Part IB Paper 6: Information Engineering

LINEAR SYSTEMS AND CONTROL

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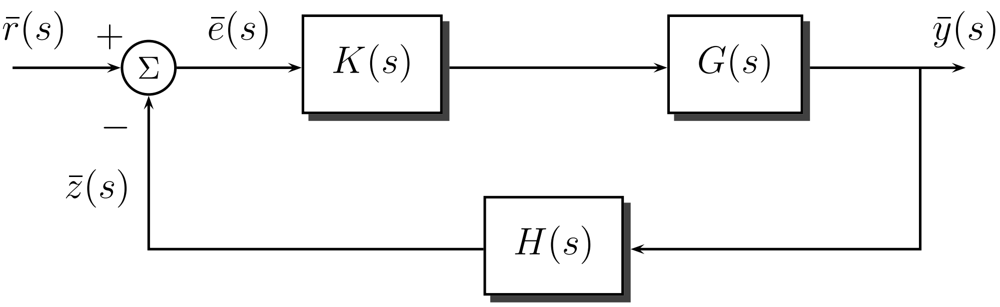
HANDOUT 5

"An Introduction to Feedback Control Systems"



$$\bar{z}(s) = \underbrace{H(s)G(s)K(s)}_{L(s)}\,\bar{e}(s)$$

Return ratio

$$\bar{e}(s) = \underbrace{\frac{1}{1+L(s)}}\,\bar{r}(s)$$

Closed-loop transfer function relating $\bar{e}(s)$ and $\bar{r}(s)$

$$\bar{y}(s) = G(s)K(s)\bar{e}(s) = \underbrace{\frac{G(s)K(s)}{1+L(s)}}\,\bar{r}(s)$$

Closed-loop transfer function relating $\bar{y}(s)$ and $\bar{r}(s)$

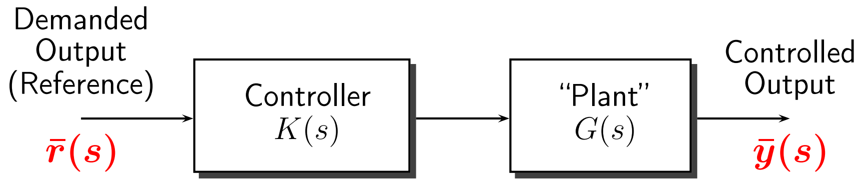
Key Points

- The *Closed-Loop Transfer Functions* are the *actual* transfer functions which determine the behaviour of a feedback system. They relate signals around the loop (such as the plant input and output) to external signals injected into the loop (such as reference signals, disturbances and noise signals).
- It is possible to infer much about the behaviour of the feedback system from consideration of the *Return Ratio* alone.
- The aim of using feedback is for the plant output $y(t)$ to follow the reference signal $r(t)$ in the presence of uncertainty. A persistent difference between the reference signal and the plant output is called a steady state error. Steady-state errors can be evaluated using the final value theorem.

- Many simple control problems can be solved using combinations of proportional, derivative and integral action:
  - Proportional action is the basic type of feedback control, but it can be difficult to achieve good damping and small errors simultaneously.
  - Derivative action can often be used to improve damping of the closed-loop system.
  - Integral action can often be used to reduce steady-state errors.

Contents

5.1 Open-Loop Control



In principle, we could could choose a "desired" transfer function $F(s)$ and use $K(s) = F(s)/G(s)$ to obtain

$$\bar{y}(s) = G(s)\frac{F(s)}{G(s)}\bar{r}(s) = F(s)\bar{r}(s)$$

In practice, **this will not work**

– because it requires an exact model of the plant and that there be no disturbances (i.e. no uncertainty).

Feedback is used to combat the effects of uncertainty

For example:

- Unknown parameters
- Unknown equations
- Unknown disturbances

By open-loop control we refer to the case where we aim to control the output of a system without feedback.

As explained on the left this will not work in practice since even small disturbances and model uncertainty will cause the output to deviate from its desired value (like trying to drive a car with your eyes closed!).

