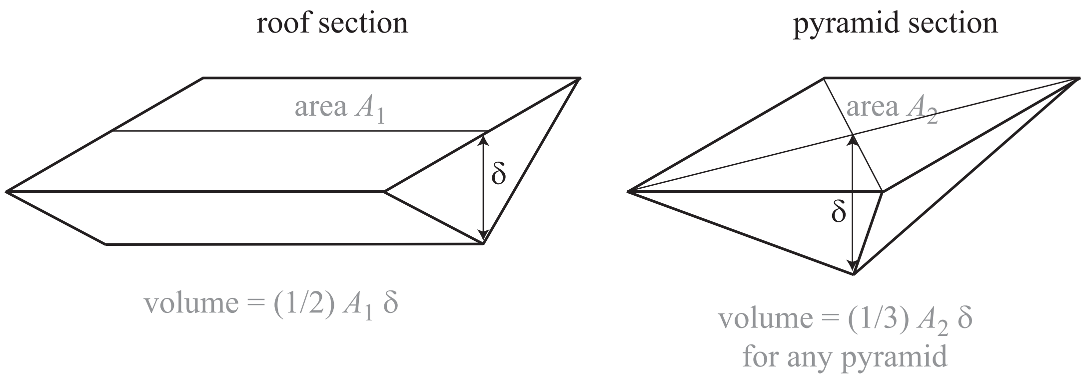
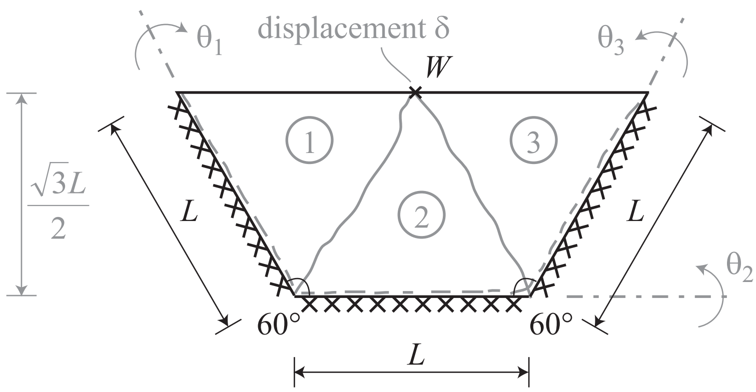
by the load on that region. A much simpler alternative, however, is to simplify the expression for work done to W.D. $= pV$, where $V = \int \delta(x,y)\, dA$ is the volume swept out by the collapsing slab.

V can be easily calculated, as most collapse mechanisms can be split into *roof* sections, and *pyramid* sections.



4.7.4 Example — Balcony

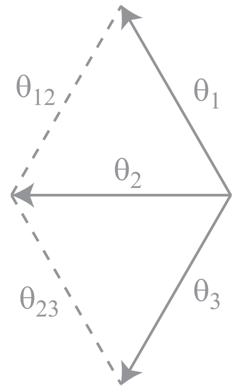
Estimate the collapse load W to cause the collapse of the balcony shown below, which has a moment capacity per unit length m.



$$l_{12} = l_{13} = L, \quad l_1 = l_2 = l_3 = L$$

$$\theta_1 = \theta_2 = \theta_3 = \frac{2\delta}{\sqrt{3}L}$$

Vector diagram of rotation



θ_{12} must be parallel with hinge 12 etc.

$$\theta_{12} = |\theta_2 - \theta_1| = \theta_{23} = \theta_1 = \frac{2\delta}{\sqrt{3}L}$$

Energy dissipated
Hinge 12 (hinge between rigid regions (1) and (2))

$$\text{E.D.} = ml_{12}\theta_{12} = mL\frac{2\delta}{\sqrt{3}L} = \frac{2m\delta}{\sqrt{3}}$$

Hinge 1 (hinge between support and rigid region (1))

$$\text{E.D.} = ml_1\theta_1 = \frac{2m\delta}{\sqrt{3}}$$

Total

$$\text{total E.D.} = \underbrace{\frac{2m\delta}{\sqrt{3}} \times 2}_{\text{hinges 12,23}} + \underbrace{\frac{2m\delta}{\sqrt{3}} \times 3}_{\text{hinges 1,2,3}}$$

