Lecture 12

The Laplace and diffusion equations

12.1 Introduction

In a few cases, particularly where the geometry is simple, some PDEs are amenable to an analytical solution. There are few formal techniques and often some trial and error is required. The ultimate test is to check that the solution both satisfies the PDE for the interior of the domain *and* also complies with the specified boundary conditions.

We will work with a single scalar function of two independent variables, $\phi = \phi(x,y)$ or $\phi = \phi(x,t)$. In general, these functions involve combinations of the two independent variables. However, if the solution can be written,

$$\phi(x,y) = X(x)Y(y) \quad , \tag{12.1}$$

or,

$$\phi(x,t) = X(x)T(t) \quad , \tag{12.2}$$

then we may attempt to solve the PDE using the "separation of variables" approach.

12.2 Separable solutions of Laplace's equation

Laplace's equation in 2-D Cartesian coordinates is,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad . \tag{12.3}$$

We seek a separable solution where, $\phi(x,y) = X(x)Y(y)$. Substitution this into the PDE and rearranging gives,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} \quad , \quad \text{since} \quad \frac{\partial\phi}{\partial x} = \frac{dX}{dx}Y \quad \text{and} \quad \frac{\partial^2\phi}{\partial x^2} = \frac{d^2X}{dx^2}Y$$

where we are using total differentials because X is a function of x only, and Y is a function of y only. The LHS is a function of x and the RHS is a function of y. This equation can only be possible if both sides are equal to the same *constant*. There are three possibilities: the constant can be negative, positive or zero.

Negative separation constant

If the separation constant is negative,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -\alpha^2 \quad , \tag{12.4}$$

and the PDE has been reduced to two simultaneous ODEs,

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0$$

$$\frac{d^2Y}{dy^2} - \alpha^2 Y = 0$$

The solutions to these are,

$$X(x) = A\sin(\alpha x) + B\cos(\alpha x) \tag{12.5}$$

$$Y(y) = C\sinh(\alpha y) + D\cosh(\alpha y) = C'\exp(\alpha y) + D'\exp(-\alpha y) \tag{12.6}$$

where A, B, C, D, C' and D' are constants. We can decide whether to use the hyperbolic or exponential forms for $Y(y)$ depending on the boundary conditions that need to be matched. A possible solution to the PDE is therefore,

$$\phi(x,y) = \underbrace{\big(A\sin(\alpha x) + B\cos(\alpha x)\big)}_{X(x)}\underbrace{\big(C\sinh(\alpha y) + D\cosh(\alpha y)\big)}_{Y(y)} \tag{12.7}$$

$$= \big(A\sin(\alpha x) + B\cos(\alpha x)\big)\big(C'\exp(\alpha y) + D'\exp(-\alpha y)\big) \tag{12.8}$$

Of the constants A, B, C, D (or A, B, C', D'), one is redundant and the others must be found using the boundary conditions.

Positive separation constant

If the separation constant is positive,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \alpha^2 \quad . \tag{12.9}$$

Hence,

$$\frac{d^2X}{dx^2} - \alpha^2 X = 0$$

$$\frac{d^2Y}{dy^2} + \alpha^2 Y = 0$$

and

$$X(x) = A\sinh(\alpha x) + B\cosh(\alpha x) = A'\exp(\alpha x) + B'\exp(-\alpha x) \tag{12.10}$$

$$Y(y) = C\sin(\alpha y) + D\cos(\alpha y) \tag{12.11}$$

So a possible solution of the PDE is,

X(x) Y(y)

$$\phi(x,y) = \big(A\sinh(\alpha x) + B\cosh(\alpha x)\big)\big(C\sin(\alpha y) + D\cos(\alpha y)\big) \tag{12.12}$$
$$= \big(A'\exp(\alpha x) + B'\exp(-\alpha x)\big)\big(C\sin(\alpha y) + D\cos(\alpha y)\big) \tag{12.13}$$

The solution is similar to that for a negative separation constant, but now the variation of ϕ with x is described by exponential functions, and the variation of ϕ with y by trigonometric functions.

Separation constant is zero

If the separation constant is zero,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \quad , \tag{12.14}$$

and the solutions are,

$$X(x) = Ax + B$$
$$Y(y) = Cy + D$$

where we must again determine the constants A, B, C and D using the boundary conditions. A possible solution is, therefore,

X(x) Y(y)

