Paper 1: Mechanics

Examples Paper 1

Revision

1 You and a friend are having a rolling race with various balls on a sloping table, as shown in Figure 1. You have a ping-pong ball, a cricket ball and a ten-pin bowling ball. Your friend has a football, a billiard ball and an ice hockey puck (on its side). You line up at the top of the slope and let them roll to the bottom. You want to figure out which ones get to the bottom fastest.

(a) Find an expression relating the speed v and the angular velocity ω for a ball of radius a rolling with no slip.

(b) Use the change in kinetic energy to find the angular velocity of a ball when it has descended a vertical height h. Express your answer in terms of k, the *radius of gyration* of the ball as given in the Mechanics Data Book.

(c) Look up the radius of gyration of each of the objects and deduce their order of arrival.

(d) What is the final velocity of a hollow cylinder descending h on a slope whose angle to the horizontal is θ where $\tan\theta = 0.1$ (assume small angles), and how long does this journey take?

(e) Where might you expect an empty, a full and a frozen can of soft drink to appear in this race? Do the experiment!

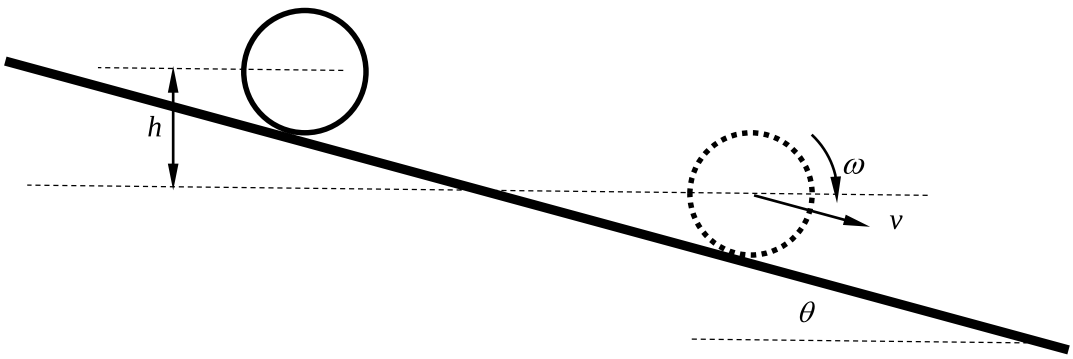

Figure 1

2 A thin uniform hoop of mass m and radius a is supported in a vertical plane by two pegs A and B at the same level, $a\sqrt{2}$ apart as shown in Figure 2. Peg B is removed and the hoop begins to fall, with no slip at peg A.

(a) Find the moment of inertia of the hoop about A and the initial angular acceleration of the hoop.

(b) Use the d'Alembert force at the hoop centre to determine the normal and tangential components of the reaction at A and hence show that there will be no initial slip if the coefficient of friction exceeds 0.5.

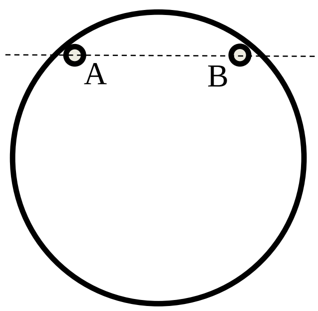

Figure 2

3 A pencil, modelled as a uniform rod of length $2a$, is standing at rest on the edge of a table as shown in Figure 3(a). An impulse I delivered to the bottom of the pencil causes it to fly through the air. The impulse is just hard enough so that the tip P of the pencil hits the edge of the table.

(a) After the impulse the centre of mass of the pencil moves with horizontal velocity v and its angular velocity is ω as shown in Figure 3(b). Find an expression for v in terms of ω.

(b) Find the angle θ of the pencil when it just hits the table, as shown in Figure 3(c). See if you can measure this angle using your phone camera.

