2P7: Probability & Statistics

Discrete Probability Distributions

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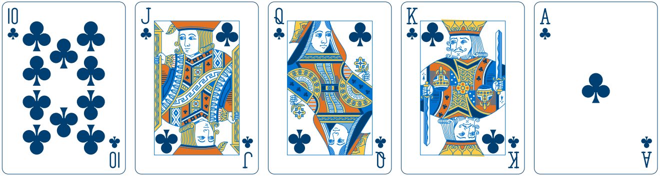

the *royal flush*, the best possible hand in poker, has a probability 0.000154%

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1. Probability Fundamentals
2. Discrete Probability Distributions
3. Continuous Random Variables
4. Manipulating and Combining Distributions
5. Decision, Estimation and Hypothesis Testing

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In the last lecture:

- We have reviewed the fundamental concepts of probability
- We have seen how discrete random variables are defined and introduced the probability mass function
- We have shown how to calculate the expectation, variance and entropy of a discrete random variable

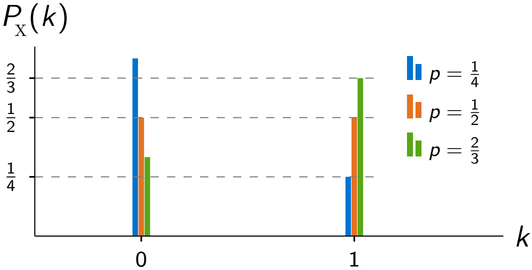
In this lecture, we will see some examples of discrete probability distributions

- There are many: see `en.wikipedia.org/wiki/List_of_probability_distributions`
- Here, a small selection of the most important ones
- A lot of discrete random variables originates from the binary random variable, and we'll start with this one

A *binary* random variable X is said to have a Bernoulli distribution[1] with parameter $p \in [0, 1]$ if:

$$\mathrm{X} \sim \mathrm{Ber}(p) \quad \Leftrightarrow \quad P_{\mathrm{X}}(k) = \begin{cases} p & \text{if } k = 1, \\ 1 - p & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The symbol "$\sim$" means "distributed as". The support of X, $\mathbb{X} = \{0, 1\}$, is discrete finite.



[1] named after the Swiss mathematician Jacob Bernoulli (1655-1705)

- Expectation $\mathbb{E}[\mathrm{X}] = p$ [DB]

$$\mathbb{E}[\mathrm{X}] = \sum_{k \in \mathbb{X}} k\, P_{\mathrm{X}}(k) = 0 \times (1-p) + 1 \times p = p \qquad \square$$

- Variance $\mathrm{Var}[\mathrm{X}] = p(1-p)$ [DB]

$$\mathrm{Var}[\mathrm{X}] = \sum_{k \in \mathbb{X}} (k-p)^2\, P_{\mathrm{X}}(k) = (-p)^2(1-p) + (1-p)^2 p = p(1-p) \qquad \square$$

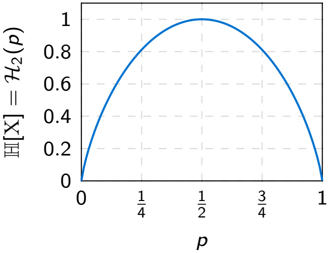
- Entropy $\mathbb{H}[\mathrm{X}] = \mathcal{H}_2(p) = -p\log_2 p - (1-p)\log_2(1-p)$

Immediate from:

$$\mathbb{H}[\mathrm{X}] = -\sum_{k \in \mathbb{X}} P_{\mathrm{X}}(k) \log_2 P_{\mathrm{X}}(k) \qquad \square$$

$\mathcal{H}_2(p)$ is known as the binary entropy function.

The max of $\mathcal{H}_2$ at $p = \frac{1}{2}$, where our uncertainty is complete.



Bernoulli distributions occur in many scenarios:

- They are *indicator* random variables for events, e.g.:
  - Will it rain tomorrow? A simple climate model may indicate rain with $\mathrm{X} \sim \mathrm{Ber}(p)$ where different values of p would be used for different areas.
  - Will the UK economy grow above expectation?
  - Will the message be received and decoded correctly?
- They occur naturally as answers to *yes/no questions*, e.g.
  - Is the product defective?
  - Did the defendant murder the victim?
- They also occur in their own right in *digital communications*, where information is often encoded into binary symbols.
- Probability textbooks often illustrate Bernoulli distributions using "biased coins". These are coins that have different probabilities of landing on "heads" or "tails".

