UNIVERSITY OF CAMBRIDGE
Department of Engineering

IB Paper 5
Electromagnetic Fields & Waves

Lecture 6
Electromagnetic Waves in Conducting Media

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Electromagnetic Waves in Conducting Media

FLE161, GER81

- In the previous lecture, we saw the four Maxwell Equations in differential form
  - These each say something about the origin and nature of electric and magnetic fields, and they are

$$\nabla.\mathbf{D} = \rho \qquad (3.12) \qquad \nabla.\mathbf{B} = 0 \qquad (3.14)$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad (3.23) \qquad \nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \qquad (3.35)$$

- When we derived the wave equation for electromagnetic waves in dielectrics, we were able to simplify the Maxwell equations as there is no free charge
  - Hence $\rho = 0$ and $\mathbf{J} = 0$
  - This is not true in conducting media, as there can clearly be a current
  - However, we will assume that we cannot get a net charge in any volume of space, as the strong attraction of charges would immediately eliminate this, and so $\rho = 0$
  - The Maxwell Equations in a conducting medium therefore become

$$\nabla.\mathbf{D} = 0 \qquad (6.1a) \qquad \nabla.\mathbf{B} = 0 \qquad (6.1b)$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad (6.1c) \qquad \nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \qquad (6.1d)$$

  - We will need to derive a new form of wave equation

- Our starting point is again the Ampère-Maxwell Equation (Eqn. 6.1d)
  - The derivation is very similar to Lecture 4, Transparencies 2 & 3
  - We will make the equations look simpler by using the terminology that

$$\varepsilon = \varepsilon_0 \varepsilon_r \qquad \mu = \mu_0 \mu_r$$

  - Hence

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H}$$

  - and so Eqn. 6.1d becomes

$$\nabla \times \mathbf{B} = \mu \mathbf{J} + \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{6.2}$$

  - We would like to have all the terms on the right hand side of this equation in terms of $\mathbf{E}$, and we can achieve this using the Ohm Law

$$\mathbf{J} = \sigma \mathbf{E} \tag{6.3}$$

  - where σ is the conductivity of the material, so Eqn. 6.2 is

$$\nabla \times \mathbf{B} = \mu\sigma\mathbf{E} + \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{6.4}$$

  - If we take the curl of the Faraday Law, we get

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t} \right) \tag{6.5}$$

  - From the Maths Data Book p16, we have the identity

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla . \mathbf{u}) - \nabla^2 \mathbf{u}$$

- Hence, Eqn. 6.3 becomes

$$\nabla(\nabla.\mathbf{E}) - \nabla^2\mathbf{E} = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) \tag{6.6}$$

- From Eqn. 6.1a, we know that $\nabla.\mathbf{D} = 0$ and $\nabla.\mathbf{E} = 0$, so substituting for $\nabla\times\mathbf{B}$ from Eqn. 6.4 gives

$$\nabla^2\mathbf{E} = \frac{\partial}{\partial t}\left(\mu\sigma\mathbf{E} + \mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)$$

$$\boxed{\nabla^2\mathbf{E} = \mu\sigma\frac{\partial\mathbf{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}} \tag{6.7}$$

- This is very similar to the wave equation that we derived for the electromagnetic wave in dielectrics (Eqn. 4.6)
  - We have ***gained a new term***, though, $\mu\sigma\,\partial\mathbf{E}/\partial t$
  - A similar equation can be generated for $\mathbf{H}$ from the Maxwell Equations
- For a plane wave propagating in the positive z-direction, a solution to Eqn. 6.7 is a wave of the form

