5. Natural Convection

In general, natural or free convection results in a smaller heat transfer than that due to forced convection. It will usually be swamped by forced convection when the latter is present, but many important flows (including all flows of air within buildings and many environmental ones are driven by natural convection).

5.1. Characteristics of Natural Convection

When heated, fluids expand and are driven upward by buoyancy forces.

For these flows there is no natural reference velocity (there is no "free stream"). There is this no natural Reynolds Number associated with the flow. There is, however, an upward velocity driven by buoyancy and examination of Schlieren pictures of naturally-convecting flows reveals what appear to be thermal boundary layers on heated objects. The fundamental questions are:- **what are the important parameters; how do we scale them, etc?**

We have all experienced natural convection. In general, we do not notice pressure forces - hold your hand over a candle (for a short time!). We notice the heat transfer but are not aware of any upward pressure force, as we would when there was a significant free stream velocity. **We will *assume that pressure variations due to convection are in general small*** and that the motion is, therefore, driven primarily by changes in density.

5.2. How big are the $\delta\rho$'s ?

We not expect large changes in density, but it appears that they are sufficient to show up optically in air due to changes in the refractive index (which is a function of density). In order to deal equally easily with liquids and gases it is convenient to use a *coefficient of expansion*.

$$\beta = \frac{1}{v}\left.\frac{\partial v}{\partial T}\right|_p \qquad \text{where } v \text{ is the specific volume.}$$

$$\text{Since } \rho v = 1 \;\Rightarrow\; \rho\, dv + v\, d\rho = 0 \;\Rightarrow\; \frac{dv}{v} = -\frac{d\rho}{\rho} \quad \text{so that} \quad \beta = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right|_p$$

(For liquids, ρ is effectively independent of pressure, so that it doesn't matter what is held constant). For small density changes (assuming constant pressure)

$$\delta\rho \approx \left.\frac{\partial \rho}{\partial T}\right|_p \delta T \;\Rightarrow\; \boxed{\delta\rho \approx -\bar{\rho}\beta\delta T}$$

where $\bar{\rho}$ is the mean value of density

For a liquid, β must be obtained from tables. For a perfect gas it can be derived from $\rho = P/RT$

$$\left.\frac{\partial \rho}{\partial T}\right|_p = -\frac{p}{RT^2} = -\frac{\rho}{T} \;\Rightarrow\; \beta = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right|_p = \frac{1}{T} \quad \text{and this is evaluated at } \bar{T}$$

Typically $\beta \approx \dfrac{1}{300}$

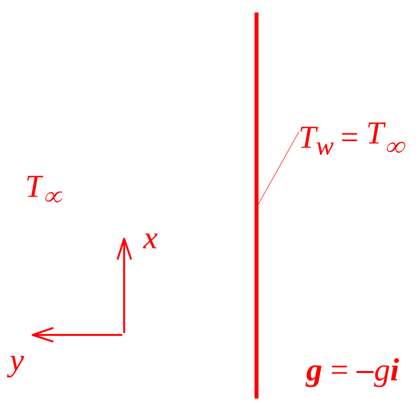
5.3. *How big are the velocities that these $\delta\rho$'s drive ?*

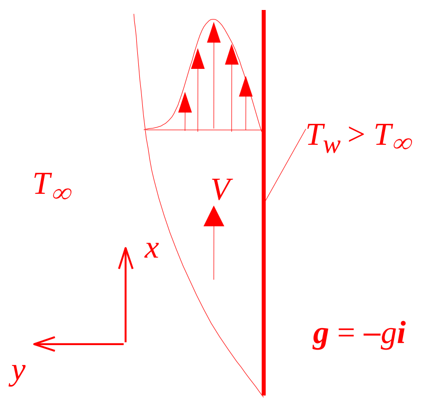
We find a very rough order of magnitude estimate by looking at a simple problem - the flow near a vertical flat plate. We expect the flow to be driven upwards near the warm plate but farther away, the flow will be at rest. In fact we are expecting some sort of boundary layer behaviour, but this time, the vertical velocity will be zero at the wall and also as we move to the edge of the boundary layer, where the shear stress will also vanish. The momentum equation is

$$\rho\left(V\frac{\partial V}{\partial s}\mathbf{e}_s - \frac{V^2}{R}\mathbf{e}_n\right) = -\frac{\partial p}{\partial s}\mathbf{e}_s - \frac{\partial p}{\partial n}\mathbf{e}_n + \rho\mathbf{g} + \textbf{viscous}$$

5.3.1. *Deriving the equation which governs the convective flow near a hot plate*

(A) Base case – No flow – V = 0, $T = T_\infty$	***(B) Flow due to temperature variation, T = T***
Horizontal scale exaggerated	Horizontal scale exaggerated