5.2 Closed-Loop Control (Feedback Control)

For Example:

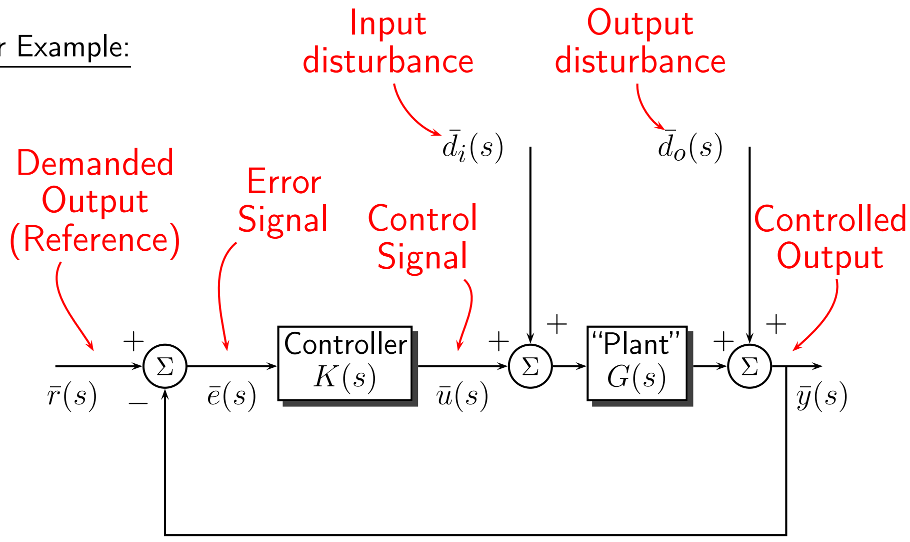


Figure 5.1

5.2.1 Derivation of the closed-loop transfer functions:

$$\bar{y}(s) = \boldsymbol{\bar{d}_o(s) + G(s)\left[\bar{d}_i(s) + K(s)\bar{e}(s)\right]}$$

$$\bar{e}(s) = \boldsymbol{\bar{r}(s) - \bar{y}(s)}$$

$$\implies \quad \bar{y}(s) = \bar{d}_o(s) + G(s)\left[\bar{d}_i(s) + K(s)\left(\bar{r}(s) - \bar{y}(s)\right)\right]$$

$$\implies \quad \left(\boldsymbol{1 + G(s)K(s)}\right)\bar{y}(s) = \bar{d}_o(s) + G(s)\bar{d}_i(s) + G(s)K(s)\bar{r}(s)$$

$$\implies \quad \bar{y}(s) = \frac{\boldsymbol{1}}{\boldsymbol{1 + G(s)K(s)}}\bar{d}_o(s) + \frac{\boldsymbol{G(s)}}{\boldsymbol{1 + G(s)K(s)}}\bar{d}_i(s) + \frac{\boldsymbol{G(s)K(s)}}{\boldsymbol{1 + G(s)K(s)}}\bar{r}(s)$$

The block diagram on the left illustrates a typical feedback control configuration. The aim is for the controlled output $\bar{y}(s)$ to follow a reference signal $\bar{r}(s)$.

The plant $G(s)$ is the system we would like to control, and $K(s)$ is the controller that needs to be designed.

Disturbances acting at the output and input of the plant are also considered, corresponding to measurement and actuation noise, respectively.

The transfer functions from $\bar{d}_0, \bar{d}_i, \bar{r}, \bar{e}$, respectively, to the output $\bar{y}$ are calculated. These are referred to as *closed-loop transfer functions* to emphasize the fact that we are in a closed-loop configuration.

Also:

$$\bar{e}(s) = \bar{r}(s) - \bar{y}(s)$$
$$= -\frac{1}{1+G(s)K(s)}\bar{d}_o(s) - \frac{G(s)}{1+G(s)K(s)}\bar{d}_i(s) + \underbrace{\left(1 - \frac{G(s)K(s)}{1+G(s)K(s)}\right)}_{\dfrac{\mathbf{1}}{\mathbf{1+G(s)K(s)}}}\bar{r}(s)$$

5.2.2 The Closed-Loop Characteristic Equation and the Closed-Loop Poles

Note: All the *C*losed-Loop Transfer Functions of the previous section have the same denominator:

$$\mathbf{1+G(s)K(s)}$$

The *Closed-Loop Poles* (*ie* the poles of the closed-loop system, or feedback system) are the zeros of this denominator.

For the feedback system of Figure 5.1, the Closed-Loop Poles are the roots of

$$\boxed{1 + G(s)K(s) = 0}$$

Closed-Loop Characteristic Equation
(for Fig 5.1)

The closed-loop poles determine:

- The stability of the closed-loop system.
- Characteristics of the closed-loop system's transient response.(e.g. speed of response, presence of any resonances etc)

The equation

$$1 + G(s)K(s) = 0$$

is referred to as the Closed-Loop Characteristic Equation. Its solutions give the location of the poles in the closed loop transfer functions.

As discussed in the previous handout, the location of the poles are very important for the behaviour of a system as they determine its stability properties, and also its transient response.

