Lecture 5

The Divergence of a Vector Field

5.1 Definition and useful identities

The divergence of a vector field $\mathbf{V}$ is obtained by taking the dot product of the vector operator ∇ with $\mathbf{V}$ and is written $\nabla \cdot \mathbf{V}$ (or 'div $\mathbf{V}$'). The divergence of a vector field is a *scalar*.

In Cartesian coordinates,

$$\nabla . \underline{V} = \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right) \cdot \left(V_x \underline{i} + V_y \underline{j} + V_z \underline{k}\right)$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

where $\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$.

We now have two uses for 'del' (∇): gradient and divergence. It might be helpful to note that:

1. The gradient ($\nabla\phi$) is a vector obtained by operating on a scalar field;
2. The divergence ($\nabla \cdot \mathbf{V}$) is a scalar obtained by operating on a vector field;
3. $\nabla \cdot \mathbf{V}$ (the divergence) is quite different from $\mathbf{V} \cdot \nabla$ (scalar operator).

Here are two useful identities involving the divergence which can be proved by expanding in Cartesian form. If $\mathbf{A}$ and $\mathbf{B}$ are vector fields and ϕ is a scalar field, then,

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \tag{5.1}$$

$$\nabla \cdot (\phi\mathbf{A}) = \phi(\nabla \cdot \mathbf{A}) + \nabla\phi \cdot \mathbf{A} \tag{5.2}$$

Example

Calculate the divergence of the field $\mathbf{V} = x^2z\,\mathbf{i} - 2y^3z^2\,\mathbf{j} + xy^2z\,\mathbf{k}$ at the point $(1, -1, 1)$.

$$\nabla \cdot \underline{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= 2xz - 6y^2z^2 + xy^2$$

@ $(1,-1,1)$, $\nabla \cdot \underline{V} = 2 - 6 + 1 = -3$

Example

Calculate the divergence of the vector field formed by the position vectors, $\mathbf{V} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

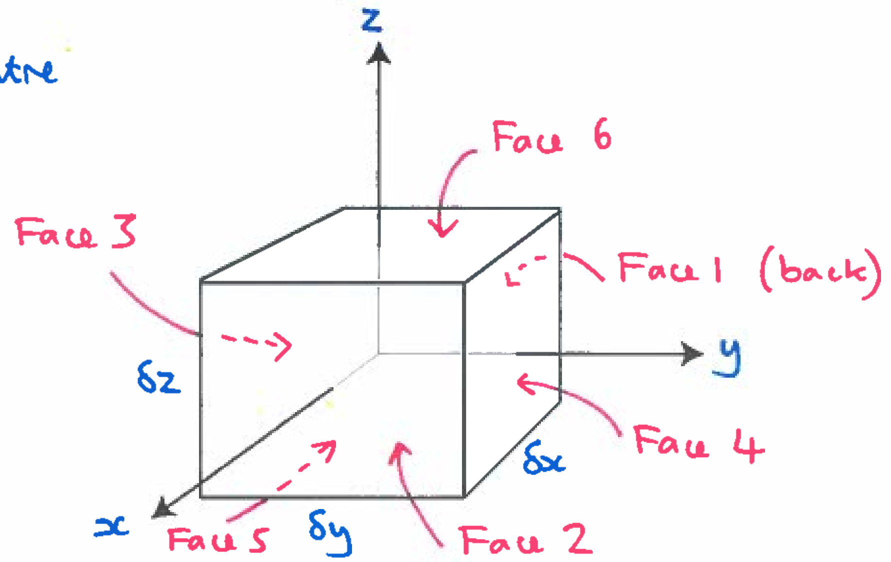
$\nabla \cdot \underline{V} = 1 + 1 + 1 = 3$ (everywhere !)

5.2 Physical interpretation of the divergence

The divergence is an important quantity that is linked to the conservation of physical properties such as mass, momentum, energy, magnetic flux, etc. With this in mind, we consider the conservation of mass in fluid mechanics.