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How many trials do I need to be successful?

- Suppose we look at a succession $\mathrm{X}_1, \mathrm{X}_2, \ldots$ of independent Bernoulli-distributed random variables (each being called a *trial*), and we measure the probability that the first "success" happens after k trials
- That is, $\mathrm{X}_k = 1$, and $\mathrm{X}_j = 0 \quad$ for all $j \leq k-1$:

$$
\begin{aligned}
& \mathbb{P}[\text{“}1^{\text{st}} \text{ success at the } k^{\text{th}} \text{ trial”}] \\
= & \mathbb{P}[\mathrm{X}_1 = 0 \ \cap \ \mathrm{X}_2 = 0 \ \cap \ldots \cap \ \mathrm{X}_{k-1} = 0 \ \cap \ \mathrm{X}_k = 1] \\
= & \mathbb{P}[\mathrm{X}_1 = 0] \times \mathbb{P}[\mathrm{X}_2 = 0] \times \ldots \times \mathbb{P}[\mathrm{X}_{k-1} = 0] \times \mathbb{P}[\mathrm{X}_k = 1] \\
= & (1-p) \times (1-p) \times \ldots \times (1-p) \times p \\
= & (1-p)^{k-1} p
\end{aligned}
$$

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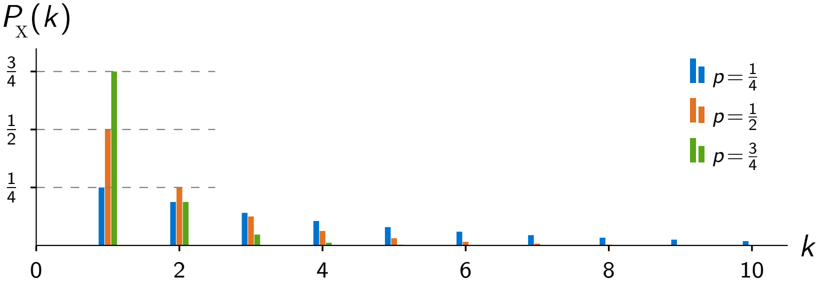
A random variable X is said to have a geometric distribution with parameter $p \in [0, 1]$ if:

$$\mathrm{X} \sim \mathrm{Geo}(p) \quad \Leftrightarrow \quad P_{\mathrm{X}}(k) = \begin{cases} p(1-p)^{k-1} & \text{if } k \in \{1, 2, \dots\}, \\ 0 & \text{otherwise.} \end{cases}$$

The support of X, $\mathbb{X} = \{1, 2, \dots\}$, is discrete infinite.



Verify that $\sum_{k \in \mathbb{X}} P_{\mathrm{X}}(k) = 1$ using the geometric series $\sum_{k=1}^{\infty} a^k = \frac{a}{1-a}$.

The Geometric Distribution

Properties of $\mathrm{X} \sim \mathrm{Geo}(p)$

- Expectation $\mathbb{E}[\mathrm{X}] = 1/p$ [DB]

$$q = 1-p,\ \mathbb{E}[\mathrm{X}] = \sum_{k=1}^{\infty} kpq^{k-1} = p\frac{\mathrm{d}}{\mathrm{d}q}\sum_{k=1}^{\infty} q^k = p\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{q}{1-q}\right) = \frac{1}{p} \qquad \square$$

- Variance $\mathrm{Var}[\mathrm{X}] = (1-p)/p^2$ [DB]

$$\mathbb{E}[\mathrm{X}^2] = \sum_{k=1}^{\infty} k^2 pq^{k-1} = qp\frac{\mathrm{d}^2}{\mathrm{d}q^2}\sum_{k=1}^{\infty} q^k + \sum_{k=1}^{\infty} kpq^{k-1} \text{ so}$$

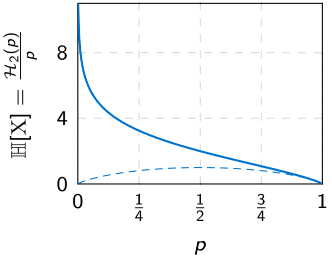
$$\mathrm{Var}[\mathrm{X}] = qp\frac{\mathrm{d}^2}{\mathrm{d}q^2}\left(\frac{q}{1-q}\right) + \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2} \qquad \square$$

- Entropy $\mathbb{H}[\mathrm{X}] = \mathcal{H}_2(p)/p$

$$\mathbb{H}[\mathrm{X}] = -\sum_{k=1}^{\infty} pq^{k-1}\log_2(pq^{k-1}) =$$

$$\frac{p}{q}\log_2\frac{q}{p}\times\sum_{k=1}^{\infty} q^k - \log_2 q \times \sum_{k=1}^{\infty} kpq^{k-1} \qquad \square$$

Diverges at $p = 0$, where success is highly unexpected...



