Lecture 9

Gauss's Divergence Theorem

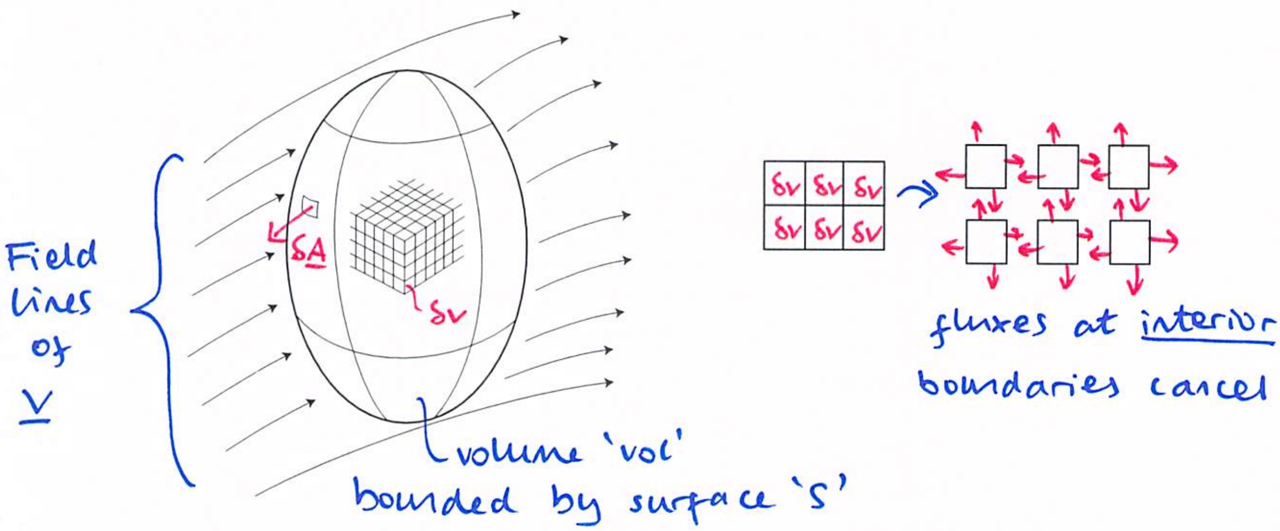
9.1 Gauss's theorem

Gauss's theorem states,

"If a finite volume vol is bounded by a *closed* surface S, and $\mathbf{V}$ is a vector field with continuous derivatives,

$$\iiint_{\text{vol}} (\nabla \cdot \mathbf{V})\, dv = \oiint_S \mathbf{V} \cdot d\mathbf{A} \quad , \tag{9.1}$$

where δv is an element of volume within vol and $\delta \mathbf{A}$ is a vector element of the area S with a direction corresponding to the outward facing normal." In words, "The volume integral of $\nabla \cdot \mathbf{V}$ over the volume vol is equal to the net flux of $\mathbf{V}$ through the surface S enclosing vol ."

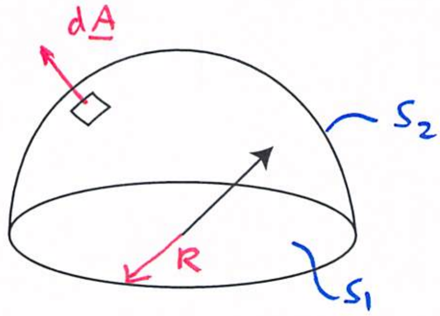


We can prove Gauss's theorem as follows: We divide our finite volume into a large number of elemental volumes δv. For each element, we know that the net efflux of $\mathbf{V}$ is given by $(\nabla \cdot \mathbf{V})\delta v$. In the interior of vol (away from the surface S), the flux out of one δv is the flux in to the adjacent δv and so all these contributions to the volume integral $\iiint_{\text{vol}} (\nabla \cdot \mathbf{V}) dv$ cancel. The only fluxes that do not cancel are on the exterior vol , i.e. on the surface S, and these sum up to give the total flux through S, $\oiint_S \mathbf{V} \cdot d\mathbf{A}$.

We can use Gauss's theorem to transform a *surface* integral over a closed surface, into a *volume* integral over the enclosed volume, and *vice-versa*.

Example

If $\mathbf{C} = \mathbf{r}$, evaluate $\oiint_S \mathbf{C} \cdot d\mathbf{A}$ where S encloses the volume V that is the hemisphere $|r| < R$ and $z > 0$.