Origin is at centre of element

At origin,
$\rho = \rho_0$
$V_x = V_{x0}$
$V_y = V_{y0}$
$V_z = V_{z0}$



The diagram shows an elemental control volume in Cartesian coordinates. The sides of the control volume have length δx, δy, δz. The fluid density is a scalar field $\rho = \rho(x,y,z)$ and the fluid velocity is a vector field $\mathbf{V} = \mathbf{V}(x,y,z)$. At the centre of the control volume, $\rho = \rho_0$ and $\mathbf{V} = V_{x0}\,\mathbf{i} + V_{y0}\,\mathbf{j} + V_{z0}\,\mathbf{k}$.

The conservation of mass, in words, is:

Net mass flowrate out of control volume

= rate of decrease of mass of control volume

We proceed by evaluating the mass flowrate on the faces of the element. For Face 1, the flow of mass *in* to the control volume is,

$$\rho_1 V_{x_1}\,\delta y\,\delta z \approx \left(\rho_0 V_0 - \frac{\delta x}{2}\frac{\partial(\rho V_x)}{\partial x}\right)\delta y\,\delta z$$

where the partial derivative is evaluated at the centre of the element. We can work out the mass flowrate on Face 2 in the same way, but this is now a flowrate *out* of the control volume,

$$\rho_2 V_{x_2}\,\delta y\,\delta z \approx \left(\rho_0 V_0 + \frac{\delta x}{2}\frac{\partial(\rho V_x)}{\partial x}\right)\delta y\,\delta z$$

The net flowrate *out* of the control volume from Faces 1 and 2 is, therefore,

$$\left(\rho_2 V_{x_2} - \rho_1 V_{x_1}\right)\delta y\,\delta z \approx \frac{\partial(\rho V_x)}{\partial x}\,\delta x\,\delta y\,\delta z$$

Similarly, the contributions from the pairs of faces perpendicular to the y and z directions are,

$$(\rho_4 V_{y4} - \rho_3 V_{y3})\,\delta z\delta x \approx \frac{\partial(\rho V_y)}{\partial y}\delta x\delta y\delta z \quad . \tag{5.3}$$

$$(\rho_6 V_{z6} - \rho_5 V_{z5})\,\delta x\delta y \approx \frac{\partial(\rho V_z)}{\partial z}\delta x\delta y\delta z \quad . \tag{5.4}$$

Taking all the faces into account, the net mass flowrate out of the control volume (the "rate of mass eflux") is

$$\text{rate of mass efflux} \quad = \left(\frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y} + \frac{\partial(\rho V_z)}{\partial z}\right)\delta v \quad , \tag{5.5}$$

where $\delta v = \delta x\delta y\delta z$ is our elemental volume. We recognise that we may write this more simply as,

$$\text{rate of mass efflux} \quad = \nabla\cdot(\rho\mathbf{V})\delta v \quad , \tag{5.6}$$

and this expression holds for any coordinate system.

The mass of the elemental volume is $\rho\delta x\delta y\delta z = \rho\delta v$. The volume is fixed, so the rate of *decrease* of mass is given by,

$$-\frac{\partial\rho}{\partial t}\,\delta v$$

We can now write our conservation of mass equation for the volume δv, in Cartesian coordinates, as

$$\left(\frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y} + \frac{\partial(\rho V_z)}{\partial z}\right)\delta v = -\frac{\partial\rho}{\partial t}\,\delta v$$

or,

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y} + \frac{\partial(\rho V_z)}{\partial z} = 0 \quad . \tag{5.7}$$

In vector form, valid for any coordinate system,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$$

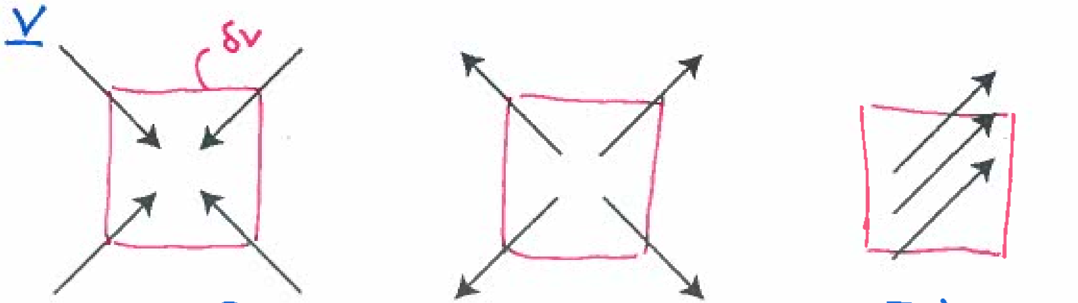
Now that the physical interpretation of the divergence has been found – the net rate of efflux of any vector field **A** from an elemental volume δv is given by $(\nabla \cdot \mathbf{A})\delta v$ – we can easily interpret the vector form of the mass conservation equation.

