1B Paper 6: Communications
Handout 4: Digital Baseband Modulation

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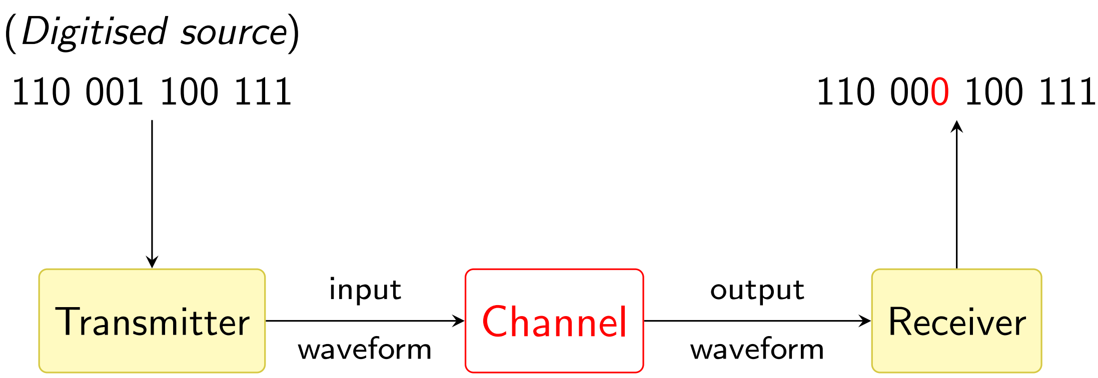
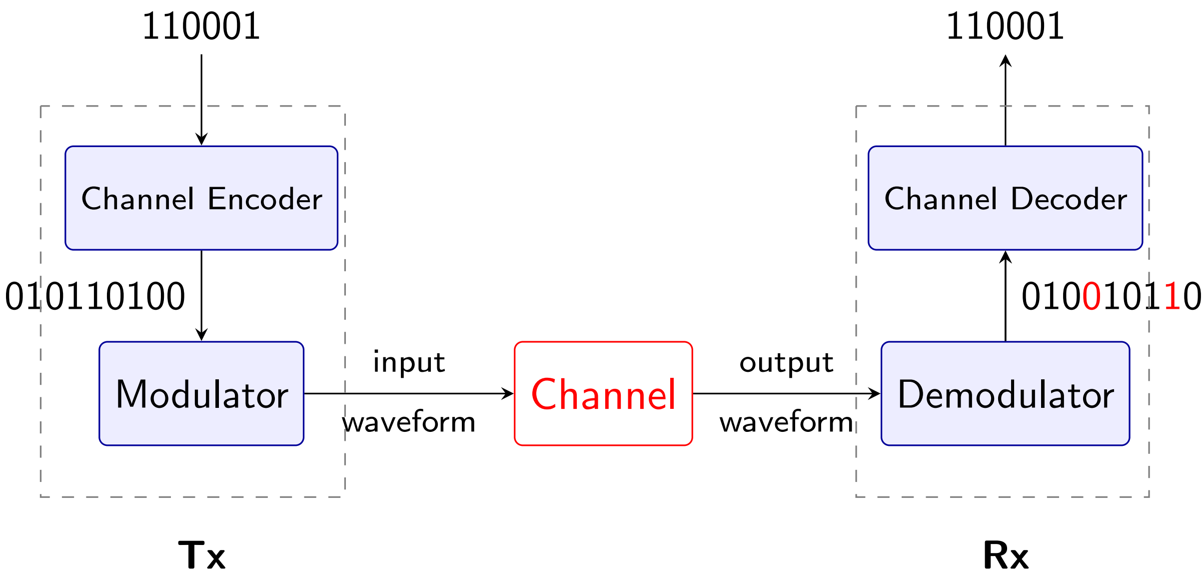
Data Transmission

We have seen how analogue sources can be digitised. E.g., An MPEG or QuickTime file is a stream of bits

$\longleftrightarrow$ $\ldots 10110010001101010 \ldots$

Now we have to transport those bits across a channel:





The transmitter (Tx) does two things:

