Lecture 1

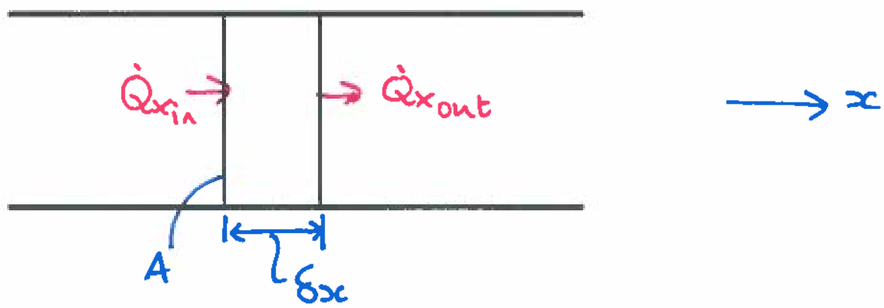
Introduction; Differentiation of Scalar Fields

1.1 Course Introduction

Modelling of the physical world is a key part of engineering. Engineers rely on such analyses during the evaluation of design concepts.

Models are often developed from an understanding of the processes at work on a small element of the problem. For example, the diagram below shows the one-dimensional heat flow for a small element of substance,

1–D
Heat Conduction



$$\dot{Q}_{x_{in}} - \dot{Q}_{x_{out}} = \frac{\partial E}{\partial t} = m c_v \frac{\partial T}{\partial t}$$

$$q_x A - \left(q_x + \frac{\partial q_x}{\partial x}\delta x\right) A = \rho A \delta x \, c_v \frac{\partial T}{\partial t}$$

$$\frac{\partial q_x}{\partial x} = -\rho c_v \frac{\partial T}{\partial t}$$

where we have used partial differentials because q_x and T are functions of both x and t.

In the above example, we have only allowed for heat flux in one direction, q_x. In 3-D, a similar analysis would lead to,

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = -\rho c_v \frac{\partial T}{\partial t}$$

where q_x, q_y and q_z are the heat fluxes in the three Cartesian coordinate directions.

It is apparent that heat flux is a *vector*, $\mathbf{q}$, with components q_x, q_y and q_z. The vector $\mathbf{q}$ is not an isolated vector, it is distributed and varies with both space (x, y, z) and time t: it is a *vector field*.

Vector calculus provides a set of rules for working with vector fields. For example, we will learn that we can write our heat flux equation concisely as,

$$\nabla \cdot \underline{q} = -\rho c_v \frac{\partial T}{\partial t}$$

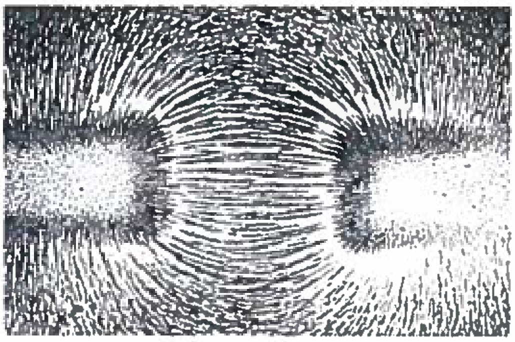
and that this equation is valid in *any* coordinate sytem.

As you become familiar with vector calculus (and the use of $\nabla\phi$, $\nabla \cdot \mathbf{V}$ and $\nabla \times \mathbf{V}$) you will be able to write down equations like $\nabla \cdot \mathbf{q} = -\rho c_v \partial T/\partial t$ without having to derive them from small elements.

