PART 1B HEAT TRANSFER

Cambridge University Engineering Department

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Recommended Books

(1)*	MORAN, M.J. & SHAPIRO, H.N.		FUNDAMENTALS OF ENGINEERING THERMODYNAMICS Wiley, 5th Edition (SI Units) 2006	VA 192
(2)	ROGERS, G.F.C & MAYHEW, Y.R.		ENGINEERING THERMODYNAMICS: WORK AND HEAT TRANSFER Harlow, 4th Edition. 1992	VA 178
(3)	HOLMAN, J.P.		HEAT TRANSFER McGraw-Hill, 7th Edition. 1990	

Copies are available in the CUED library and college libraries.

All lecture notes and additional material will be added to the moodle site.

Introduction

The study of Heat Transfer aims to help us predict the ***rate*** of energy transfer which takes place due to a difference of temperature. Therefore, this is a different subject from Thermodynamics, which is only concerned with equilibrium states. Thermodynamics tells about what will happen after a certain quantity of heat is transferred, but not how long the transfer will take. Despite this, "introductory" Heat Transfer is often taught within a Thermodynamics course, and we follow this custom here.

There are three mechanisms of heat transfer:-

- Conduction – transfer by molecular vibration, Brownian motion and electron migration,
- Convection – transfer by bulk motion
- Radiation - transfer by electro-magnetic (EM) waves.

In many practical problems, two or even all of these mechanisms may need to be considered together.

1. Conductive Heat Transfer

When a temperature gradient exists in a body, we are unsurprised to note that there is an energy transfer from the hotter to the cooler part. This process is called heat conduction, and experience shows that the rate of energy transfer is proportional to the normal temperature gradient, and therefore we can write

$$\frac{\dot{Q}}{A} \propto \frac{\partial T}{\partial x}$$

Where $\dot{Q}$ is the heat transfer (W), T is the temperature, A is the area perpendicular to the temperature gradient in the x direction. Generally we will use lower case $\dot{q}$ to indicate heat flow per unit area. The constant of proportionality is called the ***thermal conductivity*** λ (W/mK), (note many texts use k), and since the heat transfer is in the direction of decreasing temperature, (the Second Law insists on this),

$$\frac{\dot{Q}}{A} = -\lambda \frac{\partial T}{\partial x}$$

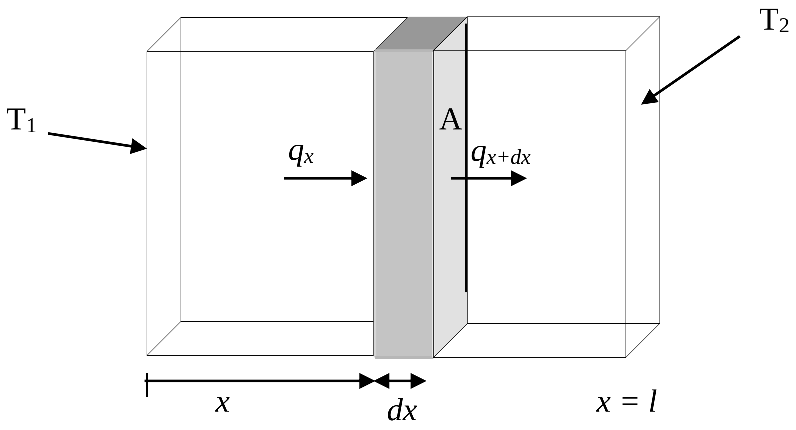
This is called Fourier's Law of heat conduction; though note, (like for example Ohm's Law), it is actually an empirical relationship. Table 1, below, gives some values for thermal conductivity at 20 ºC. There is always some variation of λ with temperature, though often it can be neglected.

Table 1: Thermal conductivity of various materials at 20 °C

Material	Thermal Conductivity W/mK	Material	Thermal Conductivity W/mK
Diamond	2300	Brick	0.72
Silver	429	Water (liquid)	0.613
Copper	401	Human Skin	0.37
Gold	317	Wood (oak)	0.17
Aluminium	237	Glass Fibre	0.043
Mild Steel	55	Air (gas)	0.026
Stainless Steel	16	Urethane (rigid foam)	0.026
Glass	0.78	Argon (gas)	0.018

1.1. Using Fourier's Law – Conduction in one dimension

Let us consider first a ***steady state***, ***one dimensional*** heat flow. Consider a slab of material as in the figure below.



For the element of thickness *dx*, the heat transferred into the face at *x,* and the heat conducted out of the face at *x+dx*, are given, per unit area, by:-

$$\dot{q}_x = -\lambda \frac{dT}{dx} \qquad \text{and}$$

Note we have dropped the partial derivatives as this is a 1D problem (x variation only), in steady state (no time derivatives). **At steady state there can be no accumulation of energy in the element, meaning that**

so $$\frac{d}{dx}\left(-\lambda \frac{dT}{dx}\right) = 0 \quad \Rightarrow \quad -\lambda \frac{dT}{dx} = constant$$

If the thermal conductivity is constant, we get the obvious result that the temperature gradient is constant. Integrating again (with constant thermal conductivity), and inserting the boundary conditions, we obtain

$$T = T_1 - \frac{x}{l}\left(T_1 - T_2\right) \quad \text{and} \quad \dot{Q} = A\lambda \frac{\left(T_1 - T_2\right)}{l}$$

It is often useful to express the last equation as:

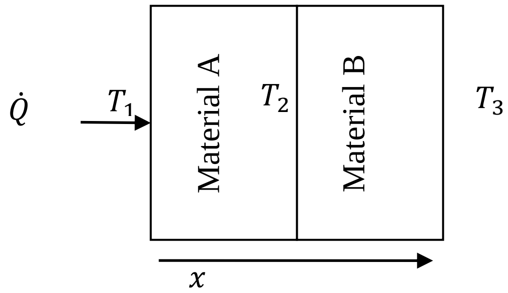
$$\dot{Q} = A\lambda \frac{(T_1 - T_2)}{l} = \frac{(T_1 - T_2)}{R_{th}},$$

where (by analogy with Ohm's law)

$$\boxed{\boldsymbol{R_{th} = \frac{l}{\lambda A}} \textbf{ is the thermal resistance}}$$

1.2. Heat transfer resistance in series

Consider heat flowing through slabs different materials:



For this ***1D, steady*** conduction problem, the heat flow (and heat flux) is not a function of x.

Material A: $\dot{Q} = \frac{T_1 - T_2}{R_A}$ $\quad R_A\dot{Q} = (T_1 - T_2)$

Material B: $\dot{Q} = \frac{T_2 - T_3}{R_B}$ $\quad R_B\dot{Q} = (T_2 - T_3)$

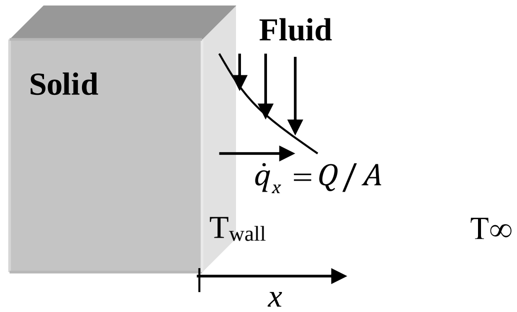
Adding (and noting that $\dot{Q}$ is the same)

$$\dot{Q}(R_A + R_B) = T_1 - T_3 \qquad \Rightarrow \quad R_{Total} = R_A + R_B$$

Heat transfer resistances in series add! This should be obvious from the Ohms law analogy.

1.3. *Another boundary condition - Convective heat losses at fluid interface*

Previously, we assumed the temperatures at the ends were known. Often heat from a surface is transferred to a flowing fluid (i.e. a convective boundary condition).



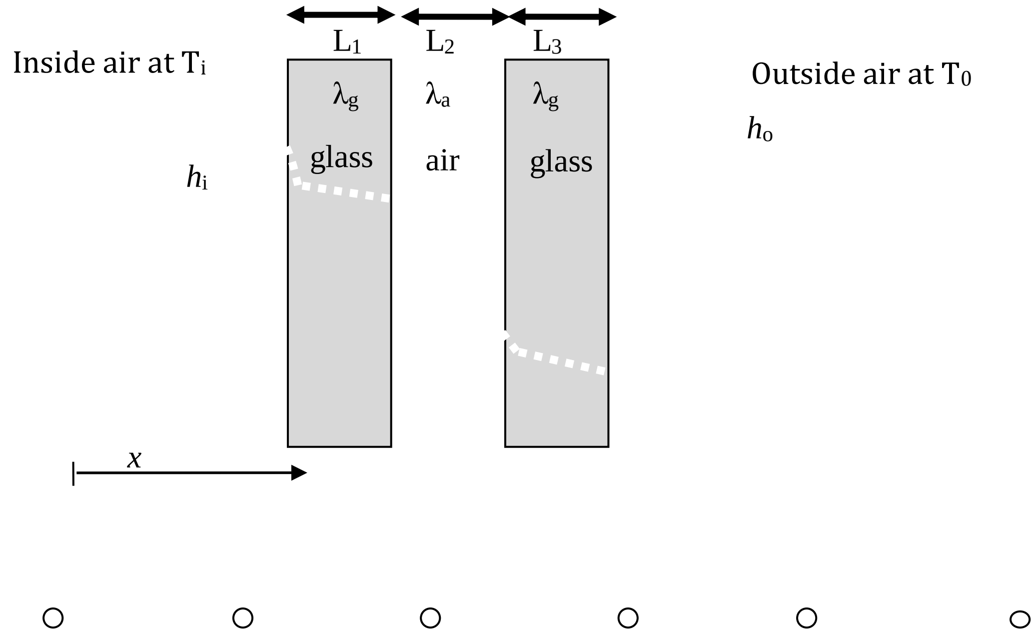
The heat flux out of the solid is proportional to the temperature difference (T_∞ - T_{wall}),

$$\dot{Q} = -hA(T_\infty - T_{wall}) =$$

Where *h* is the coefficient of heat transfer and $\boldsymbol{R_{th} = \frac{1}{hA}}$

1.4. *Example- Double glazed windows.*

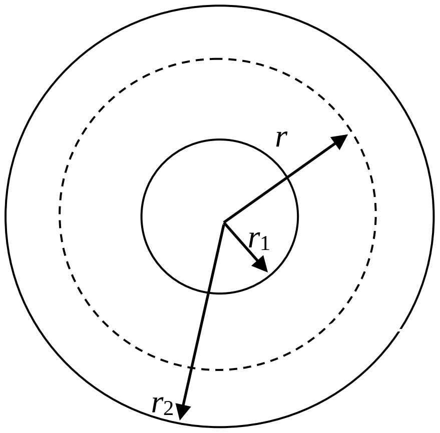
Determine the heat transfer through a window consisting of two 3 mm thick glass panes, with a gap of 7 mm between them. Take the convective heat transfer coefficient as 10 W/m^{2}K, T_0 = 5 °C, T_i = 25 °C; and assume that the air between the panes acts as a conductive layer. λ_a = 0.026 W/mK and λ_g = 0.78 W/mK.



	Thermal resistance ($\frac{m^2k}{W}$)	Temperature difference (°C)
Inside		
Glass		
Air gap		
Glass		
Outside		
Total		

Therefore the total heat loss is:

1.5. Radial Heat Transfer – Cylinders, steady state conduction



Consider a circular cylinder as shown, of length *l*. With boundary conditions:- at r = r_1, T = T_1, and at r = r_2, T = T_2.

