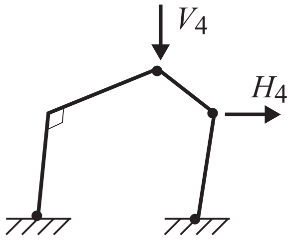
Beam mechanism 1



Beam mechanism 2

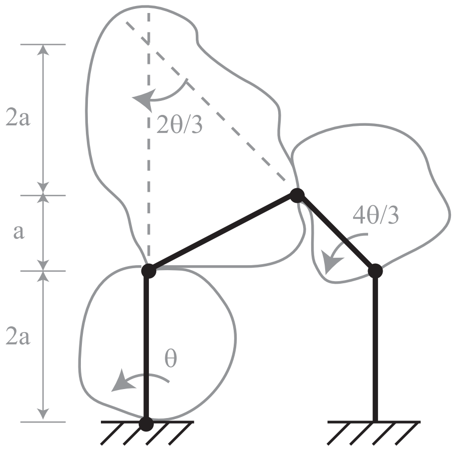


Sway mechanism



Combined mechanism

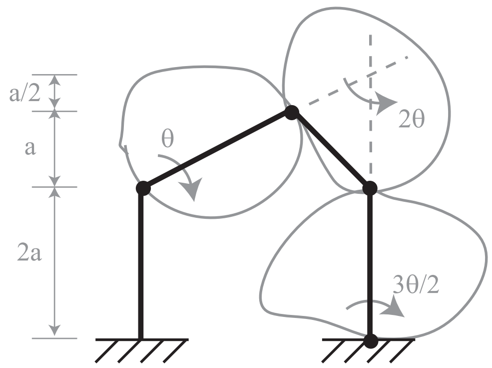
Beam mechanism 1



$$V_1\left(\frac{4\theta}{3}a\right) = M_p\left(\frac{4\theta}{3} + \left(\frac{4\theta}{3} + \frac{2\theta}{3}\right) + \left(\frac{2\theta}{3} + \theta\right) + \theta\right)$$

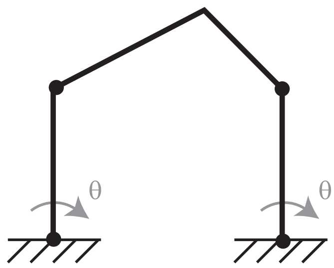
$$V_1 a = \frac{9M_p}{2}$$

Beam mechanism 2



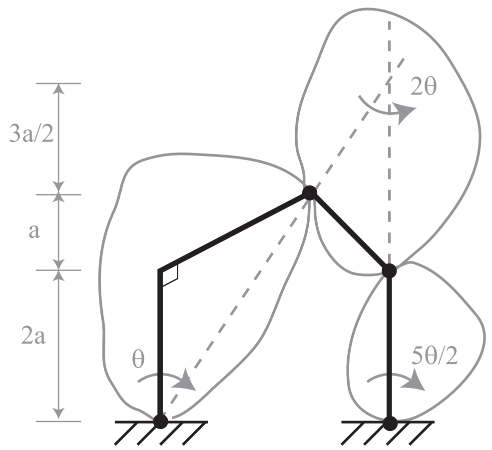
$$V_2(2\theta a) + H_2(3\theta a) = M_p\left(\theta + 3\theta + \frac{7\theta}{2} + \frac{3\theta}{2}\right)$$
$$2V_2a + 3H_2a = 9M_p$$

Sway mechanism



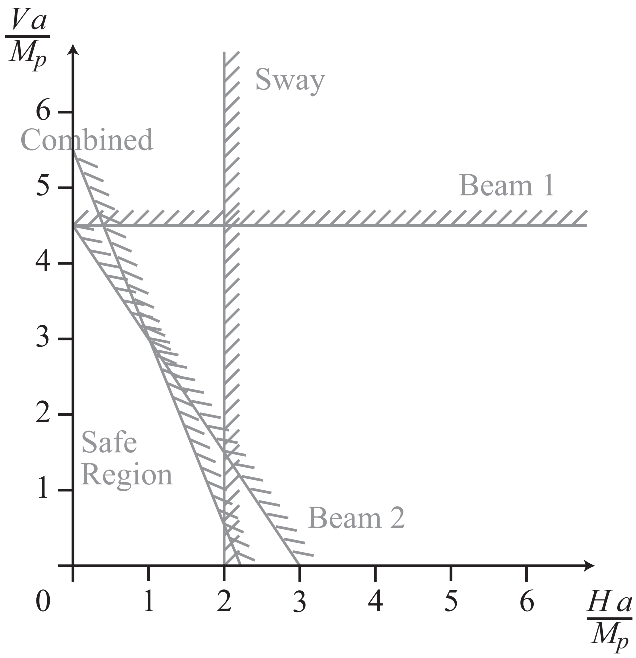
$$H_3\theta 2a = M_p 4\theta$$
$$H_3a = 2M_p$$