The techniques developed in this course are applicable to the many vector fields that are found in engineering. Two examples of vector fields are fluid flow and magnetic fields:

(H. Babinsky, CUED)

Steady velocity field $\underline{V} = \underline{V}(x, y)$



(public domain, commons.wikipedia.org)

Magnetic field $\underline{B} = \underline{B}(x, y)$

1.2 Scalar functions

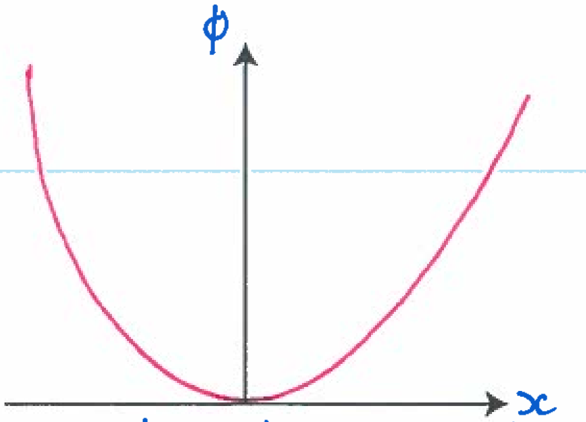
We express that ϕ is a function of one independent variable x by writing,

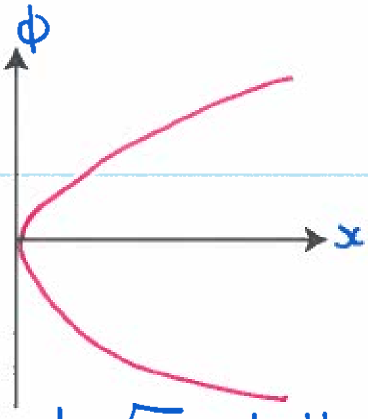
$$\phi = f(x)$$

or,

$$\phi = \phi(x)$$

This means that for a value of x there is a corresponding value of the dependent variable, ϕ, and we can represent this with a *curve* in 2-D space.





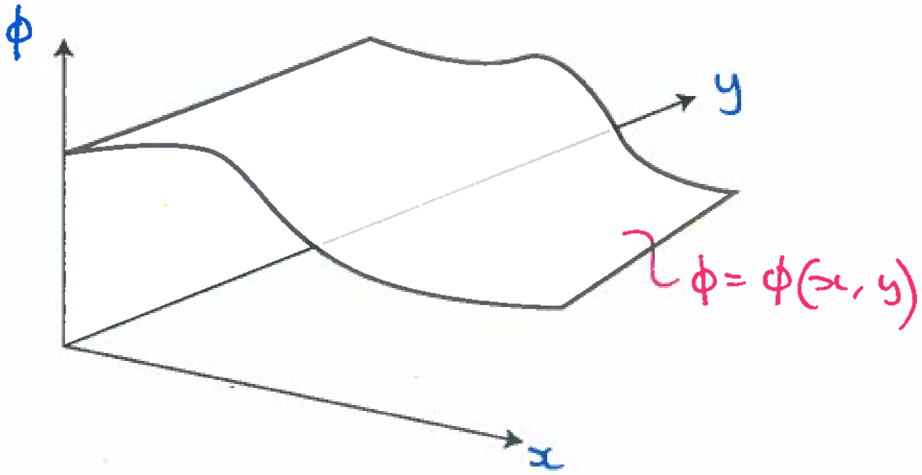
$\phi = x^2$, single-valued function

$\phi = \sqrt{x}$ double-valued function

If ϕ is a function of two independent variables, x and y, we write,

$$\phi = \phi(x, y)$$

Now, a pair of values of x and y correspond to the dependent variable ϕ. This relationship can be represented by a *surface* in 3-D space.

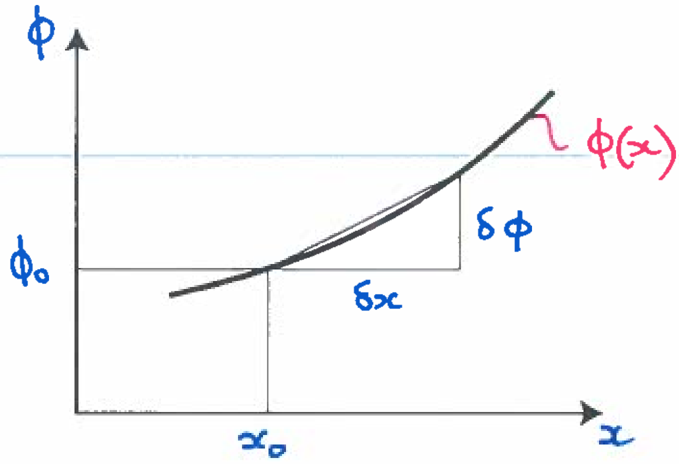


1.3 Differentiation of a scalar function of one variable

If ϕ is a function of one variable, $\phi = \phi(x)$, then we are familiar with the following definition of the derivative,