5.2.3 What if there are more than two blocks?

For Example:

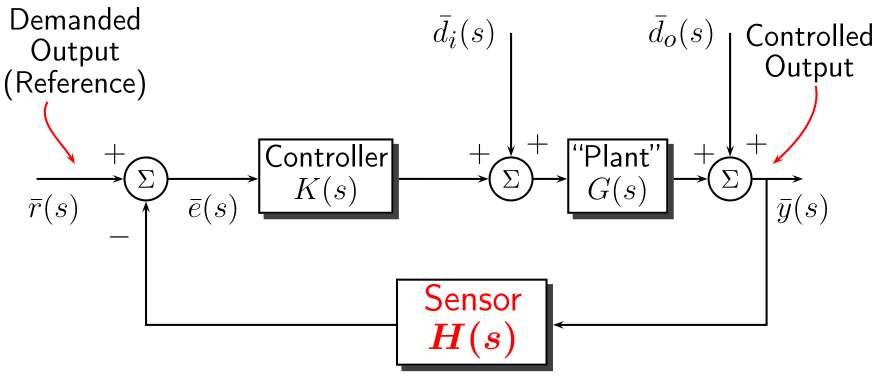


Figure 5.2

The diagram on the left illustrates a feedback configuration where in addition to $K(s)$ and $G(s)$, there is an additional transfer function $H(s)$. The latter could be part of the implementation of the control policy or could include the sensor dynamics.

We now have

$$\bar{y}(s) = \frac{G(s)K(s)}{1 + H(s)G(s)K(s)}\bar{r}(s) + \frac{1}{1 + H(s)G(s)K(s)}\bar{d}_o(s) + \frac{G(s)}{1 + H(s)G(s)K(s)}\bar{d}_i(s)$$

This time $\mathbf{1 + H(s)G(s)K(s)}$ appears as the denominator of all the closed-loop transfer functions.

Let,

$$\mathbf{L(s) = H(s)G(s)K(s)}$$

i.e. the product of all the terms around loop, not including the -1 at the summing junction. $L(s)$ is called the *Return Ratio* of the loop (and is also known as the *Loop Transfer Function*).

The *Closed-Loop Characteristic Equation* is then

$$\mathbf{1 + L(s) = 0}$$

and the *Closed-Loop Poles* are the roots of this equation.

The closed-loop transfer functions derived on the left illustrate that the denominator has a form analogous to that when only $K(s)$ and $G(s)$ were present; i.e. the closed loop poles are the solutions to the equation

$$1 + L(s) = 0$$

where $L(s)$ is the product of all the transfer functions round the loop (not including the -1 at the summing junction).

5.2.4 A note on the Return Ratio

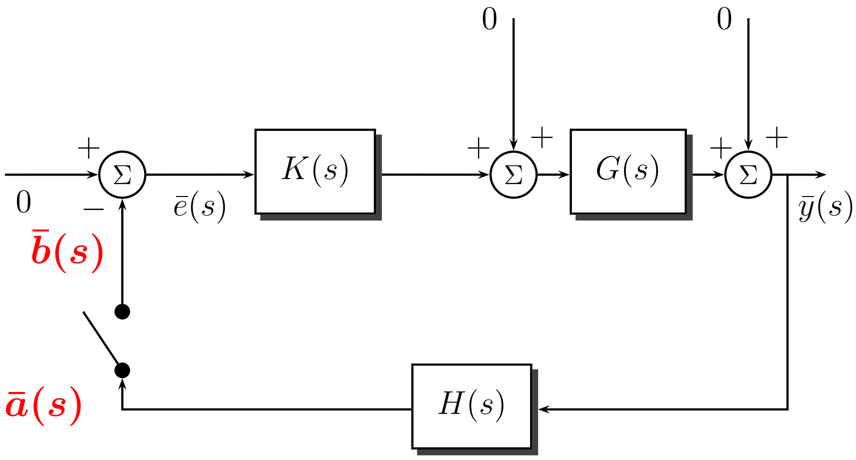


Figure 5.3

With the switch in the position shown (i.e. open), the loop is *open*. We then have

$$\bar{a}(s) = H(s)G(s)K(s) \times -\bar{b}(s) = -H(s)G(s)K(s)\bar{b}(s)$$

Formally, the *Return Ratio* of a loop is defined as -1 times the product of all the terms around the loop. In this case

$$L(s) = -1 \times -H(s)G(s)K(s) = H(s)G(s)K(s)$$

Feedback control systems are often tested in this configuration as a final check before "closing the loop" (i.e. flicking the switch to the closed position).

Note: In general, the block denoted here as $H(s)$ could include filters and other elements of the controller in addition to the sensor dynamics. Furthermore, the block labelled $K(s)$ could include actuator dynamics in addition to the remainder of the designed dynamics of the controller.

