Lecture 10

Stokes's Theorem

10.1 Stokes's theorem

Stokes's theorem states,

"If S is an open two-sided surface bounded by a closed non-intersecting curve L, and if the vector field $\mathbf{V}$ has continuous derivatives, then,

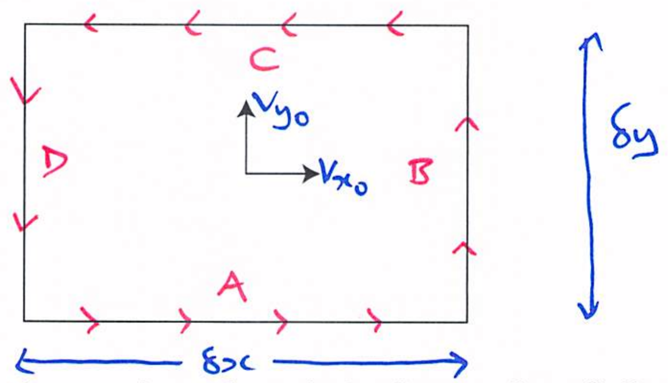
$$\oint_L \mathbf{V} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{V}) \cdot d\mathbf{A} \quad , \tag{10.1}$$

where L is traversed in the positive direction (as defined by the right-handed screw rule)."

In words, Stokes's theorem means, "the circulation of the vector field $\mathbf{V}$ around the closed curve L is equal to the flux of $\nabla \times \mathbf{V}$ passing through *any* surface S that spans the curve L."

10.2 Proof of Stokes's theorem

Stokes's theorem is, of course, a general result that applies in any coordinate system. To make our proof easier, we restrict ourself to a 2-D vector field in the (x, y) plane, $\mathbf{V} = V_x\,\mathbf{i} + V_y\,\mathbf{j}$.



We start with the small element of area shown in the diagram, $\delta A = \delta x\,\delta y\,\mathbf{k}$. We wish to evaluate the circulation $\delta\Gamma$ around the closed loop L that is the boundary of the area element,

$$\delta\Gamma = \oint_L \underline{V} \cdot d\underline{r}$$

The direction of integration must be chosen such that a right-handed screw would advance in the positive direction of the $\delta\mathbf{A}$ vector. For the element in our diagram, this is the counter-clockwise direction, hence,

$$\delta\Gamma = V_{Ax}\,\delta x + V_{By}\,\delta y - V_{Cx}\,\delta x - V_{Dy}\,\delta y$$

If we write $\mathbf{V}$ in the centre of the element as $\mathbf{V} = V_{0x}\,\mathbf{i} + V_{0y}\,\mathbf{j}$, then we can use first order Taylor expansions to evaluate the velocity components that we need,