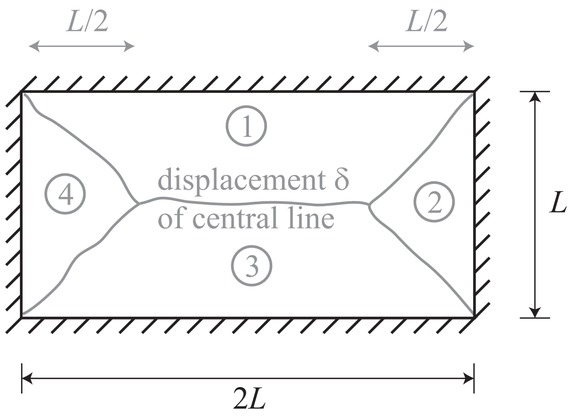
Work equation

$$W\delta = \frac{10m\delta}{\sqrt{3}}$$

$W = \dfrac{10m}{\sqrt{3}}$ is an upper bound on the collapse load.

4.7.5 Example — Simply-supported slab

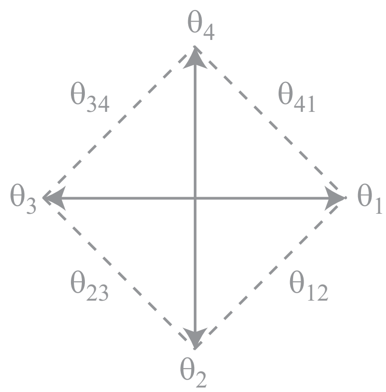
Estimate the uniform pressure p required to cause the collapse of the simply-supported slab shown below, which has a moment capacity per unit length m.



$$l_{13} = L, \quad l_{12} \text{ etc. } = \frac{L}{\sqrt{2}}$$

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{2\delta}{L}$$

Vector diagram of rotation

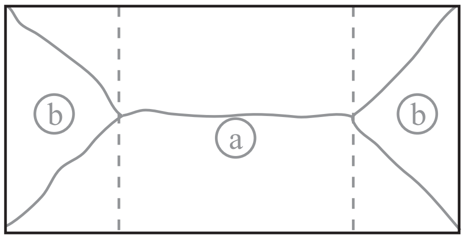


$$\theta_{13} = 2\theta_1 = \frac{4\delta}{L}, \quad \theta_{12} \text{ etc. } = \sqrt{2}\theta_1 = \frac{2\sqrt{2}\delta}{L}$$

Energy dissipated

$$\text{E.D.} = \underbrace{m\frac{L}{\sqrt{2}}\frac{2\sqrt{2}\delta}{L} \times 4}_{\text{hinges 12,23,34,41}} + \underbrace{mL\frac{4\delta}{L}}_{\text{hinge 13}} = 12m\delta$$

Work done



$$\text{Volume (a)} = \frac{1}{2} L^2 \delta$$

$$\text{Volume (b)} = \frac{1}{3} L^2 \delta$$

$$\text{W.D.} = p \frac{5}{6} L^2 \delta$$

Work equation

$$p \frac{5}{6} L^2 \delta = 12 m \delta$$

$$p = \frac{72m}{5L^2} = 14.4 m / L^2$$

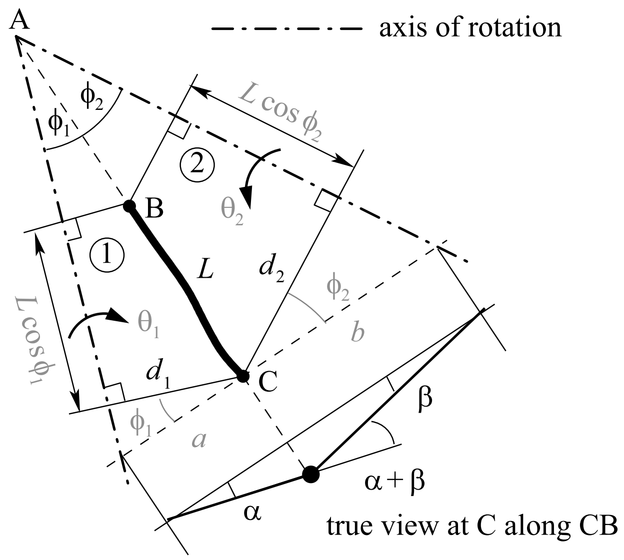
Again this is an upper bound on the true collapse load

Optimized analysis

An optimized analysis using Dr Middleton's 'Cobras' programmes, which allows the geometry of the collapse mechanism to vary, and also tries different collapse mechanisms, finds that the 'true' collapse load to be $p = 14.1m/L^2$, for a collapse mechanism similar to the one we analyzed, but with the central hinge line slightly shorter.

