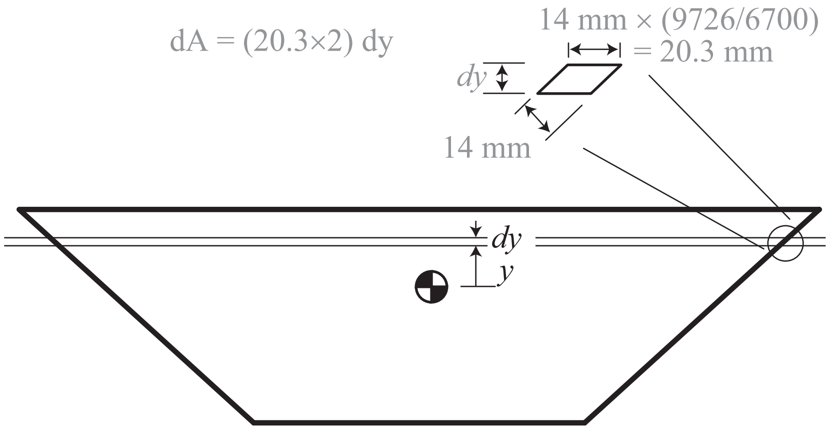
Webs — are more difficult. There are various shortcuts that we could take (talk to your supervisor), but we'll do the calculation from first principles, $I_{\text{webs}} = \int y^2\, dA$, to show that it is quite straightforward.

Consider a thin slice at a distance y from the centroid:

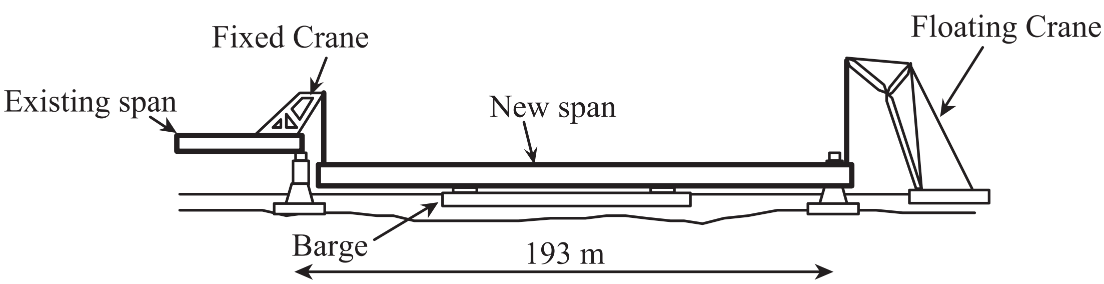


$$I_{\text{webs}} = \int y^2\, dA = \int_{-4177}^{2523} y^2 \times 40.6\, dy$$
$$= 40.6 \left[\frac{y^3}{3}\right]_{-4177}^{2523}$$
$$= 1.206 \times 10^{12}\ \text{mm}^4$$

Total

$$I = (3.355 + 4.606 + 1.206) \times 10^{12}\ \text{mm}^4$$
$$= 9.167\ \text{m}^4$$

Stresses due to lifting



$$\text{Loading due to self-weight} = 8300\ \text{kg/m} \times 9.81\ \text{N/kg} = 81420\ \text{N/m}$$

Free-body diagram of the bridge:

$$\text{Maximum bending moment, at centre} = 7.857\times10^6\times96.5 - 81420\times96.5\times48.25$$
$$= 379.1\times10^6\ \text{Nm} \qquad (= wl^2/8)$$

Longitudinal Stresses

$$\sigma_{\max} = \frac{M_{\max}y_{\max}}{I}$$

Maximum tensile stress in bottom flange — $y_{\max} = 4.177$ m

$$\sigma_{\max} = \frac{379.1\times10^6\times4.177}{9.167}$$
$$= 172.7\times10^6\ \text{N/m}^2$$

