UNIVERSITY OF CAMBRIDGE
Department of Engineering

IB Paper 5
Electromagnetic Fields & Waves

Lecture 4
Electromagnetic Waves & Antennae

https://www.vle.cam.ac.uk/course/view.php?id=70081

Electromagnetic Waves in Dielectrics

FLE116, GER71

- In the previous lecture, we saw the four Maxwell Equations in differential form
  - These each say something about the origin and nature of electric and magnetic fields, and they are

$$\nabla.\mathbf{D} = \rho \qquad (3.12) \qquad \nabla.\mathbf{B} = 0 \qquad (3.14)$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad (3.23) \qquad \nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \qquad (3.35)$$

- A major success of the Maxwell Equations was to explain the transmission of electromagnetic waves through free space, and we will now prove this
  - We will consider that we are in a dielectric (i.e. insulating) medium, such as air, glass, plastics, etc.
  - There is no free charge in these materials, so

$$\rho = 0 \qquad \mathbf{J} = 0$$

  - Therefore, the Maxwell Equations become

$$\nabla.\mathbf{D} = 0 \qquad (4.1a) \qquad \nabla.\mathbf{B} = 0 \qquad (4.1b)$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad (4.1c) \qquad \nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} \qquad (4.1d)$$

  - Let us take the curl of Eqn. 4.1c

$$\nabla\times(\nabla\times\mathbf{E}) = -\nabla\times\left(\frac{\partial\mathbf{B}}{\partial t}\right) \qquad (4.2)$$

  - From the Maths Data Book p16, we have the identity

$$\nabla\times(\nabla\times\mathbf{u}) = \nabla(\nabla.\mathbf{u}) - \nabla^2\mathbf{u}$$

- Hence, Eqn. 4.2 becomes

$$\nabla(\nabla.\mathbf{E}) - \nabla^2\mathbf{E} = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) \tag{4.3}$$

- We can use $\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$ to rewrite Eqn. 4.1a as

$$\nabla.\mathbf{E} = 0$$

- So substituting into Eqn. 4.3 gives

$$\nabla^2\mathbf{E} = \frac{\partial}{\partial t}(\nabla\times\mathbf{B}) \tag{4.4}$$

- We also know that in dielectric media $\mathbf{B} = \mu_0\mu_r\mathbf{H}$, so

$$\nabla^2\mathbf{E} = \mu_0\mu_r\frac{\partial}{\partial t}(\nabla\times\mathbf{H}) \tag{4.5}$$

- We can now use Eqn. 4.1d to substitute for $\nabla\times\mathbf{H}$ to give

$$\nabla^2\mathbf{E} = \mu_0\mu_r\frac{\partial^2\mathbf{D}}{\partial t^2}$$

$$\boxed{\nabla^2\mathbf{E} = \mu_0\mu_r\varepsilon_0\varepsilon_r\frac{\partial^2\mathbf{E}}{\partial t^2}} \tag{4.6}$$

- This is just the ***wave equation*** again, which is a 3-dimensional version of Eqn. 1.4 of the form

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \tag{4.7}$$

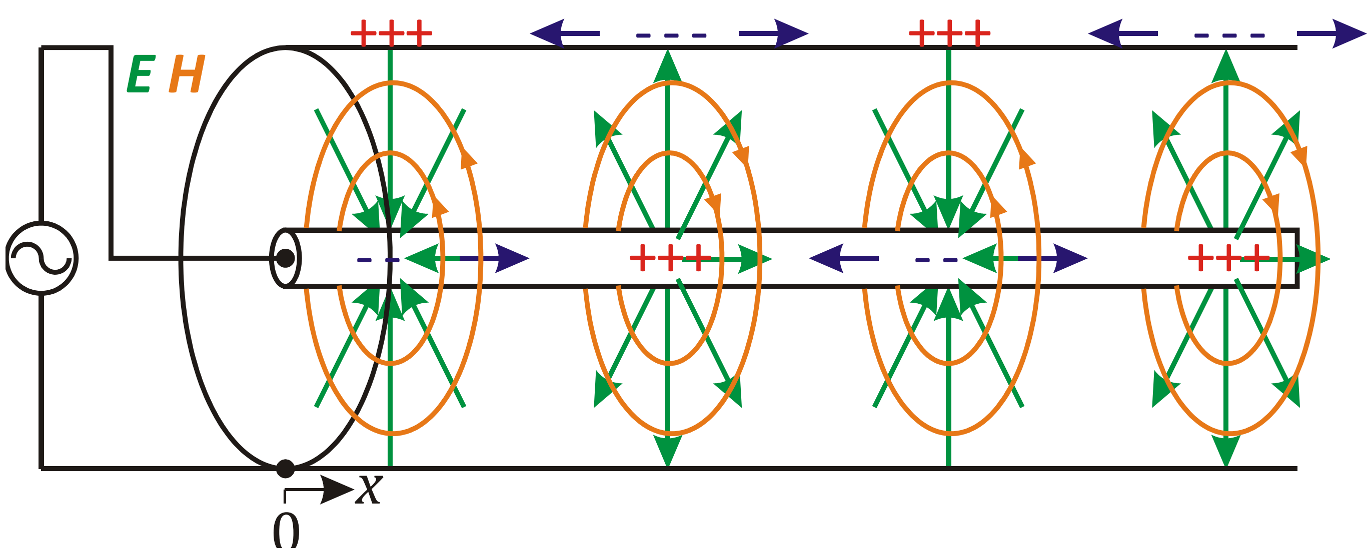
  - We saw in Lecture 1 that c is the velocity of the wave
  - Therefore, Eqn. 4.6 tells us that an electric field can propagate as a wave through a dielectric with a velocity

