Part IB Paper 6: Information Engineering

LINEAR SYSTEMS AND CONTROL

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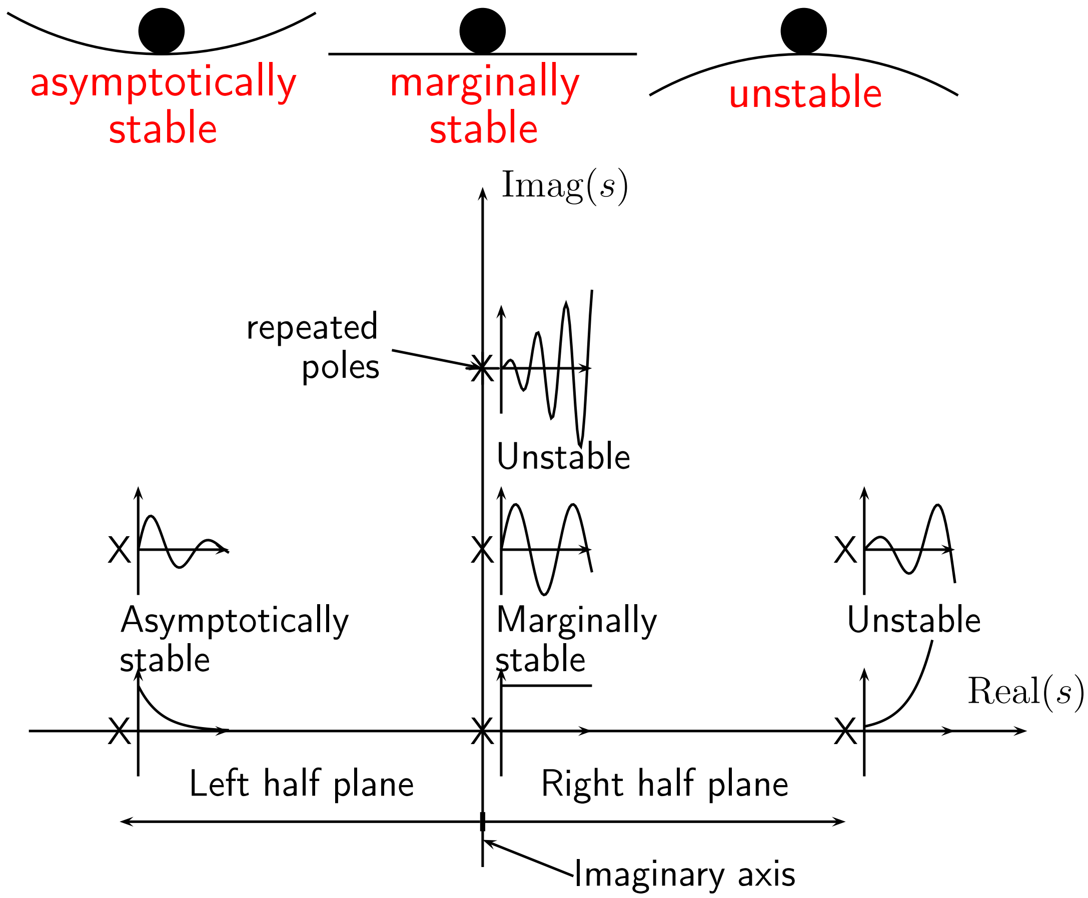
HANDOUT 3

"Stability and pole locations"



Summary

Stability, or the lack of it, is the most fundamental of system properties. When designing a feedback system the most basic of requirements is that the feedback system be stable.

There are different ways of defining stability. In this handout we shall:

- Define the following notions:
  - Asymptotic stability
  - Marginal stability
  - Instability
- Relate stability of a system to the *poles* of its transfer function

  In addition, we shall:
- Relate the *transient response* of a system to the poles of its transfer function

Contents

3.1 Asymptotic Stability

Definition:

An LTI system is *asymptotically stable* if its impulse response $g(t)$ satisfies the condition

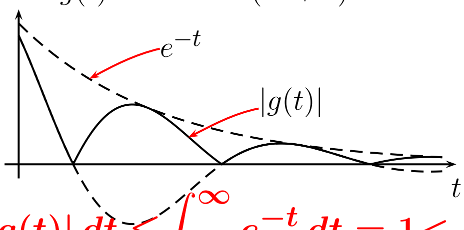
$$\int_0^\infty |g(t)|dt < \infty$$

It should be noted that this definition of asymptotic stability guarantees that $g(t) \to 0$ as $t \to \infty$ as otherwise the integral in the definition would tend to infinity.

