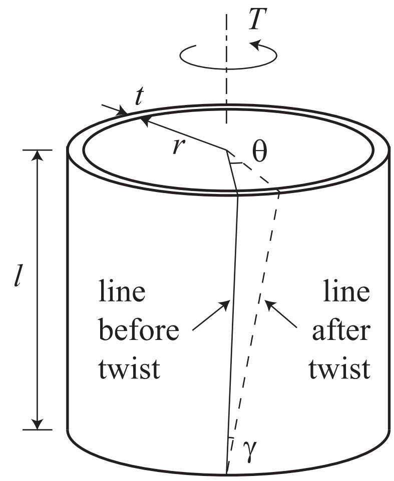
1.5 Torsional rigidity

We have already considered *equilibrium* relationships for torsion — the shear stress due to an applied torque for a thin-walled section. In this section, we will find the *stiffness* relationship between applied torque and resultant twist of a section. For general sections this is difficult, as the cross-sections will *warp* — plane sections do not remain plane. However, we will start with a simple thin-walled circular cylinder, where symmetry shows that warping will not occur.

1.5.1 Uniform thin-walled circular section

Consider a thin-walled cylinder of thickness t, radius r, length l, whose base is fixed, while a torque T is applied to the top surface. What is the resultant rotation θ? Or, better (as it doesn't depend on l), what is the resultant *twist/unit length* ϕ?



Equilibrium

$$\tau = \frac{T}{2\pi r^2 t}$$

Compatibility

$$\gamma l = \theta r$$

Define the *twist/unit length* $\phi = \theta/l$

$$\gamma = \frac{\theta}{l} r = \phi r$$

Material law

$$\tau = G\gamma$$

Substitute for τ and γ to give the required *stiffness* relationship between T and ϕ

$$\frac{T}{2\pi r^2 t} = G\phi r$$
$$T = G\,2\pi r^3 t\,\phi$$

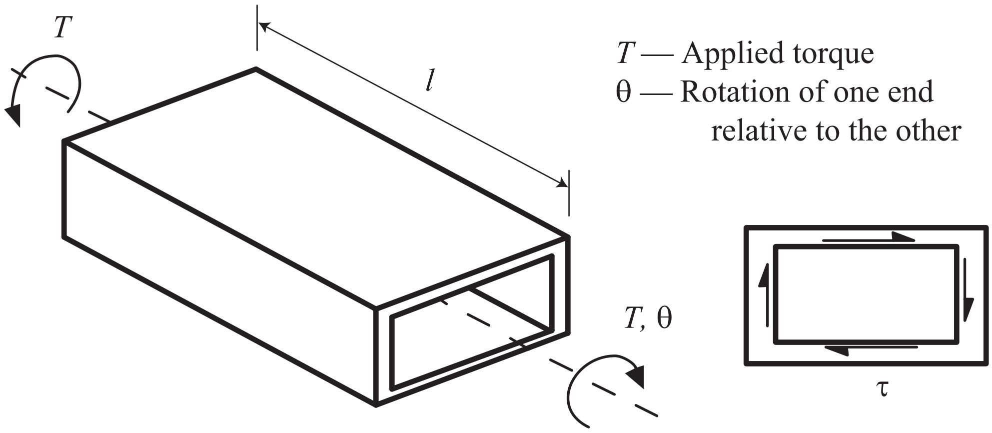
Thus the *torsional rigidity*, or torsional stiffness, T/ϕ is given by

$$\frac{T}{\phi} = G2\pi r^3 t$$

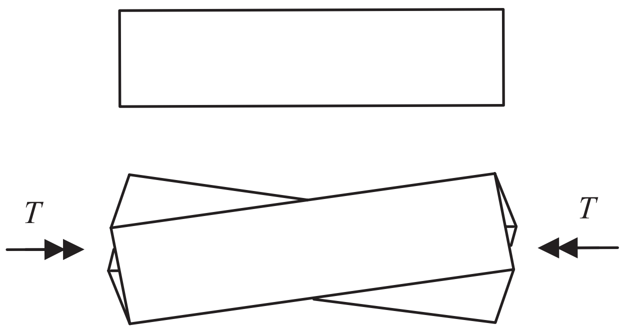
and the shear stress (constant in the section) is

$$\tau = Gr\phi$$

1.5.2 General thin-walled section



In this case, the compatibility between twist and shear strain (which was straightforward for the circular section) is tricky, because of *warping*. For non-circular cross-sections, plane sections do not remain plane when twisted — they warp.



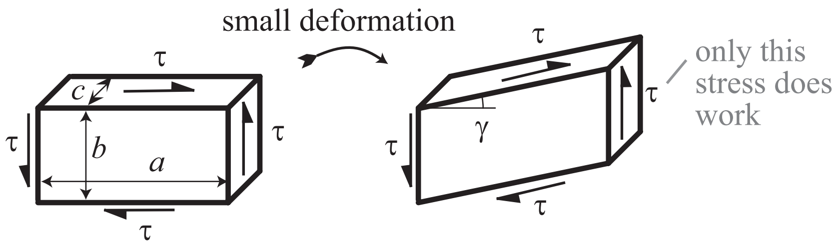
We shall use Virtual Work to find the compatibility relationship — this is possible because we have already worked out the equilibrium relationship $q = \tau t = T/2A_e$.

Virtual Work

Equilibrium set: torque T in equilibrium with shear stresses $\tau(s)$.
Compatible set: rotation θ (unknown), compatible with shear strain γ.
External Work

$$\text{External Work} = T\theta \quad \text{(torque} \times \text{rotation)}$$

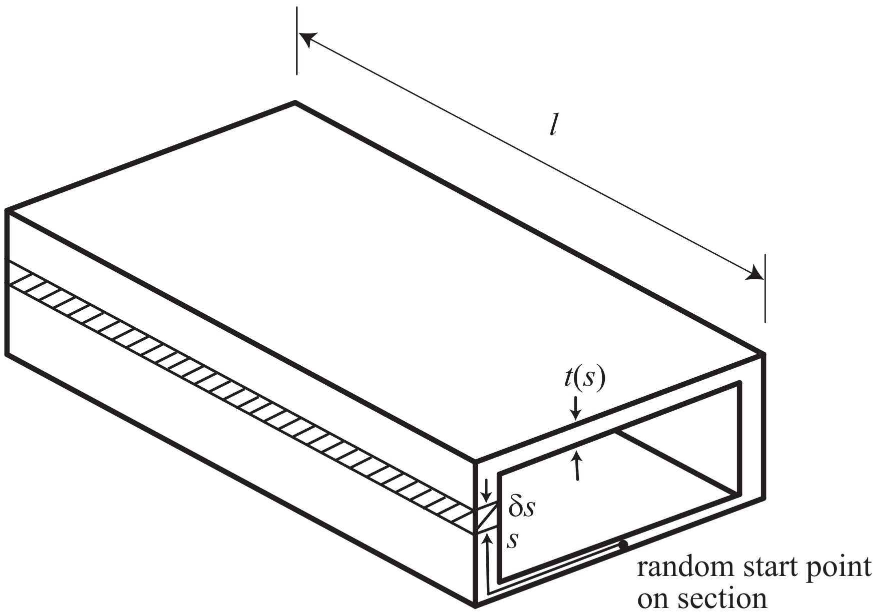
Internal Work Consider a small element, subject to a shear stress τ, that undergoes a small shear strain γ



