Lagrangian Dynamics

Part IB

Mechanical Engineering

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This is a brand new course

New handout, new examples sheets, new lab…

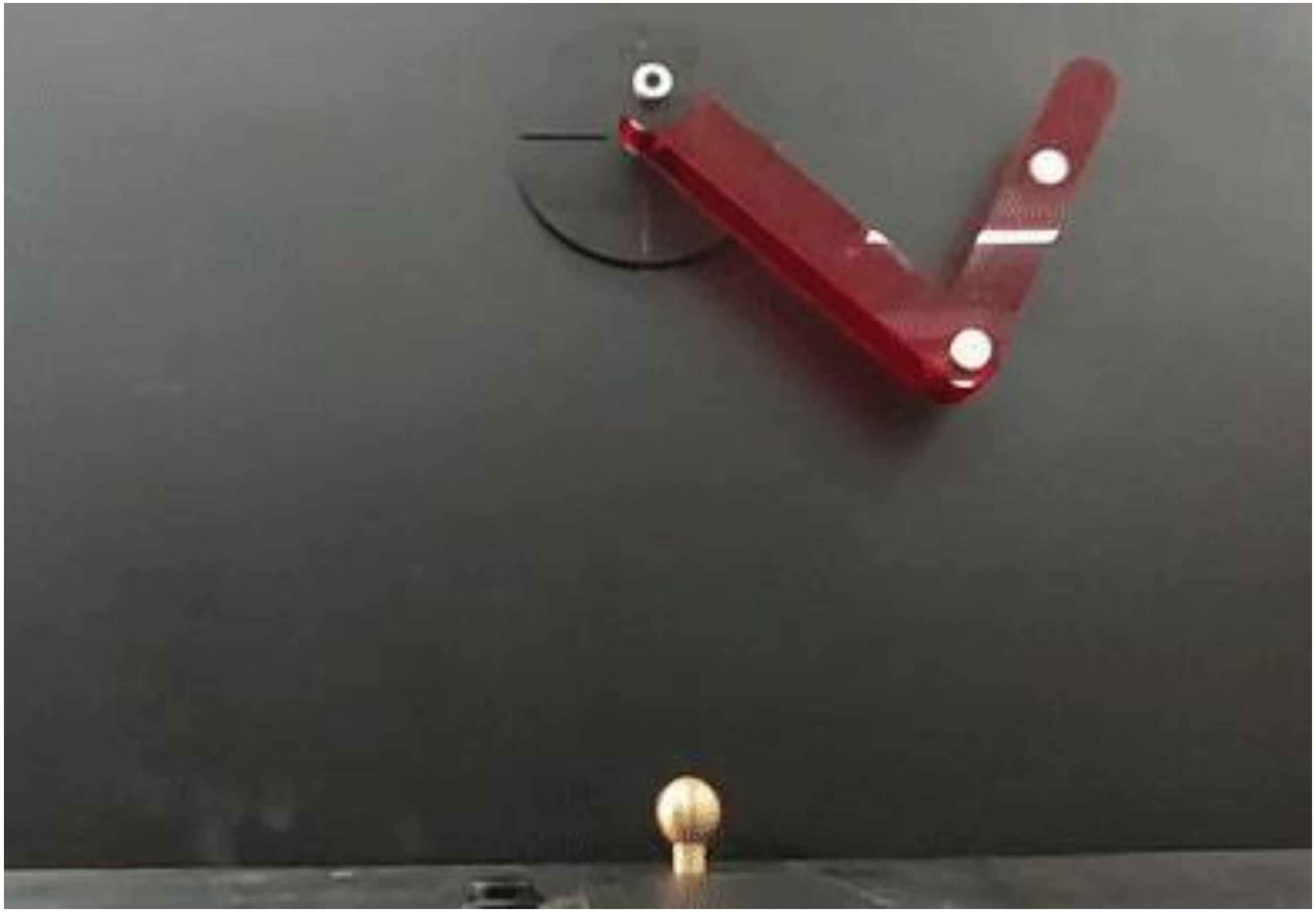
There are no IB past tripos papers, but

- Extra revision questions on examples sheets
- Sample paper issued at the end of the course
- Lots of suitable tripos questions from 3C5
- Lots of suitable books full of questions

Send typos, errors and general feedback to jsb56.

Next eight lectures:

Predicting motion of complex mechanical systems



Next eight lectures:

Predicting motion of complex mechanical systems

Three step plan.....

(1) Choose variables (*"coordinates"*) that describe the system's configuration.

Easy - next ten minutes

(2) Find *equations of motion* for these variables

New approach, $F = ma \quad \rightarrow \quad$ Lagrange's equations.

Lectures 9-11

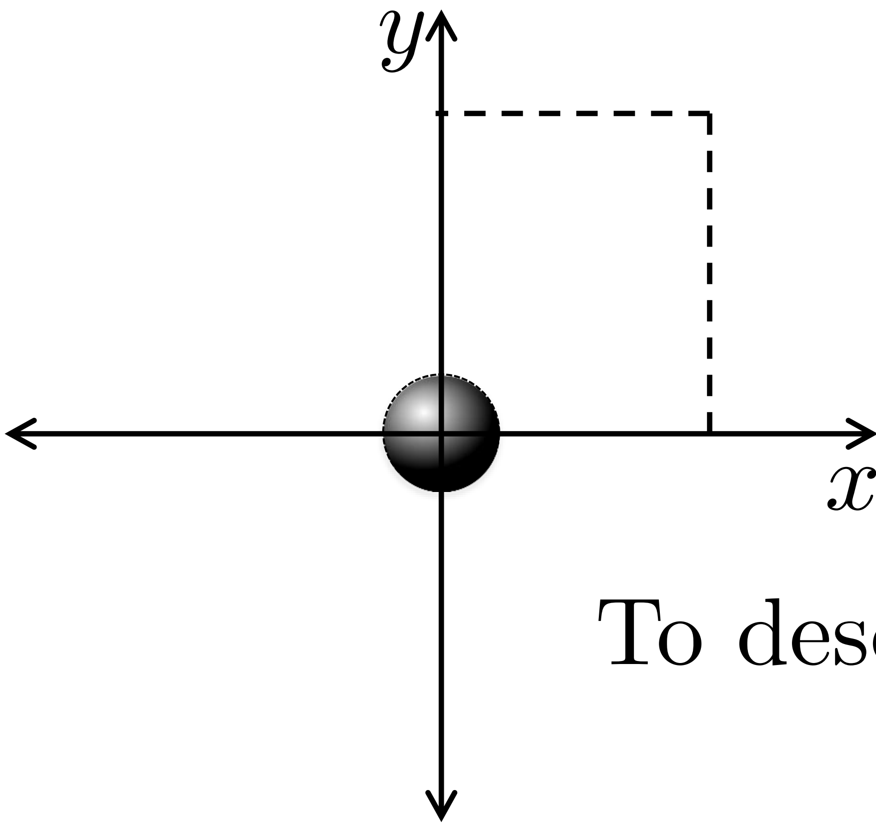
(3) Solve the equations to predict the motion.

Lectures 11-16

(1) Choose variables (*"coordinates"*) that describe the system's configuration.