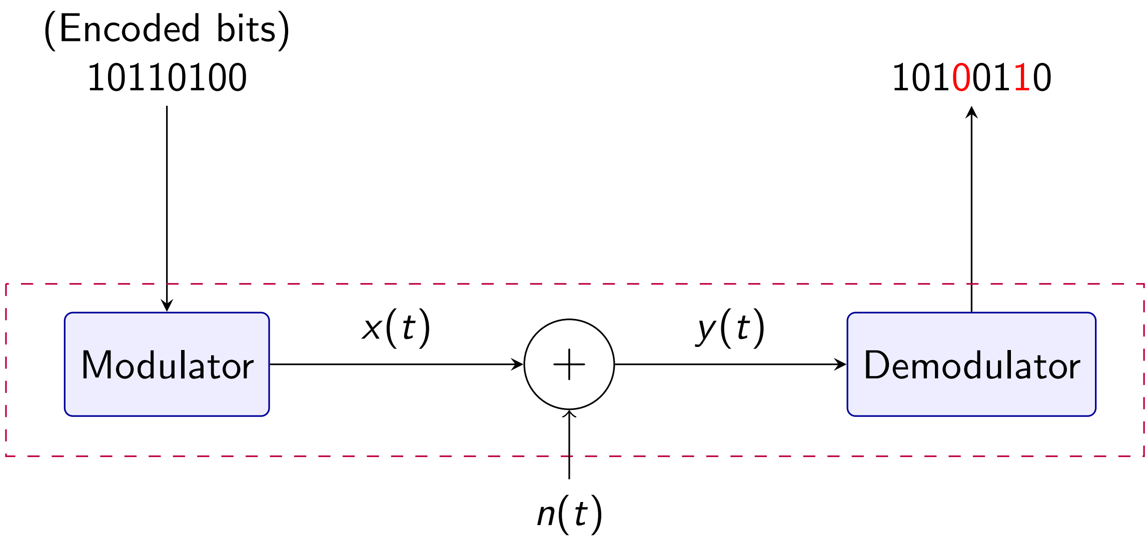
1. *Encoding*: Adding redundancy to the source bits to protect against noise
2. *Modulation*: Transforming the coded bits into waveforms.

The receiver (Rx) does:

- *Demodulation*: noisy output waveform $\longrightarrow$ output bits
- *Decoding*: Try to correct errors in the output bits and recover the source bits

Modulation/Demodulation

We'll first consider the modulation and demodulation blocks assuming that the channel encoder/decoder are fixed, and look at the design of the channel encoder and decoder later.



We now study a digital baseband modulation technique called *Pulse Amplitude Modulation* (PAM) & analyse its performance over an Additive White Gaussian Noise (AWGN) channel

The Symbol Constellation

The digital modulation scheme has two basic components.

1. The first is a mapping from bits to real/complex numbers, e.g.

$$0 \to -A, \quad 1 \to A \quad \text{(binary symbols)}$$
$$00 \to -3A, \quad 01 \to -A, \quad 10 \to A, \quad 11 \to 3A \quad \text{(4-ary symbols)}$$

The set of values the bits are mapped to is called the *constellation*, e.g., the 4-ary constellation above is $\{-3A, A, A, 3A\}$.

Once we fix a constellation, a sequence of bits can be uniquely mapped to constellation symbols. E.g., with constellation $\{-A, A\}$

$$0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0 \longrightarrow -A,\ A, -A,\ A,\ A,\ A, -A, -A,\ A, -A$$

With constellation $\{-3A, -A, A, 3A\}$, the same sequence of bits is mapped as $\underline{01}\ \underline{01}\ \underline{11}\ \underline{00}\ \underline{10} \longrightarrow -A,\ -A,\ 3A,\ -3A,\ A$

In a constellation with M symbols, each symbol represents $\log_2 M$ bits

The Pulse Shape

2. The second component of Pulse Amplitude Modulation is a unit-energy *baseband* waveform denoted $p(t)$, called the *pulse shape*. E.g., a sinc pulse or a rect pulse:

$$p(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{\pi t}{T}\right) \quad \text{or} \quad p(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } t \in (-\frac{T}{2}, \frac{T}{2}] \\ 0 & \text{otherwise} \end{cases}$$