[Video of this motion see http://www2.eng.cam.ac.uk/~hemh1/movies.htm#pencils]

[Matt Parker treatment https://www.youtube.com/watch?v=uNPDrLhXC9k]

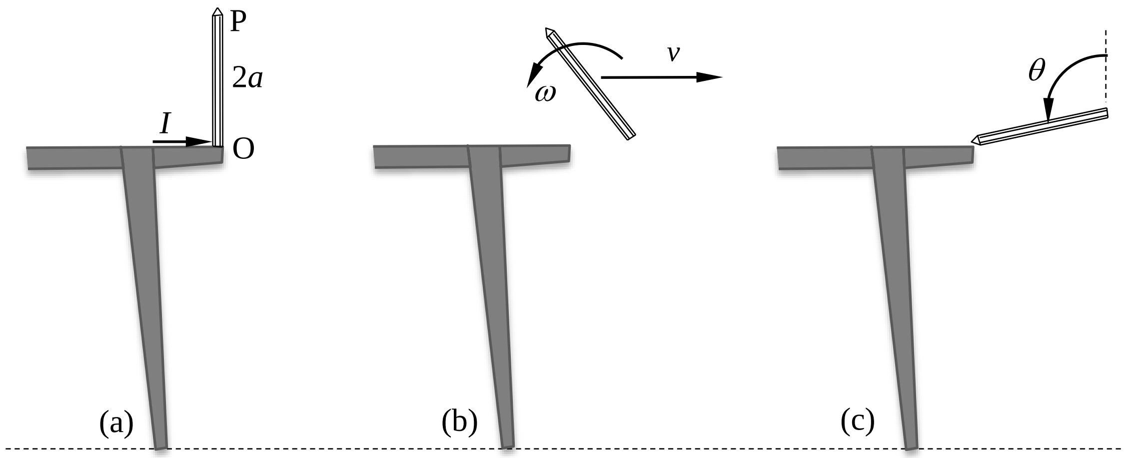

Figure 3

4 A rod OA of length $2L$ is rotating in a horizontal plane at constant angular velocity Ω as shown in Figure 4. A particle P of mass m can move on the rod and the distance OP is x, but to begin with the particle is stuck on the rod at $x = L$. At time $t = 0$ the particle becomes unstuck and then slides freely towards A.

(a) Using the rotating unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$, and coordinate x as shown find vector expressions for the position, velocity and acceleration of the particle for $t > 0$.

(b) Find a differential equation in x.

(c) Solve for x and determine the time at which the particle leaves the rod.

(d) At the instant just before the particle leaves the rod use d'Alembert principle to determine the force acting on the rod and find the torque at O required to drive the rod at this instant.

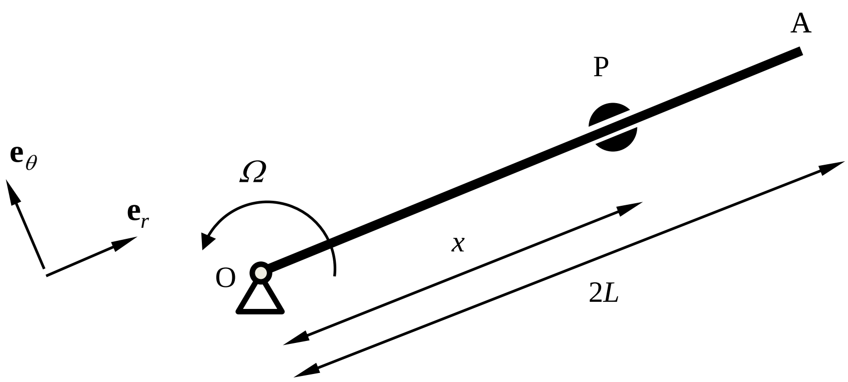

Figure 4

5 The slider-crank mechanism shown in Figure 5 has dimensions as indicated. At the instant shown link OA is rotating at a constant angular velocity ω.

(a) By the method of instantaneous centres find the angular velocity of the "coupler" ABC, the velocity of point C and the velocity of the slider at B.
(b) Sketch a velocity diagram showing the velocities of points A and B. Use the Velocity Image theorem to locate the velocity of point C and show that it agrees with your answer in (a).
(c) A gas-pressure force F acts on the slider as shown. Find the torque T acting at O that is needed to resist the action of F.
(d) A friction torque Q acts at all three pin joints O, A and B. What is the value of Q such that friction at joints reduces the torque at O to zero under the action of F?

