IB Paper 7: Linear Algebra   Handout 4

3.7 Bases for the Column Space and Row Space of A

LU decomposition gives us an immediate answer to how to generate convenient descriptions for the Row Space and Column Space of **A**. For a general $m \times n$ matrix

Column Space = all vectors formed by taking a linear combination of the *columns* of **A**

$$\lambda_1 \underline{a}_1 + \lambda_2 \underline{a}_2 + \lambda_3 \underline{a}_3 + \ldots + \lambda_n \underline{a}_n$$

as the λ's vary.

Row Space = all vectors formed by taking a linear combination of the *rows* of **A**

$$\mu_1 \underline{\tilde{a}}_1 + \mu_2 \underline{\tilde{a}}_2 + \mu_3 \underline{\tilde{a}}_3 + \ldots + \mu_m \underline{\tilde{a}}_m$$

as the μ's vary.

We shall use the matrix **A** on which we performed LU decomposition in section 3.3 denoted $\mathbf{A_I}$

$$\mathbf{A_I} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 6 & 3 & 12 \\ -1 & 2 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \mathbf{LU_I}$$

and in order to show the range of behaviour,

$$\mathbf{A_{II}} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 6 & 3 & 12 \\ -1 & 2 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{LU_{II}}$$

changed

You will see that, since it is only the bottom right-hand corner of **A** that is different, it is only the last row of **U** that changes and **L** is the same for both. You should check this by either LU or by simply multiplying out.

Now in terms of the outer products of the columns and rows of **L** and $\mathbf{U_I}$ and $\mathbf{U_{II}}$,

$$\mathbf{A_I} = \underline{l}_1 \underline{\tilde{u}}_1^T + \underline{l}_2 \underline{\tilde{u}}_2^T + \underline{l}_3 \underline{\tilde{u}}_3^T \qquad \mathbf{A_{II}} = \underline{l}_1 \underline{\tilde{u}}_1^T + \underline{l}_2 \underline{\tilde{u}}_2^T + \cancel{\underline{l}_3 \underline{\tilde{u}}_3^T} = 0$$

Basis for Column Space

Remembering that we can consider matrix multiplication as a relationship between columns (see section 2.6)

$$\begin{bmatrix} \uparrow \\ \underline{a}_1 \\ \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow \\ \underline{l}_1 \\ \downarrow \end{bmatrix} [u_{11}] + \begin{bmatrix} \uparrow \\ \underline{l}_2 \\ \downarrow \end{bmatrix} [u_{21}] + \begin{bmatrix} \uparrow \\ \underline{l}_3 \\ \downarrow \end{bmatrix} [u_{31}] \text{, i.e. } \underline{a}_1 = u_{11}\underline{l}_1 + u_{21}\underline{l}_2 + u_{31}\underline{l}_3 \text{, etc.}$$

Since all of the columns of **A** can be written in terms of them, this means that $\underline{l}_1$, $\underline{l}_2$, **…** form a *basis* for the column space of **A** (at least the set of them for which the corresponding $\underline{\tilde{u}}$ is non-zero do).

We see immediately that for matrix $\mathbf{A_I}$

$$\underline{a}_1 = \underline{l}_1 \qquad \underline{a}_2 = 2\underline{l}_1 + 2\underline{l}_2 \qquad \underline{a}_3 = \underline{l}_1 + \underline{l}_2 + \underline{l}_3 \qquad \underline{a}_4 = 3\underline{l}_1 + 6\underline{l}_2 - \underline{l}_3$$

while for matrix $\mathbf{A_{II}}$

$$\underline{a}_1 = \underline{l}_1 \qquad \underline{a}_2 = 2\underline{l}_1 + 2\underline{l}_2 \qquad \underline{a}_3 = \underline{l}_1 + \underline{l}_2 \qquad \underline{a}_4 = 3\underline{l}_1 + 6\underline{l}_2$$

Since the columns of **L** are independent (see next section),

a basis of the column space of $\mathbf{A_I}$ is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

while one for $\mathbf{A_{II}}$ is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Basis for Row Space