To describe position, we need

$$x, y$$

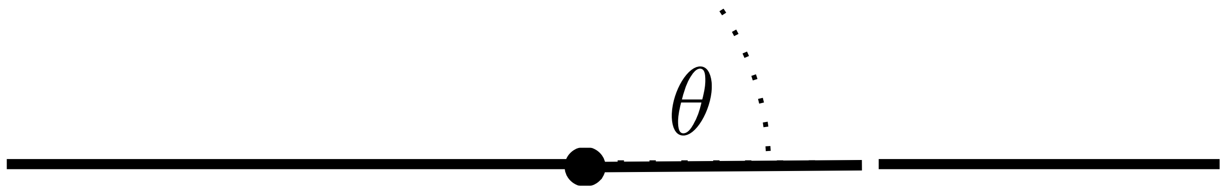


To describe motion we need

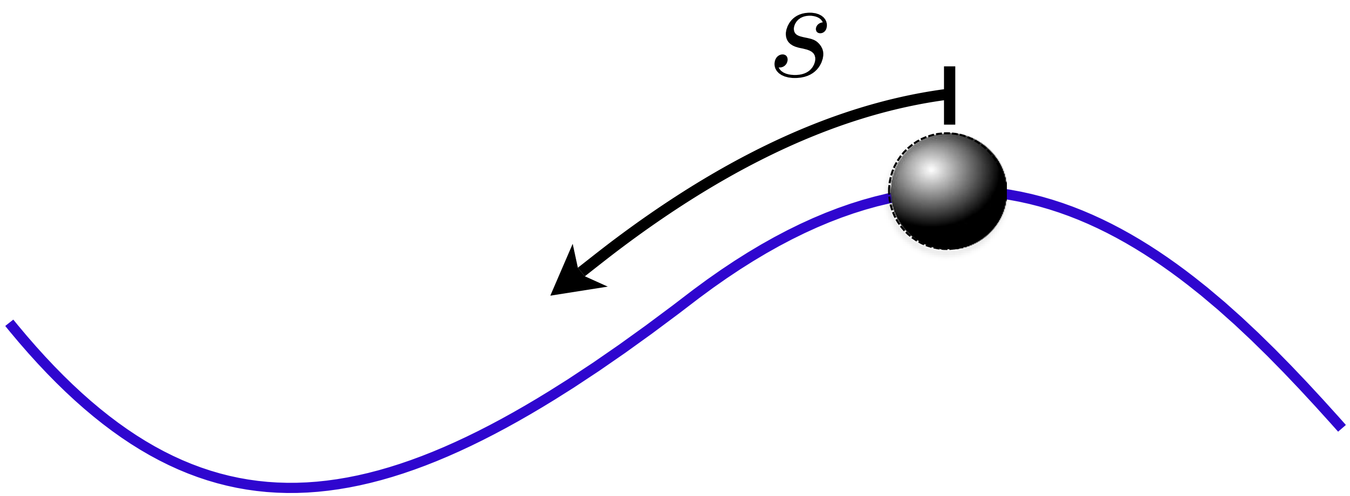
$$x(t), y(t)$$

Two coordinates

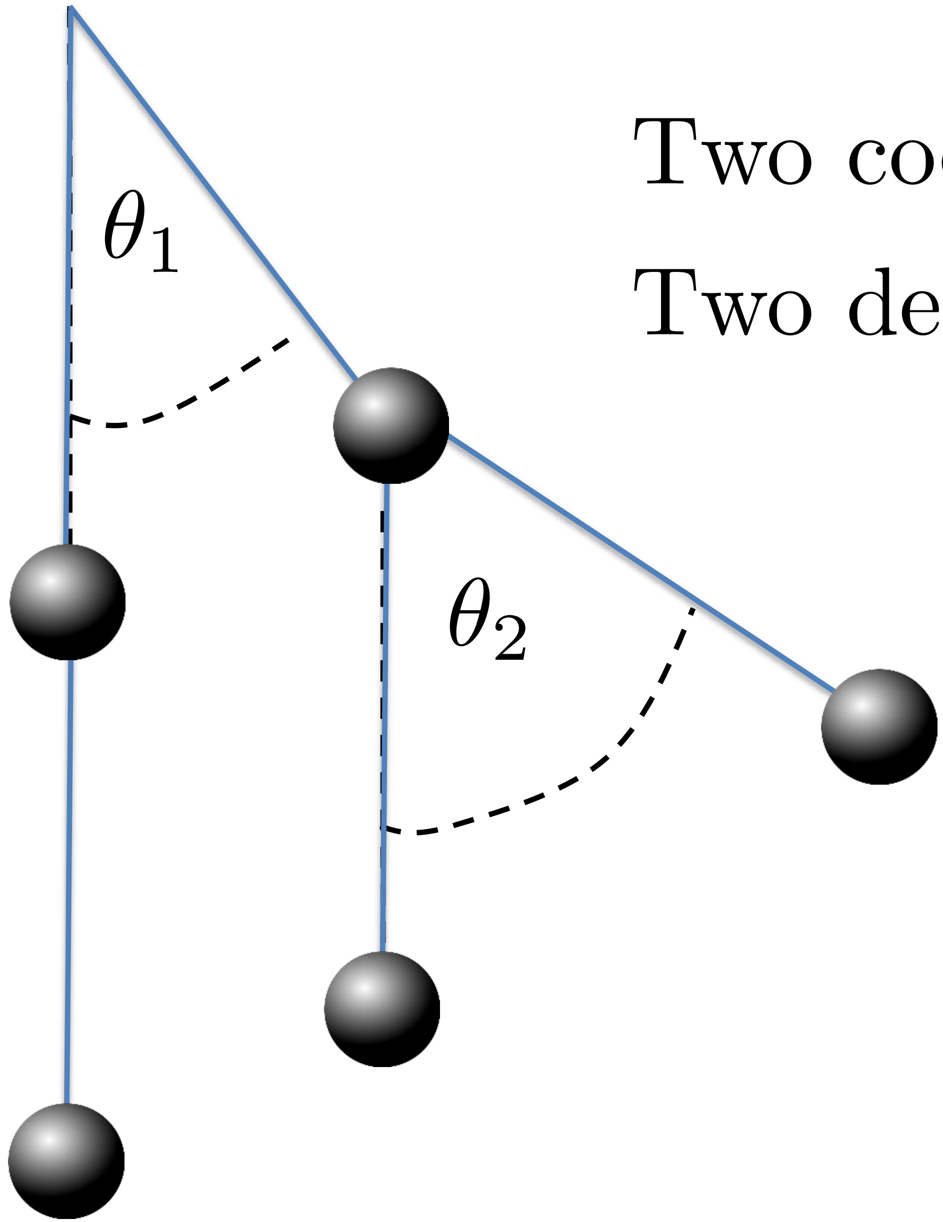
$\rightarrow$ two *degrees of freedom*



one coordinate θ
one degree of freedom



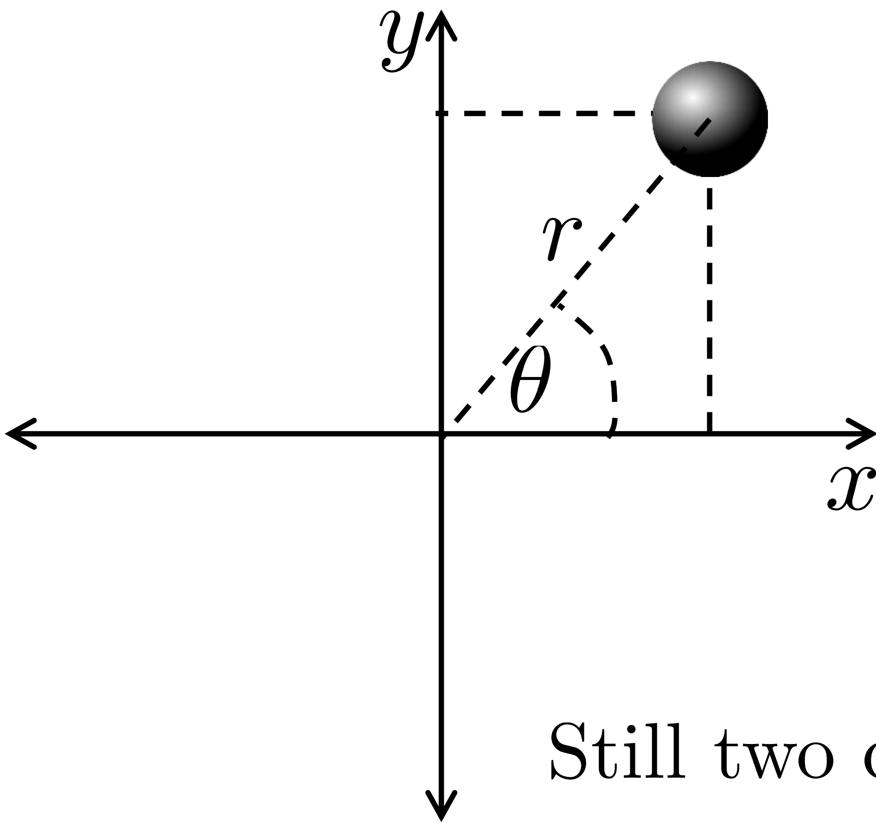
one coordinate s
one degree of freedom



Two coordinates θ_1, θ_2

Two degrees of freedom.