$$\frac{d\phi}{dx} = \lim_{\delta x \to 0} \left(\frac{\phi(x + \delta x) - \phi(x)}{\delta x} \right)$$

We can see that, as $\delta x \to 0$, the derivative becomes tangent to the curve of $\phi(x)$.



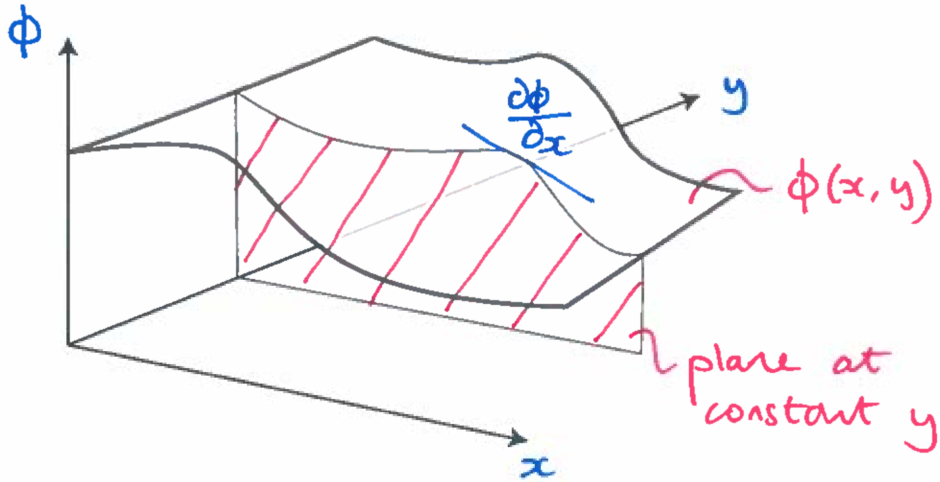
1.4 Differentiation of a scalar function of more than one variable

If ϕ is a function of two variables, $\phi = \phi(x,y)$, we now have *partial* derivatives defined as,

$$\frac{\partial \phi}{\partial x} = \lim_{\delta x \to 0} \left(\frac{\phi(x+\delta x, y) - \phi(x,y)}{\delta x} \right)$$

$$\frac{\partial \phi}{\partial y} = \lim_{\delta y \to 0} \left(\frac{\phi(x,y+\delta y) - \phi(x,y)}{\delta y} \right) \quad . \tag{1.1}$$

$\partial \phi / \partial x$ is the rate of change of ϕ with x when y is held constant. It is the slope of the curve formed by slicing the surface $\phi = \phi(x,y)$ along a plane at constant y.



The notation $\partial \phi / \partial x$ implies that y is held constant. If there is any doubt about what is being held constant, you should write,

$$\left(\frac{\partial \phi}{\partial x} \right)_y$$

If ϕ is a function of three independent variables, $\phi = \phi(x,y,z)$, the definition of the partial derivatives is similar (though there is no convenient geometrical representation),

$$\frac{\partial \phi}{\partial x} = \lim_{\delta x \to 0} \left(\frac{\phi(x+\delta x,y,z) - \phi(x,y,z)}{\delta x} \right) , \tag{1.2}$$

$$\frac{\partial \phi}{\partial y} = \lim_{\delta y \to 0} \left(\frac{\phi(x,y+\delta y,z) - \phi(x,y,z)}{\delta y} \right) , \tag{1.3}$$

$$\frac{\partial \phi}{\partial z} = \lim_{\delta z \to 0} \left(\frac{\phi(x,y,z+\delta z) - \phi(x,y,z)}{\delta z} \right) . \tag{1.4}$$

$\partial\phi/\partial x$ is now the rate of change of ϕ with x in a direction such that *both* y and z are constant.