4.7.6 Energy dissipated: alternative calculation by the Projection Method

Consider, in general, two rotating portions, 1 and 2, of a slab, with a common yield line, BC, of length L, as shown. As required by compatibility, the axes of rotation and the yield line projected intersect at the point A.



Each portion rotates about its axis by θ_1 and θ_2, respectively. However, from the true view above, the relative rotation across the yield line is $\alpha+\beta$. From the displacement of C

$$d_1\theta_1 = a\alpha; \quad d_2\theta_2 = b\beta$$

But $d_1/a = \cos\phi_1$ and $d_2/b = \cos\phi_2$, and thus

$$\alpha = \theta_1\cos\phi_1; \quad \beta = \theta_2\cos\phi_2$$

$$\Rightarrow \quad \text{E.D. along BC} = mL(\alpha+\beta) = mL\left[\theta_1\cos\phi_1 + \theta_2\cos\phi_2\right]$$

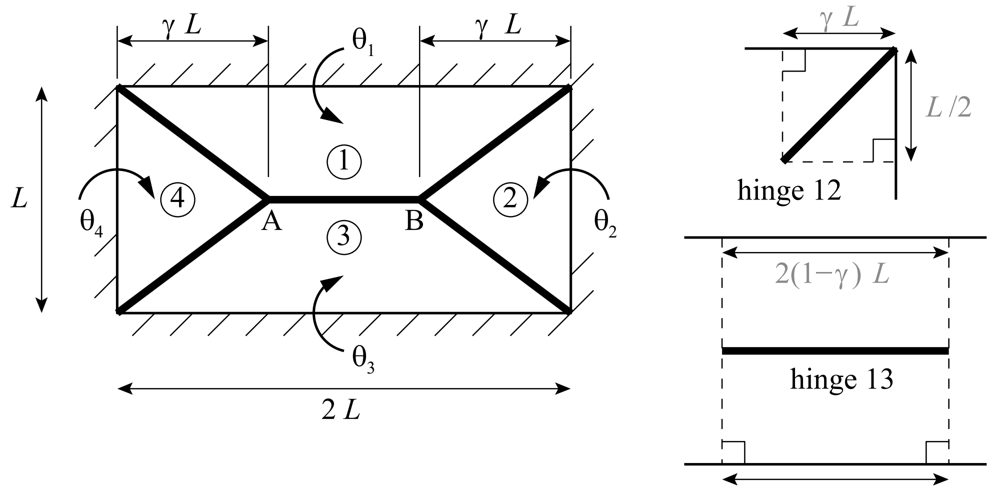
Rearrange as

$$\text{E.D. along BC} = m[\underbrace{\theta_1}_{\substack{\text{rotation}\\ \text{about 1 axis}}} \times \underbrace{(L\cos\phi_1)}_{\substack{\text{projected length}\\ \text{onto 1 axis}}} + \theta_2 \times (L\cos\phi_2)]$$

In general, for an isotropic m

$$\text{total E.D} = m\sum_{\text{all lines}} \left[\text{projected lengths of yield line onto } \textit{adjacent} \text{ axes of rotation}\right] \times \left[\text{rotation about the same axes}\right]$$

Example - simply-supported slab revisited: let the horizontal yield line, AB, have arbitrary length, as dictated by the value of γ.



If the displacement of AB is everywhere δ, then

$$\theta_1 = \frac{2\delta}{L}; \quad \theta_2 = \frac{\delta}{\gamma L}$$

and $\theta_3 = \theta_1$, $\theta_4 = \theta_2$ by symmetry. Using the Projection Method formula

$$\frac{\text{E.D.}}{m} = \underbrace{\left[\gamma L\theta_1 + \frac{L}{2}\theta_2\right]}_{\text{hinge 12}} \times \underbrace{4}_{\substack{\text{all inclined}\\ \text{yield lines}}} + \underbrace{[2(1-\gamma)L \times (\theta_1 + \theta_3)]}_{\text{hinge 13}}$$

Substitute for rotations, and rearrange to give

$$\frac{\text{E.D.}}{m} = \delta\left[\frac{2}{\gamma} + 8\right]$$

For $\gamma = 0.5$, E.D. = $12m\delta$, as before.

