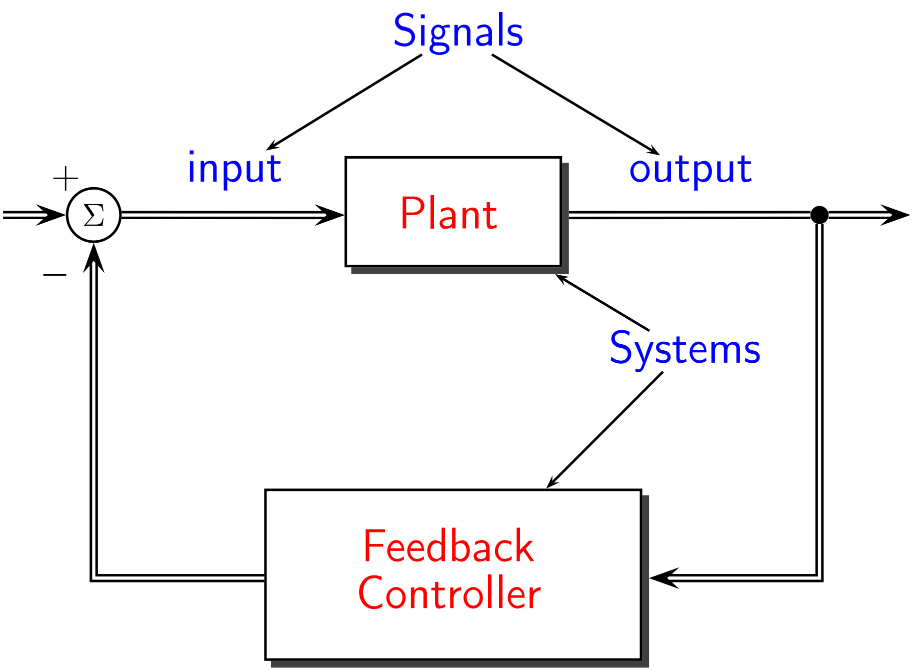
Part IB Paper 6: Information Engineering

LINEAR SYSTEMS AND CONTROL

Ioannis Lestas

HANDOUT 1

“Signals, systems and feedback”



(Filled-in and unfilled version of notes on Moodle)

Colour convention: Text in red in the filled handout is text that is animated during the lectures, and also does not appear in the unfilled handout.

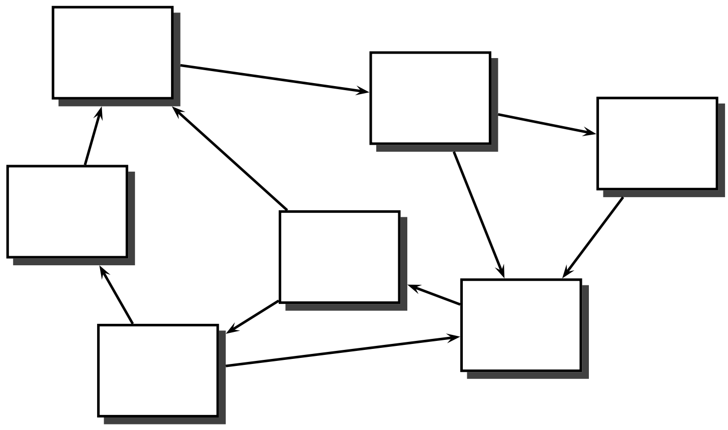
The **Aims** of the course are to:

- Introduce and motivate the use of feedback control systems.
- Introduce analysis techniques for linear systems which are used in control, signal processing, communications, and other branches of engineering.
- Introduce the specification, analysis and design of feedback control systems.
- Extend the ideas and techniques learnt in the IA Mechanical Vibrations course.

By the end of the course students should:

- Be able to develop and interpret block diagrams and transfer functions for simple systems.
- Be able to relate the time response of a system to its transfer function and/or its poles.
- Understand the term 'stability', its definition, and its relation to the poles of a system.
- Understand the term 'frequency response' (or 'harmonic response'), and its relation to the transfer function of a system.
- Be able to interpret Bode and Nyquist diagrams, and to sketch them for simple systems.
- Understand the purpose of, and operation of, feedback systems.
- Understand the purpose of proportional, integral, and derivative controller elements, and of velocity feedback.
- Possess a basic knowledge of how controller elements may be implemented using operational amplifiers, software, or mechanical devices.
- Be able to apply Nyquist's stability theorem, to predict closed-loop stability from open-loop Nyquist or Bode diagrams.
- Be able to assess the quality of a given feedback system, as regards stability margins and attenuation of uncertainty, using open-loop Bode and Nyquist diagrams.

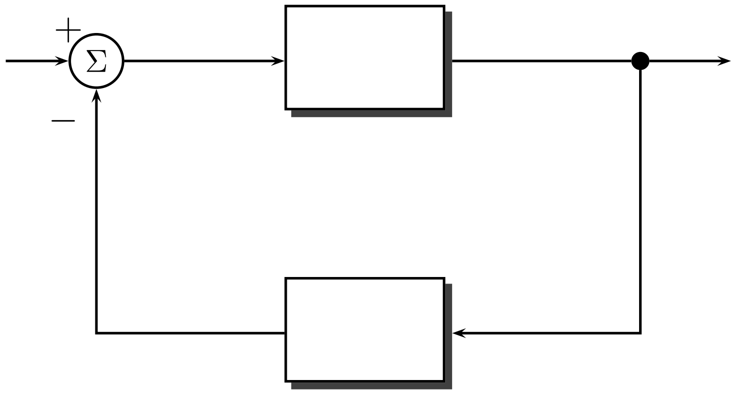
What the course is *really* about



- Understanding complex systems as an interconnection of simpler subsystems.
- Relating the behaviour of the interconnected system to the behaviour of the subsystems.
- (but we'll only consider the feedback interconnection in detail)

Part 1 (1st 7 lectures or so)

Part 2 (last 7 lectures or so)



SYLLABUS

Course material	Section numbers book 1	Section numbers book 2
Examples of feedback control systems. Use of block diagrams. Differential equation models. Meaning of 'Linear System'.	1.1-1.13 2.2-2.3	1.1-1.3 2.1-2.5
Review of Laplace transforms. Transfer functions. Poles (characteristic roots) and zeros. Impulse and step responses. Convolution integral. Block diagrams of complex systems.	2.4-2.6	3.1-3.2
Definition of stability. Pole locations and stability. Pole locations and transient characteristics.	6.1 5.6	3.3-3.6
Frequency response (harmonic response). Nyquist (polar) and Bode diagrams.	8.1-8.3	6.1-6.3
Terminology of feedback systems. Use of feedback to reduce sensitivity. Disturbances and steady-state errors in feedback systems. Final value theorem.	4.1-4.2 4.4-4.5	4.1 3.1.6
Proportional, integral, and derivative control. Velocity (rate) feedback. Implementation of controllers in various technologies.	7.7 12.6	4.3
Nyquist's stability theorem. Predicting closed-loop stability from open-loop Nyquist and Bode plots.	9.1-9.3	6.3
Performance of feedback systems: Stability margins, Speed of response, Sensitivity reduction.	8.5 9.4-9.6 12.5	6.4,6.6 6.9

References

1. Dorf,R.C, and Bishop,R.H, Modern Control Systems, 10th ed., (Addison-Wesley), 2005.

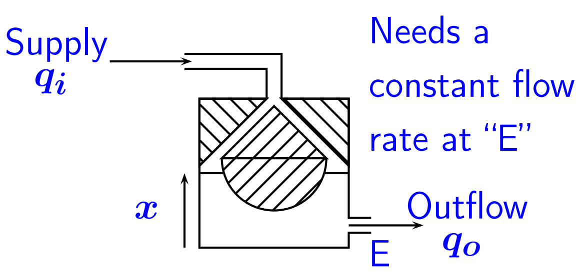
2. Franklin,G.F, Powell,J.D, and Emami-Naeini,A, Feedback Control of Dynamic Systems, 5th ed., (Addison-Wesley), 2006.