Remembering that we can also consider matrix multiplication as a relationship between *rows* (see section 2.7)

$$\begin{bmatrix} \leftarrow & \underline{\tilde{a}}_1 & \rightarrow \end{bmatrix} = [b_{11}]\begin{bmatrix} \leftarrow & \underline{\tilde{c}}_1 & \rightarrow \end{bmatrix} + [b_{12}]\begin{bmatrix} \leftarrow & \underline{\tilde{c}}_2 & \rightarrow \end{bmatrix} + [b_{13}]\begin{bmatrix} \leftarrow & \underline{\tilde{c}}_3 & \rightarrow \end{bmatrix}$$

, i.e. $\underline{\tilde{a}}_1 = l_{11}\underline{\tilde{u}}_1 + l_{12}\underline{\tilde{u}}_2 + l_{13}\underline{\tilde{u}}_3$, etc.

we see immediately that for matrix $\mathbf{A_I}$

$$\underline{\tilde{a}}_1 = \underline{\tilde{u}}_1 \qquad \underline{\tilde{a}}_2 = 2\underline{\tilde{u}}_1 + \underline{\tilde{u}}_2 \qquad \underline{\tilde{a}}_3 = -\underline{\tilde{u}}_1 + 2\underline{\tilde{u}}_2 + \underline{\tilde{u}}_3$$

while for matrix $\mathbf{A_{II}}$

$$\underline{\tilde{a}}_1 = \underline{\tilde{u}}_1 \qquad \underline{\tilde{a}}_2 = 2\underline{\tilde{u}}_1 + \underline{\tilde{u}}_2 \qquad \underline{\tilde{a}}_3 = -\underline{\tilde{u}}_1 + 2\underline{\tilde{u}}_2 \quad \cancel{(+\underline{\tilde{u}}_3)} = 0$$

A basis of the row space of $\mathbf{A_I}$ is $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ 1 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ while $\mathbf{A_{II}}$ is $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 1 \\ 6 \end{bmatrix}$

Notice that, in passing, we seem to have proved a rather remarkable theorem. The dimension of column space is equal to the columns of **L** that correspond to non-zero rows of **U**. The dimension of row space is also equal to the number of non-zero rows of **U**. For a general matrix **A**,

$$\mathbf{A} \;=\; \underset{\sim}{l}_1\,\underset{\sim}{\tilde{u}}_1^{\mathrm{T}} \;+\; \underset{\sim}{l}_2\,\underset{\sim}{\tilde{u}}_2^{\mathrm{T}} \;+\; \underset{\sim}{l}_3\,\underset{\sim}{\tilde{u}}_3^{\mathrm{T}} \;+\; \dots \;+\; \underset{\sim}{l}_m\,\underset{\sim}{\tilde{u}}_m^{\mathrm{T}}$$

and, if we throw away the zero terms

$$\mathbf{A} \;=\; \underset{\sim}{l}_1\,\underset{\sim}{\tilde{u}}_1^{\mathrm{T}} \;+\; \underset{\sim}{l}_2\,\underset{\sim}{\tilde{u}}_2^{\mathrm{T}} \;+\; \underset{\sim}{l}_3\,\underset{\sim}{\tilde{u}}_3^{\mathrm{T}} \;+\; \dots \;+\; \underset{\sim}{l}_r\,\underset{\sim}{\tilde{u}}_r^{\mathrm{T}}$$

i.e. for any matrix

$$\boxed{\text{number of independent rows} = \text{number of independent columns}}$$

The number of independent columns, you will remember, is something we called the rank of **A**.

3.8 Properties of the L & U matrices

Now, it might be obvious that the columns of **L** and the non-zero rows of **U** are independent. We could check this by assuming the opposite: that the columns of **L** are *not* independent, then we should be able to write one of them as a linear combination of the other two. This would mean

$$\alpha \underset{\sim}{l}_1 + \beta \underset{\sim}{l}_2 + \gamma \underset{\sim}{l}_3 \;=\; 0 \quad \text{i.e.} \quad \alpha\begin{bmatrix}1\\2\\-1\end{bmatrix} + \beta\begin{bmatrix}0\\1\\2\end{bmatrix} + \gamma\begin{bmatrix}0\\0\\1\end{bmatrix} \;=\; \begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\-1 & 2 & 1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix} \;=\; \begin{bmatrix}0\\0\\0\end{bmatrix}$$

and solving these by forward substitution gives $\alpha = \beta = \gamma = 0$, i.e. we can never write one as a linear combination of the other two, which is the definition of independence.