Combined mechanism



$$V_4(2\theta a) + H_4(5\theta a) = M_p\left(\theta + 3\theta + \frac{9\theta}{2} + \frac{5\theta}{2}\right)$$
$$2V_4a + 5H_4a = 11M_p$$

Interaction Diagram



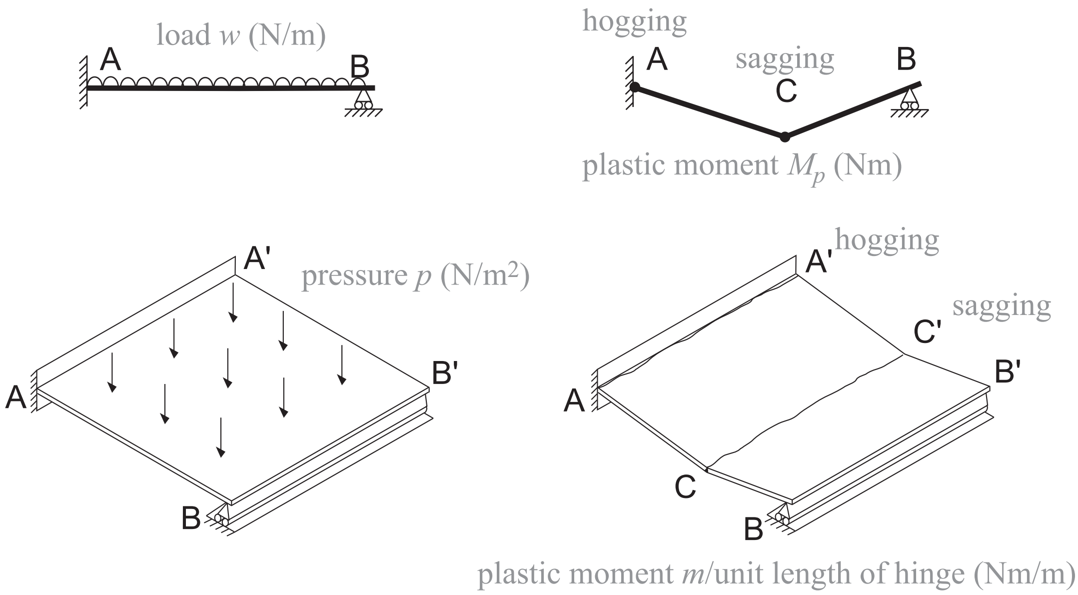
Try Questions 9 and 10, Examples Sheet 2/4