- $\dot{Q} = -2\pi r l \lambda \frac{dT}{dr}$
- $\dot{Q}$ is a constant
- $\dot{q} = \dot{Q}/A$ varies with r

Starting with

$$\dot{Q} = -2\pi r l \lambda \frac{dT}{dr}$$

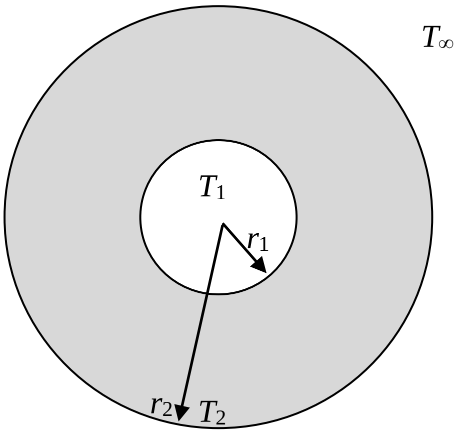
$$\dot{Q} = \frac{2\pi l \lambda (T_1 - T_2)}{\ln\left(\frac{r_2}{r_1}\right)}$$

The thermal resistance is then

$$R_{th} = \frac{T_1 - T_2}{\dot{Q}} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi l \lambda}$$

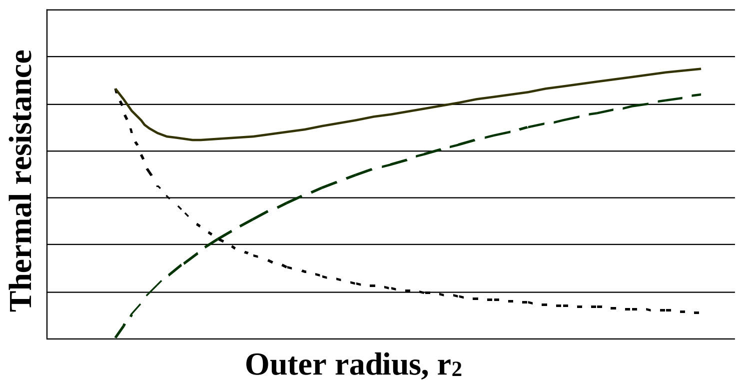
1.6. *Example- Insulation on a pipe*

Note that as the area for heat transfer increases with radius, it is possible to add material but <u>increase</u> the heat transfer. For example, consider a hot pipe, wrapped n insulating material of thermal conductivity λ. The inner and outer radii of the insulating material are r_1 and r_2, respectively. The temperature at r_1 is T_1. The outer surface of the insulator is in contact with the environment, where the convective thermal resistance is $1/hA$.



The overall thermal resistance is thus

$$R_t = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi\lambda l} + \frac{1}{2\pi r_2 l h}$$



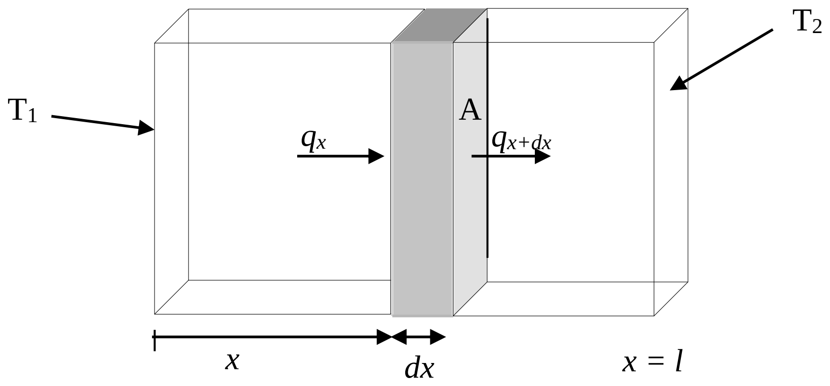
The first term on the RHS increases with insulator thickness, but the second term decreases. Therefore there can be a minimum value of R_t- i.e. there maybe some non-zero thickness of insulation that actually maximises the heat transfer! (There is a question in the exercise sheet that examines this behaviour in more detail.)

1.7. *Key points for 1D steady heat conduction*

- The rate of heat flux is given by Fourier's law, $\frac{\dot{Q}}{A} = -\lambda \frac{\partial T}{\partial x}$.
- In a planar geometry, both the heat flux and the heat flow are constant.
- In a cylindrical geometry, the heat flux is a function of position.
- We can break problems down into a series of thermal resistances, and use the analogy with ohms law to build an equivalent circuit.
- For a planar system, the thermal resistance of a material is given by $R_{th} = \frac{l}{\lambda A}$
- For a cylindrical system, the thermal resistance of a material is given by $R_{th} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi l \lambda}$.
- Where heat is transferred from a surface to a fluid, the convective thermal resistance is given by $R_{th} = \frac{1}{hA}$.

2. Unsteady Heat Conduction.

Let's return to conduction in a slab of planar material, but now relax the assumption that system is at steady state. We can also consider the case where heat is generated internally (ohmic heating for example).



Let us assume that the internal heat generation is G W/m^3, and that the temperature change of the element in time δt is δT. The net energy input to the element must then equal the change in the internal energy of the element, so,

$$\rho c A dx \frac{\delta T}{\delta t} = q_x A \qquad - \quad \left[q_x + \frac{\partial q_x}{\partial x} dx\right] A \qquad + GAdx$$

where c is the material specific heat.

$$\rho c \frac{\partial T}{\partial t} = - \frac{\partial q_x}{\partial x} \quad + G$$

$$\rho c \frac{\partial T}{\partial t} = - \frac{\partial}{\partial x}\left[-\lambda \frac{\partial T}{\partial x}\right] \quad + G$$

If the thermal conductivity is constant, then the above equation reduces to

$$\frac{\partial T}{\partial t} = \left(\frac{\lambda}{\rho c}\right) \frac{\partial^2 T}{\partial x^2} \quad + \frac{G}{\rho c}$$

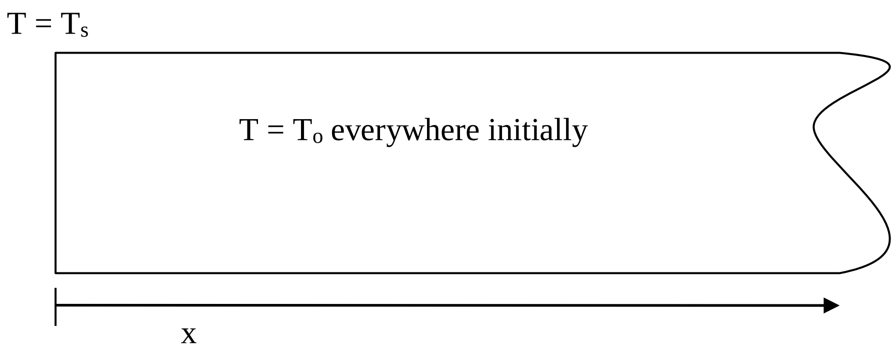
$\lambda/_{\rho c} = \alpha$ is called the thermal diffusivity (m^2/s).

In the absence of internal heat generation, we have the 1D transient heat diffusion equation,

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

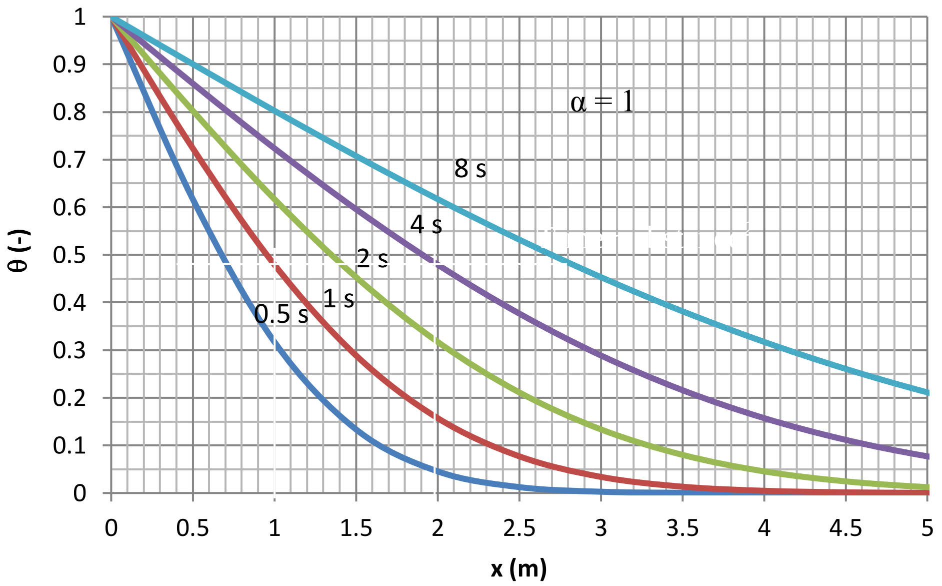
Exact solutions of this equation only exist for a few simple geometries and boundary conditions, so numerical solution is usually required.

2.1. *Example- conduction in a semi-infinite slab*



This is one case where it is possible to derive an analytical solution (you will not be asked to reproduce this, it is included here for illustrative purposes only)

$$\theta = \frac{T - T_0}{T_s - T_0} = 1 - \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$



Characteristic time for heat to penetrate a distance x is when,

$$\frac{x}{\sqrt{\alpha t}} = 1\ (say) \quad i.e. \quad t = \frac{x^2}{\alpha}$$

More generally, the "characteristic time" for heat diffusion is s^2/α, where s is a characteristic dimension of the body, e.g. (volume)/(surface area).

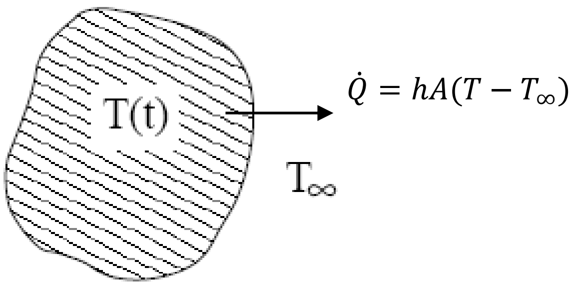
Another non-dimensional group that is useful in transient heat conduction problems is the Fourier number, *Fo*,

$$Fo = \frac{\text{time, } \tau}{\text{characteristic time for heat diffusion through body}} = \frac{\alpha \tau}{s^2}$$

A body can be considered to have heated up when Fo > 1

2.2. *Lumped Heat Capacity Analysis*

In many situations, a transient heat transfer situation involves a solid (with relatively low "internal" thermal resistance) and a solid/fluid interface with a relatively high thermal resistance. In such cases, the assumption can often be made that the solid body is at a uniform temperature. Let us consider a "lump" of material of specific heat c, density ρ, volume V, area A, with convective heat transfer coefficient h.



If the temperature is uniform throughout the body, then we can write

$$c\rho V \frac{dT}{dt} =$$

If the initial condition is $T = T_0$ at $t = 0$, the solution is

$$\frac{T - T_\infty}{T_0 - T_\infty} = \exp\left(-\frac{t}{\tau_c}\right),$$

where $\tau_c = c\rho V / hA$.

The analogy with a first order electrical RC network is clear,

2.2.1. *When is the lumped capacity model valid?*

As noted above, the assumption is that

$$\frac{internal\ thermal\ resistance}{surface\ thermal\ resistance} = \quad \text{——} \quad = \text{———} \ll 1$$

[where s = V/A is assumed to represent a characteristic length scale of the body].

The group hs/λ is called the ***Biot number (Bi)***. Though arbitrary, *Bi < 0.1* is often used as an acceptable condition for applying a lumped heat capacity analysis

2.3. Worked Example

A steel ball (specific heat 0.46 kJ/kg.K, λ = 55W/m.K, density = 7800 kg/m^3) 5 cm in diameter, initially at 450 °C, is placed in an environment at 100 °C. The convection heat transfer coefficient is 100 W/m^2K. Should a lumped heat capacity approach be used? – If so, what is the cooling time constant?

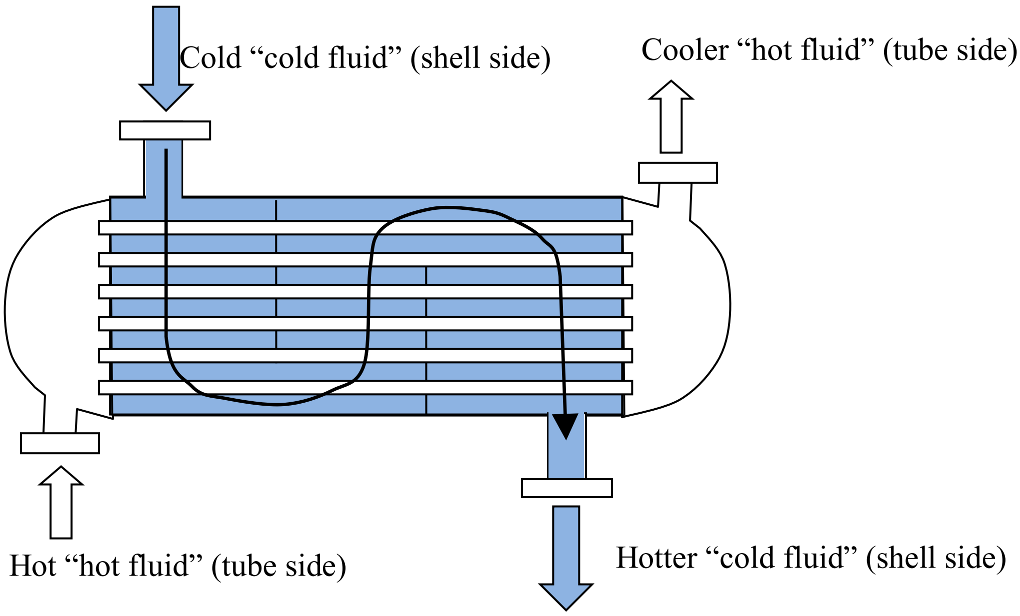
2.4. *Key Points for unsteady conduction*

- The "characteristic time" for heat diffusion through a body is s^2/α , where s is a characteristic dimension of the body, e.g. (volume)/(surface area).
- $\alpha = \lambda/\rho c$, is the thermal diffusivity.
- The Fourier number is the ratio of time, τ to the characteristic time for heat diffusion, i.e. $Fo = \frac{\alpha\tau}{s^2}$
- The Biot number is given by Bi = $hs/_{\lambda}$.
- When Bi < 0.1 conduction within the body is very fast, compared to transport of heat to the body by convection from the environment. The body can be treated as if it has a uniform temperature.