Returning to $\phi = \phi(x,y)$, higher order partial derivatives such as,

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) \qquad \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) \tag{1.5}$$

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) , \qquad \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) , \tag{1.6}$$

are defined in a similar way,

$$\frac{\partial^2 \phi}{\partial x^2} = \lim_{\delta x \to 0} \left(\frac{\partial\phi/\partial x(x+\delta x,y) - \partial\phi/\partial x(x,y)}{\delta x} \right) , \tag{1.7}$$

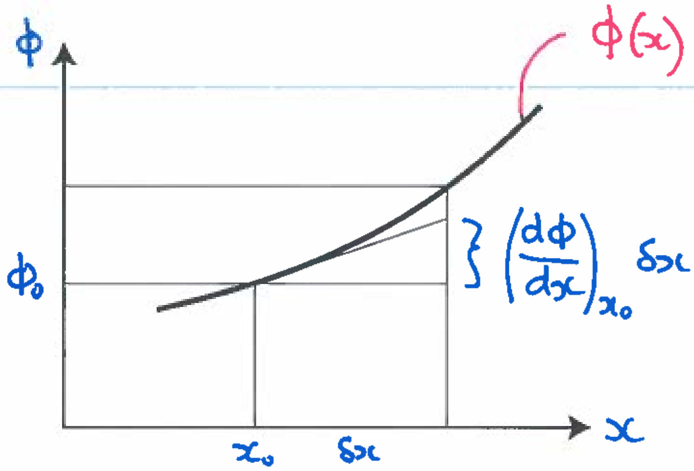
$$\frac{\partial^2 \phi}{\partial y^2} = \lim_{\delta y \to 0} \left(\frac{\partial\phi/\partial y(x,y+\delta y) - \partial\phi/\partial y(x,y)}{\delta y} \right) , \tag{1.8}$$

$$\frac{\partial^2 \phi}{\partial y \partial x} = \lim_{\delta y \to 0} \left(\frac{\partial\phi/\partial x(x,y+\delta y) - \partial\phi/\partial x(x,y)}{\delta y} \right) , \tag{1.9}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = \lim_{\delta x \to 0} \left(\frac{\partial\phi/\partial y(x+\delta x,y) - \partial\phi/\partial y(x,y)}{\delta x} \right) . \tag{1.10}$$

For example $\partial^2\phi/\partial x\partial y$ is the rate of change of $\partial\phi/\partial y$ with x, along a line of constant y.

