2P7: Probability & Statistics

Manipulating and Combining Distributions

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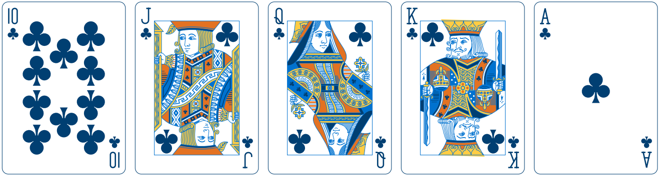
Lent 2024



the *royal flush*, the best possible hand in poker, has a probability 0.000154%

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In the last lectures:

- We have seen that discrete random variables are described by their probability mass function
- We have seen that continuous random variables are described by their probability density function
- We have given important examples of probability mass and density functions:

Discrete variables
- Bernoulli
- Geometric
- Binomial
- Poisson

Continuous variables
- Exponential
- Gaussian
- Beta

In this lecture, we will manipulate random variables, introduce important transforms of distributions, and see the Central Limit Theorem.

Consider a random variable X, and let $\mathrm{Y} = g(\mathrm{X})$ for some function $g : \mathbb{X} \to \mathbb{Y}$ mapping the support $\mathbb{X}$ of X to the domain $\mathbb{Y}$ of Y.

Can we calculate P_{Y} (or f_{Y} if continuous) from P_{X} (or f_{X})?

We'll start with the case where X is a discrete random variable. Then Y is also discrete. $\mathbb{X} = \{x_1, x_2, \dots\}$ and $\mathbb{Y} = \{y_1, y_2, \dots\}$,

$$
\begin{array}{lcll}
x_1 & \longrightarrow & y_1 & P_{\mathrm{Y}}(y_1) = P_{\mathrm{X}}(x_1) \\
x_2 & \longrightarrow & y_2 & P_{\mathrm{Y}}(y_2) = P_{\mathrm{X}}(x_2) \\
x_3 & \longrightarrow & y_3 & P_{\mathrm{Y}}(y_3) = P_{\mathrm{X}}(x_3) + P_{\mathrm{X}}(x_4) \\
x_4 & \longrightarrow & y_3 & \\
\vdots & & \vdots &
\end{array}
$$

In general,
$$P_{\mathrm{Y}}(y) = \sum_{\{x|g(x)=y\}} P_{\mathrm{X}}(x)$$

If g is *invertible* (one-to-one map between $\mathbb{X}$ and $\mathbb{Y}$), then
$$P_{\mathrm{Y}}(y) = P_{\mathrm{X}}\left(g^{-1}(y)\right)$$

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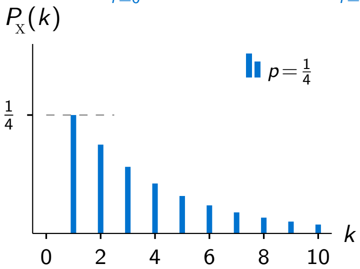
With $\mathrm{X} \sim \mathrm{Geo}(p)$ and $m \in \{1, 2, \dots\}$, what is the PMF of $\mathrm{Y} = (\mathrm{X}-1) \bmod m$? (the modulo operation "$a \bmod b$" returns the remainder of $a \div b$)

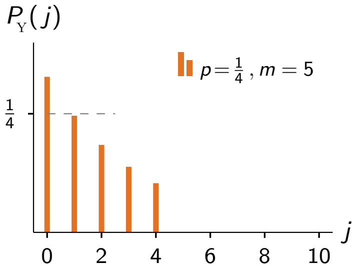
Observe that $\mathbb{Y} = \{0, 1 \dots, m-1\}$ and that

$$
\begin{array}{llllll}
\mathrm{Y} = 0 \text{ if } \mathrm{X} = & 1, & m+1, & 2m+1, & 3m+1\dots \\
\mathrm{Y} = 1 \text{ if } \mathrm{X} = & 2, & m+2, & 2m+2, & 3m+2\dots \\
\quad \dots & & & & \\
\mathrm{Y} = y \text{ if } \mathrm{X} = & \multicolumn{4}{l}{y+1, m+y+1, 2m+y+1, 3m+y+1\dots}
\end{array}
$$

Hence $\{x | g(x) = y\} = \{x = rm + y + 1\}_{r \in \{0,1,\dots\}}$ and we conclude:

$$P_{\mathrm{Y}}(y) = \sum_{r=0}^{\infty} P_{\mathrm{X}}(rm + y + 1) = \sum_{r=0}^{\infty} p(1-p)^{rm+y} = \frac{p(1-p)^y}{1-(1-p)^m} \qquad \square$$