5.2.5 Sensitivity and Complementary Sensitivity

The *Sensitivity* and *Complementary Sensitivity* are two particularly important closed-loop transfer functions. The following figure depicts just *one* configuration in which they appear.

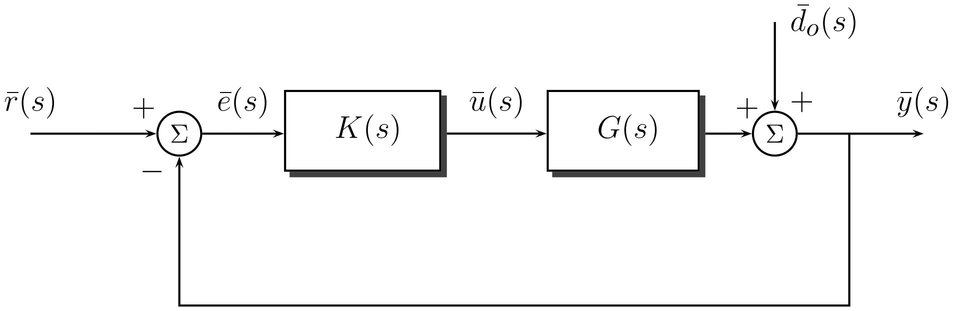


Figure 5.4

$$\boxed{L(s) = G(s)K(s)}$$

$$\begin{aligned}\bar{y}(s) &= \frac{G(s)K(s)}{1+G(s)K(s)}\bar{r}(s) + \frac{1}{1+G(s)K(s)}\bar{d}_o(s) \\ &= \underbrace{\frac{L(s)}{1+L(s)}}_{\substack{\text{Complementary} \\ \text{Sensitivity} \\ T(s)}} \bar{r}(s) + \underbrace{\frac{1}{1+L(s)}}_{\substack{\text{Sensitivity} \\ S(s)}} \bar{d}_o(s)\end{aligned}$$

Note:

$$S(s) + T(s) = \frac{1}{1+L(s)} + \frac{L(s)}{1+L(s)} = 1$$

The Sensitivity $S(s)$ and Complementary Sensitivity $T(s)$ are defined, respectively, as

$$S(s) = \frac{1}{1+L(s)}$$
$$T(s) = \frac{L(s)}{1+L(s)}$$

where $L(s)$ is the return ratio.

The block diagram on the left illustrates a configuration where they appear as particular closed-loop transfer functions.

Note that the summation of $S(s)$ and $T(s)$ is always equal to 1 irrespective of the form of $L(s)$.

5.3 Summary of notation

- The system being controlled is often called the *"plant"*.
- The control law is often called the *"controller"*; sometimes it is called the *"compensator"* or *"phase compensator"*.
- The *"demand"* signal is often called the *"reference"* signal or *"command"*, or (in the process industries) the *"set-point"*.
- The *"Return Ratio"*, the *"Loop transfer function"* always refer to the transfer function of the opened loop, that is the product of all the transfer functions appearing in a standard negative feedback loop (our $L(s)$). Figure 5.1 has $L(s) = G(s)K(s)$, Figure 5.2 has $L(s) = H(s)G(s)K(s)$.
- The *"Sensitivity function"* is the transfer function $S(s) = \dfrac{1}{1 + L(s)}$. It characterizes the sensitivity of a control system to disturbances appearing at the output of the plant.
- The transfer function $T(s) = \dfrac{L(s)}{1 + L(s)}$ is called the *"Complementary Sensitivity"*. The name comes from the fact that $S(s) + T(s) = 1$. When this appears as the transfer function from the demand to the controlled output, as in Fig 5.4 it is often called simply *the* "Closed-loop transfer function" (though this is ambiguous, as there are many closed-loop transfer functions).

5.4 The Final Value Theorem (revisited)

Consider an asymptotically stable system with impulse response $g(t)$ and transfer function $G(s)$, i.e.

$$\underbrace{g(t)}_{\text{Impulse response}} \quad \rightleftharpoons \quad \underbrace{G(s)}_{\substack{\text{Transfer Function} \\ \text{(assumed asymptotically stable)}}}$$

Let $\boldsymbol{y(t) = \int_0^t g(\tau)\, d\tau}$ denote the step response of this system and note that $\boldsymbol{\bar{y}(s) = \frac{G(s)}{s}}$.