Work done in the small element

$$\delta W_i = \underbrace{\tau bc}_{\text{force}} \times \underbrace{\gamma a}_{\text{displacement}}$$
$$= \tau\gamma \times \underbrace{\delta V}_{\text{small volume}}$$

Now consider a strip of the section as the volume δV



$$\delta V = lt(s)\delta s$$

$$\delta W_i = \tau(s)\gamma(s)lt(s)\delta s$$

γ, t and τ may all vary around the section

Total internal work

$$W_i = \oint_{\text{section}} \tau(s)\gamma(s)lt(s)\,ds$$

$\tau(s)$ and $t(s)$ may vary around the section, but $q = \tau(s)t(s)$ is constant (recall that the shear flow is uniform)

$$W_i = ql \oint \gamma(s)\, ds$$

Virtual Work The internal work must equal the external work

$$T\theta = ql \oint \gamma(s)\, ds$$

We can use this expression to find the correct *compatibility* relationship, because we already know the *equilibrium relationship* $T = 2A_e q$.

$$\therefore 2A_e\theta = l \oint \gamma(s)\, ds$$

This is the compatibility relationship for the section, which we can write as a relationship between $\phi = \theta/l$ and the shear deformation $\gamma(s)$.

Compatibility

$$\boxed{\phi = \frac{\oint \gamma(s)\, ds}{2A_e}}$$

The equilibrium and stress-strain relationship are straightforward

Equilibrium

$$q = \tau(s)t(s) = \frac{T}{2A_e}$$

Material law

$$q = \tau(s)t(s) = G\gamma(s)t(s)$$

Torsional stiffness Substitute for q in the material law to give

$$\frac{T}{2A_e} = G\gamma(s)t(s)$$

$$\gamma(s) = \frac{T}{2A_e G t(s)}$$

Substitute into the compatibility law to give a stiffness relationship between T and ϕ

$$\phi = \frac{1}{2A_e} \oint \frac{T}{2A_e G t(s)} ds$$

Only t varies around the section, so the torsional stiffness is given by:

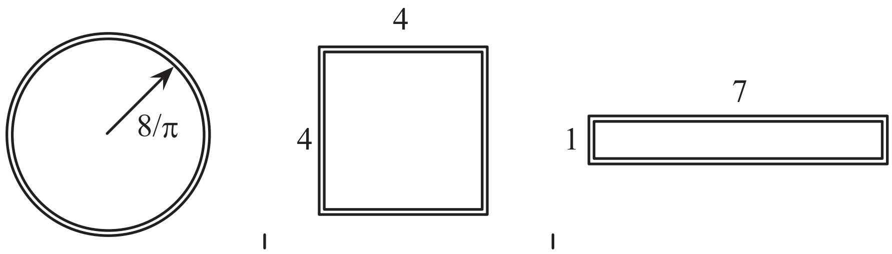
$$\boxed{\frac{T}{\phi} = \frac{G 4A_e^2}{\oint \frac{ds}{t(s)}}}$$

Torsional stiffness is often denoted as GJ, where J is the torsion constant

$$T = (GJ)\phi$$

cf. for bending, $M = EI\kappa$.

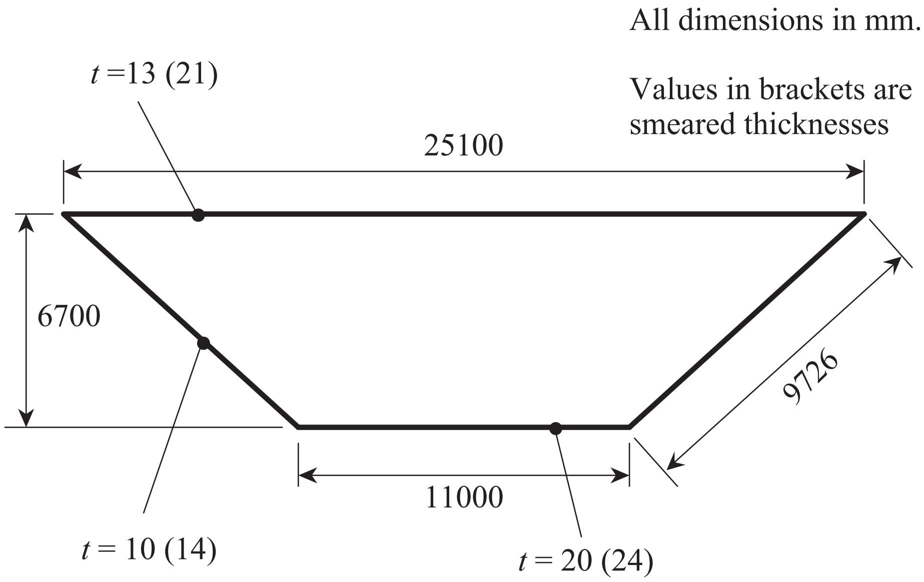
Example: Relative stiffness of closed sections with same wall length and thickness



A_e	20.4	16	7
Relative stiffness $\propto A_e^2$	1	0.62	0.12

1.5.3 Case study part 3 — torsional stiffness of a Storebælt approach span.

As well as being necessary to calculate deflections due to e.g. off-centre loads, it is essential to know the torsional stiffness of a bridge to be able to calculate its aerodynamic response — Tacoma Narrows was too flexible!