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Here are a few instances where geometric distributions occur:

- *Quality control*: how many items can be produced before having a defective one;
- *Chemistry and biology*: polymer lengths distribution during polymerisation;
- *Business*: how many attempts to make a sale will end in a success;
- *Computing*: bounding time of randomised algorithms (while loop repeated until success);
- *Surveying*: how many people do you have to ask before you find a candidate.

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How many times was I successful after n trials?

- A quick refresher on permutations and combinations
  - Permutation: how many possible ways to put n items into $k \leq n$ places?

$$n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!} = {}^{n}\mathrm{P}_{k}$$

  - Combination: how many possible ways to select k items from n available? The k places can be arranged in $k!$ ways:

$$\frac{{}^{n}\mathrm{P}_{k}}{k!} = \frac{n!}{k!(n-k)!} = {}^{n}\mathrm{C}_{k}$$

- Suppose we have n independent Bernoulli trials $\{\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n\}$, among which k are successes:

$$P_{\mathrm{X}_1\mathrm{X}_2\ldots\mathrm{X}_n}(x_1, x_2, \ldots, x_n) = p^k(1-p)^{n-k}$$

  The order doesn't matter:

$$\mathbb{P}[\text{“}k \text{ successes after } n \text{ trials”}] = {}^{n}\mathrm{C}_{k}\, p^k(1-p)^{n-k}$$

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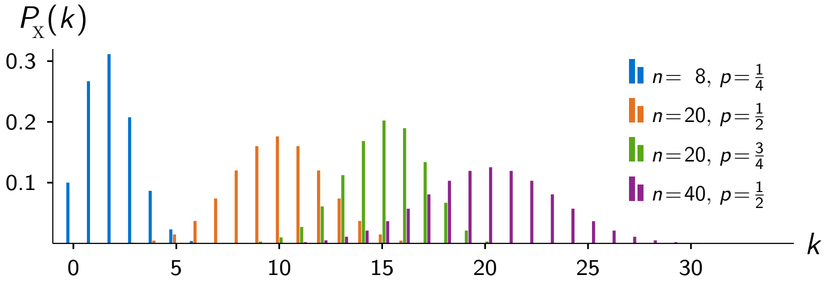
A random variable X is said to have a Binomial distribution with parameters $n \in \{1, 2, \dots\}$ and $p \in [0, 1]$ if:

$$\mathrm{X} \sim \mathrm{B}(n, p) \quad \Leftrightarrow \quad P_{\mathrm{X}}(k) = \begin{cases} {}^{n}\mathrm{C}_k \, p^k (1-p)^{n-k} & \text{if } k \in \{0, 1, 2, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The support of X, $\mathbb{X} = \{0, 1, 2, \dots, n\}$, is discrete finite.



Verify that $\sum_{k \in \mathbb{X}} P_{\mathrm{X}}(k) = 1$ using the binomial expansion $(a+b)^n = \sum_{k=0}^{n} {}^{n}\mathrm{C}_k \, a^k b^{n-k}$.

- Expectation $\mathbb{E}[\mathrm{X}] = np$ [DB]

$$\mathbb{E}[\mathrm{X}] = \sum_{k=1}^{n} {}^{n}\mathrm{C}_k \, k p^k q^{n-k} = np \sum_{k=1}^{n} {}^{n-1}\mathrm{C}_{k-1} p^{k-1} q^{n-k} \ \text{(using } {}^{n}\mathrm{C}_k \, k = {}^{n-1}\mathrm{C}_{k-1} n\text{)}$$

$$= np \sum_{k=0}^{n-1} {}^{n-1}\mathrm{C}_k p^k q^{n-k} = np(p+q)^{n-1} = np \qquad \square$$

