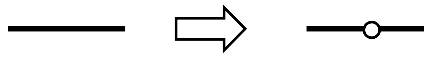
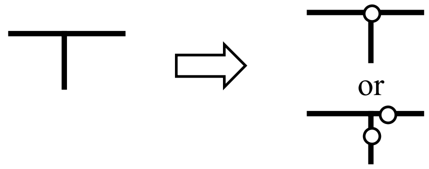
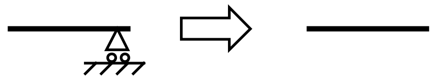
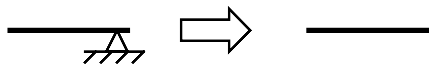
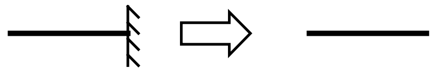
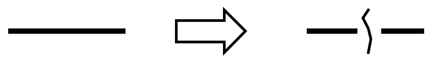
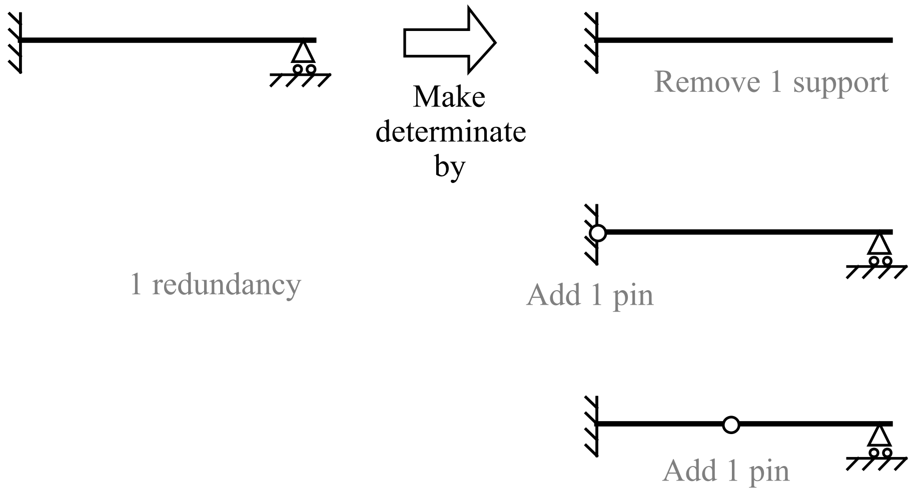
3.8.2 Examples of Removing Redundancy

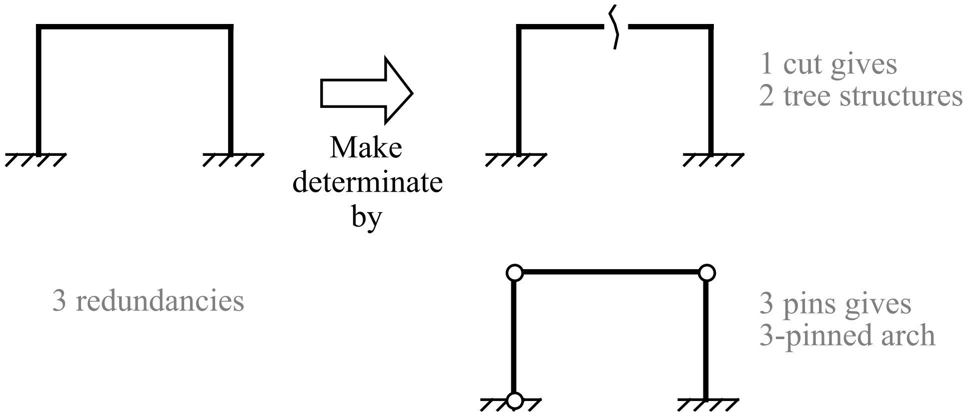
Adding a pin		destroys 1 moment(s) **removes** 1 **redundancy/ies**
	 or	destroys 2 moment(s) **removes** 2 **redundancy/ies**
Removing a support		destroys 1 force(s) **removes** 1 **redundancy/ies**
		destroys 2 force(s) removes 2 **redundancy/ies**
		destroys 1 moment(s) destroys 2 force(s) **removes** 3 **redundancy/ies**
Cutting a beam		destroys 1 bending moment destroys 1 shear force destroys 1 tension **removes** 3 **redundancy/ies**