Case A: $\quad 0 = -\left(\frac{\partial P}{\partial x}\right)_A - \rho_A g$

Case B: $\quad \rho V\frac{\partial V}{\partial x} = -\left(\frac{\partial P}{\partial x}\right)_B - \rho_B g + \mu\frac{\partial^2 V}{\partial y^2}$

When the heating is turned on, we assume that:

- there is no change in pressure,
- the density changes are fairly small
- the flow is primarily in the *x*-direction, with streamlines parallel

Now we write an equation for the difference between the base case A (no flow) and Case B (flow). Subtract A from B to give:

$$\rho V\frac{\partial V}{\partial x}=-\left(\frac{\partial P}{\partial x}\right)_B-\rho_B g+\mu\frac{\partial^2 V}{\partial y^2}-\left\{-\left(\frac{\partial P}{\partial x}\right)_A-\rho_A g\right\}$$

We assume that the pressure distribution is the same in both cases, i.e. the convective flow does not result in significant changes in pressure:

$$\left(\frac{\partial P}{\partial x}\right)_B=\left(\frac{\partial P}{\partial x}\right)_A$$

Our equation describing the flow becomes

$$\rho V\frac{\partial V}{\partial x}=-(\rho_B-\rho_A)g+\mu\frac{\partial^2 V}{\partial y^2}$$

From earlier, since the density difference between case A and case B is small,

$$(\rho_B-\rho_A)=-\bar{\rho}\,\beta\,(T_B-T_A) \qquad \text{(i.e. } \Delta\rho=-\rho\beta\Delta T\text{)}$$

$$\rho V\frac{\partial V}{\partial x}=\bar{\rho}\beta\,(T-T_\infty)g+\mu\frac{\partial^2 V}{\partial y^2}$$

This equation is sometimes referred to as the **Boussinesq** Equation.

5.3.2. *Scaling analysis*

$\Delta\rho g$ = buoyancy force per unit volume

ρ mass per unit volme ×accelaration

$$\rho V\frac{\partial V}{\partial x}=\bar{\rho}\beta(T-T_\infty)g+\mu\frac{\partial^2 V}{\partial y^2}$$

viscous term

We are not interested in solving this, just examining the likely size of the maximum velocity in the boundary layer and how we should scale it intelligently for dimensional analysis purposes. If we assume that the most important thing here is a balance between buoyancy forces and viscous forces

$$\mu\frac{\partial^2 V}{\partial y^2}\approx\bar{\rho}\,\beta(T-T_\infty)g$$

Rough scaling

$$\frac{\mu V}{\delta^2}\sim\bar{\rho}\beta(T_w-T_\infty)g$$

It seems that we should scale V as

$$V_{scale}\leftrightarrow\frac{\bar{\rho}\,\beta\,g\,\Delta T\,\delta^2}{\mu}$$ where δ is the boundary layer thickness

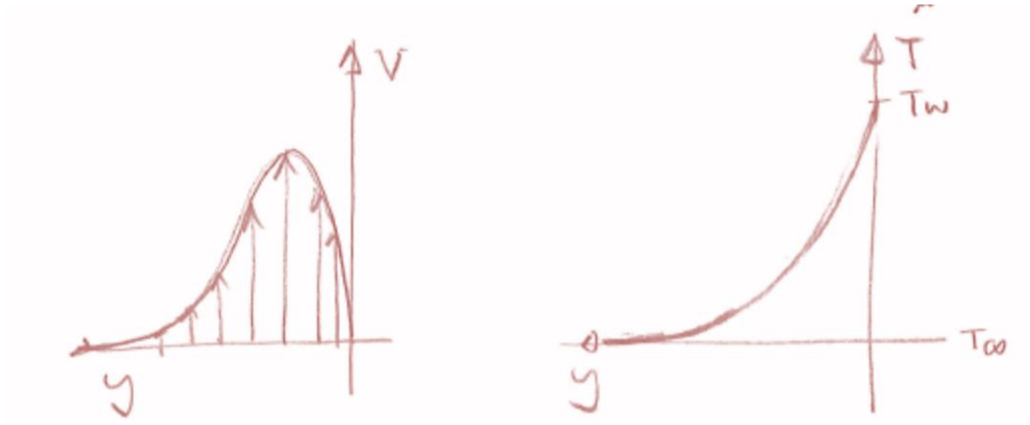
Also, for the boundary conditions we expect

$y = 0$	$y = \delta$
$V = 0$	$V = \mu \dfrac{dV}{dy} = 0$
$T = T_w$	$T = T_\infty \quad \dfrac{dT}{dy} = 0$

The simplest polynomial fits which satisfy these are conditions are

$$V = V_{scale}\frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2 \quad \text{with} \quad T - T_\infty = (T - T_\infty)\left(1-\frac{y}{\delta}\right)^2$$

where V_{scale} is some multiple of $\dfrac{\bar{\rho}\,\beta\, g\,\Delta T\,\delta^2}{\mu}$.



