Lecture 11

Introduction to PDEs

11.1 Introduction

We have seen how vector calculus provides a convenient, compact language for the mathematical modelling of physical phenomena. Equations in vector calculus are independent of any particular coordinate system. To solve these equations, the usual process is to choose an appropriate coordinate system and obtain one or more partial differential equations (PDEs). These PDEs can then be solved either analytically (in a minority of practical cases) or numerically.

11.2 Dependent and independent variables

An ordinary differential equation (ODE) has a single *independent* variable. A partial differential equation (PDE) has more than one *independent* variable. The number of simultaneous PDEs to be solved is equal to the number of *dependent* scalar variables. The following examples illustrate this:

- A single scalar variable ϕ that varies over 3-D space, $\phi = \phi(x, y, z)$.

This would yield one PDE for one dependent variable, ϕ

- A single scalar variable ϕ that varies over 2-D space, $\phi = \phi(x, y)$.

One PDE, one dependent variable ϕ

- Three components of velocity, varying over 3-D space: $V_x = V_x(x, y, z)$, $V_y = V_y(x, y, z)$, $V_z = V_z(x, y, z)$.

Three PDEs, three dependent scalar variables

We will only consider PDEs with *one dependent* scalar variable and *two independent* variables, for example:

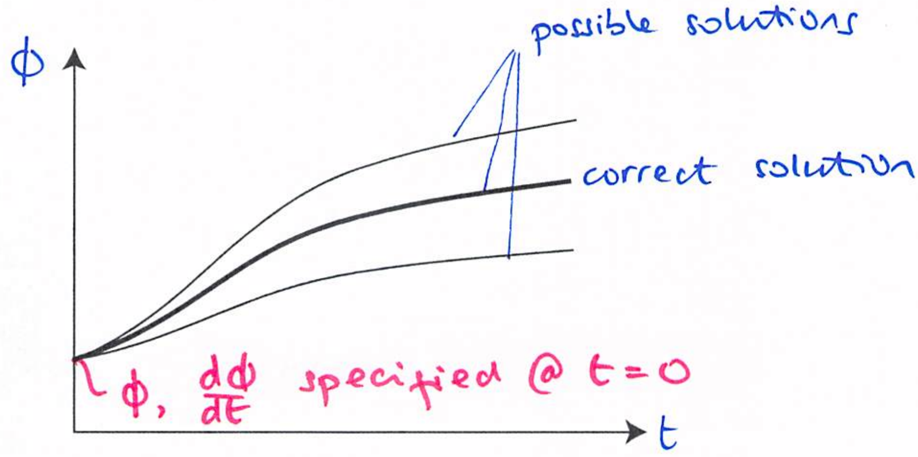
- $\phi = \phi(x, y)$ – the variation of ϕ over an area in 2-D space;
- $\phi = \phi(x, t)$ – the variation of ϕ with time along a line in 1-D space.

In each case, we will have *one* PDE to solve (one dependent variable).

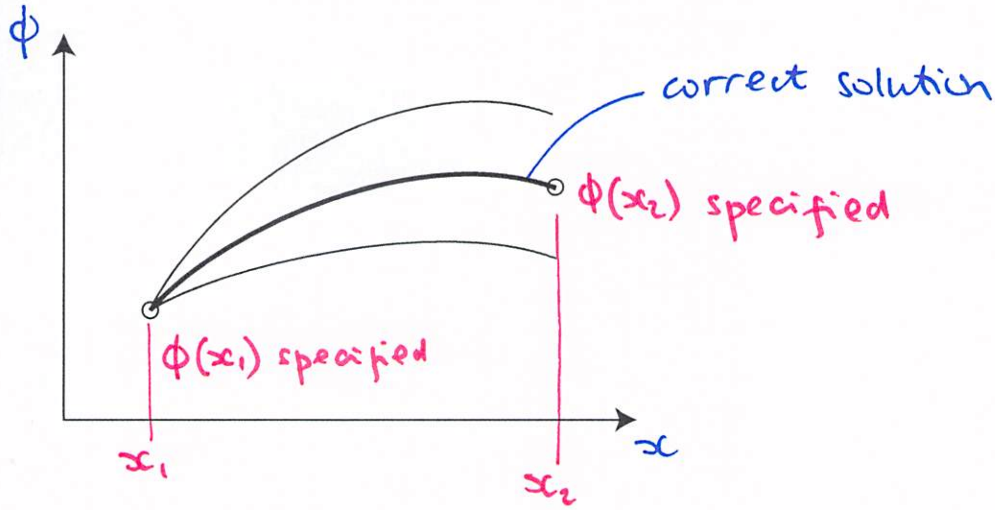
11.3 Second order *ordinary* differential equations