Examples:

1. LCR circuit:

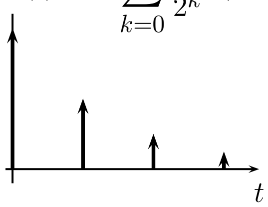
$$g(t) \;=\; e^{-t}\sin(3t+2)$$



$$\int_0^\infty |g(t)|\,dt \leq \int_0^\infty e^{-t}\,dt = 1 < \infty$$

2. Delay line with lossy reflections:

$$g(t) \;=\; \sum_{k=0}^{\infty} \frac{1}{2^k}\delta(t-kT)$$



$$\int_0^\infty |g(t)|\,dt = \int_0^\infty \sum_{k=0}^{\infty} \frac{1}{2^k}\delta(t-kT)\,dt = \sum_{k=0}^{\infty} \frac{1}{2^k} = 2 < \infty$$

$\Rightarrow$ asymptotically stable

3.2 Poles and the Impulse Response

Although stability is most easily *defined* in terms of the impulse response, it is most easily *determined* (at least for systems with rational transfer functions, the ones that come from ODE's) in terms of pole locations. To understand this, we first need to look at the relationship between the poles of a system and solutions to its differential equation – in particular its impulse response.

Example: Consider the system with input u and output y related by the ODE

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = a\frac{du}{dt} + bu.$$

The Auxillary Equation for this ODE is

$$\lambda^2 + \alpha\lambda + \beta = 0$$

with Complementary Factor
$$y_{\mathrm{CF}} = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

This decays to 0 as $t \to \infty$ only when $\lambda_1 < 0$ and $\lambda_2 < 0$ (or, if the roots are complex, when their real parts are negative).

In terms of transfer functions we have

$$\bar{y}(s) = \frac{as+b}{s^2+\alpha s+\beta}\bar{u}(s)$$

The poles of the system's transfer function are given by the roots of the denominator - that is the solutions to

$$s^2 + \alpha s + \beta = 0$$

So, for a system described by an ODE, its poles are precisely the solutions to its Auxiliary Equation.

Consider now a general LTI system described by an ODE, and consequently having a *rational* transfer function $G(s)$. That is, it can be written as the ratio of two polynomials

$$G(s) \; = \; \frac{\boldsymbol{n(s)}}{\boldsymbol{d(s)}}$$

(where the coefficients of $d(s)$ come from the LHS of the underlying ODE and the coefficients of $n(s)$ come from the RHS).

We can factorize the denominator to give

$$G(s) = \frac{n(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

We will also assume that $G(s)$ is *proper*, that is

$$\deg[\boldsymbol{n(s)}] \; \leq \; \deg[\boldsymbol{d(s)}].$$

e.g. $\boldsymbol{G(s) = s}$ (a differentiator) is *not* proper

(This condition will always *be satisfied for physically realizable systems. Moreover, any system whose transfer function violates this condition is not asymptotically stable.)*

In this case we can perform a partial fraction expansion to give

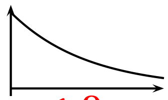
$$G(s) = \frac{\alpha_1}{s-p_1} + \frac{\alpha_2}{s-p_2} + \cdots + \frac{\alpha_n}{s-p_n} + C$$

where $\alpha_i = \lim_{s \to p_i}(s-p_i)G(s)$ is called the *residue* at $s = p_i$. *(we are assuming no repeated poles here, for simplicity of notation).* Finally, by taking inverse Laplace transforms, this means we can write the impulse response in the form

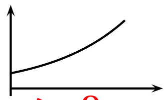
$$g(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \cdots + \alpha_n e^{p_n t} + C\delta(t)$$

Consider one of these terms, e^{pt} say. How it contributes to $g(t)$ depends on whether p is real or complex:

- If p is real: then e^{pt} is a real exponential, with time constant $\boldsymbol{|1/p|}$.



$\boldsymbol{p < 0}$



$\boldsymbol{p > 0}$

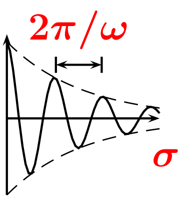
- If p is complex then we need to consider $\Re(\alpha e^{pt})$ (the imaginary part will cancel out with the contribution from p^*, which will also be a pole[†], since $g(t)$ must be real).