3. Heat Exchangers

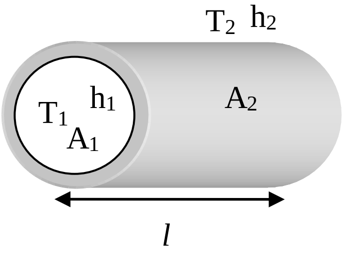
There is little need to emphasise the importance of heat exchangers – they are all around us – in vehicles (the “radiator”, the passenger compartment heater, the oil cooler), in power stations (the “boiler”, feed heaters), etc.

One of the most common forms of heat exchanger consists of banks of tubes (known as a shell and tube heat exchanger).



3.1. The overall heat transfer coefficient

Previously, in the double glazed window example, we determined an overall thermal resistance for the combination of convection $(1/hA)$ and conduction $(x/\lambda A)$ resistances, such that the heat transfer is given by $\dot{Q} = \Delta T / R_{th,overall}$. Though this formulation is to be preferred, common usage is to express the same relationship as $\dot{Q} = UA\Delta T$, where ***U* is called the *Overall Heat Transfer Coefficient***, and $UA = 1/R_{th,overall}$.



For problems with radial conduction, there is the issue of what area *A* refers to. Consider the pipe, of length *l* as shown, with inner and outer radii r_1 and r_2, respectively.

The equivalent circuit for this problem is:

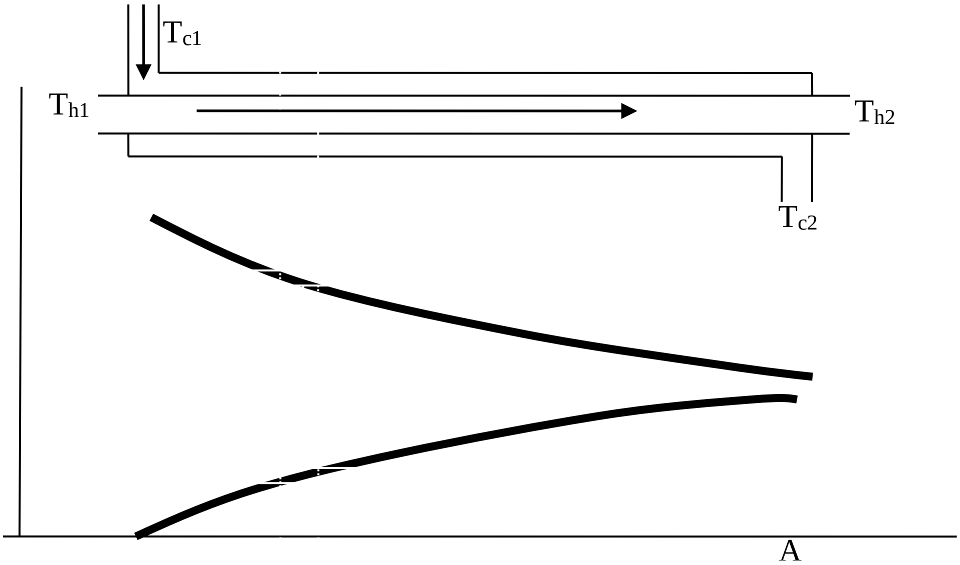
$$R_1 = \frac{1}{A_1 h_1} \qquad \mathrm{R}_2 = \ln\left(\frac{r_2}{r_1}\right) / 2\pi\lambda l \qquad R_3 = \frac{1}{A_2 h_2}$$

and so $$U_1 = \frac{1}{\frac{1}{h_1} + \frac{A_1 \ln(r_2/r_1)}{2\pi\lambda l} + \frac{A_1}{A_2}\frac{1}{h_2}} \qquad U_2 = \frac{1}{\frac{A_2}{A_1}\frac{1}{h_1} + \frac{A_2 \ln(r_2/r_1)}{2\pi\lambda l} + \frac{1}{h_2}}$$

Where U_1 is the “Overall heat transfer coefficient based on the inner diameter” and U_2 is the “Overall heat transfer coefficient based on the outer diameter”.

3.2. Performance of heat exchangers

One of the simplest heat exchanger geometries is as shown in the figure. This is called a co- or parallel-flow HX (we will use this common abbreviation for Heat eXchanger); if the fluids were flowing in opposite directions, it would be termed a counter-flow HX. Clearly the heat transfer is taking place due to a continuously changing temperature difference, so at first sight it appears a tricky problem to predict the total heat exchanged, but clearly there must be some “mean” temperature ΔT_m, which is defined by $\dot{Q}=UA\Delta T_m$ (where UA is either U_1A_1 or U_2A_2).



Consider a small element of the heat exchanger of area dA, where the heat exchanged is $d\dot{Q}$.

$$d\dot{Q}=-\dot{m}_h c_h dT_h = \dot{m}_c c_c dT_c = UdA(T_h - T_c)$$

We will assume that U, the overall heat transfer coefficient, and the fluid specific heats are constant. We can write $dT_h = -d\dot{Q}/\dot{m}_h c_h$, and $dT_c = d\dot{Q}/\dot{m}_c c_c$

$$\Rightarrow dT_h - dT_c (= d(T_h - T_c)) = -d\dot{Q}\left(\frac{1}{\dot{m}_h c_h}+\frac{1}{\dot{m}_c c_c}\right)$$

$$\therefore \quad \frac{d(T_h - T_c)}{T_h - T_c} = -U\left(\frac{1}{\dot{m}_h c_h}+\frac{1}{\dot{m}_c c_c}\right)dA.$$

Integrating from one end to the other,

$$\ln\left(\frac{T_{h2}-T_{c2}}{T_{h1}-T_{c1}}\right) = -UA\left(\frac{1}{\dot{m}_h c_h}+\frac{1}{\dot{m}_c c_c}\right).$$

Now from the first law, $\dot{Q}=\dot{m}_h c_h\left(T_{h1}-T_{h2}\right)=\dot{m}_c c_c\left(T_{c2}-T_{c1}\right)$, so finally we obtain

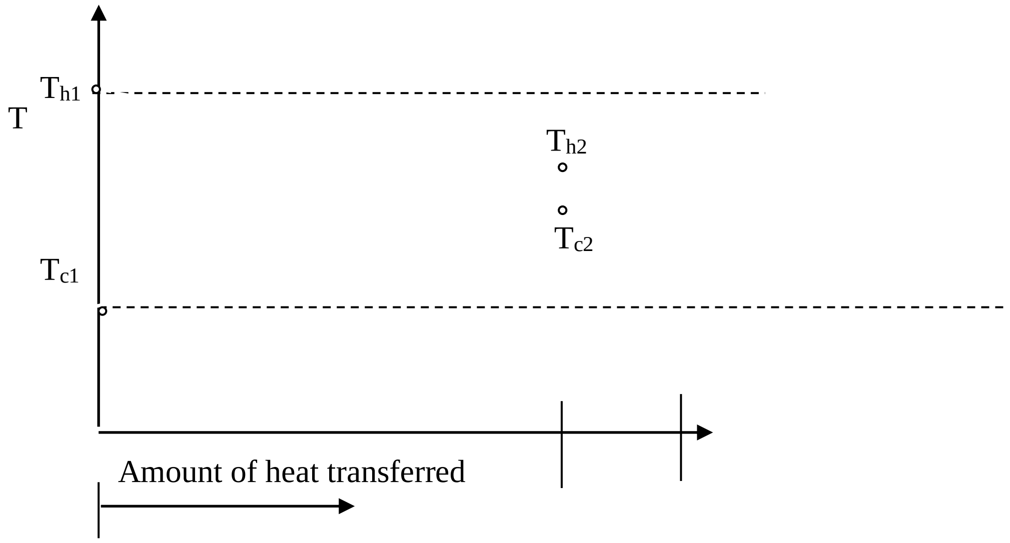
$$\dot{Q}=UA\left(\frac{\left(T_{h2}-T_{c2}\right)-\left(T_{h1}-T_{c1}\right)}{\ln\left[\left(T_{h2}-T_{c2}\right)/\left(T_{h1}-T_{c1}\right)\right]}\right)$$

$$\dot{Q}=UA\Delta T_m \text{ and } \Delta T_m=\frac{\Delta T_2-\Delta T_1}{ln\left(\frac{\Delta T_2}{\Delta T_1}\right)}$$

ΔT_m **is called the *log mean temperature difference*** and is the appropriate "average" temperature driving force for heat transfer. These relationships apply to counter-flow HX's as well[1].

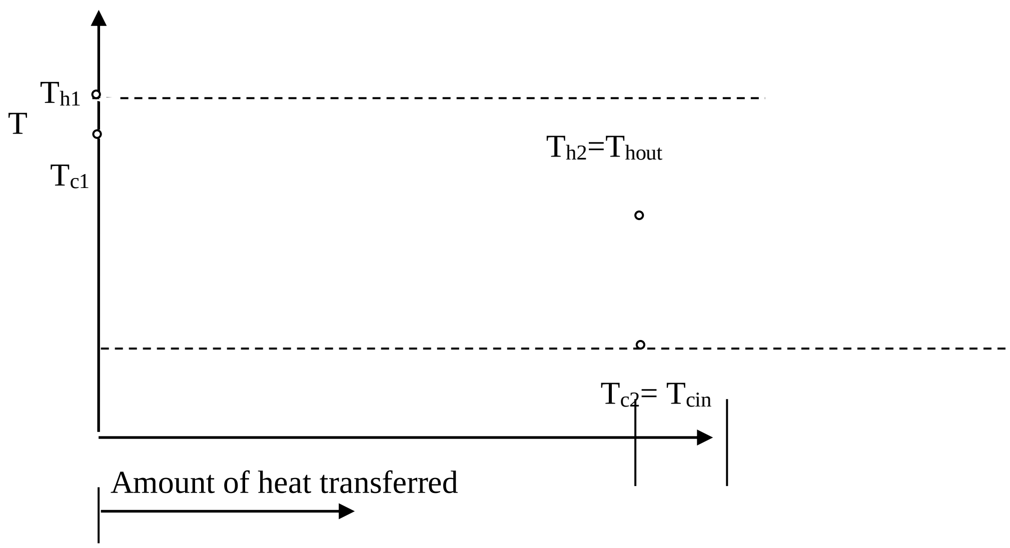
[1] You can also easily show that this also holds when the temperature is constant on one side of the heat exchanger, e.g., when steam is condensing on the shell side of an exchanger.

3.2.1. Co-flow heat exchanger T-Q plot



- $\Delta T_m = \frac{\Delta T_2 - \Delta T_1}{ln\left(\frac{\Delta T_2}{\Delta T_1}\right)}$ with $\Delta T_2 = T_{hout} - T_{cout}$, $\Delta T_1 = T_{hin} - T_{cin}$
- The gradient of a T-Q plot is $1/\{\dot{m}c\}$
- For a co – flow heat exchanger, the maximum heat transfer is when the exit temperatures become equal $(T_{h2} = T_{c2})$
- Maximum heat transfer requires infinite heat transfer area, $\Delta T_{lm} = 0$

3.2.2. Counter flow heat exchanger T-Q plot



- As you move along the heat exchanger, the cold stream can approach the inlet hot fluid temperature OR the hot stream can approach the inlet temperature of the cold fluid. It depends on which stream has the lowest $\dot{m}c$.

- At end (2) the coldest hot fluid transfers heat to the coldest cold fluid. At end (1) the hottest hot fluids transfers heat to the hottest cold fluid.
    - The lower temperature difference reduces irreversibilities.
    - More heat can be transferred.

- $\Delta T_m = \frac{\Delta T_2 - \Delta T_1}{ln\left(\frac{\Delta T_2}{\Delta T_1}\right)}$ with $\Delta T_2 = T_{hout} - T_{cin}$, $\Delta T_1 = T_{hin} - T_{cout}$

- As drawn, with the cold flow having the lowest value $\dot{m}c$, the maximum heat possible heat transfer is when $T_{cout} = T_{hin}$.