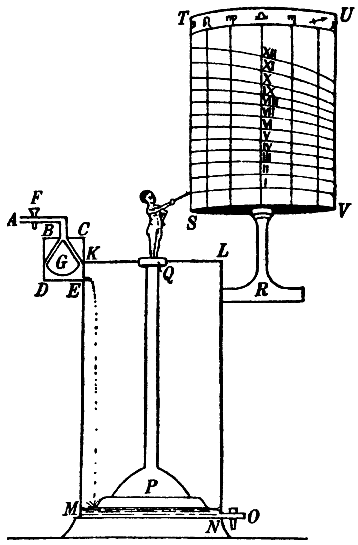
Notes based on earlier version by Prof Glenn Vinnicombe.

Contents

1.1 Examples of feedback systems

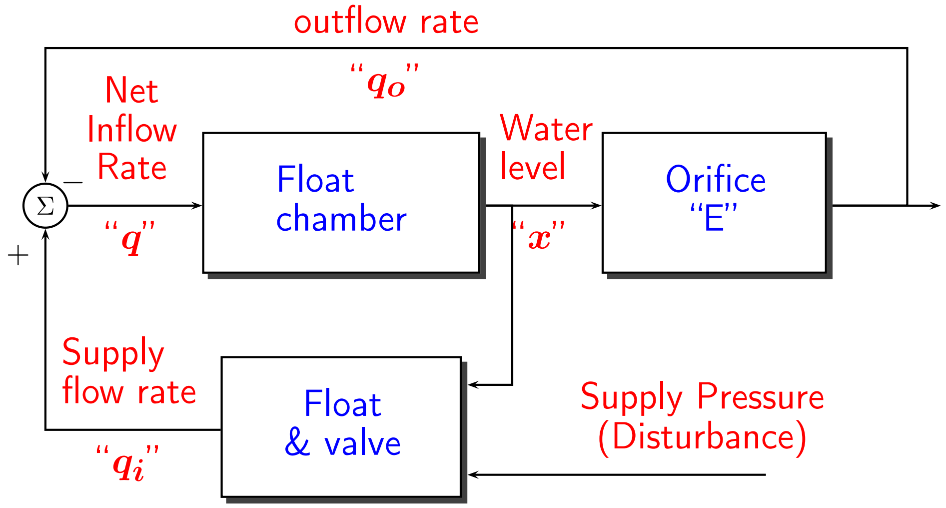
1.1.1 Ktesibios' Float Valve regulator (Water-clock, Alexandria 250BC)





is a feedback control system.

Block Diagram:



Signals have units (usually), are functions of time, and are represented by the *connections*:

e.g. Net inflow "$q(t)$" is measured in m^3/s
Water level "$x(t)$" is measured in m

The *Ktesibios' float valve regulator* is the oldest control system that has been reported in the literature. It goes back to ancient Alexandria and it is a feedback system that regulates the outflow rate q_0 from a chamber to a constant value (was used by ancient Egyptians as a water clock!).

In particular, if there is a disturbance in the supply rate q_i, e.g. decreasing the water level x, then the float in the chamber would move downwards, allowing more water to flow in and thus recover x to its desired level (similarly if the disturbance increases x).

A block diagram of the float valve regulator is illustrated on the left. Each block has *inputs* and *outputs* which are *signals* (i.e. functions of time). The block itself corresponds to a system, i.e. a differential equation the relates the inputs and outputs.

It should be noted that the block diagram is not unique as each block could be decomposed further into other sublocks.

Systems have equations, and are represented by the *blocks*:

e.g. the Float chamber is described by

$$\boldsymbol{x(t) = \frac{1}{A} \int_0^t q(\tau)\, d\tau}$$

cross-sectional area

The equation on the left is associated with the system represented by the "float chamber" block. In particular, it relates the output $x(t)$ (water level) with the input $q(t)$ (rate of water flow into chamber).
It should be noted that an equivalent way to write this equation is

$$\dot{x}(t) = \frac{1}{A} q(t)$$

1.1.2 Watt's Governor

The *Watt's Governor* is a control system that is a central part of Watt's steam engine, an invention that was at the core of the industrial revolution.

Its aim is to ensure automatically that there is a sufficient flow of steam to the engine as the load applied to the engine varies (see also next page).

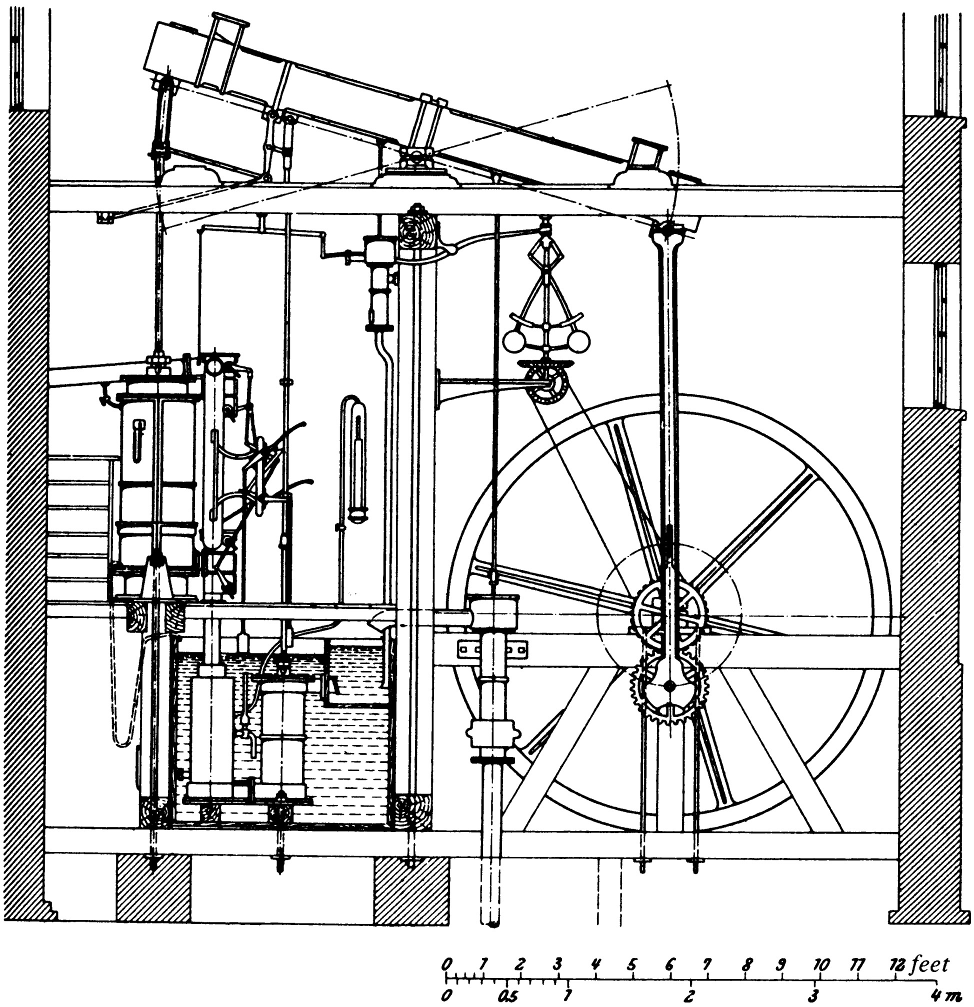
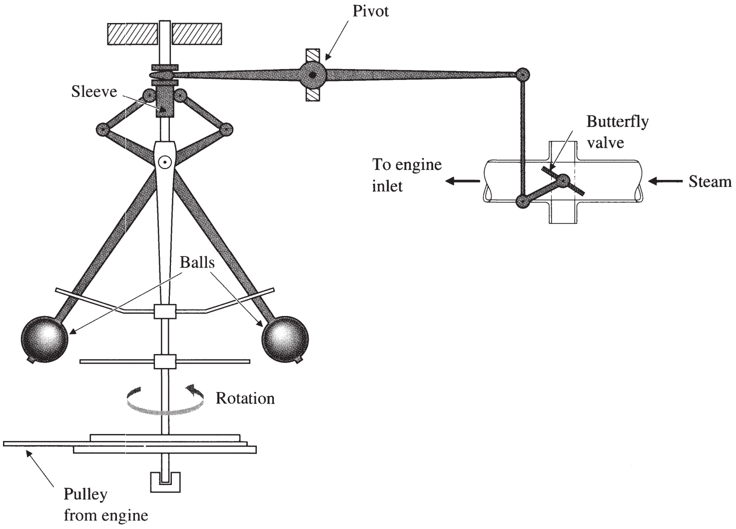


