Lecture 6

The Curl of a Vector Field

6.1 Definition

So far, we have defined two applications of the operator, ∇:

- $\nabla\phi$ - the **gradient** of a *scalar* field - the result is a new *vector* field;
- $\nabla \cdot \mathbf{V}$ - the **divergence** of a *vector* field - the results is a new *scalar* field.

We now introduce a third operation involving ∇:

- $\nabla \times \mathbf{V}$ - the **curl** of a *vector* field - the result is a new *vector* field.

The link between *curl* and the vector cross product is indicated by the $\times$ symbol. We can see the similarities by looking at the definition of the curl in Cartesian coordinates:

$$\nabla \times \mathbf{V} = \mathbf{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \tag{6.1}$$

where $\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$.

The above definition is easier to remember using the determinant form,

$$\nabla \times \underline{V} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

Despite the similarities between the algebraic cross product and the curl operation, there is one important difference. The vector $\nabla \times \mathbf{V}$ is not necessarily orthogonal to the vector $\mathbf{V}$. The direction of $\nabla \times \mathbf{V}$ can be at any angle to $\mathbf{V}$ – it can even be parallel to it.

Example

Find the field lines of the velocity field $\mathbf{V} = -\Omega y\,\mathbf{i} + \Omega x\,\mathbf{j}$ (Ω is a constant) and evaluate $\nabla \times \mathbf{V}$.

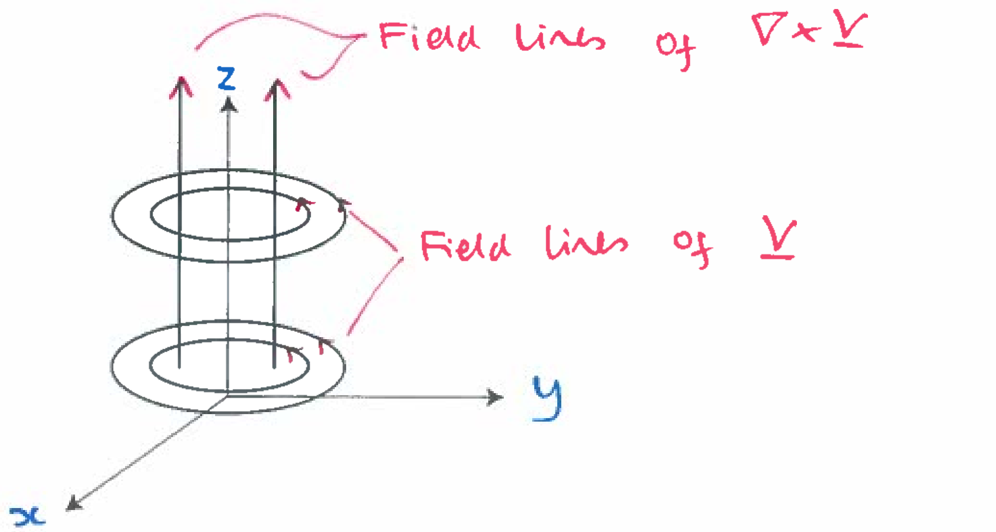
Field lines : $\frac{dy}{dx} = \frac{-\Omega x}{\Omega y} \Rightarrow x^2 + y^2 = \text{const}$

(circles centred on z axis)

$$\nabla \times \underline{V} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\Omega y & \Omega x & 0 \end{vmatrix} = \underline{k}\,(\Omega - (-\Omega)) = 2\Omega\,\underline{k}$$

Looking again at $\underline{V}$, $|\underline{V}| = \sqrt{(\Omega y)^2 + (\Omega x)^2} = r\Omega$

i.e. solid-body rotation



The vector field $\mathbf{V} = -\Omega y\,\mathbf{i} + \Omega x\,\mathbf{j}$ represents solid body rotation with angular velocity Ω. $\nabla \times \mathbf{V}$ has constant magnitude 2Ω and direction parallel to the axis of rotation. For all 2-D vector fields, the curl field is normal to the plane of the original field.

6.2 Useful identities

Two important identities are:

- $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ All curl fields are solenoidal
- $\nabla \times (\nabla \phi) = 0$ All vector fields from $\underline{V} = \nabla\phi$ have zero curl

where $\mathbf{A}$ is *any* vector field, and ϕ is any scalar field.

In the first of the above identities, we take the curl of $\mathbf{A}$ to obtain a new vector field and then take the divergence of this new field. The identity tells us that *all curl fields are solenoidal* (even if $\mathbf{A}$ is not, itself, solenoidal).