This will give either a damped or a growing sinusoid:

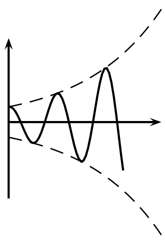
$$\begin{aligned}\Re(\underbrace{\alpha}_{\boldsymbol{Ae^{j\phi}}} e^{pt}) &= \Re(Ae^{j\phi}e^{pt}) = \Re(Ae^{\sigma t}e^{j(\omega t+\phi)}) \\ &= \boldsymbol{Ae^{\sigma t}\cos(\omega t + \phi)}\end{aligned}$$

(where we have put $p = \sigma + j\omega$ again)

time constant $\boldsymbol{= |1/\sigma|}$ frequency $\boldsymbol{= \omega}$



$\boldsymbol{\sigma = \Re(p) < 0}$



$\boldsymbol{\sigma > 0}$

[†] complex poles always appear in conjugate pairs since they are roots of a real polynomial

So each pair of complex poles contributes a term of the form

$$2Ae^{\sigma t}\cos(\omega t + \phi)$$

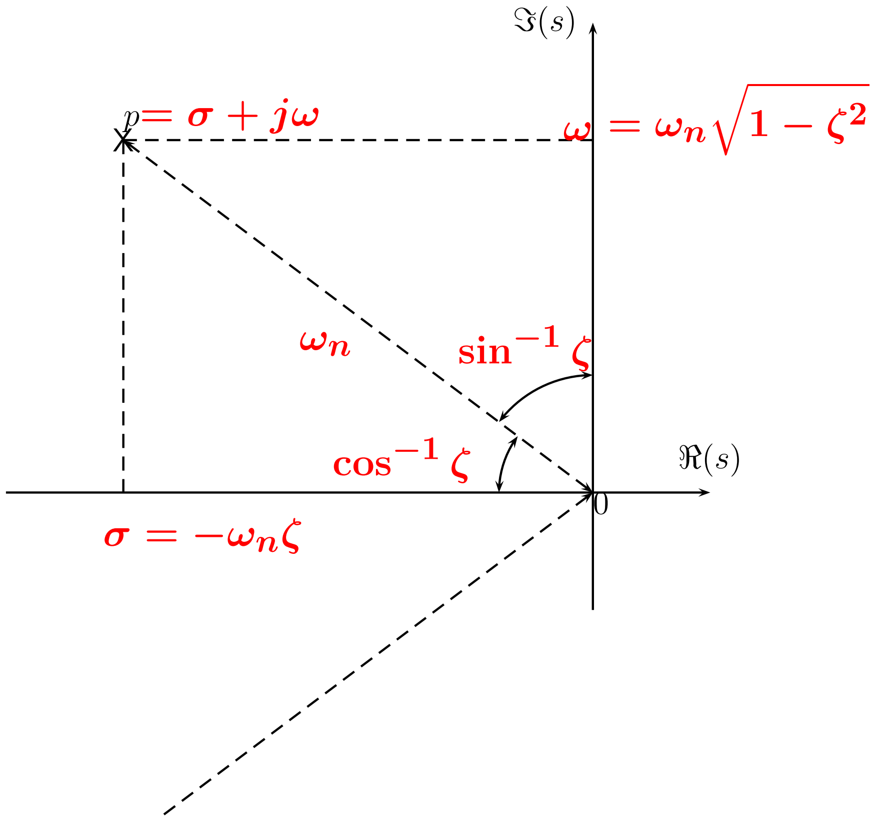
where $\boldsymbol{\sigma = \Re(p), \quad \omega = \Im(p)}$

Compare this with the impulse response of a second order system (see Mechanics data book)

$$Ce^{-\omega_n \zeta t}\sin(\omega_d t) \implies \begin{cases} \sigma = -\omega_n \zeta \\ \omega = \omega_d = \omega_n\sqrt{1-\zeta^2} \end{cases}$$

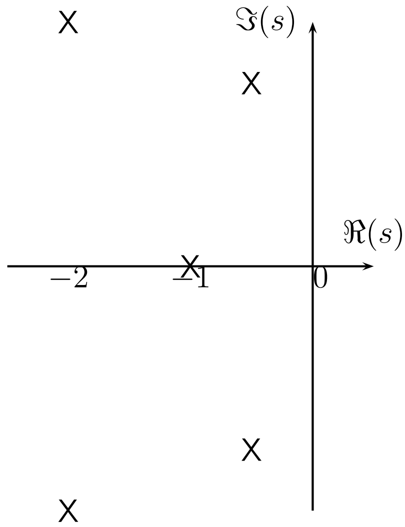
Clearly, the impulse response of any rational system can be regarded as a combination of 1st and 2nd order terms. Furthermore, the contribution of the second order terms can be understood in terms of the language of second order systems, as the following **very important** figure makes clear :