- Maximum heat transfer requires infinite heat transfer area, $\Delta T_{lm} = 0$

3.2.3. *Summary of T-Q plots and heat exchanger effectiveness*

- For a constant $\dot{m}c$, the lines of T vs Q are straight.
- The maximum possible heat exchange will always be $(\dot{m}c)_{\min}$, times the biggest temperature difference existing in the HX.
- Maximum possible heat exchange requires an infinite area for heat transfer
- The *effectiveness (ε)* of a HX is defined as the actual heat exchange, divided by the maximum possible (**achieved in counter flow mode**).

For the situation above, this will be given by

$$\varepsilon = \frac{\dot{m}_h c_h \left(T_{hin} - T_{hout}\right)}{(\dot{m}c)_{\min}\left(T_{hin} - T_{cin}\right)}.$$

3.3. *Key points for heat exchangers*

- We often use an overall heat transfer coefficient U, whose value depends on the area it is associated with.
- The temperature difference varies along the heat exchanger. The correct appropriate average driving force for heat transfer is the log mean temperature difference.
- Counter flowing heat exchangers are better than co-flowing heat exchangers.
- The *effectiveness (ε)* of a HX is defined as the actual heat exchange, divided by the maximum possible (**achieved in counter flow mode**).

4. Convective Heat Transfer (Forced Convection)

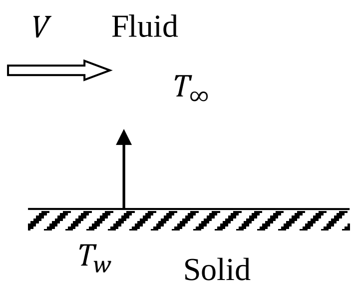
4.1. Introduction

In 1987, Cray Research Inc. released the Cray-2, then the world's fastest computer at 250MHz with 2Gb of RAM. It was contained in a cylindrical cabinet 1.15m high and 1.35m in diameter.

"We solved the problem of dissipating the 130kW it produced by immersing all the circuitry in a bath of inert liquid fluorocarbon. Computer designers are glorified cooling engineers and I am a very good plumber. "

Seymour Cray

We have already used a convective boundary condition



The heat transfer coefficient, *h,* is defined by:

$$\dot{q} = -\,h\,\Delta T$$

h may vary with position, e.g. x, (see later). Often we will use the averaged heat transfer coefficient,

$$\frac{\dot{Q}}{A} = -\,h_{av}\Delta T_{mean}$$

- h and h_{av} tend to get used interchangeably, depending upon the problem.
- Often, we will want the average heat transfer coefficient, and the fluid mechanics will be too complicated to allow us to calculate the local value.

We need to be able to estimate the values of h. There are two distinct situations we need to consider;

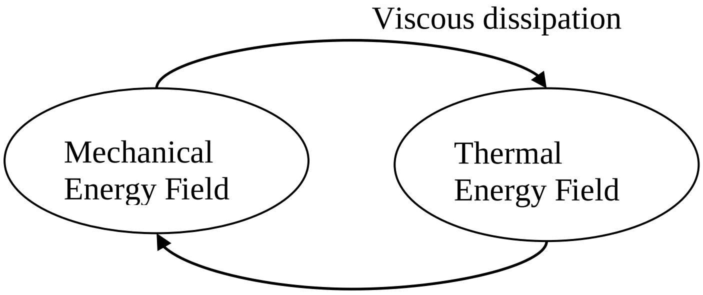
1. **Forced convection-** when the fluid is driven over the surface by e.g. a fan or pump.
2. **Free (natural) convection** – when density differences due to buoyancy drive the flow.

Forced convection is almost always the quickest way to transfer heat. In both situations, we must consider the flow of fluid over the surface.

In this lecture we will consider Forced Convection

4.2. *Heat transfer and fluid mechanics*

The thermal (temperature) field of a flow cannot affect the mechanical (pressure and velocity) fields if the density is assumed to be constant. We can calculate the mechanical fields first and then calculate their effect on the temperature field. The mechanical fields, however, always affect the thermal field, both by viscous dissipation and simply by convecting hot fluid through space (i.e. forced convection).



If the density cannot be assumed to be constant, the calculation requires feedback from the thermal to the mechanical field and is more complicated.

From the equation of state of ideal gases[1], $\rho = P/RT$, we see that $d\rho/\rho = -dT/T$. Incompressibility can be assumed when temperature fluctuations are less than approximately 0.1 of the background temperature. Incompressibility also requires that the Mach number is less than around 0.3.

i.e the enthalpy of the fluid per unit mass, c_pT, is much greater than the kinetic energy per unit mass $v^2/2$. The convective energy transfer is then just $\dot{m}c_pT$, and the steady flow energy equation becomes

$$\dot{Q} - \dot{W}_x = m_{in}\left(c_pT + \frac{1}{2}v^2 + gz\right)_{in} - m_{out}\left(c_pT + \frac{1}{2}v^2 + gz\right)_{out}$$

The properties of the fluid, such as viscosity, thermal conductivity and specific heat capacity, depend on temperature. Although this does not provide a mechanism for energy to pass from the thermal to the mechanical field, it does allow the thermal field slightly to affect the mechanical field.

If temperature differences are small, the variations in these properties are small. Very often the properties are assumed to take the values at the mean value of the temperature, the film temperature.

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[1] In other fluids, with other equations of state, we must be careful not to be near a region of sudden density change such as the boiling point.

4.3. *The analogy between heat and momentum transfer*

In earlier lectures you looked at the diffusion of **ordered** momentum between two fluid layers due to molecular motion. In the language of continuum models, this related the shear stress to the velocity gradient via viscosity.

Some molecules (randomly) swap places. Momentum exchange. Shear-stress

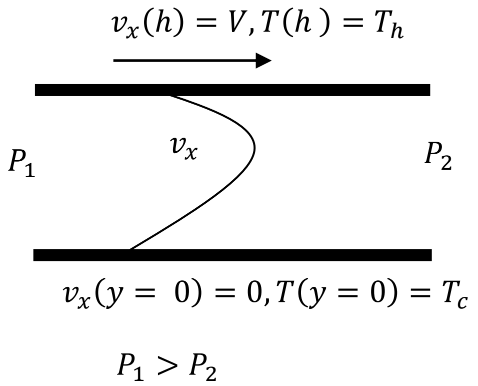
$$\tau = shear\ stress = \frac{shear\ force}{area} = \mu \frac{\partial v_x}{\partial y}$$

The kinetic energy of **disordered** motion diffuses in the same way. In the language of continuum models we measure this as a heat flux between the two layers if they are at different temperatures.

Some molecules (randomly) swap places. Thermal energy exchange. Heat flux

$$\dot{q} = heat\ flux = -\lambda \frac{\partial T}{\partial y}$$

Consider flow down a duct



Momentum conservation:

$$\rho \frac{Dv_x}{Dt} = \nabla.\left(\underline{\tau_x}\right) - \frac{\partial P}{\partial x}$$

Thermal energy conservation:

$$\rho C_p \frac{DT}{Dt} = -\nabla.\left(\underline{q}\right) + S_h$$

We are not interested in solving these equations here, just the form of the equations.

$$\frac{Dv_x}{Dt} = \frac{\mu}{\rho}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) - \frac{1}{\rho}\frac{dp}{dx}$$

$$\frac{DT}{Dt} = \frac{\lambda}{\rho C_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{1}{\rho C_p} S_h$$

The quantities **ν** and **α** both have units $m^2 s^{-1}$. **ν is the momentum diffusivity** (also known as the kinematic viscosity). **α is the thermal diffusivity**. The ratio ν/α is a property of the fluid called the Prandtl number (Pr).

	Steam 100 °C	Air	Water
Pr			

4.4. *Non-dimensional heat transfer coefficients*

In many situations, the fluid mechanics will be too complicated to allow the heat transfer coefficient to be computed directly. We will have to resort to dimensional analysis and correlations.

Nusselt Number

The Nusselt Number is found by normalising the heat flux, but a conductive heat flux.

$$Nu = \frac{Heat\ flux}{Heat\ flux\ with\ no\ flow} = \frac{\dot{q}}{\dot{q}_{noflow}} =$$

$$Nu = \frac{hD}{\lambda}$$

[For other geometries, D will be some characteristic length scale, e.g. for spheres, the diameter is used]

Stanton Number

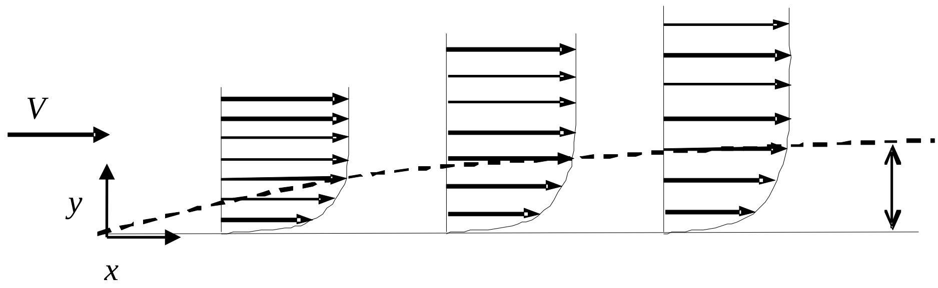
The Stanton number is found by normalising the heat flux by a characteristic convective heat flux.

$$St = \frac{Heat\ flux}{Characteristic\ convective\ heat\ flux} =$$

4.5. Reynolds' analogy (Prandtl number = 1)

If the **Prandtl number equals 1**, the momentum diffusivity equals the thermal diffusivity. This means that the non-dimensional equations of heat and momentum transfer are identical. In some situations, the non-dimensional boundary conditions are also identical. This means that the non-dimensional solutions are identical and therefore that there is a direct relationship between the mechanical and thermal fields. This is Reynolds' analogy. To demonstrate this, we will revisit the flow over a flat plate.

4.5.1. Recap of the momentum boundary layer over a flat plate



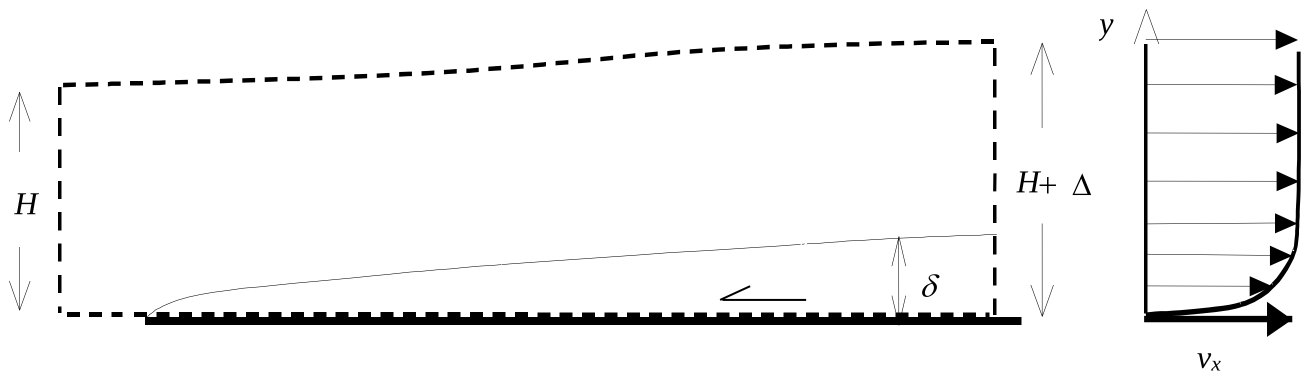
The velocity profile for the laminar boundary layer was given (approximately) by

$$\frac{v_x(y)}{V} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

We also saw that the thickness of the boundary layer grows with distance from the leading edge as

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}}$$

4.5.2. Heat transfer from a flat plate



- **Conservation of mass**

$$\dot{m}_{out} = \dot{m}_{in} \Longrightarrow \rho V(H + \Delta - \delta) + \int_0^\delta \rho v_x dy = \rho V H$$

- ***Conservation of x*-momentum** (The pressure is effectively uniform throughout the flow and its effect cancels)

Force on CV = Net Flux of Momentum Out

$$-\int_0^x \tau_w \; dx = \rho V(H + \Delta - \delta)V + \int_0^\delta (\rho v_x) v_x dy - (\rho V) H V$$

Substituting in for τ_w and dividing by V

$$-\mu \int_0^x \left. \frac{\partial \left(\frac{v_x}{V}\right)}{\partial y} \right|_{wall} dx = \rho V(H + \Delta - \delta) + \int_0^\delta (\rho v_x) \left(\frac{v_x}{V}\right) dy - (\rho V) H$$