In the second identity, we take the curl of the vector field obtained by taking the gradient of the scalar field, ϕ. The identity tells us that all vector fields obtained from this *scalar potential* process have zero curl.

The following identities are also useful and can be proved by expending in Cartesian form (all are in the Maths Data Book). If $\mathbf{A}$ and $\mathbf{B}$ are vector fields, and ϕ is a scalar field,

$$\nabla \times (\mathbf{A}+\mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \quad , \tag{6.2}$$

$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi\nabla \times \mathbf{A} \quad , \tag{6.3}$$

$$\nabla \times (\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B} \quad , \tag{6.4}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (A\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B}) \quad . \tag{6.5}$$

Here, we see again the *scalar operator* that we met in Lecture 4, e.g. $(\mathbf{B}.\nabla)\mathbf{A}$. The brackets are not always used because $\nabla\mathbf{A}$ is not a defined operation. In Cartesian coordinates, $(\mathbf{B}.\nabla)\mathbf{A}$ expands as follows,

$$(\mathbf{B}\cdot\nabla)\mathbf{A} = \left(\overbrace{B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}}^{\mathbf{B}.\nabla}\right)\mathbf{A} \tag{6.6}$$

$$= \mathbf{i}\left(B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z}\right) \tag{6.7}$$

$$+ \mathbf{j}\left(B_x\frac{\partial A_y}{\partial x} + B_y\frac{\partial A_y}{\partial y} + B_z\frac{\partial A_y}{\partial z}\right) \tag{6.8}$$

$$+ \mathbf{k}\left(B_x\frac{\partial A_z}{\partial x} + B_y\frac{\partial A_z}{\partial y} + B_z\frac{\partial A_z}{\partial z}\right) \quad . \tag{6.9}$$

6.3 Curl in non-Cartesian coordinate systems

We include here the cylindrical polar and spherical polar forms of the curl operation, $\nabla \times \mathbf{V}$.

If $\mathbf{V}$ is defined in cylindrical polar coordinates, $\mathbf{V} = \mathbf{V}(r,\theta,z)$,

$$\nabla \times \mathbf{V} = \frac{1}{r}\begin{vmatrix} \mathbf{e_r} & r\mathbf{e}_\theta & \mathbf{e_z} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ V_r & rV_\theta & V_z \end{vmatrix} \quad . \tag{6.10}$$

If $\mathbf{V}$ is defined in spherical polar coordinates, $\mathbf{V} = \mathbf{V}(r, \theta, \phi)$,

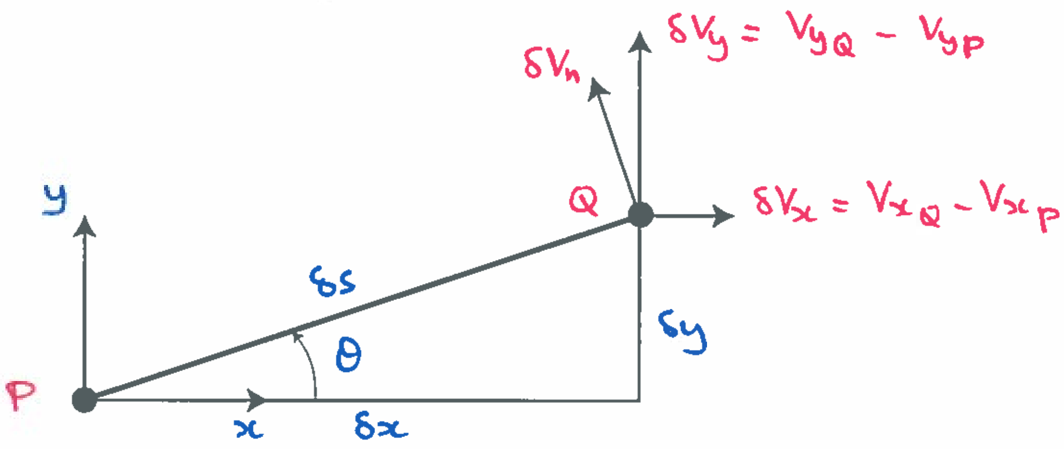
$$\nabla \times \mathbf{V} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e_r} & r\,\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ V_r & rV_\theta & r\sin\theta V_\phi \end{vmatrix} \quad . \tag{6.11}$$

6.4 Physical interpretation of curl

We have seen in the example in Section 6.1 that curl is linked to the angular velocity of a particle in the velocity vector field $\mathbf{V}$. We now explore that connection further.