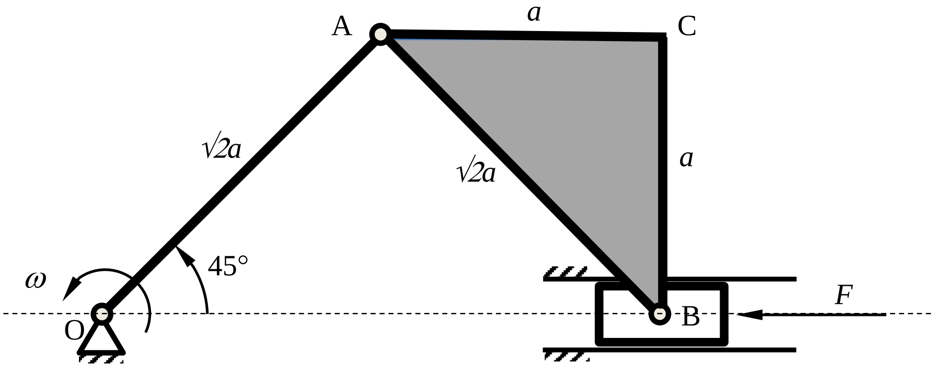

Figure 5

6 The lifting mechanism shown in Figure 6 has dimensions as indicated, with length AC$= 2a$. At the instant shown the driving link OB is rotating slowly at a constant angular velocity ω so as to lift a weight mg which is hanging from point C. A point X on the rod is instantaneously adjacent to the slider at B.

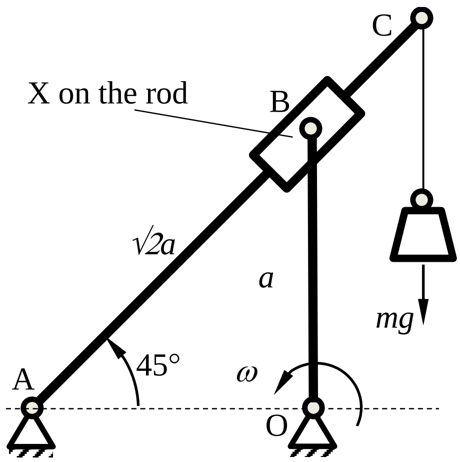

Figure 6

(a) Locate the instantaneous centre of the slider. (To do this, imagine an extension of the slider and find the direction of the velocity of that part of the extension that is instantaneously over point A).
(b) Use the method of instantaneous centres to find the angular velocity of rod AC, the vertical component of the velocity of point C and the sliding velocity at B (ie the relative velocity between B and X).
(c) Sketch a velocity diagram showing the velocities of points A, B, X and C. Check that the distances between A,X and C are consistent with the Velocity Image theorem.
(d) Find the torque T needed at O to lift the weight mg.
(e) A friction force F acts at between the rod and the slider at B and a friction torque Q acts at the pin at B. What is the additional torque required at O to overcome the effects of friction?

7 The rod AB has length $L = 2$ metres and the accelerations of A and B at a certain instant are shown in Figure 7. A point G is located at the midpoint of the rod.

(a) Sketch the acceleration diagram for the rod, and use the principle of the acceleration image to find the acceleration of point G.

(b) Find an expression for the acceleration of B relative to A using the rotating unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as shown.

(c) Hence find the angular velocity and angular acceleration of the rod.

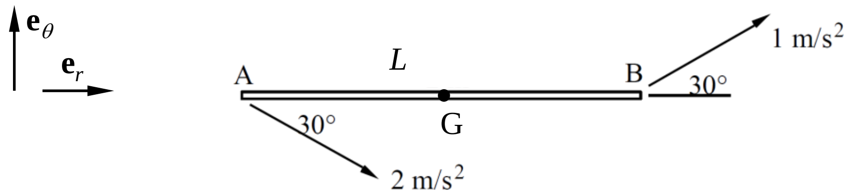

Figure 7

8 The rigid member ABC rotates in its own plane with constant angular velocity ω as shown in Figure 8 . A block D is sliding with constant velocity v relative to ABC and the distance BD is x and AC=$2a$.