- **Conservation of energy**

Heat flow = Enthalpy flow OUT – Enthalpy flow IN

$$\int_0^x \dot{q}_w \; dx = \rho V(H + \Delta - \delta) c_p (T_\infty - T_w) + \int_0^\delta (\rho v_x) c_p (T - T_w) dy \; - (\rho V) H c_p (T_\infty - T_w)$$

Substituting in for $\dot{q}_w$ and dividing by $c_p(T_\infty - T_w)$

$$-\frac{\lambda}{c_p} \int_0^x \left. \frac{\partial \left(\frac{(T - T_w)}{(T_\infty - T_w)}\right)}{\partial y} \right|_{wall} dx = \rho V(H + \Delta - \delta) + \int_0^\delta (\rho v_x) \frac{(T - T_w)}{(T_\infty - T_w)} dy - (\rho V) H$$

Writing $v' = \frac{v_x}{V}$ and $\theta = \frac{(T - T_w)}{(T_\infty - T_w)}$, we can now see the similarity between the momentum and the energy equation

$$-\mu\int_0^x \frac{\partial v'}{\partial y}\bigg|_{wall} dx = \rho V(H+\Delta-\delta)+\int_0^\delta (\rho v_x)v'dy-(\rho V)H$$

$$-\frac{\lambda}{c_p}\int_0^x \frac{\partial \theta}{\partial y}\bigg|_{wall} dx = \rho V(H+\Delta-\delta)+\int_0^\delta (\rho v_x)\theta dy-(\rho V)H$$

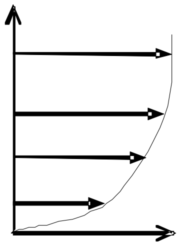
For Pr $=$ $= 1$ we conclude

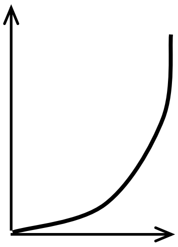
$v' = \frac{v_x}{V}$ and $\theta = \frac{(T-T_w)}{(T_\infty - T_w)}$, satisfy the same equation.

Moreover ,

$\frac{v_x}{V} = 0$ for $y = 0$ $\frac{v_x}{V} = 1$ for $y \geq \delta$

$\frac{T-T_w}{T_\infty - T_w}$ for $y = 0$, $\frac{T-T_w}{T_\infty - T_w}$ for $y \geq \delta$





i.e. is a solution of the problem for temperature!

The above relationship between v_x and T enables us to relate directly the gradient of T at the wall to that of v_x at the wall.

$$\frac{T-T_w}{T_\infty - T_w} = \frac{v_x}{V} \quad \Longrightarrow$$

$$h = \frac{\lambda \frac{\partial T}{\partial y}\Big|_{wall}}{(T_\infty - T_w)} = \frac{\lambda}{V}\frac{\partial v_x}{\partial y}\bigg|_{wall} = \frac{\lambda}{V}\frac{\tau_w}{\mu}$$

τ_w can be related to a friction factor, c_f i.e.

$$h = \frac{\lambda}{V}\frac{\tau_w}{\mu} = \frac{\lambda}{V}\frac{\frac{1}{2}\rho V^2 c_f}{\mu}$$

Dividing $\rho\, c_p V$ (and noting that we are assuming Pr = 1)

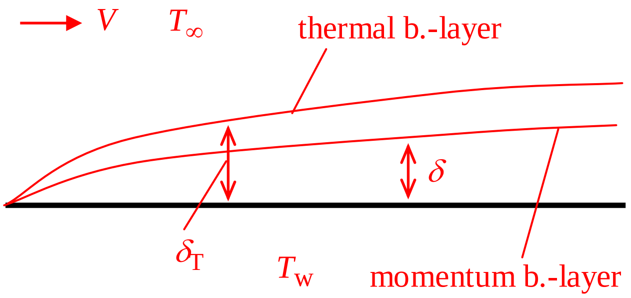
$$\frac{h}{\rho\, c_p V} = \frac{1}{2}\, c_f \frac{\lambda}{\mu c_p} = \frac{1}{2} c_f \frac{1}{Pr}$$

$$St = \frac{1}{2} c_f$$

This result indicates that we cannot have high heat transfer without having high wall friction. This is not surprising because both heat and momentum transfer have the same underlying mechanism: molecular motion or, in a turbulent flow, eddy motion

4.6. *Heat transfer if the Prandtl number ≠1*

If the Prandtl number is not equal to 1, the transport equations are no longer identical and we have two types of boundary layer: a momentum boundary layer and a thermal boundary layer. The relative thicknesses of the boundary layers are determined by the Prandtl number



$Pr << 1$ for liquid/molten metals $\Rightarrow \delta << \delta_T$

Pr about 1 for air, other gases, water $\Rightarrow \delta \approx \delta_T$

$Pr >> 1$ for thick oils, etc. $\Rightarrow \delta >> \delta_T$

We find from experiments that:

$$\frac{h}{\rho c_p V} = St = \frac{1}{2} c_f \,\mathrm{Pr}^{-2/3}$$

(For gases usually have $Pr \approx 0.7$ the correction is small and usually ignored).

4.7. *Turbulent boundary layers*

If a boundary layer becomes turbulent, there is additional transport of momentum and thermal energy due to the turbulent eddies. This can greatly increase the heat transfer.

4.8. Evaluation of St and Nu for a flat plate boundary layer

The velocity profile for the laminar boundary layer was given (approximately) by

$$\frac{v_x(y)}{V} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

with the corresponding shear stress at the wall of,

$$\tau_w = \mu \left.\frac{\partial v_x}{\partial y}\right|_{y=0} = \mu \frac{3V}{2\delta}$$

and **local** friction factor

$$c_{fx} = \frac{\tau_w}{\frac{1}{2}\rho V^2} = \mu \frac{3}{\rho \delta V}$$

From the above, $St_x = \frac{1}{2} c_{fx} Pr^{-\frac{2}{3}}$

$$St_x = \mu \frac{3}{2\rho V \delta} Pr^{-\frac{2}{3}}$$

We also saw that the thickness of the boundary layer grows with distance from the leading edge as $\delta = \frac{4.64x}{\sqrt{Re_x}}$

$$St_x = \frac{3\mu}{2\rho x V} \frac{\sqrt{Re_x}}{4.64} Pr^{-\frac{2}{3}} = 0.323 Re_x^{-\frac{1}{2}} Pr^{-\frac{2}{3}}$$

Or for the Nusselt number

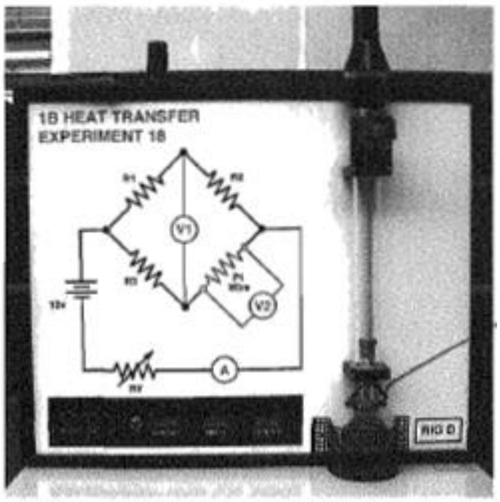
$$Nu_x = 0.323 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

These expressions give a value for the **local value** of *Nu*, *St* and *h* as a function of *x*. Suitable averages would have to be taken to give the average heat transfer coefficient over a plate of length L.

The databook gives correlations for the **averaged/overall** Nusselt number, for various geometries, from which the overall heat transfer coefficient can be derived.

4.9. The forced convection experiment

In most situations, the flow field is too complicated to be calculated by hand. Computational Fluid Dynamics (CFD) can be used but this is often expensive and may not be accurate. It can be easier to perform an experiment on a geometrically-identical situation and then scale up with the relevant non-dimensional parameters. You do this in the 1B heat transfer experiment of forced convection over a platinum wire.



1. List the dependent and independent variables

Dependent	Independent

2. Count the dimensions

3. Create dimensionless numbers

When you plot the Nusselt number as a function of Reynolds number (given that air has a fixed Prandtl number) you find that your results collapse to a single line.

The physics is encapsulated in this line. For a gas with the same Prandtl number as air, you can use the chart you created to calculated the heat transfer from a geometrically similar situation of any size, between the Reynolds number ranges that you studied. In heat transfer problems you usually have to rely on dimensional analysis and scaling.

4.10. Key Points for Forced Convection

- Heat transfer takes place in boundary layers and is strongly influenced by the state, size, etc. of the local boundary layer.
- There is a strong analogy between heat transfer and momentum transfer leading to Reynolds Analogy $\frac{h}{\rho c_p V} = St = \frac{1}{2} c_f$
- The local heat transfer coefficient for a flat plate can be derived from a simple model of the velocity profile
- Turbulent boundary layers involve higher heat transfer than laminar ones.
- Correlations (i.e. curve fits to experimental data) will be different for different states of boundary layers (laminar, transitional, turbulent) and it is important to check.
- For more general problems $Nu_{av} = \frac{h_{av} D}{\lambda} = fn(\mathrm{Re}, \mathrm{Pr})$

5. Natural Convection

In general, natural or free convection results in a smaller heat transfer than that due to forced convection. It will usually be swamped by forced convection when the latter is present, but many important flows (including all flows of air within buildings and many environmental ones are driven by natural convection).

5.1. Characteristics of Natural Convection

When heated, fluids expand and are driven upward by buoyancy forces.

For these flows there is no natural reference velocity (there is no "free stream"). There is this no natural Reynolds Number associated with the flow. There is, however, an upward velocity driven by buoyancy and examination of Schlieren pictures of naturally-convecting flows reveals what appear to be thermal boundary layers on heated objects. The fundamental questions are:- **what are the important parameters; how do we scale them, etc?**

We have all experienced natural convection. In general, we do not notice pressure forces - hold your hand over a candle (for a short time!). We notice the heat transfer but are not aware of any upward pressure force, as we would when there was a significant free stream velocity. **We will *assume that pressure variations due to convection are in general small*** and that the motion is, therefore, driven primarily by changes in density.

5.2. How big are the $\delta\rho$'s ?

We not expect large changes in density, but it appears that they are sufficient to show up optically in air due to changes in the refractive index (which is a function of density). In order to deal equally easily with liquids and gases it is convenient to use a *coefficient of expansion*.

$$\beta = \frac{1}{v}\left.\frac{\partial v}{\partial T}\right|_p \qquad \text{where } v \text{ is the specific volume.}$$

Since $\rho v = 1 \quad \Rightarrow \quad \rho\, dv + v\, d\rho = 0 \quad \Rightarrow \quad \frac{dv}{v} = -\frac{d\rho}{\rho}$ so that

(For liquids, ρ is effectively independent of pressure, so that it doesn't matter what is held constant). For small density changes (assuming constant pressure)

$$\delta\rho \approx \left.\frac{\partial\rho}{\partial T}\right|_p \delta T \quad \Rightarrow \quad \boxed{\delta\rho \approx -\bar{\rho}\beta\delta T}$$

where $\bar{\rho}$ is the mean value of density

For a liquid, β must be obtained from tables. For a perfect gas it can be derived from $\rho = P/RT$

$$\left.\frac{\partial\rho}{\partial T}\right|_p = -\frac{p}{RT^2} = -\frac{\rho}{T} \quad \Rightarrow \quad \beta = -\frac{1}{\rho}\left.\frac{\partial\rho}{\partial T}\right|_p = \frac{1}{T} \qquad \text{and this is evaluated at } \bar{T}$$

5.3. How big are the velocities that these $\delta\rho$'s drive ?

We find a very rough order of magnitude estimate by looking at a simple problem - the flow near a vertical flat plate. We expect the flow to be driven upwards near the warm plate but farther away, the flow will be at rest. In fact we are expecting some sort of boundary layer behaviour, but this time, the vertical velocity will be zero at the wall and also as we move to the edge of the boundary layer, where the shear stress will also vanish. The momentum equation is

$$\rho\left(V\frac{\partial V}{\partial s}\mathbf{e}_s - \frac{V^2}{R}\mathbf{e}_n\right) = -\frac{\partial p}{\partial s}\mathbf{e}_s - \frac{\partial p}{\partial n}\mathbf{e}_n + \rho\mathbf{g} + \textbf{viscous}$$

5.3.1. Deriving the equation which governs the convective flow near a hot plate

(A) Base case – No flow – V = 0, $T = T_\infty$	***(B) Flow due to temperature variation, T = T***
Horizontal scale exaggerated	Horizontal scale exaggerated

Case A: $$0 = -\left(\frac{\partial P}{\partial x}\right)_A - \rho_A g$$

Case B: $$\rho V\frac{\partial V}{\partial x} = -\left(\frac{\partial P}{\partial x}\right)_B - \rho_B g + \mu\frac{\partial^2 V}{\partial y^2}$$

When the heating is turned on, we assume that:

- there is no change in pressure,
- the density changes are fairly small
- the flow is primarily in the *x*-direction, with streamlines parallel

Now we write an equation for the difference between the base case A (no flow) and Case B (flow). Subtract A from B to give:

$$\rho V\frac{\partial V}{\partial x} = -\left(\frac{\partial P}{\partial x}\right)_B - \rho_B g + \mu\frac{\partial^2 V}{\partial y^2} - \left\{-\left(\frac{\partial P}{\partial x}\right)_A - \rho_A g\right\}$$

We assume that the pressure distribution is the same in both cases, i.e. the convective flow does not result in significant changes in pressure:

$$\left(\frac{\partial P}{\partial x}\right)_B = \left(\frac{\partial P}{\partial x}\right)_A$$

Our equation describing the flow becomes

$$\rho V\frac{\partial V}{\partial x} = -(\rho_B - \rho_A)g + \mu\frac{\partial^2 V}{\partial y^2}$$

From earlier, since the density difference between case A and case B is small,

$$(\rho_B - \rho_A) = -\bar{\rho}\,\beta\,(\mathrm{T_B} - \mathrm{T_A})$$

$$\rho V\frac{\partial V}{\partial x} = \bar{\rho}\beta\,(T - T_\infty)g + \mu\frac{\partial^2 V}{\partial y^2}$$

This equation is sometimes referred to as the **Boussinesq** Equation.