3.8.3 Examples of Finding Number of Redundancies

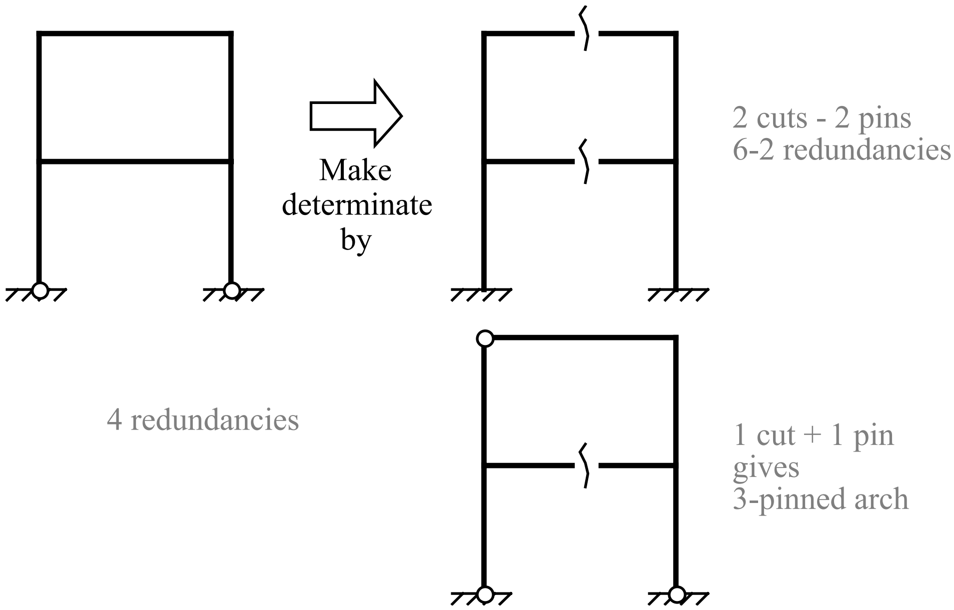
Propped Cantilever



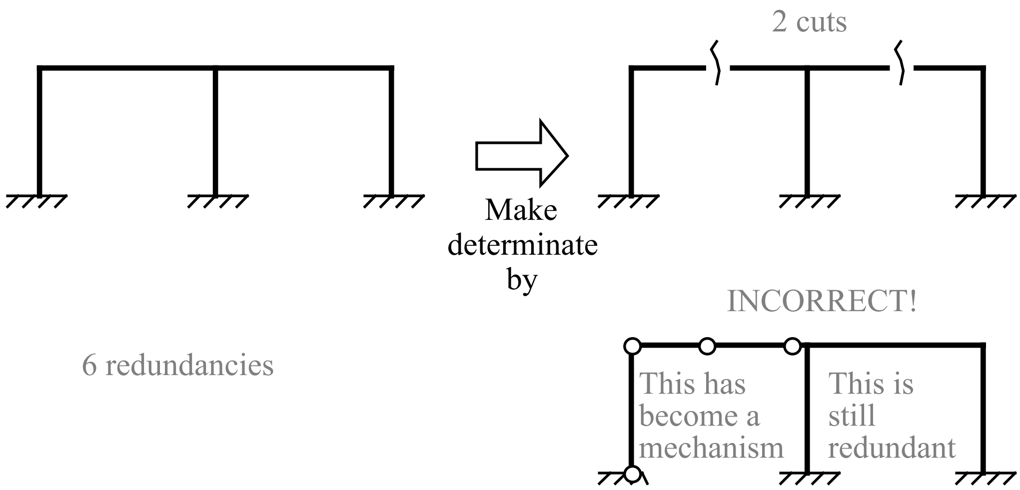
Portal Frame



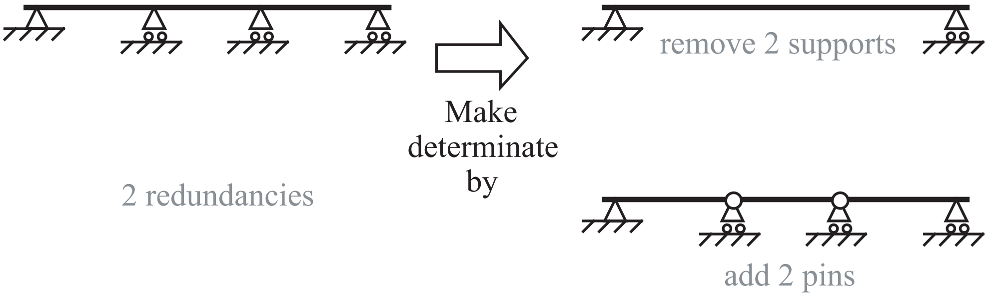
Two-Storey Portal Frame with Pinned Feet



Twin Portal Frame



Multi-Span Beams



Try Question 9, Examples Sheet 2/3

3.9 Force Method for Indeterminate Beam and Frame Structures

The method is basically the same as that described for pin-jointed truss structures, except that the method of removing indeterminacies is different.

1. Make the structure determinate by adding pins, removing supports etc. At each of these positions, explicitly add back the forces (and moments) that were destroyed. This then gives a statically determinate structure which is loaded by n unknown forces/moments, where n is the number of redundancies.

2. Use **equilibrium** to find all the internal forces in the structure (bending moments etc.) in terms of the n unknown forces.

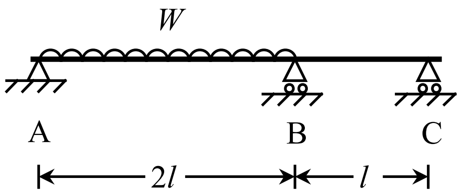
3. Use an **elastic law** to find the internal deformations in the structure (curvatures etc.) in terms of the n unknown forces.

4. Use **compatibility** to find the displacements, also in terms of the n unknown forces. It is then possible to impose continuity where pins were added, or ensure displacements are zero where supports were removed etc., thus giving n equations for the n unknown forces added during (1).

If possible, steps (3) and (4) will be combined using data book deflection coefficients, but sometimes it may be necessary to use Virtual Work (or alternatively a differential equation for bending!) The best way to proceed is with a number of examples.

3.9.1 Example 1 — A Multi-Span Beam

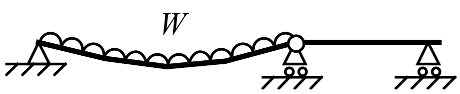
Calculate the bending moment distribution for the multi-span beam shown below, and calculate the reactions at the supports.
IMPORTANT NOTE: The pinned supports shown can provide force both up and down. Thus the structure cannot lift off the support. Assume this unless told otherwise.



Two possible approaches:

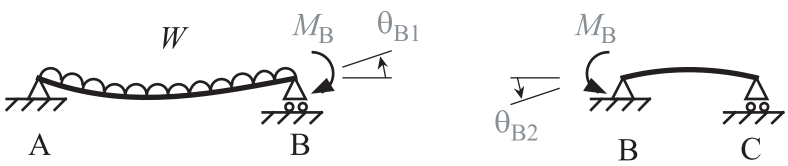
1. Remove a support, and reapply a force to make deflection there zero — difficult to analyze in terms of data book cases.

2. Add a pin over the central support — easy to analyze in terms of data book cases.



We now need to reimpose a moment at the joint to ensure continuity of the beam at the internal support, a *compatibility* condition.

It is easiest to now consider the beam as two separate spans (note that I have abandoned the clockwise +ve sign convention).



From data book:

$$\theta_{B1} = \frac{W(2l)^2}{24EI} - \frac{M_B(2l)}{3EI}$$

$$\theta_{B2} = \frac{M_B l}{3EI}$$

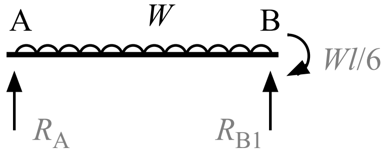
Compatibility — beam is continuous over supports

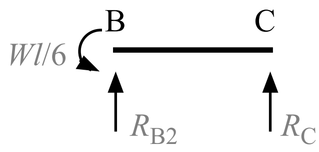
$$\theta_{B1} = \theta_{B2}$$

$$\frac{M_B l}{EI} = \frac{W l^2}{6EI}$$

$$M_B = \frac{W l}{6}$$

Now we have solved for the redundancy, we can find all other internal forces by equilibrium. Reactions:

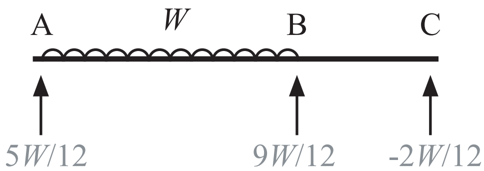


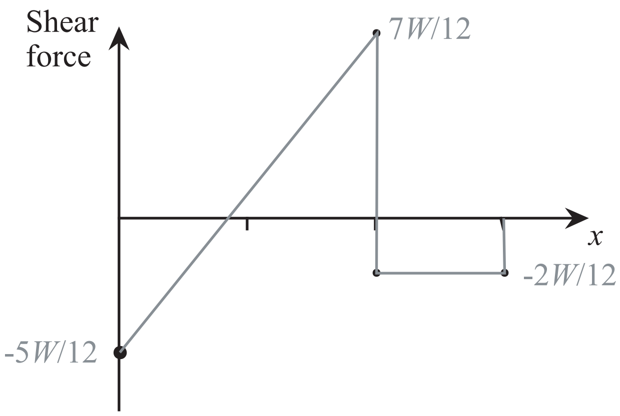


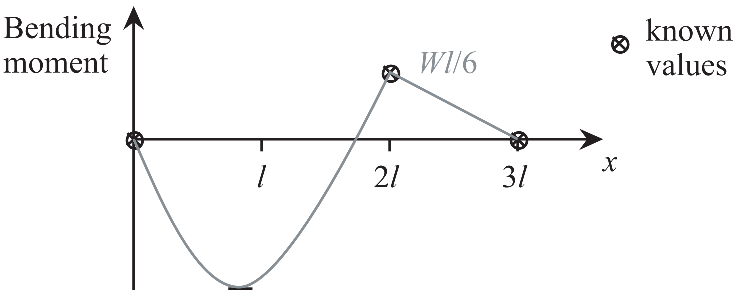
$$\begin{aligned} R_\mathrm{A} \times 2l + \frac{Wl}{6} &= Wl \\ R_\mathrm{A} &= \frac{5W}{12} \\ R_\mathrm{B1} &= \frac{7W}{12} \end{aligned}$$

$$\begin{aligned} R_\mathrm{C} \times l + \frac{Wl}{6} &= 0 \\ R_\mathrm{C} &= \frac{-2W}{12} \\ R_\mathrm{B2} &= \frac{2W}{12} \end{aligned}$$