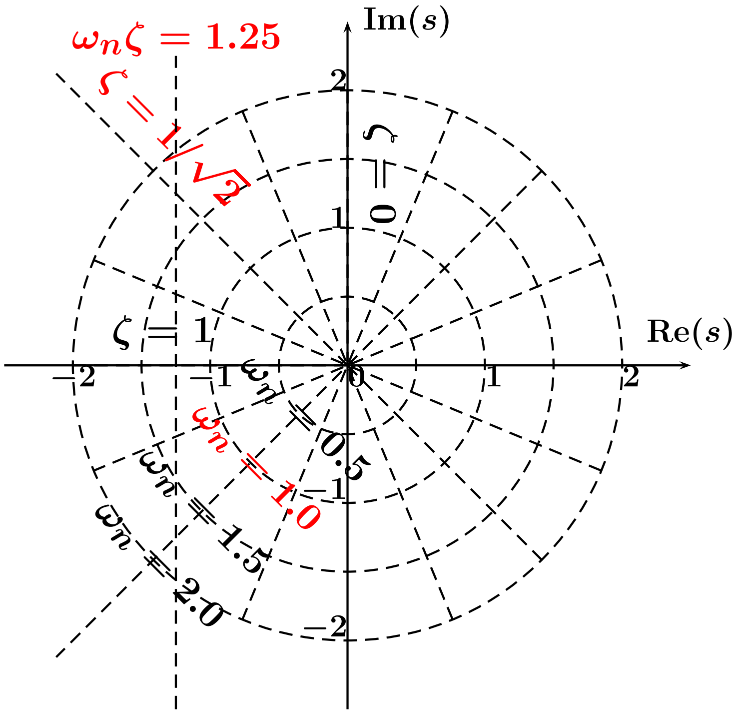
We have assumed that no poles are repeated for this discussion. Repeated poles give rise to terms of the form $t^m e^{pt}$ (or $t^m e^{\sigma t}\cos(\omega t + \phi)$), which have the same general characteristics (as the exponential dominates the polynomial term).



This figure shows that, given the pole locations, in the complex plane, of a second order system we can read off the natural frequency, the damping ratio and also $\omega_n\zeta$, the reciprocal of the time constant of the decay.

For a higher order system, we can read off the natural frequency and damping ratio of each "mode" of the system (each pair of complex poles). The poles closest to the imaginary axis are often called the *dominant* poles (their contribution dies away most slowly, and so tends to dominate the response)



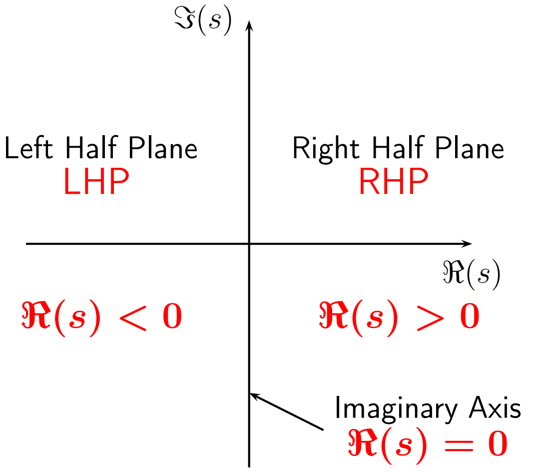


This figure shows radial contours of constant damping ratio ζ and circles of constant natural frequency ω_n as well as a vertical lines on which $\omega_n\zeta$ is constant.

3.3 Asymptotic Stability and Pole Locations

We will now show the following:

Theorem: An LTI system with rational transfer function $G(s)$ is asymptotically stable if, and only if, all poles of $G(s)$ lie in the LHP



Proof:

i) First we show that if all poles have a negative real part then the system is asymptotically stable.

For now, assume that the *poles* of $G(s)$ are distinct

i.e. that $d(s)$ has no repeated roots

(we shall remove this restriction later)

then we can write

$$\begin{aligned} G(s) &= \frac{n(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \\ &= \boldsymbol{\alpha_0 + \frac{\alpha_1}{(s-p_1)} + \frac{\alpha_2}{(s-p_2)} + \cdots + \frac{\alpha_n}{(s-p_n)}} \end{aligned}$$

by partial fraction expansion, and so

$$g(t) = \alpha_0\delta(t) + \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \cdots + \alpha_n e^{p_n t}.$$

The Theorem stated on the left is an important result in the study of linear systems and in control system design, as asymptotic stability is a property often required in the design of feedback systems.