In fact, recognising that the mass flux is $\rho\mathbf{V}$, vector calculus allows us to write down $\partial\rho/\partial t + \nabla \cdot (\rho\mathbf{V}) = 0$ and so obtain the coordinate-free equation without having to go through the lengthy foregoing analysis.

As an aside, we can see that if ρ is a constant, we must have $\nabla \cdot \mathbf{V} = 0$.

5.3 Solenoidal vector fields

A vector field where the divergence is everywhere zero is called a *solenoidal* field. Thus, for an incompressible flow, the velocity field **V** is an example of a solenoidal field. In a solenoidal field, the net efflux of the vector field from a volume element δv is zero. The flux entering the volume element is the same as flux leaving the element: there are no 'sources' or 'sinks' of the vector field within the element.



$\nabla \cdot \underline{V} < 0$ $\qquad$ $\nabla \cdot \underline{V} > 0$ $\qquad$ $\nabla \cdot \underline{V} = 0$

Example

A time steady, incompressible fluid flow field has velocity components $V_x = kx$ (k is a constant) and $V_z = 0$. Find V_y and sketch streamlines given that $V_y = 0$ on the plane $y = 0$.

$$\nabla \cdot \underline{V} = 0 \quad \text{(conservation of mass)}$$

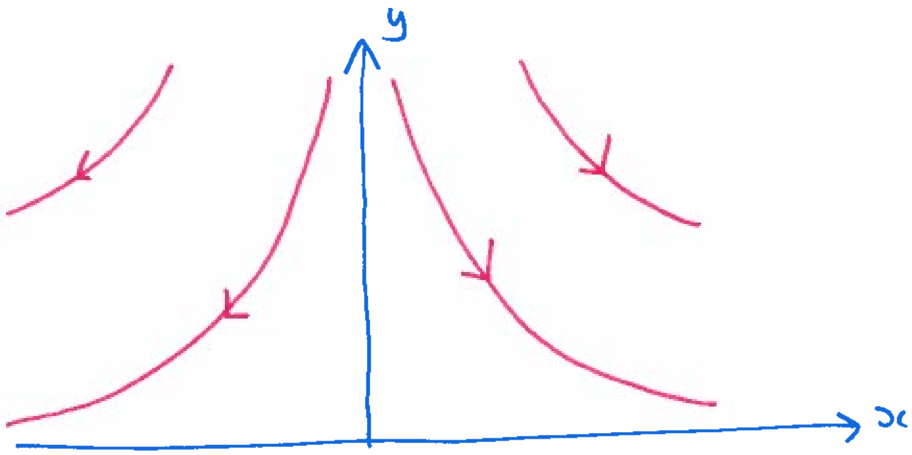
$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0$$

$$k + \frac{\partial V_y}{\partial y} + 0 = 0$$

$$\frac{\partial V_y}{\partial y} = -k \qquad V_y = -ky + f(x, z)$$

$$V_y = 0 \text{ on } y = 0 \quad \therefore f(x, z) = 0$$

$$\underline{V} = kx\,\underline{i} - ky\,\underline{j}$$



5.4 $\nabla \cdot \mathbf{V}$ in non-Cartesian coordinate systems

Cylindrical polar coordinates (r, θ, z)

The divergence of a vector field $\mathbf{V}$ is defined as $\nabla \cdot \mathbf{V}$. We have already found that ∇ in cylindrical polar coordinates is,

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \quad , \tag{5.8}$$

and so we can evaluate $\nabla \cdot \mathbf{V}$. However, we need to be careful because the base vectors are functions of position, i.e. $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are both functions of θ.

$$\nabla \cdot \mathbf{V} = \overbrace{\left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}\right)}^{\underline{\nabla}} \cdot \overbrace{\left(V_r \mathbf{e}_r + V_\theta \mathbf{e}_\theta + V_z \mathbf{e}_z\right)}^{\underline{V}} \tag{5.9}$$

