IB Paper 6: Signal and Data Analysis

Handout 1: Introduction and Preliminaries

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Introduction

Course structure

7 lectures on *Signal and Data Analysis* – continuous and discrete Fourier techniques for frequency domain analysis:

- Introduction
- Section 1: Review of energy, power, delta functions and Fourier Series.
- Section 2: The Fourier Transform and its properties.
- Section 3: Sampling and the Discrete Time Fourier Transform (DTFT)
- Section 4: The Discrete Fourier Transform (DFT)

Paper 6, Information engineering, will be divided into two sections, and candidates will be expected to answer two questions from each.

Section A will consist of three questions on *linear systems and control*.

Section B will consist of three questions on *communications and signal and data analysis*.
Section B may consist of

1. 2 communications questions and 1 signal and data analysis question, or
2. 1 communications question and 2 signal and data analysis questions, or
3. 1.5 comms questions and 1.5 signal and data analysis questions (not necessarily evenly weighted).

Recommended Books

The books given on the booklist are

- Proakis and Manolakis: Digital Signal Processing – Principles, Algorithms and Applications. Prentice Hall, 3rd Edition, 1996.
- Bracewell: The Fourier Transform – McGraw Hill, 3rd Edition, 2000.
- McLellan, Schafer and Yoder: Signal Processing First – Pearson, Prentice Hall, 1st Edition, 2003

Another short useful book is

James, J.F: A Student's Guide to Fourier Transforms, with applications in physics and engineering. CUP, 2nd Edition, 2002.

Recommended Books cont....

Note that notation and conventions differ slightly between the various books and that used in this course.

McLellan *et al.* introduces the discrete theory before the continuous theory, while in this course we do the continuous theory and lead on to the discrete theory.

Lecture notes are on Moodle

Overview of Section 1

- Introduction
  - Motivation for signal analysis.
  - Some examples of typical datasets.
- Power and Energy
- Revision and extension of δ-functions.
- Revision and extension of Fourier series
  - definition and properties
  - examples

Main body of text in handouts will be complete, some gaps will be left for derivations of equations and examples.

Motivation

What is *Signal and Data Analysis*?

Signals and data are continuously being generated and measured in all walks of life, e.g.

- Mobile phones record and generate sounds, images and video. They then analyse and code the data for efficient storage or for wireless transmission.
- In engineering environments signals can involve measurements of vibrations from mechanical structures, recording of speech or audio data, data from patient monitoring machines in hospitals, measurements of natural phenomena (temperatures, tide heights, emissions from distant galaxies....).

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Motivation cont...

- One of the most important tasks in analysing and processing of these signals is to determine *which frequency components are present in the data*.
- You already have some methods for analysing frequency content:
  - Fourier Series - but only applicable for periodic and continuous-time waveforms
  - Frequency response of linear system - more applicable, but we need to analyse signals as well as systems

This course will introduce new concepts in frequency analysis which allow you to determine the frequency content of general (non-periodic) signals, in both continuous-time (‘analogue’) and discrete-time (‘digital’).

Motivation cont...

Nowadays data will mostly be measured in digital form and often processed by computers. This gives many advantages, including repeatability, guaranteed levels of accuracy, flexibility and power of processing.

In this course we will also study the basics of digital signal processing, including the principles of digital sampling.For example, we will study a classic result – The Nyquist Sampling Theorem – which shows that it is possible to reconstruct an original analogue signal perfectly from correctly digitised data [recall (??) the debate over changing from LPs to CDs in the 80s].

Examples of Data

We may have to deal with data which is

- continuous or discrete
- real or complex / scalar or vector valued
- deterministic or random
- periodic or non-periodic

Power and Energy

We define the total **energy** content of a signal $f(t)$ as follows:

$$\boxed{E_f = \int_{-\infty}^{+\infty} |f(t)|^2 dt}$$

Often more convenient to compute as a limit:

$$E_f = \lim_{T \to \infty} \int_{-T/2}^{+T/2} |f(t)|^2 dt$$

and similarly the average **power** content of the signal is defined as

$$\boxed{P_f = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt.} \tag{1}$$

Note that P_f is the *average* power of the signal estimated over the whole time interval

Example: Consider a decaying exponential signal $f(t)$, where

$$f(t) = \begin{cases} \exp(-t) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