$$\boxed{c = \frac{1}{\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}} \tag{4.8}$$

- If the dielectric is air ($\varepsilon_r = 1$, $\mu_r = 1$) then

$$c = (8.854 \times 10^{-12}.\,4\pi \times 10^{-7})^{-1/2} = 2.998 \times 10^{8}\ \mathrm{m\,s}^{-1}$$

  - The wave travels at the speed of light
  - ***This is identical to the result that we got for the velocity of the wave on a transmission line*** (Eqn. 1.9)
  - It should be! We derived Eqn. 1.9 from the inductance and capacitance per unit length of a transmission line (e.g. the coaxial cable) which themselves are derived from the Gauss Law of Electric Fields and the Ampère Law
  - In both cases, we started with the Maxwell equations – we have gone by very different routes, but have ended at the same conclusion
  - Remember also that the wave on the transmission line was actually an electromagnetic wave in the space around the conductors, and the conductors are simply providing free charge to guide the wave



  - This derivation has set the wave free from the transmission line to roam through space!

- Apart from Eqn. 4.8 giving us the experimentally-verified velocity of light in air, it also shows that the velocity should slow down in a dielectric
  - At school, you probably learned that light slows down in water or glass, and the refractive index n quantifies this

$$c_{medium} = c_{air}/n \tag{4.9}$$

  - Glass, plastic, water and similar dielectrics are all non-magnetic, so $\mu_r = 1$, and Eqn. 4.8 becomes

$$c_{medium} = \frac{1}{\sqrt{\mu_0 \varepsilon_0 \varepsilon_r}} = \frac{c_{air}}{\sqrt{\varepsilon_r}} \tag{4.10}$$