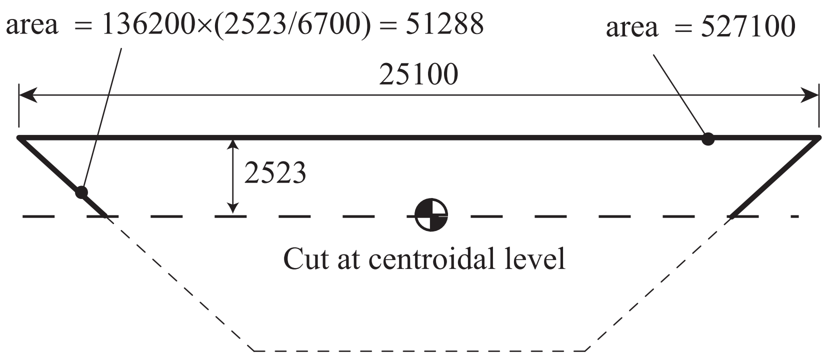
Shear Stresses

$$\text{shear force/unit length} = q = \frac{SA_s\bar{y}}{I}$$

To maximize q we should maximize S and $A_s\bar{y}$

$$S_{\max} = 7.857\times10^6\ \text{N} \quad \text{at ends of section}$$

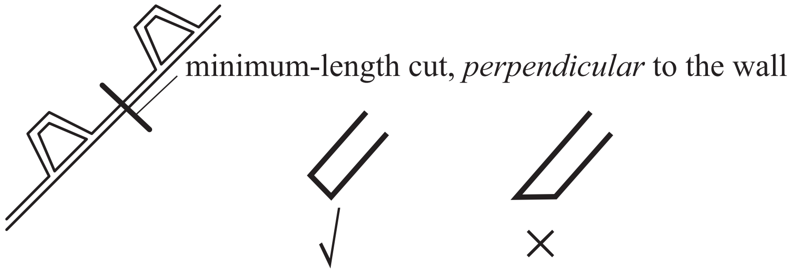
To maximize $A_s\bar{y}$ we should 'cut' the section at the centroidal level:



$$(A_s\bar{y})_{\max} = \underbrace{527100\times2523}_{\text{top flange}} + \underbrace{51288\times2}_{\text{area of two webs}} \times \underbrace{2523/2}_{\text{distance to their centroid}}$$
$$= 1.459\ \text{m}^3$$

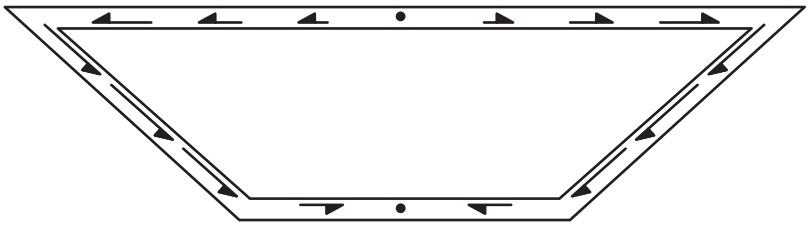
$$q_{\max} = \frac{7.857 \times 10^6 \times 1.459}{9.167} = 1.251 \times 10^6 \text{ N/m}$$

To find the maximum shear *stress*, we must cut through the wall thickness. The minimum-length cut will be perpendicular to the actual wall, hence we should use the *actual*, and not the *smeared* thickness, $t = 10$ mm $= 0.01$ m.



$$\tau_{\max} = \frac{q_{\max}}{\text{length of cut}} = \frac{1.251 \times 10^6}{\underbrace{2}_{\textit{two}\text{ webs}} \times 0.01} = 62.55 \times 10^6 \text{ N/m}^2$$

The complementary shear is perpendicular to the cut, and hence 'flows' around the section.