$$V_{Ax} = V_{x_0} - \frac{\delta y}{2}\frac{\partial V_x}{\partial y}$$

$$V_{By} = V_{y_0} + \frac{\delta x}{2}\frac{\partial V_y}{\partial x}$$

$$V_{Cx} = V_{x_0} + \frac{\delta y}{2}\frac{\partial V_x}{\partial y}$$

$$V_{Dy} = V_{y_0} - \frac{\delta x}{2}\frac{\partial V_y}{\partial x}$$

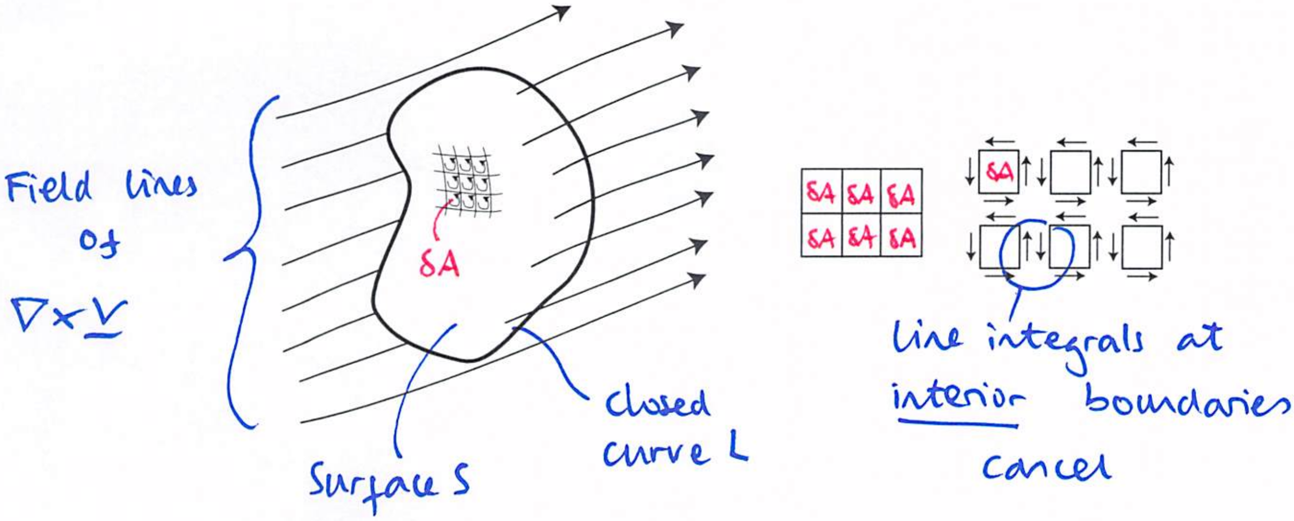
This leads to,

$$\delta\Gamma = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\delta x\,\delta y$$

We have already found that, for any 2-D vector field $\mathbf{V}$, $\nabla\times\mathbf{V} = (\partial V_y/\partial x - \partial V_x/\partial y)\,\mathbf{k}$, and so we can write,

$$\delta\Gamma = (\nabla\times\mathbf{V})\cdot d\mathbf{A} \qquad (10.2)$$

The above analysis was for a 2-D vector field, but can be extended to the general 3-D case where the element of area does not necessarily lie in the (x, y) plane; this yields the same result. The result also holds for any shape of area element, not just a Cartesian rectangle.



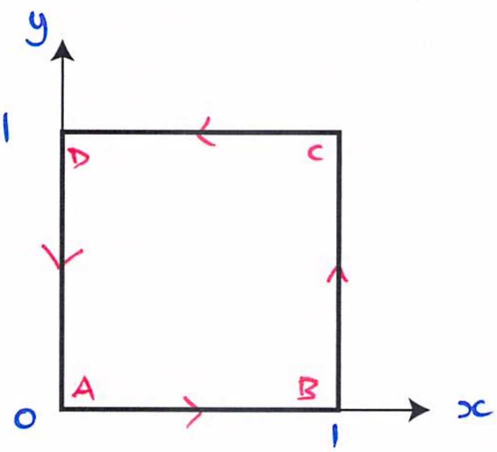
Stokes's theorem is the extension of the differential relationship $\delta\Gamma = (\nabla \times \mathbf{V}) \cdot \delta\mathbf{A}$ to integral form. The diagram shows field lines of the vector field $\nabla \times \mathbf{V}$ passing through a surface S that is bounded by the closed curve L. We can divide up the surface S into many elemental surface elements $\delta\mathbf{A}$ (which can be of arbitrary shape). For each of these elements, $\delta\Gamma = (\nabla \times \mathbf{V}) \cdot \delta\mathbf{A}$ is valid. If we add up all the elements to obtain the total circulation Γ, the line integrals at the interior boundaries between adjacent area elements will cancel (the contributions are in opposite directions); it is only along the boundary L that the contributions survive and so,

$$\oint_L \mathbf{V} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{V}) \cdot d\mathbf{A} \quad . \tag{10.3}$$

Notice that we have not specified anything about the surface S that is bounded by L. There are an infinite number of possible surfaces that would span the same closed curve, and the flux of $\nabla \times \mathbf{V}$ through each of them must be the same. This implies that there are no sources or sinks of $\nabla \times \mathbf{V}$, and we knew this already from the identity $\nabla \cdot (\nabla \times \mathbf{V}) = 0$ – all curl fields are solenoidal.

Example

For the vector field $\mathbf{V} = 2z^2\,\mathbf{i} + 3x\,\mathbf{j}$, evaluate the circulation, $\Gamma = \oint \mathbf{V} \cdot d\mathbf{r}$, around the boundary of the square in plane $z = 1$ defined by the lines $x = 0$, $x = 1$, $y = 0$ and $y = 1$.