We now calculate the final value of this step response:

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \int_0^\infty g(\tau)\, d\tau \\
&= \int_0^\infty \underbrace{\exp(-0\tau)}_{\mathbf{1}}\, g(\tau)\, d\tau = \boldsymbol{\mathcal{L}(g(t))\big|_{s=0} = G(0)}
\end{aligned}$$

Hence,

$$\boxed{\text{Final Value of Step Response} \;\equiv\; \begin{array}{c}\text{Transfer Function} \\ \text{evaluated at} \quad \boldsymbol{s = 0}\end{array}}$$

"Steady-State Gain" or "DC gain"

Note that the same result can be obtained by using the Final Value Theorem:

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{s\to 0} s\bar{y}(s) \qquad \left(\begin{array}{l}\textit{for any } y \textit{ for which} \\ \textit{both limits exist.}\end{array}\right) \\
&= \boldsymbol{\lim_{s\to 0} \; s \cdot \frac{G(s)}{s} = G(0)}
\end{aligned}$$

5.4.1 The "steady state" response – summary

The term "steady-state response" means two different things, depending on the input.

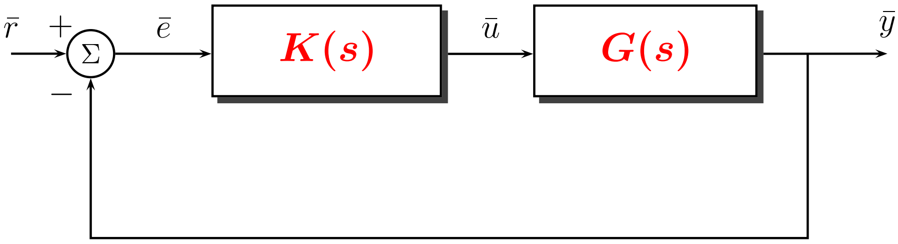
Given an asymptotically stable system with transfer function $G(s)$:

- The *steady-state* response of the system to a step input that is equal to U for $t \geq 0$, is a *constant* $G(0)U$.
- The *steady-state* response of the system to a *sinusoidal* input $\cos(\omega t)$ is the *sinusoid* $|G(j\omega)| \cos\big(\omega t + \arg G(j\omega)\big)$.

These two statements are not entirely unrelated, of course: The *steady-state gain* of a system to a step input, $G(0)$, is the same as the frequency response evaluated at $\omega = 0$ (i.e. the DC gain).

5.5 Some simple controller structures

5.5.1 Introduction – steady-state errors



Return Ratio: $L(s) = G(s)K(s)$.

CLTFs: $\bar{y}(s) = \dfrac{L(s)}{1 + L(s)}\bar{r}(s)$ and $\bar{e}(s) = \dfrac{1}{1 + L(s)}\bar{r}(s)$

Steady-state error:(for a step demand) If $r(t) = H(t)$, then
$\bar{y}(s) = \frac{L(s)}{1+L(s)} \times \frac{1}{s}$ and so

$$\lim_{t\to\infty} y(t) = s \times \frac{L(s)}{1 + L(s)} \times \frac{1}{s}\bigg|_{s\,=\,0} = \frac{L(0)}{1 + L(0)}$$

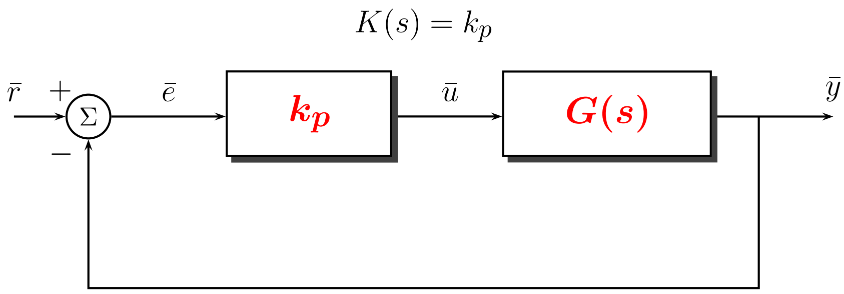
and

$$\lim_{t\to\infty} e(t) = s \times \frac{1}{1 + L(s)} \times \frac{1}{s}\bigg|_{s\,=\,0} = \underbrace{\frac{1}{1 + L(0)}}_{\text{Steady-state error}}$$

(using the final-value theorem.)

Note: These particular formulae only hold for this simple configuration – where there is a unit step demand signal and no constant disturbances (although the final value theorem can always be used).

5.5.2 Proportional Control



Proportional Control is the simplest control policy where the controller is just a multiplication with a constant k_p.

Typical result of increasing the gain k_p, (for control systems where $G(s)$ is itself stable):

- Increased accuracy of control. "good"
- Increased control action.
- Reduced damping.
- Possible loss of closed-loop stability for large k_p.

$\Big\}$ "bad"

As discussed on the left and in the next page there are tradeoffs associated with the choice of k_p. A large k_p leads a smaller steady-state error, but this is at the expense of a more oscillatory response (or even possibly instability).