For a first order ODE, we need a single boundary value to fix the solution exactly. For a second order ODE, we need two boundary values.

You will have come across two types of second order ODEs: *initial value* and *boundary value* problems. Usually, an initial value problem has time as the independent variable $\phi = \phi(t)$, and the two boundary conditions are specified at $t = 0$. The solution at $t > 0$ is obtained as the particular curve of a family that satisfies these "initial conditions". The solution can be thought of as "marching" forward from $t = 0$.



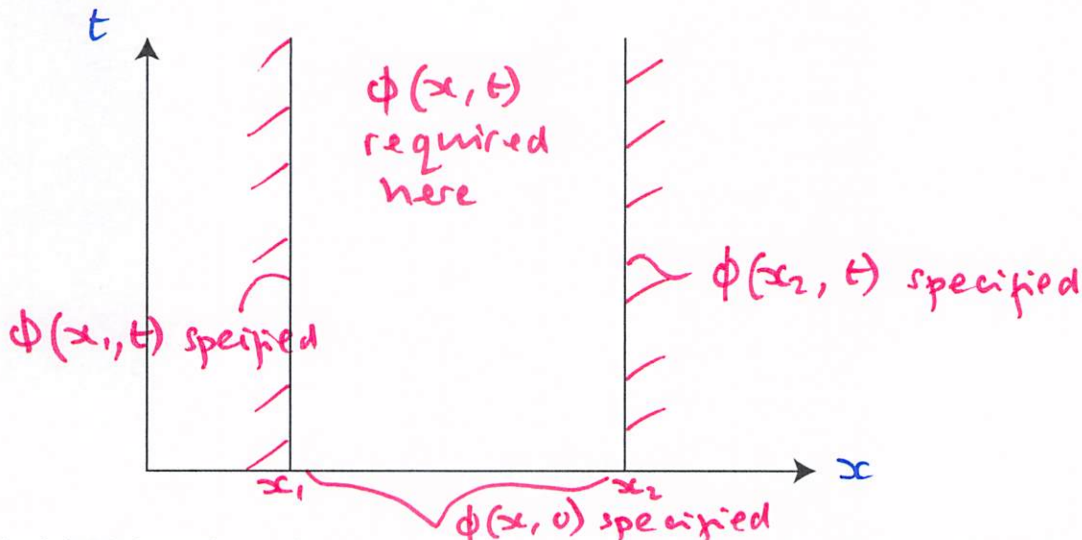
Second order boundary value ODE problems usually have a spatial coordinate as the independent variable, $\phi = \phi(x)$. Boundary conditions are specified at two locations, $\phi(x_1)$ and $\phi(x_2)$ and the solution is obtained, for $x_1 \leq x \leq x_2$ as the particular curve of a family which passes through both these points.



11.4 Boundary conditions for PDEs

11.4.1 $\phi = \phi(x, t)$

If ϕ is a function of time and a spatial coordinate, $\phi = \phi(x, t)$, we will seek, in general, a solution in a region $t \geq 0$ that is bounded by $x = x_1$ and $x = x_2$. This is related to the initial value problem for ODEs, but we must now specify initial conditions at $t = 0$ (for $x_1 \leq x \leq x_2$) *and* values of ϕ at $x = x_1$ and $x = x_2$ (for $t > 0$).