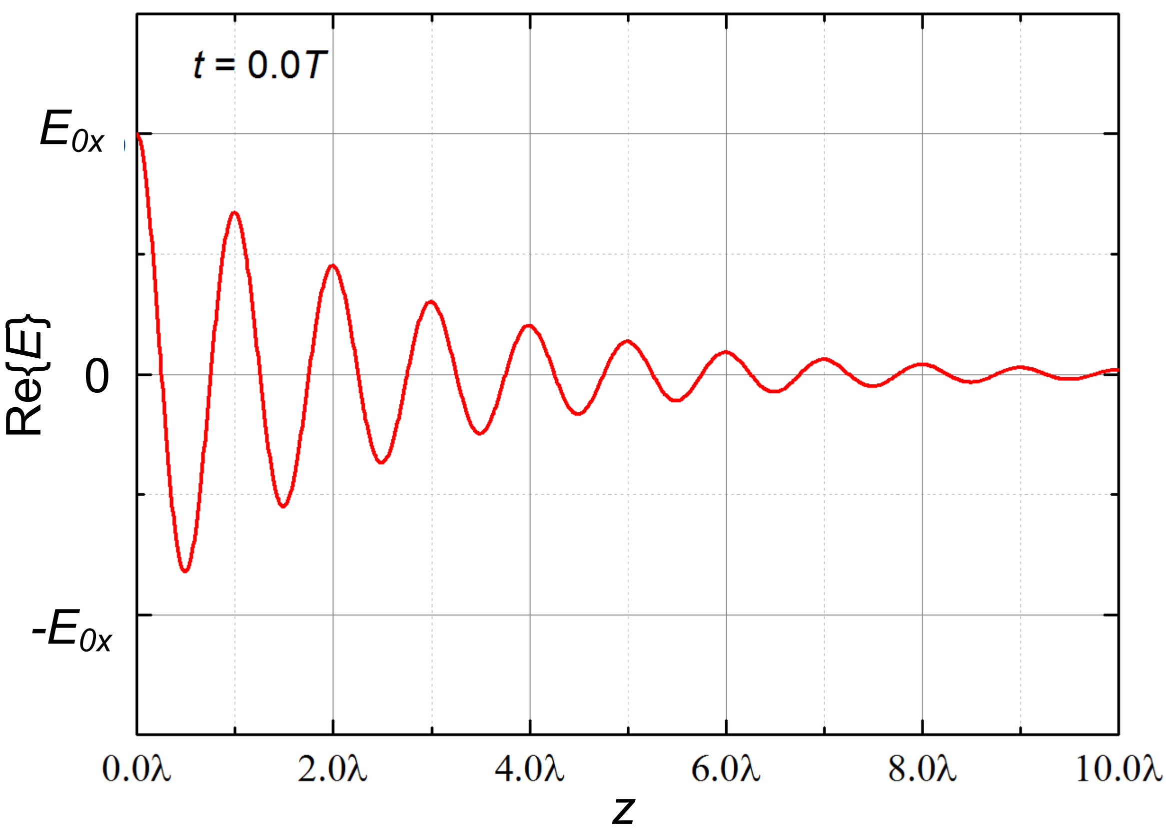
$$\boxed{\mathbf{E} = E_{0x}\mathbf{i}\{\exp(j\omega t)\exp(-\gamma z)\}} \tag{6.8}$$

  - γ is the ***Propagation Constant***, and is a complex number of the form $\gamma = \alpha + j\beta$

- Using the expanded complex form of the propagation constant, Eqn. 6.8 becomes

$$\mathbf{E} = E_{0x}\mathbf{i}\underbrace{\exp(j[\omega t - \beta z])}_{\text{forward wave}}\underbrace{\exp(-\alpha z)}_{\text{attentuation}} \tag{6.9}$$

  - The complex exponential part gives the travelling wave and the real exponential part is an attenuation term



  - We should expect the wave to attenuate as it travels through the conductor as it must be inducing a current flow
  - As the material has some resistivity, the current will result in an energy dissipation
  - It is the electromagnetic wave that is providing this energy, so it must be attenuated

The Skin Effect

FLE165, GER82

- We can understand the propagation constant by substituting this solution into Eqn. 6.7
  - Hence,

$$\frac{\partial \mathbf{E}}{\partial t} = j\omega \mathbf{E}$$

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\omega^2 \mathbf{E}$$

$$\nabla^2 \mathbf{E} = (\alpha + j\beta)^2 \mathbf{E}$$

  - and substitution into Eqn. 6.7 gives

$$(\alpha + j\beta)^2 \mathbf{E} = j\omega\mu\sigma \mathbf{E} - \mu\varepsilon\omega^2 \mathbf{E}$$

$$\boxed{\gamma = (\alpha + j\beta) = \sqrt{j\mu\omega(\sigma + j\omega\varepsilon)}} \qquad (6.10)$$