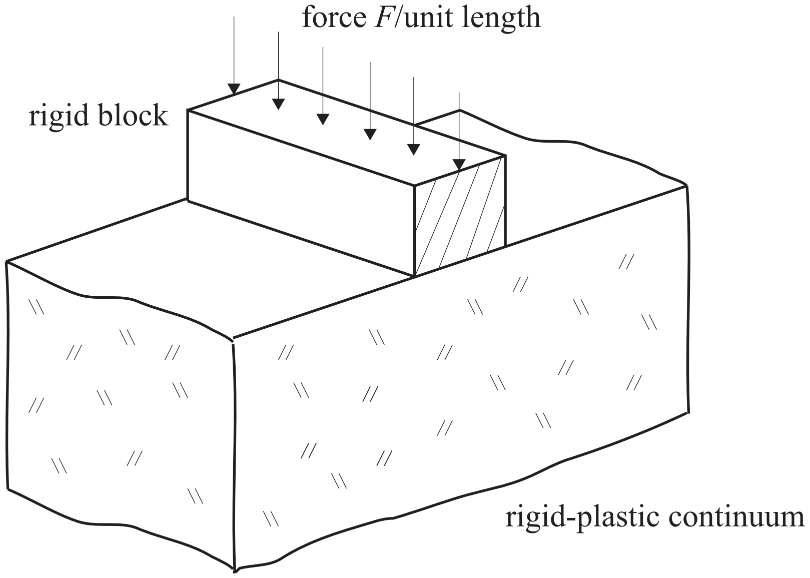
Summary: The Projection Method provides a useful shortcut for calculating the energy dissipated in the hinges of a slab — but has the disadvantage that it hides the underlying mechanics revealed by the vector diagram of rotation. However, ultimately both methods are equivalent.

Try Questions 1,2 and 3, Examples Sheet 2/5

4.8 Slip plane analysis of continua

4.8.1 Introduction

As a final example of an upper bound method, we will look at the failure of rigid-plastic blocks of material (a continuum), for example in the figure below:



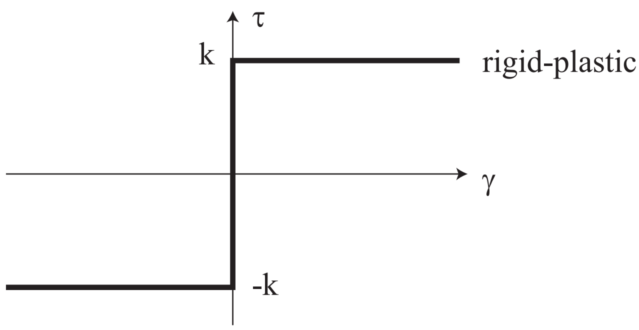
Applications

Soil mechanics: Design of foundations, embankments etc.

Metal-forming: Extrusion, indentation etc.

Assumptions

Rigid-plastic material We shall assume that the material will only fail in shear at a shearing stress k, and is otherwise rigid (This is an example of the *Tresca* yield criterion, which we examined earlier in the course).



Small deformations The usual assumption that the geometry remains essentially unchanged.

All the examples in this course will be 2D plane strain — we will take a unit depth into the page, and assume no deformation out of the plane, but the method is equally applicable to more complex geometries in 3D.

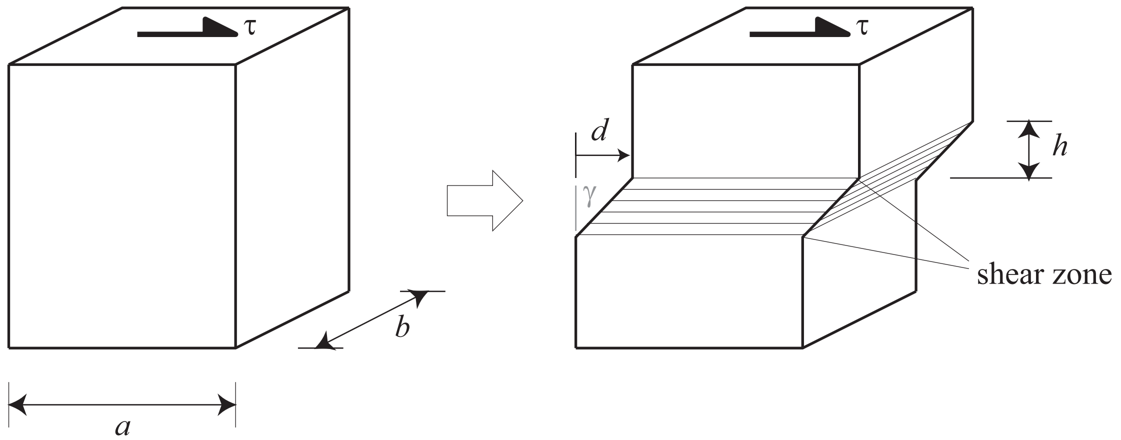
Methodology

We will again use the upper-bound work method, where we postulate a compatible mechanism, and equate the work done by the loads (or possibly self-weight, for e.g. the collapse of an embankment), with the energy dissipated in the material during an incremental deformation.