For a 2-D velocity field in Cartesian coordinates, $\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j}$, the curl of $\mathbf{V}$ is,

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & 0 \end{vmatrix} = \underline{k} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \tag{6.12}$$



What is the physical significance of $(\partial V_y/\partial x - \partial V_x/\partial y)$? Two points, P and Q, are separated by a small line element, of length δs, inclined at an angle θ to the x-axis. The difference in the x- and y- components of velocity at P and Q can be obtained from a Taylor expansion about P,

$$\delta V_x \approx \frac{\partial V_x}{\partial x}\delta x + \frac{\partial V_x}{\partial y}\delta y$$

$$\delta V_y \approx \frac{\partial V_y}{\partial x}\delta x + \frac{\partial V_y}{\partial y}\delta y$$

where the derivatives are evaluated at P. The component of $\delta\mathbf{V}$ perpendicular to PQ is,

$$\delta V_n = \delta V_y \cos\theta - \delta V_x \sin\theta$$

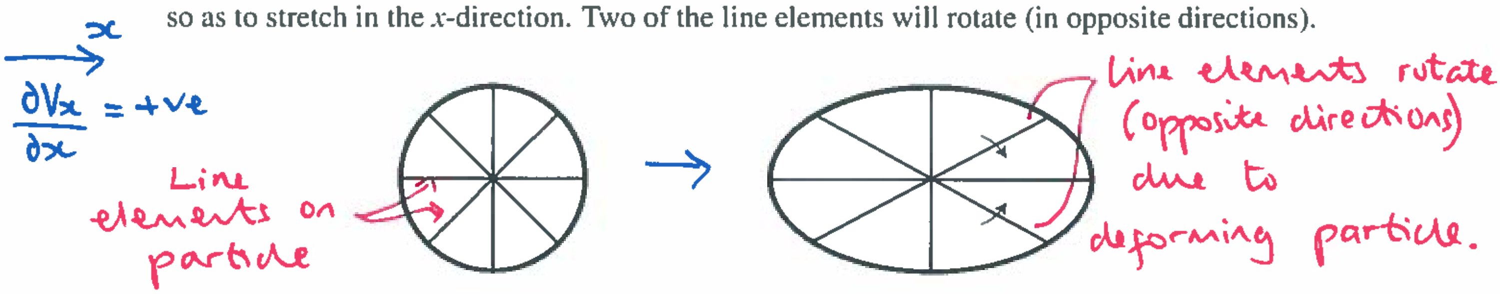
and, noting that $\delta x = \delta s\cos\theta$ and $\delta y = \delta s\sin\theta$, we have

$$\delta V_n = \left(\frac{\partial V_y}{\partial x}\delta s\cos\theta + \frac{\partial V_y}{\partial y}\delta s\sin\theta \right)\cos\theta - \left(\frac{\partial V_x}{\partial x}\delta s\cos\theta + \frac{\partial V_x}{\partial y}\delta s\sin\theta \right)\sin\theta \tag{6.13}$$

As $\delta s \to 0$, the angular velocity of the line element δs is,

$$\frac{dV_n}{ds} = \frac{\partial V_y}{\partial x}\cos^2\theta - \frac{\partial V_x}{\partial y}\sin^2\theta + \frac{1}{2}\left(\frac{\partial V_y}{\partial y} - \frac{\partial V_x}{\partial x}\right)\sin 2\theta \quad . \tag{6.14}$$

The angular velocity of our line element, therefore, depends on θ. This is because a fluid particle is, in general, deforming as well as rotating. Consider the circular fluid particle below, with 4 line elements drawn on the particle. If V_x increases with x, the fluid particle will deform so as to stretch in the x-direction. Two of the line elements will rotate (in opposite directions).



We can see that the instantaneous *mean* angular velocity of the fluid particle (centred at P) is obtained by averaging over all θ. Denoting the mean angular velocity by Ω,

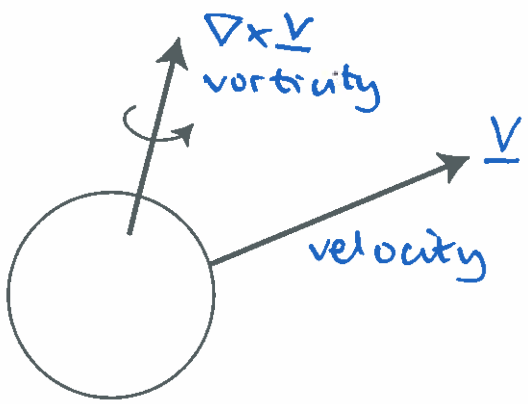
$$\Omega = \frac{1}{2\pi}\int_0^{2\pi}\frac{dV_n}{ds}\,d\theta = \frac{1}{2}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \quad , \tag{6.15}$$

since, $\frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\sin^2\theta\,d\theta = 1/2$ and $\frac{1}{2\pi}\int_0^{2\pi}\sin 2\theta\,d\theta = 0$.

