IB Structures 2023-24

Handout 4

Plastic Theory

Filled Version

Text and pictures in grey are omitted from the version in lectures

Cambridge University Engineering Department

4.1 Background and Introduction

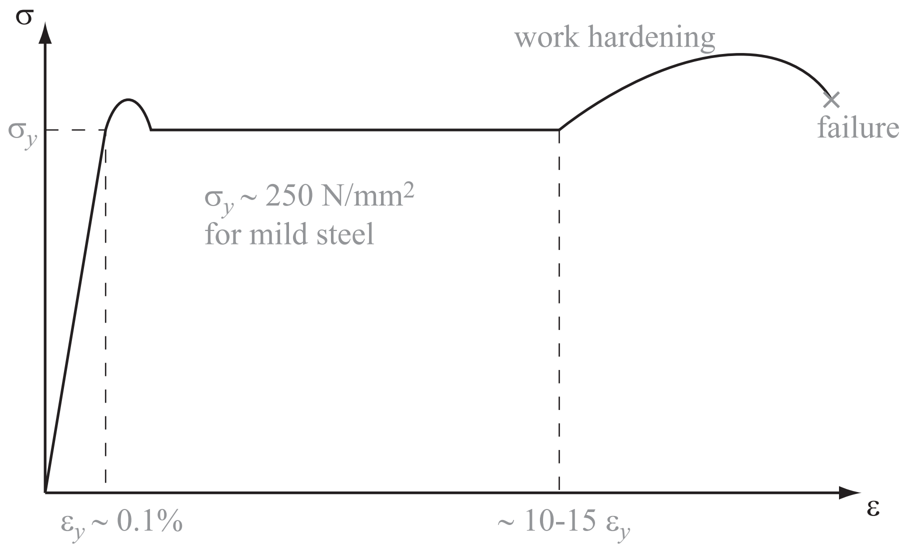
4.1.1 History

In the 1930's, experiments showed that elastic analysis was not suitable for estimating the collapse load of structures. A new method was developed that instead concentrated on the final failure mode — this method is known as plastic theory. Much of the work on plasticity theory was carried out in this Department by a team led by Lord Baker, and some of the Department's buildings are very early examples of the application of plasticity theory to design. Research on using plastic theory continues in the Department to the present day.

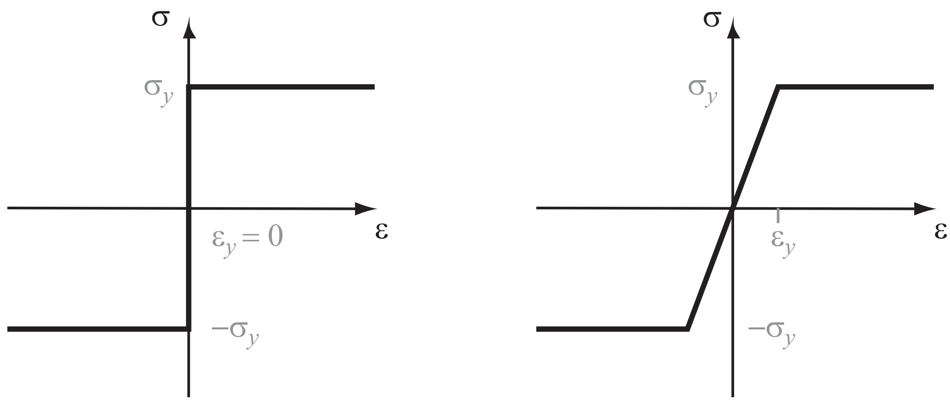
4.1.2 Material Model

Plasticity theory was originally developed for the design of steel structures. Although it is routinely applied to the analysis of structures made from other materials, we will initially consider a simple material model that is suitable for steel.

In a tension test, the stress strain curve for mild steel is approximately:



In simple plasticity theory, this is idealised as either *rigid-perfectly plastic*, or *elastic-perfectly plastic*.



rigid-perfectly plastic elastic-perfectly plastic

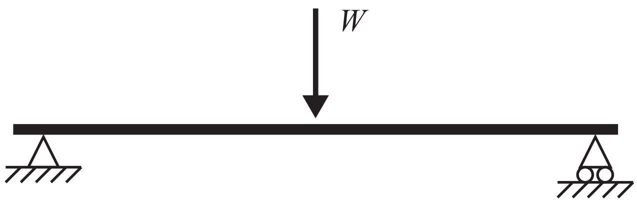
The main assumptions are:

1. Neglect work-hardening; — a safe assumption
2. The material is ductile; — a long plastic plateau
3. The behaviour is identical in tension and compression.

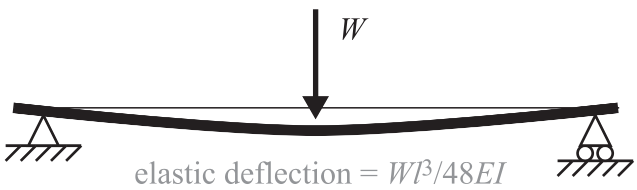
The rigid-perfectly plastic model is often used for simplicity. It neglects elastic deformation.

4.1.3 Example: failure of a simply-supported beam

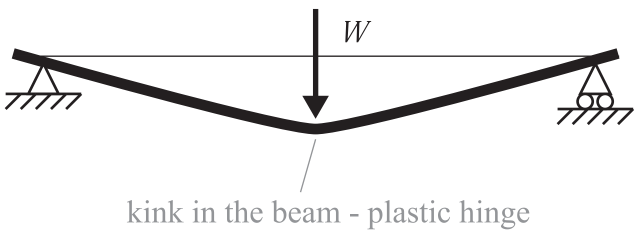
An experiment shows that as the load is increased, collapse occurs by the formation of a kink in the beam, known as a *plastic hinge*. The beam length is L.



Initially the behaviour is elastic:



Eventually further curvature becomes concentrated under the load, at the plastic hinge:



To understand the collapse, we need to examine what happens on the cross section where the plastic hinge forms.