5.3.2. Scaling analysis

$$\rho V\frac{\partial V}{\partial x} \;=\; \bar{\rho}\beta(T - T_\infty)g \;+\; \mu\frac{\partial^2 V}{\partial y^2}$$

We are not interested in solving this, just examining the likely size of the maximum velocity in the boundary layer and how we should scale it intelligently for dimensional analysis purposes. If we assume that the most important thing here is a balance between buoyancy forces and viscous forces

$$\mu\frac{\partial^2 V}{\partial y^2} \approx \bar{\rho}\,\beta(T - T_\infty)g$$

Rough scaling

$$\frac{\mu V}{\delta^2} \sim \bar{\rho}\beta(T - T_\infty)g$$

It seems that we should scale V as

where δ is the boundary layer thickness

Also, for the boundary conditions we expect

$y = 0$	$y = \delta$
$V = 0$	$V = \mu \dfrac{dV}{dy} = 0$
$T = T_w$	$T = T_\infty \quad \dfrac{dT}{dy} = 0$

The simplest polynomial fits which satisfy these are conditions are

$$V = V_{scale} \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2 \quad \text{with } \; T - T_\infty = (T - T_\infty)\left(1 - \frac{y}{\delta}\right)^2$$

where V_{scale} is some multiple of $\dfrac{\bar{\rho}\,\beta\, g\, \Delta T\, \delta^2}{\mu}$.

It is beyond the scope of this course to prove it, but a control volume analysis of the type that you did for flow in a momentum/forced convection boundary layer, shows that

$$\delta \propto x^{1/4} \;\text{ and }\; V_{scale} \propto x^{1/2}$$

In trying to scale general problems, then, we should think of using

$$V_{scale} = \frac{\bar{\rho}\,\beta\, g\, \Delta T\, \delta^2}{\mu}$$

as a typical velocity, or rather, since we don't have really know how δ behaves in general flows, we should use

$$V_{scale} = \frac{\bar{\rho}\,\beta\, g\, \Delta T\, D^2}{\mu}$$

where D is a natural length for the problem.

When talking about a parameter to replace our beloved Reynolds Number, we will then use the **Grashof Number (Gr)**

$$Gr = \frac{\bar{\rho} V_{scale} D}{\mu} = \qquad\qquad \text{where} \qquad \nu = \frac{\mu}{\rho}$$

5.4. Typical Free Convection Boundary Layer Type Flows

The figures shown previously are taken from Van Dyke's Book "An Album of Fluid Motion".

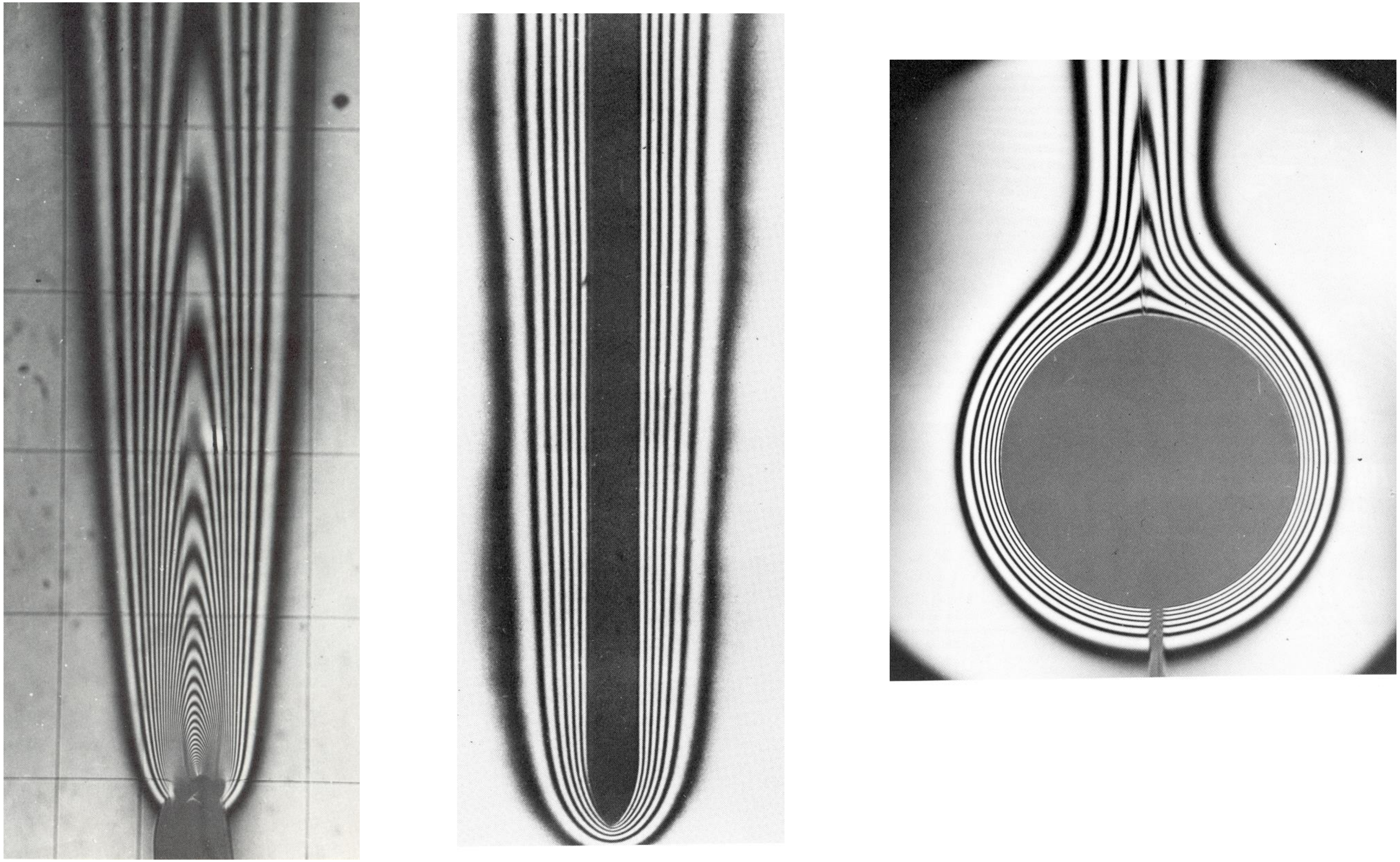
This type of picture is referred to as an interferogram, which is obtained by double exposing pictures with and without the flow. The refractive index of air is a function of its density and each fringe (a black region or a white region) corresponds to a constant density change (which is equivalent to a constant temperature change since the pressure is effectively constant in these flows).

The left hand one is a plume of hot air rising above a heated element, the middle one is the boundary layer on a uniformly heated plate and the final one is the flow over a heated cylinder.

Note:-

(a) The flows are all laminar (because the velocities involved are small).

(b) The boundary layer on the cylinder does not separate (there are no pressure gradients, therefore no adverse pressure gradients) but meets at the top and is then shed as a rising plume.



Taken from An Album of Fluid Motion by Milton Van Dyke

5.5. *Dimensional Analysis in General*

The problem above is a classic example of where dimensional analysis performed without much thought would have given a *misleading* answer.

e.g. Flow over a heated circular cylinder with constant wall temperature

Geometry	D
Phenomenon	g
Fluid Properties	$\rho \quad \mu \quad \lambda \quad \beta \quad c_p$
Operating Point	$\Delta T \quad \dot{Q}$

We would have been tempted to say that

$$\dot{Q} \quad \text{is a function of} \quad D, \rho, \mu, \lambda, c_p, \beta, \Delta T, g$$

9 dimensional variables, leading to 5 groups once the four quantities M, L, time and θ are eliminated.

$\boldsymbol{\beta \Delta T g}$ must appear **together**. i.e. they represent *one* independent dimensional quantity *not three*.

As in forced convection, use a heat transfer coefficient

$$\frac{\dot{Q}}{A\Delta T} = h_{overall} =$$

⇒ Nusselt Number

$$Nu_{overall} = \frac{h_{overall} D}{\lambda} = fn\left(\frac{\bar{\rho}^2 \beta g \, \Delta T \, D^3}{\mu^2}, \frac{\mu c_p}{\lambda}\right)$$

where

$$Gr = \text{Grashof number} \quad = \quad \frac{\bar{\rho}^2 \beta g \, \Delta T \, D^3}{\mu^2}$$

5.6. Key Points for Natural Convection

- The Grasshof Number replaces Reynolds Number.
- Pressure variations due to motion are small.
- Flow velocities are relatively modest, so that flows stay laminar for quite a long time.
- In general, $Nu_{overall} = \frac{h_{overall}D}{\lambda} = fn\ (Gr, Pr)$
- Re, Gr, Nu, Pr, etc. are all in the Data Book (page 4)

6. Heat Transfer by Radiation

Heat can be transferred by Electromagnetic waves emitted by a body *as a result of its temperature.* It does not require a medium between the bodies (e.g. sun to earth), and can occur through a medium which is colder than either body.

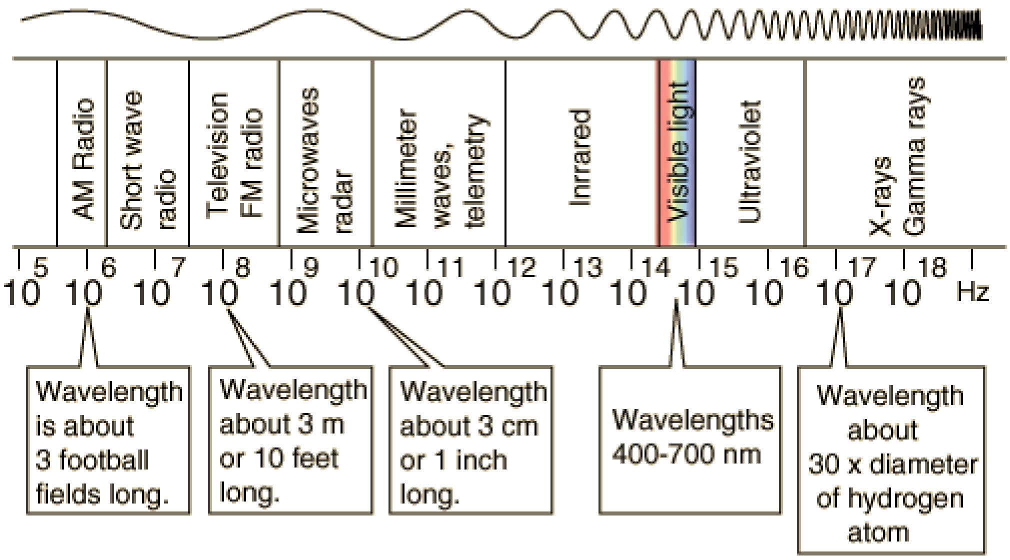


Figure 1. Electromagnetic Spectrum, source
http://en.wikipedia.org/wiki/Electromagnetic_spectrum

Thermal radiation lies approximately in the wavelength range 0.1μm to 100 μm (3.10^{15} to 3.10^{12} Hz), visible light is in the range 0.35 μm to 0.75 μm.