We first calculate the circulation by direct evaluation of the line integral:

In general, $\delta\mathbf{r} = \delta x\,\mathbf{i} + \delta y\,\mathbf{j}$

Along AB: $\underline{V} \cdot \delta\underline{r} = (2\underline{i} + 3x\underline{j}) \cdot \delta x\,\underline{i} = 2\,\delta x$

Along BC: $\underline{V} \cdot \delta\underline{r} = (2\underline{i} + 3\underline{j}) \cdot \delta y\,\underline{j} = 3\,\delta y$

Along CD: $\underline{V} \cdot \delta\underline{r} = (2\underline{i} + 3x\underline{j}) \cdot \delta x\,\underline{i} = 2\,\delta x$

Along DA: $\underline{V} \cdot \delta\underline{r} = (2\underline{i}) \cdot \delta y\,\underline{j} = 0$

$$\Gamma = \oint \underline{V} \cdot d\underline{r} = \overset{A \to B}{\int_0^1 2\, dx} + \overset{B \to C}{\int_0^1 3\, dy} + \overset{C \to D}{\int_1^0 2\, dx}$$

$$= 2 + 3 - 2$$

$$= 3$$

We now use Stokes's theorem to convert the line integral into a surface integral
S is the surface spanning the square $ABCDA$ in the plane $z = 1$

$$\delta \underline{A} = \delta x\, \delta y\, \underline{k}$$

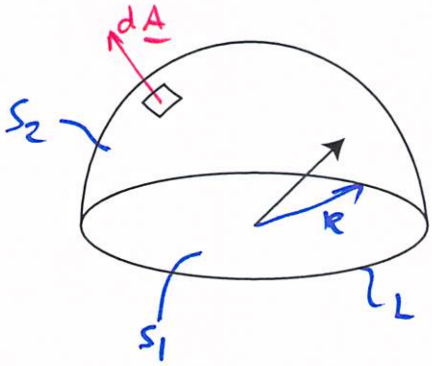
$$\nabla \times \underline{V} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z^2 & 3x & 0 \end{vmatrix} = 0\,\underline{i} + 4z\,\underline{j} + 3\,\underline{k}$$

$$\Gamma = \iint_S (\nabla \times \underline{V}) \cdot d\underline{A} = \int_0^1 \int_0^1 (4z\,\underline{j} + 3\underline{k}) \cdot \underline{k}\, dx\, dy$$

$$= \int_0^1 \int_0^1 3\, dx\, dy = 3$$

Example

If $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$, find the flux of $\mathbf{B} = \nabla \times \mathbf{F}$ through the hemisphere $|\mathbf{r}| < R$, $(z > 0)$ by: (i) using Gauss's theorem; (ii) using Stokes's theorem.



Find $\iint_{S_2} \underline{B} \cdot d\underline{A}$

$$\underline{B} = \nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 2z \end{vmatrix} = 0\underline{i} + 0\underline{j} + 2\underline{k}$$

(i) using Gauss's theorem:

$$\nabla \cdot (\nabla \times \underline{F}) = 0 \ \therefore \ \nabla \cdot \underline{B} = 0 \ \text{and} \ \oiint_S \underline{B} \cdot d\underline{A} = 0$$

$$\text{(not cyclic)} \ \cancel{\oiint}\iint_{S_2} \underline{B} \cdot d\underline{A} = -\iint_{S_1} \underline{B} \cdot d\underline{A}$$

$$= -\iint_{S_1} (2\underline{k}) \cdot (-dA\,\underline{k}) = 2\pi R^2$$

(ii) using Stokes's theorem:

$$\oint_L \underline{F} \cdot d\underline{r} = \iint_{S_2} (\nabla \times \underline{F}) \cdot d\underline{A} = \iint_{S_2} \underline{B} \cdot d\underline{A}$$

$$\oint_L \underline{F} \cdot d\underline{r} = \oint_L \left(-y\,\underline{i} + x\,\underline{j} + \overset{z=0}{\cancel{2z\,\underline{k}}}\right) \cdot \left(dx\,\underline{i} + dy\,\underline{j} + \overset{0}{\cancel{dz\,\underline{k}}}\right)$$