The decomposition **A** = **LU** will always produce a *square* matrix like

$$\mathbf{L} \;=\; \begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\-1 & 2 & 1\end{bmatrix} \qquad \boxed{L^{-1} \text{ always exists } \& \det L = 1}$$

with 0's above the diagonal and 1's down the diagonal. It should be clear that the value of the determinant of a lower diagonal matrix is the product of the diagonal terms. For **L**, which always has 1's down the diagonal, det **L** = 1 . This means that $\mathbf{L}^{-1}$ exists, rank(**L**) = no of columns , etc.

(N.B. det **L** ≠ 0, is another way of demonstrating the columns of **L** are independent)

It should also be clear that, since the column space of **L** is the whole of R^m, that *any* vector in R^m can be expressed in terms of the columns of **L**.

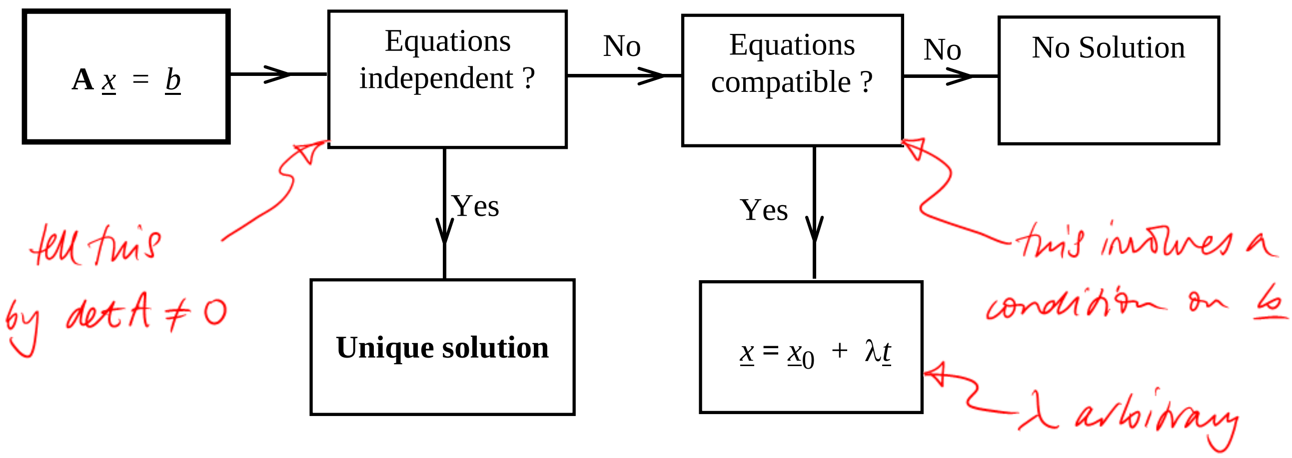
We have seen that the upper echelon matrix **U** is the same shape as **A**. **U** will be square only when **A** is. When it *is* square, it makes sense to talk about the determinant of **U** and whether **U** has an inverse. It should be clear from $A = LU \Rightarrow U = L^{-1}A$

that **U** has an inverse *if and only if* **A** has one. if A^{-1} exists $\Rightarrow UA^{-1}L = L^{-1}AA^{-1}L = I \Rightarrow U^{-1}$ exists
if U^{-1} exists $\Rightarrow U^{-1}L^{-1}A = U^{-1}L^{-1}LU = I \Rightarrow A^{-1}$ exists