T is called the *symbol* time of the pulse

A sequence of constellation symbols $X_0, X_1, X_2, \dots$ is used to generate a *baseband* signal as follows

$$x_b(t) = \sum_k X_k p(t - kT)$$

Thus we have the following **important** steps to associate bits with a baseband signal $x_b(t)$:

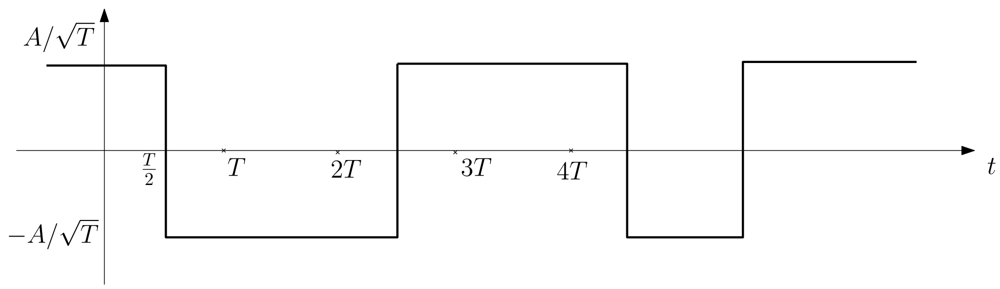
$$\dots 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0 \dots \longrightarrow X_0, X_1, X_2, \dots \longrightarrow \sum_k X_k p(t - kT)$$

Rate of Transmission

The modulated baseband signal is $x_b(t) = \sum_k X_k p(t - kT)$.
With the rect pulse shape

$$p(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } t \in \left(-\frac{T}{2}, \frac{T}{2}\right] \\ 0 & \text{otherwise} \end{cases}$$

and $X_k \in \{+A, -A\}$, $x_b(t)$ looks like



Every T seconds, a new symbol is introduced by shifting the pulse and modulating its amplitude with the symbol.

The *transmission rate* is $\frac{1}{T}$ **symbols/sec** or $\frac{\log_2 M}{T}$ **bits/second**

Desirable Properties of the Pulse Shape $p(t)$

$p(t)$ is chosen to satisfy the following important objectives:

1. We want $p(t)$ to decay quickly in time, i.e., the effect of *symbol* X_k should not start much before $t = kT$ or last much beyond $t = (k+1)T$

2. We want $p(t)$ to be approximately band-limited.
   For a fixed sequence of symbols $\{X_k\}$, the spectrum of $x_b(t)$ is

$$X_b(f) = \mathcal{F}\left[\sum_k X_k p(t - kT)\right] = P(f) \sum_k X_k \, e^{-j2\pi f k T}$$

Hence the bandwidth of $x_b(t)$ is the same as that of the pulse $p(t)$

3. The retrieval of the information sequence from the *noisy* received waveform $x_b(t) + n(t)$ should be simple and relatively reliable. In the absence of noise, the symbols $\{X_k\}_{k \in \mathbb{Z}}$ should be recovered perfectly at the receiver.

Orthonormality of pulse shifts

Consider the third objective, namely, simple and reliable detection.

To achieve this, the pulse is chosen to have the following "**orthonormal shifts**" property:

$$\int_{-\infty}^{\infty} p(t-kT)p(t-mT)\,dt = \begin{cases} 1 & \text{if } k=m \\ 0 & \text{if } k \neq m \end{cases} \qquad (1)$$

We'll see how this property makes signal detection at the Rx simple

- This property is satisfied by the rect pulse shape

$$p(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } t \in \left(\frac{-T}{2}, \frac{T}{2}\right] \\ 0 & \text{otherwise} \end{cases}$$

- The sinc pulse $p(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{\pi t}{T}\right)$ also has orthonormal shifts! (You will show this in Examples Paper 9, Q.2)