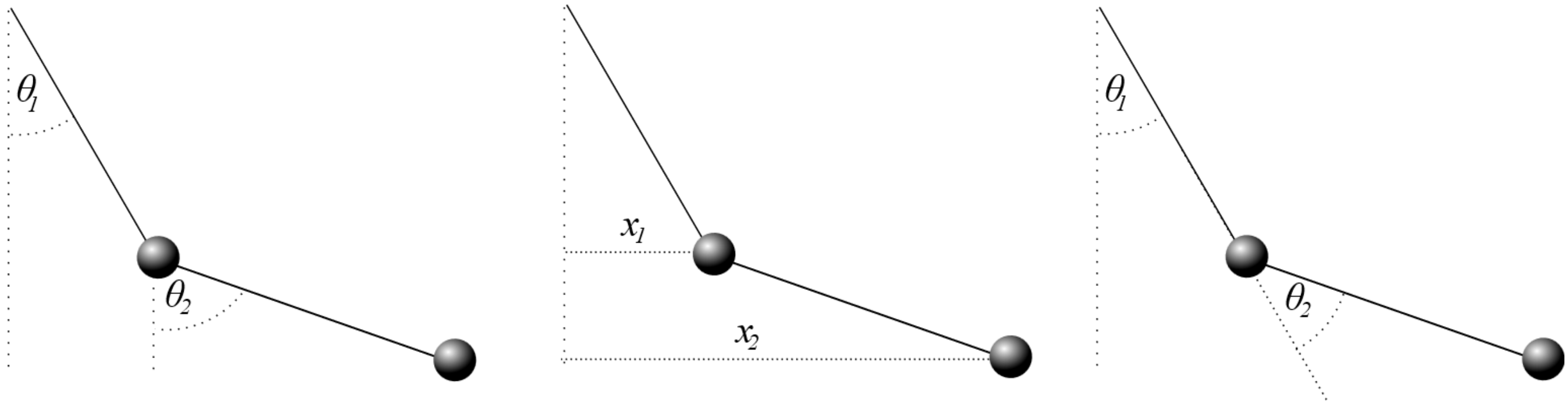
Now using (r, θ) rather than (x, y)



Still two coordinates.

as still two degrees of freedom.

Three sensible coordinate choices for the double pendulum



But always need to specify two quantities.

In general, we will need n variables for an n degree of freedom mechanism.

$$\{q_i\} = (q_1, q_2 \ldots q_n)$$

The q_i are called *generalized coordinates*.

Might be angles, positions, or something else entirely.

(2) Find *equations of motion* for these variables

Previous method: Newton's Second Law

All matter is a bunch of particles.

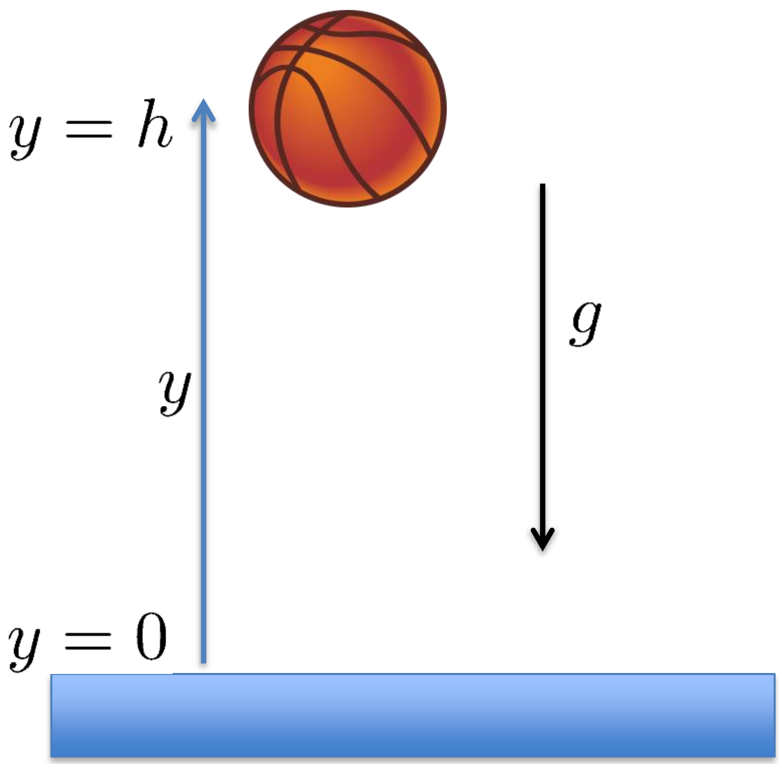
Each particle has an equation of motion:

$$F_x = m\ddot{x}$$

$$\mathbf{F} = m\mathbf{a}$$

Integrate twice, $\ddot{x} \to \dot{x} \to x$, and problem solved!

Example: Dropping a ball



$$F_y = -\cancel{m}g = \cancel{m}\ddot{y}$$

$$\ddot{y} = -g$$

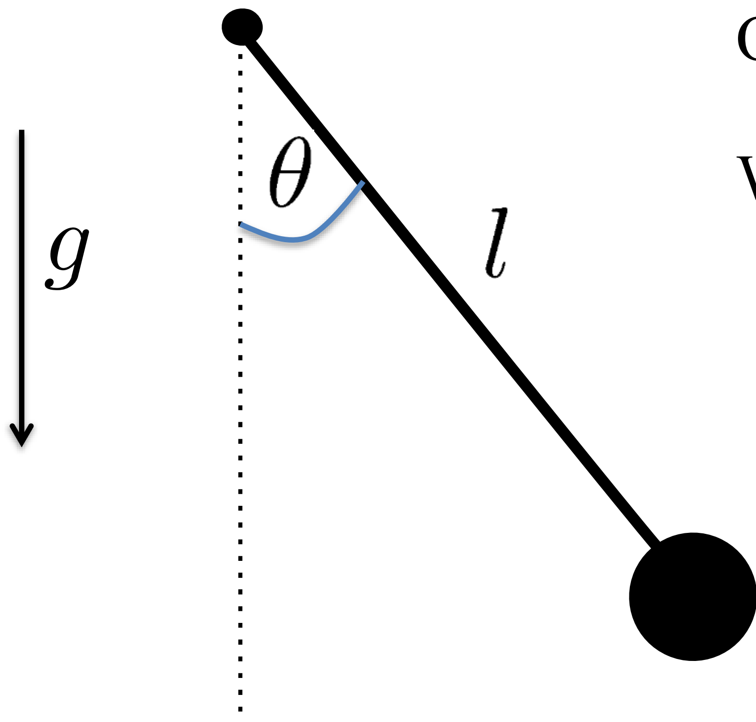
$$\dot{y} = -gt$$

$$y = h - \frac{1}{2}gt^2$$

Done!