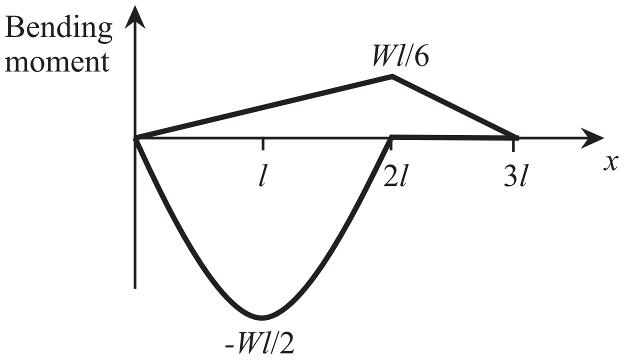
Overall:





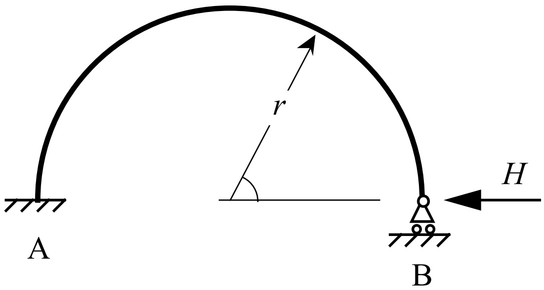


Again, the bending moments can be considered as the sum of a particular solution in equilibrium with the applied load (with $M_B = 0$), and a state of self-stress (with $M_B = Wl/6$.)

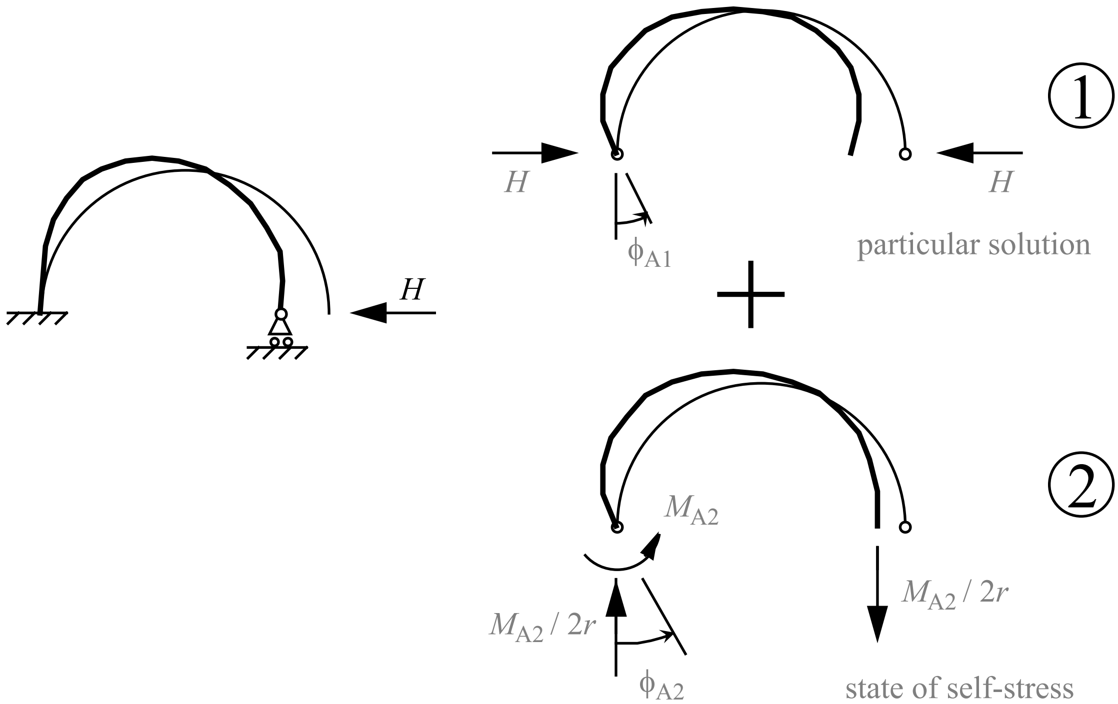


3.9.2 Example 2 — A Ring Structure

Find the bending moment distribution in the frame due to the load H.



Take as the redundancy the bending moment at the left hand support. The system then becomes:



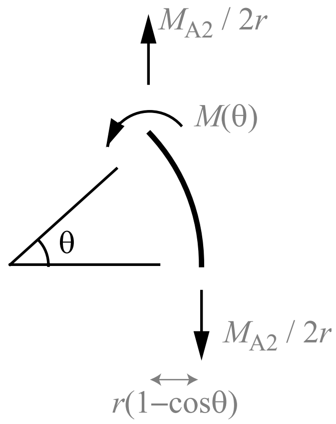
System 1 — particular solution

From our previous analysis in Section 3.6.5, $\phi_{A1} = Hr^2/EI$.

System 2 — state of self-stress

We must do a new calculation for system 2. First we will find the moments by equilibrium, then the curvatures using the elastic law, and finally we'll find the rotation that is compatible with these curvatures using Virtual Work.

Equilibrium for system 2



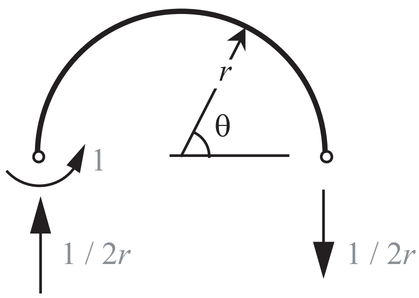
$$M(\theta) = \frac{M_{A2}}{2r} r(1-\cos\theta) = \frac{M_{A2}(1-\cos\theta)}{2}$$

Elastic Law for system 2

$$\kappa(\theta) = \frac{M_{A2}(1-\cos\theta)}{2EI}$$

Compatibility for system 2 To find the rotation ϕ_{A2} that is compatible with this curvature, use Virtual Work. Use as a virtual force system moments in equilibrium with a unit applied moment at A.

Virtual Force System



$$M^*(\theta) = \frac{1-\cos\theta}{2}$$

Virtual Work becomes

$$
\begin{aligned}
1 \times \phi_{A2} &= \int_0^{\pi} \frac{(1-\cos\theta)}{2} \frac{M_{A2}(1-\cos\theta)}{2EI} r\, d\theta \\
&= \frac{M_{A2} r}{4EI} \int_0^{\pi} (\cos^2\theta - 2\cos\theta + 1)\, d\theta \\
&= \frac{3\pi}{8} \frac{M_{A2} r}{EI}
\end{aligned}
$$

Reimpose constraint

We have now analysed the two systems to find

$$\phi_{A1} = \frac{Hr^2}{EI} \qquad \phi_{A2} = \frac{3\pi}{8}\frac{M_{A2} r}{EI}$$

Our real system had no rotation at A, so

$$\phi_{A1} + \phi_{A2} = 0$$

$$-Hr - \frac{3\pi}{8} M_{A2} = 0$$

$$M_{A2} = \frac{-8Hr}{3\pi}$$

The bending moment around the ring is therefore

$$
\begin{aligned}
M(\theta) &= Hr\sin\theta + \frac{M_{A2}}{2}(1-\cos\theta) \qquad * \\
&= Hr(\sin\theta - \frac{4}{3\pi} + \frac{4}{3\pi}\cos\theta)
\end{aligned}
$$