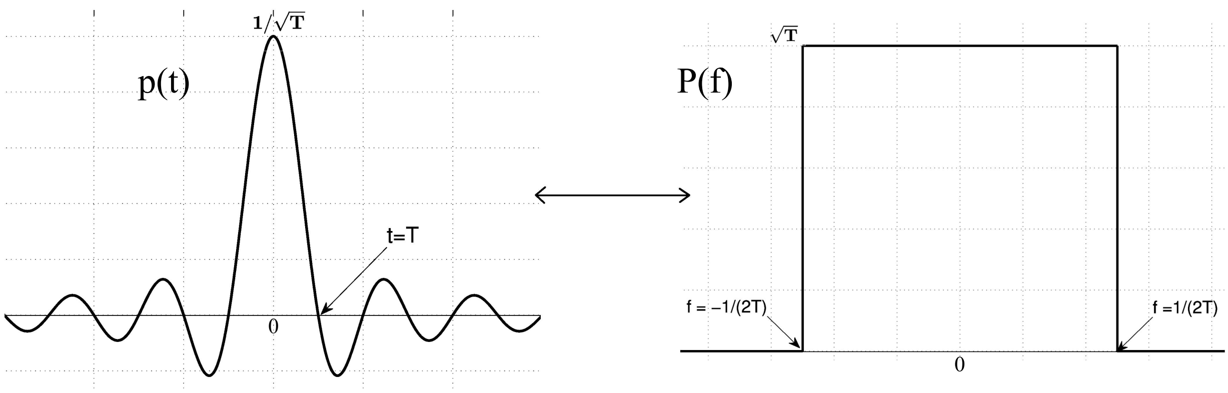
Time Decay vs. Bandwidth Trade-off

The first two objectives say that we want $p(t)$ to:

1. Decay quickly in time
2. Be approximately band-limited

But ... faster decay in time $\Leftrightarrow$ larger bandwidth

Consider the pulse $p(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{\pi t}{T}\right)$

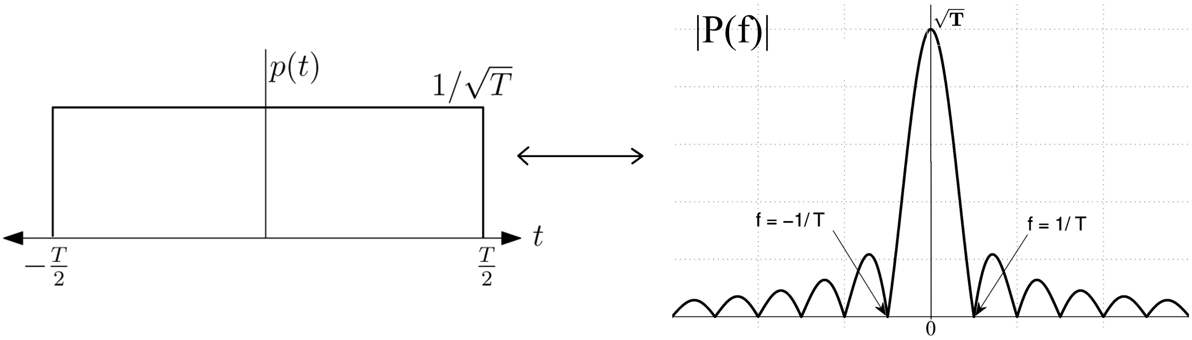


The sinc is perfectly band-limited to $W = \frac{1}{2T}$

But decays slowly in time $|p(t)| \sim \frac{1}{|t|}$

Next consider the rectangular pulse

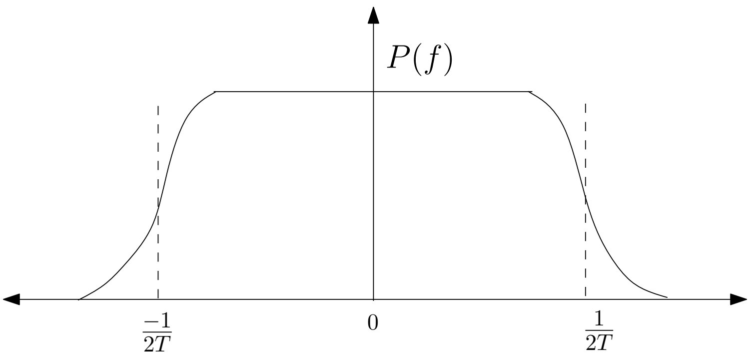
$$p(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } t \in \left(-\frac{T}{2}, \frac{T}{2}\right] \\ 0 & \text{otherwise} \end{cases}$$



This pulse is perfectly time-limited to the interval $[-T/2, T/2)$.
But ...