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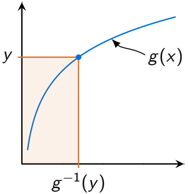
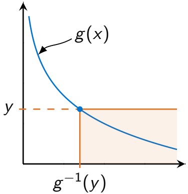
The CDF of $\mathrm{Y} = g(\mathrm{X})$ is $F_{\mathrm{Y}}(y) = \mathbb{P}[\mathrm{Y} \leq y] = \mathbb{P}[g(\mathrm{X}) \leq y]$



- If g is strictly increasing:
  $g(\mathrm{X}) \leq y \Leftrightarrow \mathrm{X} \leq g^{-1}(y)$
- $F_{\mathrm{Y}}(y) = \mathbb{P}[\mathrm{X} \leq g^{-1}(y)]$
  $= F_{\mathrm{X}}(g^{-1}(y))$
- Taking $\frac{\mathrm{d}}{\mathrm{d}y}$ on both sides,
  $$f_{\mathrm{Y}}(y) = \frac{f_{\mathrm{X}}\big(g^{-1}(y)\big)}{g'\big(g^{-1}(y)\big)}$$



- If g is strictly decreasing:
  $g(\mathrm{X}) \leq y \Leftrightarrow \mathrm{X} \geq g^{-1}(y)$
- $F_{\mathrm{Y}}(y) = \mathbb{P}[\mathrm{X} \geq g^{-1}(y)]$
  $= 1 - F_{\mathrm{X}}(g^{-1}(y))$
- Taking $\frac{\mathrm{d}}{\mathrm{d}y}$ on both sides,
  $$f_{\mathrm{Y}}(y) = -\frac{f_{\mathrm{X}}\big(g^{-1}(y)\big)}{g'\big(g^{-1}(y)\big)}$$

Hence, for g strictly monotonic, $f_{\mathrm{Y}=g(\mathrm{X})}(y) = \dfrac{f_{\mathrm{X}}\big(g^{-1}(y)\big)}{\left|g'\big(g^{-1}(y)\big)\right|}$

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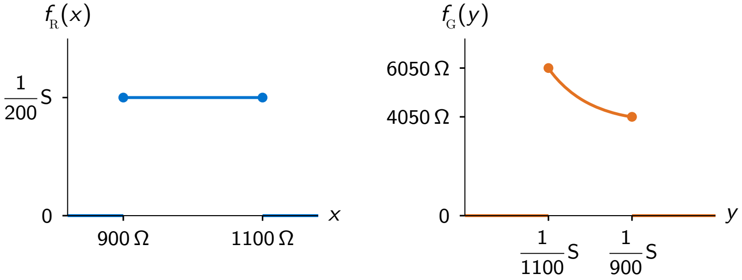
Suppose that a resistance R is uniformly distributed between 900 and 1100 Ω, what is the PDF of its conductance $\mathrm{G} = \dfrac{1}{\mathrm{R}}$?

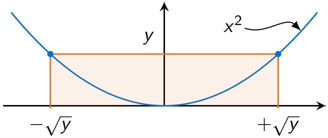
Here $g(x) = \frac{1}{x}$ is strictly decreasing, so we can use $f_{\mathrm{G}}(y) = \frac{f_{\mathrm{X}}(g^{-1}(y))}{|g'(g^{-1}(y))|}$ with

$$g'(x) = -\frac{1}{x^2} \quad \text{and} \quad g^{-1}(y) = \frac{1}{y} \quad \text{hence} \quad |g'(g^{-1}(y))| = y^2$$

$$P_{\mathrm{R}}(x) = \begin{cases} \frac{1}{200} & \text{if } x \in [900, 1100] \\ 0 & \text{otherwise} \end{cases} \Rightarrow P_{\mathrm{G}}(y) = \begin{cases} \frac{1}{200y^2} & \text{if } y \in [\frac{1}{1100}, \frac{1}{900}] \\ 0 & \text{otherwise} \end{cases} \qquad \square$$

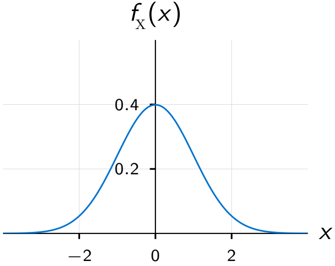


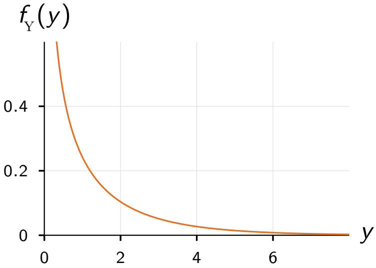
With $\mathrm{X} \sim \mathcal{N}(0,1)$, what is the PDF of $\mathrm{Y} = \mathrm{X}^2$?
Here $g(x)$ is not monotonic, and $\mathrm{Y} \geq 0$.