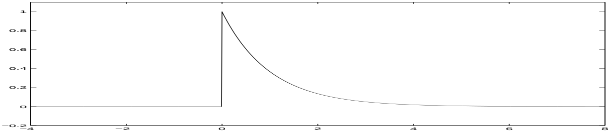


Figure 1: The function: e^{-t} for $t > 0$, and 0 for $t \leq 0$

Calculate the energy for this function

$$\text{Energy}: \quad \int_0^\infty |e^{-t}|^2 dt = \int_0^\infty e^{-2t} dt = -\frac{1}{2}\left[\exp(-2t)\right]_0^\infty = \frac{1}{2}$$

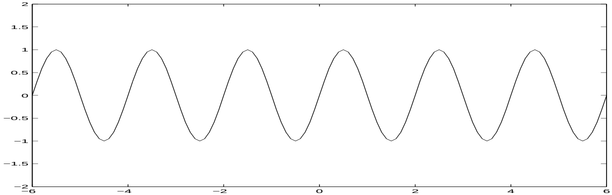
Consider now the power of this function:

$$\text{Power}: \qquad \lim_{T\to\infty} \frac{1}{T}\int_0^{T/2} e^{-2t}dt$$

$$= \lim_{T\to\infty} \frac{-1}{2T}\left[e^{-2t}\right]_0^{T/2} \longrightarrow 0$$

We see here therefore that while the *average energy* for this signal is well defined, the *average power* is zero and is therefore not a helpful quantity in this case. Looking at the form for energy it is reasonable to suppose that we might run into problems when we consider a signal such as a sine wave which does not decay as $t \longrightarrow \infty$.

Example: Consider the signal $f(t) = \sin(t)$:



Now try to evaluate the energy in this signal. First look at the energy in just one half-period of sine, from 0 to π:

$$\begin{aligned}\int_0^{\pi} |\sin(t)|^2 dt &= \frac{1}{2}\int_0^{\pi}(1-\cos(2t))dt \\ &= \frac{1}{2}\left[t - \frac{1}{2}\sin(2t)\right]_0^{\pi} \\ &= \frac{\pi}{2}\end{aligned}$$

Now, the total energy involves adding together the energy of all of the half-period intervals:

$$\begin{aligned}\int_{-\infty}^{\infty} |\sin(t)|^2 dt &= \sum_{n=-\infty}^{n=+\infty} \int_{n\pi}^{(n+1)\pi} |\sin(t)|^2 dt \\ &= \sum_{n=-\infty}^{+\infty} \pi/2 \longrightarrow \infty\end{aligned}$$

and the energy is therefore infinite.
But, is the power well-defined?

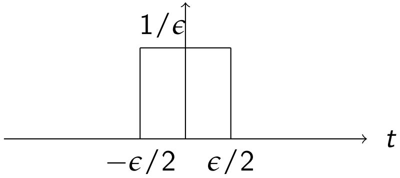
Evaluating power in this signal gives:

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|\sin(t)|^2 dt &= \lim_{T\to\infty}\frac{1}{2T}\left[t-\sin(2t)/2\right]_{-T/2}^{T/2} \\
&= \lim_{T\to\infty}\frac{1}{2T}\left[T-\sin(T)\right] \\
&= \lim_{T\to\infty}\frac{1}{2}\left[1-\frac{\sin(T)}{T}\right] = \frac{1}{2}.
\end{aligned}$$

Power is therefore well-defined. We should therefore be aware that the nature of the signal determines whether it is the energy or the power that is the useful quantity to deal with.

δ-functions revisited

δ-functions (also known as the Dirac δ-function or the impulse function) are fundamental to the understanding of signal analysis. Consider the rectangular pulse $f_1(t\,;\epsilon)$, where ϵ can vary from 0 to ∞, illustrated below;



The pulse is designed such that it has unit area, i.e.

$$\int_{-\infty}^{\infty} f_1(t\,;\epsilon)dt = 1, \ \text{ for all } \ \epsilon > 0$$

The *δ-function*, $\delta(t)$ can be defined as a limiting case: $\{f_1(t;\epsilon)\}$ as $\epsilon \to 0$:

$$\boxed{\delta(t) = \lim_{\epsilon \to 0} f_1(t\,;\epsilon).} \tag{2}$$

Two familiar properties are

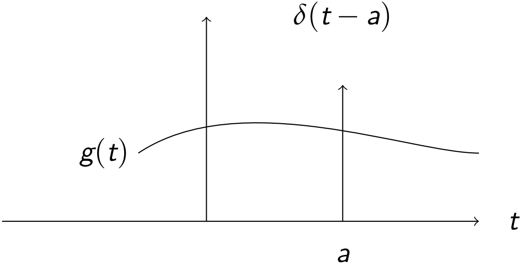
a) $\delta(t) = 0$ for $t \neq 0$.

b) $\int_{-\infty}^{\infty} \delta(t) dt = 1$.