Figure 1 Watt steam engine (1789-1800) with centrifugal governor.

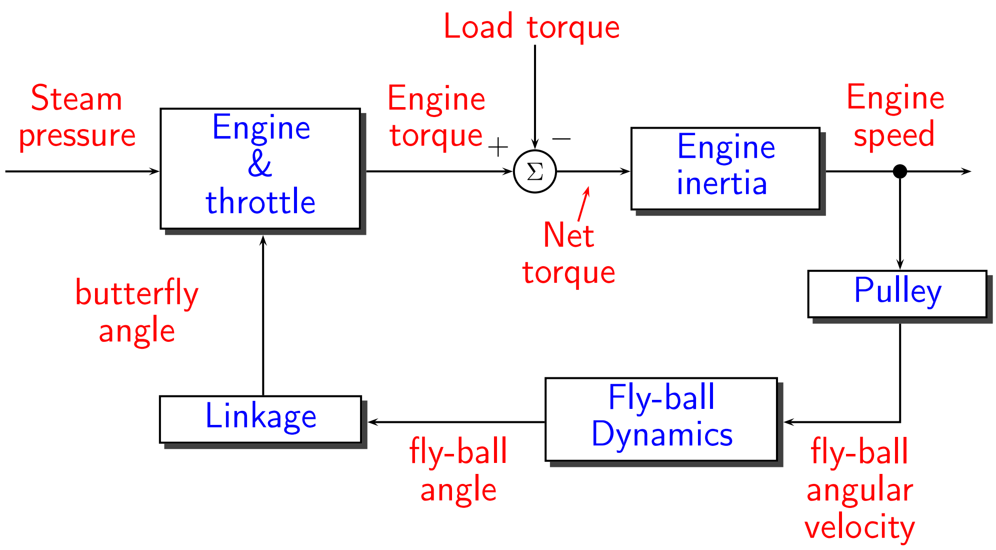
If the load upon the running engine is suddenly increased, its speed will decrease. The flyweights will swing back, and the sleeve will slide upward, causing the steam valve to open. The increase in the flow rate of steam, and hence in torque, will accelerate the engine. The centrifugal weights will fly outward again, reducing the aperture of the valve. Ultimately, the engine will reach an equilibrium at a new speed that lies somewhat below the equilibrium speed prior to the load increase. This *offset* due to lasting disturbances or changes in the command signal is a characteristic of all *proportional* control systems. The increased load requires an increased

Watt's Governor



Is a feedback control system.

Block diagram:

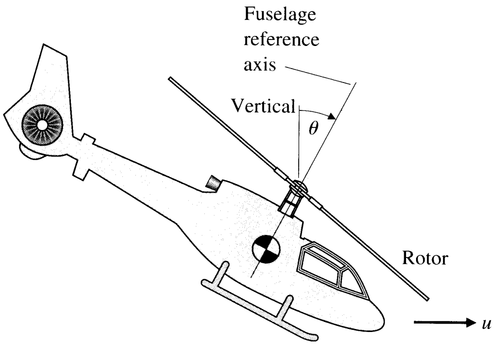


Note: it would be wrong to label the input to the feedback system as simply "steam" rather than "steam pressure". Steam in itself is not a quantity (although its pressure, temperature or flow rate is).

The operation of the Watt's Governor is based on a flyweight mechanism that increase/decreases the opening of the steam valve as the speed of the engine decreases/increases. The latter occurs due to a change in the load applied.

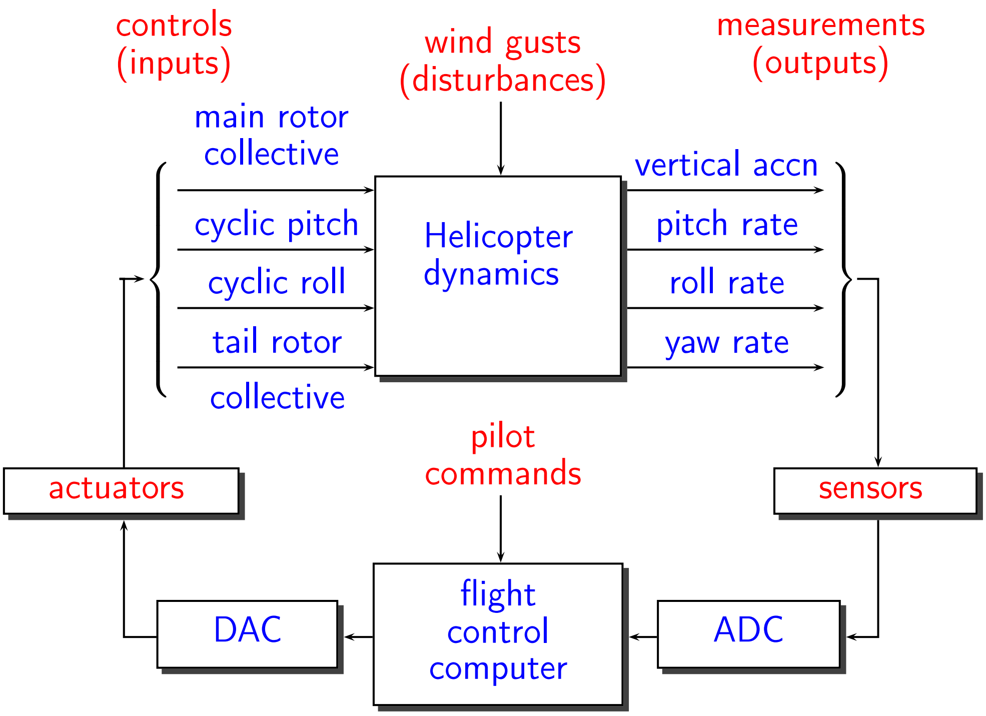
A block diagram representation of the Watt's Governor is provided on the left. As also mentioned in the previous example described, each block corresponds to a system with inputs and outputs that are signals.