4. The Solution of Ax = b

4.1 What are we expecting ?

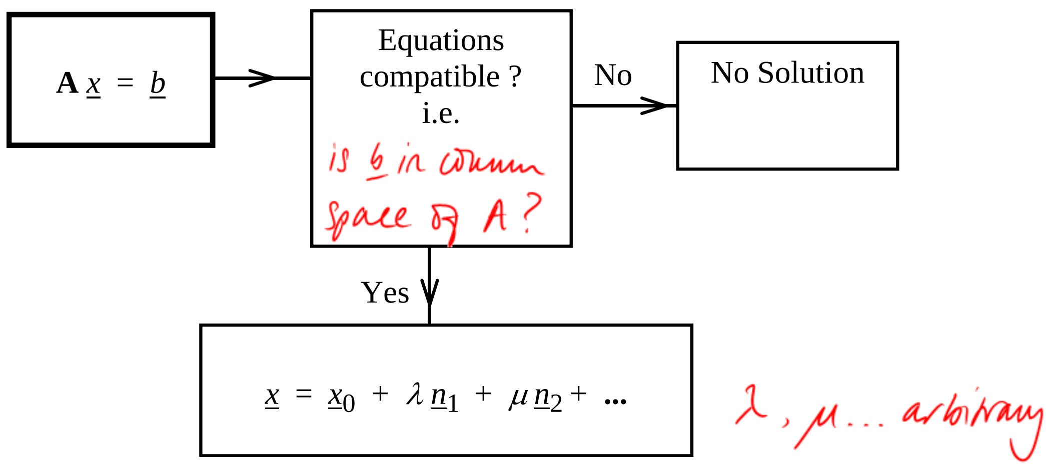
We know that for a 3×3 matrix $\mathbf{A}$



For a general $m \times n$ case, as for example the matrix associated with the structure analysed in section 1 of Handout 1, $\mathbf{A}\,\underline{t} = \underline{f}$

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 1/\sqrt{2} & 0 \\ -1 & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1 & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1 & 0 & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} t_{\mathrm{I}} \\ t_{\mathrm{II}} \\ t_{\mathrm{III}} \\ t_{\mathrm{IV}} \\ t_{\mathrm{V}} \\ t_{\mathrm{VI}} \end{bmatrix} = \begin{bmatrix} f_{\mathrm{E}x} \\ f_{\mathrm{E}y} \\ f_{\mathrm{F}x} \\ f_{\mathrm{F}y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -W \end{bmatrix}$$

we will amend this to



and the issues are now (i) how to tell if $\underline{b}$ is in the column space of $\mathbf{A}$ and (ii) how do we find $\underline{x}_0$, $\underline{n}_1$, $\underline{n}_2$, **...** (and how many of them should there be). We don't really need to consider the case of a unique solution as special; this will come out in the wash as there being zero $\underline{n}$'s.

4.2 How do we check whether $\underline{b}$ is in Column Space ?

e.g. $A\underline{x} = \underline{b} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

The answer to (i) is that we when we express $\underline{b}$ in terms of the columns of **L**, it should only need those columns that are in the column space of **A**.

$$\mathbf{A_I} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 6 & 3 & 12 \\ -1 & 2 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \mathbf{L\,U_I} \quad \text{Col Space} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{A_{II}} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 6 & 3 & 12 \\ -1 & 2 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{L\,U_{II}} \quad \text{Col Space} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

When we solve $\mathbf{A}\underline{x} = \underline{b}$ using LU decomposition, $\mathbf{LU}\underline{x} = \underline{b}$, then, we recast the problem as $\mathbf{L}\underline{c} = \underline{b}$ and $\mathbf{U}\underline{x} = \underline{c}$. The first step is to find $\underline{c}$ this is expressing $\underline{b}$ in terms of the columns of **L**.