4.1.4 Stresses on a rectangular cross-section as a plastic hinge forms

We make two assumptions about the behaviour of the beam:

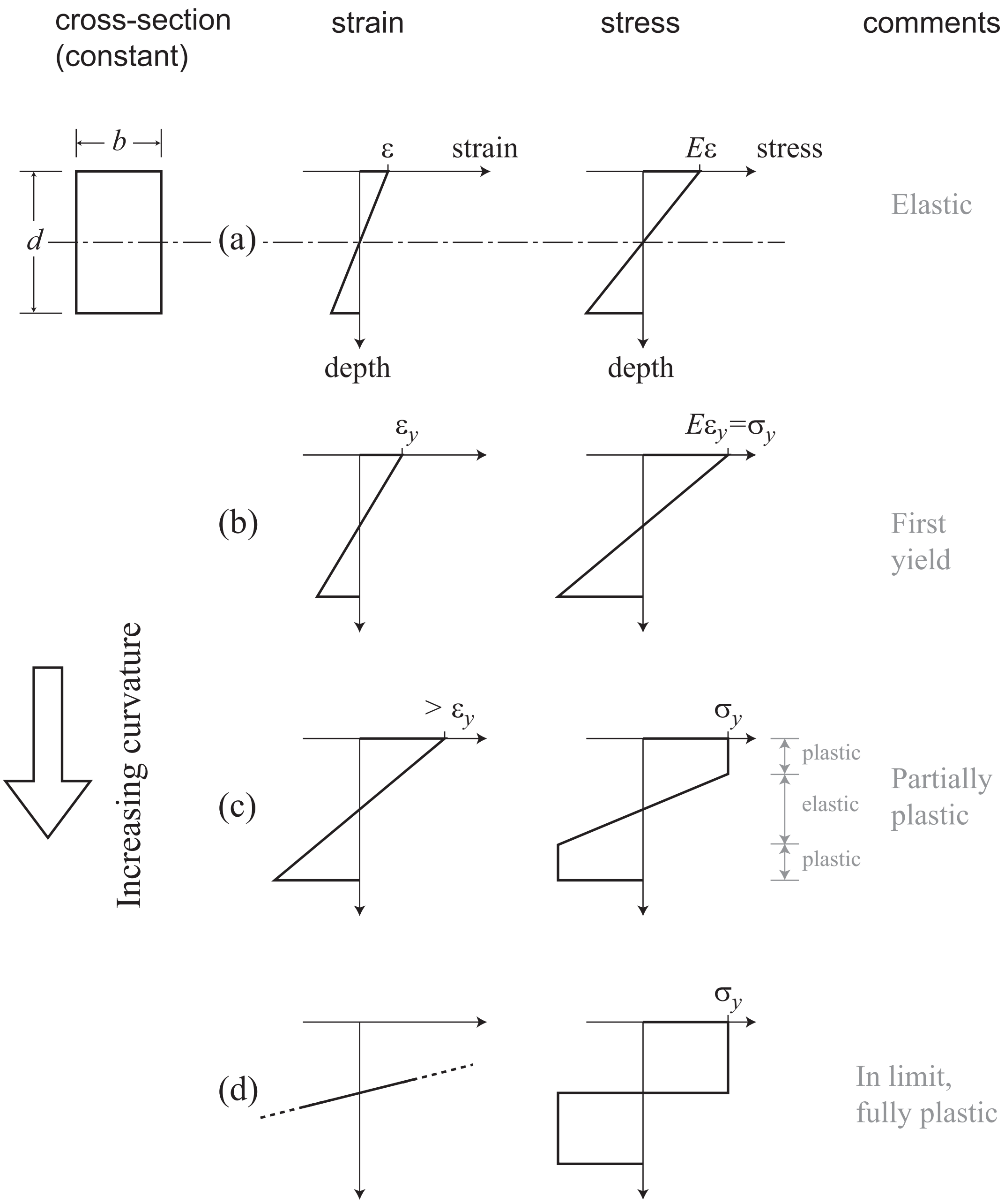
(1) Plane sections remain plane (the 'railway track' analogy). Even when the section becomes plastic this remains a good assumption. It implies that the strains on the cross-section must vary linearly with curvature. If y is the distance from the neutral axis, and κ is the local curvature, the strain is given by

$$\varepsilon = \kappa y$$

(2) The material is elastic perfectly-plastic. The stresses are related to the strains by

$$\begin{array}{lll} \sigma = E\varepsilon & \varepsilon \le \varepsilon_y & \text{— elastic} \\ \sigma = \sigma_y & \varepsilon \ge \varepsilon_y & \text{— plastic} \end{array}$$

For a simple rectangular section, this gives the behaviour shown below:



We can calculate a moment-curvature relationship for each of the regimes (a)–(d):

(a) Linear elastic range. From elastic bending theory (IA)

$$M = EI\kappa \qquad \text{where} \quad I = \text{second moment of area of section}$$
$$= \frac{bd^3}{12} \quad \text{for a rectangular section}$$

This is valid until first yield, when

$$|\sigma|_{\max} = \left|\frac{My_{\max}}{I}\right| \leq \sigma_y \qquad \text{where} \quad y_{\max} = \text{distance from the neutral axis to the extreme fibre}$$
$$= \frac{d}{2} \quad \text{for a rectangular section}$$

(b) At first yield ($M = M_y$)

$$|\sigma|_{\max} = \left|\frac{My_{\max}}{I}\right| = \sigma_y$$

$$M_y = \frac{I}{|y|_{\max}}\sigma_y$$

This is often written as $M_y = Z_e\sigma_y$, where Z_e is the *elastic section modulus*.

$$Z_e = \frac{I}{|y|_{\max}} \quad \text{only depends on the cross-section}$$

$$\sigma_y \quad \text{only depends on the material}$$

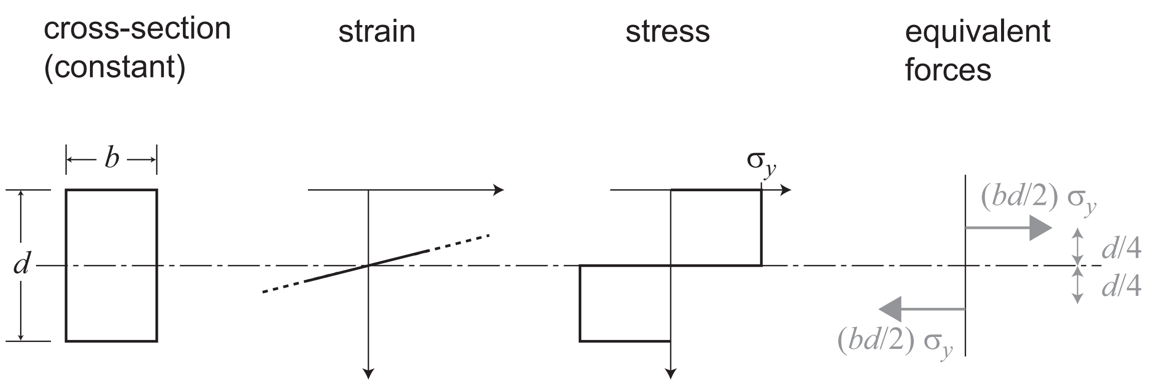
For a rectangular section,

$$Z_e = \frac{bd^3/12}{d/2} = \frac{bd^2}{6}, \qquad M_y = \frac{bd^2}{6}\sigma_y$$

