Lecture 3

Vectors and Vector Fields

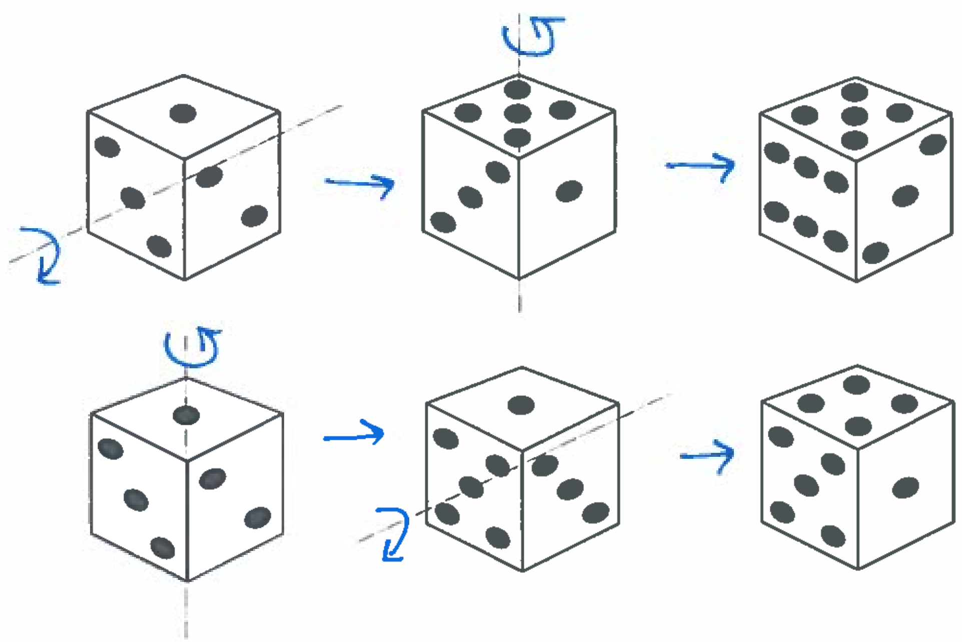
3.1 What are vectors?

A vector is a quantity that has both magnitude and direction. By 'direction', we mean that a vector has both a line of action and a sense along that line (left or right, for example). We also require that a vector obeys the rules of vector addition. In summary, a vector:

1. has a magnitude;
2. has a line of action;
3. has a sense along that line of action;
4. obeys the rules of vector addition ($\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$).

Finite rotations are the classic example of a quantity that has a magnitude, line of action and sense, but is not a vector. This is because the order in which finite rotations are performed is important, i.e. the rules of vector addition are not obeyed:

10° rotations



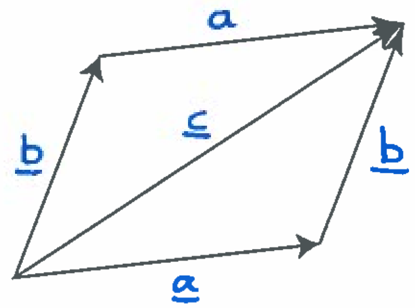
Finally, we make the distinction between *true* vectors and *pseudo* vectors. True vectors have the 4 properties listed above and, in particular, their sense is unambiguous. Examples of true vectors are force, velocity and displacement. Pseudo vectors have the 4 properties listed, but their sense is ambiguous and we can only treat them as vectors if we all agree on a convention for the sense (such as the right-hand rule). Examples of pseudo vectors are torque and angular velocity.

3.2 Properties of vectors

3.2.1 Addition

The parallelogram below shows that,

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \quad . \tag{3.1}$$



This statement is true for all coordinate systems. In Cartesian coordinates, we may write,

$$\underline{c} = \underline{a} + \underline{b} = (a_x\,\underline{i} + a_y\,\underline{j} + a_z\,\underline{k}) + (b_x\,\underline{i} + b_y\,\underline{j} + b_z\,\underline{k})$$

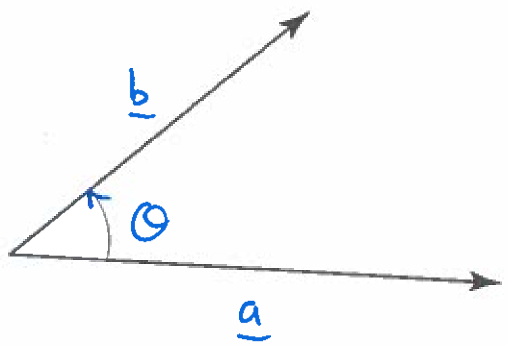
$$= (a_x + b_x)\,\underline{i} + (a_y + b_y)\,\underline{j} + (a_z + b_z)\,\underline{k}$$

3.2.2 Scalar (or dot) product

The definition of the scalar product is,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta \tag{3.2}$$

where θ is the angle between the two vectors from **a** to **b** (i.e. in the plane formed by the two vectors). The result of a scalar product is a scalar.