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} c_1 + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} c_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} c_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$c_1 = 1 \qquad c_2 = 4 - 2c_1 = 2 \qquad c_3 = 2 + c_1 - 2c_2 = -1 \qquad \Rightarrow \qquad \underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

There will be a solution for $A_{\underline{I}}$ but not for $A_{\underline{II}}$

4.3 Completing the Solution $\mathbf{U}\underline{x} = \underline{c}$ (finding the Null-Space of A)

As part of the following procedure, we will be generating $\underline{n}_1$, $\underline{n}_2$, **...** which will be the general solution of $\mathbf{A}\underline{n} = 0$. We are, in effect, generating a basis for the null space of **A**

pivots

CASE I $\qquad \mathbf{U}\underline{x} = \underline{c} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Starting with the last of these and working upwards

$$z - t = -1$$

This is the first evidence of the problem being under-determined. We can not solve this equation.

We will, instead use it to find z in terms of t.

$$z = -1 + t$$

We can now go on to solve for the other variables in terms of t, by back substitution.

$$2y = 2 - z - 6t = 2 - (-1+t) - 6t \quad \Rightarrow \quad y = \frac{3}{2} - \frac{7}{2}t$$

and $\quad x = 1 - 2y - z - 3t = 1 - 3 + 7t + 1 - t - 3t \Rightarrow x = -1 + 3t$

In *vector* form this means the solution satisfies

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -7/2 \\ 1 \\ 1 \end{bmatrix} \qquad \underline{x} = \underline{x}_0 + t\,\underline{n}$$

i.e. a *line* of solutions where t can take any value.

The variables x, y and z , which have (non-zero) pivots are called *basic variables*, while t is a *free variable*. The choice of which variable to take as the free one is a bit arbitrary, we could have regarded z as being free, rather than t. We have followed a convention. The variables with pivots are the basic variables; the variables without pivots are the free ones.

Check:

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 6 & 3 & 12 \\ -1 & 2 & 2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 6 & 3 & 12 \\ -1 & 2 & 2 & 8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1 \\ 3/2 \\ -1 \\ 0 \end{bmatrix}}_{\underline{x}_0} + \underbrace{t \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 6 & 3 & 12 \\ -1 & 2 & 2 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ -7/2 \\ 1 \\ 1 \end{bmatrix}}_{t\,A\,\underline{n}}$$

$$= \begin{bmatrix} -1+3-1 \\ -2+9-3 \\ 1+3-2 \end{bmatrix} + t \begin{bmatrix} 3-7+1+3 \\ 6-21+3+12 \\ -3-7+2+8 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}}_{\underline{b}} + t \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{0}$$

We refer to $\underline{x} = \underline{x}_0 + t\,\underline{n}$ as the *general solution*. The first vector is a *particular solution*, while the rest is the solution of $\mathbf{A}\,\underline{x} = 0$. Since $\mathbf{L}$ is always invertible, we see immediately that

$$\mathbf{A}\underline{x} = \mathbf{0} \iff LU\underline{x} = 0 \iff U\underline{x} = 0 \qquad \text{(because } \mathbf{L}^{-1} \text{ exists)}$$

so that the null space of A is the same as the null space of U (which is easier to find).

We can also approach this problem using a "a particular solution" plus "general solution of $\mathbf{A}\underline{x} = 0$" method. This would be (taken from the Maths Databook)

1. Set the free variable to zero and find a particular solution $\underline{x}_0$
2. Set the RHS to zero (i.e.$\mathbf{U}\underline{x} = 0$), put the free variable equal to the value 1 and solve to find $\underline{n}$.

There may actually be more than one free variable, when this becomes

1. Set all free variables to zero and find a particular solution $\underline{x}_0$.
2. Set the RHS to zero, give each free variable in turn the value 1 while the others are zero, and solve to find a set of vectors which span the nullspace of **A**.

CASE II $\quad$ $\mathbf{Ux} = \mathbf{c} \Rightarrow$ $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

As noted earlier, we can not solve this, unless $c_3 = 0$.

If $\underline{b}$, on the other hand, had been $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$, then $\mathbf{L}\underline{c} = \underline{b}$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \Rightarrow \underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$ and the equations *are* compatible.

$\mathbf{U}\underline{x} = \underline{c}$, is now $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Only x and y have pivots and this time both z and t are free variables.