It will be seen later in this handout and also in handouts 6 and 7, that more advanced controllers can better manage these tradeoffs.

Example:

$$G(s) = \frac{1}{(s+1)^2}$$

(A critically damped 2^{nd} order system)

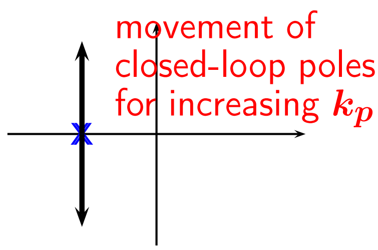
$$\bar{y}(s) = \frac{k_p G(s)}{1 + k_p G(s)}\,\bar{r}(s) = \frac{k_p \dfrac{1}{(s+1)^2}}{1 + k_p \dfrac{1}{(s+1)^2}}\,\bar{r}(s)$$

$$= \frac{k_p}{s^2 + 2s + 1 + k_p}\,\bar{r}(s)$$

So, $\omega_n^2 = \mathbf{1 + k_p}$, $\quad 2\zeta\omega_n = \mathbf{2}$

$$\implies \omega_n = \sqrt{\mathbf{1 + k_p}}, \quad \zeta = \frac{\mathbf{1}}{\sqrt{\mathbf{1 + k_p}}}$$

Closed-loop poles at $\boldsymbol{s = -1 \pm j\sqrt{k_p}}$



"root locus diagram"

A diagram that shows the closed-loop poles as parameter k_p changes is called a root locus diagram.

Steady-state errors using the final value theorem:

$$\bar{y}(s) = \frac{k_p}{s^2 + 2s + 1 + k_p}\,\bar{r}(s)$$

and

$$\bar{e}(s) = \frac{1}{1 + k_p G(s)} = \frac{(s+1)^2}{s^2 + 2s + 1 + k_p}\,\bar{r}(s).$$

final value of step response

So, if $r(t) = H(t)$,

$$\lim_{t\to\infty} y(t) = \left.\frac{k_p}{s^2 + 2s + 1 + k_p}\right|_{\boldsymbol{s\,=\,0}} = \frac{\boldsymbol{k_p}}{\boldsymbol{1 + k_p}}$$

and

$$\lim_{t\to\infty} e(t) = \left.\frac{(s+1)^2}{s^2 + 2s + 1 + k_p}\right|_{\boldsymbol{s\,=\,0}} = \underbrace{\frac{\boldsymbol{1}}{\boldsymbol{1 + k_p}}}_{\text{Steady-state error}}$$

(Note: $L(s) = k_p \frac{1}{(s+1)^2}$ $\Longrightarrow$ $\boldsymbol{L(0) = k_p \times 1 = k_p}$)

Hence, in this example, *increasing* k_p gives *smaller* steady-state errors, but a *more oscillatory* transient response .

• However, by using more complex controllers it *is* usually possible to remove steady state errors *and* increase damping at the same time:

To increase damping –
can often use derivative action (or velocity feedback).

To remove steady-state errors – can often use integral action.

For reference, the step response: (i.e. response to $\bar{r}(s) = \frac{1}{s}$*) is given by*

$$\bar{y}(s) = -\frac{\frac{k_p}{1+k_p}(2+s)}{s^2+2s+1+k_p} + \frac{\frac{k_p}{1+k_p}}{s}$$

so

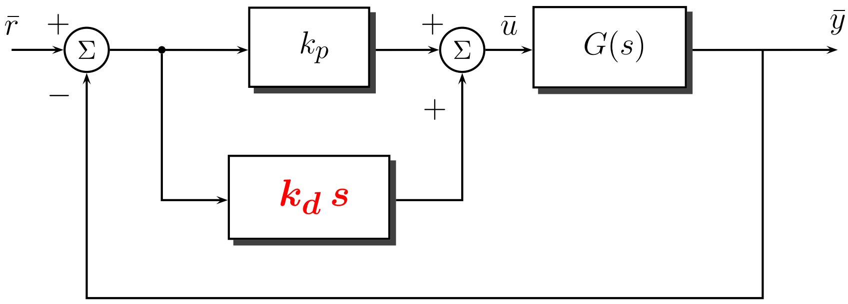
$$\begin{aligned} y(t) &= -\frac{k_p}{1+k_p}\exp(-t)\left(\cos(\sqrt{k_p}t) + \frac{1}{\sqrt{k_p}}\sin(\sqrt{k_p}t)\right) + \frac{k_p}{1+k_p} \\ &= \underbrace{-\sqrt{\tfrac{k_p}{1+k_p}}\exp(-t)\left(\cos(\sqrt{k_p}t - \phi)\right)}_{\text{Transient Response}} \quad + \quad \underbrace{\frac{k_p}{1+k_p}}_{\text{Steady-state response}} \end{aligned}$$

where $\phi = \arctan\frac{1}{\sqrt{kp}}$

But you don't need to calculate this to draw the conclusions we have made.