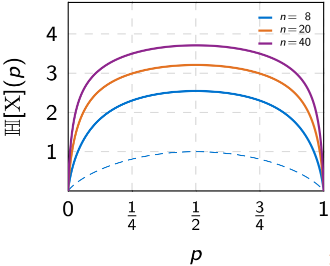
- Variance $\mathrm{Var}[\mathrm{X}] = np(1-p)$ [DB]

$$\mathbb{E}[\mathrm{X}^2] = \sum_{k=1}^{n} {}^{n}\mathrm{C}_k \, k^2 p^k q^{n-k} = \sum_{k=1}^{n} {}^{n}\mathrm{C}_k \, k p^k q^{n-k} + n(n-1)p^2 \sum_{k=2}^{n} {}^{n-2}\mathrm{C}_{k-2} p^{k-2} q^{n-k}$$

so $\mathrm{Var}[\mathrm{X}] = n(n-1)p^2(p+q)^{n-2} + np - n^2p^2 = np(1-p)$ $\square$

- Entropy $\mathbb{H}[\mathrm{X}]$

There is no general simple formula.
$\mathbb{H}[\mathrm{X}] \overset{n \gg 1}{\approx} \log_2 \sqrt{2\pi e n p(1-p)}$ using Stirling's approximation
$n! \overset{n \gg 1}{\approx} \sqrt{2\pi n}(n/e)^n$.
The max of $\mathbb{H}[\mathrm{X}]$ at $p = \frac{1}{2}$ increases with n.



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- Suppose that an aeroplane engine will fail with probability p (independently from engine to engine), and that the aeroplane makes a successful flight if at least half of its engines remain operative.
- For what values of p is a four-engine aeroplane preferable to a two-engine aeroplane?

We call X_n the number of failing engines, $\mathrm{X}_n \sim \mathrm{B}(n, p)$, with n the number of engines.

$$\begin{aligned}
\mathbb{P}[\text{"4-eng airborne"}] &= \textstyle\sum_{k=0}^{2} P_{\mathrm{X}_4}(k) \\
&= {}^4\mathrm{C}_0(1-p)^4 + {}^4\mathrm{C}_1 p(1-p)^3 + {}^4\mathrm{C}_2 p^2(1-p)^2 \\
&= 1 - 4p^3 + 3p^4
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}[\text{"2-eng airborne"}] &= \textstyle\sum_{k=0}^{1} P_{\mathrm{X}_2}(k) \\
&= {}^2\mathrm{C}_0(1-p)^2 + {}^2\mathrm{C}_1 p(1-p) \\
&= 1 - p^2
\end{aligned}$$

So $\mathbb{P}[\text{"4-eng airborne"}] \geq \mathbb{P}[\text{"2-eng airborne"}]$ for $p \leq \frac{1}{3}$. For $p = 10^{-5}$, a four-engine aeroplane is 10^{-8} % safer than a two-engine aeroplane. $\square$

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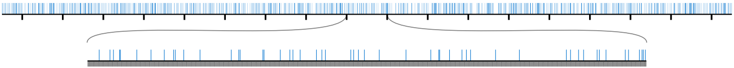
How many times was I successful given a success rate λ?

- Suppose we define a "density" of successes λ, that means we have λ successes per unit interval.

- We further divide the unit interval into n subintervals, and take n sufficiently large to see at most one success per subinterval.



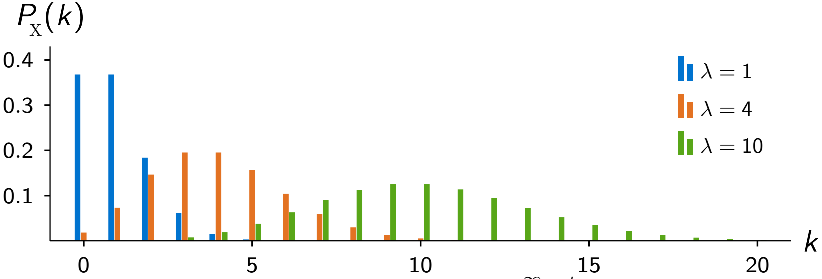
- The number of successes X in the unit interval follows a binomial distribution with n trials, $\mathrm{X} \sim \mathrm{B}(n, p)$, and average $\mathbb{E}[\mathrm{X}] = np = \lambda \Leftrightarrow p = \lambda/n$.
- The PMF is thus $\lim\limits_{n\to\infty} \mathrm{B}\left(n, \frac{\lambda}{n}\right)$: $\lim\limits_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$

$$= \lim_{n\to\infty} \underbrace{\frac{n}{n}\frac{n-1}{n}\dots\frac{n-k+1}{n}}_{=1} \left(\frac{\lambda^k}{k!}\right) \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{=e^{-\lambda}} \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{=1} = \frac{\lambda^k e^{-\lambda}}{k!} \qquad \square$$