(a) Find vector expressions for the position and velocity of point D using unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

(b) At a given instant when $x = a$

(i) what are the components of the acceleration of D along and perpendicular to BC?

(ii) what is the acceleration of the point X on the rod instantaneously adjacent to D?

(iii) sketch the accelerations of A, B, D and X on an acceleration diagram and use the acceleration image to locate the acceleration of C on your diagram.

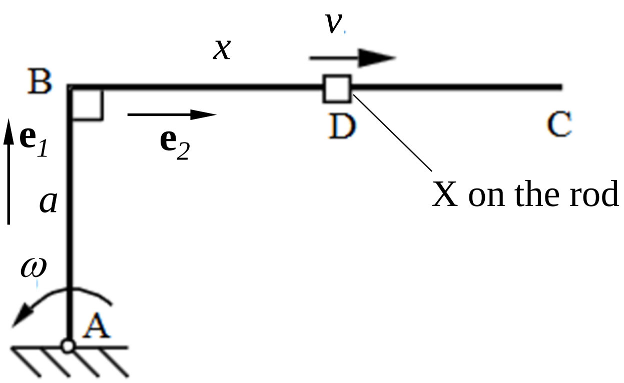

Figure 8

9 The hydraulic ram shown in Figure 9 is extending on account of oil flow. The distance between the ends A and B of the ram is x. At a certain instant when $x = 1$m the ends A and B are each moving at constant velocity and so have zero acceleration. The ram is extending so that $\dot{x} = 1$ ms^{-1} and the rate of oil flow is increasing at a rate corresponding to $\ddot{x} = 1$ ms^{-2}.

(a) What is the angular velocity and angular acceleration of the ram at this instant?

(b) What are the vertical and horizontal components of acceleration of the point G on the cylinder, a fixed distance $a = 0.5$m from B? Is it possible to determine if the acceleration of G is up or down?

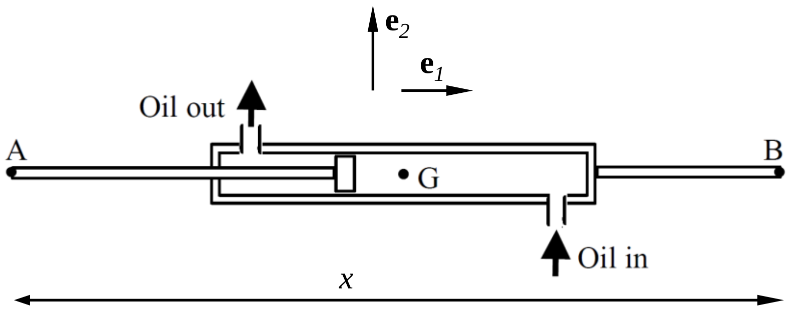

Figure 9

10 The slider-crank mechanism shown in Figure 10 has dimensions as indicated. At the instant shown link OA is rotating at a constant angular velocity ω.

(a) Find the velocity and acceleration of the slider at a general crank angle θ.

(b) Sketch an acceleration diagram for the mechanism.

(c) By considering the d'Alembert force acting on the slider find for any angle θ the torque T acting at O that is needed to drive the mechanism.