The dependence of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ on θ means we need to evaluate the $\partial/\partial\theta$ terms as follows,

$$\frac{V_r}{r}\mathbf{e}_\theta \cdot \frac{\partial \mathbf{e}_r}{\partial \theta} + \frac{V_\theta}{r}\mathbf{e}_\theta \cdot \frac{\partial \mathbf{e}_\theta}{\partial \theta} \tag{5.10}$$

$$= \frac{V_r}{r} \underbrace{\mathbf{e}_\theta \cdot \mathbf{e}_\theta}_{=1} + \frac{V_\theta}{r} \underbrace{\mathbf{e}_\theta \cdot -\mathbf{e}_r}_{=0}$$

$$= \frac{V_r}{r}$$

The expression for $\nabla \cdot \mathbf{V}$ in cylindrical polar coordinates is therefore

$$\nabla \cdot \mathbf{V} = \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} \quad , \tag{5.11}$$

and that the Maths Data Book simplifies this a little further,

$$\nabla \cdot \mathbf{V} = \frac{1}{r}\frac{\partial (rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} \quad . \tag{5.12}$$

Spherical polar coordinates (r, θ, ϕ)

We will not go through the derivation here, but $\nabla \cdot \mathbf{V}$ for spherical polar coordinates is also listed in the Maths Data Book,

$$\nabla \cdot \mathbf{V} = \frac{1}{r^2}\frac{\partial (r^2 V_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{V_\theta \sin\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi} \tag{5.13}$$

5.5 Divergence and gradient combined

We have seen that a range of physical processes can be modelled by equations of the form,

$$\mathbf{V} = -k\nabla\psi \quad . \tag{5.14}$$

Three examples given were:

1. Heat conduction ($\mathbf{V}$=heat flux vector, ψ = temperature, k = thermal conductivity)
2. Diffusion ($\mathbf{V}$=mass flux vector, ψ = concentration, k = diffusion coefficient)
3. Current flow ($\mathbf{V}$=current density, ψ = electric potential, k = electrical conductivity)

If k is constant, then $\mathbf{V} = \nabla\phi$ where ϕ is the scalar potential and $\phi = -k\psi$.

Each of the above processes involve conservation of a physical quantity (energy, the mass of the diffusing species, electric charge). If the situation to be modelled is at a steady-state, with no sources or sinks, then we know that the vector field $\mathbf{V}$ must be solenoidal and $\nabla \cdot \mathbf{V} = 0$.

When the vector field is governed by a scalar potential, and is also solenoidal, it follows that,

$$\nabla \cdot (\nabla\phi) = 0$$

Note that this operation, div (grad ϕ), is acceptable because we are taking the divergence of the *vector* field obtained by taking the gradient of a *scalar* field.

In Cartesian coordinates,

$$\nabla \cdot (\nabla \phi) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) \cdot \left(\mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z} \right) \tag{5.15}$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{5.16}$$

The operator $\nabla \cdot \nabla$ is a second order scalar differential operator and is usual written ∇^2 ('del squared') and is known as the Laplacian (the equation $\nabla^2 \phi$ is called Laplace's equation). In Cartesian coordinates, the Laplacian is, $\nabla^2 \phi = 0$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The Data Book contains formulae for the Laplacian in cylindrical polar coordinates (r, θ, z),

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} \quad , \tag{5.17}$$

and in spherical polar coordinates (r, θ, ϕ) the Laplacian is,

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial \psi}{\partial r} \right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial \psi}{\partial \theta} \right] + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} \tag{5.18}$$

Example

Derive the equation governing heat conduction in a solid.

rate of energy increase in δv = − net flux of heat out of δv

$$\rho c \frac{\partial T}{\partial t} \delta v = -(\nabla \cdot \underline{q})\, \delta v$$

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (\lambda \nabla T)$$

If steady and $\lambda = \text{const} \Rightarrow \nabla^2 T = 0$

e.g. for 2-D Cartesian : $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

or, for time-dependent, radial symmetry, with λ = const

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{\rho c}{\lambda}\frac{\partial T}{\partial t}$$

You can now do Examples Paper 2: Q1, 2 and 3