Lecture 13

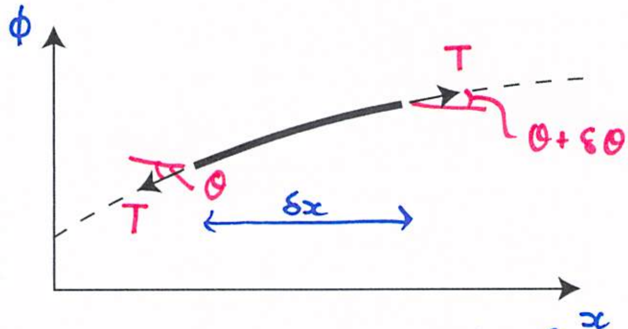
The wave equation

13.1 Introduction

Wave motion is one of the most fundamental and common phenomena that occur in nature: surface waves radiating outward from a stone dropped in a pond, sound waves propagating from a source, electromagnetic waves propagating through space – there are many examples. In all cases, energy is propagated at a finite velocity to distant points, and the wave travels through the medium that supports it without giving rise to a permanent change (e.g. no permanent displacement).

13.2 The 1-D wave equation

We derive the 1-D wave equation by considering small displacements $\phi = \phi(x,t)$ of a ~~taught~~ taut! string (tension T and mass per unit length m_L).



$F=ma \uparrow \quad -T\sin\theta + T\sin(\theta+\delta\theta) = m_L\,\delta x\,\dfrac{\partial^2\phi}{\partial t^2}$

small angles : $\sin\theta \approx \tan\theta = \dfrac{\partial\phi}{\partial x}$

$$-T\frac{\partial\phi}{\partial x} + T\left(\frac{\partial\phi}{\partial x} + \frac{\partial^2\phi}{\partial x^2}\delta x\right) = m_L\,\delta x\,\frac{\partial^2\phi}{\partial t^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} \tag{13.1}$$

where $c^2 = T/m_L$. The positive constant, c, will be shown to be the wave speed.

We will be able to solve the 1-D wave equation via the separation of variables approach, but first we look at "D'Alembert's solution."

13.3 D'Alembert's solution

We start with a first oder PDE that also exhibits wave-like behaviour,

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \tag{13.2}$$

We construct a compound variable, $\eta = ct - x$, and express ϕ as,

$$\phi = f(\eta) = f(ct - x) \tag{13.3}$$

By the chain rule, $$\frac{\partial \phi}{\partial t} = \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = c\frac{df}{d\eta}$$

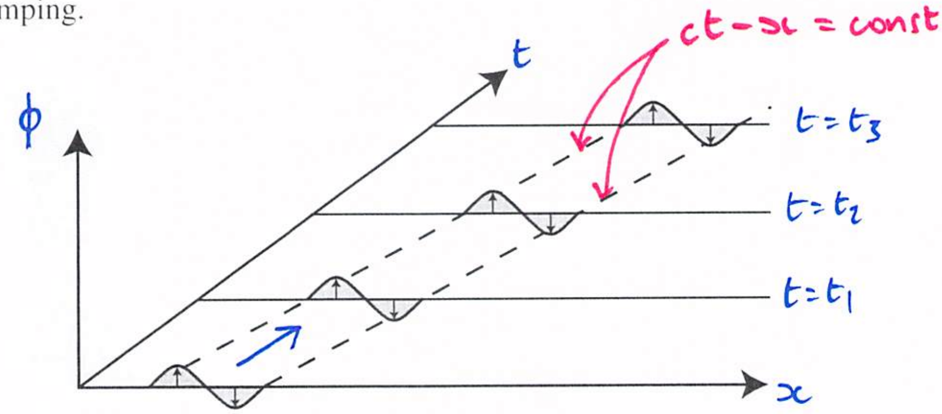
and, $$\frac{\partial \phi}{\partial x} = \frac{df}{d\eta}\frac{\partial \eta}{\partial x} = -\frac{df}{d\eta}$$

Adding these two results,

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \quad , \tag{13.4}$$

so that $\phi = f(ct - x)$ is therefore a solution of the PDE.

$\phi = f(ct - x)$ is an undamped wave, of arbitrary shape, travelling at velocity c in the positive x direction. We can see this because along lines of constant $\eta = ct - x$, $\phi = f(\eta)$ does not change. The wave travels unchanged, without distortion, because the simple wave equation does not include damping.



wave propagates, without distortion, at speed c

Now consider the first order PDE,

$$\frac{\partial \phi}{\partial t} - c\frac{\partial \phi}{\partial x} = 0, \tag{13.5}$$

where c is again a positive constant. A similar analysis shows that the solution is,

$$\phi = g(ct + x) \tag{13.6}$$

which represents a wave of arbitrary shape travelling at speed c, without distortion, in the *negative* x direction.

We can show that the solution to our second order wave equation is a linear superposition of a right and left travelling wave:

$$\phi(x,t) = f(ct - x) + g(ct + x) \quad , \tag{13.7}$$

and this is known as D'Alembert's solution of the wave equation. We can check that this is correct by differentiation. Writing $\eta = ct - x$ and $\xi = ct + x$,

$$\frac{\partial \phi}{\partial x} = -\frac{df}{d\eta} + \frac{dg}{d\xi} \qquad \therefore \frac{\partial^2 \phi}{\partial x^2} = \frac{d^2 f}{d\eta^2} + \frac{d^2 g}{d\xi^2}$$

and, $$\frac{\partial \phi}{\partial t} = c\frac{df}{d\eta} + c\frac{dg}{d\xi} \qquad \therefore \frac{\partial^2 \phi}{\partial t^2} = c^2\frac{d^2 f}{d\eta^2} + c^2\frac{d^2 g}{d\xi^2}$$

and hence,

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} \quad . \tag{13.8}$$

The lines $(ct - x) =$ constant and $(ct + x) =$ constant, along which the right and left running waves move in the (x,t) plane, are known as *characteristics* of the PDE. The wave equation is the archetypal hyperbolic PDE. All hyperbolic PDEs (with two independent variables) have two different real characteristics.