1.5 Total differentials

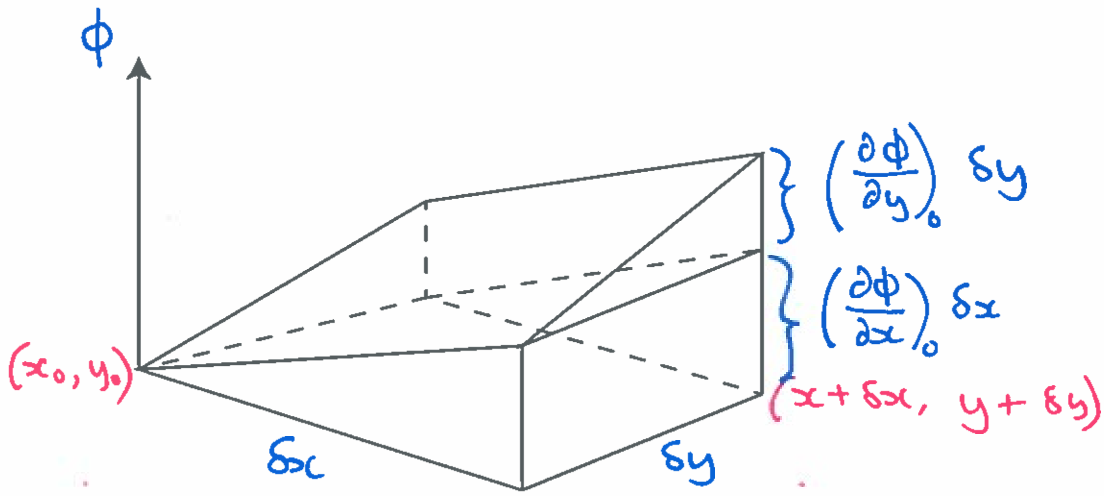


If $\phi = \phi(x)$, the definition of the derivative tells us that the change in ϕ, $\delta\phi$, when we change x by δx is given by,

$$\phi(x_0+\delta x) - \phi(x_0) = \delta\phi \approx \left(\frac{d\phi}{dx}\right)_{x_0} \delta x$$

and in the limit at $\delta x \to 0$ we have,

$$d\phi = \frac{d\phi}{dx}\, dx$$



If $\phi = \phi(x,y)$ then we can use partial derivatives to evaluate the contribution to $\delta\phi$ from both δx and δy,

$$\phi(x_0+\delta x, y_0+\delta y) - \phi(x_0,y_0) = \delta\phi \approx \left(\frac{\partial\phi}{\partial x}\right)_0 \delta x + \left(\frac{\partial\phi}{\partial y}\right)_0 \delta y$$

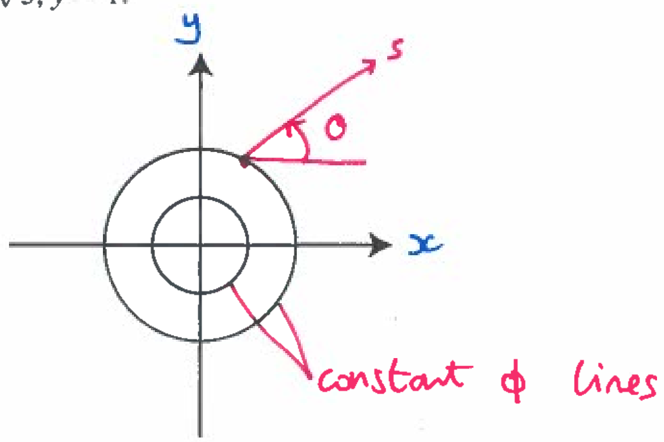
As $\delta x \to 0$ and $\delta y \to 0$ we may write this neatly as,

$$d\phi = \frac{\partial\phi}{\partial x}\, dx + \frac{\partial\phi}{\partial y}\, dy$$

and $d\phi$ is called the *total differential*.

Example

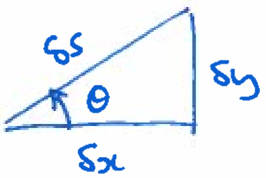
A type of vortex has a two-dimensional, steady in time, pressure field $p = p_0 + (x^2 + y^2)$. Find an expression for the rate of change of pressure with distance in a direction at an arbitrary angle θ to the x-axis at the point $x = \sqrt{3}$, $y = 1$.



$$\delta p = \frac{\partial p}{\partial x}\delta x + \frac{\partial p}{\partial y}\delta y = 2x\,\delta x + 2y\,\delta y$$

δs is small distance in s direction:

$$\frac{\delta p}{\delta s} = \frac{\partial p}{\partial x}\frac{\delta x}{\delta s} + \frac{\partial p}{\partial y}\frac{\delta y}{\delta s}$$