- This form of the Propagation Constant is quite insightful
  - It shows us that it is the magnitude of the conductivity of a material σ ***relative to*** the product of the permittivity and angular frequency $\omega\varepsilon$ that tells us whether we need to worry about the attenuation effect
- If the material is **highly resistive**, so that $\sigma \ll \omega\varepsilon$,

$$\gamma = (\alpha + j\beta) = \sqrt{(j\omega)^2 \mu\varepsilon} = j\omega\sqrt{\mu\varepsilon} \qquad (6.11)$$

  - Therefore,

$$\alpha = 0 \qquad (6.12a)$$

$$\beta = \omega\sqrt{\mu\varepsilon} \qquad (6.12b)$$

- This is saying that the wave is not attenuated
- Also, remembering that $\beta = 2\pi/\lambda$,

$$\frac{\omega}{\beta} = \frac{2\pi f}{2\pi/\lambda} = f\lambda = c = \frac{1}{\sqrt{\mu\varepsilon}}$$

- and we have got back the velocity of the wave in a dielectric that we derived in Eqn. 4.8
- In other words, we have the solution for the electromagnetic wave in an insulating dielectric again

- If the material is **highly conductive**, so that $\sigma \gg \omega\varepsilon$,

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$$\gamma = (\alpha + j\beta) = \sqrt{j\mu\omega\sigma} \tag{6.13}$$

- Remembering that $\sqrt{j} = (1+j)/\sqrt{2}$, we can see that

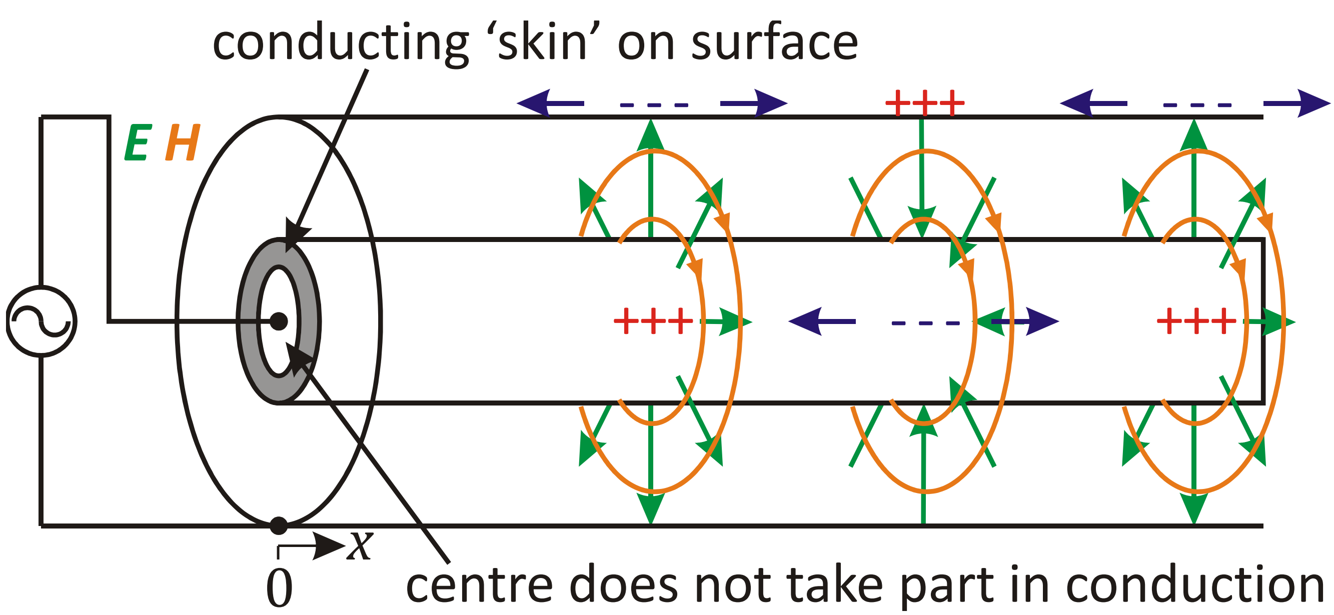
$$\alpha = \beta = \sqrt{\frac{\mu\omega\sigma}{2}} \tag{6.14}$$