5.5.3 Proportional + Derivative (PD) Control

$$\boxed{K(s) = k_p + k_d\, s}$$



Typical result of increasing the gain k_d, (when $G(s)$ is itself stable):

- Increased Damping.
- Greater sensitivity to noise.

(It is usually better to measure the rate of change of the error directly if possible – i.e. use velocity feedback)

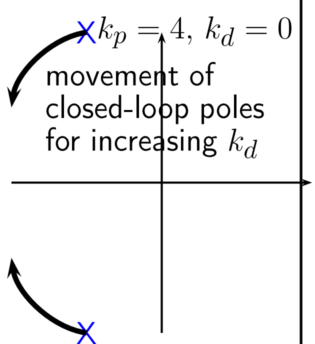
Example: $$G(s) = \frac{1}{(s+1)^2}, \qquad K(s) = k_p + k_d\, s$$

$$\bar{y}(s) = \frac{K(s)G(s)}{1 + K(s)G(s)}\,\bar{r}(s) = \frac{(k_p + k_d\, s)\frac{1}{(s+1)^2}}{1 + (k_p + k_d\, s)\frac{1}{(s+1)^2}}\,\bar{r}(s)$$

$$= \frac{k_p + k_d s}{s^2 + (2 + k_d)s + 1 + k_p}\,\bar{r}(s)$$

So, $\omega_n^2 = 1 + k_p$, $\quad 2c\omega_n = 2 + k_d \implies$

$$\boldsymbol{\omega_n = \sqrt{1 + k_p}}\;, \quad \boldsymbol{c = \frac{2 + k_d}{2\sqrt{1 + k_p}}}$$



In this case, in addition to the multiplication of the input to the controller with a constant gain k_p, we have the addition of a term where the input to the controller is differentiated and multiplied by a constant k_d.

As with proportional control there are trade-offs associated with the choice of the gain k_d within the derivative term. In particular, increasing k_d makes the response less oscillatory, but this is at the expense of amplifying noise.

5.5.4 Proportional + Integral (PI) Control

In the absence of disturbances, and for our simple configuration,

$$\bar{e}(s) = \frac{1}{1 + G(s)K(s)}\,\bar{r}(s)$$

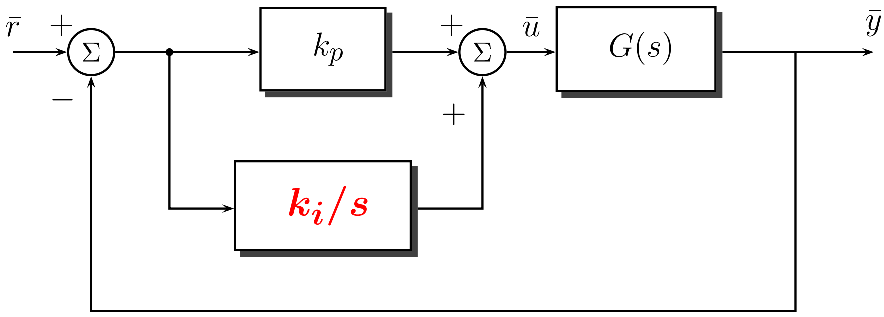
Hence,

$$\begin{array}{l}\text{steady-state error,}\\ \text{(for step demand)}\end{array} = \left.\frac{1}{1+G(s)K(s)}\right|_{s=0} = \frac{1}{1+G(0)K(0)}$$

To remove the steady-state error, we need to make $K(0) = \infty$ (assuming $G(0) \neq 0$).

e.g

$$\boxed{K(s) = k_p + \frac{k_i}{s}}$$



Example:

$$G(s) = \frac{1}{(s+1)^2}, \quad K(s) = k_p + k_i/s$$

$$\bar{y}(s) = \frac{K(s)G(s)}{1+K(s)G(s)}\,\bar{r}(s) = \frac{(k_p + k_i/s)\frac{1}{(s+1)^2}}{1+(k_p+k_i/s)\frac{1}{(s+1)^2}}\,\bar{r}(s)$$

$$= \frac{k_p\, s + k_i}{s(s+1)^2 + k_p\, s + k_i}\,\bar{r}(s)$$

$$\bar{e}(s) = \frac{1}{1+K(s)G(s)}\,\bar{r}(s) = \frac{1}{1+(k_p+k_i/s)\frac{1}{(s+1)^2}}\,\bar{r}(s)$$