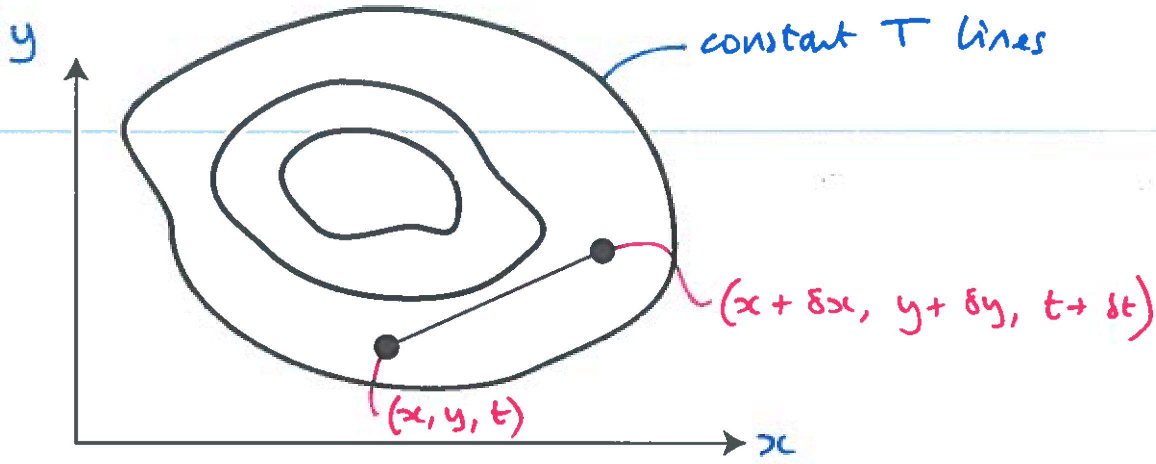


as $\delta s \to 0$ $$\frac{dp}{ds} = \frac{\partial p}{\partial x}\cos\theta + \frac{\partial p}{\partial y}\sin\theta$$

At $x = \sqrt{3}$, $y = 1$ $$\frac{dp}{ds} = 2\sqrt{3}\cos\theta + 2\sin\theta$$

1.6 Substantive or material derivative

Consider a temperature field T that is a function of two-dimensional space and also time, $T = T(x, y, t)$



Imagine we have a probe to measure the temperature and we move the probe from a point (x,y,t) to a point $(x+\delta x, y+\delta y, t+\delta t)$. We can use the total differential to work out the change in T between these two points,

$$\delta T = \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial t}\delta t$$

Dividing through by the small change in time, δt, we obtain,

$$\frac{\delta T}{\delta t} = \frac{\partial T}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial T}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial T}{\partial t}$$

In the limit as $\delta t \to 0$, $\delta x/\delta t \to dx/dt = V_x$ where $V_x = V_x(t)$ is the x-component of the velocity of the probe. Similarly, $\delta y/\delta t \to dy/dt = V_y$ where $V_y = V_y(t)$ is the y-component of the velocity of the probe,

$$\frac{dT}{dt} = V_x\frac{\partial T}{\partial x} + V_y\frac{\partial T}{\partial y} + \frac{\partial T}{\partial t}$$

dT/dt is a total derivative because it is associated with the specific path taken by our probe; it is the rate of change of the temperature as 'seen' by the probe as it moves through the temperature field.

Note that $\partial T/\partial t$ in the above expression is the rate of change of temperature in time with x and y held constant; this is the rate of change of temperature at a fixed point in space and would be the rate of change seen by a stationary probe ($V_x = V_y = 0$). Even if the temperature field was steady ($\partial T/\partial t = 0$) a moving probe would still see a rate of change of temperature given by $V_x\partial T/\partial x + V_y\partial T/\partial y$.