In Cartesian coordinates,

$$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$$

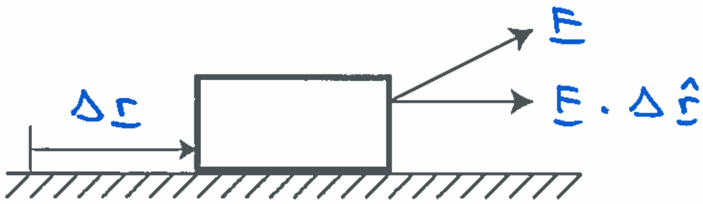
The scalar product is both distributive,

$$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$$

and commutative,

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$$

An example of the use of the scalar product in engineering is the calculation of the work done W by a force $\mathbf{F}$ that moves through a displacement $\Delta\mathbf{r}$,



$$W = \mathbf{F} \cdot \Delta\mathbf{r} \quad . \tag{3.3}$$

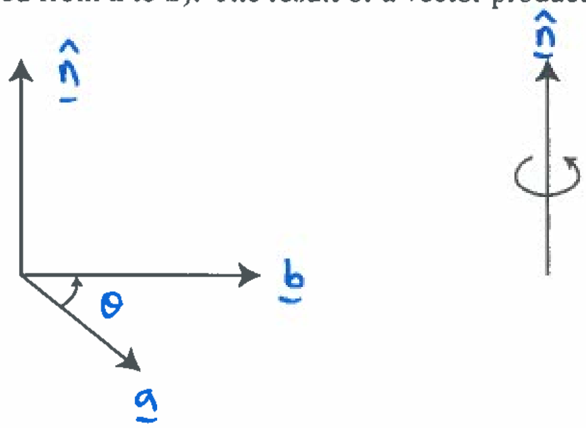
What we are evaluating is the component of $\mathbf{F}$ in the direction $\Delta\hat{\mathbf{r}}$ (the unit vector in the direction of $\Delta\mathbf{r}$), multiplied by the distance moved in that direction.

3.2.3 Vector (or cross) product

The definition of the vector product is,

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\, \hat{\mathbf{n}} \quad , \tag{3.4}$$

where θ is the angle between the two vectors and $\hat{\mathbf{n}}$ is the unit vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$ with direction determined by the right-hand rule (i.e. direction of a right-handed screw when turned from $\mathbf{a}$ to $\mathbf{b}$). The result of a vector product is, therefore, a vector.



In Cartesian coordinates,

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

The vector product is distributive,

$$\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$$

but *is not commutative*,

$$\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a}$$

An example of the use of the vector product in engineering is the calculation of the moment $\mathbf{M}$, about the origin, caused by the application of a force $\mathbf{F}$ at a point $\mathbf{r}$,

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \quad . \tag{3.5}$$

3.2.4 Triple products

The two types of triple product are: the scalar triple product,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \quad , \tag{3.6}$$

Result is a scalar

and the vector triple product,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \quad . \tag{3.7}$$

Result is a vector

3.3 Vector fields

So far, we have looked at scalar functions of multiple variables (and their differentiation and integration), and we have reminded ourselves of some fundamental properties of vectors. This course is principally about *vector fields*. A vector field is a region of space in which a vector property varies with position, and, in general, with time. Some examples will help us here.

In fluids, pressure is a scalar variable which varies with space and time. In Cartesian coordinates, we write,

$$p = p(x, y, z, t)$$

and we can apply the scalar differentiation and integration rules discussed in the first two lectures. The velocity of the fluid is a vector quantity which also varies with space and time. In Cartesian coordinates,

$$\underline{V} = \underline{V}(x, y, z, t)$$

and we now seek rules for working with this vector field. These new rules are called Vector Calculus.

But you might say we don't need new rules, because the velocity field has three components which are each scalar fields,

$$V_x = V_x(x,y,z,t) \qquad V_y = V_y(x,y,z,t) \qquad V_z = V_z(x,y,z,t) \quad , \tag{3.8}$$

so can't we just work with these scalars? However, the manipulation of these inter-related scalar fields becomes complicated very quickly (you can imagine lines and lines of partial derivatives that look similar for each component - it would be easy to make mistakes).