1.1.3 A Helicopter Flight Control System



Is a feedback control system

Block Diagram:



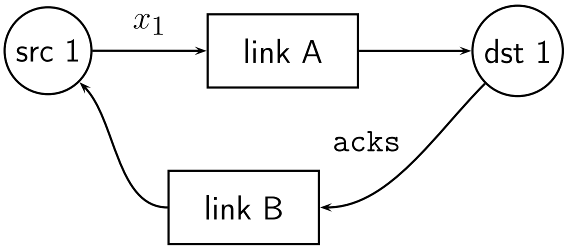
Designing the flight control system of a helicopter is in general a non-trivial problem as heicopters are, so called, open loop unstable systems, i.e. the helicopter will crash if its controllers are not in operation.

As illustrated in the block diagram the flight control policy will be implemented by a computer, hence an ADC is needed to convert the analogue measurements to digital signals to be processed by the flight control computer. A DAC will then convert the digital outputs of the computer to appropriate analogue signals for the actuators that determine the control inputs.

It should be noted that external disturbances can also affect the dynamic behaviour of the helicopter.

1.1.4 Internet congestion control (TCP)

Internet congestion control is a protocol that determines the rates with which computers/devices send data, based on congestion signals they receive from the network.



Is a feedback control system

– in fact, the largest man made one in the world.

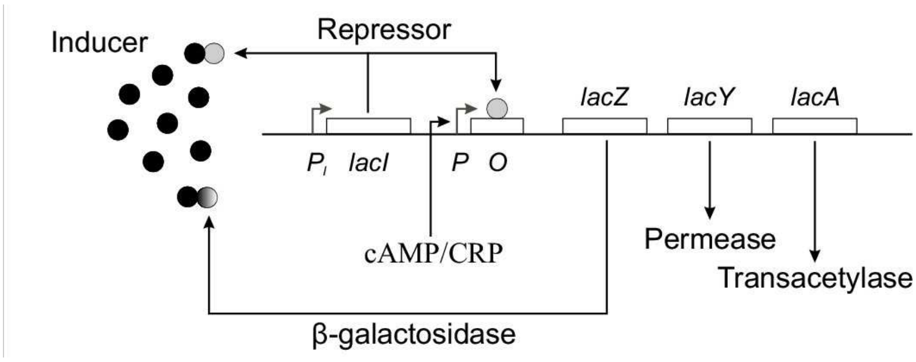
(in reality, of course, there are many source/destination pairs competing for bandwidth over many links)

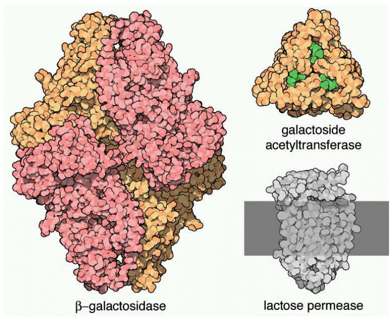
Note: This is NOT a block diagram

– it shows the flow of "stuff" (in this case packets) not information.

Files to be transferred across the Internet using the Transmission Control Protocol (TCP) - eg a download from the web - are broken into packets of size typically around 1500bytes, with headers specifying the destination and the number of the packet amongst other information. These packets are sent one by one into the network, with the recipient sending acknowledgements back to the source whenever one is received. Routers in the network typically operate a drop tail queue. If a packet is received when the queue is full then it is simply discarded. Packet loss thus indicates congestion. If a packet is received out of order, it is assumed that intervening packets have been lost. The recipient sends a duplicate acknowledgement to signal this and the source lowers its rate (in response to the congestion) and resends the lost packet(s). Whilst a steady stream of successive acknowledgements is being received the source gradually increases its sending rate. In normal operation sources are thus constantly increasing and decreasing their rates in an attempt to make use of the available bandwidth. Congestion (ie full queues and the resulting packet loss) can occurr anywhere in the network - at the edges (eg your adsl modem, or at the exchange), in the core (eg a big transatlantic link) or, very often, at peering points, which are the connections between the networks that make up the Internet.

1.1.5 The *lac* operon – E.Coli (≈ 130 million years BC!)





(Only read this if you're interested!) The diagram above illustrates 4 genes and some control regions along the DNA of E.Coli. E.Coli's favourite sugar is glucose, but it will quite happily "eat" lactose if there's no glucose around. If there *is* glucose around *or* if there is *no* lactose around then there is no need to produce β-galactosidase (the enzyme which breaks down lactose, first into allolactose and then glucose) or the permease (which transports lactose into the cell). In addition, when it is metabolising lactose, it wants to regulate the amount of enzyme production to match the available lactose. This is the control system which achieves this: The lacI gene codes for a protein (the repressor) which binds to the operator (O) and stops the lacZ,Y and A genes being transcribed (ie "read"). If there's lactose in the cell, and at least some β-galactosidase, then there will also be allolactose (the inducer). In this case the repressor binds with it instead, and falls off the DNA. In the absence of glucose, the cAMP/CRP complex binds at the promoter (P), this encourages RNA polymerase to bind and initiate transcription of lacZ,Y and A.

– for more details, see 3G1 next year …

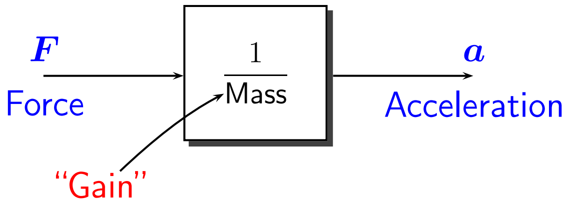
Feedback control is also ubiquitous in biological processes. A significant part of the DNA has a regulatory role, controlling which chemical reactions take place and at what rates. This is crucial for the various functionalities of the cells.

1.2 Block Diagrams

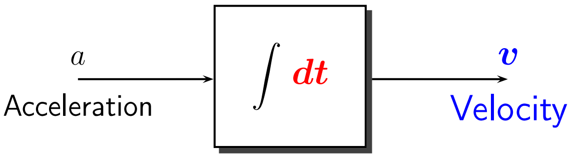
1.2.1 What goes in the blocks?

Some of them act like "amplifiers" or "attenuators"

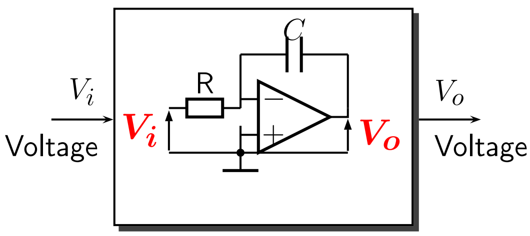
$$\boldsymbol{a} = \frac{\mathbf{1}}{\boldsymbol{Mass}} \times \boldsymbol{F}$$



But many are dynamic processes described by Ordinary Differential Equations (ODEs).



$$\dot{\boldsymbol{v}} = \boldsymbol{a}$$



$$\dot{V}_o = -\frac{1}{RC} V_i$$

(We shall (later) describe these by *transfer functions*.)

Note: By drawing this circuit as a block, we are implicitly assuming that any current it draws has negligible effect on the preceding block and that the following block draws insignificant current from it (i.e. that R is large and the op-amp is close to ideal).

The figures on the left illustrates various examples of systems and the corresponding equations that describe them.

In the first block the output is equal to the input multiplied by a constant.

In the second block the output is the integral of the input.

The third block corresponds to an op-amp circuit where the input and output voltage are related via an analogous differential equation.