Local stiffeners will not greatly affect the torsional rigidity — the flexible sections between stiffeners will dominate the calculation. Hence we will use actual, not smeared, thicknesses.

$$A_e = 120.9\ \mathrm{m}^2$$

$$\oint \frac{ds}{t} = \underbrace{\frac{25100}{13}}_{\text{top flange}} + \underbrace{2\times\frac{9726}{10}}_{\text{two webs}} + \underbrace{\frac{11000}{20}}_{\text{bottom flange}} = 4426$$

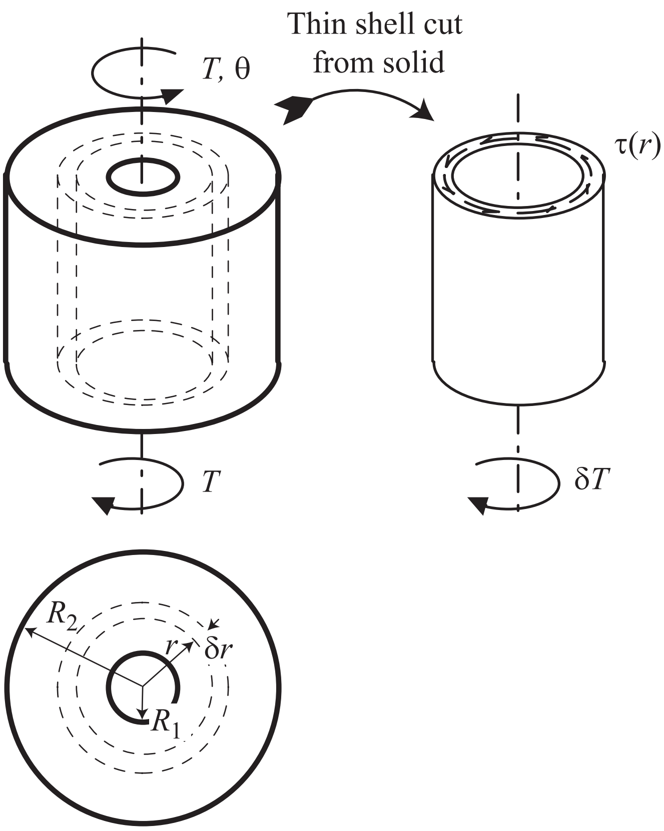
$$G = 81\times 10^9\ \mathrm{N/m^2}\ \text{for steel}$$

$$\frac{T}{\phi} = \frac{G\,4A_e^2}{\oint \frac{ds}{t(s)}} = \frac{81 \times 10^9 \times 4 \times 120.9^2}{4426}$$
$$= 1.07 \times 10^{12} \frac{\text{Nm}}{\text{radians/m}}$$

$$\text{Torsion constant } J = \frac{1.07 \times 10^{12}}{81 \times 10^9} = 13.2\ \text{m}^4$$

1.5.4 Axi-symmetric shafts

For the special case of shafts with circular symmetry, the results for thin sections can be used to make calculations for thick, or solid sections.



Torsional rigidity

For the thin shell

$$\delta T = 2\pi r^3\, \delta r\, G\phi$$

Integrate over the whole shaft

$$T = 2\pi G\phi \int_{R_1}^{R_2} r^3 dr$$
$$= \frac{2\pi G\phi}{4}(R_2^4 - R_1^4)$$

For circular shafts, the torsional rigidity is given by

$$\boxed{\frac{T}{\phi} = G\frac{\pi}{2}(R_2^4 - R_1^4)}$$

For circular shafts only, the torsion constant J is given by the polar second moment of area, $J = \int r^2\, dA$

Shear stress due to torque

For the thin shell

$$\tau = Gr\phi$$

For circular shafts, the shear stress is given by

$$\boxed{\frac{\tau}{r} = G\phi = \frac{T}{J}}$$

cf. for bending, $\frac{\sigma}{y} = E\kappa = \frac{M}{I}$

Try Questions 8 and 9, Examples Sheet 2/1

IB Structures 2023-24

Handout 2

Analysis of stress and strain. Yield criteria

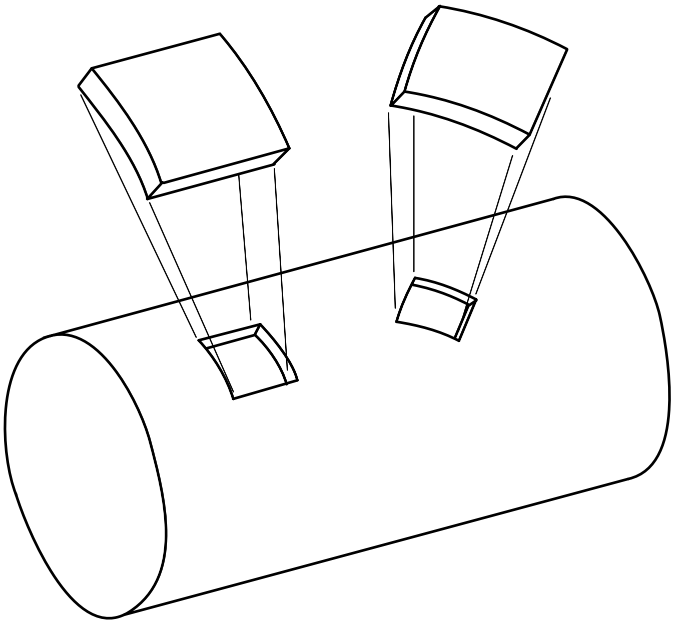
Filled Version

Text and pictures in grey are omitted from the version in lectures

Cambridge University Engineering Department

2.1 Introduction

We have been examining the state of stress at points in thin-walled structures by cutting through the structure, and using equilibrium to examine the stresses that must exist on the cut face. What would happen if we instead cut through the structure at a different orientation? How is the stress at a point affected by a change in axes? Similarly, how is the strain affected?

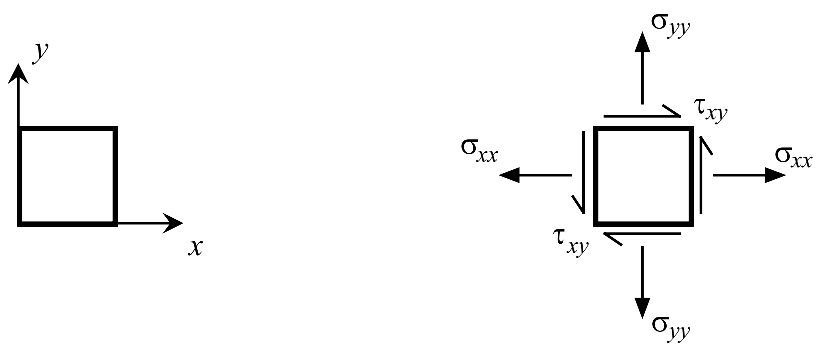


In Handout 1 we were examining stress in thin-walled structures, when we assumed all the stress on the through-thickness face were zero. This is a common assumption for thin structures, and is called *plane stress* (all stress components lie in one plane). We will examine stresses for this case first, and then the general 3D case, and then move on to strains.