$F_{\mathrm{Y}}(y) = \mathbb{P}[\mathrm{Y} \leq y] = \mathbb{P}[-\sqrt{y} \leq \mathrm{X} \leq +\sqrt{y}] = F_{\mathrm{X}}(+\sqrt{y}) - F_{\mathrm{X}}(-\sqrt{y})$.

Taking $\frac{\mathrm{d}}{\mathrm{d}y}$ on both sides, $f_{\mathrm{Y}}(y) = \frac{f_{\mathrm{X}}(\sqrt{y})}{2\sqrt{y}} + \frac{f_{\mathrm{X}}(-\sqrt{y})}{2\sqrt{y}} = \frac{e^{-y/2}}{\sqrt{2\pi y}}$ for $y \geq 0$ $\square$





- What is the PDF of $\mathrm{Y} = a\mathrm{X} + b$?
  Here $g(x)$ is strictly monotonic and we can use the formula $f_{\mathrm{Y}}(y) = \frac{f_{\mathrm{X}}(g^{-1}(y))}{|g'(g^{-1}(y))|}$. Hence,

$$f_{\mathrm{Y}}(y) = \frac{1}{|a|} f_{\mathrm{X}}\left(\frac{y-b}{a}\right)$$

  Note that $\mathbb{E}[a\mathrm{X} + b] = a\mathbb{E}[\mathrm{X}] + b$ and $\mathrm{Var}[a\mathrm{X} + b] = a^2\mathrm{Var}[\mathrm{X}]$.
- With $\mathrm{X} \sim \mathcal{N}(\mu, \sigma^2)$, what is the PDF of $\mathrm{Y} = \frac{\mathrm{X}-\mu}{\sigma}$?
  With $a = 1/\sigma$ and $b = -\mu/\sigma$, we find

$$f_{\mathrm{Y}}(y) = \sigma f_{\mathrm{X}}(\sigma y + \mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \quad \square$$

  so $\mathrm{Y} \sim \mathcal{N}(0, 1)$ standard Gaussian (as stated in the previous lecture).

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Let X and Y be two random variables. We are interested in the probability distribution of the random variable $\mathrm{S} = \mathrm{X} + \mathrm{Y}$.

- We have already seen $\mathbb{E}[\mathrm{S}] = \mathbb{E}[\mathrm{X}] + \mathbb{E}[\mathrm{Y}]$.
- One can easily show that
$$\mathrm{Var}[\mathrm{S}] = \mathrm{Var}[\mathrm{X}] + \mathrm{Var}[\mathrm{Y}] + 2\mathrm{Cov}[\mathrm{X}, \mathrm{Y}]$$
where $\mathrm{Cov}[\mathrm{X}, \mathrm{Y}] = \mathbb{E}[\mathrm{XY}] - \mathbb{E}[\mathrm{X}]\mathbb{E}[\mathrm{Y}]$ is called the covariance.
- We also define the correlation coefficient:
$$\rho = \frac{\mathrm{Cov}[\mathrm{X}, \mathrm{Y}]}{\sqrt{\mathrm{Var}[\mathrm{X}]\mathrm{Var}[\mathrm{Y}]}}$$
which satisfies $-1 < \rho < 1$.
- If $\rho < 0$, X and Y are *anticorrelated*
If $\rho > 0$, X and Y are *correlated*
If $\rho = 0$, X and Y are *uncorrelated* (and $\mathbb{E}[\mathrm{XY}] = \mathbb{E}[\mathrm{X}]\mathbb{E}[\mathrm{Y}]$)

Let X and Y be two discrete random variables, with joint PMF $P_{\mathrm{XY}}(x,y)$, and $\mathrm{S}=\mathrm{X}+\mathrm{Y}$. Using the law of total probability

$$P_{\mathrm{S}}(s)=\sum_{y\in\mathbb{Y}}P_{\mathrm{S|Y}}(s|y)P_{\mathrm{Y}}(y)$$

with

$$\begin{aligned}P_{\mathrm{S|Y}}(s|y)=\mathbb{P}[\mathrm{S}=s|\mathrm{Y}=y]&=\mathbb{P}[\mathrm{X}+\mathrm{Y}=s|\mathrm{Y}=y]\\&=\mathbb{P}[\mathrm{X}=s-y|\mathrm{Y}=y]\\&=P_{\mathrm{X|Y}}(s-y|y)\end{aligned}$$