Physical phenomena described by the wave equation are of the initial condition type. For example, our vibrating string is fixed at two points, $x = 0$ and $x = L$ and may be distorted into an initial shape $\phi(x,0) = F(x)$, and released from rest at time $t = 0$. The domain of interest is open-ended in the time direction. There is no damping in the equation and so the string would continue vibrating forever. The boundary conditions are, therefore,

initial shape $\phi = F(x)$ for $0 \leq x \leq L$ @ $t = 0$

initially at rest $\frac{\partial \phi}{\partial t} = 0$ for $0 \leq x \leq L$ @ $t = 0$

$$\left.\begin{aligned} \phi &= 0 \text{ at } x = 0 \\ \phi &= 0 \text{ at } x = L \end{aligned}\right\} \text{ for } t \geq 0$$

When the disturbances are sinusoidal in shape, they are known as harmonic waves. A train of harmonic waves travelling to the right are represented by,

right running

$$\phi = A_1 \cos(\omega t - kx) + A_2 \sin(\omega t - kx) = A\cos(\omega t - kx + \alpha) \quad , \tag{13.9}$$

and harmonic waves travelling to the left are,

left running

$$\phi = B_1 \cos(\omega t + kx) + B_2 \sin(\omega t + kx) = B\cos(\omega t + kx + \beta) \quad . \tag{13.10}$$

In these equations, ω is the angular frequency ($\omega = 2\pi f$) and the time taken for a complete wave to pass a fixed point is the period, T ($f = 1/T$). k is the *wave number* and is the number of waves in a distance of 2π. A and B are the amplitudes of the two waves and α and β are the phase angles. The wave speed is $c = \omega/k$.

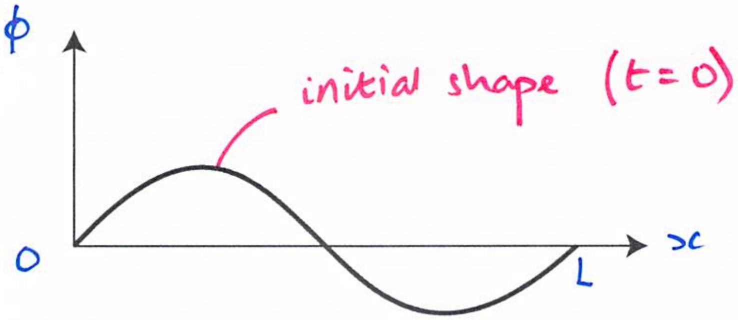
Standing wave patterns can be obtained by the superposition of left and right running waves (the wave equation is linear, so we can add solutions). If we have two wave trains of equal frequency and amplitude (and zero phase angles), but travelling in opposite directions, we obtain:

$$\phi(x,t) = A\cos(\overrightarrow{\omega t - kx}) + A\cos(\overleftarrow{\omega t + kx}) \tag{13.11}$$

$$= 2A\cos(kx)\cos(\omega t) \quad . \tag{13.12}$$

Each point vibrates at angular frequency ω but the amplitude varies along the string according and is equal to $2A\cos(kx)$.

Example



A taut string of length L is secured at $x = 0$ and $x = L$. It is given an initial displacement and then released, from rest, at $t = 0$. $\phi(x,t)$ is the transverse displacement of the string and satisfies the wave equation,

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial \phi}{\partial t^2} \quad , \tag{13.13}$$

where c is a constant. Investigate the modes of vibration of the string for $t \geq 0$.

We attempt a solution using separation of variables:

$$\phi(x,t) = X(x)\,T(t)$$

substitute in wave equation:

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2 T}\frac{d^2T}{dt^2} = -\alpha^2$$

where we have chosen a negative separation constant.

The solutions of these ODEs are,

$$X(x) = A \cos \alpha x + B \sin \alpha x$$

$$T(t) = C \cos \alpha c t + D \sin \alpha c t$$

so that the solution for ϕ is of the form,

$$\phi(x,t) = \underbrace{(A\cos\alpha x + B\sin\alpha x)}^{X(x)}\underbrace{(C\cos\alpha ct + D\sin\alpha ct)}^{T(t)} \quad (13.14)$$

We now apply boundary conditions to find the constants A, B, C, D (one of which is redundant).

at $x = 0$, $\phi = 0$ for $t \geqslant 0$ $\therefore$ $A = 0$

String is released from rest $\therefore \partial\phi/\partial t = 0$ for all x at $t = 0$.

$$\frac{\partial \phi}{\partial t} = \alpha c B \sin \alpha x \left(-C \sin(\alpha c t) + D \cos(\alpha c t)\right)$$

$$\therefore D = 0 \Rightarrow \phi(x,t) = B' \sin(\alpha x) \cos(\alpha c t)$$

The remaining boundary condition is that $\phi = 0$ at $x = L$ for $t \geq 0$. Hence,

$$\alpha L = n\pi \quad (n = 1, 2, 3, 4 \ldots)$$

Hence the solutions for ϕ are standing waves,

$$\phi_n = B'_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) \quad , \quad (13.15)$$

where each n is a different harmonic. The frequency of vibration is $f = \omega/(2\pi)$ where $\omega = n\pi c/L$. The amplitude varies with x and equals $B'_n \sin(n\pi x/L)$. The shape of the initial distribution $\phi(x,0)$ determines the proportions of each harmonic that are present. Using a Fourier series, any specified initial shape can be matched.