- Decays slowly in freq. $|P(f)| \sim \frac{1}{|f|}$
- Main-lobe bandwidth $= \frac{1}{T}$

In practice, the pulse shape is often chosen to have a *root raised cosine* spectrum



Bandwidth slightly larger than $\frac{1}{2T}$; decay in time $|p(t)| \sim \frac{1}{|t|^2}$

A happy compromise!

- More on raised cosine pulses in 3F4
- For intuition, it often helps to envision $x_b(t)$ with a rect pulse, though it is never used in practice

Reminder

Data → constellation symbols → continuous waveform

Thus we have the following **important** steps to associate bits with a baseband signal $x_b(t)$:

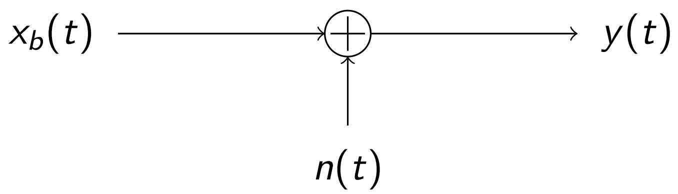
$$\ldots 0\,1\,0\,1\,1\,1\,0\,0\,1\,0 \ldots \longrightarrow X_0, X_1, X_2, \ldots \longrightarrow \sum_k X_k p(t-kT)$$

PAM Demodulation

Now, assume that we have picked a constellation and a pulse shape satisfying the objectives, and we transmit the baseband waveform

$$x_b(t) = \sum_k X_k p(t - kT)$$

over a *baseband* channel $y(t) = x_b(t) + n(t)$



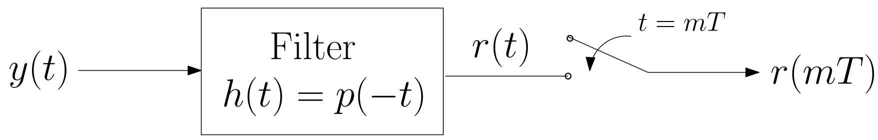
How does the receiver recover the information symbols $\{X_0, X_1, X_2, \ldots\}$ from $y(t)$?

- This process is called *demodulation*
- We will see that the orthonormal shift property of $p(t)$ leads to a simple and elegant demodulator

Matched Filter Demodulator

Let us first understand the operation assuming no noise, i.e.,

$$y(t) = x_b(t) = \sum_k X_k p(t - kT)$$

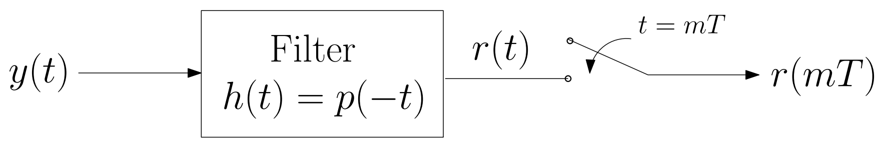


$y(t)$ is passed through a filter with impulse response $h(t) = p(-t)$

This is called a **matched filter**. The filter output is

$$\begin{aligned} r(t) = y(t) \star h(t) &= x_b(t) \star h(t) \quad \text{(assuming no noise)} \\ &= \int_{-\infty}^{\infty} x_b(\tau) h(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} x_b(\tau) p(\tau - t) d\tau \\ &= \sum_k X_k \int_{-\infty}^{\infty} p(\tau - kT) p(\tau - t) d\tau \end{aligned}$$

Matched filter output



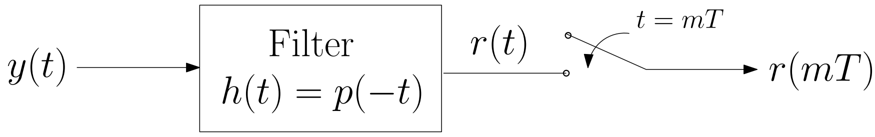
$$\begin{aligned} r(t) &= \sum_k X_k \int_{-\infty}^{\infty} p(\tau - kT) p(\tau - t) d\tau \\ &= \sum_k X_k \int_{-\infty}^{\infty} p(u + t - kT) p(u)\, du \quad \text{(using } u = \tau - t\text{)} \\ &= \sum_k X_k g(t - kT) \end{aligned}$$

where

$$g(t) = \int_{-\infty}^{\infty} p(u + t) p(u)\, du$$

Matched filter output $r(t)$ is of the form as the PAM signal, except that the 'pulse' is now $g(t)$