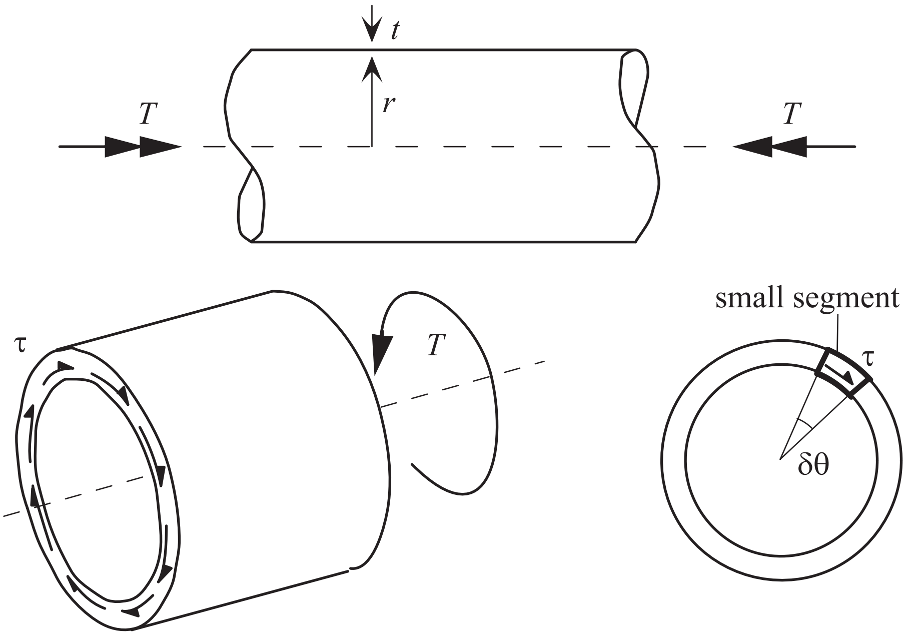


Try Questions 3 and 4, Examples Sheet 2/1

1.2.5 Torsion

Uniform thin-walled circular cylinders

The shear stress will be constant because of the axi-symmetry



Calculate the torque about the centre, δT, in equilibrium with the small segment shown

$$\delta T = \underbrace{\tau \underbrace{t r\,\delta\theta}_{\text{area}}}_{\text{force}} \times \underbrace{r}_{\text{lever arm}}$$

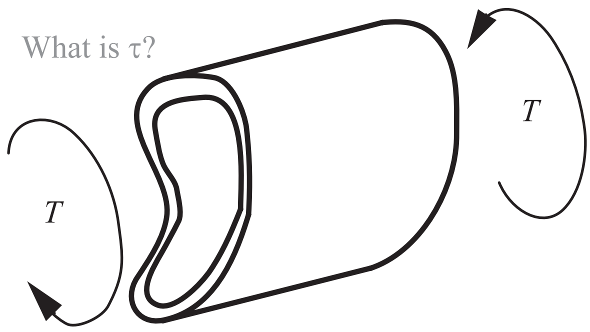
Integrate around the circle to find T

$$\begin{aligned} T &= \int_0^T dT = \int_0^{2\pi} \tau r^2 t\, d\theta \\ &= \tau r^2\, 2\pi t \end{aligned}$$

$$\tau = \frac{T}{2\pi r^2 t}$$

General thin-walled cross-section

Because of the lack of symmetry, we can no longer assume that the shear stress is constant around the section.

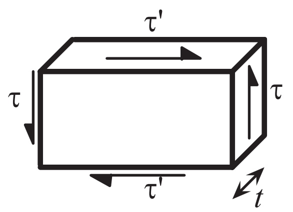


Shear Flow

For thin-walled structures, the concept of a *shear flow*, q, is useful. The shear flow is the *shear force per unit wall length* of the structure. It is the shear stress times the wall thickness:

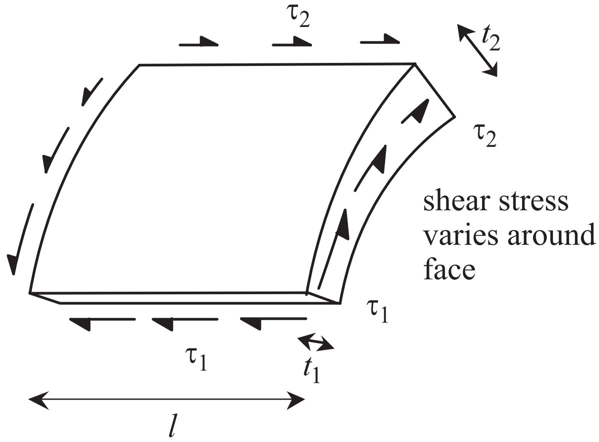
$$q = \tau \times t$$

For example, the shear flow in a thin-walled circular cylinder is $q = T/2\pi r^2$.
Just as there must always be a complementary shear stress, there must also always be a complementary shear flow.



$$\tau' = \tau \Rightarrow \tau' t = \tau t \Rightarrow q' = q$$

Consider a section of thin-walled structure, of varying thickness, in a state of pure shear (no normal stresses)



Equilibrium $\leftrightarrow$

$$\tau_2 t_2 l = \tau_1 t_1 l$$

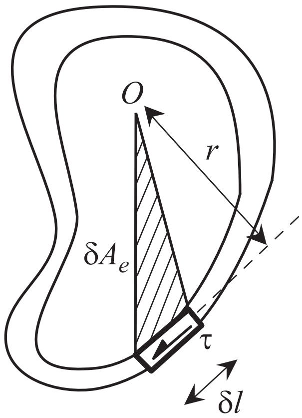
but

$$q_1 = \tau_1 t_1, \quad q_2 = \tau_2 t_2,$$

$$\therefore q_1 = q_2$$

$\boxed{\text{Shear flow due to applied torque is constant around a thin-walled section}}$