1. To find $\underline{x}_0$, set the free variables to zero. This gives (using back substitution)

$2y = 2 \Rightarrow y = 1$

& $x = 1 - 2y \Rightarrow x = -1$

i.e. $\underline{x}_0 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

2. Set the RHS to zero, give each free variable in turn the value 1 while the others are zero, and solve to find a set of vectors which span the nullspace of **A**.

Put $t = 1$, $z = 0$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By back substitution, $2y = -6 \Rightarrow y = -3$ and $x = -3 - 2y = 3$ i.e. $\underline{n}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

Put $t = 0$, $z = 1$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By back substitution,

$2y = -1 \Rightarrow y = -\frac{1}{2}$

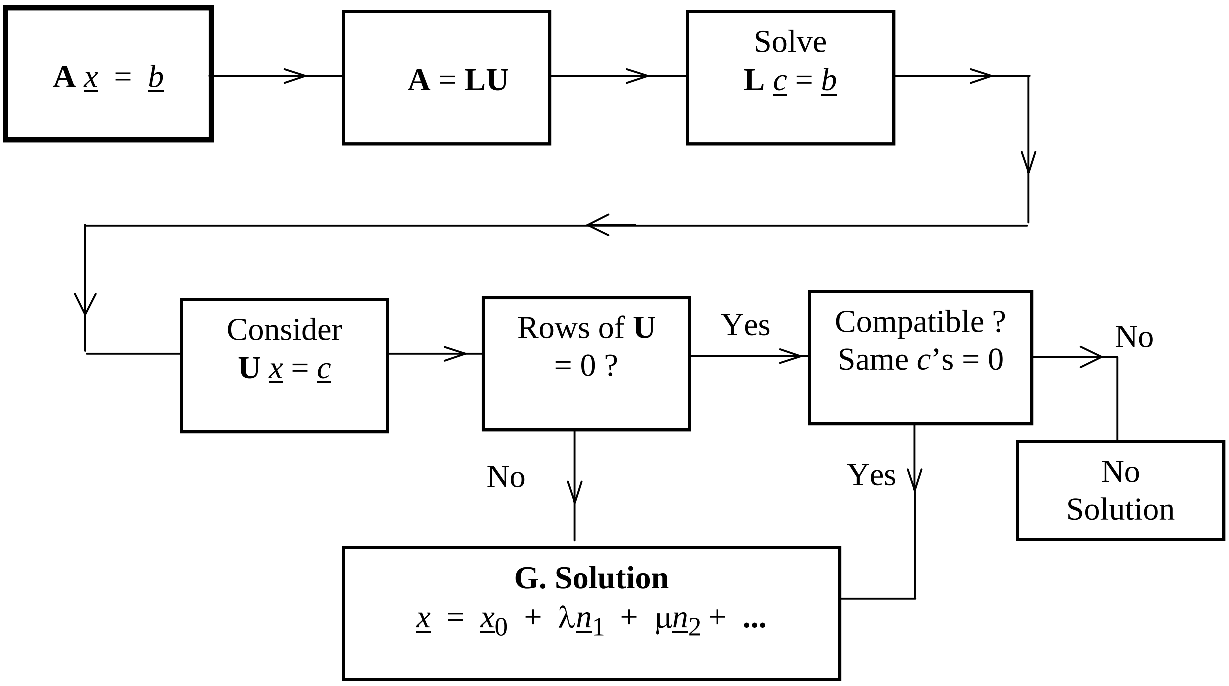
& $x = -1 - 2y \Rightarrow x = 0$

i.e. $\underline{n}_2 = \begin{bmatrix} 0 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$

The general solution is, therefore, $\underline{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$

You can now do Examples Paper 1 Question 8.

Key Points from Lecture



1. Set all free variables to zero and find a particular solution $\underline{x}_0$.
2. Set the RHS to zero, give each free variable in turn the value 1 while the others are zero, and solve to find a set of vectors which span the nullspace of **A**.

Basis for Column Space

The first r columns of **L** are a convenient basis for column space where

r = no of non-zero rows of **U**

Basis for Null Space of A

Generate by Note 2 above.

Basis for Row Space of A

The non-zero rows of **U**.