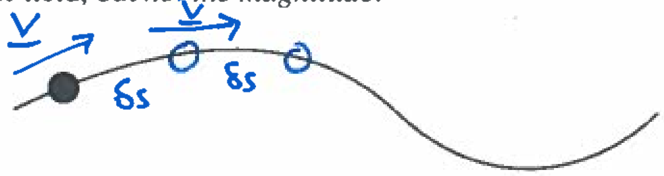
If we specify a vector field using the general position vector $\mathbf{r}$, e.g.

$$\mathbf{V} = \mathbf{V}(\mathbf{r},t) \quad , \tag{3.9}$$

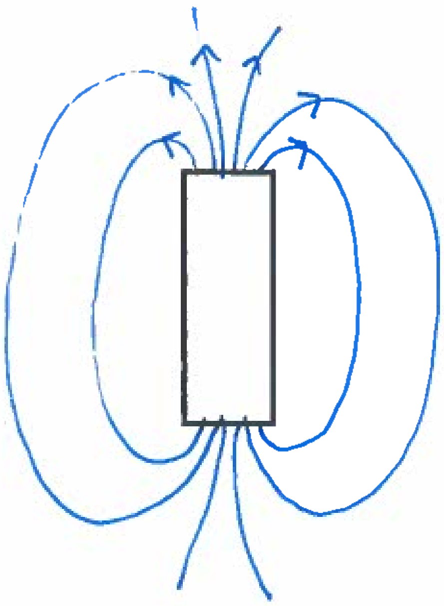
then we don't need to restrict ourself to one particular coordinate system; equation (??) can be applied to *any* coordinate system. But remember that if we use $\mathbf{V} = \mathbf{V}(\mathbf{r},t)$, then $\mathbf{V}$ is still a function of four (not two) independent scalar variables.

3.4 Field lines

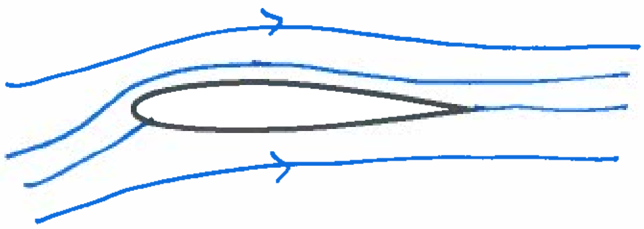
Field lines are a convenient way to represent a vector field. Field lines are constructed such that they are tangential to the vector field at a particular instant in time. Field lines therefore show the *direction* of the vector field, but *not the magnitude*.



Field lines can be constructed from small line elements as follows. Suppose we have a vector field $\mathbf{V}$ that varies in time but is two-dimensional in space, $\mathbf{V} = \mathbf{V}(x,y,t)$. We freeze the vector field at a particular instant in time. At a chosen point in space, we draw a small line of length δs to represent the direction (not the magnitude) of $\mathbf{V}$ *at that point and time*. We then move to the end of our short line and repeat the process. As $\delta s \to 0$ we end up with a curve that is tangential to $\mathbf{V}$ at every point. This is a field line.



Magnetic Field



Fluid Flow

Field line = "streamline"

If $\mathbf{V}$ is a vector field with components V_x, V_y, V_z then the equation of a field line projected onto the (x,y) plane must satisfy the condition that the slope of the field line is parallel to the velocity direction in the (x,y) plane,

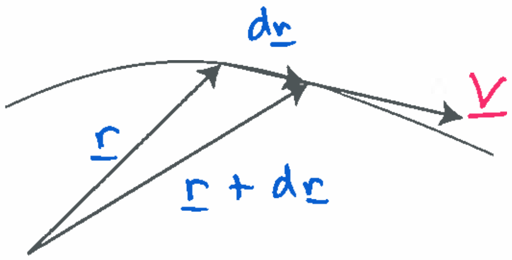
$$\frac{dy}{dx} = \frac{V_y}{V_x} \quad . \tag{3.10}$$

Similarly,

$$\frac{dz}{dy} = \frac{V_z}{V_y} \quad \text{and} \quad \frac{dx}{dz} = \frac{V_x}{V_z} \quad . \tag{3.11}$$

A neater way of expressing this is,

$$\frac{dx}{V_x} = \frac{dy}{V_y} = \frac{dz}{V_z}$$



Written in a way that is not coordinate system specific, an element of field line that is represented by vector $d\mathbf{r}$ must be parallel to V so that,

$$d\mathbf{r} = k\mathbf{V} \quad \text{where k is a constant, and,} \quad \mathbf{V} \times d\mathbf{r} = 0 \quad . \tag{3.12}$$

Example

Find the equation for, and sketch, the field lines of the 2-D, steady in time, vector field,

$$\mathbf{V} = (kx)\,\mathbf{i} - (ky)\,\mathbf{j}$$

where $y \geq 0$ and k is a constant.