What applied torque, T, is in equilibrium with this constant shear flow? Consider the torque in equilibrium with a small element of the wall:



$$\delta T = \underbrace{\tau \underbrace{t\delta l}_{\text{area}}}_{\text{force}} \times \underbrace{r}_{\text{lever arm}}$$

$$\begin{aligned} \text{but} \quad & \tau t = q = \text{constant} \\ \text{and} \quad & \delta l r = 2\delta A_e \quad \text{— the shaded area enclosed to \underline{mid-thickness}} \\ \therefore \; & \delta T = 2q\delta A_e \end{aligned}$$

Integrate around the section

$$T = \oint dT = \oint 2q\, dA_e$$

q is constant, $\oint dA_e = A_e$,

$$T = 2qA_e$$

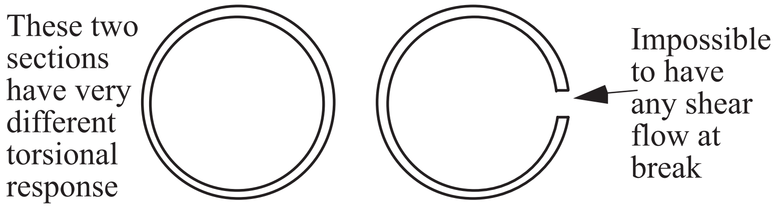
Thus, for an arbitrary thin-walled closed section subject to an applied torque,

$$\boxed{q = \frac{T}{2A_e}}$$

or, as $q = \tau \times t$

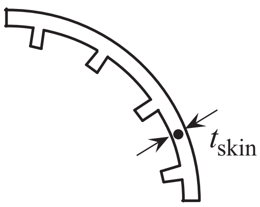
$$\boxed{\tau = \frac{T}{2A_e t}}$$

Important: These formulae only apply to *closed sections*:



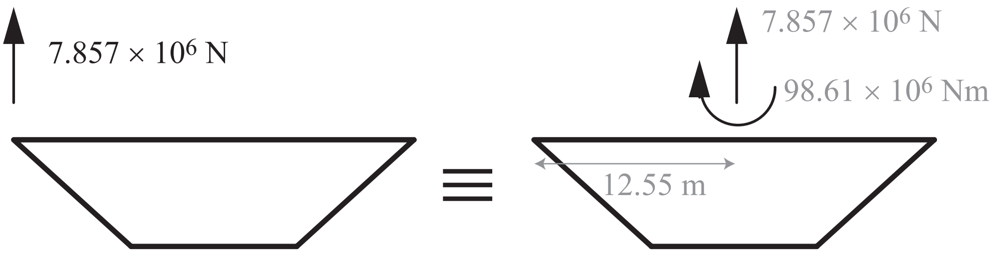
Effect of stiffeners

The stiffeners will not affect A_e , the area enclosed by the section, and hence the shear flow q will be unaffected. Locally, at a point as shown below, the shear stress $\tau = q/t_{\text{skin}}$ will be unaffected by the stiffeners.



1.2.6 Case study part 2 — stresses due to twisting of a Storebælt approach span

As a worst case, consider what would happen if one end of the span was lifted by only one corner.



$$q = \frac{T}{2A_e}$$

$$T = 98.61 \times 10^6 \text{ Nm}$$
$$A_e = \frac{25.1 + 11}{2} \times 6.7 = 120.9 \text{ m}^2$$

$$\therefore q = 407.8 \times 10^3 \text{ N/m}$$

To find the maximum shear *stress*, cut at the thinnest section ($\tau = q/t$). In the web, $t = 10$ mm $=$ 0.01 m,

$$\tau = 40.78 \times 10^6 \text{ N/m}^2$$

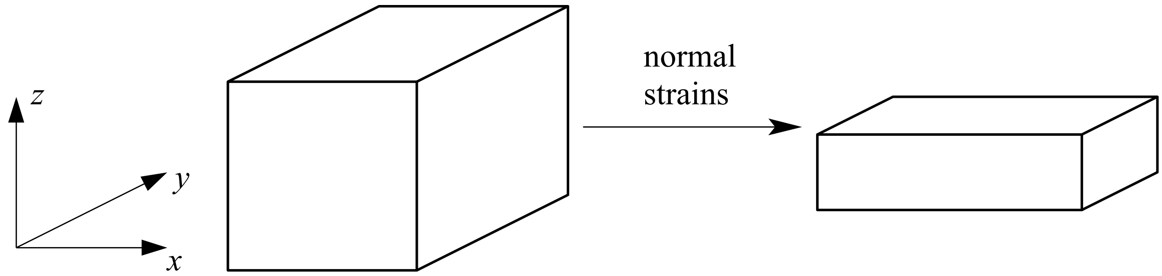
n.b. this is due to torque alone — a complete answer requires us to also superimpose stresses due to the shear force. In the webs, the stresses will reinforce in one web, and tend to cancel in the other web.