It is beyond the scope of this course to prove it, but a control volume analysis of the type that you did for flow in a momentum/forced convection boundary layer, shows that

$$\delta \propto x^{1/4} \quad \text{and} \quad V_{scale} \propto x^{1/2}$$

In trying to scale general problems, then, we should think of using

$$V_{scale} = \frac{\bar{\rho}\,\beta\, g\,\Delta T\,\delta^2}{\mu}$$

as a typical velocity, or rather, since we don't have really know how δ behaves in general flows, we should use

$$V_{scale} = \frac{\bar{\rho}\,\beta\, g\,\Delta T\,D^2}{\mu}$$

where D is a natural length for the problem.

When talking about a parameter to replace our beloved Reynolds Number, we will then use the **Grashof Number (Gr)**

$$Gr = \frac{\bar{\rho} V_{scale} D}{\mu} = \frac{\bar{\rho}^2 \beta g \, \Delta T \, D^3}{\mu^2} = \frac{\beta g \, \Delta T \, D^3}{\nu^2} \quad \text{where} \quad \nu = \frac{\mu}{\rho}$$

5.4. Typical Free Convection Boundary Layer Type Flows

The figures shown previously are taken from Van Dyke's Book "An Album of Fluid Motion".

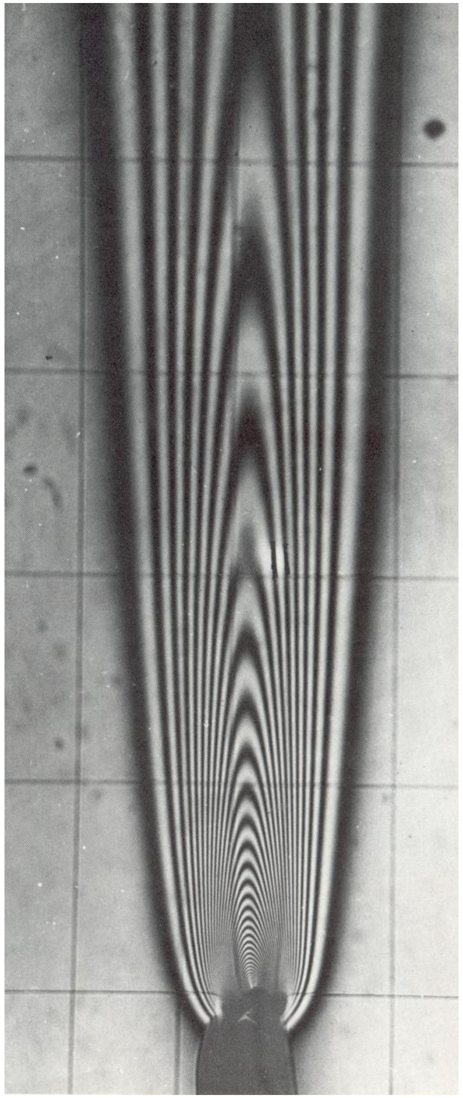
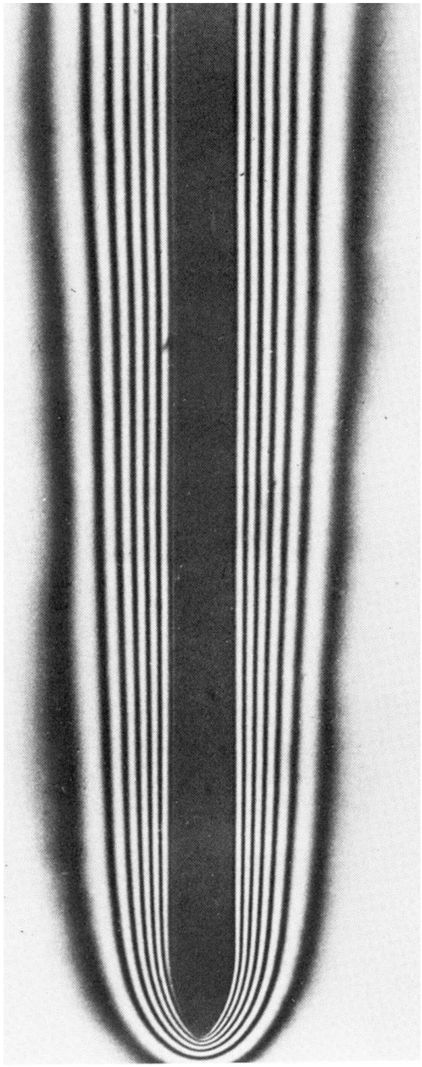
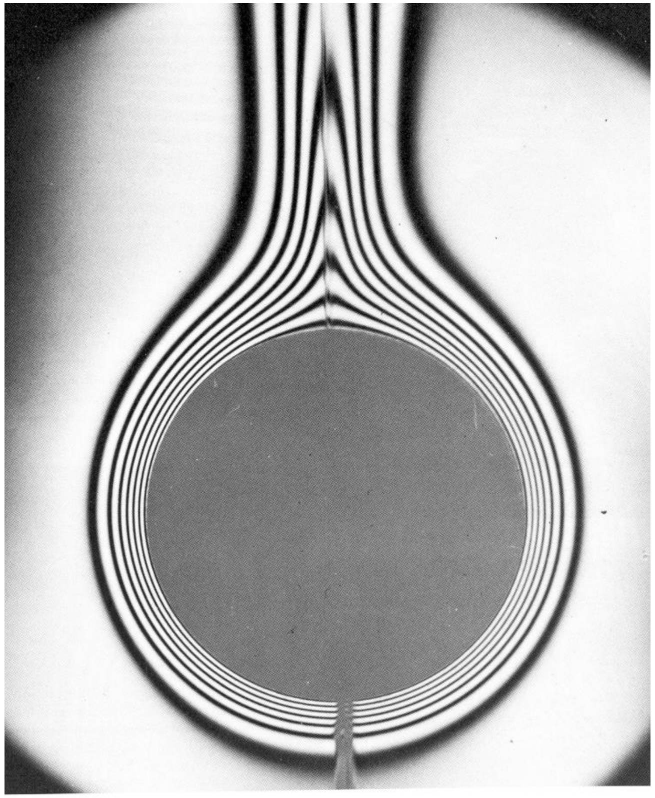
This type of picture is referred to as an interferogram, which is obtained by double exposing pictures with and without the flow. The refractive index of air is a function of its density and each fringe (a black region or a white region) corresponds to a constant density change (which is equivalent to a constant temperature change since the pressure is effectively constant in these flows).