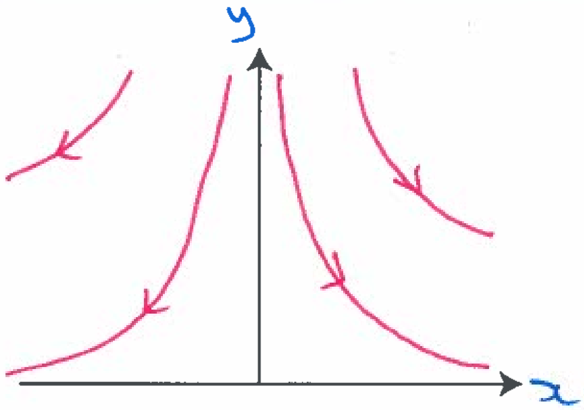
Field lines: $\dfrac{dy}{dx} = \dfrac{V_y}{V_x} = \dfrac{-y}{x}$

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

$$\ln x + \ln y = \ln c$$

$$xy = c$$

$$\left.\begin{array}{l} x > 0 \Rightarrow V_x > 0 \\ x < 0 \Rightarrow V_x < 0 \end{array}\right\}$$ for arrows on field lines



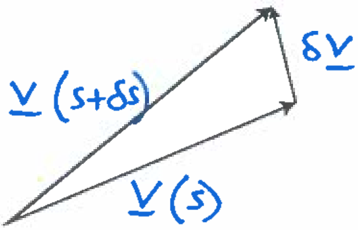
3.5 Differentiation of vectors

3.5.1 Ordinary derivatives

If a vector $\mathbf{V}$ is dependent on only one scalar variable, $\mathbf{V} = \mathbf{V}(s)$, then,

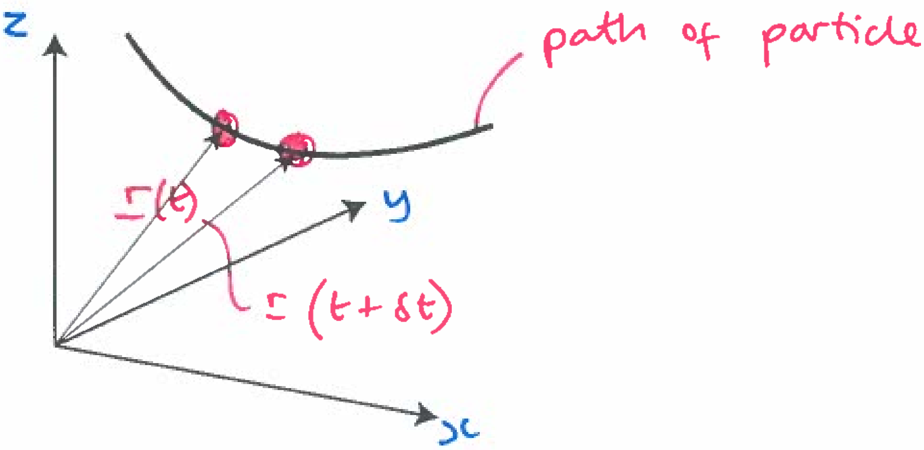
$$\frac{d\mathbf{V}}{ds} = \lim_{\delta s \to 0} \left(\frac{\mathbf{V}(s+\delta s) - \mathbf{V}(s)}{\delta s} \right) \quad . \tag{3.13}$$

$d\mathbf{V}/ds$ is a vector, and $\delta\mathbf{V} = (d\mathbf{V}/ds)\delta s$ is not normally in the same direction as $\mathbf{V}$.



$$\frac{d\underline{V}}{ds} = \frac{\delta\underline{V}}{\delta s} \text{ as } \delta s \to 0$$

As an example, consider a particle moving along a specified curve in 3-D space. The position of the particle is then only a function of time $\mathbf{r} = \mathbf{r}(t)$.



The velocity vector of the particle is given by,

$$\underline{v}(t) = \lim_{\delta t \to 0} \left(\frac{\underline{r}(t+\delta t) - \underline{r}(t)}{\delta t} \right) = \frac{d\underline{r}}{dt}$$

and this must be tangential to the specified curve.