- The attenuation term in our equation for the electromagnetic wave (Eqn. 6.9) is $\exp(-\alpha z)$, and so the wave is attenuated by a factor of e over a distance of α^{-1}
- We therefore define the ***Skin Depth*** δ to be

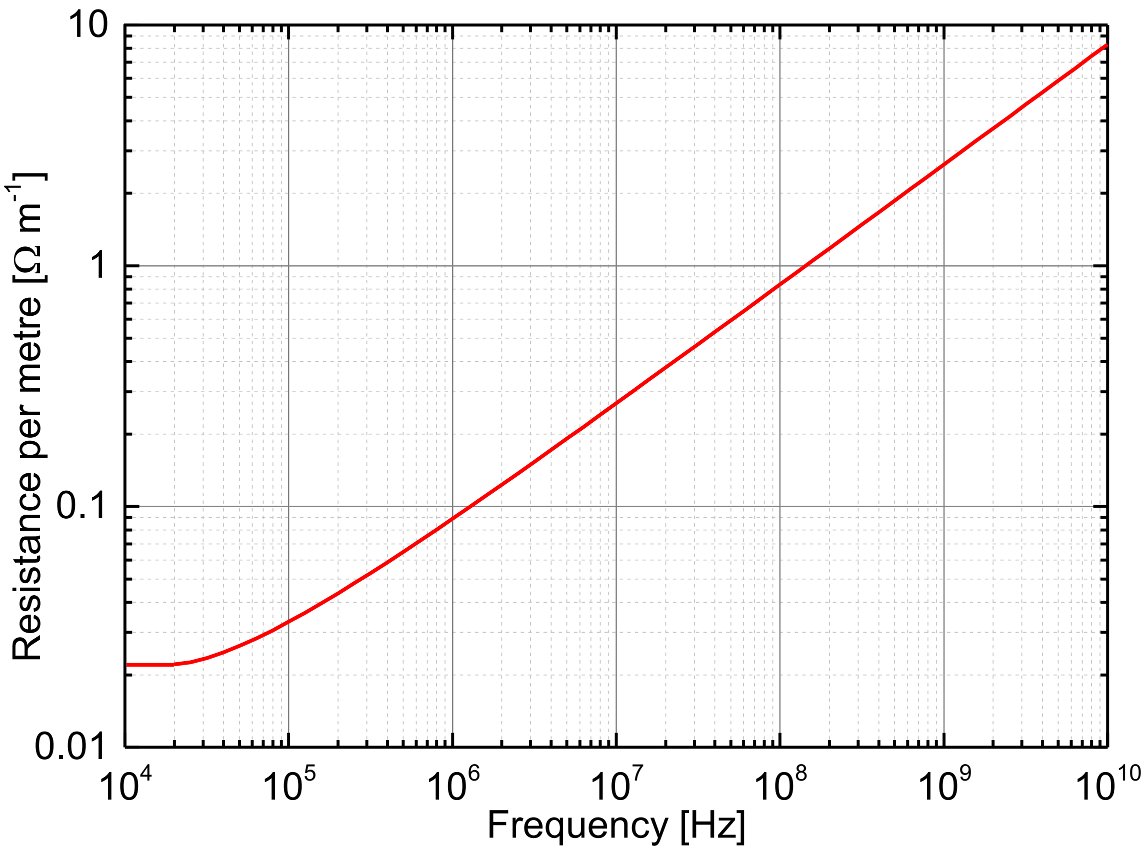
$$\boxed{\delta = \alpha^{-1} = \sqrt{\frac{2}{\mu\omega\sigma}}} \tag{6.15}$$

- It is a measure of the distance that an electromagnetic wave can penetrate into a conducting medium
- It decreases with ω as greater oscillation frequency of charge would be expected to lead to greater energy loss