Instead of single boundary values, we now have boundary *functions*. The full set of boundary conditions for the problem is:

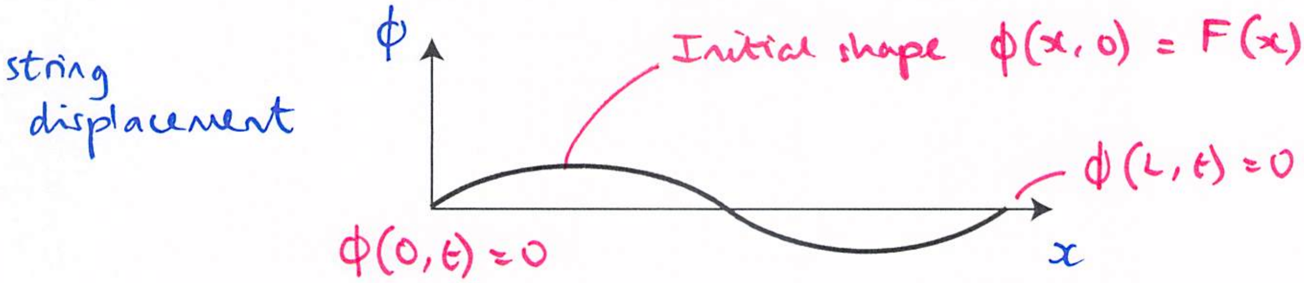
$$\phi(x,0) = F(x) \qquad x_1 \le x_2$$

$$\phi(x_1,t) = G_1(t) \qquad t > 0$$

$$\phi(x_2,t) = G_2(t) \qquad t > 0$$

Our boundary conditions are therefore *functions* that define ϕ along a *line* in (x,t) space. This illustrates the additional complexity of solving a PDE as compared to an ODE. ϕ is a surface in (x,t) space and we need to find this surface such that it not only satisfies the PDE for ϕ at all points in the interior of our (x,t) domain, but also matches the functions prescribed at the boundaries.

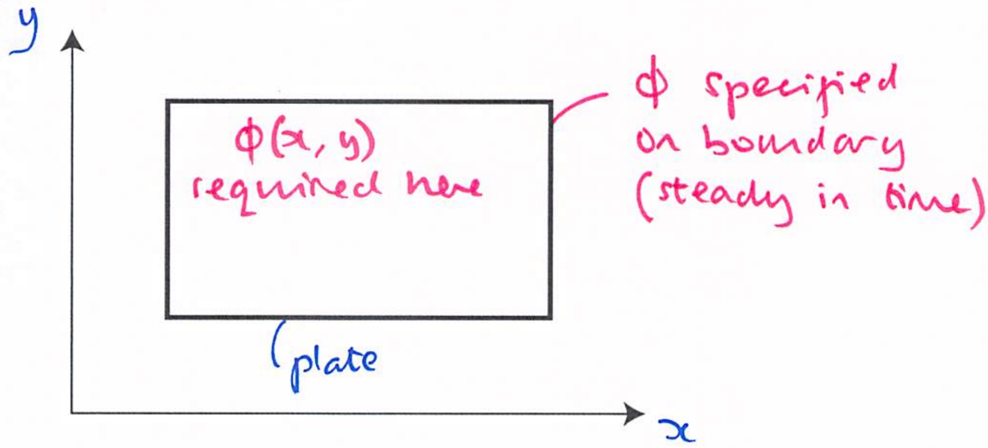
An example of a $\phi = \phi(x,t)$ problem that we will solve is a taut vibrating string, where ϕ is the displacement of the string. Appropriate boundary conditions are: the initial displacement of the string, $\phi(x,0) = F(x)$; and the end conditions $\phi(0,t) = G_1(t) = 0$ and $\phi(L,t) = G_2(t) = 0$



11.4.2 $\phi = \phi(x,y)$

If ϕ is a function of 2-D space, $\phi = \phi(x,y)$, then the boundary conditions for the PDE are usually specified around a closed loop in the (x,y) plane and the solution we seek is the value of ϕ at all points within this loop. This is the PDE equivalent of a boundary value ODE problem. If s is the distance around the loop, the boundary condition is $\phi(x_B, y_B) = F(s)$.

An example of a $\phi = \phi(x,y)$ problem is the steady condition of heat in a 2-D plate (i.e. constant thickness in the z direction). If ϕ is the temperature of the plate, appropriate boundary conditions would be the temperature around the edge of the plate. The solution to the PDE will provide the temperature field at all points in the interior of the plate.