1.2.2 Signals and systems

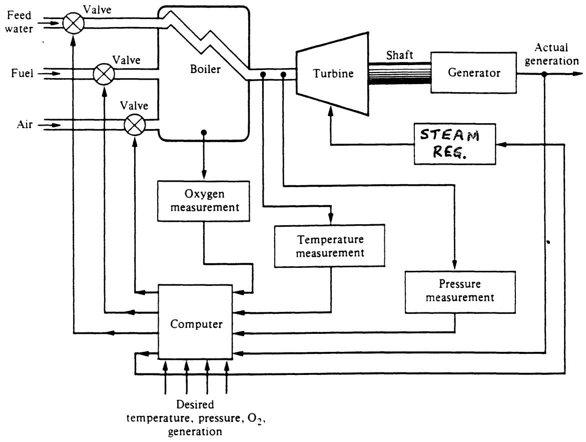
Block diagrams represent the flow of information, not the flow of "stuff".

(taking a numeric value as a function of time)

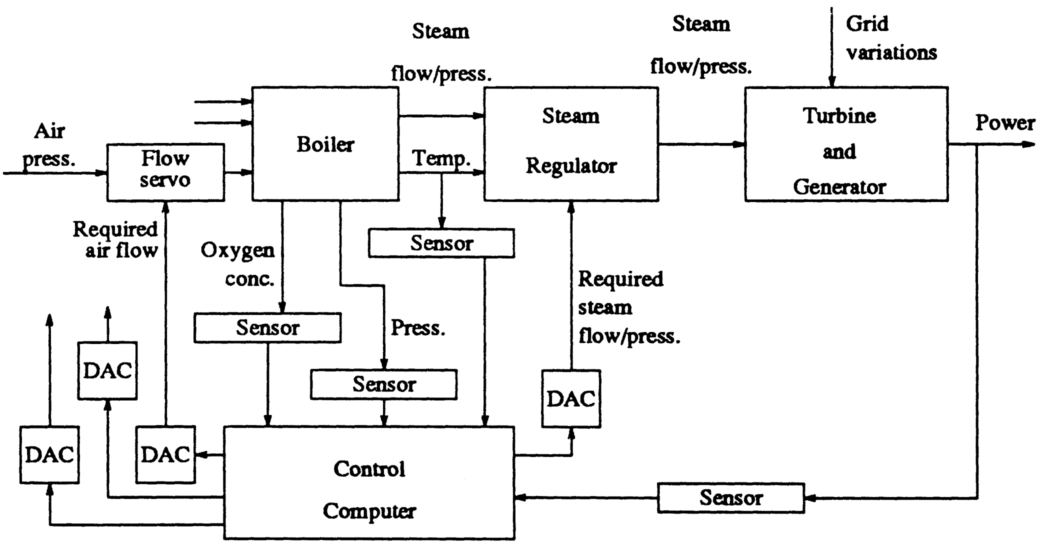
Blocks represent "systems ", whose inputs and outputs are "signals ".

(equations mapping inputs into outputs)

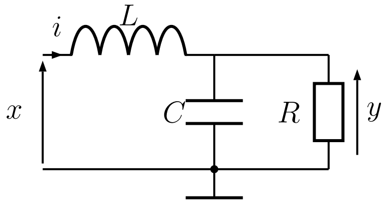
This is **NOT** a block diagram (in our sense)



This **IS** a block diagram



1.2.3 ODE models – A circuits example



A more involved system described by an electrical circuit is illustrated on the left.

The voltage difference x is the input of the system and the voltage difference y is the output. A differential equation relating x and y is derived using the standard differential equations satisfied by the voltage and current of an inductor and capacitor, respectively (part IA).

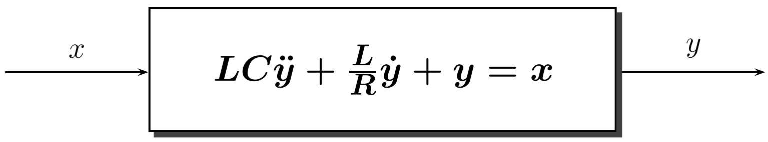
$$x - y = L\frac{di}{dt}$$

$$i = C\dot{y} + \frac{y}{R}$$

$$\Longrightarrow \qquad \boldsymbol{x - y = L\left(C\ddot{y} + \frac{\dot{y}}{R}\right)}$$

which gives a 2nd-order *linear* Ordinary Differential Equation:

$$\Longrightarrow \qquad \boldsymbol{LC\ddot{y} + \frac{L}{R}\dot{y} + y = x}$$



1.2.4 Block diagrams and the control engineer

For the control engineer:

Some blocks are given (fixed)

eg

- Steam Engine Dynamics
- Aircraft Dynamics

(the "plant")

while other blocks are to be designed

e.g.

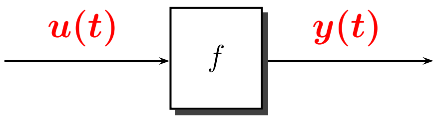
- Geometry of fly-ball mechanism in Watt governor.
- The program in an aircraft's flight control computer.

(the "controller")

1.3 Linear Systems

1.3.1 What is a "linear system"

Consider a "system" f mapping dynamic inputs u into outputs y



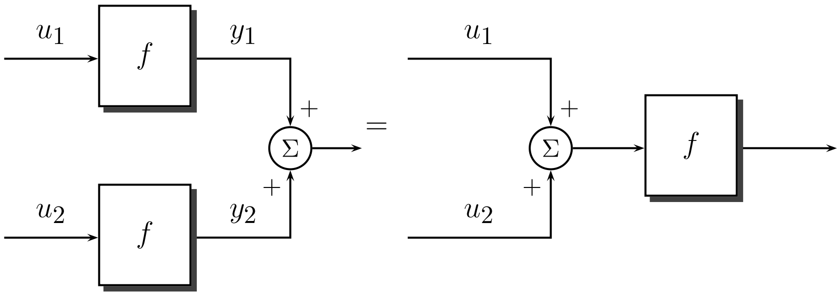
$$\boldsymbol{y = f\Big(u\Big)}$$

the "system" f is *linear* if superposition holds, that is, if

$$\underbrace{f\Big(u_1\Big)}_{} + \underbrace{f\Big(u_2\Big)}_{} = f\Big(\boldsymbol{u_1 + u_2}\Big)$$

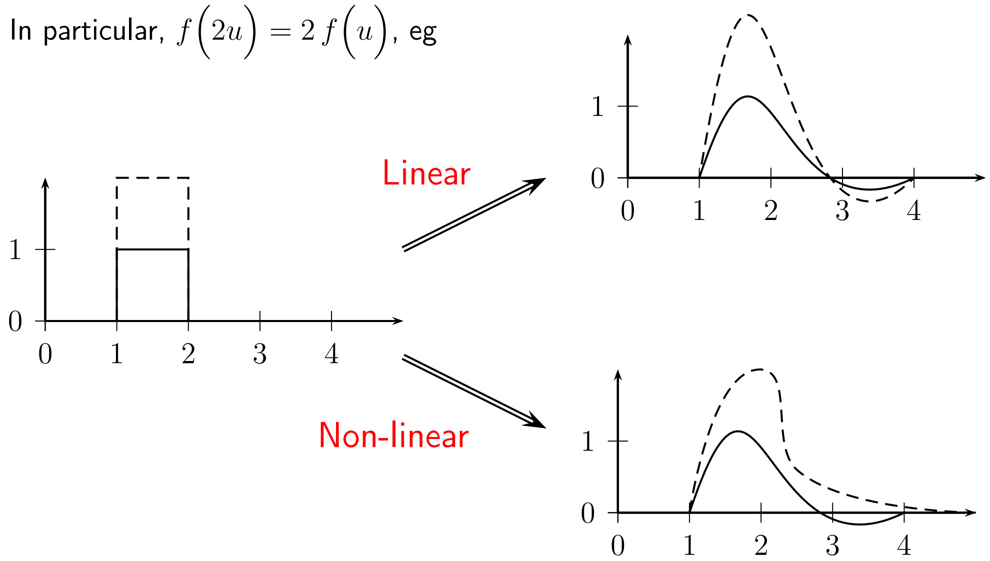
for any u_1 and u_2.