The path taken by the temperature probe is arbitrary, but if the temperature field is actually a property of a fluid and we are interested in the temperature of a *fluid particle* as it moves through this field, then $V_x = V_x(x,y,t)$ and $V_y(x,y,t)$ are now the components of the fluid's velocity. In this case, the total derivative is referred to as the substantive or material derivative, and is usually written as DT/Dt ("big d by dt"). This derivative is very useful in fluid mechanics. For example, in Paper 4, you will apply D/Dt to the velocity vector field $\mathbf{V}$ to obtain $D\mathbf{V}/Dt$ - the acceleration (vector) of a fluid particle as it moves through a velocity field,

$$\frac{DV}{Dt} = \frac{\partial V}{\partial t} + V_x \frac{\partial V}{\partial x} + V_y \frac{\partial V}{\partial y}$$

1.7 Chain rule for functions of more than one variable

Suppose that we know the relationship between ϕ and an independent variable, x. We now wish to change the independent variable to u where $x = x(u)$ and find $d\phi/du$. We start from,

$$\delta\phi = \frac{d\phi}{dx}\delta x$$

Since $x = x(u)$, we may write,

$$\delta x = \frac{dx}{du}\delta u$$

and so,

$$\delta\phi = \frac{d\phi}{dx}\frac{dx}{du}\delta u \quad . \tag{1.13}$$

In the limit as $\delta u \to 0$ we obtain,

$$\frac{d\phi}{du} = \frac{d\phi}{dx}\frac{dx}{du} \quad , \tag{1.14}$$

which is the familiar 'chain rule' for a scalar function of one variable.

We can follow a similar process if ϕ is a function of two variables, $\phi = \phi(x, y)$. We now seek $\partial\phi/\partial u$ and $\partial\phi/\partial v$ where $x = x(u, v)$ and $y = y(u, v)$. We start with the change in ϕ corresponding to small changes in x and y,

$$\delta\phi = \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y$$

From $x = x(u, v)$ and $y = y(u, v)$, we know that δx and δy are related to δu and δv by,

$$\delta x = \frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v$$

$$\delta y = \frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v \quad . \tag{1.15}$$

Substituting these into the expression for $\delta\phi$ we obtain,

$$\delta\phi = \frac{\partial\phi}{\partial x}\left(\frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v\right) + \frac{\partial\phi}{\partial y}\left(\frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v\right) \quad , \tag{1.16}$$

and we may collect terms in δu and δv so that,

$$\delta\phi = \left(\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u}\right)\delta u + \left(\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial v}\right)\delta v \quad . \tag{1.17}$$

Now, we know that we can also find the change in ϕ corresponding to small changes in u and v using,

$$\delta\phi = \frac{\partial\phi}{\partial u}\delta u + \frac{\partial\phi}{\partial v}\delta v$$

Comparing the previous two expressions, we see that,

$$\frac{\partial\phi}{\partial u} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u} \quad \text{, and} \tag{1.18}$$

$$\frac{\partial\phi}{\partial v} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial v} \quad . \tag{1.19}$$

1.8 Taylor series for functions of more than one variable

For a function of one independent variable, $\phi = \phi(x)$, we can use the following Taylor expansion to find the value of $\phi(x)$ close to some known value $\phi(x_0)$,

$$\phi(x) = \phi(x_0) + (x - x_0)\left(\frac{d\phi}{dx}\right)_0 + \frac{(x-x_0)^2}{2!}\left(\frac{d^2\phi}{dx^2}\right)_0 + \ldots \tag{1.20}$$

where the derivatives are evaluated at $x = x_0$.

For a function of two independent variables, $\phi = \phi(x,y)$, we can use a similar Taylor expansion to find $\phi(x,y)$ given knowledge of ϕ, and the derivatives of ϕ, at (x_0, y_0),

$$\begin{aligned}\phi(x,y) = \phi(x_0,y_0) + (x-x_0)\left(\frac{\partial\phi}{\partial x}\right)_0 + (y-y_0)\left(\frac{\partial\phi}{\partial y}\right)_0 + \\ \frac{(x-x_0)^2}{2!}\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + (x-x_0)(y-y_0)\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_0 + \frac{(y-y_0)^2}{2!}\left(\frac{\partial^2\phi}{\partial y^2}\right)_0 + \ldots\end{aligned} \tag{1.21}$$

Note that we must now include the "cross-terms" such as $\partial^2\phi/\partial x\partial y$.

You can now do Examples Paper 1: Q1