11.5 Order and linearity

The *order* of a PDE refers to the highest order partial derivative in the equation. Hence, if $\phi = \phi(x, y)$, an example of a second order equation is,

$$\frac{\partial^2 \phi}{\partial x^2} + \left(\frac{\partial \phi}{\partial y}\right)^3 = 0$$

An extremely important characteristic of a PDE is whether it is *linear* or *non-linear*. The test for linearity is straightforward. If $\phi = \phi_1$ and $\phi = \phi_2$ are both solutions of the PDE, the equation is linear if $\phi = A\phi_1 + B\phi_2$ is also a solution.

Example

Are the following equations linear or non-linear?

(i) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial y} = 0$

If ϕ_1 is a solution : $\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial \phi_1}{\partial y} = 0$ (1)

If ϕ_2 is a solution : $\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial \phi_2}{\partial y} = 0$ (2)

(1) $\times A$ + (2) $\times B$: $\frac{\partial^2}{\partial x^2}(A\phi_1 + B\phi_2) + \frac{\partial}{\partial y}(A\phi_1 + B\phi_2) = 0$

$\therefore (A\phi_1 + B\phi_2)$ is a solution. PDE is linear.

(ii) $\frac{\partial^2 \phi}{\partial x^2} + \left(\frac{\partial \phi}{\partial y}\right)^3 = 0$

ϕ_1 is a solution: $\dfrac{\partial^2 \phi_1}{\partial x^2} + \left(\dfrac{\partial \phi_1}{\partial y}\right)^3 = 0$ ①

ϕ_2 is a solution: $\dfrac{\partial^2 \phi_2}{\partial x^2} + \left(\dfrac{\partial \phi_2}{\partial y}\right)^3 = 0$ ②

$$① \times A + ② \times B: \quad \frac{\partial^2}{\partial x^2}\left(A\phi_1 + B\phi_2\right) + \underbrace{A\left(\frac{\partial \phi_1}{\partial y}\right)^3 + B\left(\frac{\partial \phi_2}{\partial y}\right)^3}_{\text{i.e. not } \left[\frac{\partial}{\partial y}(A\phi_1 + B\phi_2)\right]^3} = 0$$

$\therefore$ PDE is non-linear

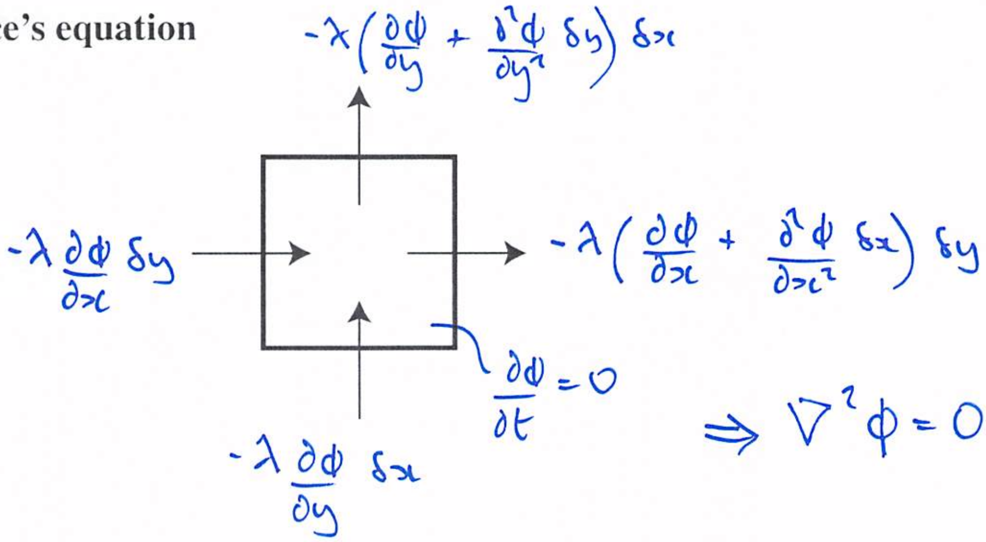
If a PDE is linear, new solutions can be generated by linear combinations of known simple solutions. In some cases, it is possible to match any prescribed boundary function by summing a linear combination of simple solutions that are known to satisfy the PDE. This is reminiscent of the Fourier series technique where an arbitrary function can be represented by a linear combination of simple trigonometric functions.