2.2 2-D Stress state

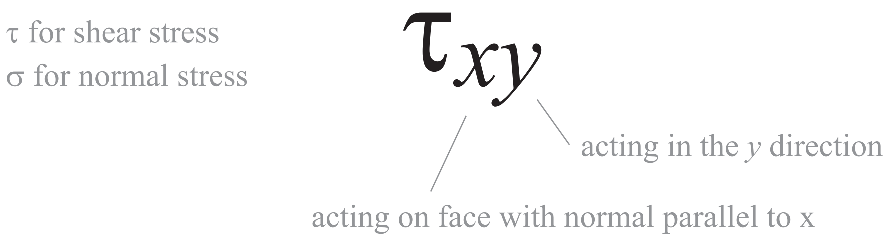
2.2.1 Introduction

The following square shows a general state of stress in 2D, defined using axes x and y.



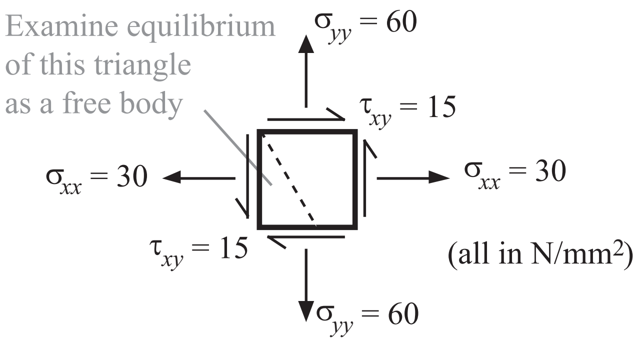
What are the stresses if we use another set of axes, a and b at an angle θ to the original set?

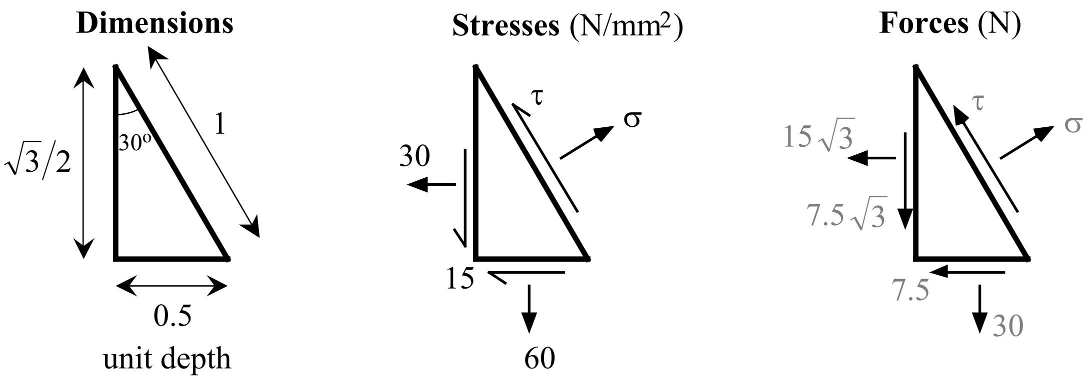
Notation used for stresses



Example

A pressurised cylinder subject to a torque has wall stresses as shown in below (x is the longitudinal direction, and y the hoop direction). What are the stresses on a face at 30° to the x-face?





Equilibrium ↗

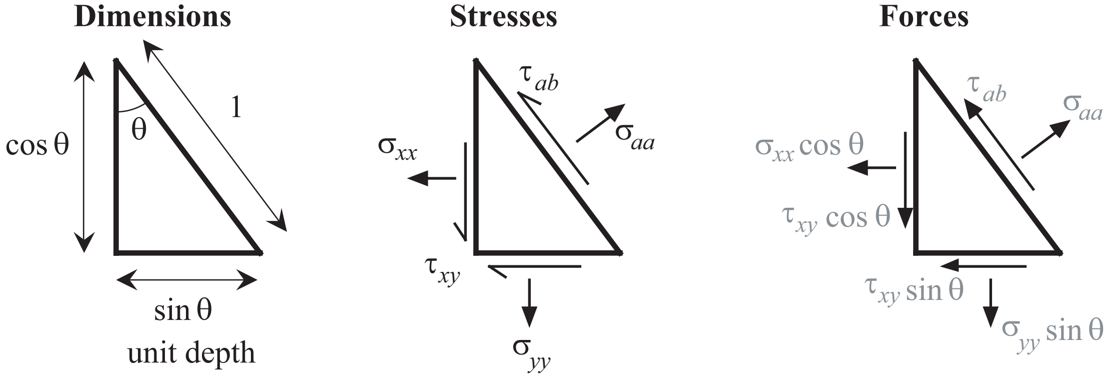
$$\sigma = (15\sqrt{3} + 7.5)\cos(30^\circ) + (7.5\sqrt{3} + 30)\sin(30^\circ)$$
$$= 50.5\ \mathrm{N/mm^2}$$

Equilibrium ↖

$$\begin{aligned}\tau &= -(15\sqrt{3}+7.5)\sin(30^\circ)+(7.5\sqrt{3}+30)\cos(30^\circ)\\ &= 20.5\ \mathrm{N/mm^2}\end{aligned}$$

2.2.2 Equilibrium equations

Repeat the calculation for a general case



Equilibrium ↗

$$\sigma_{aa} = (\sigma_{xx}\cos\theta + \tau_{xy}\sin\theta)\cos\theta + (\sigma_{yy}\sin\theta + \tau_{xy}\cos\theta)\sin\theta$$

Equilibrium ↖

$$\tau_{ab} = -(\sigma_{xx}\cos\theta + \tau_{xy}\sin\theta)\sin\theta + (\sigma_{yy}\sin\theta + \tau_{xy}\cos\theta)\cos\theta$$

$$\boxed{\begin{aligned}\sigma_{aa} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\cos\theta\sin\theta\\ \tau_{ab} &= -\sigma_{xx}\sin\theta\cos\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)\end{aligned}}$$

These equations will be found in the Structures Data Book, page 2.