But much harder with constrained particles and non-Cartesian coordinates

Simple pendulum



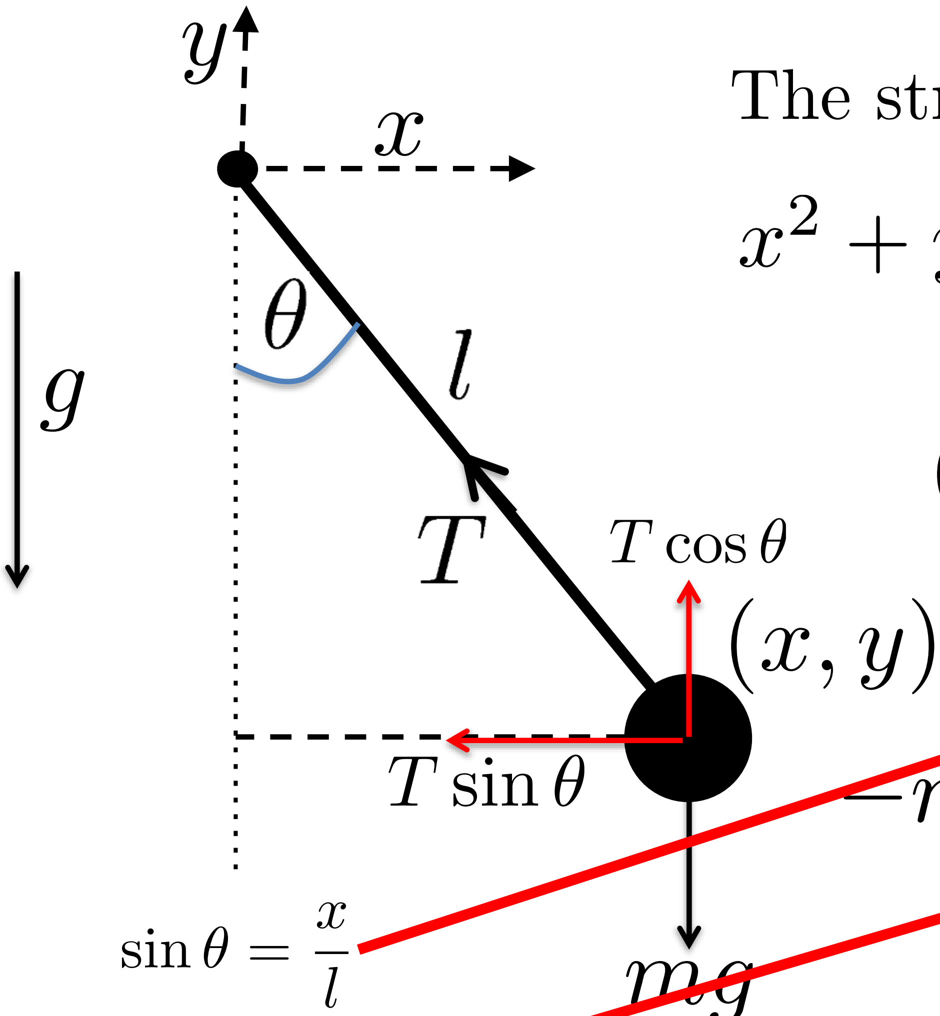
One degree of freedom,
One coordinate θ

We all know the equation of motion is

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

but how do we derive this?

Simple pendulum: Cartesians



The string enforces a constraint

$$x^2 + y^2 = l^2$$

Newton's second law

(in $x - y$ Cartesians)

$$F_x = m\ddot{x}$$

$$-T\sin\theta = m\ddot{x}$$

$$-T\frac{x}{l} = m\ddot{x}$$

$$F_y = m\ddot{y}$$

$$-mg + T\cos\theta = m\ddot{y}$$

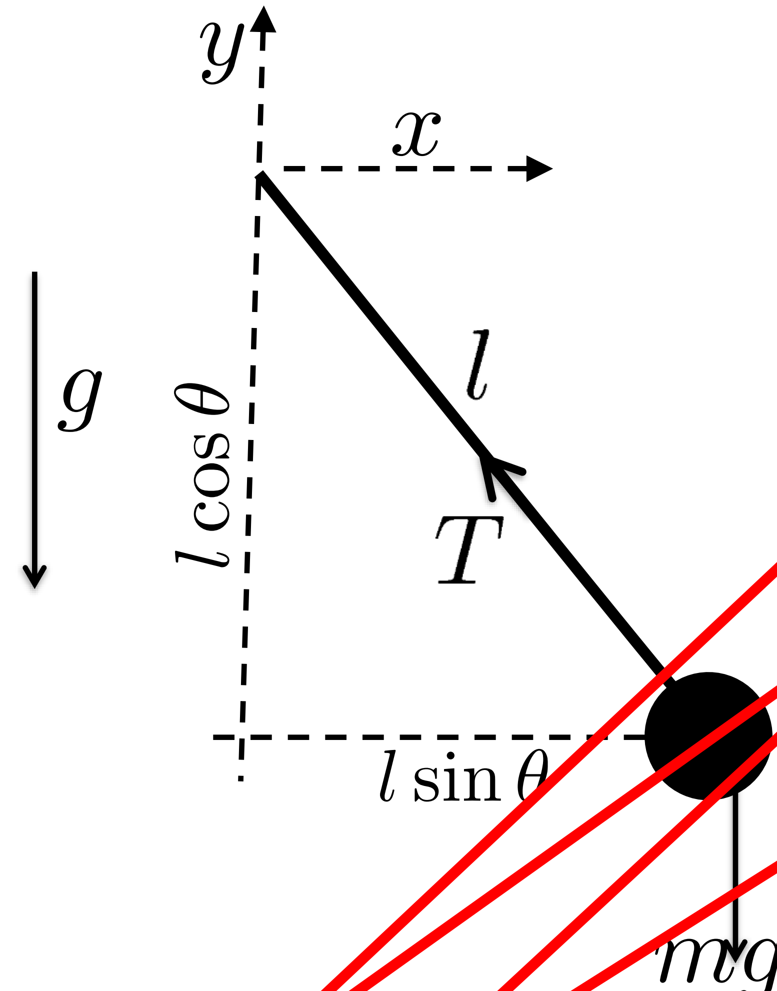
$$-mg - T\frac{y}{l} = m\ddot{y}$$

$$\sin\theta = \frac{x}{l}$$

$$\cos\theta = -\frac{y}{l}$$

Not much help.

Simple pendulum: Cartesians



Three equations.

$$x^2 + y^2 = l^2$$

$$-T\frac{x}{l} = m\ddot{x}$$

$$-mg - T\frac{y}{l} = m\ddot{y}$$

Three unknowns:

$$x(t),\ y(t),\ T(t)$$

$$\boxed{\begin{aligned}\ddot{\theta} &= -\frac{g}{l}\sin\theta \\ T &= mg\cos\theta - ml\dot{\theta}^2\end{aligned}}$$