In terms of block diagrams. If f is a linear system,



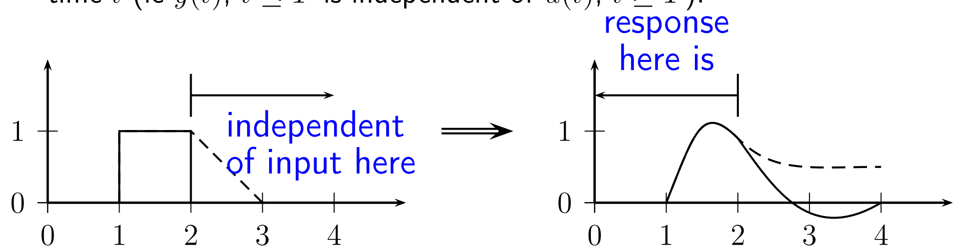
A system is linear if it satisfies the *principle of superposition*, i.e. if input u_1 gives output y_1, and input u_2 gives output y_2, then the input $u_1 + u_2$ will have as output $y_1 + y_2$.

In particular, $f\left(2u\right) = 2\,f\left(u\right)$, eg



In addition, we shall also assume that all systems are:

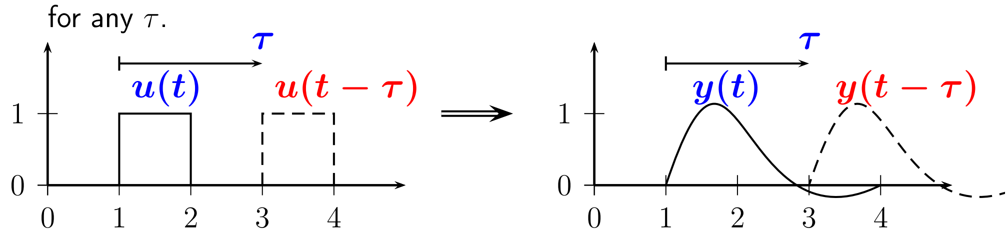
- *causal* – the output at time T, $y(T)$, depends only on the input up to time t (ie $y(t)$, $t \leq T$ is independent of $u(t)$, $t \geq T$).



- *time-invariant* – the response of the system to a particular input doesn't depend on when that input is applied, ie if

$$u(t) \to y(t), \text{ then } \boldsymbol{u(t-\tau) \to y(t-\tau)}$$

for any τ.



The principle of superposition is illustrated by the diagram on the left. Here the input is multiplied by 2, hence for a linear system the output is also multiplied by the same factor. If this is not the case then the system is non-linear.

The notions of *causality* and *time invariance* are also introduced on the left. These properties will be assumed throughout the course and will not be discussed further.

In particular, causality is satisfied by all physical systems. It captures the property that "the current output depends only on past inputs and not on future inputs".

Time invariance can be violated in some advanced models (e.g. a fault occurs on an aircraft). These models are though much more difficult to analyse and will not be covered in this course.

Note that, if a system can be described by linear differential equations with constant coefficients, and possibly delays, then it is necessarily linear (and time invariant).

For example:

$$\frac{d^2x(t)}{dt^2} + x(t-T) = \frac{du(t)}{dt} + 2u(t)$$

describes a linear system, as if

$$\frac{d^2x_1(t)}{dt^2} + x_1(t-T) = \frac{du_1(t)}{dt} + 2u_1(t)$$

and

$$\frac{d^2x_2(t)}{dt^2} + x_2(t-T) = \frac{du_2(t)}{dt} + 2u_2(t)$$

then

$$\begin{aligned}\frac{d^2}{dt^2}\big(x_1(t)+x_2(t)\big) + \big(x_1(t-T)+x_2(t-T)\big)\\ = \frac{d}{dt}\big(u_1(t)+u_2(t)\big) + 2\big(u_1(t)+u_2(t)\big)\end{aligned}$$

(because $\boldsymbol{\frac{d}{dt}\big(u_1(t)+u_2(t)\big) = \frac{du_1}{dt} + \frac{du_2}{dt}}$)

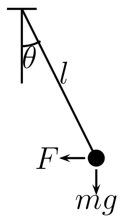
which is just the *superposition* of solutions. If there are x^2 terms or $\sin(x)$ terms, for example, then this doesn't work.

Almost all the linear systems considered in this course will be of this form, although it is convenient to develop the theory in more generality.

1.3.2 Linearization

All real systems are actually nonlinear, but many of these behave approximately linearly for small perturbations from equilibrium.

e.g. Pendulum:



$$Fl\cos\theta + mlg\sin\theta = -ml^2\ddot{\theta}$$

But, for *small* θ

$$Fl + mlg\theta \approx -ml^2\ddot{\theta}$$

or

$$\boxed{l\ddot{\theta} + g\theta \approx -F/m} \text{ which is a linear ODE}$$

General case

Suppose a system is described by an ODE of the form

$$\dot{x} = f(x,u)$$

where f is a smooth function. Assume that this system has an *equilibrium* at (x_0,u_0), by which we mean that

$$f(x_0,u_0) = 0.$$

where x_0 and u_0 are constants.

Let $x = x_0 + \delta x$, $u = u_0 + \delta u$,

and use a Taylor series expansion to obtain:

$$\cancelto{0}{\dot{x}_0} + \dot{\delta x} = f(x_0 + \delta x, u_0 + \delta u)$$

$$= \cancelto{0}{f(x_0, u_0)} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x_0,u_0}}_{\boldsymbol{A}} \delta x + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{x_0,u_0}}_{\boldsymbol{B}} \delta u + \cancelto{\text{neglect}}{\text{higher order terms}}$$

which results in the linear ODE

$$\boxed{\dot{\delta x} = A\delta x + B\delta u}$$

This is a simple example of a state-space model. *This procedure can be generalized to higher order systems with many inputs and outputs - see 3F2 next year.*

1.3.3 When can we use linear systems theory?

Linearity is often desirable:

- Hi-Fi audio system (non-linearities are called distortion).
- Aircraft fly-by-wire system (for predictable response)

If we are going to design a controller to keep a system near equilibrium then we can ensure that perturbations are small (and hence that behaviour is approximately linear). This justifies the use of linear theory for the design!

– so linear systems theory is often very useful even when the underlying systems are actually nonlinear

However: some systems are designed to behave nonlinearly:

- Switch or relay (because it is either on or off).
- Automated air traffic control system.
  (either have a collision or not) .

In such cases linear theory is of little use in itself.

1.4 Laplace Transforms

Laplace transforms are an extremely convenient tool for the analysis of linear, time-invariant, causal systems. We shall now briefly review some pertinent facts that you learnt at Part IA and introduce some new ideas.

DEFINITION:

$$\bar{y}(s) \;=\; \int_{0^-}^{\infty} y(t)e^{-st}\,dt$$

(provided the integral converges for sufficienty large and positive values of s.)

Note, a Laplace transform

- is *NOT* a function of t
- *IS* a function of s.