The curvature at this point is

$$\kappa_y = \frac{M_y}{EI} = \frac{\sigma_y}{Ey_{\max}}$$
$$= \frac{2\sigma_y}{Ed} \quad \text{for a rectangular section}$$

(d) Fully plastic ($M = M_p$) We wish to calculate the moment on the section when it becomes fully plastic (as the curvature becomes very large)



The fully plastic moment M_p is in equilibrium with the resultant forces

$$M_p = \frac{bd}{2}\sigma_y \times \frac{d}{4} + \frac{bd}{2}\sigma_y \times \frac{d}{4}$$
$$= \frac{bd^2}{4}\sigma_y$$

This is often written as $M_p = Z_p \sigma_y$, where Z_p is the *plastic section modulus*.

Z_p only depends on the cross-section

σ_y only depends on the material

For a rectangular section,

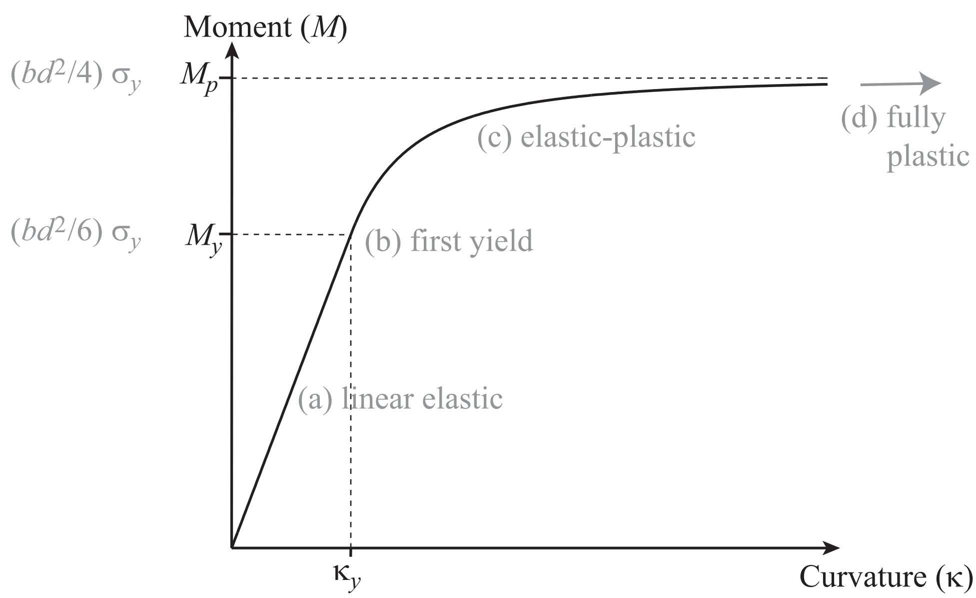
$$Z_p = \frac{bd^2}{4}$$

(c) Partially plastic It can be shown that for a rectangular section in the partially plastic range

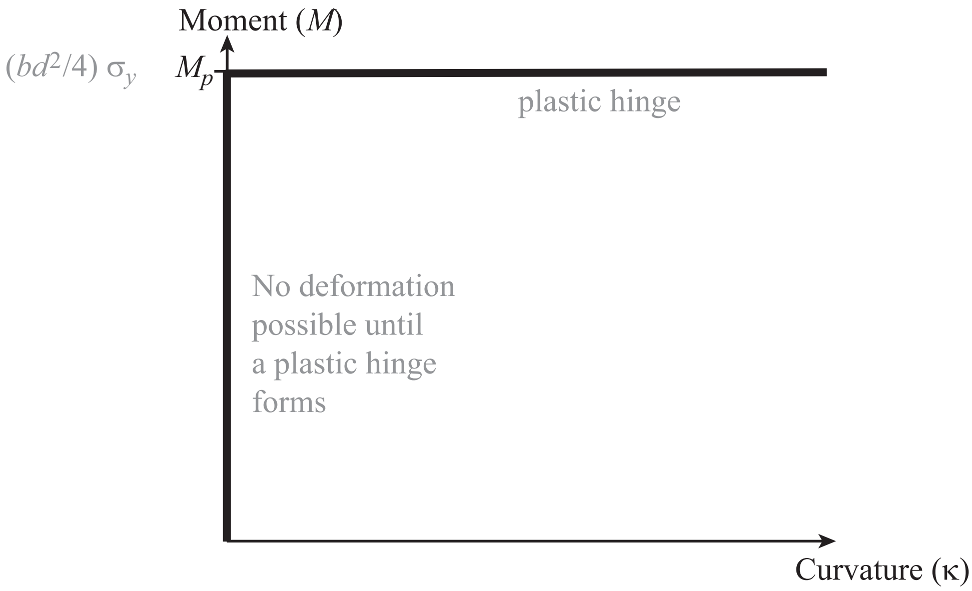
$$M = M_p \left[1 - \frac{1}{3} \left(\frac{\kappa_y}{\kappa} \right)^2 \right]$$

Moment-Curvature plot

The results (a)–(d) can be plotted to show the formation of a plastic hinge in a rectangular cross-section:

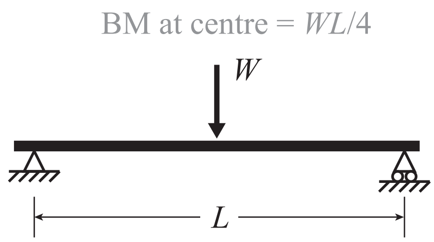


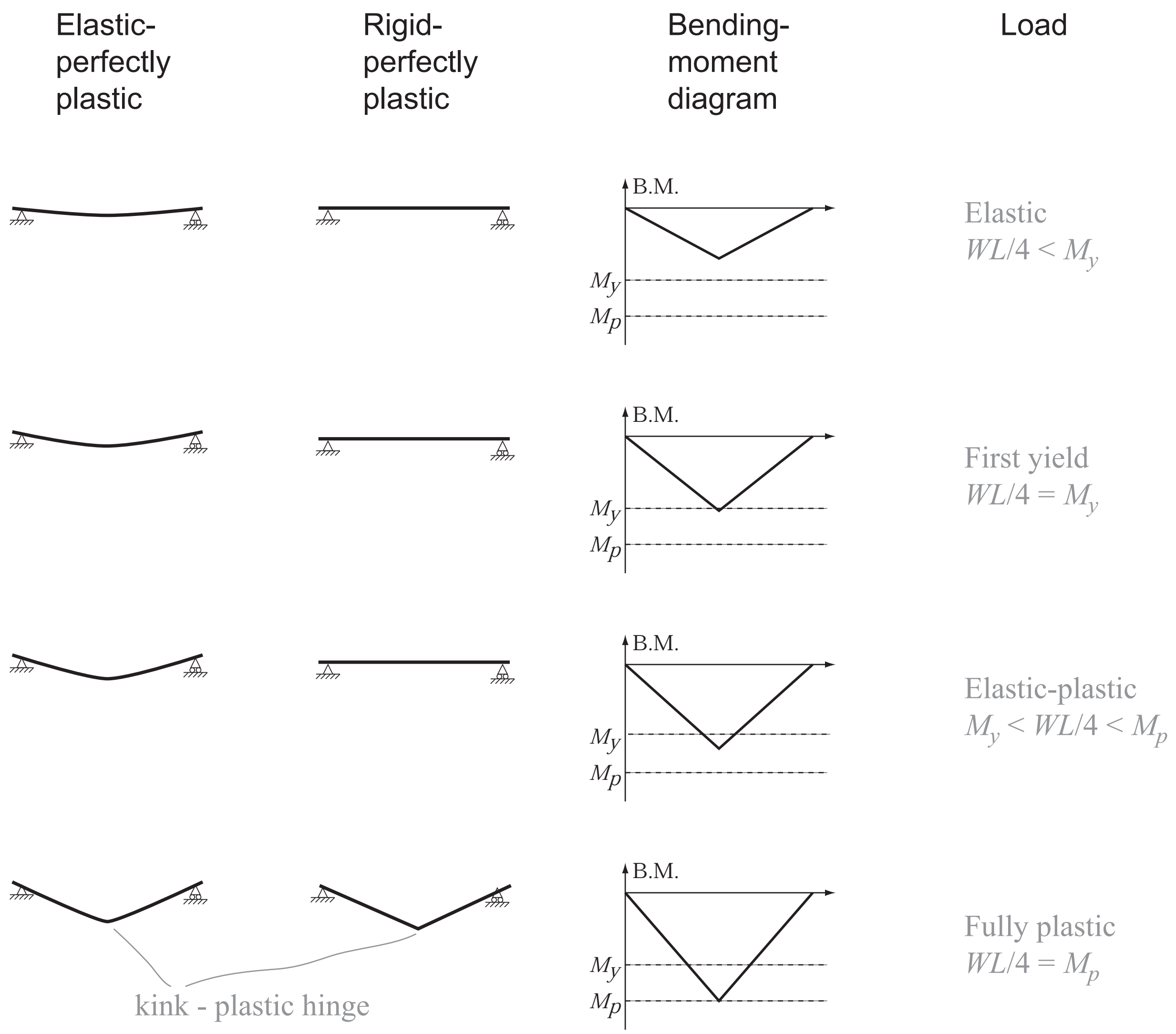
If we had instead assumed that the material was rigid-perfectly plastic, there would have been no deformation at all until the entire section was plastic, giving a moment-curvature plot:



4.1.5 Example: failure of a simply-supported beam revisited

Now we know the moment-curvature relationship in the hinge, we can examine the collapse of the beam more carefully:



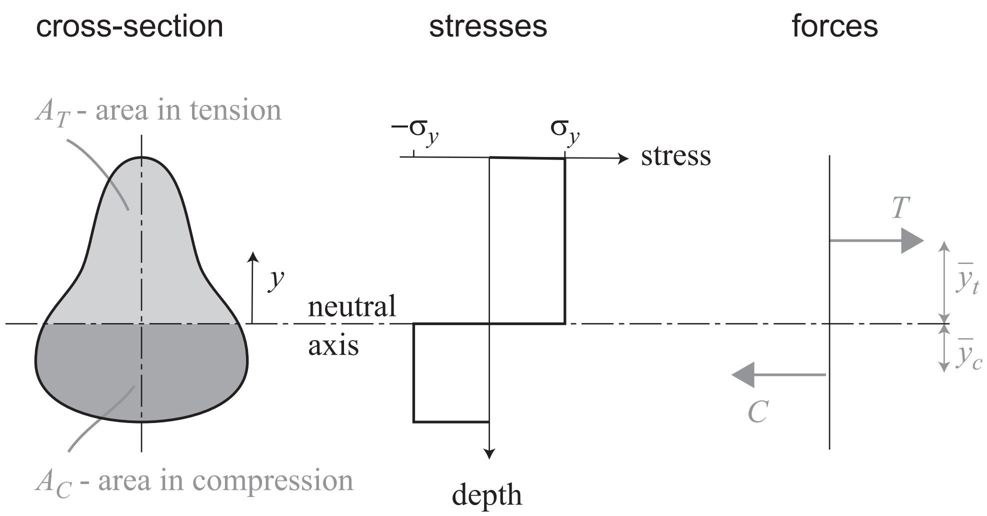


If collapse occurs when $W = W_{\text{collapse}}$,

$$W_{\text{collapse}} = \frac{4M_p}{L}$$

4.2 Plastic section modulus

This section will show how the calculation carried out to find the plastic moment of a rectangular beam can be generalised to almost any cross-section. The only assumption that we shall make is that there is an axis of symmetry of the cross-section in the direction of the applied loads. We shall again aim to write the plastic moment as $M_p = Z_p\sigma_y$, where Z_p is the *plastic section modulus*, which only depends on the cross-section.