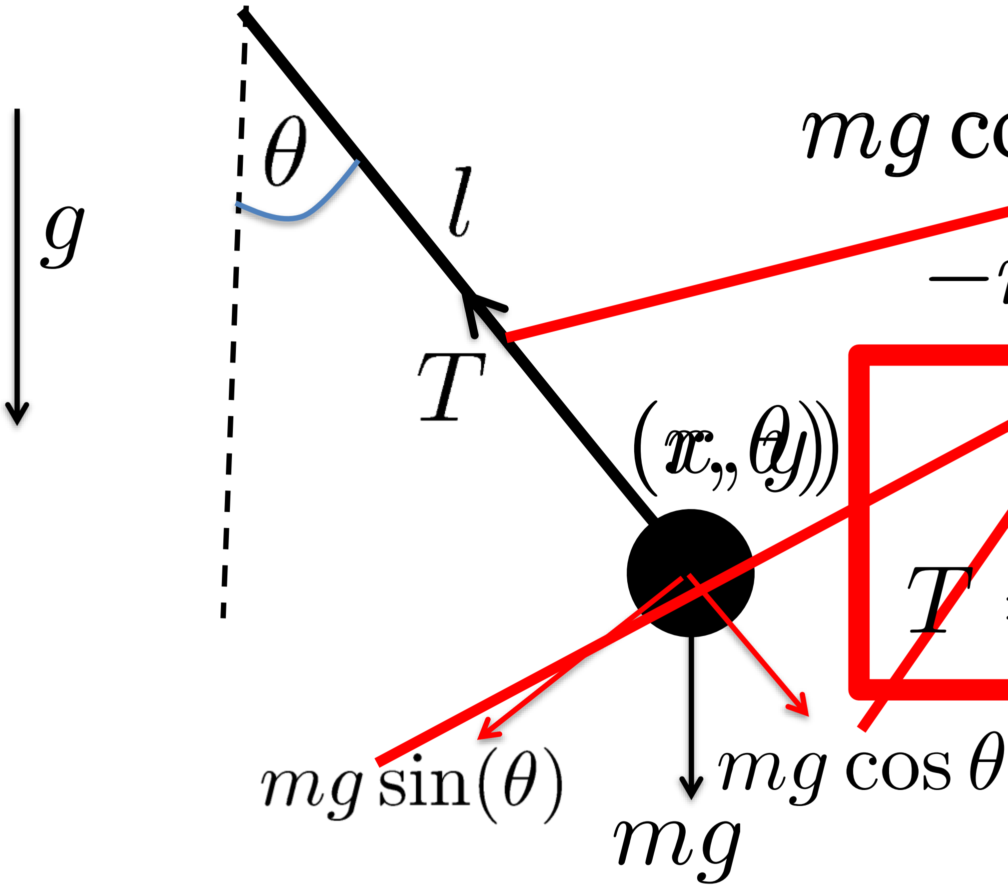
$$x = l\sin\theta \qquad \dot{x} = l\dot{\theta}\cos\theta \qquad \ddot{x} = l(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$

$$y = -l\cos\theta \qquad \dot{y} = l\dot{\theta}\sin\theta \qquad \ddot{y} = l(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$$

Simple pendulum: polars

The string enforces a constraint

$$r = l$$



$$F_r = m a_r$$

$$mg\cos\theta - T = m l\dot{\theta}^2$$

$$F_\theta = m a_\theta$$

$$-mg\sin\theta = m l\ddot{\theta}$$

$$a_r = \cancel{\ddot{r}} - r\dot{\theta}^2 = -l\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + \cancel{2\dot{r}\dot{\theta}} = l\ddot{\theta}$$

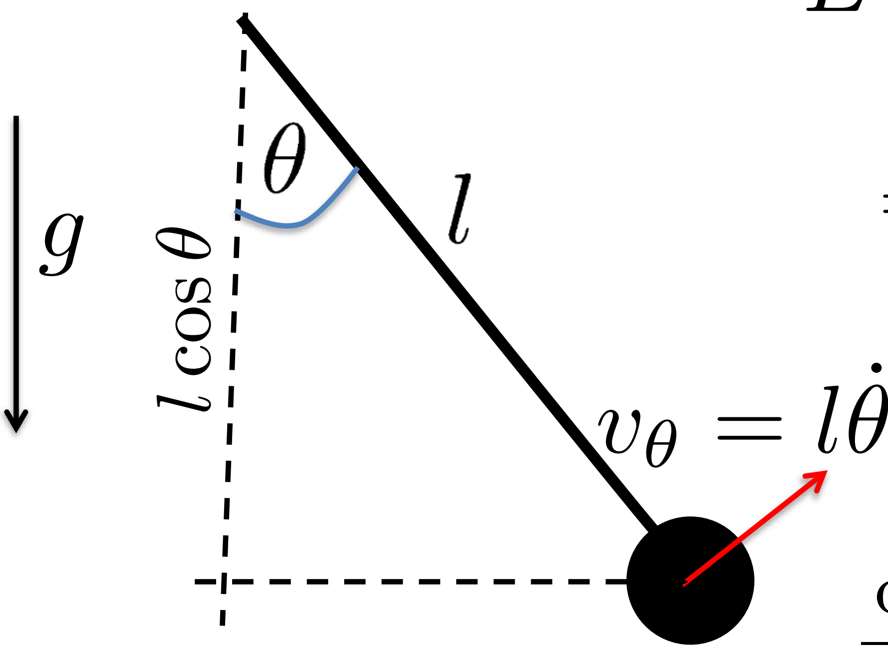
$$\boxed{\begin{aligned}\ddot{\theta} &= -\frac{g}{l}\sin\theta \\ T &= mg\cos\theta - ml\dot{\theta}^2\end{aligned}}$$

✓

Good, but relies on spotting a convenient coordinate system where the constraint is trivial.

Also, half the work is going into finding *T*, which we don't actually care about.

Simple pendulum: Energy



$$E = T + V$$

$$E = \frac{1}{2}mv^2 + mgy$$

$$= \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

Conservation of energy

$$\dot{E} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta\right) = 0$$

The energy method has several advantages.

Quick

No worrying about tension

Direct to eqn of motion for θ

Also, no worrying about vectors and signs

Example 2: Ladder sliding down a wall

$$F_x = m\ddot{x}$$

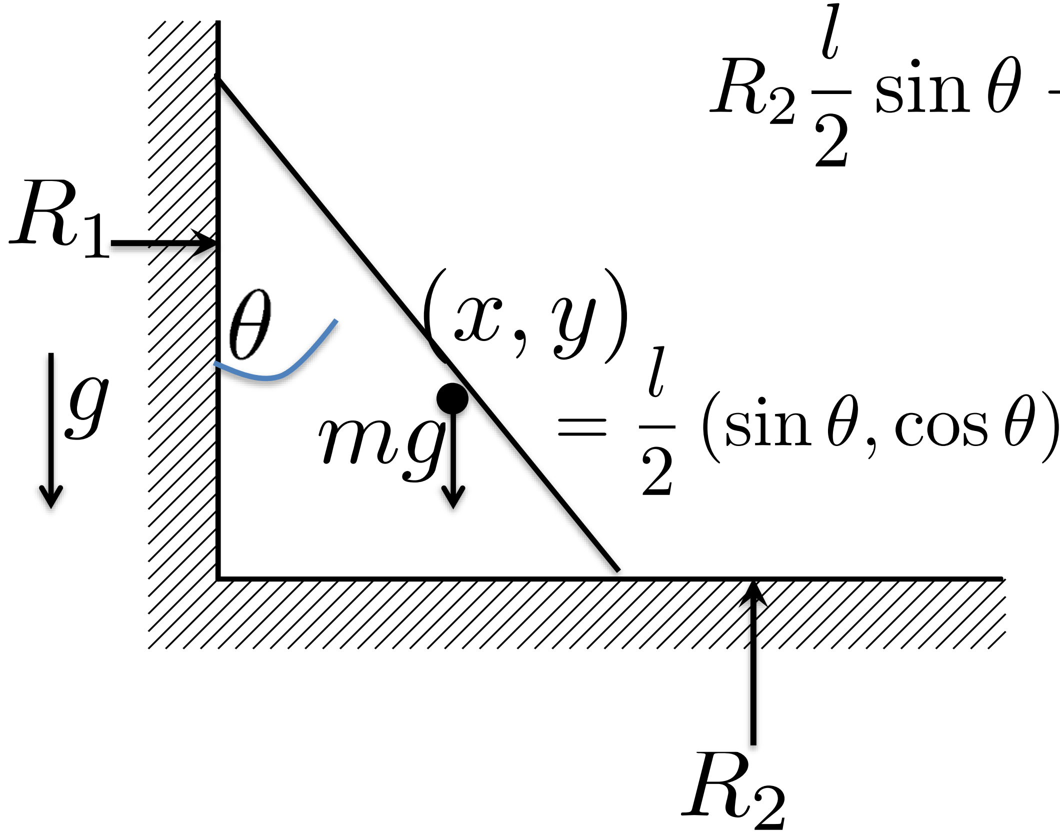
$$R_1 = m\ddot{x}$$

$$F_y = m\ddot{y}$$

$$R_2 - mg = m\ddot{y}$$

$$G = I\ddot{\theta}$$

$$R_2 \frac{l}{2}\sin\theta - R_1 \frac{l}{2}\cos\theta = \frac{1}{12} m l^2 \ddot{\theta}$$