The left hand one is a plume of hot air rising above a heated element, the middle one is the boundary layer on a uniformly heated plate and the final one is the flow over a heated cylinder.

Note:-

(a) The flows are all laminar (because the velocities involved are small).

(b) The boundary layer on the cylinder does not separate (there are no pressure gradients, therefore no adverse pressure gradients) but meets at the top and is then shed as a rising plume.

Taken from An Album of Fluid Motion by Milton Van Dyke

5.5. *Dimensional Analysis in General*

The problem above is a classic example of where dimensional analysis performed without much thought would have given a *misleading* answer.

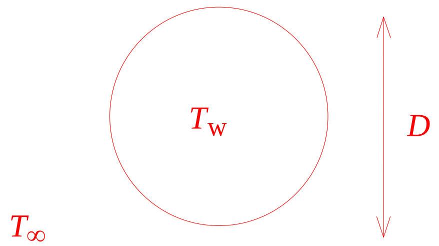
e.g. Flow over a heated circular cylinder with constant wall temperature

Geometry D

Phenomenon g

Fluid Properties $\rho \quad \mu \quad \lambda \quad \beta \quad c_p$

Operating Point $\Delta T \quad \dot{Q}$



We would have been tempted to say that

$$\dot{Q} \quad \text{is a function of} \quad D, \rho, \mu, \lambda, c_p, \beta, \Delta T, g$$

9 dimensional variables, leading to 5 groups once the four quantities M, L, time and θ are eliminated.

WRONG

Correct Way

$\boldsymbol{\beta \Delta T g}$ must appear **together**. i.e. they represent *one* independent dimensional quantity *not three*.

As in forced convection, use a heat transfer coefficient

$$\frac{\dot{Q}}{A\Delta T} = h_{overall} = fn(D, \rho, \mu, \lambda, c_p, \beta\Delta T g)$$

$\Rightarrow$ Nusselt Number

$$Nu_{overall} = \frac{h_{overall}D}{\lambda} = fn\left(\frac{\bar{\rho}^2 \beta g \,\Delta T\, D^3}{\mu^2}, \frac{\mu c_p}{\lambda}\right) = fn\,(Gr, Pr)$$

where

$$Gr = \text{Grashof number} \quad = \quad \frac{\bar{\rho}^2 \beta g \,\Delta T\, D^3}{\mu^2}$$

5.6. Key Points for Natural Convection

- The Grasshof Number replaces Reynolds Number.
- Pressure variations due to motion are small.
- Flow velocities are relatively modest, so that flows stay laminar for quite a long time.
- In general, $Nu_{overall} = \frac{h_{overall}D}{\lambda} = fn\ (Gr, Pr)$
- *Re*, *Gr*, *Nu*, *Pr*, etc. are all in the Data Book (page 4)