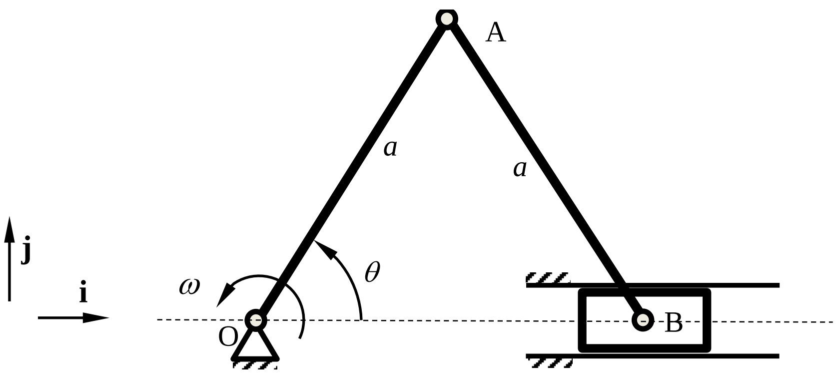

Figure 10

Suitable past Tripos questions

Revision: Part IB: 2021 Q2, 2018 Q1, 2017 Q1 Q5 Q6, 2016 Q2, 2015 Q1(a)(b), 2012 Q6, 2011 Q5. Part IA: 2018 Q7, 2015 Q10, 2014 Q10, 2013 Q10, 2012 Q8
Mechanisms: Instant Centres, Velocity Diagrams, Velocity Image, Forces, Friction: Part IB: 2019 Q2, 2014 Q4(a)(bi). Part IA: 2018 Q8, 2016 Q7, 2014 Q8, 2011 Q8,Q9, 2010 Q11
Mechanisms: Accelerations, Acceleration Diagrams, Acceleration Image: Part IB: 2021 Q1, 2018 Q2, 2017 Q4, 2016 Q3, 2015 Q2, 2013 Q2 Q4, 2012 45. Part IA: 2017 Q7, 2016 Q9

Answers

1(a). $v = a\omega$

1(b). $\omega^2 = \frac{2gh}{a^2+k^2}$

1(c). $(k/a)^2 = 0.4, 0.5, 2/3, 1$ for solid sphere, solid cylinder, hollow sphere, hollow cylinder respectively. Hence this is the order of arrival

1(d). $v = \sqrt{gh}$, $t \approx 20\sqrt{h/g}$

1(e). fastest to slowest: full (liquid), frozen, empty

2(a). $2ma^2$,$\sqrt{2}g/4a$

2(b). $mg\sqrt{2}/2$,$mg\sqrt{2}/4$

3(a). $v = a\omega/3$

3(b). θ ≈2.279 (131°), being the solution to $3\sin\theta = \theta$

4(a). $\mathbf{r} = x\mathbf{e}_r$, $\dot{\mathbf{r}} = \dot{x}\mathbf{e}_r + x\Omega\mathbf{e}_\theta$, $\ddot{\mathbf{r}} = (\ddot{x} - \Omega^2 x)\mathbf{e}_r + 2\dot{x}\Omega\mathbf{e}_\theta$

4(b). $\ddot{x} - \Omega^2 x = 0$

4(c). $x = L\cosh(\Omega t)$, $t = \cosh^{-1}(2)/\Omega$

4(d). $4\sqrt{3}L^2\Omega^2 m$

5(a). ω clockwise, $a\omega$ to the left, $2a\omega$ to the left

5(c). $2aF$

5(d). $Fa/2$

6(b). $\omega/2$ anticlockwise, $a\omega/\sqrt{2}$ upwards, $a\omega/\sqrt{2}$ with B sliding towards A

6(d). $mga/\sqrt{2}$

6(e). $Fa/\sqrt{2} + Q/2$

7(a). $(3\sqrt{3}\mathbf{i}\text{-}\mathbf{j})/4$ m/s^2

7(b). $\ddot{\mathbf{r}}_{B/A} = -L\dot{\theta}^2\mathbf{e}_r + L\ddot{\theta}\mathbf{e}_\theta$

7(c). 0.658 rad/s clockwise or anticlockwise, 0.75 rad/s^2 anticlockwise.

8(a). $\mathbf{r} = a\mathbf{e}_1 + x\mathbf{e}_2$, $\dot{\mathbf{r}} = (-a\omega + v)\mathbf{e}_2 + x\omega\mathbf{e}_1$

8(b)(i). $-a\omega^2$, $2v\omega - a\omega^2$

8(b)(ii). $a\sqrt{2}\omega^2$ towards A.

9(a). $\pm$1rad/s, $\mp$ 2rad/s^2

9(b). $\pm$1m/s^2, 0.5m/s, No

10(a). $-2a\omega\sin\theta\mathbf{i}$,$-2a\omega^2\cos\theta\mathbf{i}$

10(c). $2ma^2\omega^2\sin(2\theta)$