⋮

3 eqns, 3 unknowns
R_1, R_2 and θ

⋮

$$\ddot{\theta} = \frac{3g\sin\theta}{2l}.$$

Example 2: Ladder sliding down a wall

$$E = T + V$$

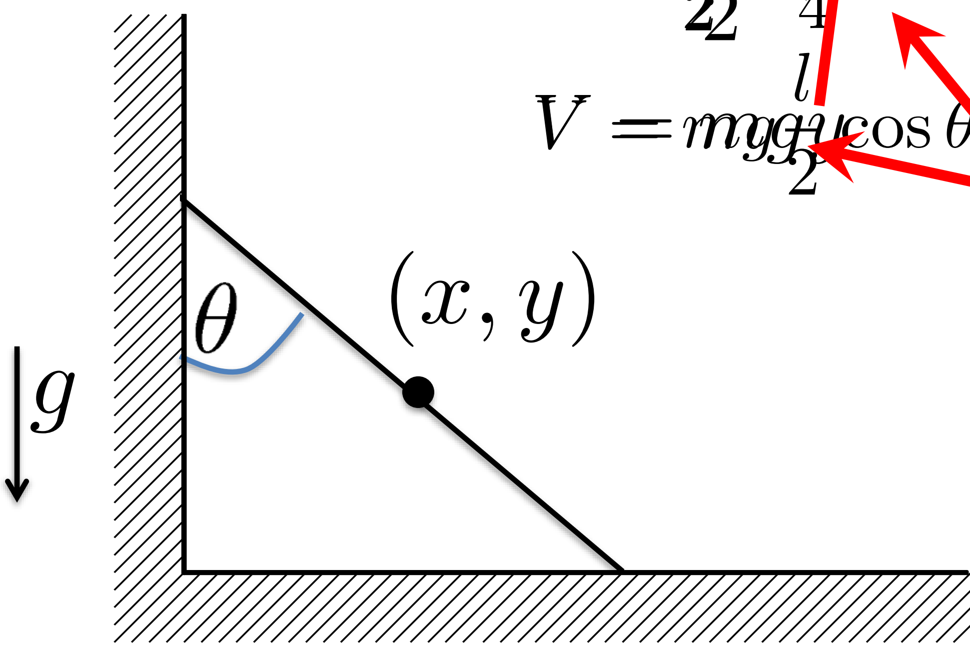
$$E = \frac{1}{6} m l^2 \dot{\theta}^2 + mg\frac{l}{2}\cos\theta$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta}^2$$

$$T = \frac{1}{2} m \frac{l^2}{4} \dot{\theta}^2 + \frac{1}{2}\frac{1}{12} m l^2 \dot{\theta}^2 = \frac{1}{6} m l^2 \dot{\theta}^2$$

$$V = mgy$$

$$V = mg\frac{l}{2}\cos\theta$$



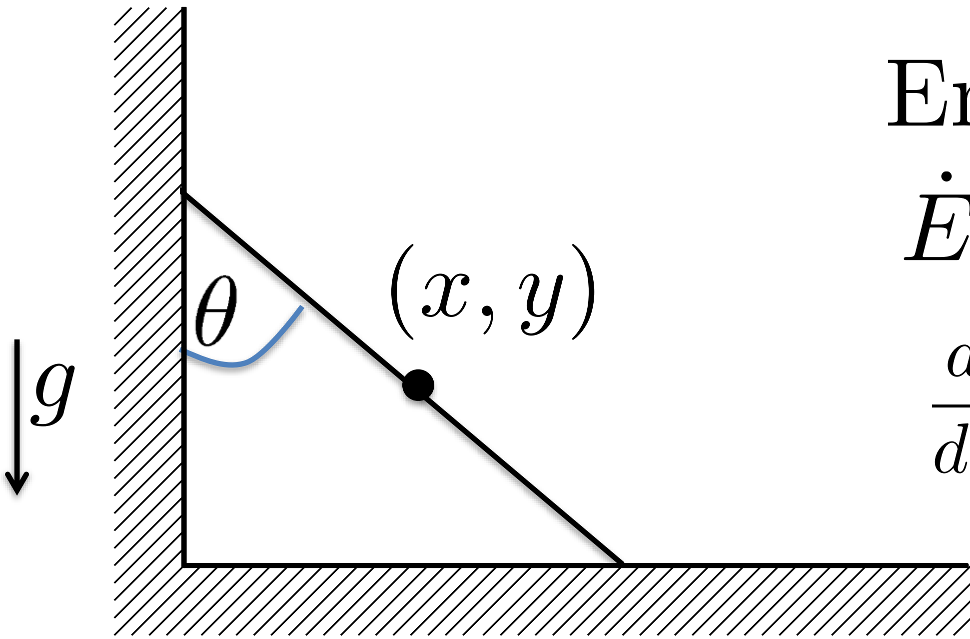
$$(x, y) = \frac{l}{2}(\sin\theta, \cos\theta)$$

$$(\dot{x}, \dot{y}) = \frac{l}{2}(\cos\theta, -\sin\theta)\dot{\theta}$$

$$\dot{x}^2 + \dot{y}^2 = \frac{l^2}{4}\dot{\theta}^2\left(\cos^2\theta + \sin^2\theta\right)$$

Example 2: Ladder sliding down a wall

$$E = \frac{1}{6}ml^2\dot{\theta}^2 + mg\frac{l}{2}\cos\theta$$



Energy conservation

$$\dot{E} = 0$$

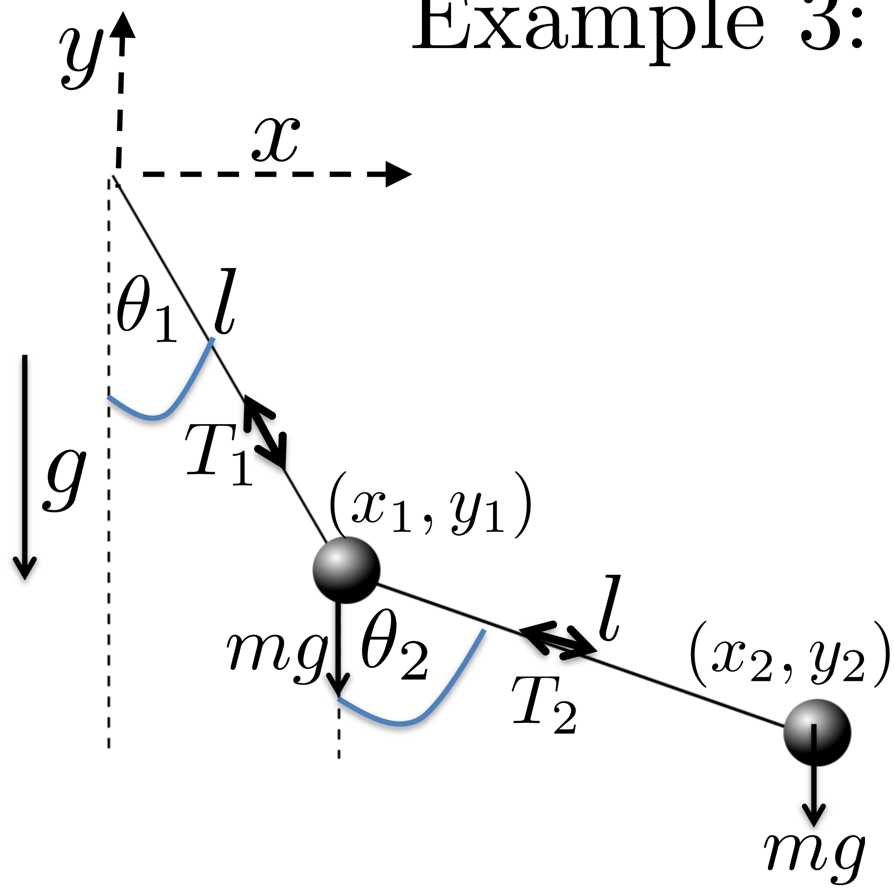
$$\frac{d}{dt}\left(\frac{1}{6}ml^2\dot{\theta}^2 + mg\frac{l}{2}\cos\theta\right) = 0$$