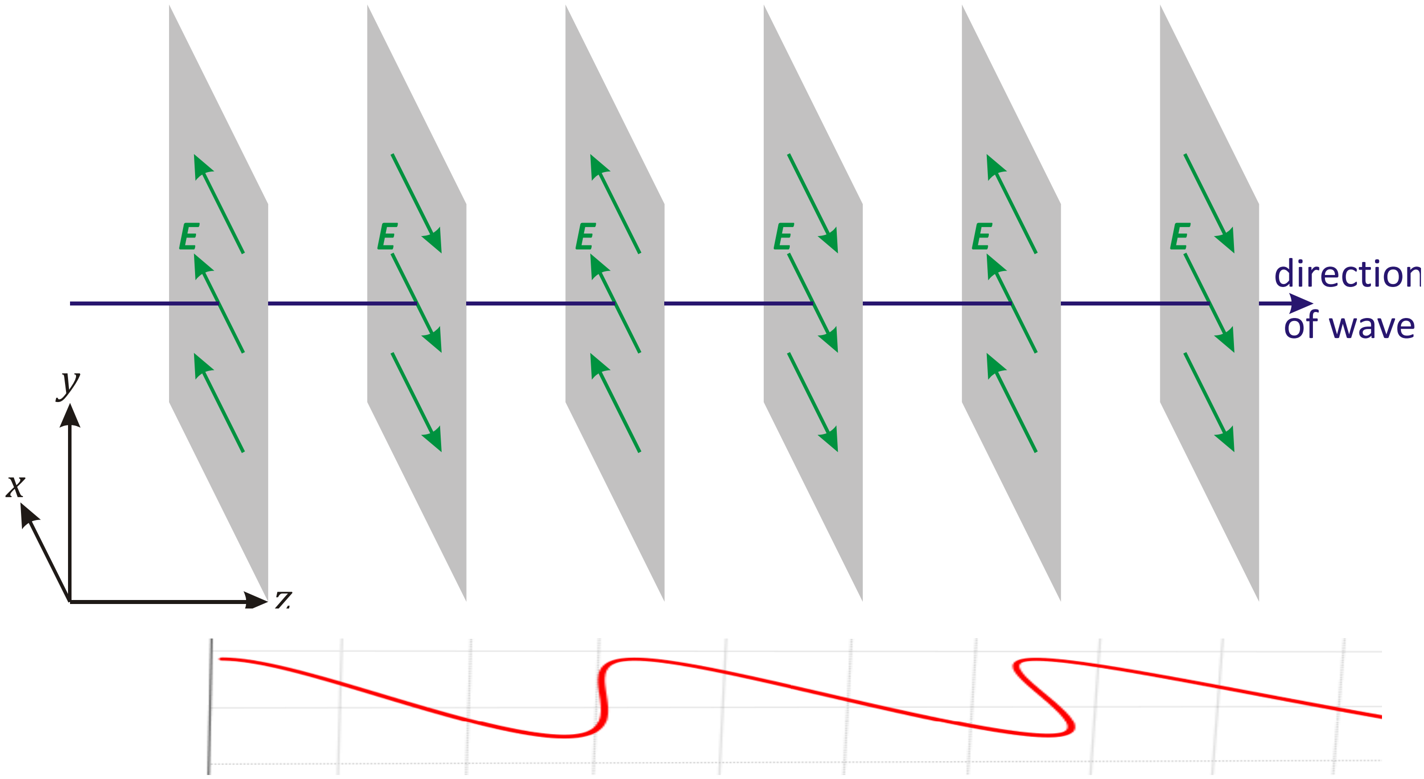
  - Comparing Eqns. 4.9 and 4.10, we can see that the refractive index is actually related to the permittivity by

$$\boxed{n = \sqrt{\varepsilon_r}} \tag{4.11}$$

- A similar wave equation to Eqn. 4.6 can be derived for magnetic fields by taking the curl of Eqn. 4.1d

$$\boxed{\nabla^2 \mathbf{H} = \mu_0 \mu_r \varepsilon_0 \varepsilon_r \frac{\partial^2 \mathbf{H}}{\partial t^2}} \tag{4.12}$$

- In order to think further about the nature of the wave, we are going to consider a ***plane wave***
  - This is a wave that propagates in a specific direction and which is uniform in the plane perpendicular to the propagation direction
  - Examples are light from the sun, a radio wave a long distance from the transmitter or a sound wave a long distance from its source



- A valid plane wave solution to Eqn. 4.6 is

$$\mathbf{E} = \left(E_{0x}\mathbf{i} + E_{0y}\mathbf{j} + E_{0z}\mathbf{k}\right)\exp\{j(\omega t - \beta z)\} \qquad (4.13)$$

  - $\mathbf{i}$,$\mathbf{j}$ and $\mathbf{k}$ are the unit vectors in each of the x, y and z directions respectively
  - β is the propagation constant $(2\pi/\lambda)$
  - This wave is therefore travelling in the positive z-direction
- You have probably been told at A-level that the electric field vector is perpendicular to the direction of propagation, but we can prove this from Eqn. 4.1a
  - As $\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$, this becomes

$$\nabla.\mathbf{E} = 0$$

- Hence, writing $\nabla.\mathbf{E}$ in Cartesian coordinates

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0 \tag{4.14}$$

- However, for a plane wave which is uniform in the xy plane perpendicular to propagation

$$\frac{\partial E_x}{\partial x}=\frac{\partial E_y}{\partial y}=0$$

- So Eqn. 4.14 becomes

$$\frac{\partial E_z}{\partial z}=0 \tag{4.15}$$

- Substituting in the plane wave (Eqn. 4.13) gives

$$-j\beta E_{oz}\exp\{j(\omega t-\beta z)\}=0 \tag{4.16}$$

- This has only two possible solutions
- Either $\beta=0$, but this implies an infinite λ, which is not a wave, or

$$E_{oz}=0 \tag{4.17}$$

- ***There is no component of electric field in the direction of propagation of a plane electromagnetic wave***
- We arrive at a similar conclusion for **H** from Eqn. 4.1b