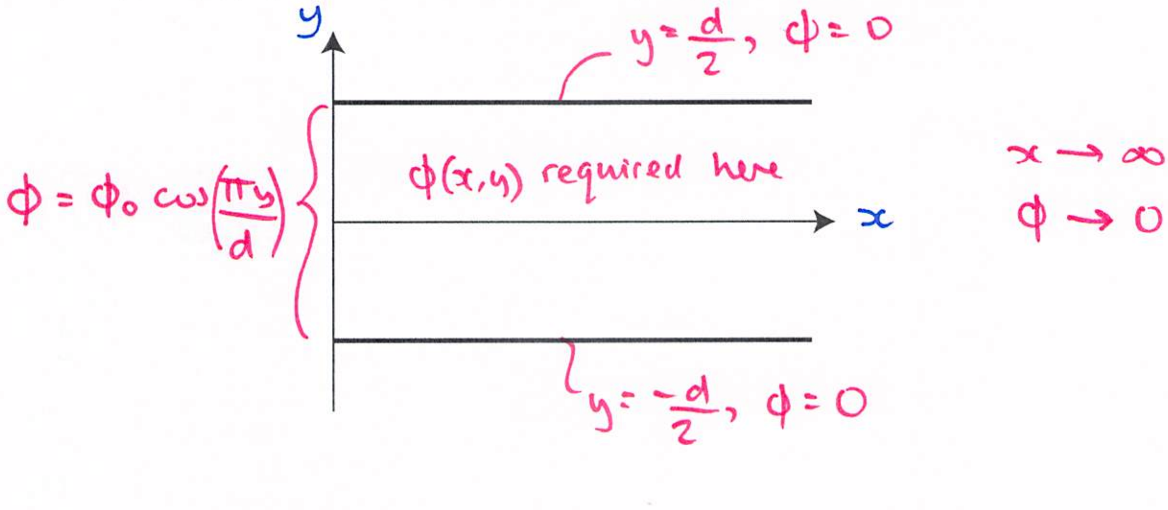
$$\phi(x,y) = (Ax + B)(Cy + D) \tag{12.15}$$

Which, if any, of three types of separation constant will ultimately yield the solution to Laplace's equation that we seek will be determined by whether the solution can be matched to the boundary conditions.

Example

A cooling fin has a temperature field $\phi = \phi(x,y)$ where $\phi = T - T_a$ and T_a is the ambient temperature. The region of interest is $x \geq 0$, $-d/2 \leq y \leq d/2$. The boundary conditions are:

1. $\phi \to 0$ as $x \to \infty$ (independently of y)
2. $\phi = 0$ on $y \pm d/2$ for $x \geq 0$
3. $\phi = \phi_0 \cos(\pi y/d)$ on $x = 0$ for $-d/2 \leq y \leq d/2$ (ϕ_0 is a constant)



$$\nabla^2\phi = 0 \quad \text{(steady heat conduction)}$$

in 2-D Cartesian: $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$

Try $\phi(x,y) = X(x)Y(y)$

+ve separation constant: $\dfrac{1}{X}\dfrac{d^2X}{dx^2} = -\dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = \alpha^2$

$$\phi(x,y) = \left(A\exp(\alpha x) + B\exp(-\alpha x)\right)\left(C\sin(\alpha y) + D\cos(\alpha y)\right)$$

Boundary conditions: $\phi \to 0$ as $x \to \infty$ $\therefore A = 0$

$x = 0$, $\phi = \phi_0\cos\dfrac{\pi y}{d}$ $\therefore C = 0$, $\alpha = \dfrac{\pi}{d}$

$$\phi = BD\exp\left(\frac{-\pi x}{d}\right)\cos\left(\frac{\pi y}{d}\right)$$

$\therefore BD = \phi_0$ $\quad \phi = \phi_0\exp\left(\dfrac{-\pi x}{d}\right)\cos\left(\dfrac{\pi y}{d}\right)$

Our temperature distribution for the cooling fin at $x = 0$ seems rather artificial, $T = T_a + \phi_0\cos(\pi y/d)$. However, by summing solutions (the PDE is linear) and using a Fourier series, we can match *any* prescribed ϕ distribution on this boundary.

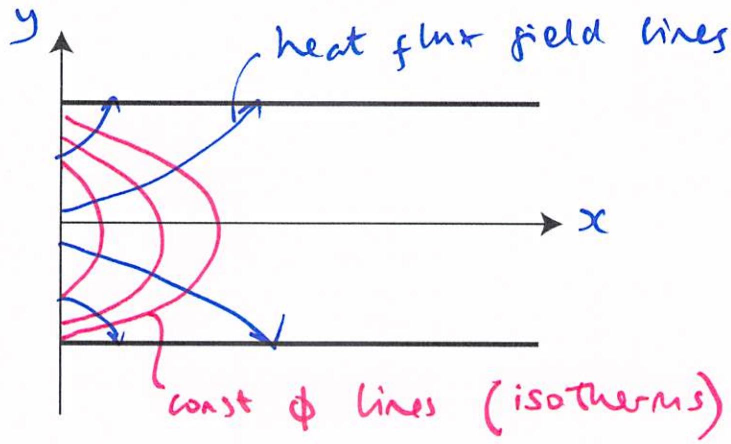
The heat flux vector field is given by $\mathbf{q} = -\lambda\nabla\phi$,

$$\underline{q} = -\lambda\nabla\phi = -\lambda\left(\underline{i}\frac{\partial\phi}{\partial x} + \underline{j}\frac{\partial\phi}{\partial y}\right)$$

$$= \frac{\pi\lambda}{d}\phi_0\exp\left(\frac{-\pi x}{d}\right)\left(\underline{i}\cos\left(\frac{\pi y}{d}\right) + \underline{j}\sin\left(\frac{\pi y}{d}\right)\right)$$

The differential equation for the heat flux field lines (which are always orthogonal to the constant ϕ lines) is

$$\frac{dy}{dx} = \tan\left(\frac{\pi y}{d}\right)$$



12.3 The unsteady heat conduction (or diffusion) equation

Diffusive processes are very common (heat conduction, mass diffusion of chemical species, momentum diffusion of a viscous fluid, magnetic fields). Here, our example will be the 1-D unsteady heat conduction equation, but the methodology also applies to other diffusion processes.