- Let us think about the effect of this on a transmission line
  - If we imagine trying to drive an alternating current down a wire of circular cross section in a coaxial cable, we know that an electromagnetic wave exists around the wire
  - If δ is large compared with the radius of the wire, then the wave penetrates all the way to the centre, there is current flow through the entire cross-section, and the resistance of the wire will be determined by its geometry and resistivity
  - However, if δ is less than the radius, then the wave will only penetrate a small distance into the wire
  - Conduction will therefore be limited to the surface 'skin'
  - This is where the ***Skin Effect*** gets its name from



- The consequence of this is that the conductivity of a wire appears to change with frequency



- The approximate resistance of a 1 mm diameter copper wire as a function of frequency is shown above (assuming all conduction is only in the skin)
- The Skin Effect means that the effective resistance of the wire increases with frequency
- This causes attenuation in the wire with the result that the signal amplitude decays over much shorter distances at high frequencies
- This is one of the reasons why optical fibres are generally better than copper wires for high speed data transmission over long distances

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Intrinsic Impedance of a Conductive Medium

FLE168

- In Lecture 4, we derived that there is no component of electric (or magnetic) field in the direction of propagation of a plane electromagnetic wave
  - This was because $\nabla.\mathbf{E} = 0$ (and $\nabla.\mathbf{B} = 0$)
  - This is still true in conducting media
- We can work out the magnetic field in a wave from the electric field using the Faraday Law (Eqn. 6.1c) in the Maxwell Equations

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

  - Using the form of our electric field in Eqn. 6.8

$$\mathbf{E} = E_{0x}\mathbf{i}\exp(j[\omega t - \beta z])\exp(-\alpha z)$$

  - we have that

$$\nabla\times\mathbf{E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_{0x}\exp\{j(\omega t - \beta z)\}\exp(-\alpha z) & 0 & 0 \end{vmatrix}$$

$$\mathbf{j}\frac{\partial}{\partial z}[E_{0x}\exp\{j(\omega t - \beta z)\}\exp(-\alpha z)]$$

$$-\mathbf{k}\frac{\partial}{\partial y}[E_{0x}\exp\{j(\omega t - \beta z)\}\exp(-\alpha z)] = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\mathrm{j}(\alpha + j\beta)E_{0x}\exp\{j(\omega t - \beta z)\}\exp(-\alpha z) = \frac{\partial \mathbf{B}}{\partial t}$$

- Hence

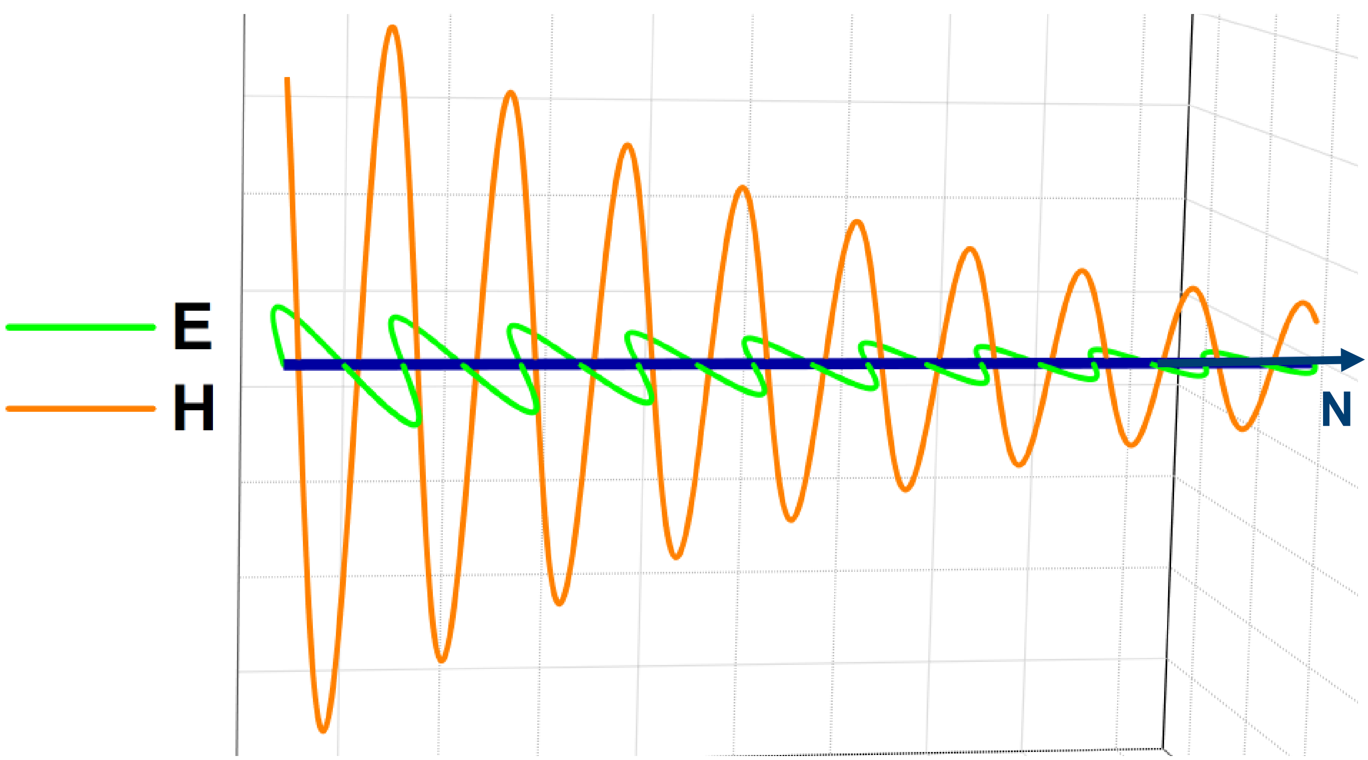
$$\mathbf{B} = \mathrm{j}\frac{(\alpha + j\beta)}{j\omega}E_{0x}\exp\{j(\omega t - \beta z)\}\exp(-\alpha z)$$