The important result is that, for 2-D fields,

$$\nabla \times \mathbf{V} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{k} = 2\Omega\,\mathbf{k} \quad . \tag{6.16}$$

The local magnitude of the curl of the velocity field is equal to twice the instantaneous mean angular velocity of a fluid particle at that point. We have shown this for 2-D flows, but the same is true in the general 3-D case: the curl vector points in the direction of the axis of rotation of the fluid particle. In fluid mechanics, $\nabla \times \mathbf{V}$ is called the vorticity.



Since all curl fields are solenoidal, $\nabla \cdot (\nabla \times \mathbf{V}) = 0$, then there can be no sources or sinks of vorticity within the fluid flowfield (vorticity must be generated at solid boundaries).

6.5 Irrotational vector fields and the scalar potential

A vector field **V** which has $\nabla \times \mathbf{V} = 0$ everywhere is called an *irrotational* field.

If a 3-D vector field **V** is irrotational then all three components of $\nabla \times \mathbf{V}$ must be zero. In Cartesian coordinates,

$$\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) = \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) = 0 \tag{6.17}$$

There is a close connection between an irrotational field, where $\nabla \times \mathbf{V} = 0$, and the scalar potential, ϕ. We found in Lecture 4 that, although it is always possible to obtain a vector field **V** from a given scalar field ϕ using the gradient ($\mathbf{V} = \nabla\phi$), it is not always possible to find a scalar field that will yield a given vector field using $\mathbf{V} = \nabla\phi$. i.e., only certain types of vector fields are associated with scalar potentials.

We have already mentioned the identity,

$$\nabla \times (\nabla\phi) = 0$$

which is true for all differentiable scalar fields, ϕ. This implies,

- If $\mathbf{V} = \nabla\phi$, then $\nabla \times \mathbf{V} = 0$ and **V** is irrotational.
- Conversely, if **V** is irrotational, we can find a scalar potential ϕ such that $\mathbf{V} = \nabla\phi$.

Example

If $\mathbf{V} = (kx)\,\mathbf{i} - (ky)\,\mathbf{j}$, (for $y > 0$), determine if a scalar potential exists and, if so, find the scalar potential function $\phi = \phi(x, y)$.

Check to see if scalar potential exists:

$$\nabla \times \underline{V} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ kx & -ky & 0 \end{vmatrix} = 0\underline{i} + 0\underline{j} + 0\underline{k}$$

Field is irrotational $\therefore$ scalar potential exists

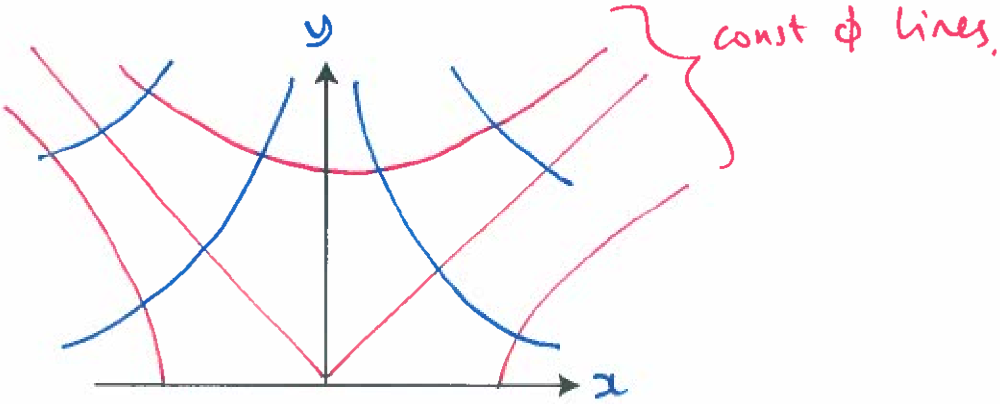
$$\underline{V} = \nabla \phi$$

$$\frac{\partial \phi}{\partial x} = V_x = kx \quad \therefore \quad \phi = \frac{kx^2}{2} + f(y,z)$$

$$\frac{\partial \phi}{\partial y} = V_y = -ky \quad \therefore \quad \phi = \frac{-ky^2}{2} + g(x,z)$$

$$\frac{\partial \phi}{\partial z} = V_z = 0 \quad \therefore \quad \phi = h(x,y)$$

$$\phi = \frac{kx^2}{2} - \frac{ky^2}{2} + c$$



You can now do Examples Paper 2: Q4, 5, 6 and 7