Find $\oiint_S \underline{C} \cdot d\underline{A}$

By surface integral: $\oiint \underline{C} \cdot d\underline{A} = \iint_{S_1} \underline{r} \cdot d\underline{A} + \iint_{S_2} \underline{r} \cdot d\underline{A}$

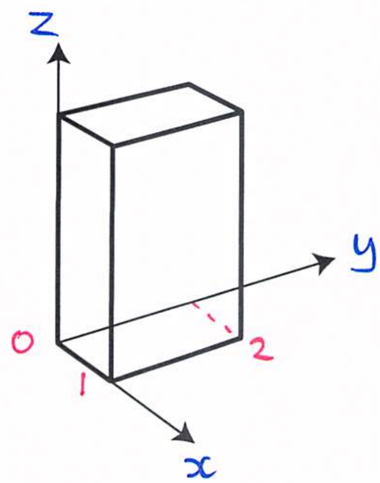
$$= \iint_{S_2} R \, dA = 2\pi R^3$$

By Gauss: $\nabla \cdot \underline{C} = 3$

$$\oiint_S \underline{C} \cdot d\underline{A} = \iiint_{vol} (\nabla \cdot \underline{C}) \, dv = 3V = 2\pi R^3$$

Example

If S is the surface of the volume bounded by planes at $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$, $z = 3$, evaluate $\oiint_S \mathbf{V} \cdot d\mathbf{A}$ where $\mathbf{V} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$.



Method 1

Transform to a volume integral using Gauss's theorem:

$$\nabla \cdot \underline{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= 4z - 2y + y = 4z - y$$

$$\oiint_S \underline{V} \cdot d\underline{A} = \iiint_{vol} (\nabla \cdot \underline{V})\, dv$$

$$= \int_{z=0}^{3} \int_{y=0}^{2} \int_{x=0}^{1} (4z - y)\, dx\, dy\, dz$$

$$= \int_0^3 \int_0^2 \left[4xz - xy\right]_0^1 dy\, dz = \int_0^3 \int_0^2 (4z - y)\, dy\, dz$$

$$= \int_0^3 \left[4yz - \frac{y^2}{2}\right]_0^2 dz = \int_0^3 8z - 2\, dz = \left[4z^2 - 2z\right]_0^3 = 30$$

Method 2

Evaluation of fluxes through each part of the surface S:

Face A at $x = 0$. $\delta \mathbf{A} = -\delta y \delta z\, \mathbf{i}$; $\mathbf{V} = -y^2\, \mathbf{j} + yz\mathbf{k}$, hence,

$$\iint_A \underline{V} \cdot d\underline{A} = 0$$

Face B at $x = 1$. $\delta \mathbf{A} = +\delta y \delta z\, \mathbf{i}$; $\mathbf{V} = 4z\, \mathbf{i} - y^2\, \mathbf{j} + yz\mathbf{k}$, hence,

$$\iint_B \underline{V} \cdot d\underline{A} = \int_{z=0}^{3} \int_{y=0}^{2} 4z\, dy\, dz = 36$$

Face C at $y = 0$. $\delta \mathbf{A} = -\delta x \delta z\, \mathbf{j}$; $\mathbf{V} = 4xz\, \mathbf{i}$, hence,

$$\iint_C \underline{V} \cdot d\underline{A} = 0$$

Face D at $y = 2$. $\delta\mathbf{A} = +\delta x \delta z\,\mathbf{j}$; $\underline{\mathbf{V}} = 4xz\,\mathbf{i} - 4\,\mathbf{j} + 2z\,\mathbf{k}$, hence,

$$\iint_D \underline{V} \cdot d\underline{A} = \int_{z=0}^{3} \int_{x=0}^{1} -4 \, dx \, dz = -12$$

Face E at $z = 0$. $\delta\mathbf{A} = -\delta x \delta y\,\mathbf{k}$; $\mathbf{V} = -y^2\,\mathbf{j}$, hence,

$$\iint_E \underline{V} \cdot d\underline{A} = 0$$

Face F at $z = 3$. $\delta\mathbf{A} = +\delta x \delta y\,\mathbf{k}$; $\mathbf{V} = 12x\,\mathbf{i} - y^2\,\mathbf{j} + 3y\,\mathbf{k}$, hence,

$$\iint_F \underline{V} \cdot d\underline{A} = \int_{y=0}^{2} \int_{x=0}^{1} 3y \, dx \, dy = 6$$

So the net flux through S is

$$\oiint_S \underline{V} \cdot d\underline{A} = 0 + 36 + 0 - 12 + 0 + 6 = 30$$

The result from Method 1 is the same as Method 2, in agreement with Gauss's theorem.

9.2 Solenoidal fields

From Gauss's theorem,

- If $\mathbf{V}$ is a solenoidal field, $\nabla \cdot \mathbf{V} = 0$ everywhere, then $\oiint_S \mathbf{V} \cdot d\mathbf{A} = 0$: the net efflux of $\mathbf{V}$ from *any finite volume* is zero.

9.3 Conservation equations in integral and differential form

In engineering, a key application of Gauss's theorem is in allowing us to move between integral and differential forms of conservation-type statements without the need to re-derive the equations or specify a coordinate system.