4.7 Yield Line analysis of slabs and plates

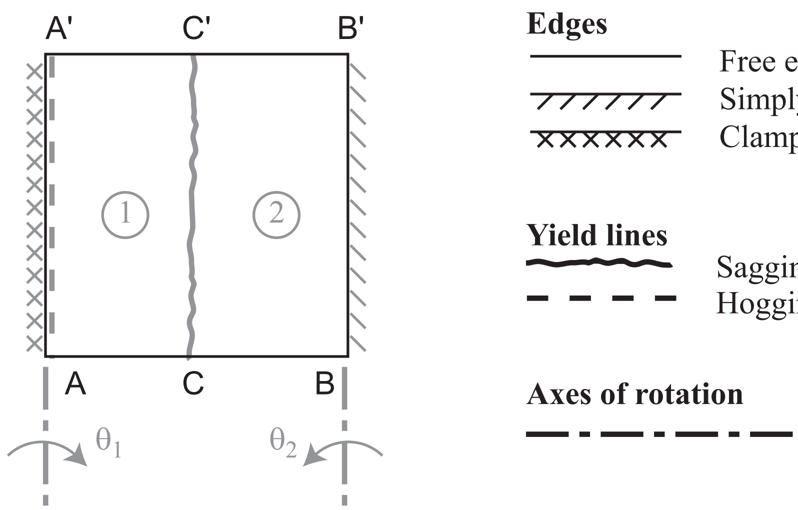
4.7.1 Introduction

So far, plasticity theory has been applied to 1D elements such as beams and frames. In this section, we will extend the same ideas to 2D plate structures. An elastic analysis of these structures is very difficult, but a plastic analysis just follows the principles we have seen before. The main difficulty is to ensure that we use a *compatible* collapse mechanism.

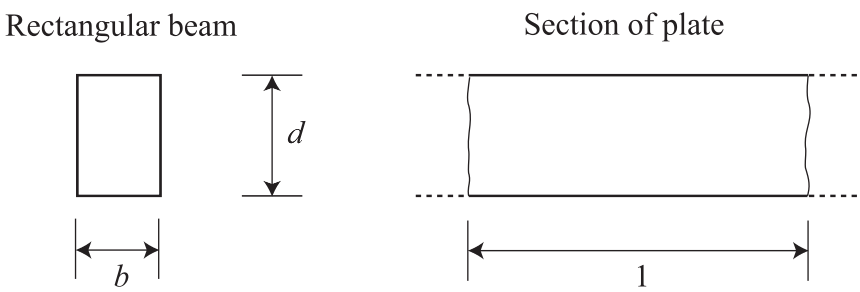
Simple extension of beam example to plate



Notation



Calculation of the plastic moment of a yield line



In Section 4.1.4 we calculated that the plastic moment in a rectangular beam was

$$M_p = Z_p \sigma_y = \frac{bd^2}{4} \sigma_y$$

For a plate made from e.g. steel, the moment capacity *per unit length* will therefore be

$$m = \frac{d^2}{4}\sigma_y$$

Many plates are in fact made of reinforced concrete, and are then referred to as slabs. Concrete slabs do fail by yield lines, but the energy absorbed is all in the steel reinforcement, and the above calculation for moment capacity is not valid. Calculating the plastic moment for reinforced concrete beams and slabs will be covered later in the course. However, one important point is that, depending on the layout of the steel, the plastic moment capacity m can be different in sagging or hogging, and can also be different in different directions.

The 1B course will always assume that the value of m is isotropic, apart from one examples paper question.

Assumptions of yield line theory

1. Collapse is due to *ductile flexural* failure, and is not due to other modes such as shear failure.

2. In-plane forces are ignored.

3. The deformations are small compared to the overall dimensions of the structure.

4.7.2 Compatible Yield Line Patterns

An essential requirement of an upper-bound collapse analysis is that the failure mechanism is *compatible*. A few simple guidelines help to ensure that a chosen yield line pattern is compatible.

1. The yield lines divide the slab into several rigid regions, which must remain planar. Each region must have a unique axis about which it rotates.