6.1. *The black body*

A "***black body***" is an idealised object, which will

1. completely absorb all radiation incident upon it.
2. emit the largest amount of energy per unit area possible for a given temperature

The maximum rate at which a body can radiate energy is given by the ***Stefan-Boltzmann Law***[1]

$$E_b = \sigma T^4$$

where σ (= 5.67×10^{-8} $Wm^{-2}K^{-4}$) is the ***Stefan-Boltzmann constant*** and E_b is called the *black body* emissive power.

[1] The Stefan-Boltzmann Law can be derived from statistical mechanics, which is beyond the scope of this course.

The monochromatic emissive power of a black body[2] (i.e. the power emitted between wavelengths λ and λ+dλ) is given by the relationship

$$E_{b\lambda} = \frac{c_1 \lambda^{-5}}{e^{c_2/\lambda T} - 1}$$

where λ is the wavelength, c_1 = 3.743x10^8 Wμm/m^2, and c_2 = 1.4387x10^4 μmK. If one integrates this expression over all wavelengths we obtain

The figure below shows plots from the equation for $E_{b\lambda}$ for a range of temperatures.

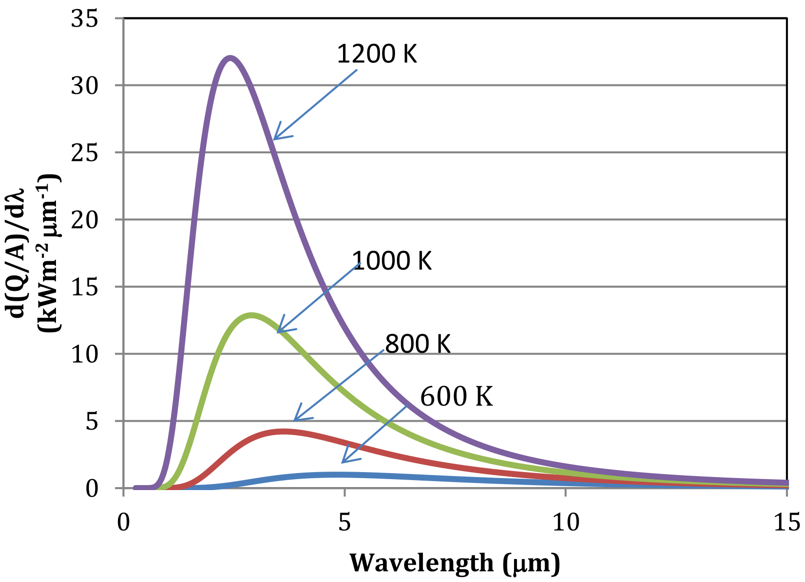


Figure 2. Spectrum of radiation from a black body

Note that the peak moves to shorter wavelengths with increasing temperature – the visible spectrum is between 0.4 μm (violet) and 0.7μm (red), so we would expect a body being heated to appear dull red first, then a colour change through orange to yellow etc.

The peak value of $E_{b\lambda}$ is at a wavelength $\lambda_{\max} = \dfrac{2897.6}{T}\,\mu m$

How hot is the surface of the sun?

[2] This is a result from statistical thermodynamics and its derivation is beyond the scope of this course.

6.2. *Real surfaces*

Real surfaces emit less radiation than a black body. The monochromatic emissivity ε_λ characterises the surface

$$\varepsilon_\lambda = \frac{\text{radiation emmited per unit area between } \lambda \text{ and } \lambda + d\lambda}{E_{b\lambda}}$$

The monochromatic emissivity is a function of the wavelength of radiation, λ. The total emissivity (usually just called the emissivity) is given by

$$\varepsilon = \frac{}{} = \frac{}{} = \frac{}{}$$

In general the emissivity is a function of temperature.

Surface	**T(K)**	**Emissivity ε**	**Surface**	**T(K)**	**Emissivity ε**
Polished aluminium	500, 1000	0.039, 0.066	Polished copper	353	0.018
Polished iron	450	0.052	Oil paints, all colours	373	0.92-0.96
Oxidized iron	373	0.74	Water	273	0.95

6.2.1. The Grey Body

A grey body is defined such that the monochromatic emissivity, ε_λ is independent of wavelength, i.e. $\varepsilon_\lambda$is constant $\Longrightarrow$

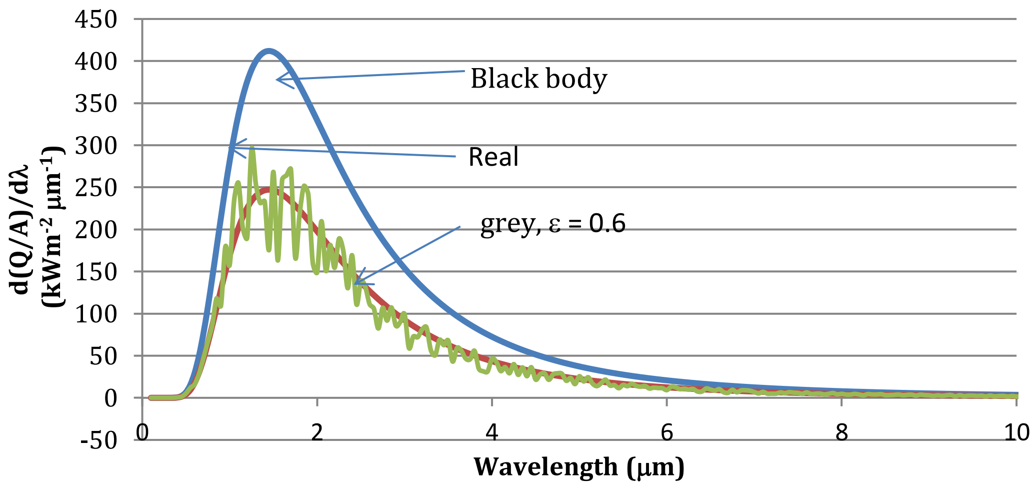
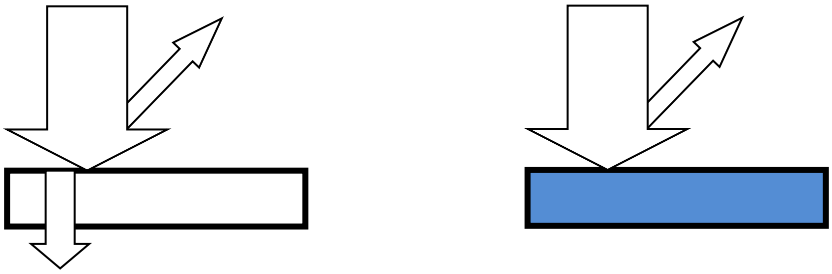


Figure 3. Spectrum of emissive power from a black, grey (ε=0.6) and a real surface at 2000 K

6.2.2. Absorption, transmission and reflection

When radiation strikes a surface, a fraction (= ρ, the reflectivity) will be reflected, a fraction (= α, the absorbtivity) will be absorbed, and a fraction (= τ, the transmissivity) will be transmitted. Thus $\rho+\alpha+\tau=1$. Most solid bodies do not transmit thermal radiation, so the transmissivity can be taken as zero, hence $\rho+\alpha=1$.



The reflected part of the radiation may be "specular" (i.e. as from a mirror), diffuse (i.e. scattered equally in all directions), or some combination of these. ***Most surfaces are predominately diffuse reflectors.***

6.2.3. Kirchhoff's identity

Due to processes occurring at the quantum scale (very very small!), the probability of a photon being absorbed on hitting the surface, is equal to the probability of a photon being emitted. This gives rise to Kirchhoff's identity.

Kirchhoff's identity, $\varepsilon = \alpha$, i.e. the absorptivity is equal to the emissivity.

There is a question on the example sheet which "proves" this must be true for a body exchanging radiation with black enclosure, at equilibrium. However, Kirchhoff's identity is generally true.

6.2.4. Radiation to and from a non -black bodies.

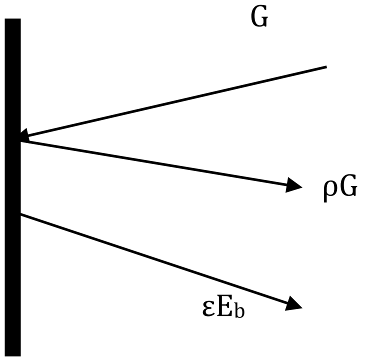
The total radiative emission from a non-black body surface not only has an εE_b component (the radiation leaving a surface due to its temperature), but also a ρ times the incident radiation' component.

Let us define two new terms:

***G* = Irradiation.** The total radiation incident on a surface per unit time and area

***J* = Radiosity.** The total radiation leaving a surface per unit time and area.

We further assume that these quantities are uniform over each surface: this needs to be kept in mind when applying the results to come.



$$J = \varepsilon E_b + \rho G$$

$$J = \varepsilon E_b + \rho G$$

The radiation leaving a surface is the sum of the radiation originating from the body due to its temperature, and the reflected part of the incoming radiation.

Since $\rho = 1 - \alpha = 1 - \varepsilon$ (the surfaces are assumed to be non-transmissive), the expression for J may be written

Now the net radiation leaving a surface per unit area per unit time $\left(\dot{Q}/A\right)$ is the difference between the radiosity and the irradiation, so

If we substitute for G, we obtain

$$\dot{Q} = \frac{E_b - J}{(1-\varepsilon)/\varepsilon A}.$$

The expression above can be interpreted using an "ohm's law" analogy, with $E_b - J$ as a potential difference, and $\frac{1-\varepsilon}{\varepsilon A}$ as a 'surface' resistance.

i.e.

6.3. *Radiative exchange between surfaces*

Let us examine the heat transfer between two surfaces A_1 and A_2 (both diffuse reflectors)

We define

F_{12} **as the fraction of energy leaving surface 1 that arrives at surface 2**

F_{21} **as the fraction of energy leaving surface 2 that arrives at surface 1**

- $F_{i,j}$ is called the shape factor (or sometimes the view factor)
- $F_{i,j}$ is purely geometric and is related to how much of surface j can be seen by surface i

Since electromagnetic rays travel in straight lines, if radiation can travel from 1 to 2 along a path, it can also travel from 2 to 1 along the same path. This means that view factors are linked by the ***reciprocity relationship***.

$$A_1 F_{12} = A_2 F_{21}$$

sinceit is ***purely geometric*** this does not depend the nature of the radiation emitted or reflected from the surface.

Of the total radiation that leaves surface 1 ($J_1 A_1$), a quantity $J_1 A_1 F_{12}$ reaches surface 2. Likewise from 2 to 1, a quantity $J_2 A_2 F_{21}$ reaches surface 2 from surface 1. The net exchange is thus given by

$$\dot{Q}_{12} = J_1 A_1 F_{12} - J_2 A_2 F_{21}$$

But $A_1 F_{12} = A_2 F_{21}$, so

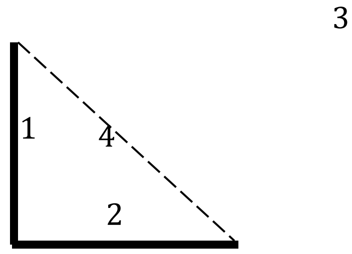
$$\dot{Q}_{12} = \frac{J_1 - J_2}{1/A_1 F_{12}}$$

We interpret this expression in an "Ohm's Law" way, regarding $\dot{Q}_{12}$ as a current flow, $J_1 - J_2$ as a potential difference, and $\dfrac{1}{A_1 F_{12}}$ as a 'space' resistance.

The determination of shape factors for other than the simplest geometries is tedious and normally done by computer or taken from graphical data. However we shall now see, certain useful results can be deduced by physical reasoning.

6.3.1. Shape factor algebra

Consider the radiation exchange situation shown in the figure – the L-section is extends infinitely into the page, and for simplicity, let us assume surfaces 1 and 2 are of equal length in the plane of the page. Typically we will be interested in the radiation exchange between the three surfaces, 3 representing the environment. Ignore for the moment the dashed line (surface 4).



Shape factors for a surface add up to unity

By definition, the shape factor is the fraction of energy leaving one surface and hitting another, therefore

$$F_{11} + F_{12} + F_{13} = 1$$

Note that we have included the self shape factor, F_{11}, since a surface may be able to see itself.

Convex surfaces cannot see themselves – self view factor = 0

In the above figure, surface 1 cannot see itself, so $F_{11} = 0$.

Reciprocity

As demonstrated above, for example, $F_{12}A_1 = F_{21}A_2$, where $A_1 = A_2$ in this example.