- Let us now *assume* that the plane electromagnetic wave is polarised so that the electric field vector is pointing in the x-direction, so Eqn. 4.13 has become

$$\boxed{\mathbf{E}=E_{0x}\mathbf{i}\exp\{j(\omega t-\beta z)\}} \tag{4.18}$$

- What is the equation for the magnetic field?
- We can use Eqn. 4.1c

$$\nabla\times\mathbf{E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_{0x}\exp\{j(\omega t - \beta z)\} & 0 & 0 \end{vmatrix} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{j}\frac{\partial}{\partial z}[E_{0x}\exp\{j(\omega t - \beta z)\}] - \mathbf{k}\frac{\partial}{\partial y}[E_{0x}\exp\{j(\omega t - \beta z)\}]$$

$$= -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{j}j\beta E_{0x}\exp\{j(\omega t - \beta z)\} = \frac{\partial \mathbf{B}}{\partial t}$$

- Hence,

5/7 Q3

$$\mathbf{B} = \mathbf{j}\frac{\beta}{\omega}E_{0x}\exp\{j(\omega t - \beta z)\}$$

$$\boxed{\mathbf{H} = H_{0y}\mathbf{j}\exp\{j(\omega t - \beta z)\}} \qquad (4.19)$$

- where

$$H_{0y} = \frac{\beta}{\omega}\frac{E_{0x}}{\mu_0\mu_r} \qquad (4.20)$$

- Therefore, $\mathbf{H}$ ***is perpendicular to both*** $\mathbf{E}$ ***and the direction of propagation of the wave***
- Also, considering the amplitude of the wave using Eqn. 4.8

$$\beta/\omega = (2\pi/\lambda)/(2\pi f) = 1/f\lambda = 1/c = \sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}$$

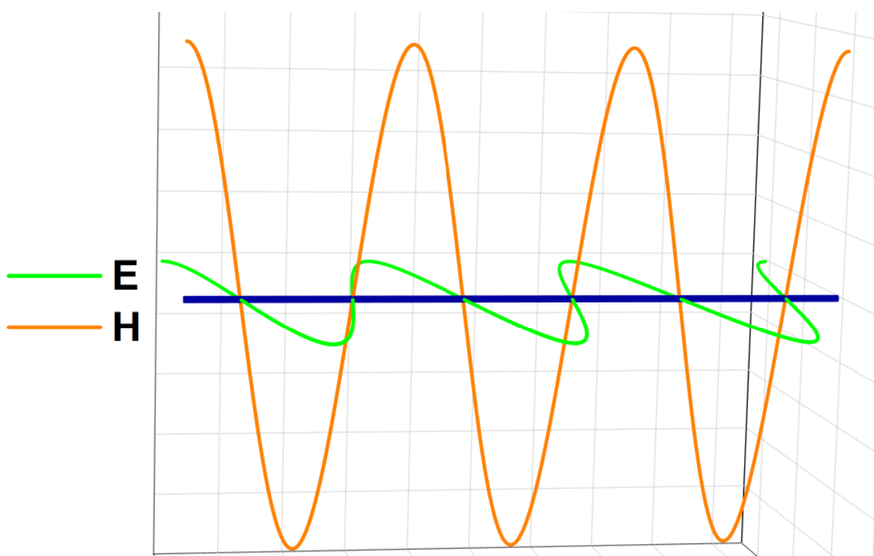
$$H_{0y} = E_{0x}\sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0\mu_r}} \qquad (4.21)$$