2.2.3 Mohr's circle of stress in 2D

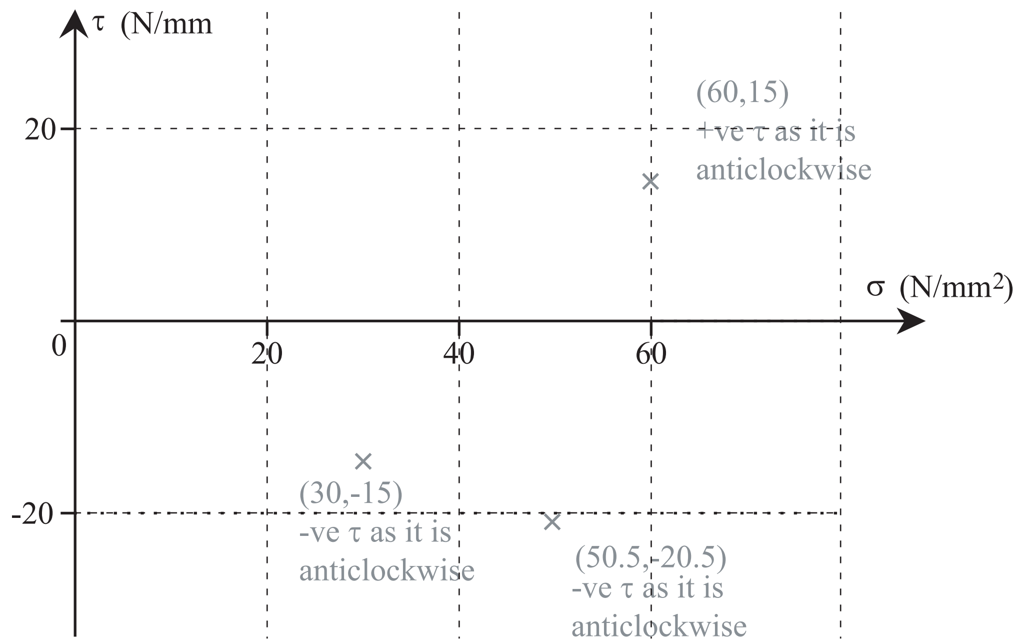
A very useful construction can be generated from the equations of equilibrium. For various angles θ, plot a graph of normal stress on a face against shear stress, but for shear stress use a special new sign convention:

For Mohr's circle, shear stress is plotted positive when it is acting clockwise.

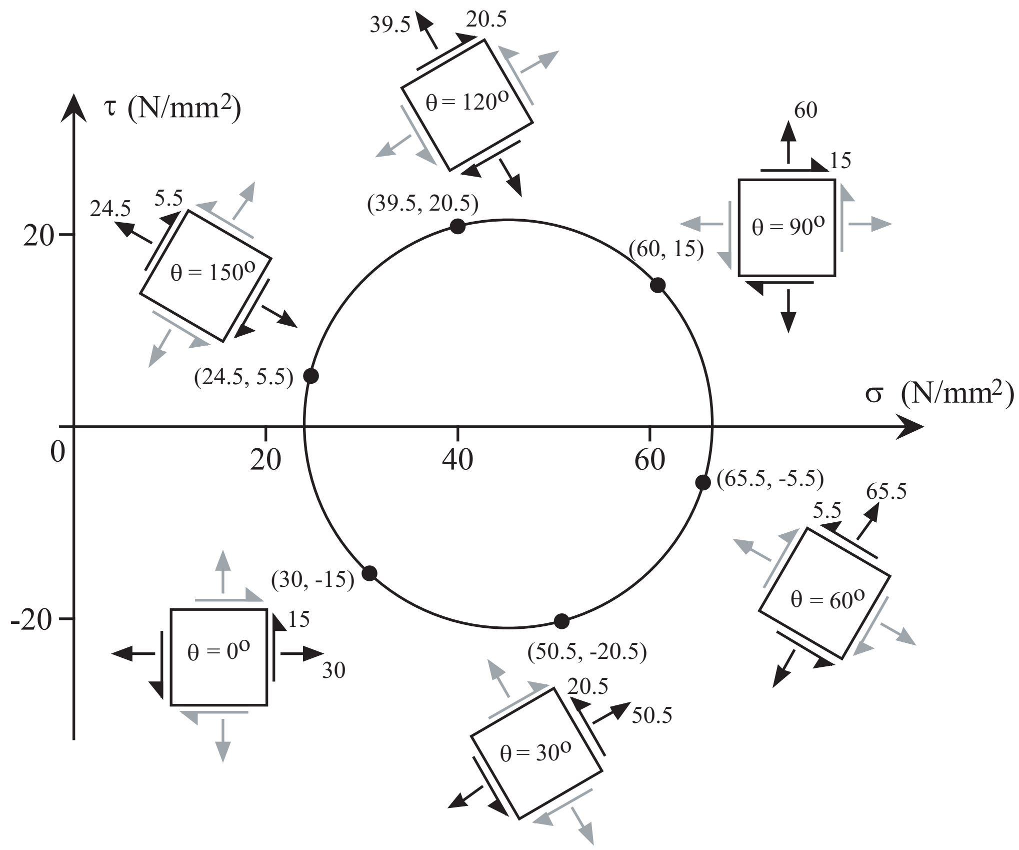
Consider our earlier example. We have calculated three pairs of normal and shear stress:

$$\begin{aligned}(\sigma_{xx},\tau_{xy}) &= (30,15) \text{ — shear stress anticlockwise}\\ (\sigma_{yy},\tau_{yx}) &= (60,15) \text{ — shear stress clockwise}\\ (\sigma,\tau) &= (50.5,20.5) \text{ — shear stress anticlockwise}\end{aligned}$$

Plot these with the special sign convention for τ:



If we repeated the calculation for many different angles θ, we would obtain the following plot. Each point is plotting the stresses on one pair of opposite face — the ones with the bold arrows. The stresses on the other faces will be found elsewhere on the plot.



In this case, the results plot as a circle. Can we show that this will always be the case?