The sifting property

Now consider the following integral

$$\int_{-\infty}^{\infty} g(t)\delta(t-a)dt.$$



Since $\delta(t-a)$ is *zero* for all values of t *except* $t = a$, we have:

$$\int_{-\infty}^{\infty} g(t)\delta(t-a)dt = \int_{-\infty}^{\infty} g(a)\delta(t-a)dt$$
$$= g(a)\int_{-\infty}^{\infty} \delta(t-a)dt = g(a).$$

We therefore have the **VERY USEFUL** result that integrating a function with $\delta(t-a)$ picks out the value of the function at a:

$$\boxed{\int_{-\infty}^{+\infty} g(t)\delta(t-a)dt = g(a)}$$

This will be referred to as the **Sifting Property**

Other definitions of the delta function

The delta-function can be defined as limiting cases of other functions as well as the family of rectangular pulses $f_1(t; \epsilon)$ considered above.
Consider for example the two functions:

$$\begin{aligned} f_2(t\,; \epsilon) &= \frac{\epsilon}{\epsilon^2 \pi^2 + t^2} \\ f_3(t\,; a) &= \frac{\sin(at)}{\pi t} \end{aligned}$$

It can be shown that we can also produce a delta function using f_2 or f_3:

$$\delta(t) = \lim_{\epsilon \to 0} f_2(t\,; \epsilon) = \lim_{a \to \infty} f_3(t\,; a)$$

Let us concentrate on the family of functions $f_3(t\,; a)$. The function $\sin(t)/t$, is known as the *sinc* function or $\text{sinc}(t)$, which oscillates like a sine-wave, but decays to zero as t goes to $\pm\infty$, see figure 2.

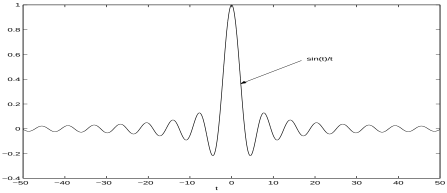


Figure 2: The sinc function $\sin(t)/t$

Its integral (not derived here) is equal to π:

$$\int_{-\infty}^{+\infty} \frac{\sin(t)}{t} dt = \pi \tag{3}$$

Hence, integrating by substitution, the pulse $f_3(t\,;a)$ has area 1, as required.
Now see figure 3 to see $\frac{\sin(at)}{\pi t}$ converging to a δ-function.

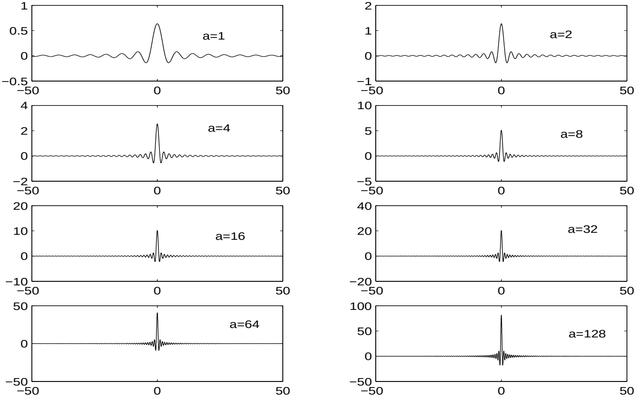


Figure 3: $\frac{\sin(at)}{\pi t}$ for various values of a

A useful integral

We can use the previous result to evaluate the following integral, which will be used several times later in the course:

$$I = \int_{-\infty}^{\infty} \exp(j\omega t)d\omega = \lim_{A\to\infty} \int_{-A}^{A} \exp(j\omega t)d\omega \tag{4}$$

Standard integration gives

$$I = \lim_{A\to\infty} \left[\frac{e^{j\omega t}}{jt}\right]_{-A}^{A} = \lim_{A\to\infty} \left(2\frac{\sin(At)}{t}\right) = \lim_{A\to\infty} 2\pi f_3(t;A) = 2\pi\delta(t) \tag{5}$$

where the last line follows from the definition of the δ-function as the limit of a sequence of functions $f_3()$ (see previous section).

This gives the useful result:

$$\lim_{A \to \infty} \int_{-A}^{A} \mathrm{e}^{j\omega t} d\omega = 2\pi\delta(t) \tag{6}$$

or

$$\boxed{\int_{-\infty}^{\infty} \exp(j\omega t) d\omega = 2\pi\delta(t)} \tag{7}$$