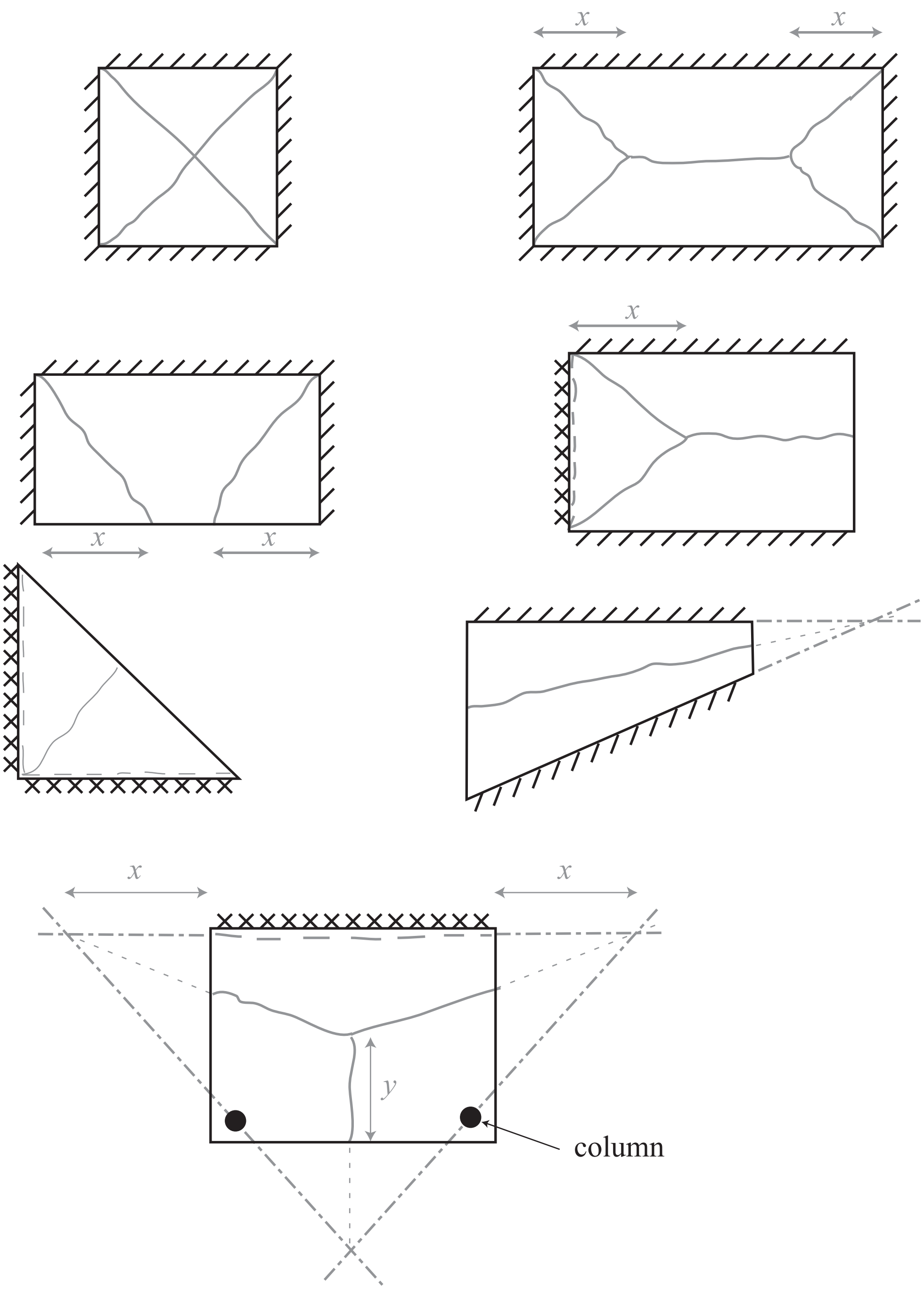
2. The yield lines must be straight between intersections.

3. A yield line between two rigid regions must pass through the intersection of the axis of rotation of those regions.

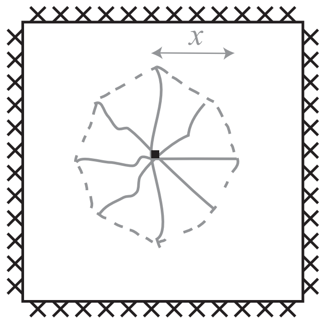
If these rules are broken, the chosen collapse mechanism will be incompatible, for instance it may require some of the rigid regions to twist. A good way of visualising a collapse mechanism is to imagine it projected into 3D. Could you make a simple cardboard model of the collapsed structure?

Example collapse mechanisms for uniformly distributed loading

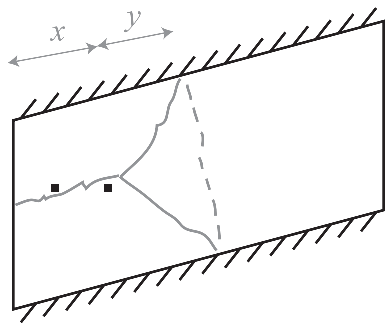
x, y etc. are variable parameters


x
x
x
x
x
x
x
x
y
column

Example collapse mechanisms for point loads



single point load



skew bridge with axle loading
(2 point loads)

4.7.3 Work equation

An upper bound for the collapse load of a slab can be calculated in the same way as for beam structures, by assuming a compatible collapse mechanism, and equating the work done and the energy dissipated during collapse.