The total tensile force is given by

$$T = A_T \sigma_y$$

The total compressive force is given by

$$C = A_C \sigma_y$$

In the absence of any overall axial force (the usual assumption), these forces must balance

$$T = C \qquad \Rightarrow \qquad A_T = A_C$$

Thus the neutral axis for fully plastic bending is the *equal-area* axis.
The plastic moment is given by

$$\begin{aligned} M_p &= \int_{\text{tensile}} y\, \sigma_y\, dA + \int_{\text{compressive}} (-y)\,(-\sigma_y)\, dA \\ &= \int_{\text{section}} |y| \sigma_y\, dA \end{aligned}$$

and hence the plastic section modulus is given by

$$Z_p = \int_{\text{section}} |y|\, dA$$

In practice, sections can usually be split into a number of simple regions, and the formula for plastic section modulus reduces to

$$Z_p = \sum_{\text{regions}} A_r |\bar{y}_r|$$

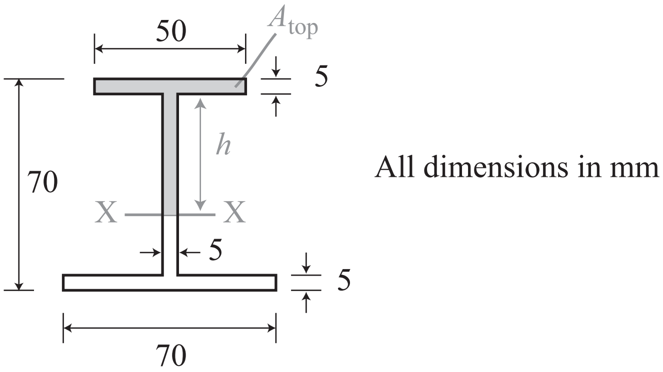
where: A_r is the area of a region yielded fully in tension, or fully in compression; $|\bar{y}_r|$ is the distance from the equal area axis to the centroid of A_r.

For our example

$$Z_p = A_T \bar{y}_T + A_C \bar{y}_C$$

4.2.1 Example

Find Z_p and M_p for the following section, given $\sigma_y = 250\ \mathrm{N/mm^2}$.

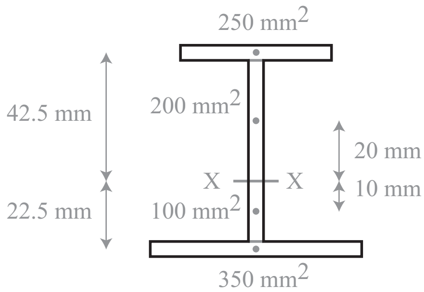


Locate position of equal area axis X-X

$$\text{Total area} = 50\times 5 + 60\times 5 + 70\times 5 = 900\ \mathrm{mm^2}$$

$$A_{\text{top}} = h\times 5 + 50\times 5 = 900/2 \qquad \Rightarrow \qquad h = 40\ \mathrm{mm}$$

Divide the tension zone into simple shapes and identify their centroids w.r.t. X-X axis. Repeat for the compression zone.



Evaluate $Z_p = \sum_{\text{regions}} A_r|\bar{y}_r|$, $M_p = Z_p\sigma_y$

$$Z_p = 250\times 42.5 + 200\times 20 + 100\times 10 + 350\times 22.5 = 23500\ \mathrm{mm^3}$$

$$M_p = 23500\ \mathrm{mm^3}\times 250\ \mathrm{N/mm^2} = 5.88\times 10^6\ \mathrm{N\,mm} = 5.88\ \mathrm{kNm}$$

4.2.2 Summary of plastic section modulus

1. In the limit, the whole section is yielded either in tension or compression.
2. If there is no axial force, longitudinal equilibrium forces the neutral axis to be at the equal area axis. Note that this may not be the centroid of the section — when a beam yields in bending, the neutral axis may move.

3. The plastic section modulus is given by $Z_p = \int_{\text{section}} |y|\, dA$. Note that this is *not* the first moment of area, as all of the beam makes a *positive* contribution. In practice, Z_p is more easily calculated by splitting the beam into a number of simple regions which are fully yielded in either tension or compression, and applying $Z_p = \sum_{\text{regions}} A_r |\bar{y}_r|$.

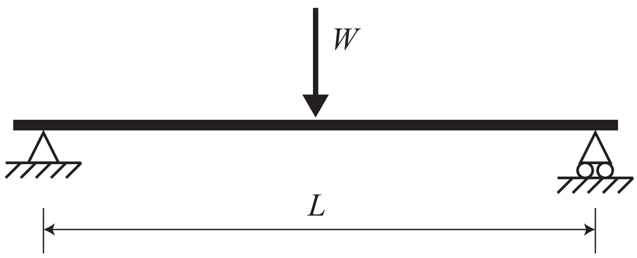
Try Question 4, Examples Sheet 2/4

4.3 Plastic collapse for determinate beams

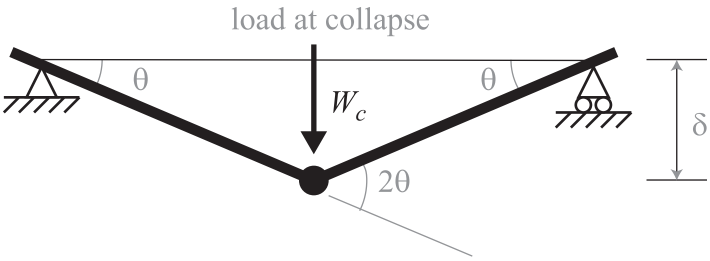
An alternative way of calculating the collapse load of a beam is to consider an energy balance of the beam as it fails. The key idea is that during plastic collapse, the energy dissipated as the beam yields must balance the work done by the load.