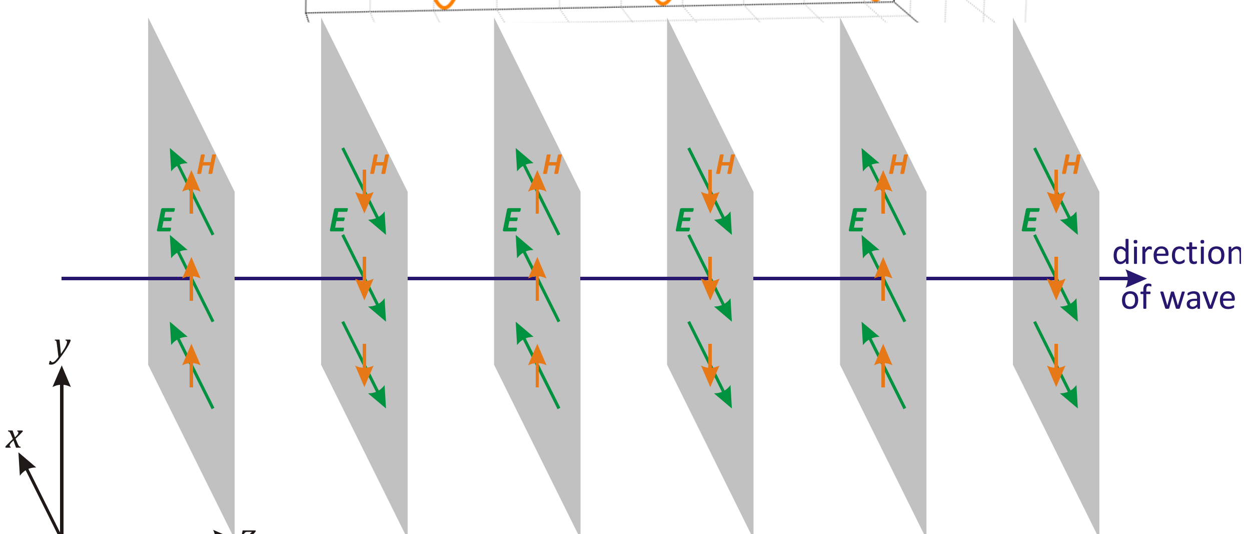
5/7 Q2

- Eqn. 4.21 tells us that the amplitude of the electric and magnetic fields in an electromagnetic wave are related
- The ratio of the two amplitudes is only dependent on the properties of the medium through which the wave is travelling, and we call this the ***impedance*** of the medium

$$\boxed{\eta = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}}} \tag{4.22}$$

- Note that this is the characteristic impedance of an ideal transmission line (Eqn. 2.4) with no geometry factor
- For air, the characteristic impedance is 377 Ω





The Power in a Wave & the Poynting Vector

FLE126, GER72

- If a field occupies a volume of space, then there must be potential energy stored
  - This is true for gravity, as if we put a mass in a gravitational field then a force will act on the mass and any motion will result in an energy change
  - As charges experience a force in an electric field and moving charges experience a force in a magnetic field, then these fields must also store energy
  - An electromagnetic wave is therefore transmitting energy, and so must have a power density (power per unit area)
  - Poynting considered this problem, and using the Maxwell equations, he defined the ***Poynting Vector*** $\mathbf{N}$ as

$$\boxed{\mathbf{N} = \mathbf{E} \times \mathbf{H}} \tag{4.23}$$

  - The direction of $\mathbf{N}$ is the ***direction of propagation of the wave***, which is consistent with Eqns. 4.18 and 4.19
  - The magnitude of $\mathbf{N}$ is the ***instantaneous power density*** (in W m^{-2}) in the wave – this will be time-varying
  - As $\mathbf{E}$ and $\mathbf{H}$ are perpendicular, then we can evaluate the magnitude of $\mathbf{N}$ for the case of our plane wave
  - In this case it is easier to express $\mathbf{E}$ and $\mathbf{H}$ as real numbers, Eqns. 4.18 and 4.19 become

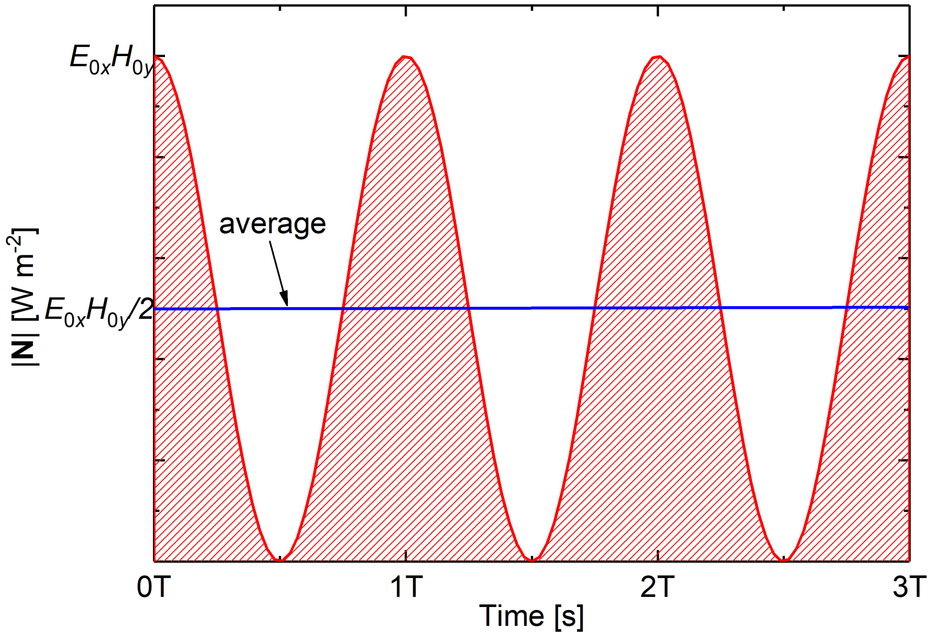
$$\mathbf{E} = E_{0x}\mathbf{i}\cos(\omega t - \beta z) \tag{4.24a}$$

