UNIVERSITY OF CAMBRIDGE
Department of Engineering

Fundamentals

- A continuous random variable X takes values from a continuous set (typically a subset of $\mathbb{R}$)
- The cumulative distribution function (CDF) is $F_{\mathrm{X}}(x) = \mathbb{P}[\mathrm{X} \leq x]$. It has the following properties:
  - Non-decreasing: $F_{\mathrm{X}}(a) \leq F_{\mathrm{X}}(b)$ if $a \leq b$
  - Limits: $\lim_{x\to-\infty} F_{\mathrm{X}}(x) = 0$ and $\lim_{x\to\infty} F_{\mathrm{X}}(x) = 1$
  - Interval: $F_{\mathrm{X}}(b) - F_{\mathrm{X}}(a) = \mathbb{P}[a < \mathrm{X} \leq b]$
- The joint CDF is defined as $F_{\mathrm{XY}}(x, y) = \mathbb{P}[\mathrm{X} \leq x \cap \mathrm{Y} \leq y]$
- If X and Y independent, $F_{\mathrm{XY}}(x, y) = F_{\mathrm{X}}(x)F_{\mathrm{Y}}(y)$ for all x, y

Probability Density Function

- The probability density function (PDF) is defined as $f_{\mathrm{X}}(x) = \dfrac{\mathrm{d}F_{\mathrm{X}}(x)}{\mathrm{d}x}$. It has the following properties:
  - Positive: $f_{\mathrm{X}}(x) \geq 0$ for all $x \in \mathbb{R}$ (the support may be extended to $\mathbb{R}$ by setting $f_{\mathrm{X}}(x) = 0$ for $x \notin \mathbb{X}$)
  - Integral: $\int_a^b f_{\mathrm{X}}(x)\mathrm{d}x = F_{\mathrm{X}}(b) - F_{\mathrm{X}}(a) = \mathbb{P}[a < \mathrm{X} \leq b]$
  - Normalised: $\int_{-\infty}^{+\infty} f_{\mathrm{X}}(x)\mathrm{d}x = 1$
- The joint PDF is defined as $f_{\mathrm{XY}}(x, y) = \dfrac{\partial^2 F_{\mathrm{XY}}(x, y)}{\partial x \partial y}$, and:
  - The conditional PDF is $f_{\mathrm{X|Y}}(x|y) = \dfrac{f_{\mathrm{XY}}(x, y)}{f_{\mathrm{Y}}(y)}$, and the product rule $f_{\mathrm{XY}}(x, y) = f_{\mathrm{X|Y}}(x|y)f_{\mathrm{Y}}(y)$
  - Marginalisation: $\int_{-\infty}^{+\infty} f_{\mathrm{XY}}(x, y)\mathrm{d}y = f_{\mathrm{X}}(x)$
  - Bayes' rule: $f_{\mathrm{Y|X}}(y|x) = \dfrac{f_{\mathrm{X|Y}}(x|y)f_{\mathrm{Y}}(y)}{f_{\mathrm{X}}(x)} = \dfrac{f_{\mathrm{X|Y}}(x|y)f_{\mathrm{Y}}(y)}{\int_{-\infty}^{+\infty} f_{\mathrm{X|Y}}(x|y)f_{\mathrm{Y}}(y)\mathrm{d}y}$
- Independence: X and Y independent iff $f_{\mathrm{XY}}(x, y) = f_{\mathrm{X}}(x)f_{\mathrm{Y}}(y)$ for all $x, y \in \mathbb{X} \times \mathbb{Y}$ (hence $f_{\mathrm{X|Y}}(x|y) = f_{\mathrm{X}}(x)$)

Exponential Density

- What is the time/distance between two successive λ-rate successes? $\mathrm{X} \in \mathbb{X} = [0, \infty)$ and $\lambda \in (0, \infty)$
- $\mathrm{X} \sim \mathrm{Exp}(\lambda) \quad \Leftrightarrow \quad f_{\mathrm{X}}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$
- $\mathbb{E}[\mathrm{X}] = 1/\lambda$ and $\mathrm{Var}[\mathrm{X}] = 1/\lambda^2$

Gaussian (or Normal) Density

- $\mathrm{X} \in \mathbb{X} = \mathbb{R}$, and $\mu, \sigma \in \mathbb{R} \times \mathbb{R}$
- $\mathrm{X} \sim \mathcal{N}(\mu, \sigma^2) \quad \Leftrightarrow \quad f_{\mathrm{X}}(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad$ for all $x \in \mathbb{R}$
- $\mathbb{E}[\mathrm{X}] = \mu$ and $\mathrm{Var}[\mathrm{X}] = \sigma^2$

Beta Density

- What is the PDF of the trial probability if we observe $\alpha - 1$ successes and $\beta - 1$ fails? $\mathrm{X} \in \mathbb{X} = [0, 1]$, and $\alpha, \beta > 0$
- $\mathrm{X} \sim \mathrm{Beta}(\alpha, \beta) \;\Leftrightarrow\; f_{\mathrm{X}}(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if } x \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$ where $\Gamma(a) = \int_0^\infty \xi^{a-1} e^{-\xi} \mathrm{d}\xi$
- $\mathbb{E}[\mathrm{X}] = \dfrac{\alpha}{\alpha + \beta}$ and $\mathrm{Var}[\mathrm{X}] = \dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Gaussian CDF

- For $\mathrm{Y} \sim \mathcal{N}(0, 1)$, the CDF $F_{\mathrm{Y}}(y) = \Phi(y) = \frac{1}{2}\left[1 + \mathtt{erf}\left(\frac{y}{\sqrt{2}}\right)\right]$, with `erf` the "error function" found in computing
- For $\mathrm{X} \sim \mathcal{N}(\mu, \sigma^2)$, $F_{\mathrm{X}}(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$