4.3.1 Example 1

Consider our earlier simple example:



Neglecting elastic deformation (or assuming rigid-perfectly plastic material), during collapse the beam will look like:



Compatibility using small angles

$$\theta = \frac{\delta}{L/2} = \frac{2\delta}{L}$$

Work done by external load

$$\text{W.D.} = W_c \times \delta$$

Energy dissipated by plastic hinge

$$\text{E.D.} = \text{moment} \times \text{rotation} \quad = M_p \times 2\theta$$

Calculate plastic collapse load — equate work done and energy dissipated:

$$W_c \times \delta = M_p \times 2\theta = M_p \times \frac{4\delta}{L}$$

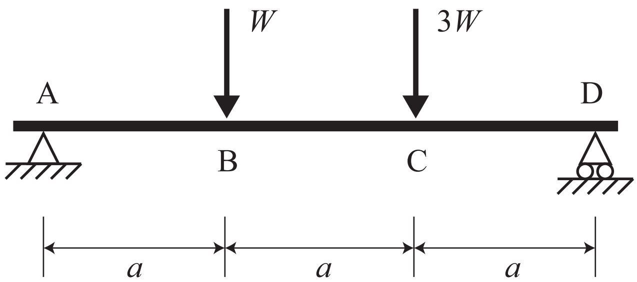
δ cancels, i.e. collapse load remains constant as beam deforms

$$W_c = \frac{4M_p}{L}$$

Note that we have calculated the same collapse load as before *without using any statement of equilibrium*.

4.3.2 Example 2

Find the collapse load of the simply supported beam shown below. The plastic moment of the beam is M_p.

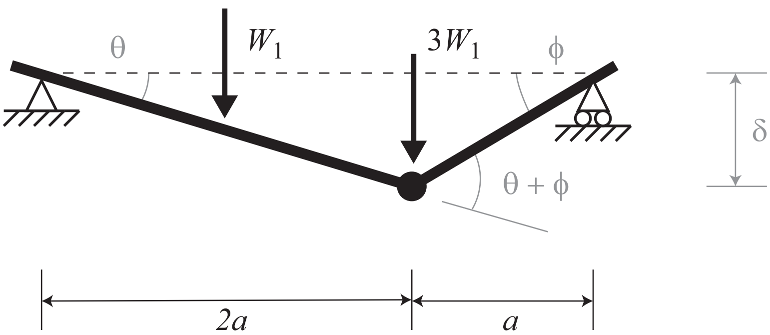


For this example, we might consider that two failure mechanisms were possible, either a plastic hinge would form at B, or it would form at C. We will examine the effect of these assumptions on the estimate of collapse load.

Mechanism (1): plastic hinge at C

Collapse mechanism:

Collapse load for Mechanism 1



Compatibility

$$\theta = \frac{\delta}{2a}, \quad \phi = \frac{\delta}{a}, \quad \theta + \phi = \frac{3\delta}{2a}$$

Work done by external load

$$\text{W.D.} = W_1 \times \theta a + 3W_1 \times \phi a$$

Energy dissipated by plastic hinge

$$\text{E.D.} = \text{moment} \times \text{rotation} \quad = M_p \times (\theta + \phi)$$

Calculate plastic collapse load

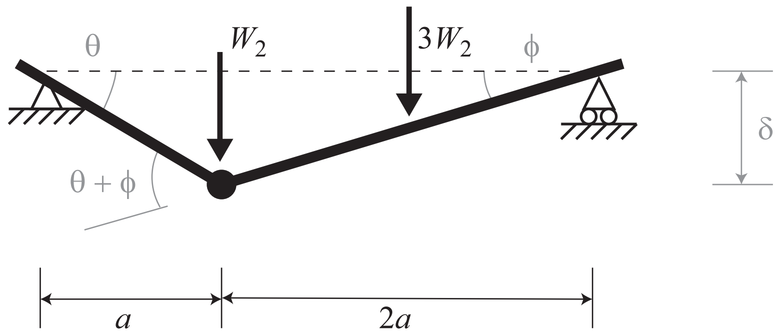
$$W_1 a \left(\frac{\delta}{2a} + \frac{3\delta}{a} \right) = M_p \frac{3\delta}{2a}$$

$$W_1 = \frac{3M_p}{7a}$$

Mechanism (2): plastic hinge at B

Collapse mechanism:

Collapse load for Mechanism 2



Compatibility

$$\theta = \frac{\delta}{a}, \quad \phi = \frac{\delta}{2a}, \quad \theta + \phi = \frac{3\delta}{2a}$$

Work done by external load

$$\text{W.D.} = W_2 \times \theta a + 3W_2 \times \phi a$$

Energy dissipated by plastic hinge

$$\text{E.D.} = \text{moment} \times \text{rotation} \quad = M_p \times (\theta + \phi)$$

Calculate plastic collapse load

$$W_2 a \left(\frac{\delta}{a} + \frac{3\delta}{2a} \right) = M_p \frac{3\delta}{2a}$$

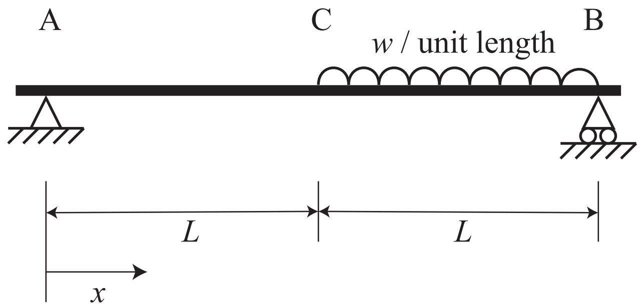
$$W_2 = \frac{3M_p}{5a} \qquad (\text{n.b.} > W_1)$$

Choosing correct collapse mechanism

As W increases from zero, the beam will collapse when W reaches W_1, and the failure mechanism will be mechanism (1). The load will never reach W_2, and failure mechanism (2) will never occur.

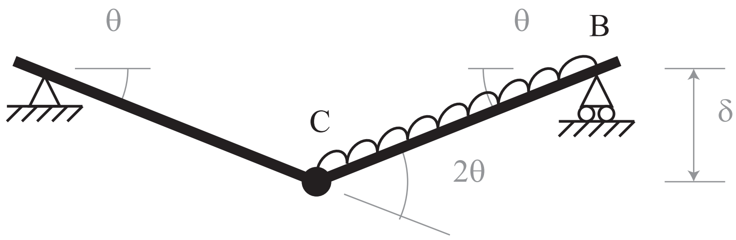
4.3.3 Example 3 — Distributed loading

In the previous examples it was clear that hinges would only form where a point load was applied. With distributed loading, it is not clear where the hinge will form. Consider the following beam, with plastic moment M_p



Initial estimate

For simplicity, initially assume that the hinge forms in the centre.
Collapse mechanism:



Compatibility

$$\theta = \frac{\delta}{L}$$

Work done by external load — on average the load moves down a distance $\delta/2$.