For slip plane mechanisms (and also for plastic hinges and yield lines), work methods are often written in books using velocities and equating rates of work, and rates of deformation. However we will continue to consider small deformations — our formulation is identical to a rate of work formulation where everything is multiplied by some small time δt.

4.8.2 Energy dissipated in shearing

The energy dissipated in slip plane mechanisms is dissipated in narrow bands of intense shearing, as shown below

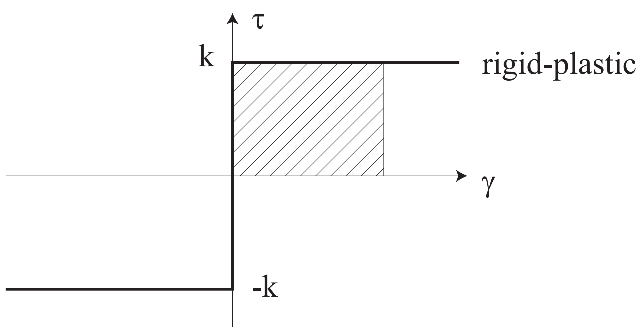


Shear strain in the shear zone

Engineering shear strain is measured as the change in an angle that was originally a right-angle in the material, the angle γ shown above.

$$\gamma = \frac{d}{h}$$

Energy dissipated per unit volume of shear zone



$$\text{E.D.} = \tau\gamma = k\gamma = \frac{kd}{h}$$

Total energy dissipated

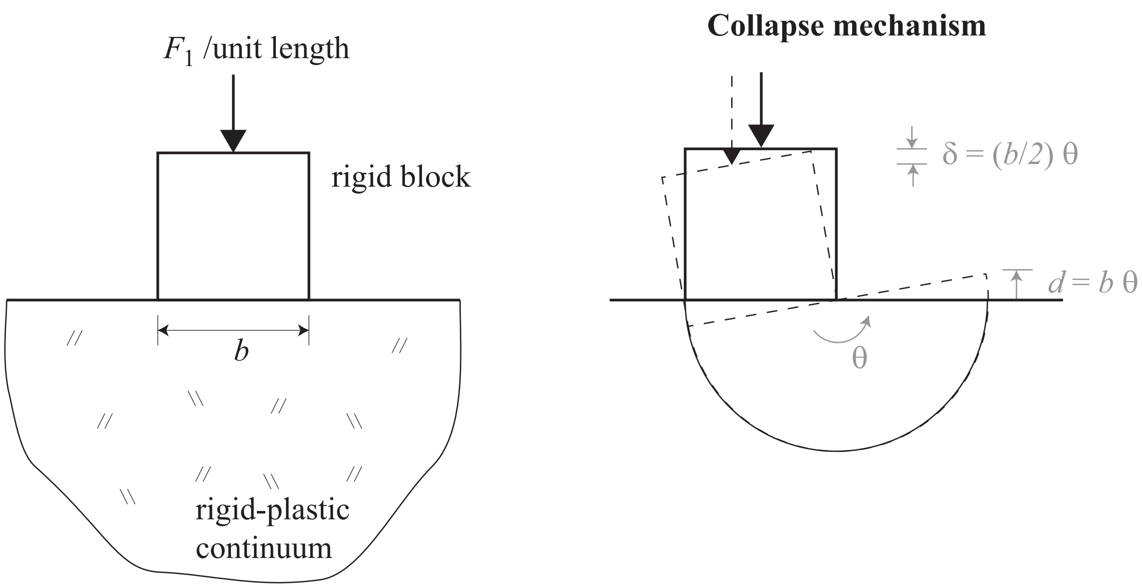
$$\text{Volume of shear zone} = abh$$

$$\text{E.D} = abh \times \frac{kd}{h} = \underbrace{ab}_{\text{area}} \times k \times \underbrace{d}_{\text{displacement}}$$

Note that the energy dissipated does not depend on the thickness of the shear zone — we shall assume that it is infinitesimally thin.

4.8.3 Slip-circle mechanism

One way that the example shown in Section 4.8.1 might fail is by the formation of a slip circle, which is certainly a compatible mechanism:



Consider a unit depth into the page.