In particular, the significance of the Theorem is that it allows to check the asymptotic stability of a linear system without explicitly evaluating its impulse response, but by simply checking the position of the poles of its transfer function.

More precisely, the system is asymptotically stable if and only if the real part of the poles is negative.

Now, let

$$\sigma_k = \Re(p_k) \text{ and } \omega_k = \Im(p_k)$$

so $\boldsymbol{p_k = \sigma_k + j\omega_k}$, for each $k = 1 \ldots n$, then

$$|e^{p_k t}| = |e^{(\sigma_k + j\omega_k)t}| = |e^{\sigma_k t} e^{j\omega_k t}| = |e^{\sigma_k t}| \underbrace{|e^{j\omega_k t}|}_{\mathbf{1}} = e^{\sigma_k t}$$

and so

$$|g(t)| \leq |\alpha_0|\delta(t) + |\alpha_1|e^{\sigma_1 t} + |\alpha_2|e^{\sigma_2 t} + \cdots + |\alpha_n|e^{\sigma_n t}.$$

Now,

$$\int_0^\infty e^{\sigma t}\, dt = \left[\frac{1}{\sigma}e^{\sigma t}\right]_0^\infty = \begin{cases} -\dfrac{\mathbf{1}}{\boldsymbol{\sigma}}, & \text{if } \sigma < 0 \\ \boldsymbol{\infty}, & \text{if } \sigma \geq 0 \end{cases}$$

and furthermore, since $\boxed{\textit{every}}$ pole has $\sigma_k < 0$, then

$$\int_0^\infty |g(t)|\, dt \leq |\alpha_0| + \left|\frac{\alpha_1}{\sigma_1}\right| + \left|\frac{\alpha_2}{\sigma_2}\right| + \cdots + \left|\frac{\alpha_n}{\sigma_n}\right| \boldsymbol{< \infty}$$

and consequently the system is asymptotically stable as required.

REPEATED POLES: If $G(s)$ has repeated poles, i.e.

$$G(s) = \frac{\cdots}{\cdots (s-p)^l \cdots},$$

where l denotes the *multiplicity* of the pole at $s = p$, then the partial fraction expansion of $G(s)$ will be of the form

$$G(s) = \cdots + \frac{\beta_1}{(s-p)} + \frac{\beta_2}{(s-p)^2} + \cdots + \frac{\beta_l}{(s-p)^l} + \cdots.$$

Hence, the impulse response $g(t)$ will be of the form

$$g(t) = \cdots + \beta_1 e^{pt} + \beta_2 t e^{pt} + \cdots + \frac{\beta_l}{(l-1)!} t^{l-1} e^{pt} + \cdots$$

However, if $p = \sigma + j\omega$ and $\sigma < 0$ (ie $\Re(p) < 0$), then

$$\int_0^\infty |t^{k-1} e^{pt}|\, dt = \int_0^\infty t^{k-1} e^{\sigma t}\, dt < \infty$$

for any k. Hence the conclusion remains valid.

ii) Now we show the converse, that if a system is asymptotically stable then all poles have a negative real part.

For all values of s for which $\Re(s) \geq 0$, we have

$$\begin{aligned} |G(s)| &= \left| \int_0^\infty e^{-st} g(t)\, dt \right| \leq \int_0^\infty \left| e^{-st} \right| |g(t)|\, dt \\ &\leq \int_0^\infty |g(t)|\, dt \quad (\text{ since } \left| e^{-st} \right| \leq 1 \text{ for } \mathrm{Re}(s) \geq 0 \text{ and } \boldsymbol{t > 0}) \\ &\boldsymbol{= A < \infty}. \end{aligned}$$

since the system is asymptotically stable. This means that $G(s)$ cannot have any poles on the imaginary axis or in the right half of the complex plane. So any poles it does have must have a negative real part, as required. .

So far, we have divided systems into two classes: those that are asymptotically stable and those that are not. We shall now further classify the systems that are not asymptotically stable into two classes: those that are marginally (i.e. "almost") stable and those that are unstable.