1.3 Strain in 3-dimensions

We assume in the definitions in this section that the strains are small, of the order of 10^{-3}. Note that strains are dimensionless.

1.3.1 Normal strain

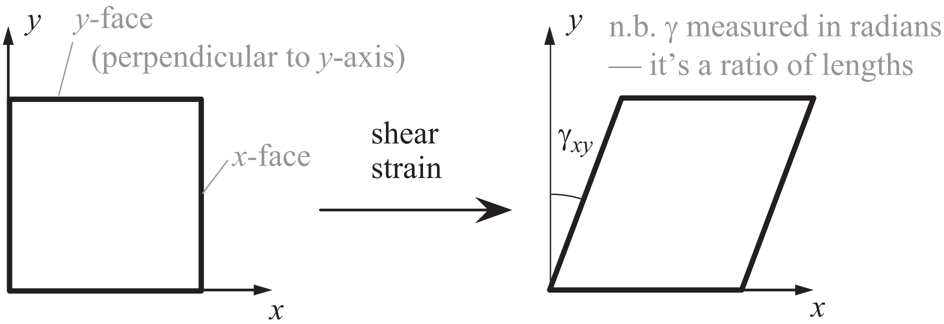
You are already familiar with one-dimensional strain as being extension/original length. In 3 dimensions, there are three such strains, in three orthogonal directions, called the *normal strains*. A cube, subject to normal strains in directions perpendicular to its faces, will become some general brick shape. The angle between every face will remain at 90°.



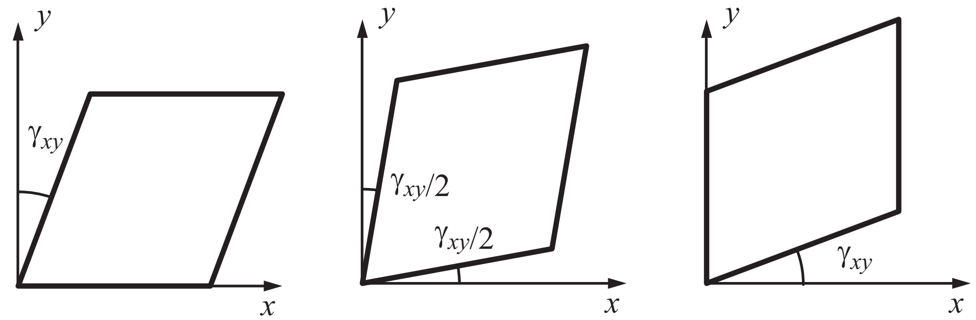
The normal strains in the x, y, z directions are called, respectively, ε_{xx}, ε_{yy}, and ε_{zz}.

1.3.2 Shear strain

In addition to normal strains there are also *shear strains*, e.g. in two dimensions:

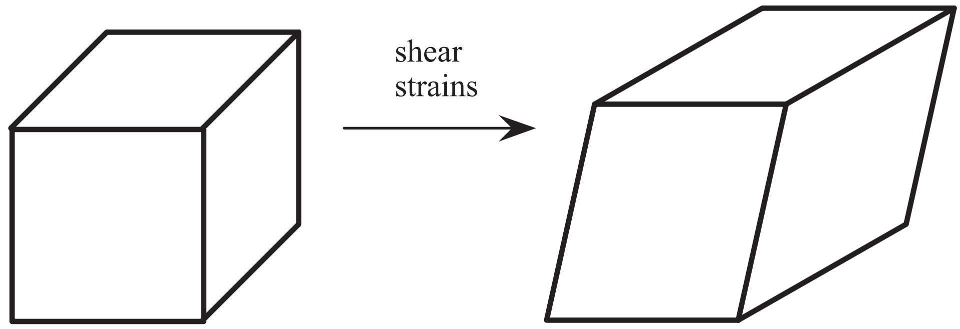


For small strains, the shear strain γ_{xy} is the change in angle between faces that were originally perpendicular, in this example the x and y faces. Note that the same shear strain could have drawn in a number of ways, by superposing a rotation (a rigid body motion which causes *no* strain). Obviously, $\gamma_{yx} = \gamma_{xy}$.