$$\boxed{\mathbf{H} = H_{0y}\mathrm{j}\exp\{j(\omega t - \beta z)\}\exp(-\alpha z)} \quad (6.16)$$

- where

$$H_{0y} = \frac{(\alpha + j\beta)}{j\omega\mu}E_{0x} \quad (6.17)$$

- Therefore, H ***is perpendicular to both*** E ***and the direction of propagation of the wave*** (6.10)



- Let us now consider the relative amplitude of the electric and magnetic fields
  - We have already shown that the propagation constant

$$\gamma = (\alpha + j\beta) = \sqrt{j\mu\omega(\sigma + j\omega\varepsilon)} \tag{6.10}$$

  - so substituting this into Eqn 6.17 gives

$$H_{0y} = \sqrt{\frac{\sigma + j\omega\varepsilon}{j\omega\mu}} E_{0x}$$

  - The electric and magnetic fields are therefore perpendicular to each other and related by the intrinsic impedance which is defined to be

$$\boxed{\eta = \frac{E_{0x}}{H_{0y}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}} \tag{6.18}$$

  - This is very similar to the case for the dielectric medium
  - Indeed, for an insulating medium where $\sigma \ll j\omega\varepsilon$ then this simplifies to $\eta = \sqrt{\mu/\varepsilon}$, which is the intrinsic impedance that we derived for the dielectric (Eqn. 4.22)
  - However, whereas this was just a real number for a dielectric, the ***intrinsic impedance of a conducting medium is complex***
  - This means that there is a phase difference between the magnetic and electric field waves in a conductor
  - The peaks in electric field and magnetic field do not occur at the same point in space and time