$$\frac{1}{3}\cancel{m}l^2\cancel{\dot{\theta}}\ddot{\theta} - \cancel{m}g\frac{l}{2}\sin\theta\cancel{\dot{\theta}} = 0$$

$$\ddot{\theta} = \frac{3g}{2l}\sin\theta$$

Quick, direct, no worrying about R_1, R_2

Example 3: Double pendulum



Now two constraints

$$x_1^2 + y_1^2 = l^2$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2$$

And four $F = ma$ equations

$$-T_1 \sin\theta_1 + T_2 \sin\theta_2 = m\ddot{x}_1$$

$$T_1 \cos\theta_1 - T_2 \cos\theta_2 - mg = m\ddot{y}_1$$

$$-T_2 \sin\theta_2 = m\ddot{x}_2$$

$$T_2 \cos\theta_2 - mg = m\ddot{y}_2,$$

$$\vdots$$

$$(x_1, y_1) = l(\sin\theta_1, \cos\theta_1)$$

$$\begin{aligned}(x_2, y_2) &= (x_1, y_1) + l(\sin\theta_2, \cos\theta_2) \\ &= l(\sin\theta_1, -\cos\theta_1) + l(\sin\theta_2, \cos\theta_2)\end{aligned}$$

$$\vdots$$

$$T_1 = -\frac{2m\left(2g\cos(\theta_1) + 2l\dot{\theta}_1^2 + l\dot{\theta}_2^2\cos(\theta_1-\theta_2)\right)}{\cos(2(\theta_1-\theta_2)) - 3}$$

$$T_2 = \frac{-2m\cos(\theta_1-\theta_2)\left(g\cos(\theta_1) + l\dot{\theta}_1^2\right) - 2lm\dot{\theta}_2^2}{\cos(2(\theta_1-\theta_2)) - 3}$$

$$\ddot{\theta}_1 = \frac{g(\sin(\theta_1 - 2\theta_2) + 3\sin(\theta_1)) + 2l\sin(\theta_1-\theta_2)\left(\dot{\theta}_1^2\cos(\theta_1-\theta_2) + \dot{\theta}_2^2\right)}{l(\cos(2(\theta_1-\theta_2)) - 3)}$$

$$\ddot{\theta}_2 = -\frac{2\sin(\theta_1-\theta_2)\left(2g\cos(\theta_1) + 2l\dot{\theta}_1^2 + l\dot{\theta}_2^2\cos(\theta_1-\theta_2)\right)}{l(\cos(2(\theta_1-\theta_2)) - 3)}$$

We can't use the energy method in this case as $\dot{E} = 0$ is only one equation of motion but we need two, one for θ_1 and one for θ_2.

We need an analogue of the energy method,

that delivers equations of motion for generalized coordinates directly,
without worrying about constraint forces,

but which works on systems with multiple degrees of freedom.

Such a method was discovered in 1788 by Lagrange.

First helpful preliminary observations:

$$F = ma \quad \rightarrow \quad \frac{d}{dt}(p) = F$$

Second helpful preliminary observations:

$$T = \frac{1}{2}m\dot{x}^2 \qquad V = V(x)$$

$$p = m\dot{x} = \frac{\partial T}{\partial \dot{x}} \qquad F = -\frac{\partial V}{\partial x}$$

Joseph-Louis (Giuseppe Luigi), comte de Lagrange

General Recipe

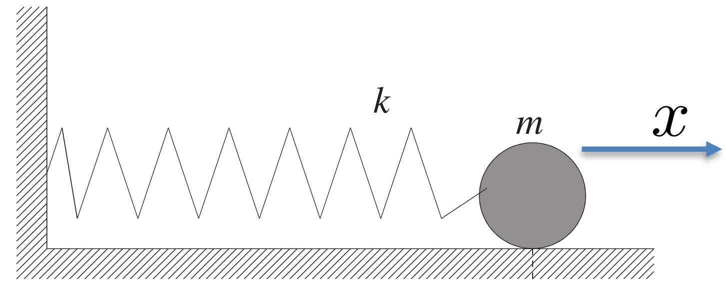
1) Find a convenient set of generalized coordinates

$$q_1, q_2 \ldots . q_n$$

2) Calculate kinetic energy T and potential energy V in terms of the q_i and $\dot{q}_i$

3) Calculate the *Lagrangian* $\mathcal{L}(q_i, \dot{q}_i) = T - V$

Mass on a spring



1) One coordinate, $q_1 = x$
Take $x = 0$ at equilibrium.

2) $T = \frac{1}{2}m\dot{x}^2$, $V = \frac{1}{2}kx^2$

3) $\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

General Recipe

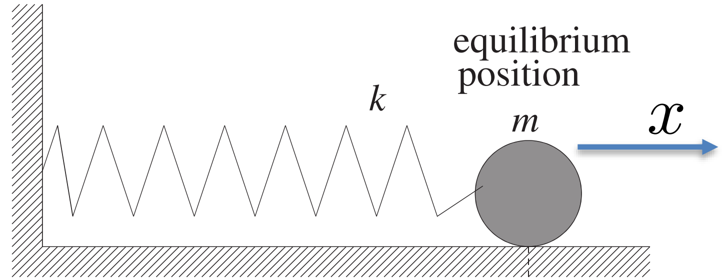
3) Calculate the *Lagrangian*
$\mathcal{L}(q_i, \dot{q}_i) = T - V$

4) For each generalized coordinate q_i
calculate a generalized momentum and force

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad F_i = \frac{\partial \mathcal{L}}{\partial q_i}$$

∂ is a partial derivative
all the other q_i, $\dot{q}_i$ constant

Mass on a spring



3) $\mathcal{L}(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$

4) $$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \right)$$
$$= m\dot{x}$$
Actual momentum

$$F_x = \frac{\partial \mathcal{L}}{\partial x} \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \right)$$
$$= -kx$$
Actual force

General Recipe

3) Calculate the *Lagrangian*
$\mathcal{L}(q_i, \dot{q}_i) = T - V$

4) Calculate

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad F_i = \frac{\partial \mathcal{L}}{\partial q_i}$$

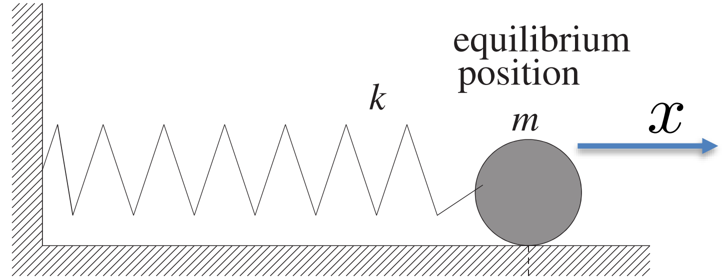
5) For each generalized coordinate q_i
the equation of motion is