For the example, $\theta = 180°$ took us round the circle by $360°$, giving us a clue that the way to proceed is with double angle formulae. Starting with the equilibrium equations

$$\begin{aligned}\sigma_{aa} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\cos\theta\sin\theta \\ \tau_{ab} &= -\sigma_{xx}\sin\theta\cos\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)\end{aligned}$$

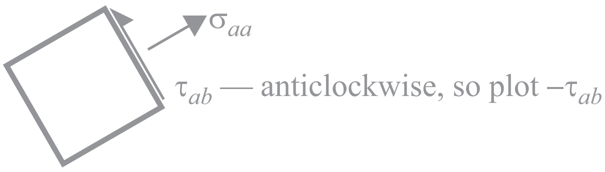
substitute $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, $\sin\theta\cos\theta = \frac{1}{2}(\sin 2\theta)$

$$\begin{aligned}2\sigma_{aa} &= \sigma_{xx} + \sigma_{xx}\cos 2\theta + \sigma_{yy} - \sigma_{yy}\cos 2\theta + 2\tau_{xy}\sin 2\theta \\ 2\tau_{ab} &= -\sigma_{xx}\sin 2\theta + \sigma_{yy}\sin 2\theta + 2\tau_{xy}\cos 2\theta\end{aligned}$$

Rewrite as:

$$\begin{aligned}[3]\sigma_{aa} &= \frac{1}{2}(\sigma_{xx}+\sigma_{yy}) + \frac{1}{2}(\sigma_{xx}-\sigma_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{ab} &= \qquad\qquad\qquad\quad -\frac{1}{2}(\sigma_{xx}-\sigma_{yy})\sin 2\theta + \tau_{xy}\cos 2\theta\end{aligned}$$

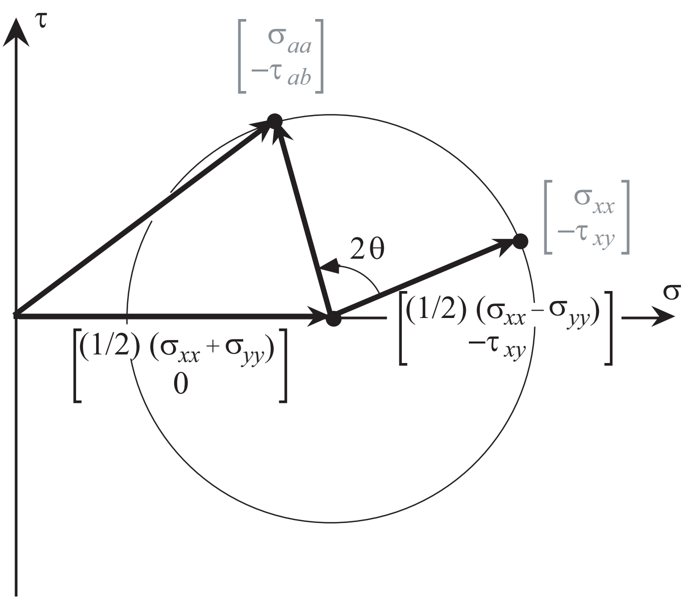
For this to plot as a circle, we need to plot the point $(\sigma_{aa}, -\tau_{ab})$ as θ varies.



Write the expressions as a vector equation

$$\begin{bmatrix}\sigma_{aa} \\ -\tau_{ab}\end{bmatrix} = \begin{bmatrix}(1/2)(\sigma_{xx}+\sigma_{yy}) \\ 0\end{bmatrix} + \begin{bmatrix}\cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta\end{bmatrix}\begin{bmatrix}(1/2)(\sigma_{xx}-\sigma_{yy}) \\ -\tau_{xy}\end{bmatrix}$$

Plot these vectors.



2.2.4 Properties of Mohr's circle of stress

1. The stress state on two perpendicular faces (e.g. x and y, a and b) lie at opposite ends of a diameter of the circle.

2. The centre of the circle lies on the σ axis, at the mean value of the normal stresses on perpendicular faces, $\frac{1}{2}(\sigma_{xx} + \sigma_{yy})$. This mean value doesn't depend on the choice of axes.

3. To find the stresses on faces at an orientation θ to the original faces, rotate around Mohr's circle by 2θ in the same direction.

4. There are two directions in which there is no shear stress on the perpendicular face. These are called *principal directions of stress*.

2.2.5 Mohr's circle teaching package

A teaching package to help you become familiar with Mohr's circle is available on the teaching system. Log on, and type

```
start mohr_circle
```

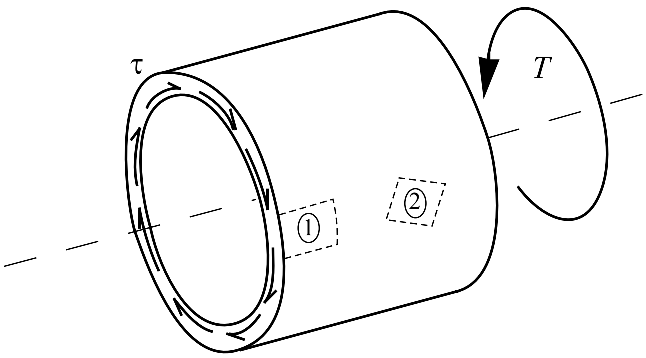
Alternatively, if you are familiar with the package matlab, type `mohr_circle` in a matlab window.

The teaching package plots Mohr's circle for a set of stresses that you specify. It also allows you to rotate the axes that you are using to define the stresses. You can do this by entering values, or by 'clicking and dragging' either the axes, or the stresses plotted on the Mohr's circle. There are also some simple 'trial and error' questions for you to try.

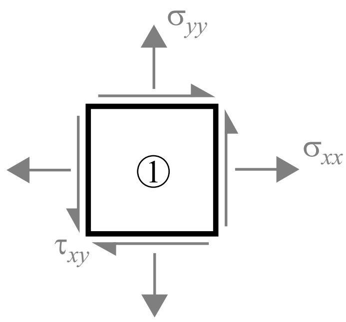
The package is aimed to help you become familiar with the new concepts of a 2D stress state, and the Mohr's circle construction. Don't forget, though, that sketching Mohr's circle is meant to be a simple way of doing stress calculations by hand. Make sure that that you are able to sketch Mohr's circle by hand!

2.2.6 Example

A cylinder of radius 50 mm, and wall thickness 1 mm is subject to a torque of 785 Nm. Find the stresses on faces at an angle 30° to the longitudinal and hoop directions, as shown below.