The equation governing a 1-D diffusion process is

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \phi}{\partial t} \tag{12.16}$$

If a separable solution is possible, then $\phi(x,t) = X(x)T(t)$. Substitution into the PDE gives,

$$\frac{1}{X} \frac{d^2 \phi}{dx^2} = \frac{1}{\alpha T} \frac{d\phi}{dt}$$

The LHS is a function only of x and the RHS is a function only of t. This is only possible if both sides of the equation are equal to a constant. We first take the constant to be a negative real number, and write it as $-\beta^2$ (so as not to confuse it with the diffusivity, α).

$$\frac{1}{X} \frac{d^2 \phi}{dx^2} = \frac{1}{\alpha T} \frac{d\phi}{dt} = -\beta^2$$

We have therefore transformed our PDE into a second order ODE for $X(x)$ and a first order ODE for $T(t)$,

$$\frac{d^2 X}{dx^2} + \beta^2 X = 0$$

$$\frac{dT}{dt} + \alpha \beta^2 T = 0$$

to which the solutions are,

$$X(x) = A\sin(\beta x) + B\cos(\beta x) \tag{12.17}$$

$$T(t) = C\exp(-\alpha\beta^2 t) \tag{12.18}$$

where the constants A, B and C (one of which is redundant) are fixed by the boundary conditions. One possible solution of the unsteady diffusion equation is, therefore,

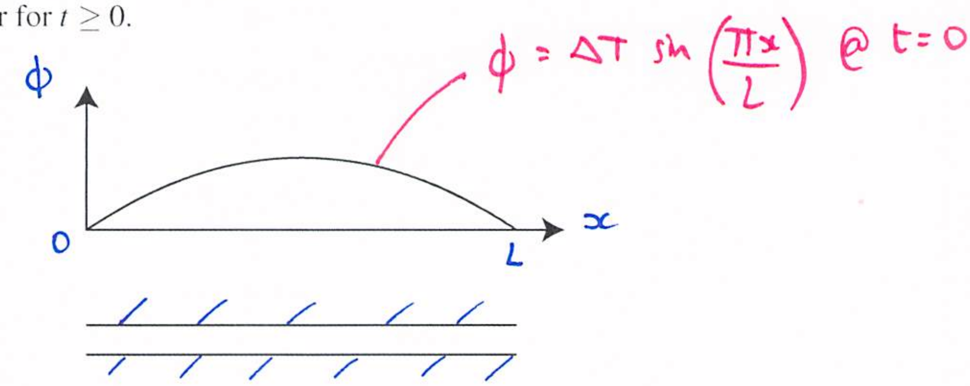
$$\phi(x,t) = \overbrace{\exp(-\alpha\beta^2 t)}^{T(t)}\overbrace{\big(A\sin(\beta x) + B\cos(\beta x)\big)}^{X(x)} \quad . \tag{12.19}$$

Solutions of this type correspond to problems that are open-ended in time, but have prescribed values of ϕ at two boundary points.

A positive separation constant would lead to exponential growth in time and this is unlikely to be a realistic situation. If the separation constant is zero, the time dependence drops out, and this, also, is not of interest to us.

Example

A metal bar extends from $x = 0$ to $x = L$. The bar is insulated along its length, but not at the ends. At time $t = 0$, the bar has a temperature distribution given by $T = T_0 + \Delta T\sin(\pi x/L)$. The ends of the bar are held at a fixed temperature, T_0. If $\phi = T - T_0$, find an expression for the distribution of $\phi(x,t)$ along the bar for $t \geq 0$.



ϕ is governed by 1-D unsteady heat conduction

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\phi}{\partial t}$$

If $\phi(x,t) = X(x)\,T(t)$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\alpha T}\frac{dT}{dt}$$

-ve separation constant

$$\phi(x,t) = \exp\left(-\alpha\beta^2 t\right)\left(A\sin(\beta x) + B\cos(\beta x)\right)$$

$$\phi(x,0) = \Delta T \sin\left(\frac{\pi x}{L}\right) \quad \therefore \quad A = \Delta T, \quad B = 0, \quad \beta = \frac{\pi}{L}$$

$$\phi(x,t) = \Delta T \exp\left(\frac{-\alpha\pi^2 t}{L^2}\right)\sin\left(\frac{\pi x}{L}\right)$$