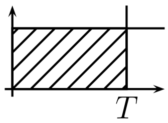
3.4 Marginal Stability

Definition: An LTI system is *marginally stable* if it is *not* asymptotically stable, but there nevertheless exist numbers A, $B < \infty$ such that

$$\int_0^T |g(t)|dt < A + BT \qquad \text{for all } T$$

Examples:

1. Integrator:

$$g(t) \ = \ H(t) \ =$$ 

$$\boldsymbol{\int_0^T |g(t)|\, dt = T}$$

$$\boldsymbol{G(s) = 1/s} \implies j\omega\text{-axis pole at } \boldsymbol{s = 0}$$

2. Undamped spring-mass system:

$$g(t) \ = \ \cos(3t) \qquad \implies \int_0^T |g(t)|\, dt \le \boldsymbol{\int_0^T 1\, dt = T}$$

$$G(s) = \boldsymbol{s/(s^2+9)} \implies j\omega\text{-axis poles at } \boldsymbol{s = \pm 3j}$$

3. Delay line with lossless reflections:

$$g(t) \ = \ \sum_{k=0}^{\infty} \delta(t-k), \qquad \implies \int_0^T |g(t)|\, dt \le \boldsymbol{T+1}$$

4. Something which cannot arise as the impulse response of any ODE: ($g(t) \to 0$, but system is *not* asymptotically stable)

$$g(t) \ = \ \frac{1}{t+1} \qquad \implies \int_0^T |g(t)|\, dt = \log(T+1) \quad \boldsymbol{\le T}$$

3.5 Instability

Definition: A system is *unstable* if it is neither asymptotically stable nor marginally stable.

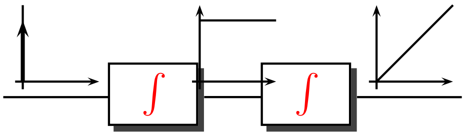
Examples:

1. Inverted pendulum:

$$g(t) = e^{4t} + e^{-4t}$$

$$G(s) = \frac{\mathbf{1}}{\boldsymbol{s-4}} + \cdots \implies \text{pole at} \quad \boldsymbol{s = 4}$$

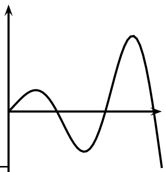
2. Two integrators in series:



$$g(t) = t$$

$$G(s) = \frac{\mathbf{1}}{\boldsymbol{s^2}} \implies \text{double pole at} \quad \boldsymbol{s = 0}$$

3. Oscillation of badly designed control system:



$$g(t) = e^{0.01t}\sin(0.3t)$$

$$G(s) = \frac{0.3}{((s-0.01)^2 + 0.3^2)}$$

$$\implies \text{poles at} \quad \boldsymbol{s = 0.01 \pm 0.3j}$$

Warning: Different people use different definitions of stability. In particular, systems which we have defined to be marginally stable would be regarded as stable by some, and unstable by others. For this reason we avoid using the term "stable" without qualification.

It should be noted that the impulse response $g(t)$ of an unstable system will tend to infinity as $t \to \infty$.

This is also evident in the examples illustrated on the left.

3.6 Stability Theorem

It should be clear from these examples that

- if *any* of the poles of $G(s)$ have a positive real part then the impulse response will have a term that blows up exponentially (consider the partial fraction expansion of $G(s)$).
- Also, if $G(s)$ has a repeated imaginary axis pole then the impulse response will have a term that still blows up, although more slowly.

In either of these cases, the system is unstable.

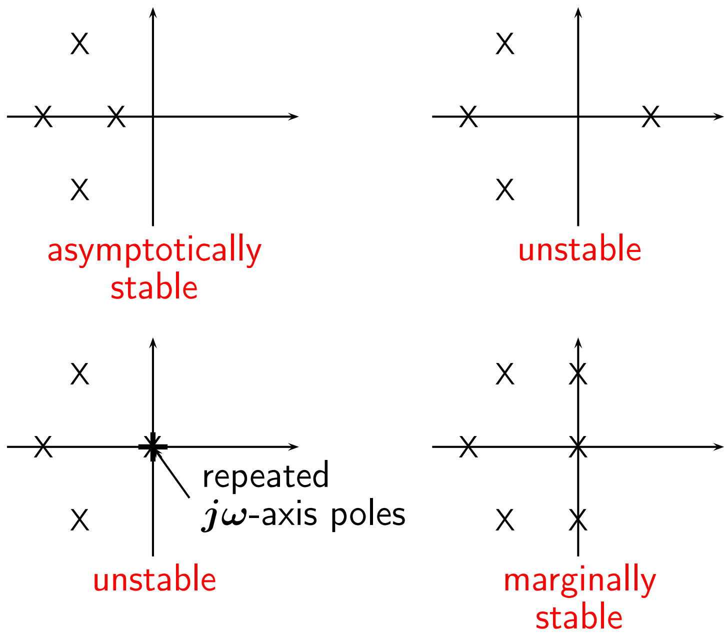
- Isolated poles on the imaginary axis, on the other hand, give rise to terms in the impulse response which remain bounded (e.g. steps or sinusoids).