$$\mathbf{H} = H_{0y}\mathbf{j}\cos(\omega t - \beta z) \tag{4.24b}$$

- Hence

$$|\mathbf{N}| = E_{ox} H_{oy} \cos^2(\omega t - \beta z)$$

- If we plot $|\mathbf{N}|$ at a particular point in space, say $z = 0$, then we find that the instantaneous power is time varying



- Note the similarity with a.c. power, but here there is no phase difference between the electric and magnetic fields
- The average power per unit area $|\overline{\mathbf{N}}|$ is only half the peak power, so

$$\boxed{|\overline{\mathbf{N}}| = \frac{|\mathbf{E}||\mathbf{H}|}{2} = \frac{\mathbf{E}\times\mathbf{H}^*}{2}} \tag{4.25}$$

- ***This is the equation that we most frequently use*** to actually work out the average power per unit area (i.e. the intensity) of an electromagnetic wave
- Alternatively, if we define rms values of E and H, then this becomes

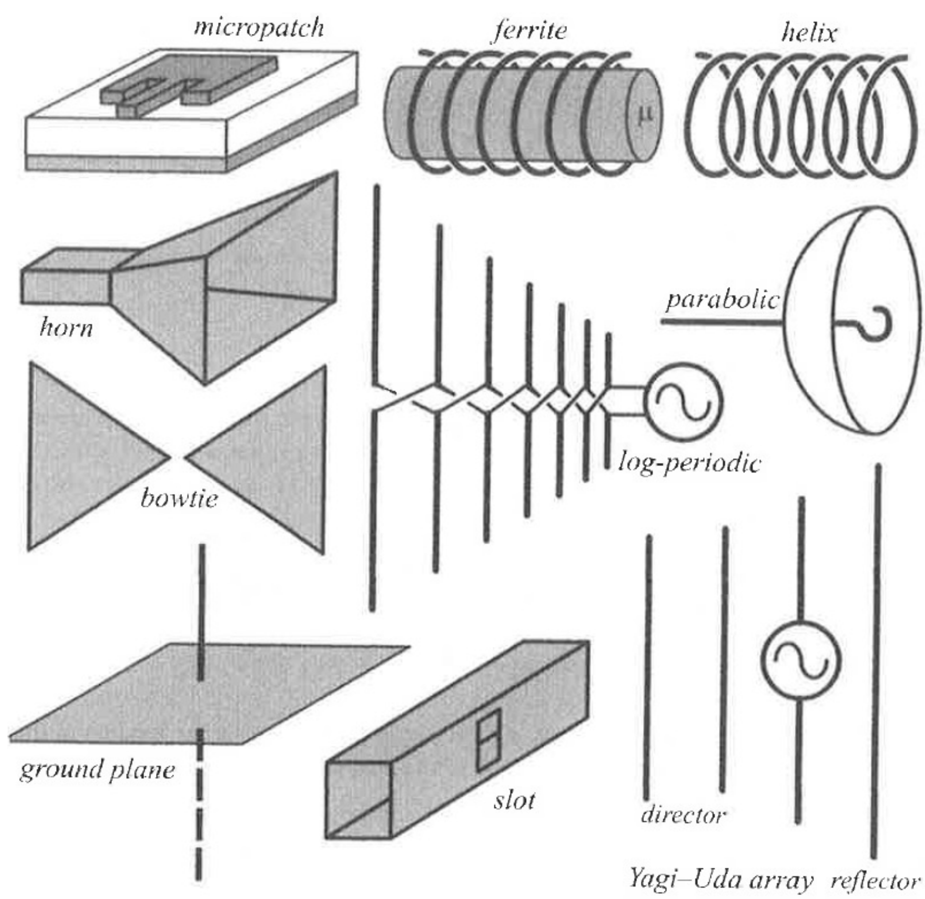
$$|\overline{\mathbf{N}}| = E_{rms} H_{rms} \tag{4.26}$$