We need to find the stresses on body 2, but we cannot do that directly. Instead, we initially find the stresses on body 1, which we can do easily:



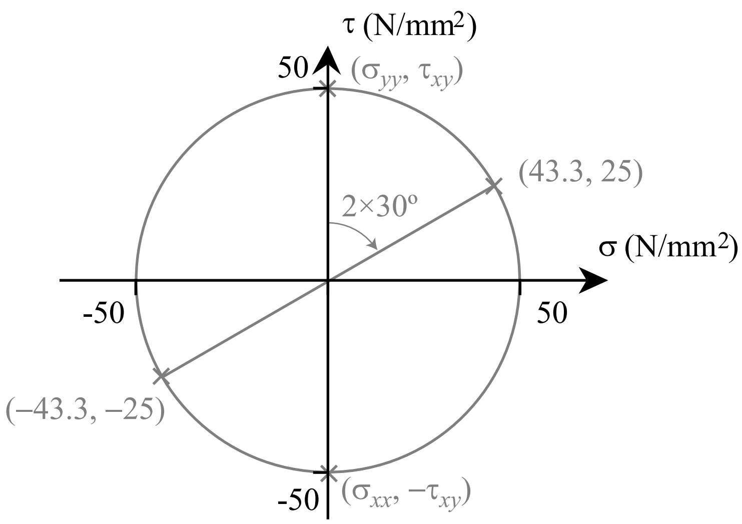
$$\sigma_{xx} = \sigma_{yy} = 0 \text{ — no internal pressure}$$

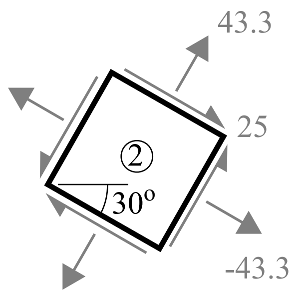
$$\tau_{xy} = \frac{T}{2A_e t} = 50 \text{ N/mm}^2$$

To find the stresses on a free body at another orientation, we can use Mohr's Circle. We know the pairs of normal and shear stress on two perpendicular faces already:

$$\text{Plot points: } \left.\begin{array}{c}(\sigma_{xx}, -\tau_{xy}) = (0, -50) \\ (\sigma_{yy}, \tau_{xy}) = (0, 50)\end{array}\right\} \text{ opposite ends of a diameter}$$

To find stresses at 30°, rotate around circle by $2 \times 30°$





2.3 Principal Stress Directions

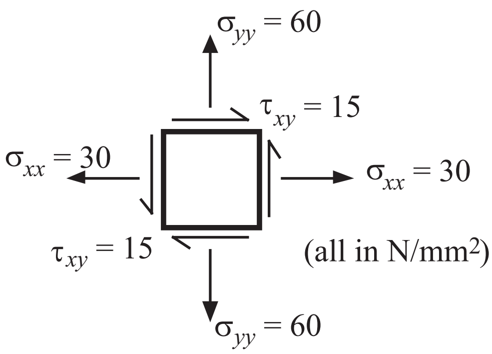
It is possible to show that: (but not proved here)

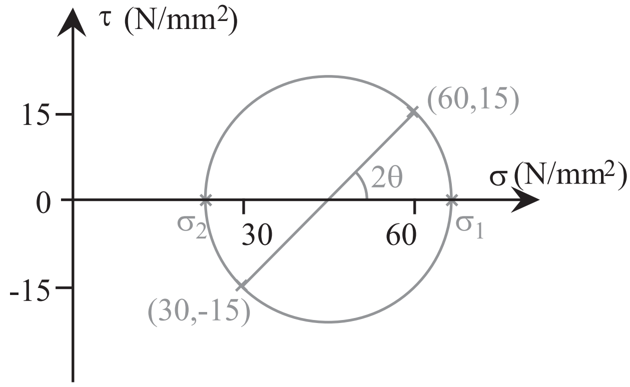
> For any state of stress, there will be three perpendicular directions on which faces there are no shear stresses. These directions are called the principal stress directions. On these faces the only stress is a normal stress, and these three stresses are called the principal stresses.

We saw previously the two principal directions and stresses that could be found earlier by Mohr's circle, but of course the normal direction in these examples is unstressed, and is hence the third principal direction with a principal stress of zero.

Example

Find the principal stresses and principal directions for the earlier example.



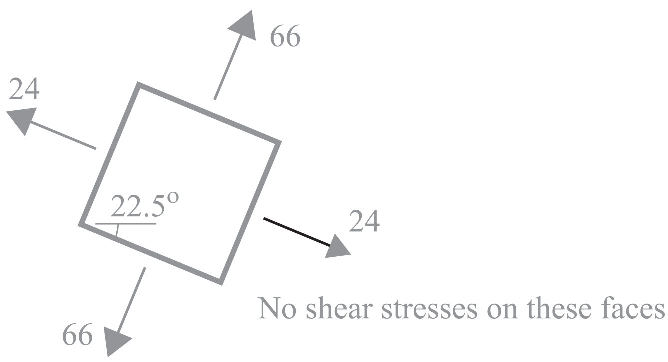


Centre of Mohr's circle, $(30+60)/2 = 45\ \mathrm{N/mm}^2$

Radius of Mohr's circle, $R^2 = 15^2 + 15^2$, $R = 21\ \mathrm{N/mm}^2$

$$\sigma_1 = 45 + 21 = 66\ \mathrm{N/mm}^2$$
$$\sigma_2 = 45 - 21 = 24\ \mathrm{N/mm}^2$$

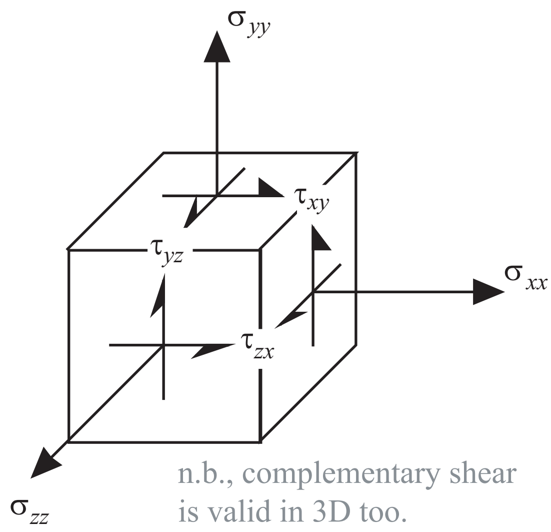
Orientation, $2\theta = \tan^{-1}(15/15) = 45°$, $\theta = 22.5°$ Also, σ_3 = 0 in through-thickness direction (again, no shear stress on this face)



> *Try Questions 1, 2 and 3, Examples Sheet 2/2*

2.4 3-D Stress State

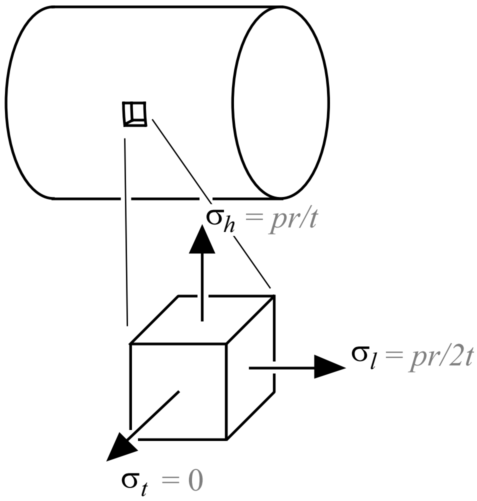
By considering only a two-dimensional world, we have considerably over-simplified the analysis of stress. On a general elemental cube, there will be *six* stresses to consider, three normal stresses, and three shear stresses