Using these rules (and simple logic), we can for example, calculate F_{12}. By drawing the imaginary surface 4, we note that $F_{14} + F_{12} = 1$, so if we can find F_{14} the problem is solved.

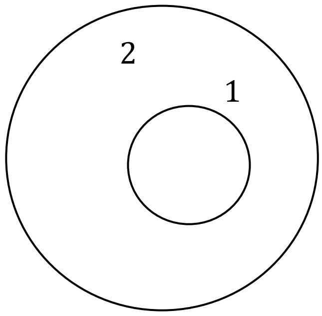
By reciprocity:

And by symmetry

Note that this relationship applies to black and non-black bodies, but the radiation must be diffuse.

6.3.2. Example – radiation exchange between concentric cylinders.

The radiation network is



Thus, the overall "resistance" is

So that the heat exchanged between 1 and 2 is,

Since $F_{12}=1$, and multiplying top and bottom by A_1,

$$\dot{Q} = \frac{A_1(\sigma T_1^4 - \sigma T_2^4)}{\left(\frac{1-\varepsilon_1}{\varepsilon_1} + 1 + \left(\frac{A_1}{A_2}\right)\frac{(1-\varepsilon_2)}{\varepsilon_2}\right)}$$

Note that this expression is valid whether the cylinders are concentric or not; in fact either cylinder can be any shape as long as $F_{11}=0$ - i.e. the inner cylinder cannot see itself.

A very useful result is obtained when $A_2 \to \infty$ (or $\varepsilon_2 \to 1$), which is that

This expression is quite general – it says (in words) that when an object is in a large enclosure, the net radiation exchange is independent of the emissivity of the enclosure, and it is as if the enclosure were a black body.

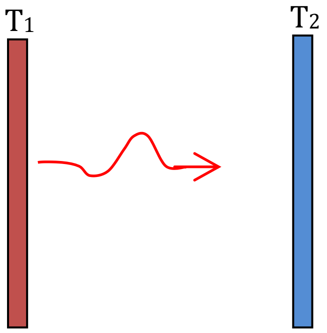
A couple of lines of manipulation of the relationships above shows that $G_1 \approx E_{b2}$ for $A_2 >> A_1$, i.e. all the radiation striking surface 1, appears as if it has come from a black body.

6.3.3. Example - Radiation Shields.

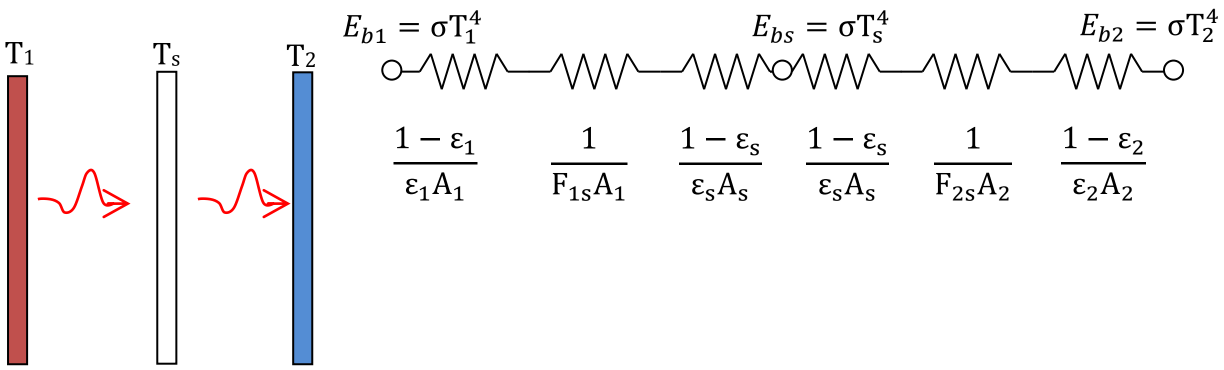
Two very large parallel planes with emissivities of 0.3 and 0.8 exchange heat. Find the reduction in heat transfer per unit area, if a shield with emissivity 0.04 is placed between them.

[Note that all $A_1 = A_2 = 1$ and all the shape factors are unity, $F_{12} = F_{21}=1$ etc..]

No shield



Shield



Since

$$\dot{Q} = \frac{\sigma(T_1^4 - T_2^4)}{R}$$

6.4. *Radiation in the environment*

Solar radiation is very different to most terrestrial radiation, in that it is at much shorter wavelengths, and many important phenomena that affect us directly relate to this.

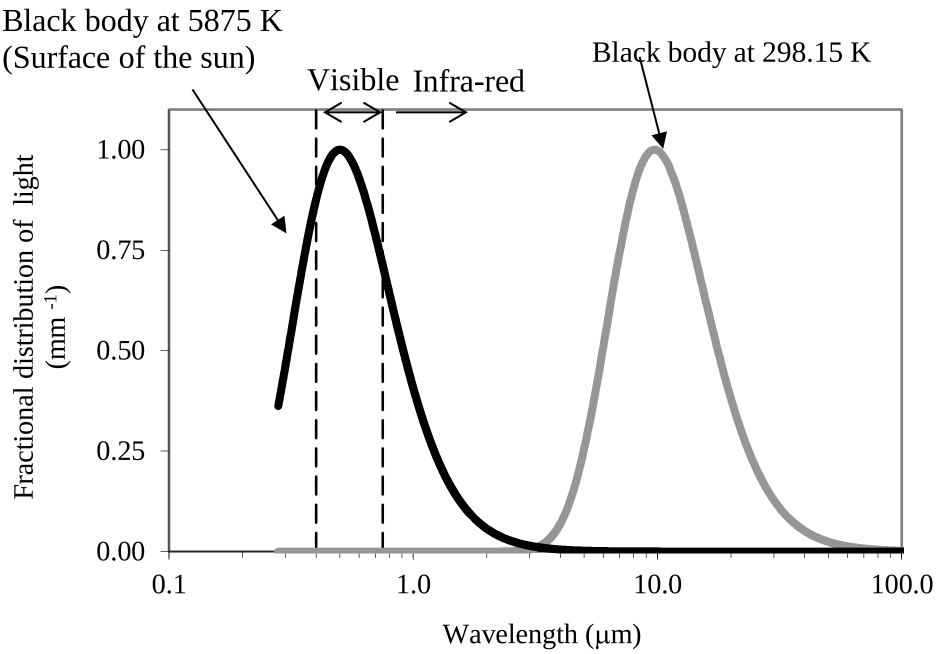


Figure 4. Spectrum for black bodies at 5780 K (i.e. the sun) and 298.15 K (i.e. the earth). Note that the spectrums have been normalised so that the area under each curve is unity.

Table 1. Major contributors to the greenhouse effect and which wavelengths of radiation they absorb

Gas	Main Absorption bands
Carbon Dioxide, CO_2	4.3 – 4.4 μm, 14 – 16 μm
Water Vapour, H_2O	2.6 -2.8 μm, 5.2 – 7.4 μm, 21 – 116 μm,
Methane, CH_4	3.2-3.5 μm, 7.4 – 8.1 μm
Nitrous Oxide, N_2O	4.5-4.6 μm, 7.6-8 μm

- Incoming UV rays are absorbed by O_3 and O_2.

- Outgoing infra-red radiation is absorbed by gases such as CO_2, H_2O and CH_4.

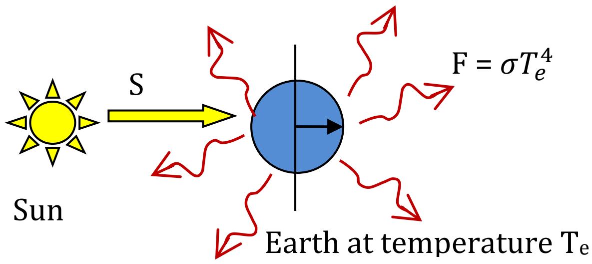
.

6.4.1. Example- a simple model of the greenhouse effect

A very simple model of the green house gas effect shows how the wave length difference between solar and terrestrial radiation is central. Previously we assumed that the absorbance, reflectivity and emissivity were not functions of wavelength, when considering the greenhouse effect, this approximation is no longer valid.

Albedo, A = the fraction of the incident (i.e. short wave) radiation which is reflected

First consider a planet, radius R, with a solar flux (short wave radiation) of S Wm^{-2}, and an albedo of A. For long wave radiation, the planet can be taken to be black.



The overall energy (short wave) absorbed is projected area×S × fraction absorbed, i.e.

$$\dot{Q}_S = S\pi R^2(1-A)$$

If the planet's surface is at T_e, then the (mainly long wave) energy emitted is

$$\dot{Q}_p = 4\pi R^2 \sigma T_e^{\,4}.$$

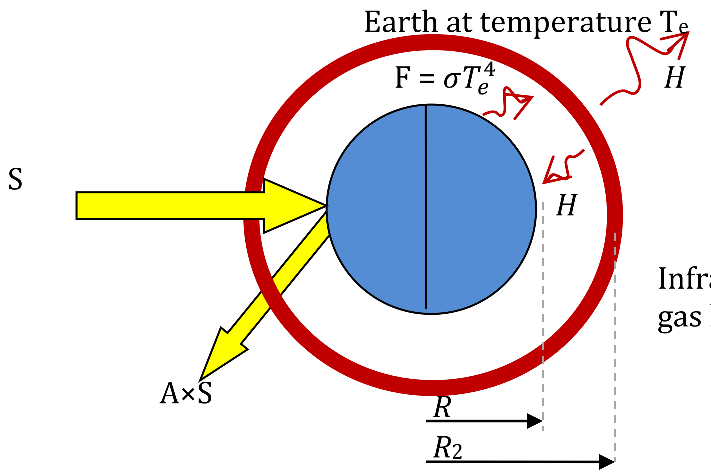
At steady state $\dot{Q}_s = \dot{Q}_p$ giving

$$\sigma T_e^4 = \frac{S(1-A)}{4}$$

For S = 1000 Wm^{-2}, and average value of the albedo, A = 0.1, we get T_e = 250 K, **too cold to support life on earth**!

We have neglected fact that the Earth's atmosphere contains gases that absorb the longwave radiation emitted from the Earth's surface.

Consider the (simplified) situation shown below, where we have (very crudely) assumed that infra-red absorbing gases form a layer of some thickness above the surface, spread equally over the entire planet.



Note that the diagram exaggerates the distance of the layer form the planet, in practice $R_2 \approx R$

The overall energy (short wave) absorbed is projected area×S × fraction absorbed, i.e.

$$\dot{Q}_s = S\pi R^2(1-A)$$

The gas layer absorbs F, and itself emits a radiative (infra-red) flux of H. Since the layer is elevated from the surface, it emits radiation through both its upper and lower surface in equal amounts. Thus overall the layer emits a total flux of $2H$ $W\ m^{-2}$. In this new situation the outer surface of the absorbing layer becomes the outer surface of the planet**,** so that the amount of energy radiated by the planet becomes,

$$\dot{Q}_p = 4\pi R^2 H\ .$$

At steady sate $\dot{Q}_s = \dot{Q}_p$ giving

$$H = \frac{(1-A)S}{4} \qquad (*)$$

At the surface, the energy balance is:

$$4\pi R^2 F = H\ 4\pi R^2 + (1-A)S\pi R^2$$

i.e. $$F = H + \frac{(1-A)S}{4}$$

Using this result to eliminate H from (*) gives:

$$F = \sigma T_e^{\ 4} = \frac{(1-A)S}{2}$$

The addition of one IR absorbing layer has thus changed the balance of energy at the ground. Using the same values of S (=1000 Wm^{-2}) and A (=0.1) as before, we now get T_e = 298.5 K.

Without the greenhouse effect, life on earth would not exist!

6.5. *Key points for Radiative heat transfer*

- The power per unit area emitted by a black body is $E_b = \sigma T^4$
- Emissivity is the ratio of actual power emitted to that emitted by a black body at the same temperature.
- A grey surface has a constant emissivity which does not depend on wavelength. For a real (grey) surface the power emitted is $E = \varepsilon\sigma T^4$
- $\rho+\tau+\alpha = 1$ (reflectivity + transmissivity + absorptivity = 1)
- The absorptivity is equal to the emissivity (Kirchhoff's law), $\varepsilon = \alpha$
- The irradiation, G, is the total power arriving on a unit area of a surface
- The radiosity, J, is the total power leaving a unit area of a surface. $J = \rho G + E$
- The shape/view factor between two surfaces is a geometric quantity which gives the amount of radiation leaving one surface and hitting the other.
- We can construct an electrical circuit analogy for a radiation problem.
- Without the greenhouse effect, the earth would be too cold to support life. Too much infra-red absorbing gas in the atmosphere will cause global warming.