Various notations:

$$\mathcal{L}\{y(t)\} = \mathcal{L}y = \bar{y}(s) = \int_{0^-}^{\infty} y(t)e^{-st}\,dt$$

Notation for the *inverse transform*:

$$y(t) \;=\; \mathcal{L}^{-1}\bar{y}(s)$$

EXAMPLES

Find $\bar{y}(s)$ if $y(t) = C$ (a constant)

$$\bar{y}(s) \;=\; \int_{0}^{\infty} Ce^{-st}dt = C\left[\frac{-e^{-st}}{s}\right]_0^{\infty} = \frac{C}{s} \quad (\text{taking Real}(s) > 0\ ).$$

Find $\bar{y}(s)$ if $y(t) = e^{-at}$

$$\bar{y}(s) = \int_{0}^{\infty} e^{-(s+a)t}dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty} = \frac{1}{s+a} \quad (\text{taking Real}(s) > -a\ ).$$

Addition or Superposition property

If

$$y(t) = Ay_1(t) + By_2(t)$$

then

$$\bar{y}(s) = A\bar{y}_1(s) + B\bar{y}_2(s)$$

(A, B constants)

Proof:

$$\begin{aligned}\bar{y} &= \int_0^\infty (Ay_1 + By_2)e^{-st}dt \\ &= A\int_0^\infty y_1 e^{-st}dt + B\int_0^\infty y_2 e^{-st}dt \\ &= A\bar{y}_1 + B\bar{y}_2\end{aligned}$$

$\Longrightarrow$ The operation of taking a Laplace transform is linear.

Transforms of derivatives

$$\begin{aligned}\mathcal{L}\dot{y}(t) &= \int_0^\infty \frac{dy}{dt}e^{-st}dt \\ &= \left[y(t)e^{-st}\right]_0^\infty + s\int_0^\infty y(t)e^{-st}dt \\ &= s\bar{y} - y(0)\end{aligned}$$

$$\begin{aligned}\mathcal{L}\ddot{y} &= \int_0^\infty \frac{d^2y}{dt^2}e^{-st}dt \\ &= \left[\frac{dy}{dt}e^{-st}\right]_0^\infty + s\int_0^\infty \frac{dy}{dt}e^{-st}dt \\ &= -\dot{y}(0) + s(s\bar{y} - y(0)) \\ &= s^2\bar{y} - sy(0) - \dot{y}(0)\end{aligned}$$

Obvious pattern:

$$\boxed{\begin{array}{rcl} \mathcal{L}y & = & \bar{y} \\ \mathcal{L}\dot{y} & = & s\bar{y} - y(0) \\ \mathcal{L}\ddot{y} & = & s^2\bar{y} - sy(0) - \dot{y}(0) \\ \vdots & \vdots & \vdots \\ \mathcal{L}\dfrac{d^n y}{dt^n} & = & s^n\bar{y} - s^{n-1}y(0) - s^{n-2}\dot{y}(0) - \\ & & - \ldots - \left(\frac{d^{n-1}y}{dt^{n-1}}\right)(0) \end{array}}$$

In particular, if $y(0) = \dot{y}(0) = \ddot{y}(0) = \cdots = 0$, then

$$\begin{array}{rcl} \mathcal{L}y & = & \bar{y} \\ \mathcal{L}\dot{y} & = & s\bar{y} \\ \mathcal{L}\ddot{y} & = & s^2\bar{y} \\ \vdots & \vdots & \vdots \\ \mathcal{L}\dfrac{d^n y}{dt^n} & = & s^n\bar{y} \end{array}$$

differentiation (in the time domain) corresponds to multiplication by s (in the s domain)

Laplace Transform of t^n

Define $\bar{y}_n(s) \;=\; \mathcal{L}\,\dfrac{t^n}{n!}$.

$$\begin{aligned}
\bar{y}_n &= \int_0^\infty \frac{t^n}{n!} e^{-st} dt \\
&= \left[-\frac{1}{s}\frac{t^n}{n!}e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty \frac{nt^{n-1}}{n!}e^{-st}dt \\
&= \frac{1}{s}\int_0^\infty \frac{t^{n-1}}{(n-1)!}e^{-st}dt \\
&= \frac{1}{s}\bar{y}_{n-1},
\end{aligned}$$

(since for Real$(s) > 0$, and as $t \to \infty$, then $\left|e^{-st}\right| \to 0$ faster than $t^n \to \infty$).

Thus we have

$$\begin{aligned}
\bar{y}_0 &= \mathcal{L}\,1 = \frac{1}{s} \\
\bar{y}_1 &= \mathcal{L}\,t = \frac{1}{s^2} \\
\bar{y}_2 &= \mathcal{L}\,\frac{t^2}{2} = \frac{1}{s^3} \\
\bar{y}_3 &= \mathcal{L}\,\frac{t^3}{3 \times 2} = \frac{1}{s^4} \\
\text{Similarly} \quad \bar{y}_n &= \mathcal{L}\,\frac{t^n}{n!} = \frac{1}{s^{n+1}}
\end{aligned}$$

integration (in the time domain) corresponds to
division by s (in the s domain)

Poles and Zeros

Suppose $G(s)$ is a *rational* function of s, by which we mean

$$G(s) = \frac{n(s)}{d(s)}$$

where $n(s)$ and $d(s)$ are polynomials in s.

Then the roots of $n(s)$ are called the *zeros* of $G(s)$

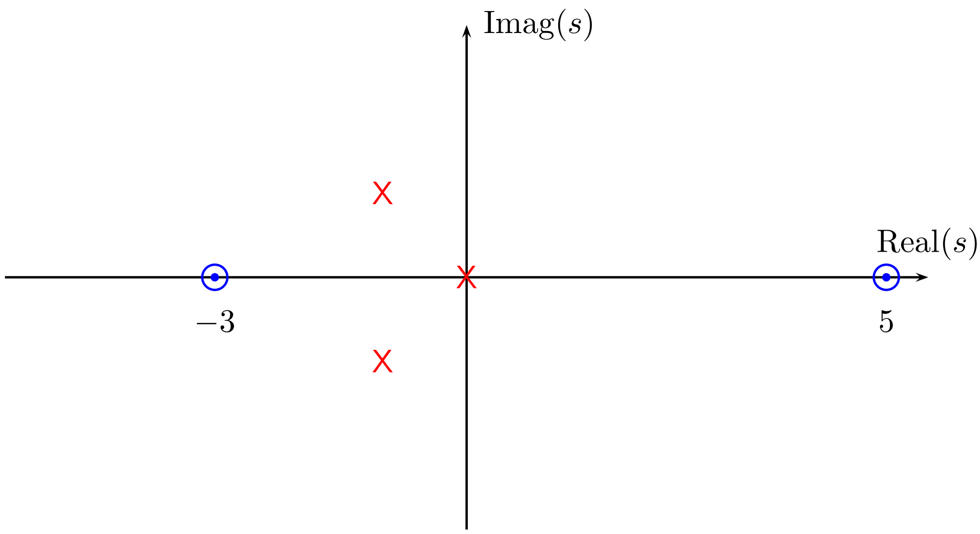
and the roots of $d(s)$ are called the *poles* of $G(s)$

Example:

$$\begin{aligned} G(s) &= \frac{4s^2 - 8s - 60}{s^3 + 2s^2 + 2s} \\ &= \frac{\mathbf{4(s+3)(s-5)}}{\mathbf{s(s+1+j)(s+1-j)}} \end{aligned}$$