Equal shear strains

In three dimensions, there will be three shear strains, γ_{xy}, γ_{yz}, γ_{zx}. A cube, subjected only to shear strains, will become a parallelepiped, and the lengths of its sides will be unchanged.

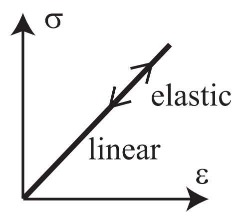


1.4 Stress-strain relationships

Real material behaviour is complex, but the key features can often be captured in a simple model. We will use a linear-elastic, homogeneous, isotropic, time-independent model. What do these terms mean?

1.4.1 Material model

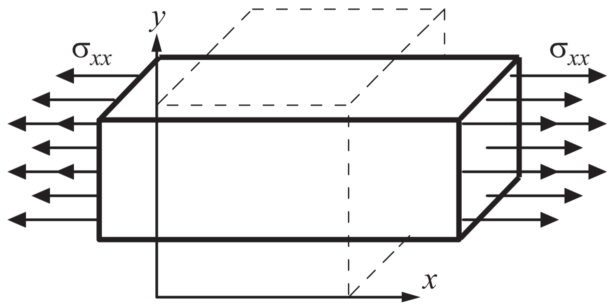
Linear-elastic



elastic — deformation disappears when load is removed.
linear — stress-strain curve is a straight line.
Homogeneous Doesn't vary with position.
Isotropic Doesn't vary with direction (*not* true for e.g. fibre-reinforced materials.)

Time-independent No creep.

1.4.2 Normal strain due to uniaxial normal stress



E = Young's Modulus, ν = Poisson's Ratio

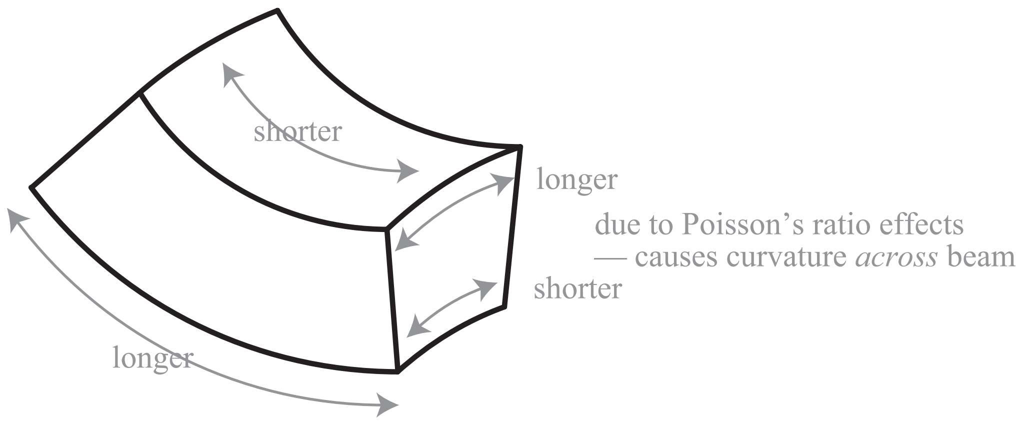
$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$$

$$\begin{aligned}\varepsilon_{yy} = \varepsilon_{zz} &= -\nu\varepsilon_{xx} \\ &= -\nu\left(\frac{\sigma_{xx}}{E}\right)\end{aligned}$$

Typical values for E: Steel, 210×10^9 N/m^2; Aluminium alloy, 70×10^9 N/m^2; Rubber, 0.1×10^9 N/m^2.
Typical values for ν: Steel, 0.3; Aluminium alloy, 0.33; Cork, ≈ 0.

Example: Anticlastic curvature

Bend an eraser between your fingers — note the *anticlastic* curvature:



1.4.3 Normal strain due to temperature change

An unrestrained body subject to a temperature rise will undergo a uniform expansion, without changing shape. Thus the body undergoes a uniform normal strain in all directions, and no shear strain. The normal strain is (approximately) linear with change in temperature ΔT.

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \alpha\,\Delta T$$

α is the coefficient of linear expansion.

Typical values for α: Steel $11 \times 10^{-6}\ \mathrm{K}^{-1}$; Aluminium Alloy, $20 \times 10^{-6}\ \mathrm{K}^{-1}$.