An Alternative Approach

We could have completed the entire analysis of the system in one Virtual Work equation, instead of considering two systems. If we had started with the equation for the moments, $*$ above, where M_{A2} was still unknown, we could have used the following Virtual Work equation:
Real Compatible System

This is now for the whole system, and so is

$$\kappa(\theta) = \frac{1}{EI}(Hr\sin\theta + \frac{M_{A2}}{2}(1-\cos\theta))$$

Virtual Force System

We can reuse the system from the previous analysis.

$$M^*(\theta) = \frac{1-\cos\theta}{2}$$

For the complete system, the rotation at A is zero. Therefore the virtual moment of 1 at A is multiplied by the actual rotation of zero, and Virtual Work becomes:

$$1 \times 0 = \int \frac{1-\cos\theta}{2} \frac{1}{EI} \left(Hr\sin\theta + \frac{M_{\mathrm{A2}}}{2}(1-\cos\theta)\right) ds$$

Completing the integration gives

$$0 = \frac{r}{EI}\left(Hr + \frac{3\pi}{8} M_{\mathrm{A2}}\right)$$

$$M_{\mathrm{A2}} = -\frac{8Hr}{3\pi}$$

This is the equivalent of the Virtual Work method that we used for pin-jointed trusses, and is a very powerful approach. We still have to identify our redundancies, and write down the real equilibrium system in terms of these unknowns, and then use the elastic law to find the real curvatures. However, by carefully choosing a virtual equilibrium system, we can directly find an equation for the unknown forces at the redundancies we have identified.

Try Questions 10,11 and 12, Examples Sheet 2/3

3.10 Implications of Redundancy

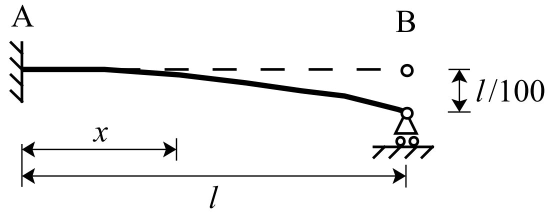
An important aspect of redundant structures is that it is possible to have internal forces within the structure, with no external loading being applied. These may exist because of:

- Settlement of supports;
- The structure not fitting together before it was assembled ('lack of fit');
- Temperature changes.

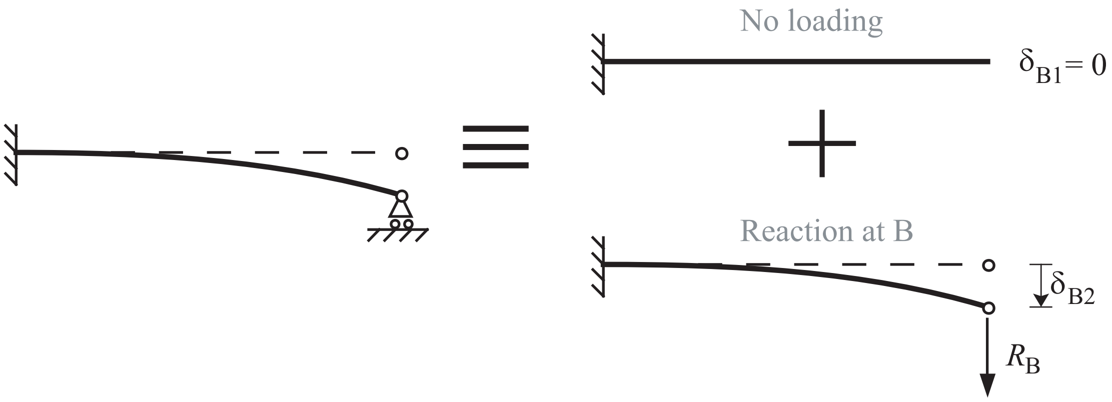
In a determinate structure, the structure could deform to take account of these effects. In an indeterminate structure, the structure cannot freely adjust, and so a state of self stress results.

3.10.1 Example — Support Settlement

A propped cantilever of length l is initially stress-free. Find the resultant stresses in the structure if the support drops by $l/100$.



Split system in two



From the Structures data book,

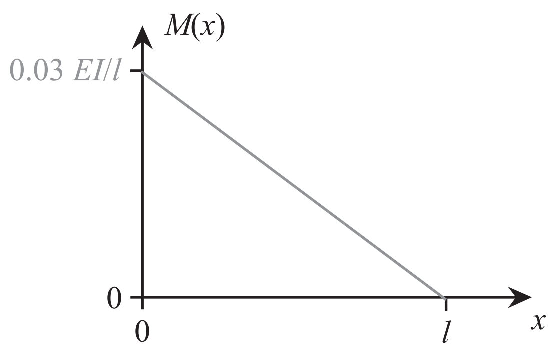
$$\delta_{B2} = \frac{R_B l^3}{3EI}$$

Compatibility at the support

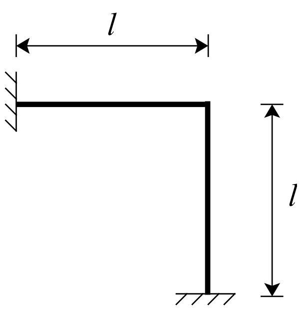
$$\frac{R_B l^3}{3EI} = \frac{l}{100}, \qquad R_B = 0.03\frac{EI}{l^2}$$

Bending moments

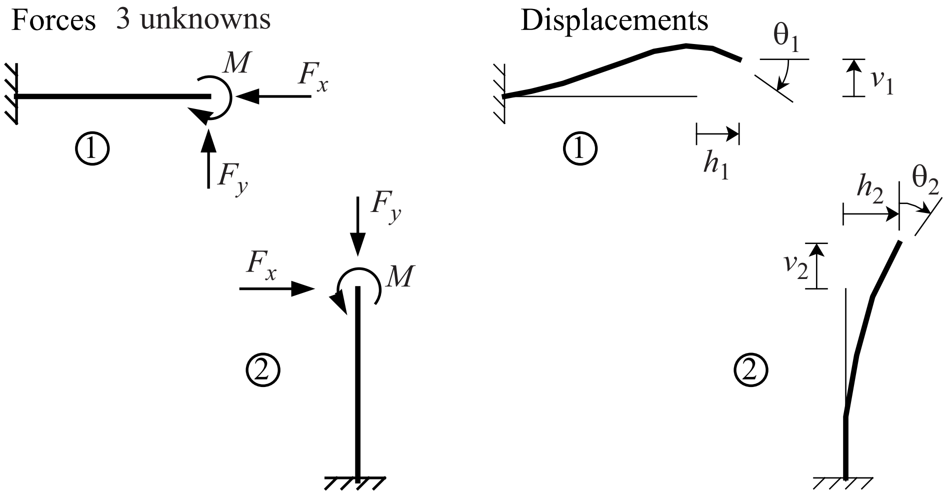
$$M(x) = 0.03\frac{EI}{l^2}(l - x)$$



3.10.2 Example — Temperature Rise



The structure shown is initially stress-free. There is then a temperature increase of T. If the structure has coefficient of thermal expansion α, and bending stiffness EI, what will be the reactions at the supports?