$$\frac{d}{dt}(p_i) = F_i$$

d is a full derivative

$$\frac{d}{dt}(q) = \dot{q}$$

Mass on a spring



3) $\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

4) $p_x = m\dot{x} \qquad F_x = -kx$

5) $$\frac{d}{dt}(p_x) = F_x$$
$$\frac{d}{dt}(m\dot{x}) = -kx$$

$$m\ddot{x} = -kx$$

Actual $F = ma$

1) Choose any convenient generalized coordinates

$$q_1, q_2$$

2) Calculate total

3) Make the Lagr

$$\mathcal{L}(q_i,$$

4) Calculate the g
for each generaliz

$$p_i = $$

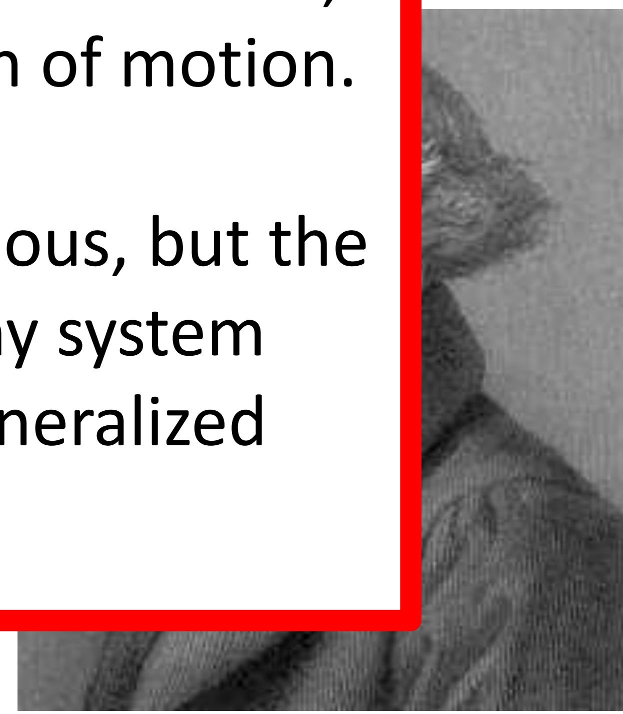
5) The equation of motion for each coordinate is

$$\frac{d}{dt}(p_i) = F_i$$

For the mass on a spring, we had an actual coordinate x, leading to the actual momentum, the actual force, and F=ma as the equation of motion.

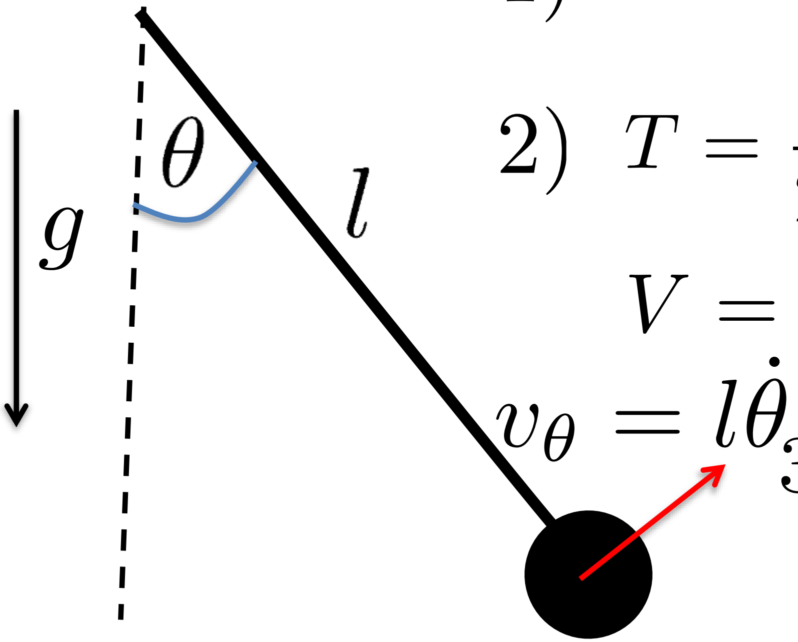
…

It won't always be so obvious, but the approach works for any system and with any set of generalized coordinates.



Joseph-Louis (Giuseppe Luigi), comte de Lagrange

Simple pendulum



1) Convenient coordinate $q_1 = \theta$.

2) $T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$

$V = mgh = mgl(1 - \cos\theta)$

3) $\mathcal{L}(\theta, \dot{\theta}) = T - V$

$= \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos(\theta))$

4) $p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}}$

Angular momentum!

$F_\theta = \frac{\partial \mathcal{L}}{\partial \theta}$

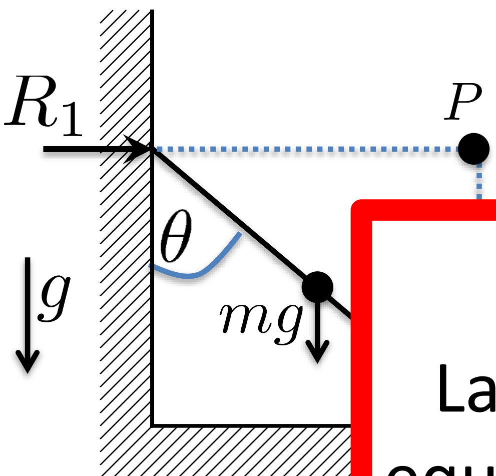
Torque!

5) $\frac{d}{dt}\left(p_\theta\right) = F_\theta$

$\frac{d}{dt}\left(ml^2\dot{\theta}\right) = -mgl\sin\theta$

$\ddot{\theta} = -\frac{g}{l}\sin\theta$

Example 2: Ladder sliding down a wall



1) Convenient coordinate $q_1 = \theta$.

2) $T = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2$

$T = \frac{1}{6}ml^2\dot{\theta}^2$

Lagrangian mechanics delivered the equation of motion directly, without us spotting this trick.

$- mg\frac{l}{2}\cos\theta$

4) $p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}}$

Angular momentum about P

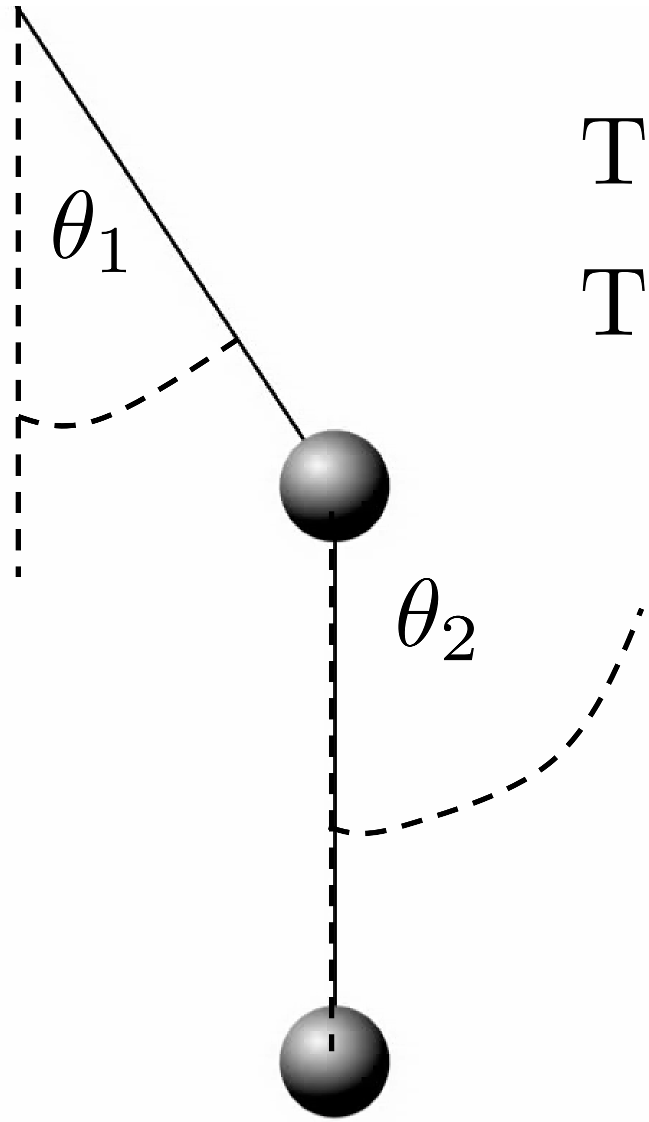
Torque about P

5) $\frac{d}{dt}(p_\theta) = F_\theta$

$\frac{1}{3}ml^2\ddot{\theta} = \frac{1}{2}mgl\sin\theta$

$\ddot{\theta} = \frac{3g}{2l}\sin\theta$

Correct, but what are p_θ and F_θ?



Two coordinates θ_1, θ_2

Two degrees of freedom.