1.4.4 Normal strain due to 3D normal stress

The strain due to normal stress in three perpendicular directions can be found by superposition:

$$\varepsilon_{xx} = \underbrace{\frac{1}{E}\sigma_{xx}}_{\text{due to } \sigma_{xx}} - \underbrace{\nu\frac{1}{E}\sigma_{yy}}_{\text{due to } \sigma_{yy}} - \underbrace{\nu\frac{1}{E}\sigma_{zz}}_{\text{due to } \sigma_{zz}}$$

If a temperature change is also superposed, then we get the expression on page 1 of the data book:

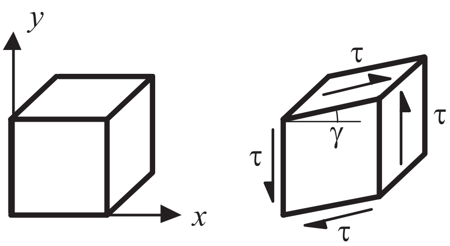
$$\boxed{\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}) + \alpha\Delta T}$$

with similar expressions in the other directions.

In an isotropic material, there is never any *shear* strain due to normal stress and temperature change alone

1.4.5 Shear strain

A shear stress on the x-face acting in the y direction (and its complementary stress on the y-face in the x direction) will cause a shear strain in the x-y plane, but no shear strains on the y-z or z-x planes.



$$\boxed{\gamma_{xy} = \frac{1}{G}\tau_{xy}}$$

G is the shear modulus. G is related to E and ν by the formula $G = E/2(1+\nu)$ (proved in Examples sheet 2/2).

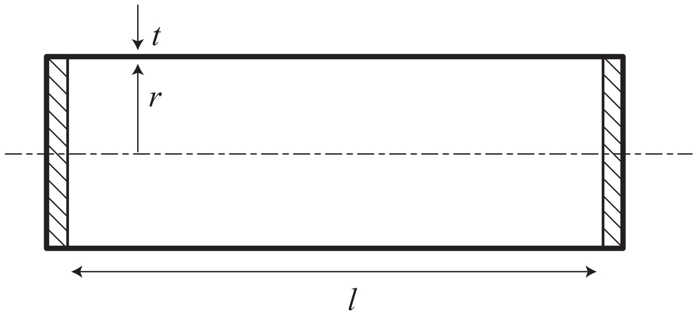
In an isotropic material, a shear stress will cause no normal strain in any direction,

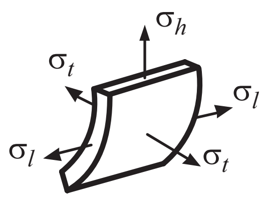
$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 0$$

For an isotropic material:

- Shear stress only causes shear strain in its own plane;
- Normal stress only causes normal strain

Example 1: Change of dimension of a closed pressurised cylinder





Stresses

$$\sigma_t \approx 0, \quad \sigma_l = \frac{pr}{2t}, \quad \sigma_h = \frac{pr}{t}$$

Strains

Longitudinal

$$\begin{aligned}\varepsilon_l &= \frac{1}{E}(\sigma_l - \nu\sigma_h - \nu\sigma_t) \\ &= \frac{pr}{2Et}(1-2\nu)\end{aligned}$$

Hoop

$$\begin{aligned}\varepsilon_h &= \frac{1}{E}(\sigma_h - \nu\sigma_l - \nu\sigma_t) \\ &= \frac{pr}{2Et}(2-\nu)\end{aligned}$$

Through-wall

$$\begin{aligned}\varepsilon_t &= \frac{1}{E}(\sigma_t - \nu\sigma_h - \nu\sigma_l) \\ &= -\frac{pr}{2Et}(3\nu) \quad \text{— becomes thinner}\end{aligned}$$

Note: zero stress in a direction *doesn't* imply zero strain.

Change in dimension

Length

$$\frac{\Delta l}{l} = \varepsilon_l, \qquad \Delta l = \varepsilon_l l$$

Circumference

$$\frac{\Delta C}{C} = \varepsilon_h, \qquad \Delta C = \varepsilon_h C$$

Radius — *Note that this is nothing to do with the strain in the through-wall direction, ε_t!*

$$\frac{\Delta r}{r} = \frac{2\pi\Delta r}{2\pi r} = \frac{\Delta C}{C} = \varepsilon_h, \qquad \Delta r = \varepsilon_h r$$

Change in volume

Define a *Volumetric Strain* as $\dfrac{\Delta V}{V}$.