For System 1

$$h_1 = \alpha T l \qquad \text{(neglect effect of } F_x\text{)}$$

$$v_1 = \frac{F_y l^3}{3EI} - \frac{Ml^2}{2EI}$$

$$\theta_1 = -\frac{F_y l^2}{2EI} + \frac{Ml}{EI}$$

For System 2

$$h_2 = \frac{F_x l^3}{3EI} - \frac{Ml^2}{2EI}$$

$$v_2 = \alpha T l$$

$$\theta_2 = \frac{F_x l^2}{2EI} - \frac{Ml}{EI}$$

Compatibility

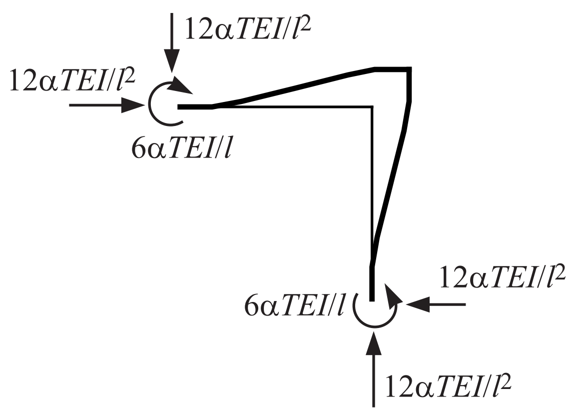
$$h_1 = h_2, v_1 = v_2, \theta_1 = \theta_2$$

Solution is

$$F_x = F_y = \frac{12\alpha TEI}{l^2}$$

$$M = \frac{6\alpha TEI}{l}$$

Reactions at supports



3.10.3 Applications of Self-Stress

In a carefully designed system, setting up a state of self-stress can be an important aspect of structural behaviour. It enable parts of the structure to be *prestressed*, either in tension or compression.

Cable-stiffened deployable structures

Deployable structures are commonly used on satellites to allow the final structure to be much larger than the restricted space within the launch vehicle. One approach to deployable structure design is to have a set of cables which can be prestressed at the end of deployment. These cables don't affect the deployment, but when prestressed at the end of deployment, they make the structure much stiffer. Without prestress, a cable cannot take any compressive load. With prestress, this compressive load is superimposed on an initial tension, and so the cable acts as a structural member.

Bicycle Wheels

The hub on a bicycle wheel is supported by very thin members, the spokes. The spokes would buckle at a very low load if they weren't prestressed, so they are tightened to ensure that they are always in a state of tension - only possible because it is a redundant structure.

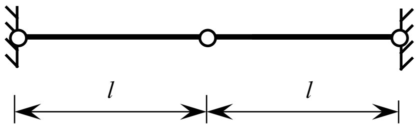
Prestressed concrete beams

Concrete is a useful material in compression, but will crack and fail at very low tensile stresses. A common structural element is a prestressed concrete beam, where steel tendons within the beam are tensioned, thus setting up a state of self-stress with the concrete in compression. Subsequent bending stresses will then not cause the concrete to crack.

Tensegrity Systems, Fabric Roofs and Balloons

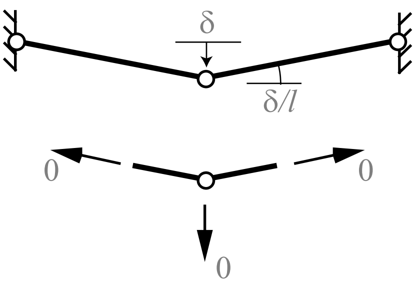
Tensegrity (Tension-Integrity) structures were popularised by Buckminster Fuller. They are structures where compression members are connected together by a web of cable tension members so that none of the compression members touch one another: many famous sculptures, such as the “Triple Crown“ below, were designed by Kenneth Snelson, a student of Buckminster Fuller:

In common with fabric roofs, and balloons, they only have stiffness because they are initially prestressed - a state of self-stress exists in the structure. In tensegrity systems, turnbuckles are included to deliberately introduce lack of fit. In fabric roofs the fabric is pretensioned by cables, and a balloon is prestressed by internal pressure. We will look at a simple ill-conditioned structure to explain the concept.

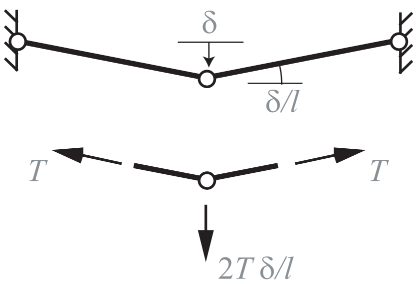


Ignoring second order effects (we are considering very small movements away for the initial position, and so the bars do not change length, and the internal force remains constant), what is the stiffness of the structure to lateral loads? Consider the joint moves by a small distance δ.

No prestress



Prestress *T*



3.11 Analysis of Symmetric Structures

Many structures possess some symmetry. This may be just a simple plane of symmetry, or the more complex symmetry of a radio telescope. It is possible to make use of symmetry to simplify the analysis of structures. We will concentrate on the simplest case, bilateral symmetry, but similar techniques can be applied to more complex symmetry.

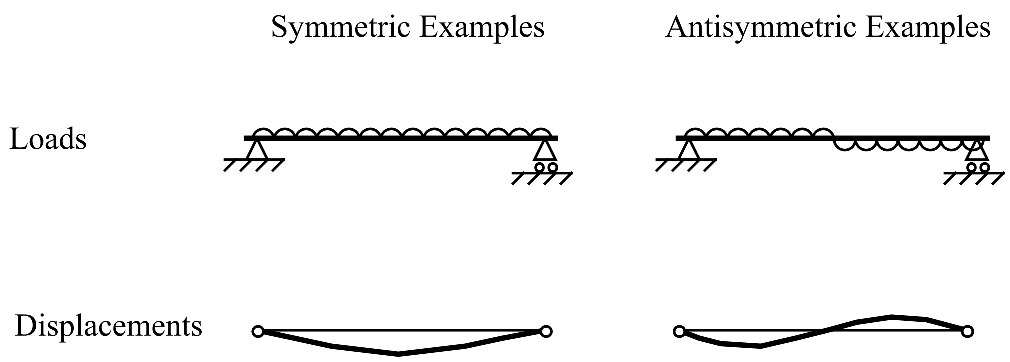
3.11.1 Symmetry Properties for Bilateral Symmetry

A structure with bilateral symmetry has a single plane of reflection.

Symmetry and Antisymmetry

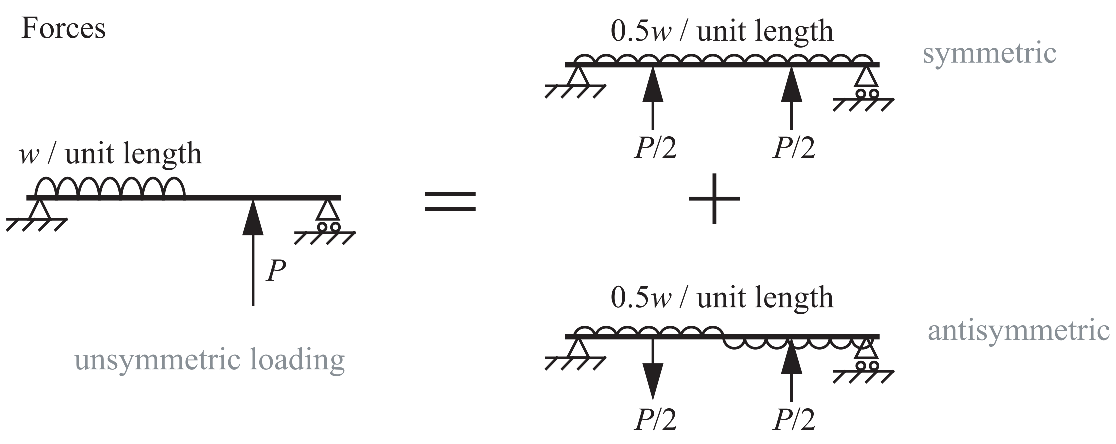
During this section we will often use the terms *symmetric* and *antisymmetric*. The definitions of these terms are:

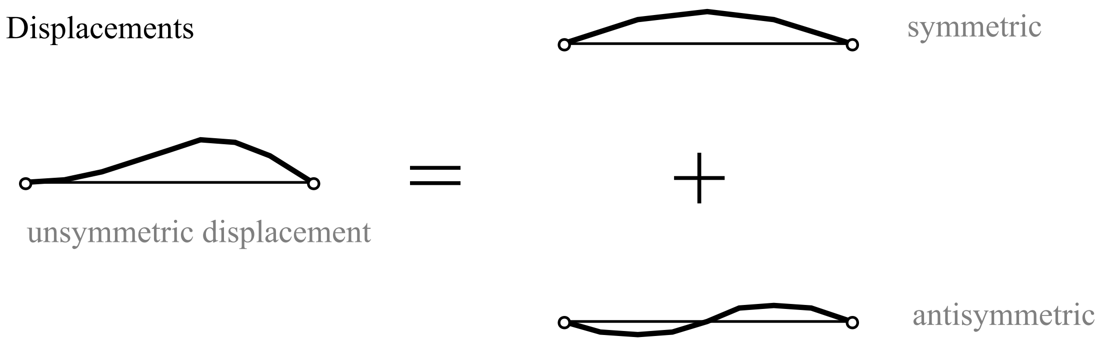
- Anything *symmetric* is *preserved* by reflection of the structure in its plane of symmetry.
- Anything *antisymmetric* is *reversed* by reflection of the structure in its plane of symmetry.



Splitting Unsymmetric into Symmetric and Antisymmetric

A very useful property is that any unsymmetric load or displacement can be split into a symmetric and an antisymmetric component. For example:



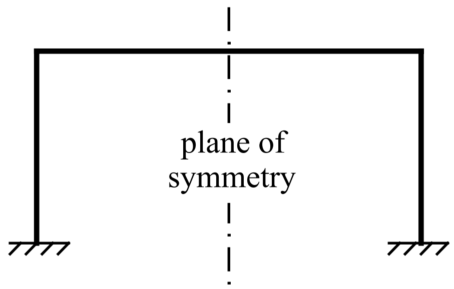


Response to Symmetric and Antisymmetric Loading

The reason that symmetry can help with structural analysis is because:

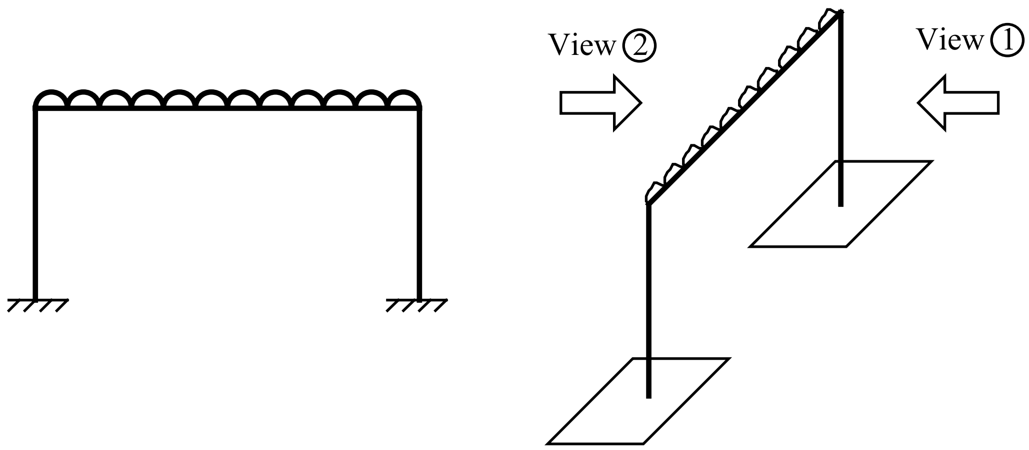
1. A symmetric structure, subject to symmetric loads
   (a) Will only have symmetric internal forces;
   (b) Will only undergo symmetric displacements.
2. Similarly, a symmetric structure, subject to antisymmetric loads:
   (a) Will only have antisymmetric internal forces;
   (b) Will only undergo antisymmetric displacements.

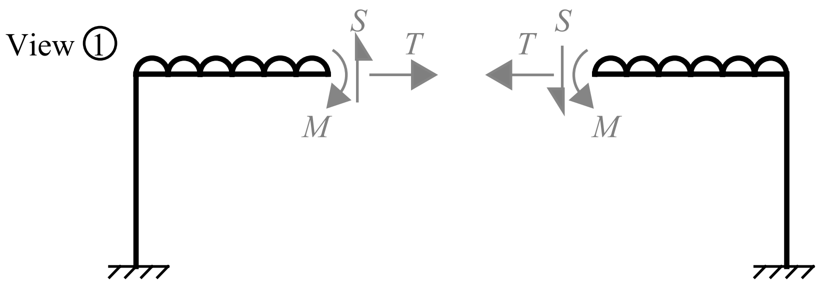
We will prove this to be true for two of the four possibilities, but similar arguments could be used for all four. In each case we will examine the simple portal frame shown below.

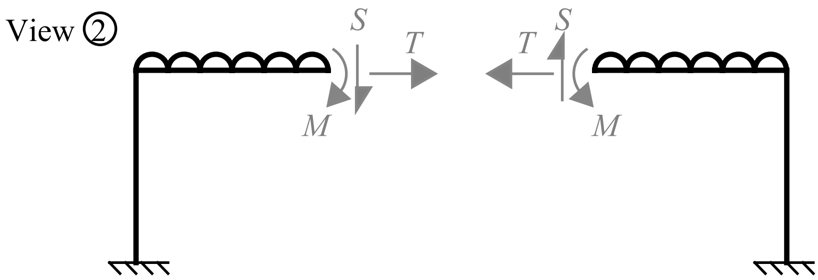


Internal Forces due to Symmetric Loading

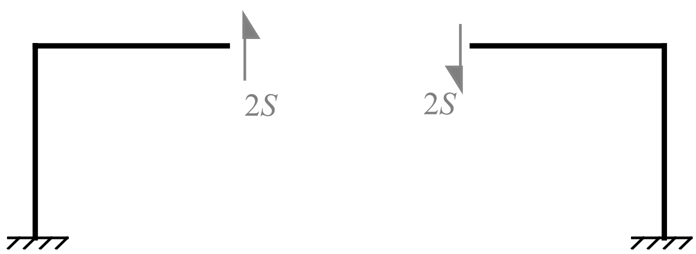
Consider the frame subject to a uniform loading across the beam. We will look at the structure from both sides, views 1 and 2, and carefully examine the stress resultants at the centre of the beam.