5/7 Q4

Antennas

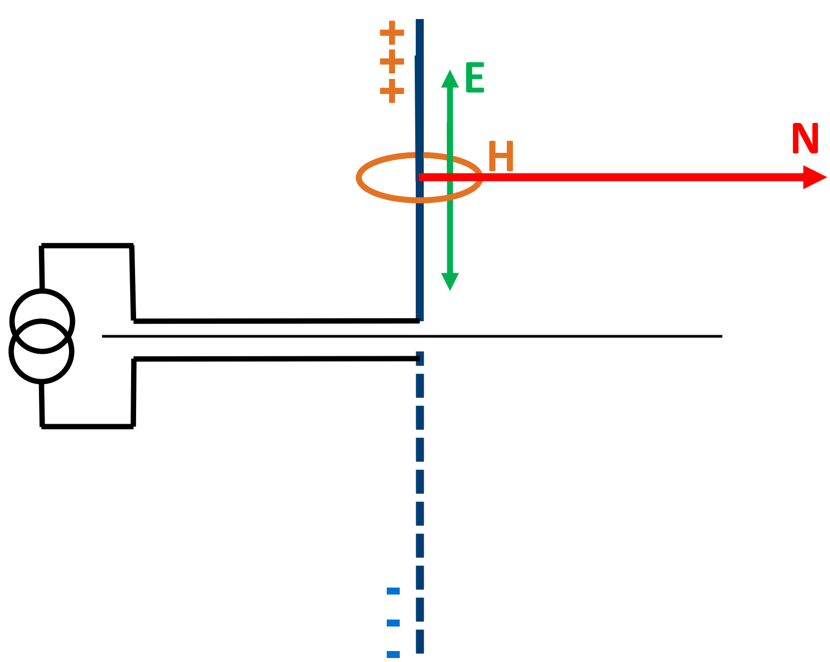
FLE138, GER102, GER109

- One of the most important applications of free space electromagnetic waves in to broadcast information (e.g. radio, terrestrial TV, satellite TV, mobile telephones, wifi, bluetooth, etc.)
  - In order to do this we need to be able to both generate a free-space electromagnetic wave from an electrical signal and convert an electromagnetic wave back into an electrical signal again
  - This is achieved using antennas, of which there are a huge diversity

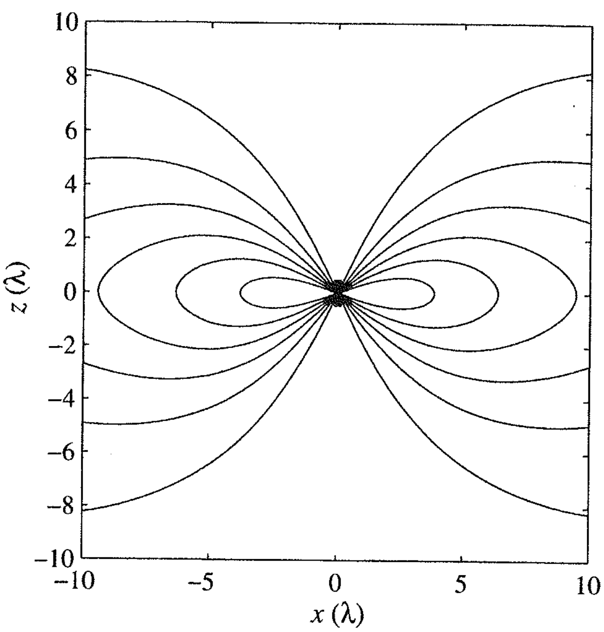


GER109

- Which antenna is selected to send or receive a particular signal depends on the wavelength, distance and directionality required
  - We will not consider lots of different types of antennas, but instead the metrics by which they may be evaluated
- One of the simplest antenna designs for broadcasting is the half-dipole (or grand plane) antenna
  - This consists of a conductor of height $\lambda/4$ sticking out of a ground plane
  - Radio antennas are of this type
  - An a.c. signal is applied to the antenna, which results in an oscillating dipole and current (remembering that the method of images means that there is an effective opposite charge being produced under the ground plane)