So

$$\begin{aligned}P_{\mathrm{S}}(s)=\sum_{y\in\mathbb{Y}}P_{\mathrm{X|Y}}(s-y|y)P_{\mathrm{Y}}(y)&=\sum_{y\in\mathbb{Y}}P_{\mathrm{XY}}(s-y,y)\\&=\sum_{x\in\mathbb{X}}P_{\mathrm{XY}}(x,s-x)\end{aligned}$$

If X and Y are independent,

$$P_{\mathrm{S}}(s)=\sum_{y\in\mathbb{Y}}P_{\mathrm{X}}(s-y)P_{\mathrm{Y}}(y)=\sum_{x\in\mathbb{X}}P_{\mathrm{X}}(x)P_{\mathrm{Y}}(s-x)$$

that is, $P_{\mathrm{X+Y}}=P_{\mathrm{X}}*P_{\mathrm{Y}}$ the discrete convolution product

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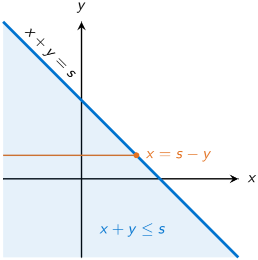
Let X and Y be two continuous random variables, with joint PDF $f_{\mathrm{XY}}(x,y)$, and $\mathrm{S} = \mathrm{X} + \mathrm{Y}$.

$$\begin{aligned} F_{\mathrm{S}}(s) &= \mathbb{P}[\mathrm{X} + \mathrm{Y} \leq s] \\ &= \iint_{x+y \leq s} f_{\mathrm{XY}}(x,y)\mathrm{d}x\mathrm{d}y \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{s-y} f_{\mathrm{XY}}(x,y)\mathrm{d}x\mathrm{d}y \\ &\overset{u=x+y}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{s} f_{\mathrm{XY}}(u-y,y)\mathrm{d}u\mathrm{d}y \end{aligned}$$



Take $\frac{\mathrm{d}}{\mathrm{d}s}$ on both sides

$$f_{\mathrm{S}}(s) = \int_{-\infty}^{\infty} f_{\mathrm{XY}}(s-y,y)\mathrm{d}y = \int_{-\infty}^{\infty} f_{\mathrm{XY}}(x,s-x)\mathrm{d}x$$

If X and Y are independent,

$$f_{\mathrm{S}}(s) = \int_{-\infty}^{\infty} f_{\mathrm{X}}(s-y)f_{\mathrm{Y}}(y)\mathrm{d}y = \int_{-\infty}^{\infty} f_{\mathrm{X}}(x)f_{\mathrm{Y}}(s-x)\mathrm{d}x$$

that is, $f_{\mathrm{X+Y}} = f_{\mathrm{X}} * f_{\mathrm{Y}}$ the convolution product

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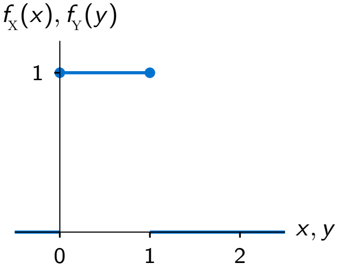
Let X and Y be independent and uniform between 0 and 1, that is

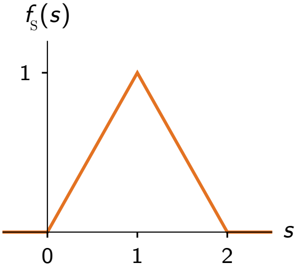
$$f_{\mathrm{X}}(x) = f_{\mathrm{Y}}(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

and $\mathrm{S} = \mathrm{X} + \mathrm{Y}$. Find $f_{\mathrm{S}}(s)$.

$$f_{\mathrm{S}}(s) = \int_{-\infty}^{\infty} f_{\mathrm{X}}(s-y) f_{\mathrm{Y}}(y)\mathrm{d}y = \int_{0}^{1} f_{\mathrm{X}}(s-y)\mathrm{d}y \overset{u=s-y}{=} \int_{s-1}^{s} f_{\mathrm{X}}(u)\mathrm{d}u$$

$$= \begin{cases} \int_0^s \mathrm{d}u = s & \text{if } s \in [0,1] \\ \int_{s-1}^1 \mathrm{d}u = 2-s & \text{if } s \in [1,2] \\ 0 & \text{otherwise} \end{cases} \qquad \square$$