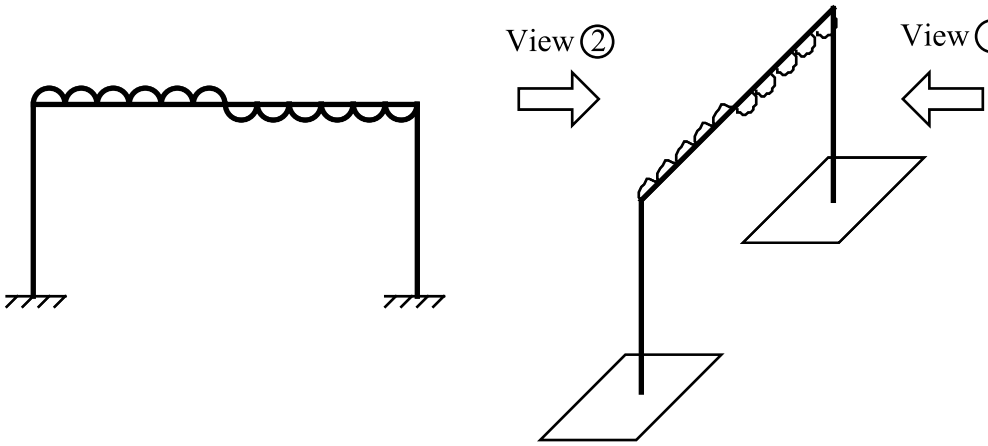
Subtracting ② from ① gives a structure that is unloaded



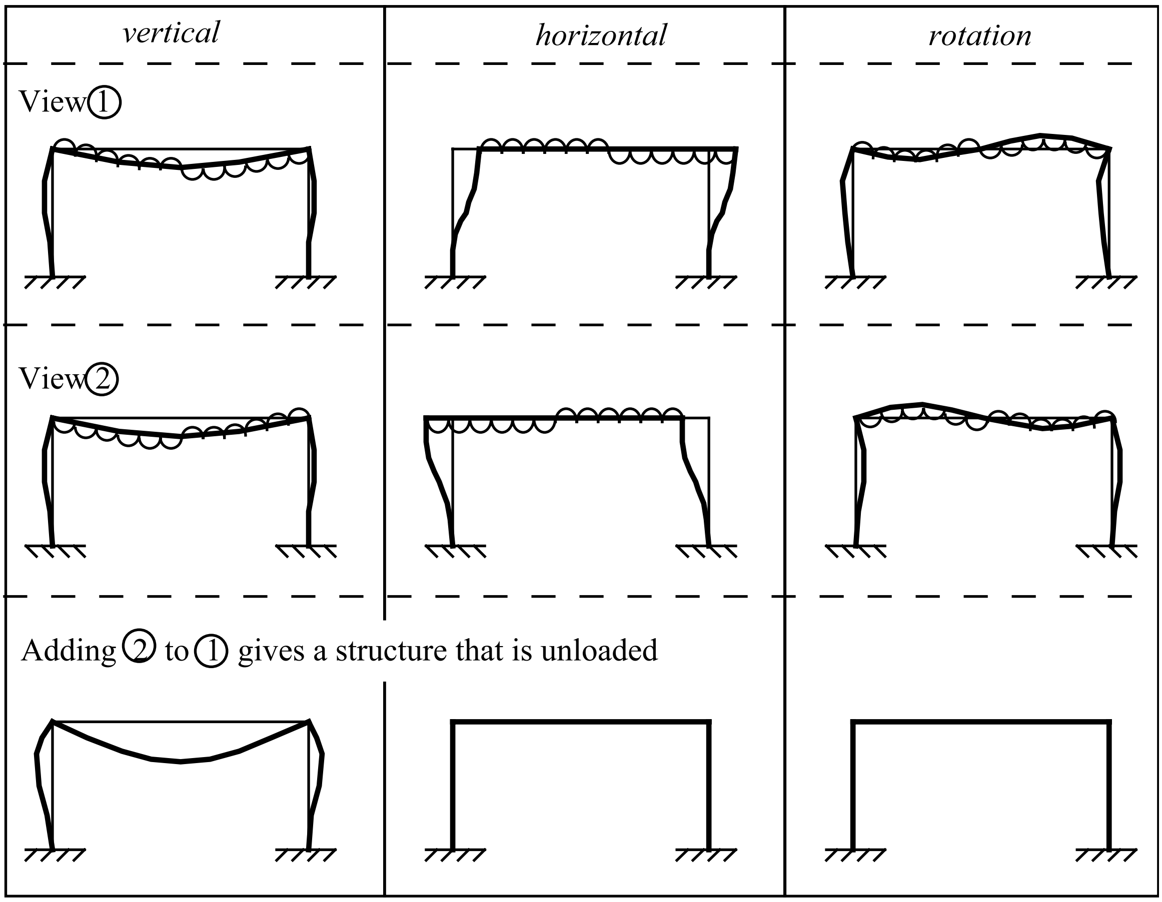
Although there is no external load, our analysis has shown a resultant shear force of $2S$. Since this cannot be true, the shear force S must be zero. We could have repeated the analysis with any symmetric load, any symmetric/antisymmetric stress resultants. We would always find the same result. Symmetric loads on a symmetric structure can only give symmetric internal forces.

Displacements due to Antisymmetric Loading

Consider the frame subject to an antisymmetric distributed loading across the beam. We will look at the structure from both sides, views 1 and 2, and examine three possible modes of deformation.



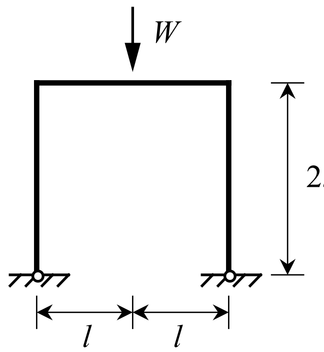
Examine the possible displacements. The actual displacement will be some combination of these three.



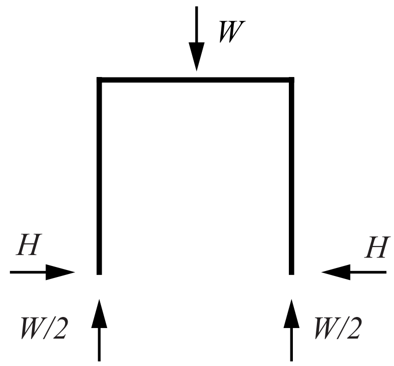
Although there is no external load, our analysis has shown a resultant displacement. This cannot be true, and hence the symmetric displacement due to the antisymmetric load must be zero. We could have repeated the analysis with any antisymmetric load, any symmetric displacement. We would always find the same result. Antisymmetric loads on a symmetric structure can only give antisymmetric displacements.

3.11.2 Example — Symmetric Analysis

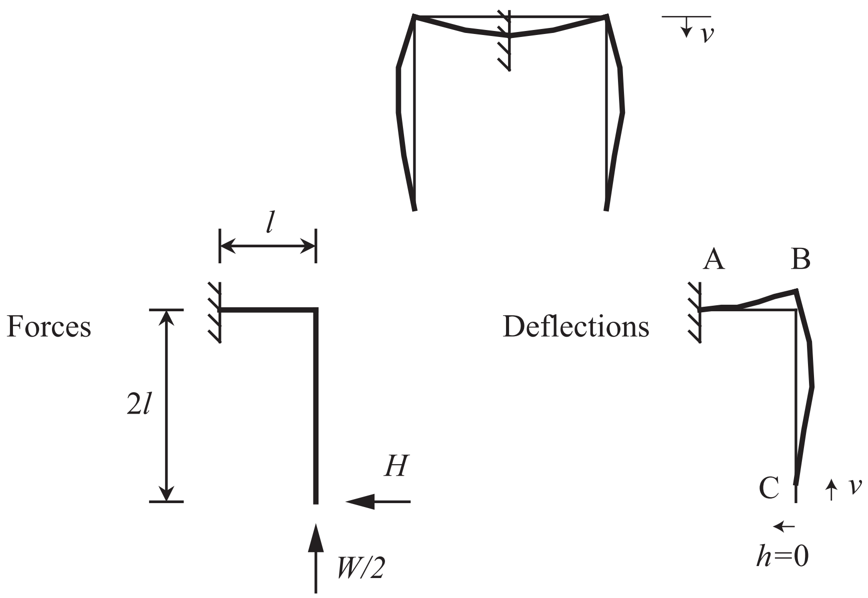
Find the displacement of the centre of the beam in the portal frame due to the load shown.