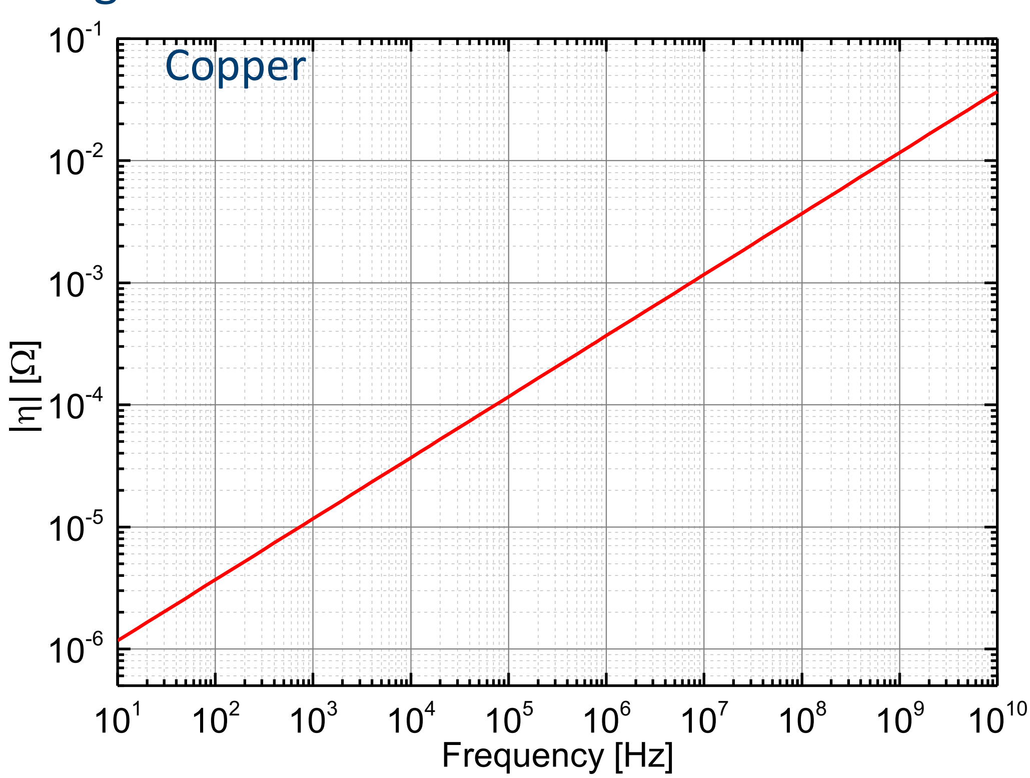
- For a good conductor, where $\sigma \gg j\omega\varepsilon$, Eqn. 6.18 simplifies to

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}}$$

  - and so remembering that $\sqrt{j} = (1+j)/\sqrt{2}$,

$$\eta = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} \qquad (6.18)$$

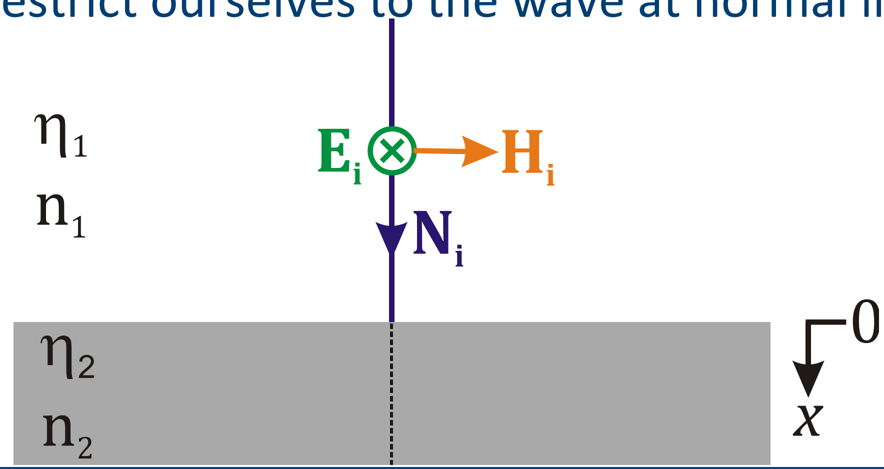
  - Therefore, the electric field leads the magnetic field by $\pi/4$
  - If we work out the magnitude of the characteristic impedance for a typical metal, such as copper, we find that it is much smaller than the characteristic impedances we have seen for dielectrics, so the wave is dominated by the magnetic field



Waves at Conducting Interfaces

FLE171

- In Lecture 5, we considered the interaction of electromagnetic waves at interfaces, and we derived a series of equations for the transmission and reflection of different polarisations of electromagnetic waves
  - The equations that we derived (Eqns. 5.11 and 5.15) are all still valid for conducting media, as they were based on the ***conservation of the normal components of* $\mathbf{D}$ *and* $\mathbf{B}$ *and the tangential components of* $\mathbf{E}$ *and* $\mathbf{H}$ *at interfaces***
  - We just need to remember that the conductor has a complex intrinsic impedance, whereas the impedance of a dielectric medium is real
- In practice, the most common situation that we are interested in is when a wave in a dielectric medium '1' (e.g. air) is incident on a conducting medium '2'
  - We will restrict ourselves to the wave at normal incidence