Work done by load

$$\text{W.D.} = F_1 \times 1 \times \delta = \frac{Fb\theta}{2}$$

Energy dissipated along slip circle

Area of slip circle $= \pi b \times 1$

Relative displacement along slip circle $= b\theta$

Total energy dissipated $= (\pi b 1) \times k \times (b\theta) = \pi b^2 k\theta$

Work equation

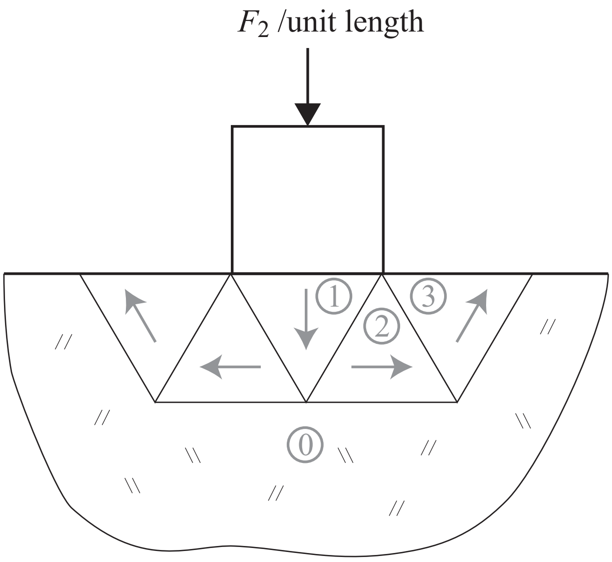
$$\pi b^2 k\theta = \frac{F_1 b\theta}{2}$$

$$F_1 = 2\pi bk$$

$$F_c \leq F_1 = 6.28bk$$

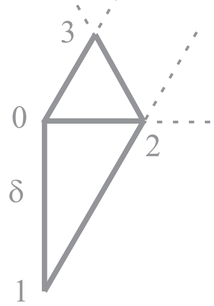
4.8.4 Triangular blocks mechanism

Another class of mechanisms commonly considered is where the slip planes split the material into rigid triangular blocks. In this case, it is necessary to use a displacement diagram to ensure that the mechanism is compatible. Consider a mechanism for the indentation problem where all of the blocks are equilateral triangles.



$$l_{12} = l_{23} = b, \quad l_{02} = l_{03} = b$$

Displacement diagram — Assume the indenter moves down a distance δ.



Relative displacement along slip planes

$$d_{12} = \delta \times \frac{2}{\sqrt{3}}, \quad d_{02} = d_{03} = d_{23} = \delta \times \frac{1}{\sqrt{3}},$$

Consider unit depth into the paper

Work done by load

$$\text{W.D.} = F_2 \times 1 \times \delta$$

Energy dissipated along slip planes

$$E.D. = \underbrace{2\times}_{\text{symmetric}} \left[\underbrace{(1b) \times k \times \frac{2\delta}{\sqrt{3}}}_{12} + \underbrace{(1b) \times k \times \frac{\delta}{\sqrt{3}}}_{02} + \underbrace{(1b) \times k \times \frac{\delta}{\sqrt{3}}}_{03} + \underbrace{(1b) \times k \times \frac{\delta}{\sqrt{3}}}_{23} \right]$$

$$= \frac{10bk\delta}{\sqrt{3}}$$

Work equation

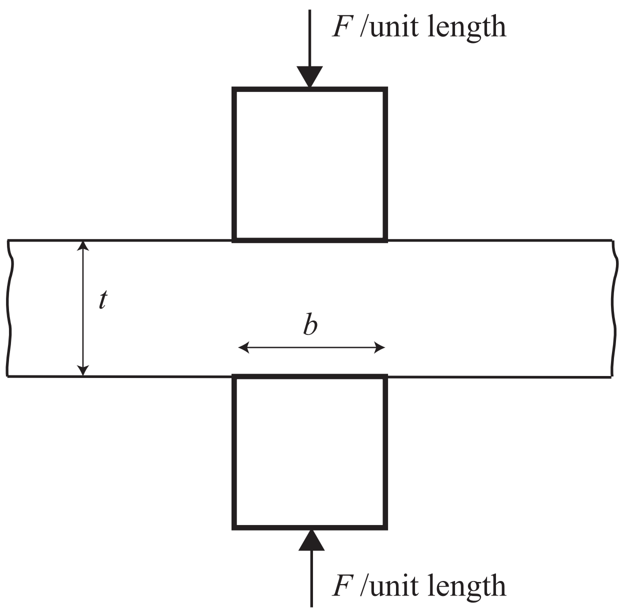
$$F_2\delta = \frac{10bk\delta}{\sqrt{3}}$$