Sampled matched filter output



By sampling the filter output at time $t = mT$, $m = 0, 1, 2, \ldots$, you get

$$r(mT) = \sum_k X_k \, g((m-k)T)$$

Because of the *orthonormal shifts* property of $p(t)$

$$g((m-k)T) = \int_{-\infty}^{\infty} p(u + (m-k)T) \, p(u) \, du = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}$$

Therefore,

$$r(mT) = \sum_k X_k \, g((m-k)T) = X_m$$

Orthonormal shifts property is crucial for this demodulator to work!

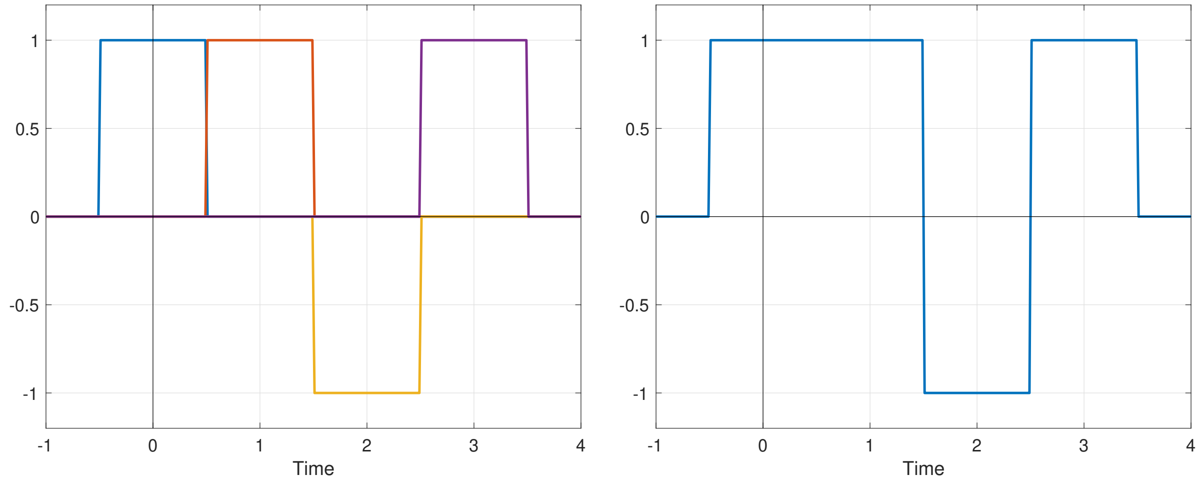
For two different choices of pulse $p(t)$, we now visualize

- the transmitted PAM signal $\sum_k X_k p(t - kT)$, and
- the matched filter output $r(t) = \sum_k X_k g(t - kT)$

Example 1: Transmitted signal

Rectangular pulse: $p(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } t \in \left(-\frac{T}{2}, \frac{T}{2}\right] \\ 0 & \text{otherwise} \end{cases}$

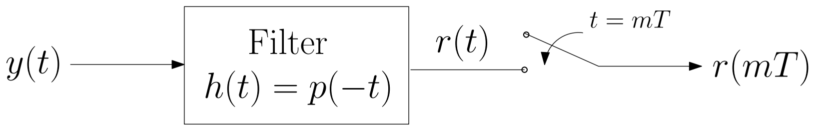
Assume $T = 1$ and the symbols $\{X_0, X_1, X_2, X_3\} = \{1, 1, -1, 1\}$