For curve L: $x = R\cos\theta \Rightarrow dx = -R\sin\theta\, d\theta$

$y = R\sin\theta \Rightarrow dy = R\cos\theta\, d\theta$

$$\oint_L \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \Big[\underbrace{(-R\sin\theta)}_{F_x}\underbrace{(-R\sin\theta)}_{dx} + \underbrace{(R\cos\theta)}_{F_y}\underbrace{(R\cos\theta)}_{dy}\Big]\, d\theta = 2\pi R^2$$

10.3 Coordinate-free definition of curl

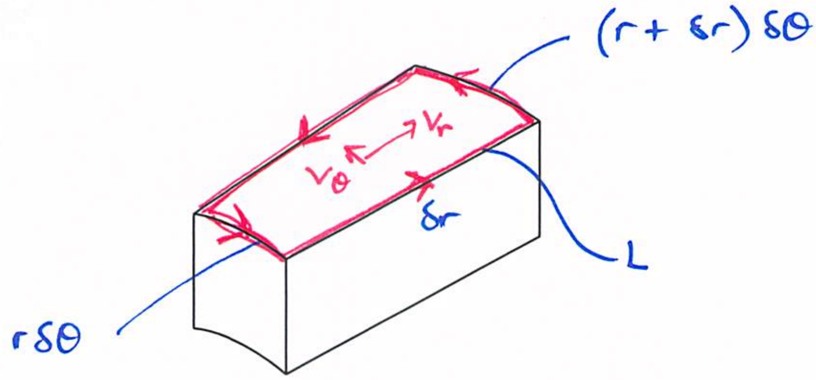
Stokes's theorem can be used to provide a coordinate-free definition of curl,

$$(\nabla \times \mathbf{V}) \cdot \delta\mathbf{A} = \oint_L \mathbf{V} \cdot d\mathbf{r} \qquad (10.4)$$

where L is the closed curve forming the boundary of the small element δA. We can then use different orientations of $\delta\mathbf{A}$ to pick out different components of $\nabla \times \mathbf{V}$.

Example

Find the z component of $\nabla \times \mathbf{V}$, ω_z, where $\mathbf{V}$ is defined in cylindrical polar coordinates $\mathbf{V} = \mathbf{V}(r, \theta, z)$.



$$\omega_z \, r\,\delta\theta\,\delta r = \oint_L \underline{V} \cdot d\underline{r}$$

$$\omega_z \, r\,\delta\theta\,\delta r = \delta r\left(V_r - \frac{\partial V_r}{\partial \theta}\frac{\delta\theta}{2}\right) + \delta\theta(r+\delta r)\left(V_\theta + \frac{\partial V_\theta}{\partial r}\frac{\delta r}{2}\right)$$

$$- \delta r\left(V_r + \frac{\partial V_r}{\partial \theta}\frac{\delta\theta}{2}\right) - \delta\theta\, r\left(V_\theta - \frac{\partial V_\theta}{\partial r}\frac{\delta r}{2}\right)$$

$$\omega_z = \frac{1}{r}\left(-\frac{\partial V_r}{\partial \theta} + V_\theta + r\frac{\partial V_\theta}{\partial r}\right)$$

10.4 Conservative fields, a summary

If a vector field, $\mathbf{V}$, is *conservative*:

- $\oint_C \mathbf{V} \cdot d\mathbf{r} = 0$ for *any* closed curve, C;
- $\mathbf{V}$ is also *irrotational* ($\nabla \times \mathbf{V} = 0$) (the terms *conservative* and *irrotational* are synonymous);
- a 'scalar potential' ϕ exists such that $\mathbf{V} = \nabla\phi$.

10.5 Solenoidal fields, a summary

If a vector field, $\mathbf{V}$, is *solenoidal*:

- $\nabla \cdot \mathbf{V} = 0$;
- $\oiint_S \mathbf{V}.d\mathbf{A} = 0$ for any closed surface, S;
- a 'vector potential' $\mathbf{C}$ exists such that $\mathbf{V} = \nabla \times \mathbf{C}$.

The final point comes from $\nabla \cdot (\nabla \times \mathbf{C}) = 0$ for any continuously differentiable vector field, $\mathbf{C}$. Hence, if $\nabla \cdot \mathbf{V} = 0$, we can always express $\mathbf{V}$ as the curl of the 'vector potential' field, $\mathbf{C}$.

You can now do Examples Paper 3: Q8 and 9