$$F_c \leq F_2 = 5.77bk$$

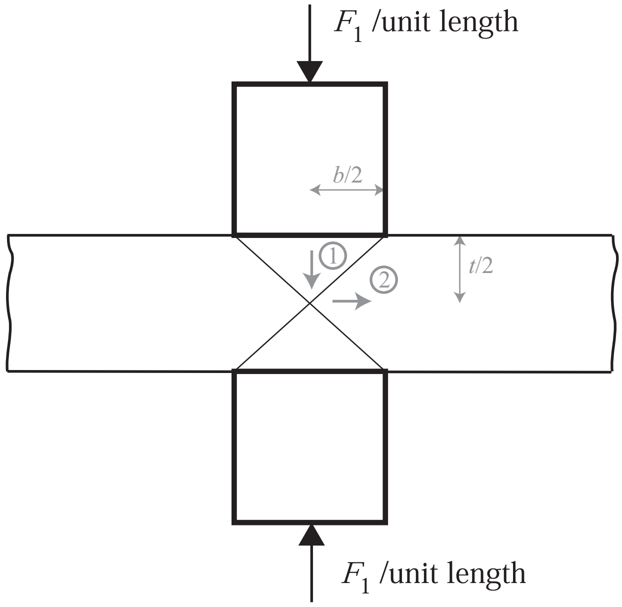
8% lower (better) than F_1 — this is more critical.

4.8.5 Forming Process

Estimate the force/unit length required to indent a rigid-plastic material with yield stress in shear k between two long rigid anvils of breadth b.

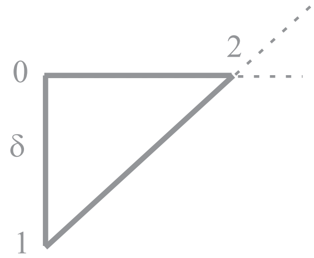


4.8.6 Mechanism 1



$$l_{12} = \left(\frac{b^2+t^2}{4}\right)^{\frac{1}{2}}$$

Displacement diagram — Assume anvils move down/up by δ



By similar triangles with the slip mechanism, above:

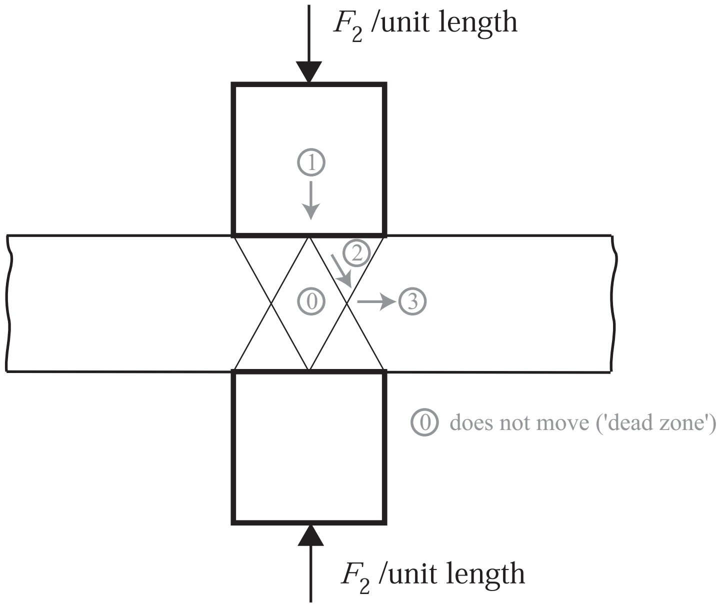
$$d_{02} = \frac{\delta}{t/2} \times \frac{b}{2} = \frac{b\delta}{t}$$

$$d_{12} = \frac{\delta}{t/2} \times l_{12} = \frac{\delta}{t}(b^2+t^2)^{\frac{1}{2}}$$

Work equation (Considering unit depth into the paper)

$$\begin{aligned} 2 \times F_1\delta &= 4 \times l_{12}d_{12}k \\ &= 4\left(\frac{b^2+t^2}{4}\right)^{\frac{1}{2}} \frac{\delta}{t}(b^2+t^2)^{\frac{1}{2}}k \\ F_1 &= \frac{(b^2+t^2)k}{t} \end{aligned}$$