In this case the system is not asymptotically stable but is nevertheless marginally stable (provided it has no RHP or repeated imaginary axis poles).

In fact, (**for systems with proper rational transfer functions**) it can be shown that

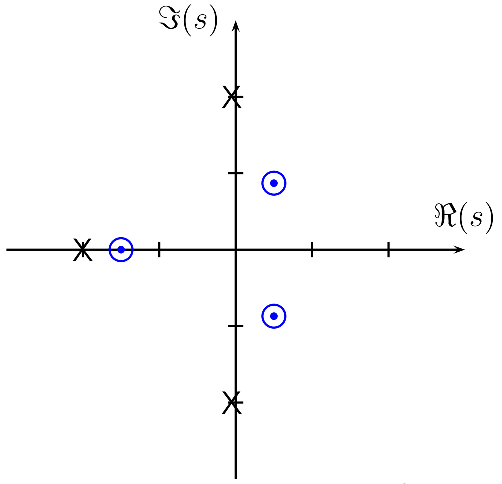
Stability Theorem:
1. A system is asymptotically stable if all its poles have negative real parts.
2. A system is unstable if any pole has a positive real part, or if there are any repeated poles on the imaginary axis.
3. A system is marginally stable if it has one or more distinct poles on the imaginary axis, and any remaining poles have negative real parts.

Note: we proved part 1, and the converse statement that a system is *not* asymptotically stable if any of its poles have a zero or positive real part, on page 6) The refinement of "not asymptotically stable" into marginal stability and instability has only been illustrated by examples. The proof of parts 2 and 3 is not difficult, but is messy (and so is omitted).



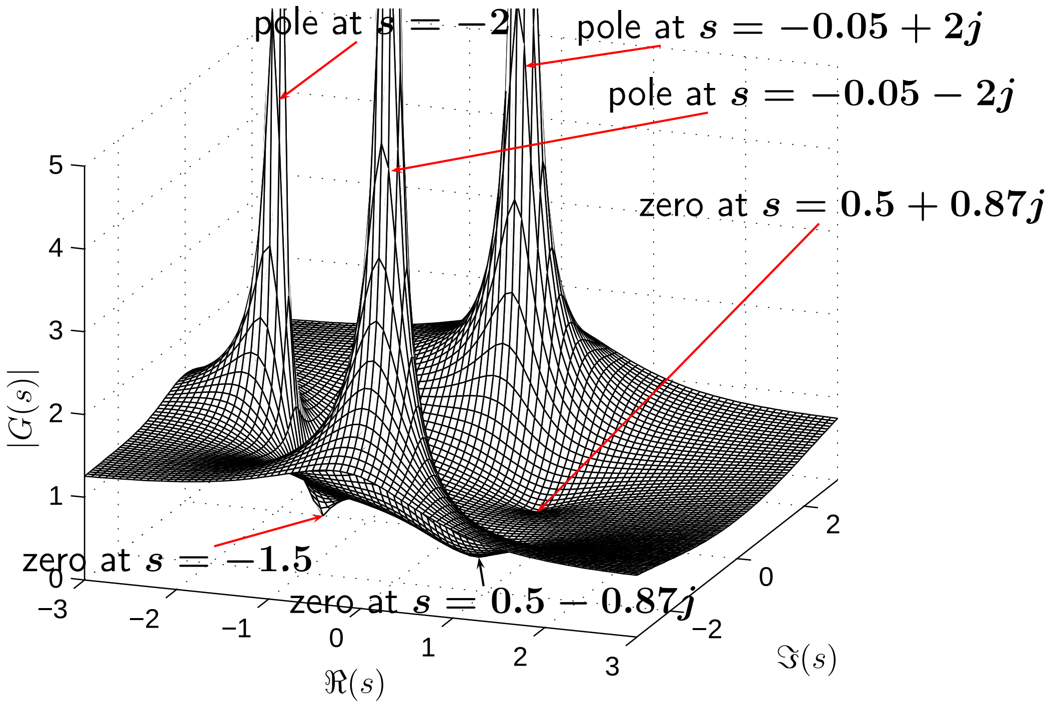
- Note: it's the "worst" poles that determine the stability properties

The examples on the left illustrate the Stability Theorem stated in the previous page, i.e. how the location of the poles determines the stability properties of a system.