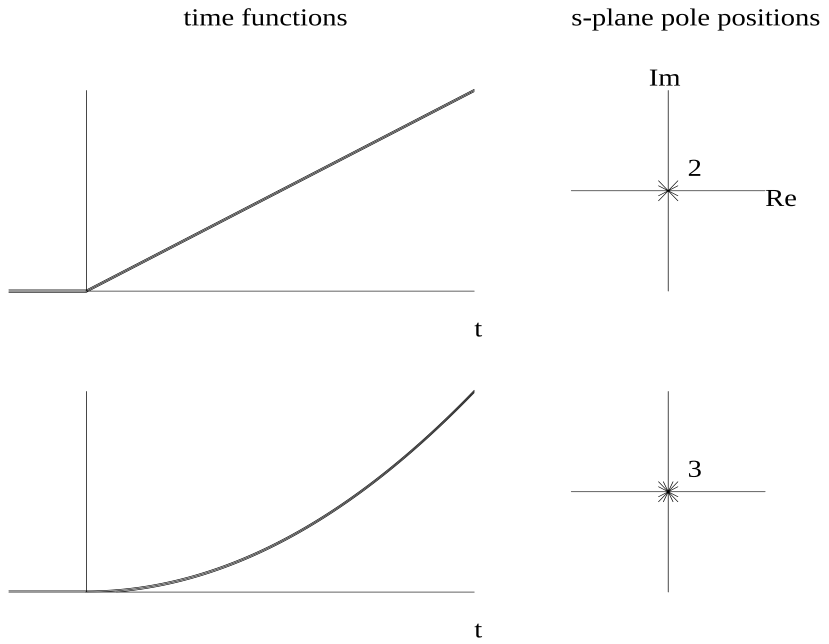
Zeros of $G(s)$ are $\mathbf{-3, +5}$.

Poles of $G(s)$ are $\mathbf{-1-j}$, $\mathbf{-1+j}$, $\mathbf{0}$



X – denote poles

⊙ – denote zeros



Time functions and pole positions for $y(t) = t$ and $y(t) = t^2$

Laplace Transforms of Sines and Cosines

$$y = e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$\begin{aligned} \bar{y} &= \frac{1}{s - i\omega} = \mathcal{L}\cos\omega t + i\mathcal{L}\sin\omega t \\ &= \frac{s + i\omega}{s^2 + \omega^2} \end{aligned}$$

Equating reals : $\boxed{\mathcal{L}\cos\omega t = \frac{s}{s^2 + \omega^2}}$

and similarly : $\boxed{\mathcal{L}\sin\omega t = \frac{\omega}{s^2 + \omega^2}}$

poles at $\boldsymbol{s = \pm\omega i}$ in both cases

NOTE: Results like this are tabulated in the Maths and Electrical Data Books.

Shift in s theorem

$$\boxed{\begin{aligned} \text{If } \mathcal{L}y(t) &= \bar{y}(s) \\ \text{then } \mathcal{L}e^{at}y(t) &= \bar{y}(s-a). \end{aligned}}$$

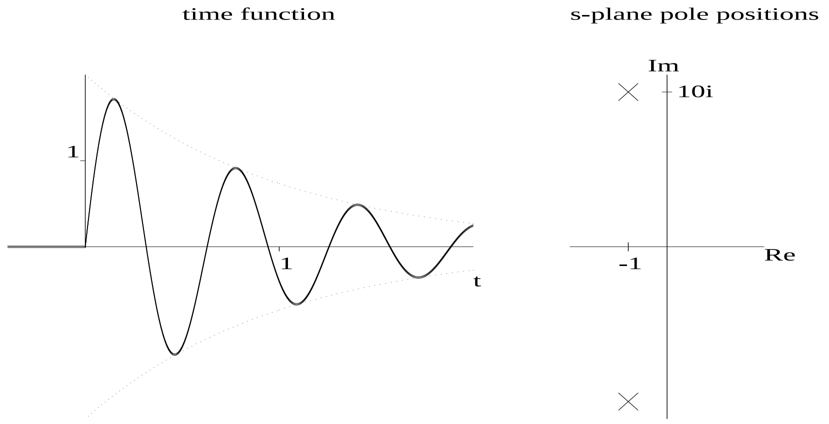
Proof:

$$\begin{aligned} \mathcal{L}e^{at}y(t) &= \int_0^\infty e^{-(s-a)t}y(t)\ dt \\ &= \bar{y}(s-a), \end{aligned}$$

Example of use: **poles at $s = -1 + 10j, -1 - 10j$**

$$\begin{aligned} \mathcal{L}^{-1}\frac{20}{s^2+2s+101} &= \mathcal{L}^{-1}\frac{20}{(s+1)^2+100} \\ &= 2e^{-t}\sin 10t \end{aligned}$$

because $\mathcal{L}^{-1}\dfrac{10}{s^2+100} = \sin 10t$



Time functions and pole positions for $y(t) = 2e^{-t}\sin 10t$

Initial and Final Value Theorems

If $\bar{y}(s) = \mathcal{L}\, y(t)$ then *whenever the indicated limits exist* we have

Final Value Theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\bar{y}(s)$$

Initial Value Theorem:

$$\lim_{t\to 0^+} y(t) = \lim_{s\to\infty} s\bar{y}(s)$$

Proofs omitted (as it's a little tricky to prove these properly.)

However, for rational functions of s it is easy to demonstrate that these relationships hold:

Let a partial fraction of $\bar{y}(s)$ be given as:

$$\bar{y}(s) = \frac{b_0}{s} + \sum_{i=1}^{n} \frac{b_i}{s+a_i} \quad \text{and so} \quad y(t) = b_0 + \sum_{i=1}^{n} b_i e^{-a_i t}.$$

Hence

$$y(0) = b_0 + \sum_{i=1}^{n} b_i \quad \text{and, provided } a_i > 0, \quad y(\infty) = b_0.$$

On the other hand,

$$s\bar{y}(s) = b_0 + \sum_{i=1}^{n} \frac{sb_i}{s+a_i}$$

hence

$$s\bar{y}(s)|_{s=\infty} = b_0 + \sum_{i=1}^{n} b_i \quad \text{and, provided } a_i \neq 0, \quad s\bar{y}(s)|_{s=0} = b_0$$

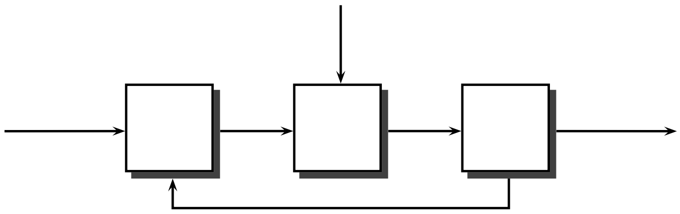
which are the same expressions as above.

The *Final Value Theorem* is often used in control system design and will be discussed extensively later within the course.

In particular, it allows to quantify the behaviour of a signal as time tends to infinity, from its Laplace transform.

It should be noted that this theorem cannot be used for signals that do not tend to a constant value as $t \to \infty$ (e.g. sinusoids, or signals that tend to infinity).

1.5 Key points



- Feedback is used to reduce sensitivity.
- We use block diagrams to represent feedback interconnections.
- Each block represents a "system".
- Each connection carries a "signal".
- We shall assume that systems are described by *linear*, *time-invariant* and *causal* ODE's.

- We distinguish between *causes* (the input signals) and *effects* (the output signals).
- Large and complex systems can be constructed by connecting together simpler sub-systems.
- Laplace transforms are central to the study of linear, time-invariant systems.