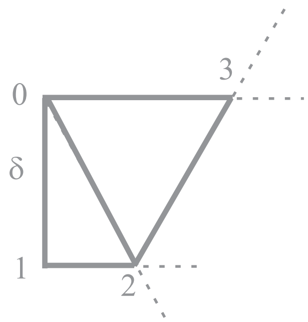
4.8.7 Mechanism 2



$$l_{12} = \frac{b}{2}$$

$$l_{02} = l_{23} = \left[\left(\frac{b}{4}\right)^2 + \left(\frac{t}{2}\right)^2\right]^{\frac{1}{2}} = \frac{(b^2+4t^2)^{\frac{1}{2}}}{4}$$

Displacement diagram — Assume anvils move down/up by δ



By similar triangles with the slip mechanism, above:

$$d_{03} = \frac{\delta}{t/2} \times \frac{b}{2} = \frac{b\delta}{t}$$

$$d_{12} = \frac{\delta b}{2t}$$

$$d_{02} = d_{23} = \frac{\delta}{t/2} \times l_{02} = \frac{\delta}{2t}(b^2+4t^2)^{\frac{1}{2}}$$

A new feature of this mechanism is that it includes sliding between the anvil and the material (interface 12) — and we don't know the properties of this interface. However, we can consider two limiting situations:

Zero friction no energy is dissipated in the interface;

Infinite friction material directly next to the anvil will shear as a normal slip plane.

These limiting cases give us some idea of the effect of friction on the forming process.
Work equation (i) Zero interface friction (considering unit depth into the paper)

$$2 \times F_{2(\mathrm{i})}\,\delta = 4 \times \left[\underbrace{\frac{\delta(b^2+4t^2)k}{8t}}_{02} + \underbrace{\frac{\delta(b^2+4t^2)k}{8t}}_{23} \right]$$

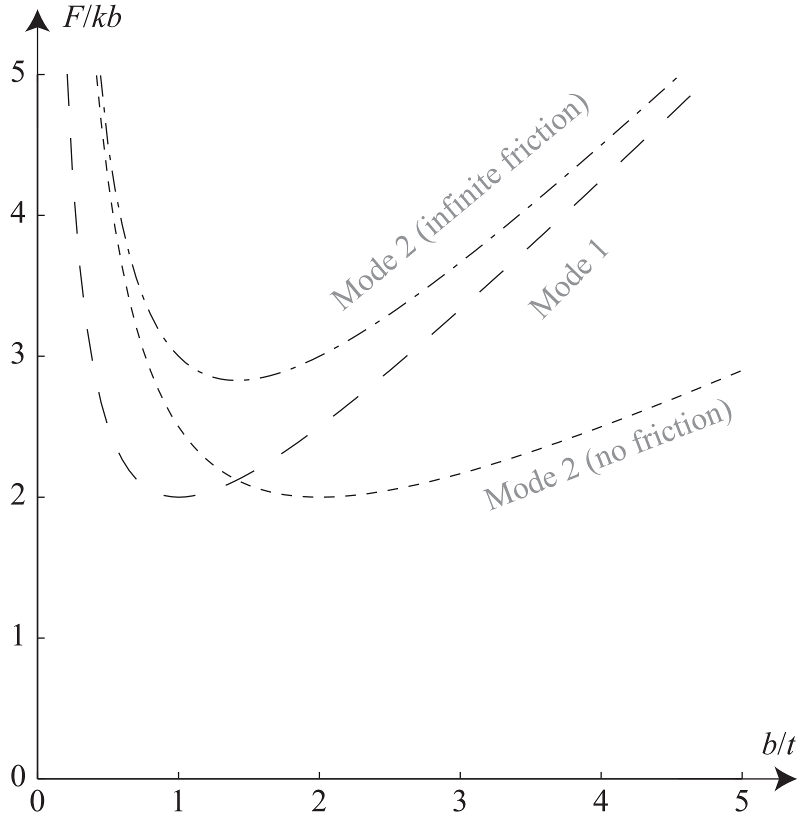
$$F_{2(\mathrm{i})} = \frac{(b^2+4t^2)k}{2t}$$

Work equation (ii) Infinite interface friction (considering unit depth into the paper)

$$2 \times F_{2(\mathrm{ii})}\,\delta = 4 \times \left[\underbrace{\frac{\delta(b^2+4t^2)k}{8t}}_{02} + \underbrace{\frac{\delta(b^2+4t^2)k}{8t}}_{23} + \underbrace{\frac{b}{2}\frac{\delta b}{2t}}_{12}k \right]$$

$$F_{2(\mathrm{ii})} = \frac{(2b^2+4t^2)k}{2t}$$

For different thicknesses of material, these upper-bounds are summarized on the following non-dimensional graph:



Try Questions 4,5 and 6, Examples Sheet 2/5