The acceleration vector of the particle is given by,

$$\mathbf{a}(t) = \lim_{\delta t \to 0} \left(\frac{\mathbf{v}(t+\delta t) - \mathbf{v}(t)}{\delta t} \right) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \quad , \tag{3.14}$$

and this will only be tangential to the specified curve if the curve has zero curvature.

3.5.2 Differentiation formulae

If **A**, **B** and **C** are differentiable vector functions of a scalar variable s, and ϕ is a differentiable scalar function of s, then the following formulae apply (note that, as always, the order in the vector product terms is important):

$$\frac{d}{ds}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{ds} + \frac{d\mathbf{B}}{ds} \tag{3.15}$$

$$\frac{d}{ds}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{ds} + \frac{d\mathbf{A}}{ds}\cdot\mathbf{B} \tag{3.16}$$

$$\frac{d}{ds}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{ds} + \frac{d\mathbf{A}}{ds}\times\mathbf{B} \tag{3.17}$$

$$\frac{d}{ds}(\phi\mathbf{A}) = \phi\frac{d\mathbf{A}}{ds} + \frac{d\phi}{ds}\mathbf{A} \tag{3.18}$$

$$\frac{d}{ds}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}\times\frac{d\mathbf{C}}{ds} + \mathbf{A}\cdot\frac{d\mathbf{B}}{ds}\times\mathbf{C} + \frac{d\mathbf{A}}{ds}\cdot\mathbf{B}\times\mathbf{C} \tag{3.19}$$

$$\frac{d}{ds}(\mathbf{A}\times(\mathbf{B}\times\mathbf{C})) = \mathbf{A}\times\left(\mathbf{B}\times\frac{d\mathbf{C}}{ds}\right) + \mathbf{A}\times\left(\frac{d\mathbf{B}}{ds}\times\mathbf{C}\right) + \frac{d\mathbf{A}}{ds}\times(\mathbf{B}\times\mathbf{C}) \tag{3.20}$$

The above relationships are true in any coordinate system, but it is usually easiest to prove them in Cartesian coordinates. For example,

$$\frac{d}{ds}(\underline{A}\cdot\underline{B}) = \frac{d}{ds}(A_x B_x + A_y B_y + A_z B_z)$$

$$= A_x \frac{dB_x}{ds} + A_y \frac{dB_y}{ds} + A_z \frac{dB_z}{ds} + B_x \frac{dA_x}{ds} + B_y \frac{dA_y}{ds} + B_z \frac{dA_z}{ds}$$

$$= \underline{A}\cdot\frac{d\underline{B}}{ds} + \underline{B}\cdot\frac{d\underline{A}}{ds}$$

$$\left(\text{since } \frac{d\underline{A}}{ds} = \frac{dA_x}{ds}\underline{i} + \frac{dA_y}{ds}\underline{j} + \frac{dA_z}{ds}\underline{k}\right)$$

3.5.3 Partial derivatives

If **V** is a function of more than one scalar variable, for example $\mathbf{V} = \mathbf{V}(x,y,z)$ then the partial derivative of **V** with respect to x is defined by,

$$\frac{\partial \mathbf{V}}{\partial x} = \lim_{\delta x \to 0} \left(\frac{\mathbf{V}(x+\delta x, y, z) - \mathbf{V}(x,y,z)}{\delta x} \right) \quad . \tag{3.21}$$

$\partial \mathbf{V}/\partial x$ is therefore a *vector* representing (in magnitude and direction) the rate of change of **V** with x when y and z are both kept constant. We can evaluate $\partial \mathbf{V}/\partial x$ from the components of **V**,

$$\frac{\partial \underline{V}}{\partial x} = \frac{\partial V_x}{\partial x}\underline{i} + \frac{\partial V_y}{\partial x}\underline{j} + \frac{\partial V_z}{\partial x}\underline{k}$$

There are similar expressions for $\partial \mathbf{V}/\partial y$ and $\partial \mathbf{V}/\partial z$,

$$\frac{\partial \mathbf{V}}{\partial y} = \lim_{\delta y \to 0} \left(\frac{\mathbf{V}(x, y+\delta y, z) - \mathbf{V}(x,y,z)}{\delta y} \right) \quad , \tag{3.22}$$

$$\frac{\partial \mathbf{V}}{\partial z} = \lim_{\delta z \to 0} \left(\frac{\mathbf{V}(x, y, z+\delta z) - \mathbf{V}(x,y,z)}{\delta z} \right) \quad . \tag{3.23}$$

The rules for partial derivatives are similar to those given above for ordinary derivatives. For example,

$$\frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\cdot\mathbf{B} \quad . \tag{3.24}$$

You can now do Examples Paper 1: Q5, 6 and 7