PART IB Paper 7: Mathematics

PROBABILITY

Examples paper 5

Elementary exercises are marked: †, Tripos standard, but not necessarily Tripos length, are marked: ∗.

Probability Fundamentals

1. Suggest appropriate sample spaces to model the following applications as random experiments:
   (a) A microprocessor has 16 memory banks it can distribute write requests to. The write requests arrive at random intervals and the number of bytes to be stored for each request is random.
   (b) A gas composed of 7 types of particles in known concentrations fills a cylinder. A model is required for the behaviour of the molecules as the volume of the cylinder changes.
   (c) A mobile phone network has base stations covering overlapping areas and a fluctuating number of users moving in and out of coverage zones.
   (d) The Department for Transport requires a traffic model of the road network in view of planning future road works and expansions to the network.

2. Use the three axioms of probability (non-negativity, unit probability of the certain event and additivity for disjoint events) to prove for arbitrary events, $\mathcal{A}$ and $\mathcal{B}$, the following four statements:
   (a) complement rule: $\mathbb{P}[\mathcal{A}^{\mathsf{c}}] = 1 - \mathbb{P}[\mathcal{A}]$.
   (b) impossible event: $\mathbb{P}[\varnothing] = 0$.
   (c) if $\mathcal{A}$ is contained in $\mathcal{B}$, $\mathcal{A} \subseteq \mathcal{B}$, then $\mathbb{P}[\mathcal{A}] \leq \mathbb{P}[\mathcal{B}]$.
   (d) general addition rule: $\mathbb{P}[\mathcal{A} \cup \mathcal{B}] = \mathbb{P}[\mathcal{A}] + \mathbb{P}[\mathcal{B}] - \mathbb{P}[\mathcal{A} \cap \mathcal{B}]$.

3. † The probability that a scheduled flight departs on time is 0.83 and the probability that it arrives on time is 0.92. The probability that it both departs and arrives on time is 0.78. Find the probability that
   (a) the plane arrives on time given that it departed on time
   (b) the plane did not depart on time given that it failed to arrive on time

4. A function g is said to be *convex* if, for any x_1, x_2 and any $0 \leq \lambda \leq 1$,
$$g\big(\lambda x_1 + (1-\lambda)x_2\big) \leq \lambda g(x_1) + (1-\lambda)g(x_2).$$
   In other words, the function values between x_1 and x_2 lie below the straight line going from $g(x_1)$ to $g(x_2)$.
   (a) Draw a picture to visualise the definition of convexity in an example.
   (b) Consider a binary random variable X with $P_{\mathrm{X}}(1) = p$ and $P_{\mathrm{X}}(0) = 1 - p$. Derive *Jensen's inequality*
$$g\left(\mathbb{E}[\mathrm{X}]\right) \leq \mathbb{E}[g(\mathrm{X})]$$
   for any convex function g.
   (c) Jensen's inequality applies to all random variables, not just binary random variables. Derive Jensen's inequality for ternary random variables.
   Hint: you could use the following trick: let X and Y be independent binary random variables and Z be a ternary random variable such that Z = X if Y = 0 and Z = 2 if Y = 1. Show that $\mathbb{E}[g(\mathrm{Z})] = P_{\mathrm{Y}}(0)\mathbb{E}[g(\mathrm{X})] + P_{\mathrm{Y}}(1)g(2)$ and hence show that Jensen's inequality applies to Z.

5. † Calculate the *entropy* (or uncertainty) for the following random variables:
   (a) the number X of supervisions in a random Lent term week, given that there are 3 weeks when you have 1 supervision, 3 weeks when you have 2 supervisions, and 2 weeks when you have 3 supervisions;
   (b) the number X′ of supervisions in a week after you complained to your DoS and she shifted two supervisions to avoid 3-supervision weeks;
   (c) the number Y of students studying in IB every year, uniformly distributed[1] between 301 and 316.

[1] this is fictitious and not based on actual figures...

Discrete distributions

6. $*$ Commercial airline pilots need to pass four out of five separate tests for certification. Assume that the tests are equally difficult, and that the performance on separate tests are independent.
   (a) If the probability of failing each separate test is $p = 0.15$, then what is the probability of failing certification?
   (b) To improve safety, new more stringent regulations require that pilots pass all five tests. In order to be able to meet the demand for new pilots, the individual tests are made easier. What should the new individual failure rate be if the overall certification probability should remain unchanged?

7. Cabs arrive independently at a taxi stand at an average rate of 2 per minute.
   (a) What is the probability that no taxi arrives in a given minute?
   (b) What is the probability that no taxi arrives in a given two-minute interval?
   (c) What would the answers be to the two preceding questions, if the cabs would arrive equidistantly spaced in time (with the same average rate), instead of independently?
   (d) What is the probability of observing at least two arrivals in a given minute?

Continuous distributions

8. † Assuming that the height of clouds above ground at some location is a Gaussian random variable X with a mean of 1830 m and a standard deviation of 460 m, what is the probability that the clouds will be higher than 2750 m?

9. A production process for nominal 500Ω resistors produces values which are Normally distributed with mean 501Ω and standard deviation 3Ω. Resistors are rejected if their resistance is less than 498Ω or greater than 508Ω. Find
   (a) the proportion rejected
   (b) the proportion rejected if the mean value is adjusted to minimise the wastage (leaving the standard deviation as 3Ω)
   (c) the value to which the standard deviation would have to be set (leaving the mean at 501Ω) to reduce the wastage to half of the level in the first question.

Previous Tripos questions

2022, 6. a) & b)	2019, 6.	2016, 6.	2011, 4. a)	2009, 4.
2021, 6.	2018, 6.	2013, 4.	2010, 4. a)	2006, 6. b), c) & d).

Answers

3. a) 0.94, b) 0.375.

5. a) 1.56 bits, b) 0.54 bits, c) 4 bits

6. a) combined probability of failure 0.16, b) new individual failure rate 0.035.

7 a) 0.14, b) 0.018, c) zero in both cases, d) 0.59.

8. 0.0228.

9. a) 0.1685, b) 0.0956, c) $\sigma \simeq 2.17$.

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