- The oscillating dipole produces an electric field in the direction of the antenna, whilst the current produces a magnetic field that is perpendicular to the electric field
- These are the conditions to set up an electromagnetic wave that will propagate in a direction perpendicular to both waves
- It is clear that the wave will be preferentially transmitted in the plane of the ground, which for a radio transmitter is what is required



GER105

- Such anisotropy in the radiated signal is always true of antennas, and we could use the Maxwell Equations and Poynting Vector to calculate the power in the transmitted signal as a function of spherical polar coordinates
- However, a good engineering figure of merit is the gain of the antenna which is defined as

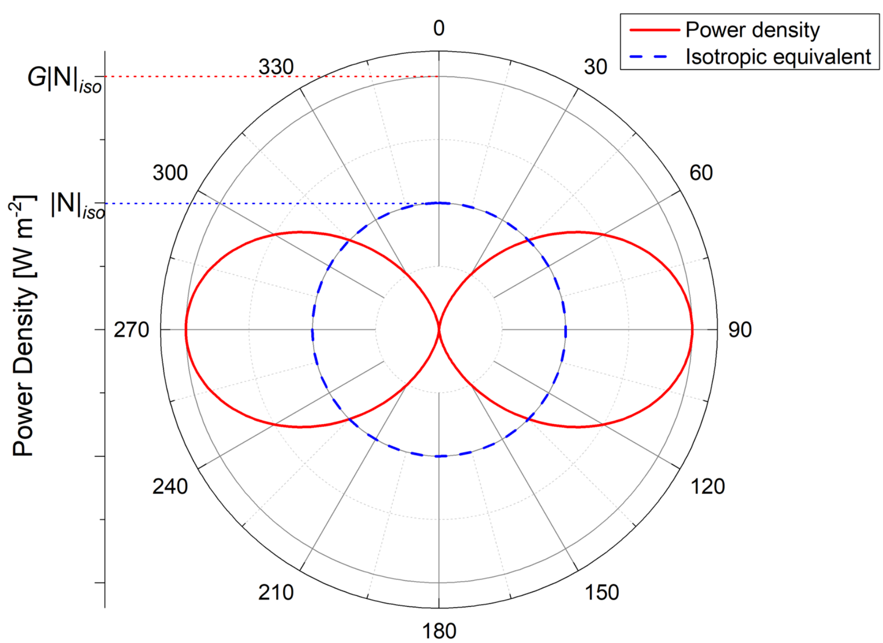
$$\boxed{\text{Antenna Gain} = \frac{\text{Maximum power density}}{\text{Isotropic power density}}} \qquad (4.27)$$

- In other words, let us say that a transmitting antenna is outputting a total power P
- If this power was being radiated isotropically, then at some distance r from the antenna

$$|\bar{N}|_{iso} = \frac{P}{4\pi r^2} \tag{4.28}$$

- However, if the antenna has a gain G, then the power density in the maximum direction of propagation will be

$$\boxed{|\bar{N}| = G|\bar{N}|_{iso} = \frac{GP}{4\pi r^2}} \tag{4.29}$$



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- We also define what is called the Radiation Resistance
- This is the resistance R_a that would have to be put in the place of the transmitting antenna that would dissipate as much power as the antenna radiates

$$\boxed{R_a = P/I_{rms}^2} \tag{4.30}$$

- I_{rms} is the rms current driving the antenna

- Receiving antennas attempt to harvest energy from an electromagnetic wave and convert this back into an electrical signal again
  - Common examples are the parabolic dish antenna used to receive satellite signals or a log-periodic antenna for terrestrial TV reception
  - Simple dipole antennas are often used for radio reception

  - Once again, the Maxwell equations can be used to determine how much power in an electromagnetic wave is absorbed by the antenna
  - A good engineering figure of merit is the ***effective area*** A_{eff} which is the area of the electromagnetic wave that would give the power P_{abs} absorbed by the antenna

$$A_{eff} = \frac{P_{abs}}{|\bar{\mathbf{N}}|} \tag{4.31}$$

  - In other words, it is the area than an ideal antenna would have to occupy to absorb the same power

Home-made antenna for receiving data on a Scottish island

(courtesy Prof Woodhouse)