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You have seen Fourier and Laplace Transforms in other courses.
You appreciate their usefulness in sidestepping tedious calculations.
Here we introduce two transforms with similar benefits:

- For a discrete random variable X with support $\mathbb{X}$, the Probability-Generating Function is defined by

$$G_{\mathrm{X}}(z) = \sum_{k \in \mathbb{X}} z^k P_{\mathrm{X}}(k) = \mathbb{E}[z^{\mathrm{X}}] \quad \text{[DB p.27]}$$

- For a continuous random variable X, the Moment-Generating Function is defined by

$$g_{\mathrm{X}}(s) = \int_{-\infty}^{+\infty} f_{\mathrm{X}}(x) e^{sx} \mathrm{d}x = \mathbb{E}[e^{s\mathrm{X}}] \quad \text{[DB p.28]}$$

- **Moments**
  Since $\frac{\mathrm{d}^k}{\mathrm{d}z^k}(z^{\mathrm{X}}) = \mathrm{X}(\mathrm{X}-1)(\mathrm{X}-2)\ldots(\mathrm{X}-k+1)z^{\mathrm{X}-k}$, we verify that

$$\mathbb{E}[\mathrm{X}] = G'_{\mathrm{X}}(1) \quad \text{[DB p.27]}$$
$$\mathrm{Var}[\mathrm{X}] = G''_{\mathrm{X}}(1) + G'_{\mathrm{X}}(1) - G'_{\mathrm{X}}(1)^2 \quad \text{[DB p.27]}$$

- **Sum of independent random variables**
  For X and Y two independent discrete random variables, $\mathbb{E}[z^{\mathrm{X}+\mathrm{Y}}] = \mathbb{E}[z^{\mathrm{X}} z^{\mathrm{Y}}] = \mathbb{E}[z^{\mathrm{X}}]\mathbb{E}[z^{\mathrm{Y}}]$ so

$$G_{\mathrm{X}+\mathrm{Y}}(z) = G_{\mathrm{X}}(z) \times G_{\mathrm{Y}}(z)$$

  More generally, for $\{\mathrm{X}_1, \mathrm{X}_2 \ldots \mathrm{X}_n\}$ mutually independent,

$$G_{\sum_{i=1}^n \mathrm{X}_i}(z) = \prod_{i=1}^{n} G_{\mathrm{X}_i}(z)$$

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With $p \in [0,1]$, $q = 1-p$, $n \in \{1,2,\dots\}$ and $\lambda > 0$:

Distribution of X	Support of X	$P_X(k)$	$G_X(z)$
$\mathrm{Ber}(p)$	$\{0,1\}$	$p^k q^{1-k}$	$q + pz$
$\mathrm{Geo}(p)$	$\{1,2\dots\}$	pq^{k-1}	$\dfrac{pz}{1-qz}$ $(\lvert z\rvert < q^{-1})$
$\mathrm{B}(n,p)$	$\{0,1\dots,n\}$	${}^n\mathrm{C}_k\, p^k q^{n-k}$	$(q+pz)^n$
$\mathrm{Pois}(\lambda)$	$\{0,1\dots\}$	$\dfrac{\lambda^k e^{-\lambda}}{k!}$	$e^{\lambda(z-1)}$

- Note the short form for the Bernoulli PMF...
- Using the expressions of G_X and $G_{X+Y} = G_X \times G_Y$, one gets

same p

$$X \sim \mathrm{B}(n_X, p), \quad Y \sim \mathrm{B}(n_Y, p) \quad \Rightarrow \quad X+Y \sim \mathrm{B}(n_X+n_Y, p)$$

$$X \sim \mathrm{Pois}(\lambda_X), \quad Y \sim \mathrm{Pois}(\lambda_Y) \quad \Rightarrow \quad X+Y \sim \mathrm{Pois}(\lambda_X+\lambda_Y)$$

when X and Y are independent.

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- $\mathrm{X} \sim \mathrm{Ber}(p)$

  Immediate from definition $\displaystyle\sum_{k=0}^{1} z^k P_{\mathrm{X}}(k) = (1-p)z^0 + pz^1$ $\square$

- $\mathrm{X} \sim \mathrm{Geo}(p)$

  From definition

  $$\sum_{k=1}^{\infty} pq^{k-1} z^k = pz \sum_{k=1}^{\infty} (qz)^{k-1} \overset{j=k-1}{=} pz \sum_{j=0}^{\infty} (qz)^j \overset{|qz|<1}{=} \frac{pz}{1-qz} \qquad \square$$