Example

Unsteady diffusion of heat. Consider a solid that is heated and cooled at its edges. For any volume V within the solid that is bounded by the closed surface S, In words,

- rate of change of energy in V = net heat flow out of S .

which we may write as,

$$-\frac{\partial}{\partial t}\left(\iiint_V \rho c T \, dv\right) = \iint_S \mathbf{q}\cdot d\mathbf{A} \quad . \tag{9.2}$$

If V is fixed then we can take the time derivative inside the integral,

$$-\iiint_V \frac{\partial}{\partial t}(\rho c T)\, dv = \iint_S \mathbf{q}\cdot d\mathbf{A} \quad . \tag{9.3}$$

If the temperature field is $T = T(x,y,z,t)$, the heat flux vector field is given by, $\mathbf{q} = -\lambda\nabla T$, so we have

$$-\iiint_V \frac{\partial}{\partial t}(\rho c T)\, dv = \iint_S -\lambda\nabla T\cdot d\mathbf{A} \quad . \tag{9.4}$$

Using Gauss's theorem,

$$\iiint_V \left(\frac{\partial}{\partial t}(\rho c T) - \nabla\cdot(\lambda\nabla T)\right) dv = 0 \quad . \tag{9.5}$$

If we shrink V to a point ($V \to 0$), the equation must still be valid, but now in differential form,

$$\frac{\partial}{\partial t}(\rho c T) - \nabla\cdot(\lambda\nabla T) = 0 \quad . \tag{9.6}$$

If ρ, c and λ are constant,

$$\frac{\partial T}{\partial t} = \alpha\nabla\cdot(\nabla T) = \alpha\nabla^2 T \tag{9.7}$$

where $\alpha = \lambda/(\rho c)$. This is referred to as the diffusion equation.

Example

Derive the mass continuity equation in integral and differential forms.

The integral formulation of the conservation of mass is concerned with a *finite* control volume (CV) bounded by a closed surface S:

Rate of increase of mass in CV = − net efflux of mass out of CV

$$\frac{\partial}{\partial t}\left(\iiint_{vol} \rho \, dv\right) = \iiint \frac{\partial \rho}{\partial t}\, dv = -\oiint \rho \underline{v}\cdot d\underline{A}$$

Continuity eq^n in integral form:

$$\iiint_{vol} \frac{\partial \rho}{\partial t}\, dv + \oiint_S \rho \underline{V} \cdot d\underline{A} = 0$$

Using Gauss:
$$\iiint_{vol} \frac{\partial \rho}{\partial t}\, dv + \iiint_{vol} \nabla \cdot (\rho \underline{V})\, dv = 0$$

$$\iiint_{vol} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) \right) dv = 0$$

Shrink vol to a point:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$$

9.4 Coordinate-free definition of the divergence

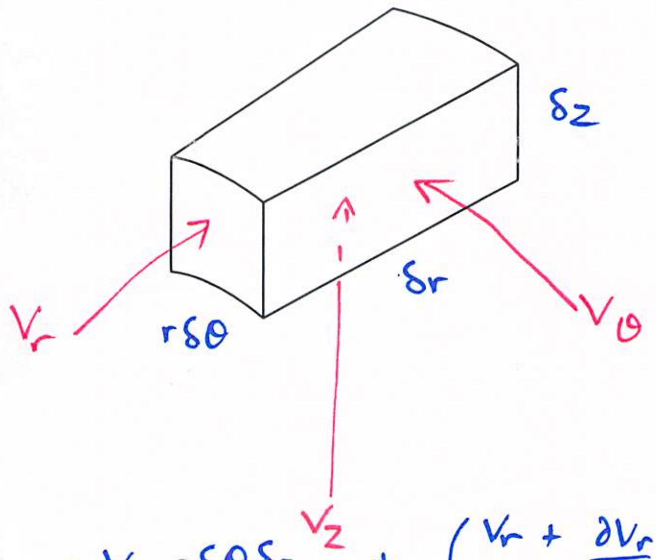
Gauss's divergence theorem can be used to provide a coordinate-free definition of the divergence,

$$\nabla \cdot \mathbf{V} = \lim_{\delta v \to 0} \frac{1}{\delta v} \oiint_S \mathbf{V} \cdot d\mathbf{A} \tag{9.8}$$

where S is the surface of the elemental volume δv.

Example

Obtain the divergence of $\mathbf{V}$ when $\mathbf{V}$ is defined in cylindrical polar coordinates, $\mathbf{V} = \mathbf{V}(r, \theta, z)$:



$$\oint\!\!\oint \underline{V}\cdot d\underline{A} = -V_r\, r\delta\theta\delta z + \left(V_r + \frac{\partial V_r}{\partial r}\delta r\right)(r+\delta r)\delta\theta\,\delta z$$
$$-V_\theta\,\delta r\,\delta z + \left(V_\theta + \frac{\partial V_\theta}{\partial \theta}\delta\theta\right)\delta r\,\delta z$$
$$-V_z\, r\delta\theta\delta r + \left(V_z + \frac{\partial V_z}{\partial z}\delta z\right) r\,\delta\theta\delta r$$

$$\nabla\cdot\underline{V} = \frac{\oint\!\!\oint \underline{V}\cdot dA}{r\delta\theta\,\delta r\,\delta z} = \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_z}{\partial z}$$

You can now do Examples Paper 3: Q4, 5, 6 and 7