$$\text{W.D.} = w_1 L \times \delta/2$$

Energy dissipated by plastic hinge

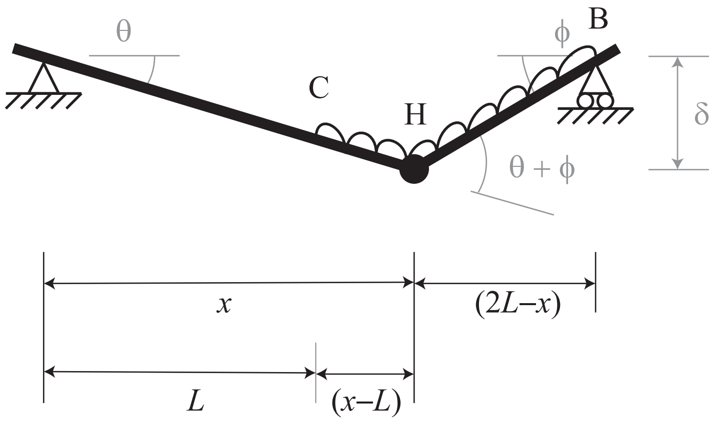
$$\text{E.D.} = M_p \times 2\theta$$

Calculate plastic collapse load

$$\frac{w_1 L\delta}{2} = \frac{2M_p\delta}{L} \qquad \Rightarrow \qquad w_1 = \frac{4M_p}{L^2}$$

More careful analysis

Consider a general collapse mechanism, where the hinge forms at some unknown position x.



Compatibility

$$\theta = \frac{\delta}{x}, \quad \phi = \frac{\delta}{2L-x}, \quad \theta+\phi = \frac{\delta\, 2L}{x(2L-x)}$$

Work done by external load

in HB

$$\text{the load moves down an average distance } \frac{\phi(2L-x)}{2} = \frac{\delta}{2}$$

$$\text{W.D.} = w(2L-x)\times\delta/2$$

in CH

$$\text{the load moves down an average distance } \theta(L+\frac{x-L}{2}) = \frac{L+x}{2x}\delta$$

$$\text{W.D.} = w(x-L)\times\frac{L+x}{2x}\delta$$

in total

$$\text{W.D.} = \frac{wL(2x-L)}{2x}\delta$$

Energy dissipated by plastic hinge

$$\text{E.D.} = M_p\times(\theta+\phi) = M_p\frac{2L}{x(2L-x)}\delta$$

Calculate plastic collapse load

$$\text{total W.D.} = \text{E.D.}$$

$$\frac{wL(2x-L)}{2x}\delta = M_p \frac{2L}{x(2L-x)}\delta$$

$$w_{\text{mech}} = \frac{4M_p}{(2x-L)(2L-x)}$$

Choose correct collapse mechanism — to find the *actual* collapse load, and the *actual* position of the hinge, we require the lowest possible value of w_{mech}. We can find the minimum by considering when $dw_{\text{mech}}/dx = 0$.

In this case, however, the expression for w_{mech} has all the terms involving x on the bottom, and hence it is easier to find the position of the maximum of $1/w_{\text{mech}}$ (the same thing), by considering when $d(1/w_{\text{mech}})/dx = 0$.

$$\frac{d}{dx}\left(\frac{1}{w_{\text{mech}}}\right) = \frac{1}{4M_p}[2(2L-x)-(2x-L)] = 0$$

$$5L - 4x = 0$$

$$x = 5L/4$$

Substitute back to give the critical load w_c

$$w_c = \frac{4M_p}{(3L/2)(3L/4)} = \frac{32M_p}{9L^2}$$

Note the relatively small improvement in the estimated collapse load compared with the initial estimate, $w_1 = 36M_p/9L^2$.

Try Questions 5 and 6, Examples Sheet 2/4

4.4 Upper Bound Theorem of Plasticity

An estimate of the plastic collapse load, W_{mech}, calculated for any arbitrary compatible mechanism by equating the work done by the applied load, and the plastic energy dissipated, will either be greater than, or equal to, the actual collapse load W_c.

$$W_{\text{mech}} \geq W_c$$

General methodology

1. Postulate compatible collapse mechanisms;
2. Evaluate W_{mech} for each;
3. Select the collapse mechanism that gives the *lowest* upper bound on W_c.

Note that we have only used two of the three basic principles of structural analysis. We have to postulate a *compatible* collapse mechanism, and we use a *material law* to find the energy dissipated, but we have not considered *equilibrium*.

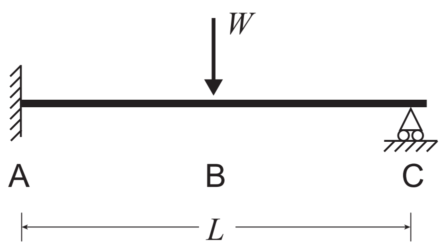
Later, we shall also see the Lower Bound theorem of Plasticity, that considers only *equilibrium* and a *material law*, but neglects *compatibility*.

A formal proof of the bound theorems of plasticity will be given at the end of the lectures on plastic theory.

4.5 Plastic Collapse of Statically Indeterminate Beams

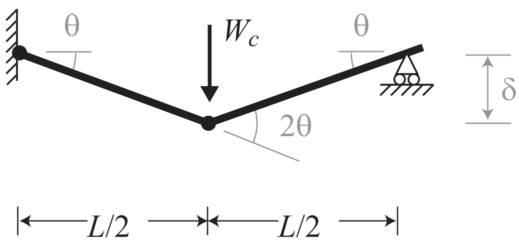
We saw in Handout 1 that an *elastic* analysis of a statically indeterminate structure is harder than the analysis of a determinate structure: it was necessary to simultaneously solve equations of *equilibrium* and *compatibility*. For a *plastic collapse analysis*, however, it is no more difficult to analyse an indeterminate structure than a determinate one. The only distinction is that failure must occur at more than one point. A statically determinate beam can collapse when one hinge forms. A statically indeterminate beam will require more than one hinge.

4.5.1 Example — Propped Cantilever



For this example, two hinges are required for collapse to occur. It is evident that the only sensible places that they may occur are under the load or at the root, and hence we can be confident that our *upper bound analysis* in this case will give the correct collapse load.