- $\mathrm{X} \sim \mathrm{B}(n, p)$

  $$\sum_{k=0}^{n} {}^n\mathrm{C}_k \, p^k q^{n-k} z^k = \sum_{k=0}^{n} {}^n\mathrm{C}_k \, (zp)^k q^{n-k} \overset{\text{Binomial expansion}}{=} (q + pz)^n \qquad \square$$

- $\mathrm{X} \sim \mathrm{Pois}(\lambda)$

  $$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} z^k = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda z)^k}{k!} \overset{\text{Power series of } e^{\lambda z}}{=} e^{-\lambda} e^{\lambda z} \qquad \square$$

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- **Moments**
  Since $\frac{\mathrm{d}^k}{\mathrm{d}s^k}(e^{sX}) = X^k e^{sX}$, we verify that $g_X^{(k)}(0) = \mathbb{E}[X^k]$. In particular:
  $$\mathbb{E}[X] = g_X'(0) \quad \text{[DB p.28]}$$
  $$\mathrm{Var}[X] = g_X''(0) - g_X'(0)^2 \quad \text{[DB p.28]}$$

- **Sum of independent random variables**
  For X and Y two independent continuous random variables, $\mathbb{E}[e^{s(X+Y)}] = \mathbb{E}[e^{sX}e^{sY}] = \mathbb{E}[e^{sX}]\mathbb{E}[e^{sY}]$ so
  $$g_{X+Y}(s) = g_X(s) \times g_Y(s)$$
  More generally, for $\{X_1, X_2 \ldots X_n\}$ mutually independent,
  $$g_{\sum_{i=1}^n X_i}(s) = \prod_{i=1}^n g_{X_i}(s)$$

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With $a < b \in \mathbb{R}$, $\lambda > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, $\alpha, \beta > 0$ and $\gamma = \alpha + \beta$:

Distribution	$\mathbb{X}$	$f_{\mathrm{X}}(x)$	$g_{\mathrm{X}}(s)$
$\mathrm{U}(a, b)$	$[a, b]$	$\dfrac{1}{b-a}$	$\dfrac{e^{bs} - e^{as}}{s(b-a)}$
$\mathrm{Exp}(\lambda)$	$\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$\dfrac{\lambda}{\lambda - s}$ $(s<\lambda)$
$\mathcal{N}(\mu, \sigma^2)$	$\mathbb{R}$	$\dfrac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$	$e^{\mu s + \frac{\sigma^2 s^2}{2}}$
$\mathrm{Beta}(\alpha, \beta)$	$[0, 1]$	$\dfrac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$1 + \sum_{k=1}^{\infty} \left[\prod_{r=0}^{k-1} \dfrac{\alpha+r}{\gamma+r}\right] \dfrac{s^k}{k!}$

- Using the expressions of g_{X} and $g_{\mathrm{X+Y}} = g_{\mathrm{X}} \times g_{\mathrm{Y}}$, one gets
$$\mathrm{X} \sim \mathcal{N}(\mu_{\mathrm{X}}, \sigma_{\mathrm{X}}^2),\ \mathrm{Y} \sim \mathcal{N}(\mu_{\mathrm{Y}}, \sigma_{\mathrm{Y}}^2) \Rightarrow \mathrm{X+Y} \sim \mathcal{N}(\mu_{\mathrm{X}} + \mu_{\mathrm{Y}}, \sigma_{\mathrm{X}}^2 + \sigma_{\mathrm{Y}}^2)$$
when X and Y are independent.

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- $\mathrm{X} \sim \mathrm{U}(a, b)$

  From definition $g_{\mathrm{X}}(s) = \frac{1}{b-a}\int_a^b e^{sx}\mathrm{d}x = \frac{1}{b-a}\left[\frac{e^{sx}}{s}\right]_a^b$ □

- $\mathrm{X} \sim \mathrm{Exp}(\lambda)$

  From definition

  $g_{\mathrm{X}}(s) = \lambda\int_0^\infty e^{(s-\lambda)x}\mathrm{d}x = \lambda\left[\frac{e^{(s-\lambda)x}}{s-\lambda}\right]_0^\infty \overset{s<\lambda}{=} \frac{\lambda}{\lambda - s}$ □

- $\mathrm{X} \sim \mathcal{N}(\mu, \sigma^2)$

  For the standard Gaussian, $g_{\mathrm{X}}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}e^{sx}\mathrm{d}x =$

  $e^{s^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-s)^2/2}\mathrm{d}x = e^{s^2/2}$ and note that