A random variable X is said to have a Poisson distribution[2] with parameter $\lambda \in \mathbb{R}$ $(\lambda > 0)$ if:

$$\mathrm{X} \sim \mathrm{Pois}(\lambda) \quad \Leftrightarrow \quad P_{\mathrm{X}}(k) = \begin{cases} \dfrac{\lambda^k e^{-\lambda}}{k!} & \text{if } k \in \{0, 1, 2, \ldots\}, \\ 0 & \text{otherwise.} \end{cases}$$

The support of X, $\mathbb{X} = \{0, 1, 2, \ldots\}$, is discrete infinite.



Verify that $\sum_{k \in \mathbb{X}} P_{\mathrm{X}}(k) = 1$ using the power series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

[2] named after the French mathematician Siméon Denis Poisson (1781-1840)

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- Expectation $\mathbb{E}[X] = \lambda$ [DB]

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} e^{\lambda} = \lambda \qquad \square$$

- Variance $\mathrm{Var}[X] = \lambda$ [DB]

$$\mathbb{E}[X^2] = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!} = \lambda^2 e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \text{ so}$$

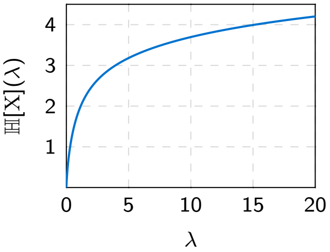
$$\mathrm{Var}[X] = \lambda^2 e^{-\lambda} e^{\lambda} + \lambda e^{-\lambda} e^{\lambda} - \lambda^2 = \lambda \qquad \square$$

- Entropy $\mathbb{H}[X]$

There is no general simple formula.

$\mathbb{H}[X] \overset{\lambda \gg 1}{\approx} \log_2 \sqrt{2\pi e \lambda}$

$\mathbb{H}[X]$ increases with λ.



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Poisson-distributed events are common. Here are a few instances:

- Time events
  - *Telecommunication*: telephone calls arriving in a system; internet traffic;
  - *Astronomy*: photons arriving at a telescope;
  - *Management*: customers arriving at a counter;
  - *Finance and insurance*: number of losses or claims;
  - *Seismology*: seismic risk in a given period of time;
  - *Radioactivity*: number of decays in a radioactive sample;
  - *Optics*: number of photons emitted in a single laser pulse.
- Spacial events
  - *Biology*: number of mutations on a strand of DNA;
  - *Medicine*: number of bacteria in a certain amount of liquid;
  - *Materials*: number of surface defects on a new refrigerator;
  - *Edition*: number of typographical errors found in a manuscript;
  - *Warfare*: targeting of flying bombs on London in World War II.

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Sum of independent Bernoulli trials:

- For n independent Bernoulli trials $\{\mathrm{X}_j \sim \mathrm{Ber}(p)\}_{j=1\dots n}$

  $$\sum_{j=1}^{n} \mathrm{X}_j \sim \mathrm{B}(n,p)$$

  Since the sum of the trials with support $\{0,1\}$ gives the number of successes.
- We verify that $\mathbb{E}\left[\sum_{j=1}^{n} \mathrm{X}_j\right] = np = \sum_{j=1}^{n} \mathbb{E}[\mathrm{X}_j]$, but also that $\mathrm{Var}\left[\sum_{j=1}^{n} \mathrm{X}_j\right] = np(p-1) = \sum_{j=1}^{n} \mathrm{Var}[\mathrm{X}_j]$. In general:

  $$\mathrm{Var}[\mathrm{X}+\mathrm{Y}] = \mathrm{Var}[\mathrm{X}] + \mathrm{Var}[\mathrm{Y}] \quad \text{if X, Y independent}$$

Approximating the Binomial distribution:

- We have seen $\mathrm{B}(n, \lambda/n) \xrightarrow{n\to\infty} \mathrm{Pois}(\lambda)$
- For large n, small p, and intermediate np, $\mathrm{B}(n,p) \approx \mathrm{Pois}(np)$ can be a convenient approximation.
- We will see a famous limit when $n \to \infty$ but p is fixed.

You can attempt Problems 1 to 7 of Examples Paper 5