Right panel shows the transmitted PAM signal $\sum_{k=0}^{3} X_k p(t - kT)$

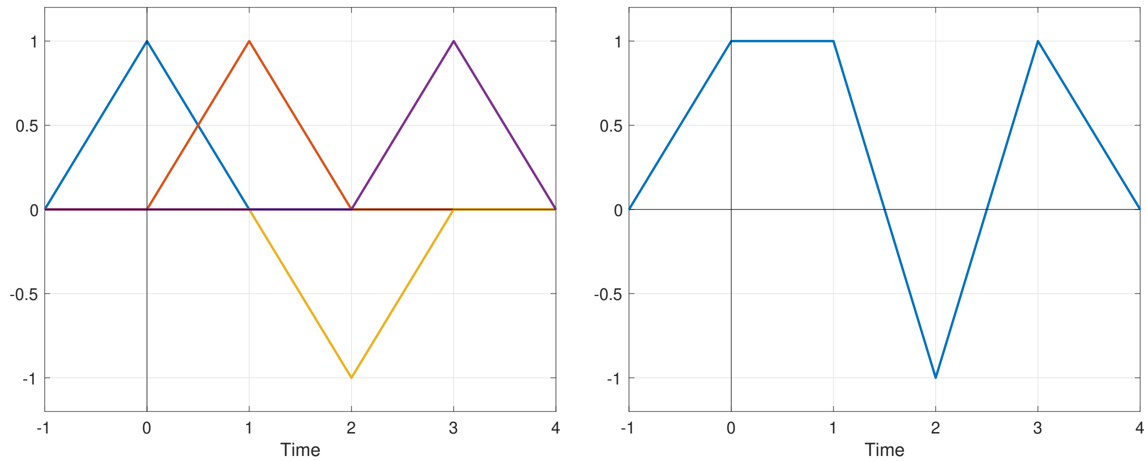
Left panel shows each component of the sum separately

Example 1: Matched filter output



Matched filter output $r(t) = \sum_{k=0}^{3} X_k g(t - kT)$

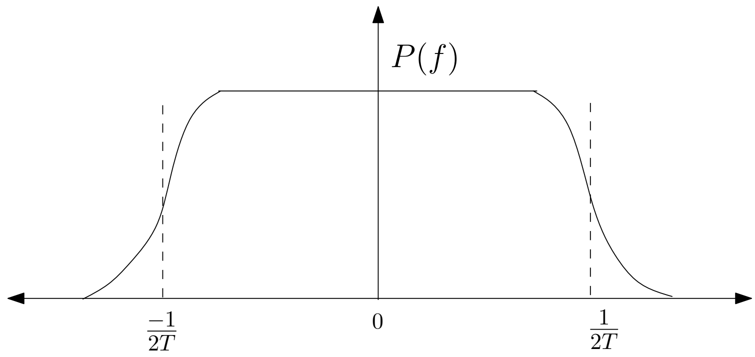
$$g(t) = \int_{-\infty}^{\infty} p(u+t)p(u)du = \begin{cases} 1 + \frac{t}{T}, & -T \le t \le 0, \\ 1 - \frac{t}{T}, & 0 \le t \le T \end{cases}$$