Poles/zeros for $G(s) = \dfrac{(s+1.5)(s^2-s+1)}{(s+2)(s^2+0.1s+4)}$

Note: this is an asymptotically stable system.



3.7 Poles and the Transient Response

The term *Transient Response* refers to the initial part of the (time domain) response of a system to a general input (before the "transients" have died out). To a very large extent, these transients are a characteristic of the system itself rather than the input.

If, for example a system with transfer function

$$G(s) = \frac{n(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

is given an input $u(t)$, with Laplace transform $\bar{u}(s)$, then the response is given by

$$\begin{aligned}\bar{y}(s) = G(s)\bar{u}(s) &= \frac{n(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}\bar{u}(s) \\ &= \frac{\gamma_1}{s-p_1} + \frac{\gamma_2}{s-p_2} + \cdots + \frac{\gamma_n}{s-p_n} + \text{other stuff}\end{aligned}$$

and so

$$y(t) = \gamma_1 e^{p_1 t} + \gamma_2 e^{p_2 t} + \cdots + \gamma_n e^{p_n t} + \text{other stuff}$$

That is, the response $y(t)$ contains the same terms as the impulse response (although with different amplitudes) plus some extra terms due to particular characteristics of the input.

3.8 Key Points

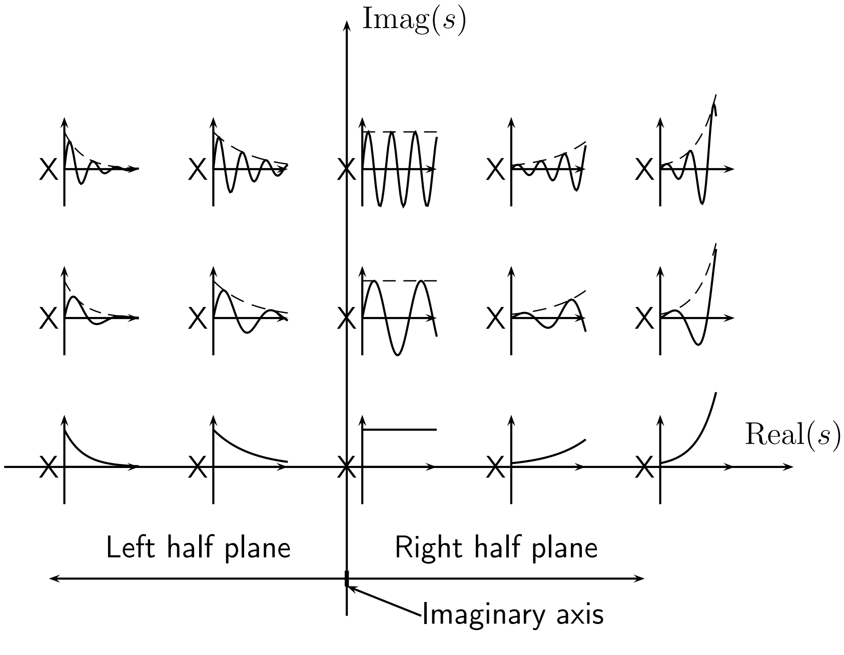
- The impulse response of an LTI system is a sum of terms due to each real pole, or pair of complex poles.
- The system's response to *any* input will also include these features.

The following figure shows a selection of pole locations, with their corresponding contribution to the total response.

This again is an important figure.

Note:

- The real part of the pole, σ, determines both stability and the time constant, $|1/\sigma|$.
- The imaginary part of the pole, ω, determines the damped natural frequency (actual frequency of oscillation) in rad/sec.
- The magnitude of the pole determines the natural frequency.
- The argument of the pole determines the damping ratio.



Pole locations and corresponding transient responses

The figure on the left illustrates how the location of the poles of the transfer function of a system (denoted by X) affect its impulse response (and hence also its transient response when a general input is applied as explained in the previous page).

In particular, as the imaginary part of the poles increases the response becomes more oscillatory. Also, as the real part of the poles becomes more negative, the response decays more quickly to zero.

Note also that the existence of at least one pole in the RHP implies instability.