11.6 Classification of second order PDEs

Many important PDEs which model physical phenomena are second order. Second order PDEs can be classified as "elliptic", "parabolic" or "hyperbolic" – we will not discuss the reasoning behind these names. Elliptic PDEs are of the boundary value type. Parabolic and hyperbolic PDEs are of the initial condition type, and usually have time as an independent variable.

11.6.1 Laplace's equation

e.g. steady 2-D heat conduction



$\Rightarrow \nabla^2 \phi = 0$

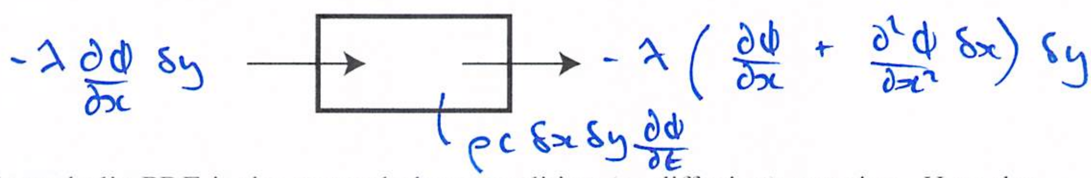
We have already come across Laplace's equation, $\nabla^2\phi = 0$. This is the archetypal elliptic PDE. It arises where the flux vector, $\mathbf{V}$, can be written $\mathbf{V} = \nabla\phi$ and there are no sources or sinks of $\mathbf{V}$, so that $\nabla\cdot\mathbf{V} = 0$. In 2-D, Cartesian form,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

If there are source terms ($\mathbf{V}$ is not solenoidal) such that $\nabla\cdot\mathbf{V} = S(x,y)$, then we have Poisson's equation (also elliptic):

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = S(x,y)$$

11.6.2 Unsteady heat conduction (or diffusion) equation

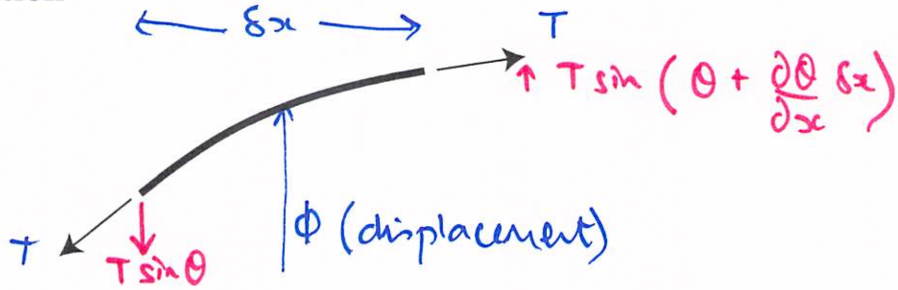


The archetypal parabolic PDE is the unsteady heat condition (or diffusion) equation. Here the flux $\mathbf{V}$ is again given by $\mathbf{V} = \nabla\phi$, but the field is no longer steady so that a net flux into a small element will change ϕ, i.e. $\nabla\cdot\mathbf{V}$ is proportional to $\partial\phi/\partial t$. In 1-D Cartesian space:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\phi}{\partial t}$$

where α is the diffusivity and is a positive constant. For heat conduction, $\alpha = \lambda/(\rho c)$ where λ is the thermal conductivity, ρ the density, and c the specific heat capacity.

11.6.3 Wave equation



The archetypal hyperbolic PDE is the wave equation. The wave equation arises when $\partial\mathbf{V}/\partial t$ (rather than $\mathbf{V}$) is proportional to $\nabla\phi$ and $\nabla\cdot\mathbf{V}$ is proportional to $\partial\phi/\partial t$. In 1-D Cartesian space,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

where c is a constant with the physical interpretation of the speed of propagation of the wave.

Laplace's equation, the unsteady heat conduction (or diffusion) equation, and the wave equation are all *linear*. (The nature of the $S(x,y)$ function needs to be specfied before we can say if a specific Poisson's equation is linear).