2.4.1 3-D Mohr's circle

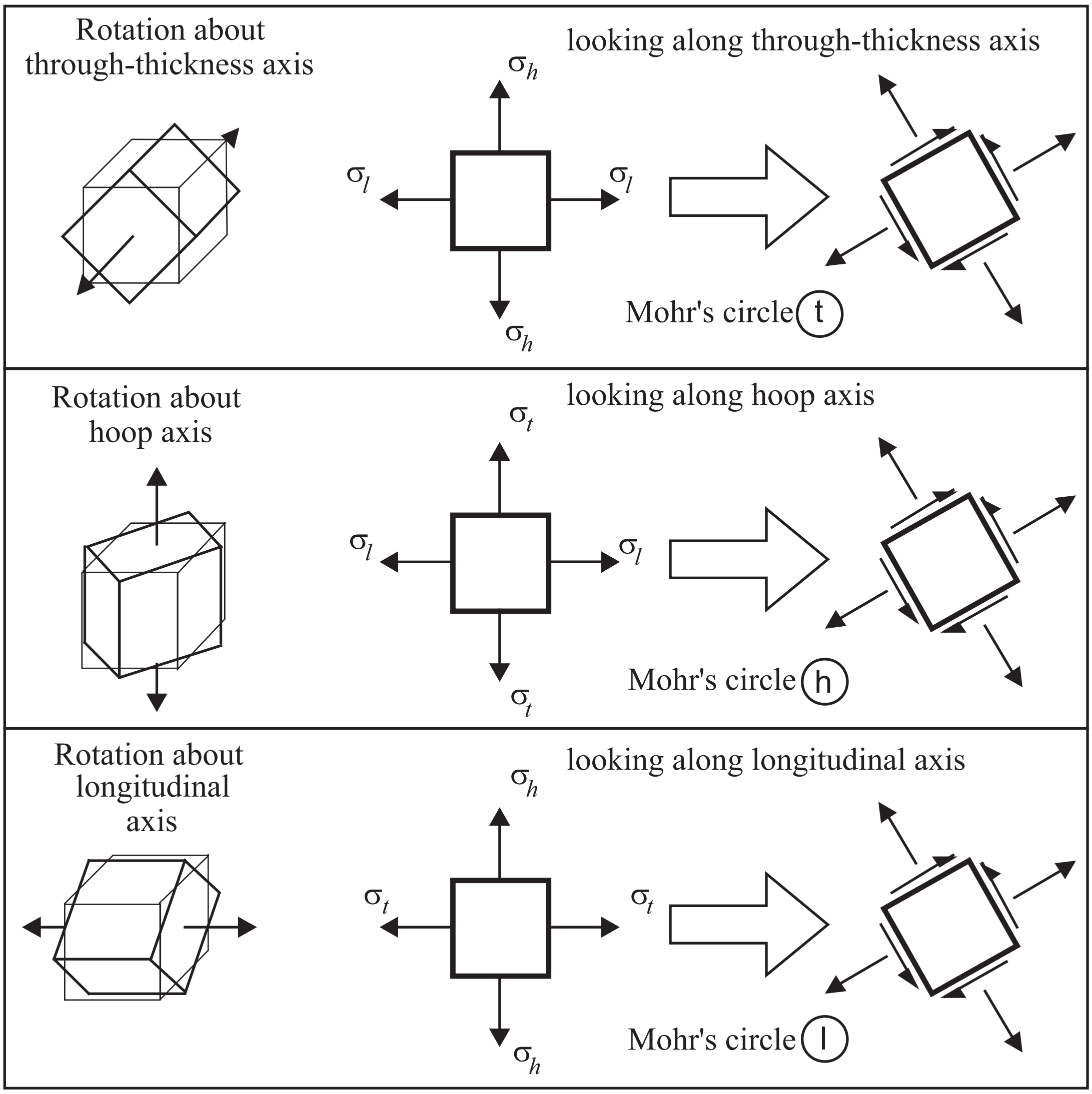
Example

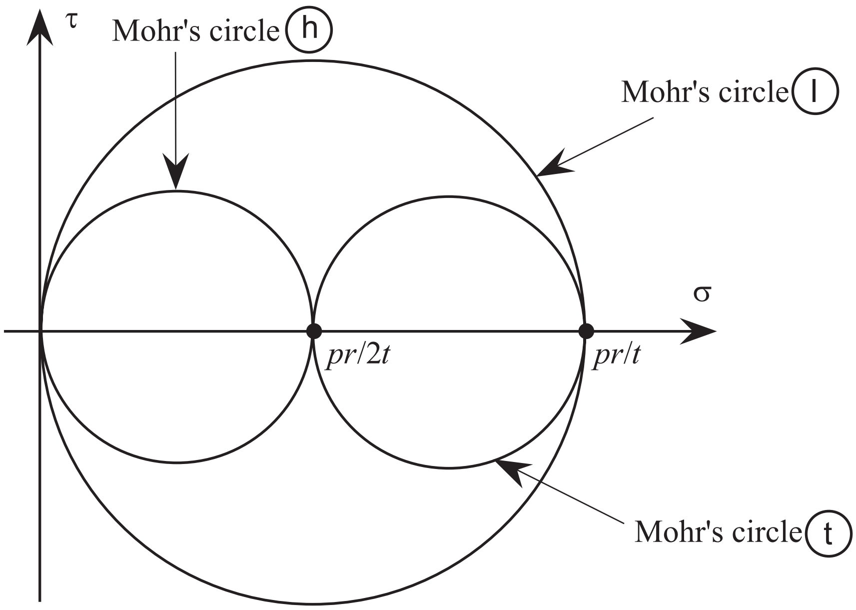
Find the maximum shear stress in a pressurised cylinder. On which planes do these stresses act?



If we looked at the cube along the through-thickness axis, we would see a square with the stresses σ_l and σ_h acting on its sides. If we now *rotated* this square (about the through-thickness direction), we would still get the Mohr's Circle we have plotted previously, representing the relationship between the stresses in that plane.

Now, however, this is not the only Mohr's Circle that we can plot. We could have rotated about the hoop, or the longitudinal, direction, giving a total of three Mohr's Circles.





If we had only considered the 2D case, for rotation about the through-thickness direction, we would have got the wrong answer! The maximum shear stress is found by rotating around the longitudinal axis by 45°