Volume enclosed by cylinder, $V = \pi r^2 l$

$$\frac{\partial V}{\partial l} = \pi r^2, \quad \frac{\partial V}{\partial r} = 2\pi r l$$

For small variations around the current position,

$$\Delta V = \frac{\partial V}{\partial l}\Delta l + \frac{\partial V}{\partial r}\Delta r = \pi r^2\,\Delta l + 2\pi r l\,\Delta r$$

$$\begin{aligned}\text{Volumetric strain} = \frac{\Delta V}{V} &= \frac{\Delta l}{l} + \frac{2\,\Delta r}{r} \\ &= \varepsilon_l + 2\varepsilon_h = \frac{pr}{2Et}(5-4\nu)\end{aligned}$$

$$\Delta V = \frac{pr}{2Et}(5-4\nu) \times \pi r^2 l = \frac{p\pi r^3 l}{2Et}(5-4\nu)$$

Volume of metal in wall, $V_m = 2\pi r t l$

$$\frac{\partial V_m}{\partial l} = 2\pi r t, \quad \frac{\partial V_m}{\partial r} = 2\pi t l, \quad \frac{\partial V_m}{\partial t} = 2\pi r l$$

For small variations around the current position,

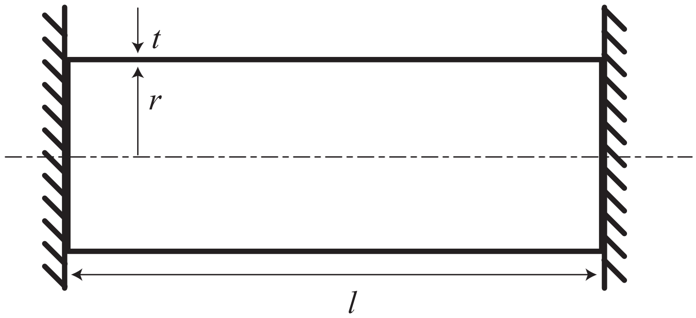
$$\Delta V_m = 2\pi r t\,\Delta l + 2\pi t l\,\Delta r + 2\pi r l\,\Delta t$$

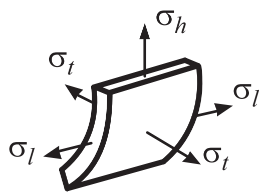
$$\begin{aligned}\text{Volumetric strain} = \frac{\Delta V_m}{V_m} &= \frac{\Delta l}{l} + \frac{\Delta r}{r} + \frac{\Delta t}{t} \\ &= \varepsilon_l + \varepsilon_h + \varepsilon_t = \frac{pr}{2Et}(3-6\nu)\end{aligned}$$

$$\Delta V_m = \frac{pr}{2Et}(3-6\nu) \times 2\pi r t l = \frac{p\pi r^2 l}{E}(3-6\nu)$$

Example 2: Thermal Stress in a Restrained Cylinder

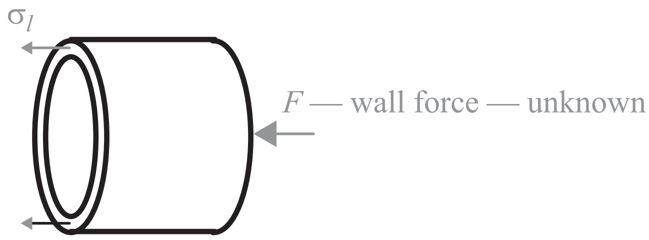
A cylinder is initially unstressed, and just fits between two rigid walls. The temperature of the cylinder then increases by 100°C. Calculate the stresses in the cylinder





In the longitudinal direction this is a *statically indeterminate* problem. There are two unknowns, and only one equation of equilibrium, and so is impossible to calculate the stresses from equilibrium alone. (There will be much more about statically indeterminate problems later.)

Free-body diagram



The key to statically indeterminate problems is to use *compatibility*. In this case, we know that there is zero longitudinal *strain* — the cylinder cannot change in length.

$$\varepsilon_l = \frac{1}{E}(\sigma_l - \nu\sigma_h - \nu\sigma_t) + \alpha\Delta T = 0$$

$\sigma_h = 0$ — no internal pressure

$$\therefore \sigma_l = -E\alpha\Delta T$$

e.g. for Al-alloy, $E = 70 \times 10^9$ N/m^2, $\quad \alpha = 22 \times 10^{-6}$

$$\Rightarrow \sigma_l = -154 \times 10^6 \text{ N/m}^2$$

- Could cause a weak alloy to yield.
- Would cause a thin cylinder ($r/t \approx 1000$) to buckle.

The skin of Concorde (Al-alloy) heated up by approximately 100°C during flight — this would cause problems if the outer skin was rigidly connected to the interior of the aircraft!

Try Questions 5, 6 and 7, Examples Sheet 2/1