  $g_{\sigma\mathrm{X}+\mu}(s) = \mathbb{E}[e^{(\sigma s)\mathrm{X}+\mu s}] = e^{\mu s}g_{\mathrm{X}}(\sigma s)$ □

- $\mathrm{X} \sim \mathrm{Beta}(\alpha, \beta)$

  Tedious, we won't do it here. □

Let X_1, X_2 ... be independent random variables with means μ_1, $\mu_2 \ldots$ and variances σ_1^2, σ_2^2 ... Then the random variable

$$\mathrm{S}_n = \mathrm{X}_1 + \mathrm{X}_2 + \cdots + \mathrm{X}_n$$

tends to a Gaussian random variable S,

$$\mathrm{S} \sim \mathcal{N}(\mu_1 + \mu_2 + \cdots + \mu_n, \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2)$$

as n tends to infinity, regardless of the actual individual distributions of X_i.

- As expected from independence, means and variances add up.
- If all X_i are *identically* distributed with mean μ and variance σ^2, then the theorem is equivalent to

$$\mathrm{Y}_n = \frac{\mathrm{X}_1 + \mathrm{X}_2 + \cdots + \mathrm{X}_n - n\mu}{\sigma\sqrt{n}} \quad \overset{n\to\infty}{\sim} \quad \mathcal{N}(0, 1)$$

Let us consider X_1, $X_2 \dots X_n$ be independent random variables that have been *shifted/rescaled* to have means $\mu_1 = \mu_2 = \cdots = \mu_n = 0$ and variances $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_2^2 = 1$.

We will show that $Y_n = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}$ tends to a standard Gaussian random variable when $n \to \infty$.

We write the MGF of Y_n as

$$g_{Y_n}(s) = \prod_{k=1}^{n} g_{X_k}\left(\tfrac{s}{\sqrt{n}}\right) \overset{\text{Taylor series}}{=} \prod_{k=1}^{n} \left[g_{X_k}(0) + g'_{X_k}(0)\tfrac{s}{\sqrt{n}} + g''_{X_k}(0)\tfrac{s^2}{2n} + o\left(\tfrac{s^3}{n^{3/2}}\right) \right]$$

Since $g_X^{(k)}(0) = \mathbb{E}[X^k]$, $g_{X_k}(0) = 1$, $g'_{X_k}(0) = \mu_k = 0$ and $g''_{X_k}(0) = \sigma_k^2 = 1$ and we are left with

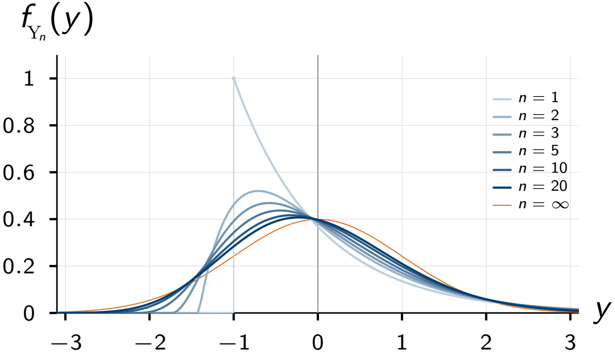
$$g_{Y_n}(s) = \prod_{k=1}^{n} \left[1 + \tfrac{s^2}{2n} + o\left(\tfrac{s^3}{n^{3/2}}\right) \right] = \left(1 + \tfrac{s^2}{2n}\right)^n + o\left(\tfrac{s^3}{n^{3/2}}\right)$$

and $\lim_{n\to\infty} g_{Y_n}(s) = \lim_{n\to\infty} \left(1 + \frac{s^2}{2n}\right)^n = e^{s^2/2}$ which is the MGF of the standard Gaussian. Hence $Y_\infty \sim \mathcal{N}(0, 1)$. $\square$

There are subtle restrictions on the distribution of X_k for the theorem to work. But it works in most cases. . .

Let us consider that $\mathrm{X}_i \sim \mathrm{Exp}(1)$ for all $i \in \{1, \ldots, n\}$
(ie., $\mu_i = \mathbb{E}[\mathrm{X}_i] = 1 = \mu$ and $\sigma_i^2 = \mathrm{Var}[\mathrm{X}_i] = 1 = \sigma^2$ for all i).

Let us monitor the PDF of $\mathrm{Y}_n = \dfrac{\mathrm{X}_1 + \mathrm{X}_2 + \cdots + \mathrm{X}_n - n\mu}{\sigma\sqrt{n}}$ as n increases. (Note that $\mathrm{Y}_1 = \mathrm{X}_1 - 1$ is shifted exponential).