Collapse mechanism



Compatibility

$$\theta = \frac{2\delta}{L}$$

Work done by external load

$$\text{W.D.} = W \times \delta$$

Energy dissipated by plastic hinges

$$\text{E.D.} = M_p \times 2\theta + M_p \times \theta = M_p \frac{6\delta}{L}$$

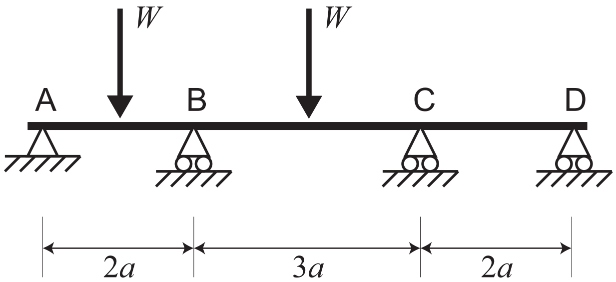
Calculate plastic collapse load

$$W \times \delta = M_p \frac{6\delta}{L}$$

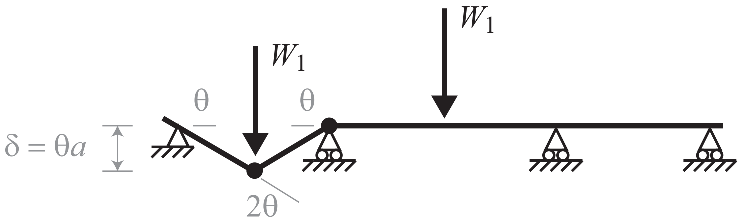
$$W_c = \frac{6M_p}{L}$$

4.5.2 Example — Multi-span Beam

find the collapse load W_c of the following structure:

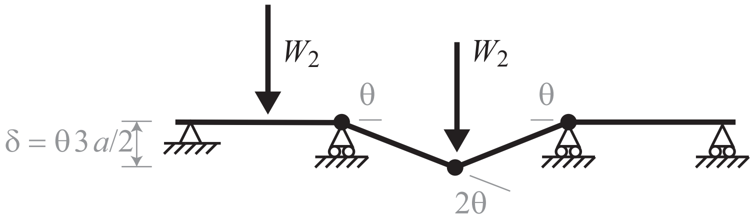


Collapse mechanism 1



$$\text{W.D.} = \text{E.D.}$$
$$W_1 \times \theta a = M_p \times (2\theta + \theta)$$
$$W_1 = \frac{3M_p}{a}$$

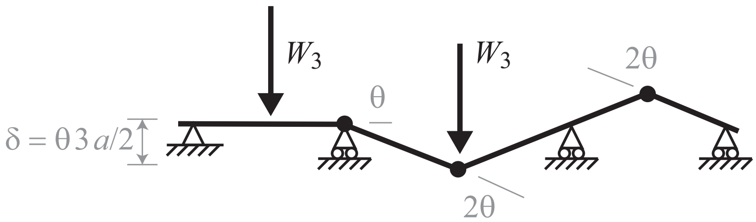
Collapse mechanism 2



$$\text{W.D.} = \text{E.D.}$$
$$W_2 \times \theta\ 3a/2 = M_p \times (2\theta + \theta + \theta)$$
$$W_2 = \frac{8M_p}{3a}$$

These are the only two sensible mechanisms — for uniform structures subjected to point loads, hinges will only form at the load or support points. For instance, the third mechanism shown below has the same work done by the load, but more energy dissipated.

Collapse mechanism 3



$$W_3 = \frac{10M_p}{3a}$$

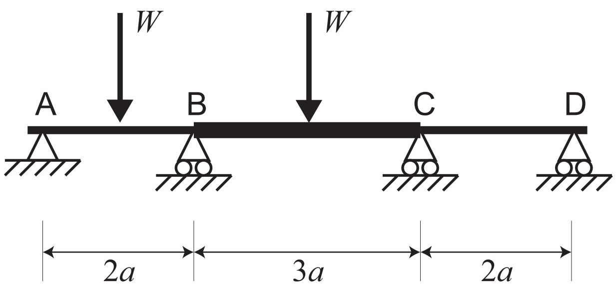
Choose correct collapse mechanism

By the upper-bound theorem, the actual collapse load will be the lowest collapse load for any collapse mechanism. Because there are no other sensible mechanisms, W_2 must be the actual collapse load.

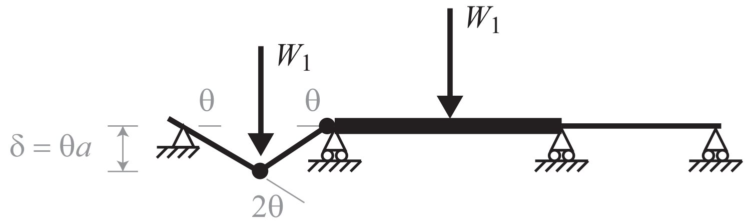
$$W_c = W_2 = \frac{8M_p}{3a} < W_1 < W_3$$

4.5.3 Example — Beam of non-uniform section

The structure of the previous example is to be reinforced, so the plastic moment capacity of the middle span is now kM_p. Calculate k to ensure that collapse occurs in the spans AB and BC simultaneously.

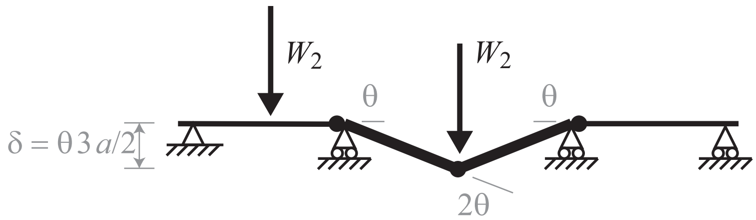


Collapse mechanism 1



$$W_1 \times \theta a = M_p \times 3\theta$$
$$W_1 = \frac{3M_p}{a}$$

Collapse mechanism 2



$$W_2 \times \theta\, 3a/2 = \theta M_p + \theta M_p + 2\theta\, kM_p$$
$$W_2 = \frac{4M_p}{3a}(1+k)$$

To ensure collapse occurs simultaneously, equate W_1 and W_2.

$$\frac{4M_p}{3a}(1+k) = \frac{3M_p}{a}$$

$$1+k = \frac{9}{4}$$

$$k = \frac{5}{4}, \quad \text{i.e. strengthen by 25\%}$$

Try Questions 7 and 8, Examples Sheet 2/4