- Using the same notation as in Lecture 5, we find that the proportion of the electric field that is transmitted into the metal is given by Eqns. 5.11a and 5.15a to be

$$\frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2} \tag{6.19}$$

- Therefore, if we know that $\eta_2 \ll \eta_1$, then we can see that almost all of the wave is reflected
- For example, for the case of a 2.45 GHz wifi signal incident on a copper surface from air, the impedance of the air is 377 Ω whilst that of the copper has a magnitude of 10^{-2} Ω
- This means that only ~0.001% of the electric field is transmitted, and the surface behaves like an almost perfect reflector

- Even less highly conducting media will be highly reflecting
  - In the example paper, you are asked to consider trying to send a signal to a submarine

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  - You will see that the conducting nature of sea water leads to significant reflection of an electromagnetic wave at its surface
  - The wave is then significantly attenuated as it propagates into the water, meaning that communication is only possible when the submarine is close to the surface
  - It is for this reason that 'black box' flight recorders on aircraft have acoustic transmitters

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Electromagnetic Waves & Transmission Lines

- For guided waves on transmission lines
  - We used the equivalent circuit of a short section of a line to derive the ***Telegrapher's Equations***
  - We combined these to give a ***wave equation***
  - The transmission line guides an electromagnetic wave in the dielectric, which defines the velocity
  - The fields penetrate into the conductors to give a local current and voltage
  - The ratio of voltage and current is called the characteristic impedance, and is determined by the dielectric and a geometry factor
  - We can use impedances to understand reflection and transmission of signals on the line
- For free-space waves
  - We reviewed the Maxwell equations which describe the origin and nature of electric and magnetic fields
  - We combined these to give a ***wave equation***
  - The velocity of the wave is dependent on the medium
  - The ratio of the electric and magnetic fields is called the intrinsic impedance, and is determined by the medium
  - We can use impedance to understand reflection and transmission of waves
  - Electromagnetic waves are attenuated by conductors

- Transmission Lines
  - Telegrapher's Equations

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}$$

  - Wave equation

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}$$

  - Wave velocity

$$c = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\varepsilon_0\varepsilon_r\mu_0\mu_r}}$$

  - Characteristic impedance

$$Z_0 = \sqrt{L/C} = \mathrm{GF}\times\sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}$$

  - Reflection coefficient

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

- Electromagnetism
  - Maxwell Equations

$$\nabla.\mathbf{D} = \rho$$

$$\nabla.\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$$

  - Wave equation

$$\nabla^2\mathbf{E} = \mu_0\mu_r\varepsilon_0\varepsilon_r\frac{\partial^2\mathbf{E}}{\partial t^2}$$

  - Wave velocity

$$c = \frac{1}{\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}$$

  - Intrinsic impedance

$$\eta = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}$$

  - Reflection coefficient

$$\left(\frac{E_r}{E_i}\right)_\perp = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$\left(\frac{E_r}{E_i}\right)_\parallel = \frac{\eta_1\cos\theta_i - \eta_2\cos\theta_t}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}$$

- Electromagnetism is a particularly elegant branch of physics which impacts across engineering
- The Maxwell Equations are at the heart of the subject, and predict a diverse range of phenomena
  - Speed of light
  - Inductance and capacitance
  - Transmission of waves on lines
  - Reflection, refraction and attenuation of waves in media
- The Maxwell Equations allow us to understand how to engineer the best possible solutions to common problems
  - Solar cell efficiency
  - Losses on power distribution systems
  - Maximum data rates on copper wires and optical fibres
  - Broadcasting of TV, radio and wifi signals
  - Computational power of microprocessors
- We have shown that even a basic analytic manipulation of the Maxwell Equations allow us to achieve a deep understanding
- They are also amenable to numerical analysis using finite element methods, and this enables us to tackle extremely complex problems, and so act as a gateway to great engineering