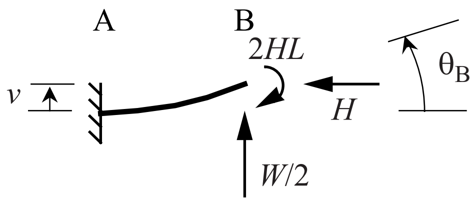
This frame has one redundancy, which we could take as the horizontal force at the base

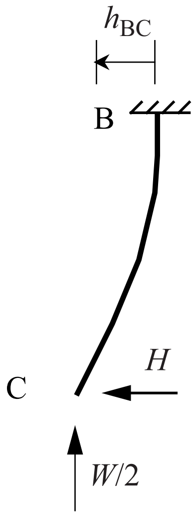


Because of symmetry, we can consider only half the structure



Examine each part separately, and use data book coefficients





$$v = \frac{W}{2}\frac{l^3}{3EI} - 2Hl\frac{l^2}{2EI} = \frac{Wl^3}{6EI} - \frac{Hl^3}{EI}$$

$$\theta_B = \frac{W}{2}\frac{l^2}{2EI} - 2Hl\frac{l}{EI} = \frac{Wl^2}{4EI} - \frac{2Hl^2}{EI}$$

$$h_{BC} = H\frac{(2l)^3}{3EI} = \frac{8Hl^3}{3EI}$$

Horizontal deflection

$$h = -\theta_B \times 2l + h_{BC}$$
$$= -\frac{Wl^3}{2EI} + \frac{4Hl^3}{EI} + \frac{8Hl^3}{3EI}$$

To satisfy compatibility, $h = 0$

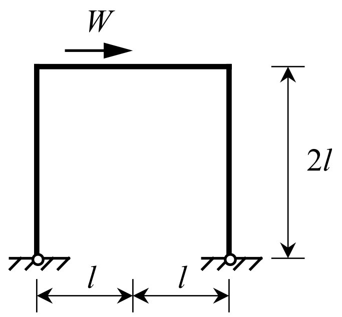
$$\frac{20}{3}H = \frac{W}{2}$$
$$H = \frac{3W}{40}$$

Vertical Deflection

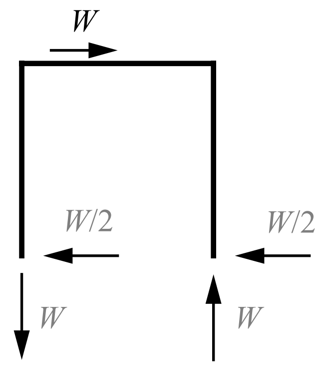
$$v = \frac{Wl^3}{6EI} - \frac{3W}{40}\frac{l^3}{EI}$$
$$= \frac{11}{120}\frac{Wl^3}{EI}$$

3.11.3 Example — Antisymmetric Load

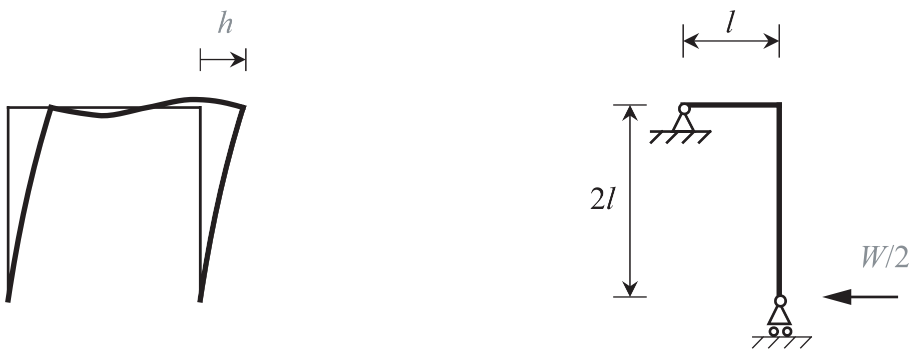
Find the displacement of the centre of the beam in the portal frame due to the load shown.



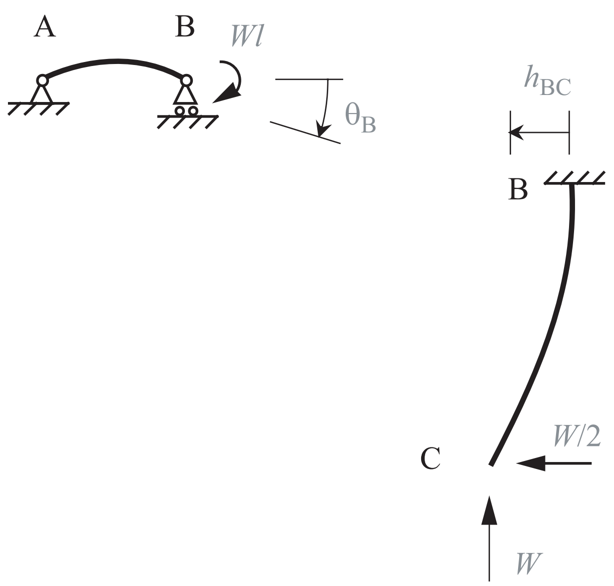
Although the frame has one redundancy, as we saw in the previous example, the redundancy was symmetric. For our antisymmetric analysis, we can calculate all forces using antisymmetry and equilibrium.



Again because of symmetry, we can consider only half the structure



Examine each part separately, and use data book coefficients



$$\theta_B = Wl\frac{l}{3EI}$$

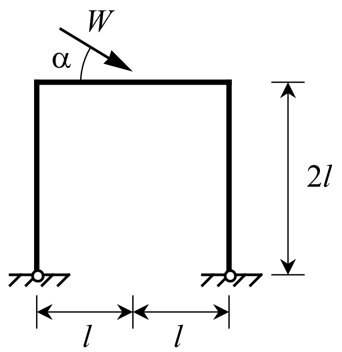
$$h_{BC} = \frac{W}{2}\frac{(2l)^3}{3EI} = \frac{4Wl^3}{3EI}$$

Horizontal deflection

$$\begin{aligned} h &= \theta_B 2l + h_{BC} \\ &= \frac{2Wl^3}{3EI} + \frac{4Wl^3}{3EI} = \frac{2Wl^3}{EI} \end{aligned}$$

3.11.4 Example — General Load

Find the displacement of the centre of the beam in the portal frame due to the load shown.



Horizontal displacement is only due to horizontal load (antisymmetric)

$$h = 2\frac{W\cos\alpha\, l^3}{EI}$$

Vertical displacement is only due to vertical load (symmetric)

$$v = \frac{11}{120}\frac{W \sin\alpha\, l^3}{EI}$$

Try Questions 1,2 and 3, Examples Sheet 2/4