- We haven't seen any example of a *joint* PDF;
- We know that the Gaussian distribution is important;
- We can concisely write the joint PDF $f_{\mathrm{X}_1\mathrm{X}_2\ldots\mathrm{X}_n}(x_1, x_2 \ldots, x_n)$ of n random variables as $f_{\mathbf{X}}(\boldsymbol{x})$ using the vector notation.

Hence, the n-dimensional random vector $\mathbf{X} = [\mathrm{X}_1, \mathrm{X}_2 \ldots, \mathrm{X}_n]^\mathsf{T}$ is multivariate Gaussian if

$$\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \;\Leftrightarrow\; f_{\mathbf{X}}(\boldsymbol{x}) = \frac{\exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\mathsf{T}\,\boldsymbol{\Sigma}^{-1}\,(\boldsymbol{x}-\boldsymbol{\mu})\right)}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}}$$

evaluated at $\boldsymbol{x} = [x_1, x_2 \ldots, x_n]^\mathsf{T}$, with

- $\boldsymbol{\mu} = [\mu_1, \mu_2 \ldots, \mu_n]^\mathsf{T}$ is the mean vector:
  $\mu_i = \mathbb{E}[\mathrm{X}_i]$ for $i = 1 \ldots n$
- $\boldsymbol{\Sigma}$ is the symmetric $n \times n$ covariance matrix:
  $$\Sigma_{ij} = \mathbb{E}[(\mathrm{X}_i - \mu_i)(\mathrm{X}_j - \mu_j)] = \begin{cases} \mathrm{Cov}[\mathrm{X}_i, \mathrm{X}_j] & \text{if } i \neq j \\ \mathrm{Var}[\mathrm{X}_i] = \sigma_i^2 & \text{if } i = j \end{cases}$$

For $n = 2$, we have $\begin{bmatrix} \mathrm{X}_1 \\ \mathrm{X}_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right)]$ where $\rho = \frac{\mathrm{Cov}[\mathrm{X}_1,\mathrm{X}_2]}{\sigma_1\sigma_2}$ is the correlation. The full expression

$$f_{\mathrm{X}_1\mathrm{X}_2}(x_1, x_2) = \frac{\exp\left(-\frac{1}{2(1-\rho^2)} \left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right] \right)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

In particular, we can show:

- The marginals are Gaussian: $f_{\mathrm{X}_1} = \mathcal{N}(\mu_1, \sigma_1^2)$, $f_{\mathrm{X}_2} = \mathcal{N}(\mu_2, \sigma_2^2)$.
  This is also true for any "partial" marginals (integration over $k < n$ components of $\mathbf{X}$) of $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.
- If $\rho = 0$, then $f_{\mathrm{X}_1\mathrm{X}_2} = f_{\mathrm{X}_1} \times f_{\mathrm{X}_2}$ and $\mathrm{X}_1, \mathrm{X}_2$ are independent.
  The components of $\mathbf{X}$ are mutually independent if $\boldsymbol{\Sigma}$ is diagonal.
- The conditional $f_{\mathrm{X}_1|\mathrm{X}_2 = x_2} = \mathcal{N}\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2), (1-\rho^2)\sigma_1^2\right)$ is also Gaussian.
  The multivariate conditionals $f_{\mathrm{X}_1 \ldots \mathrm{X}_k|\mathrm{X}_{k+1} \ldots \mathrm{X}_n}$ are also Gaussian.

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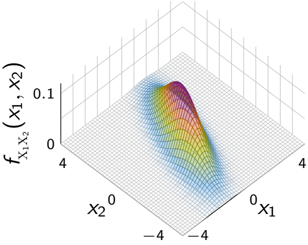
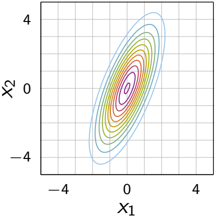
$\rho = 0.7$

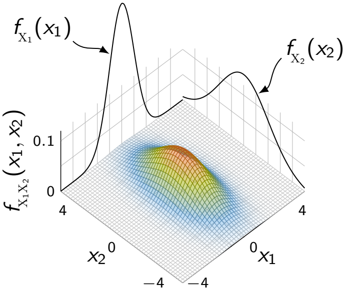
$\mu_1 = 0$
$\mu_2 = 0$

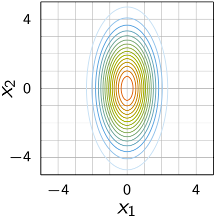
$\sigma_1^2 = 1$
$\sigma_2^2 = 4$

$\rho = 0$









You can attempt Problems 1 to 8 of Examples Paper 6