$$= \frac{s(s+1)^2}{s(s+1)^2 + k_p\, s + k_i}\,\bar{r}(s)$$

Hence, for $r(t) = H(t)$,

$$\lim_{t\to\infty} y(t) = \left.\frac{k_p\, s + k_i}{s(s+1)^2 + k_p\, s + k_i}\right|_{\boldsymbol{s=0}} \boldsymbol{= 1}$$

and

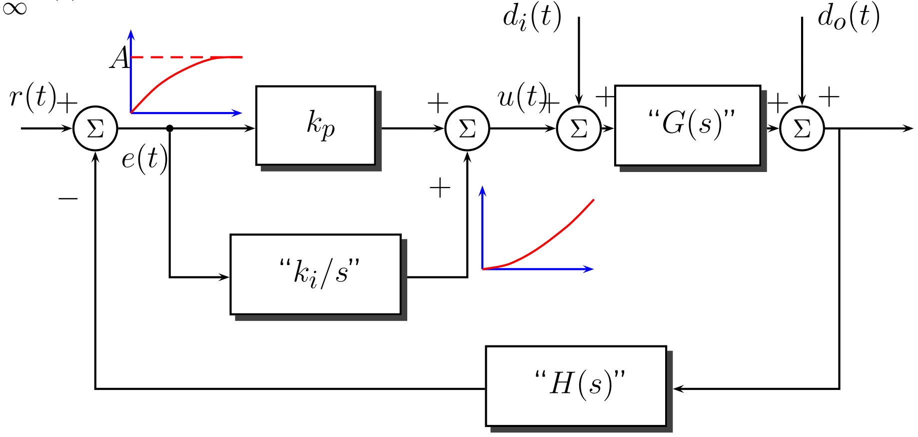
$$\lim_{t\to\infty} e(t) = \left.\frac{s(s+1)^2}{s(s+1)^2 + k_p\, s + k_i}\right|_{\boldsymbol{s=0}} \boldsymbol{= 0}$$

$\Longrightarrow$ no steady-state error

PI control – General Case

In fact, integral action (if stabilizing) always results in zero steady-state error, in the presence of constant disturbances and demands, as we shall now show.

Assume that the following system settles down to an equilibrium with $\lim_{t\to\infty} e(t) = A \neq 0$, then:



$\implies$ Contradiction

(as system is not in equilibrium)

Hence, with PI control the only equilibrium possible has

$$\lim_{t\to\infty} e(t) = 0.$$

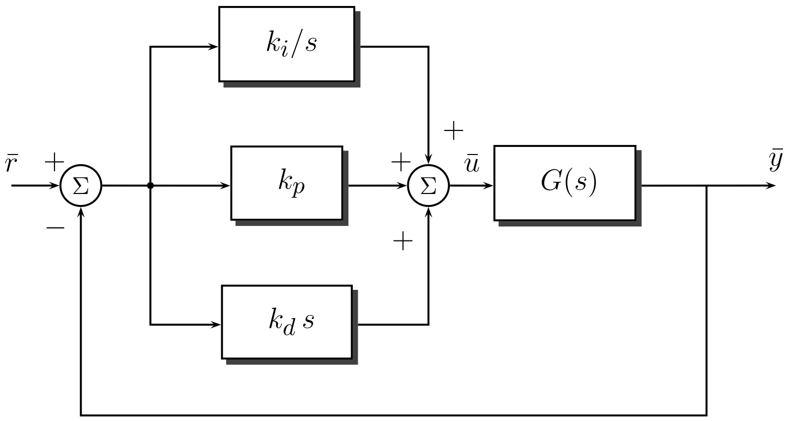
That is, $\lim_{t\to\infty} e(t) = 0$ provided the closed-loop system is asymptotically stable.

The presence of an integrator in the controller, ensures that there is no steady-state error if the closed-loop system is asymptotically stable.

This follows by contradiction. In particular, if the error $e(t)$ tends to a constant value that is non-zero, the output of the integrator would tend to infinity, which contradicts the asymptotic stability of the system.

5.5.5 Proportional + Integral + Derivative (PID) Control

$$\boxed{K(s) = k_p + \frac{k_i}{s} + k_d\, s}$$



Characteristic equation:

$$1 + G(s)(k_p + k_d\, s + k_i/s) = 0$$

• can potentially combine the advantages of both derivative and integral action:

but can be difficult to "tune".

There are many empirical rules for tuning PID controllers (Ziegler-Nichols for example) but to get any further we really need some more theory . . .

A PID controller can combine the advantages of derivative and integral action. Appropriate tuning of the controller parameters is though important.

Systematic methods for designing such dynamic controllers will be discussed towards the end of the course.