Energy dissipated:

For a hinge between two rigid regions, i and j, the total plastic moment in the hinge will be

$$M_p = m\,l_{ij}$$

where: m is the plastic moment per unit length of the slab;
l_{ij} is the length of the hinge.

The energy dissipated in this hinge will therefore be:

$$\text{E.D. in one hinge} = m\,l_{ij}\,\theta_{ij}$$

where: θ_{ij} is the rotation of the hinge between rigid regions i and j.

The total energy dissipated will therefore be:

$$\text{total E.D.} = \sum_{\text{all hinges}} m\,l_{ij}\,\theta_{ij}$$

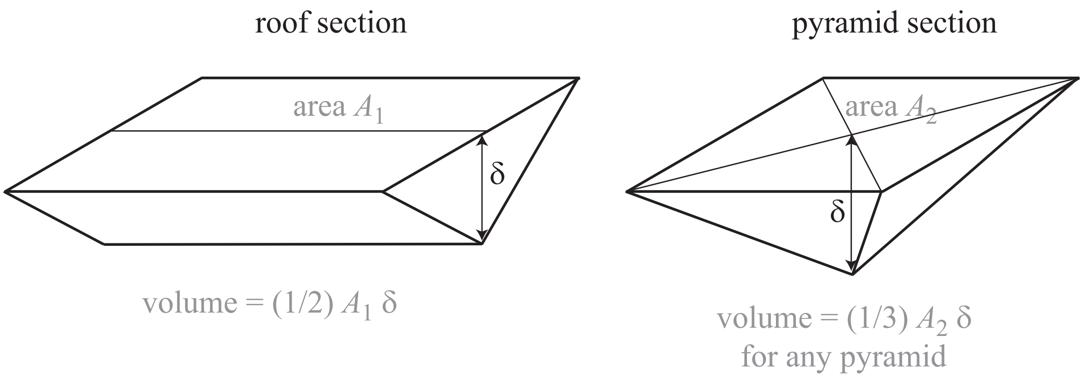
A principal difficulty with yield line analysis is finding a compatible set of hinge rotations, θ_{ij}. However, because we assume that all rotations are small, the rotations of each rigid region can be plotted as vectors (using a right-hand screw-rule), and the difference between these vectors is the rotation along the hinges.

Work done by loads:

The work done by a point load W displacing a distance δ is still $W\delta$.

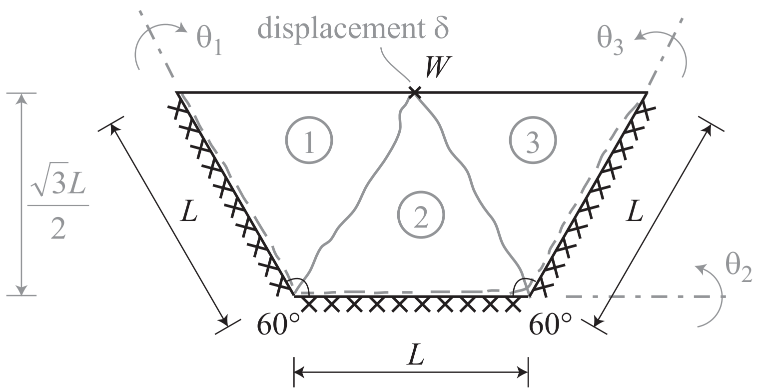
The work done by a uniform pressure appears to be much harder to calculate. Formally it could be written as W.D. $= \int p\,\delta(x,y)\,dA$, where $\delta(x,y)$ is the displacement at a point on the slab. An alternative would be to find the centroid of each rigid region, as the average distance moved by the load on that region. A much simpler alternative, however, is to simplify the expression for work done to W.D. $= pV$, where $V = \int \delta(x,y)\,dA$ is the volume swept out by the collapsing slab.

V can be easily calculated, as most collapse mechanisms can be split into *roof* sections, and *pyramid* sections.



4.7.4 Example — Balcony

Estimate the collapse load W to cause the collapse of the balcony shown below, which has a moment capacity per unit length m.



$$l_{12} = l_{13} = L, \quad l_1 = l_2 = l_3 = L$$

$$\theta_1 = \theta_2 = \theta_3 = \frac{2\delta}{\sqrt{3}L}$$