Left: each component of the sum separately, Right: $r(t)$

Example 2

A practical choice: Root-raised cosine pulse $p(t)$

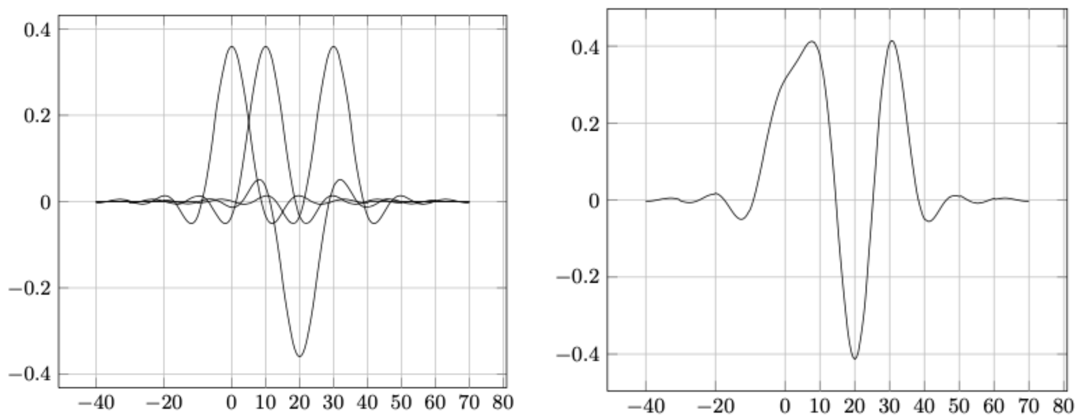


Assume $T = 10$ and the symbols $\{X_0, X_1, X_2, X_3\} = \{1, 1, -1, 1\}$

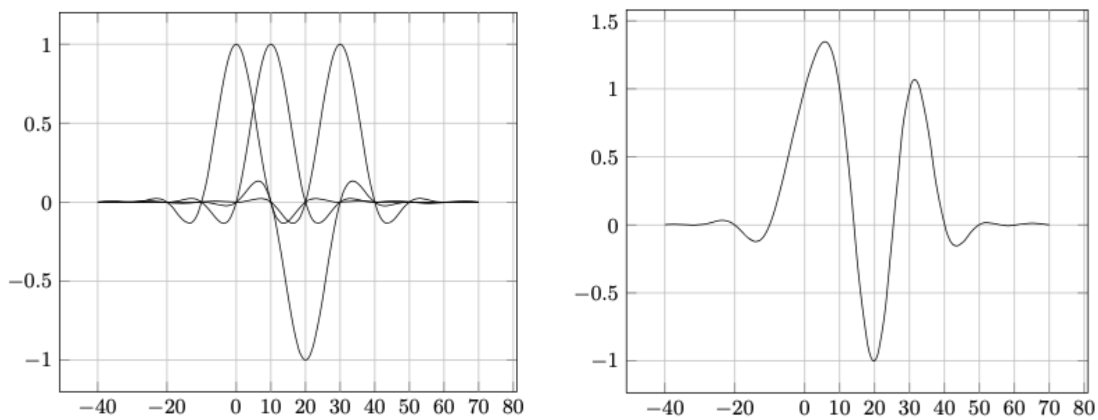
Let us visualise :

- the transmitted PAM signal $\sum_{k=0}^{3} X_k p(t - kT)$, and
- the matched filter output $r(t) = \sum_{k=0}^{3} X_k g(t - kT)$

Transmitted PAM signal $\sum_{k=0}^{3} X_k p(t - kT)$:



Matched filter output $r(t) = \sum_{k=0}^{3} X_k g(t - kT)$:

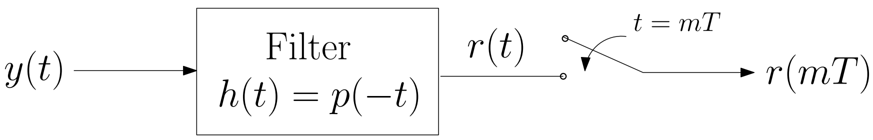


Figures from *Principles of Digital Communication* by B. Rimoldi, CUP 2016.

What happens when there is noise at the receiver?

Demodulation with Noisy $y(t)$

Now consider the noisy case. The receiver gets $y(t) = x(t) + n(t)$



The matched filter output is

$$r(t) = y(t) \star h(t) = x_b(t) \star h(t) \; + \; n(t) \star h(t)$$
$$= \sum_k X_k \, g(t - kT) \; + \; \int_{-\infty}^{\infty} n(\tau) p(\tau - t) d\tau$$

Sampling at $t = mT$, $m = 0, 1, 2, \ldots$, we now get

$$r(mT) = X_m + N_m$$

where N_m is noise part of the filter output at time mT:

$$N_m = \int_{-\infty}^{\infty} n(\tau) p(\tau - mT) d\tau$$

Properties of the Noise

Let us denote $r(mT)$, the sampled output at time mT, by Y_m.

$$Y_m = X_m + N_m, \quad m = 0, 1, 2, \ldots$$

Note that this is a *discrete-time channel*. We have converted the continuous-time problem into a discrete-time one of detecting the symbols X_m from the noisy outputs Y_m.

- To do this, we first need to understand the properties of the noise N_m. Recall that

$$N_m = \int_{\infty}^{\infty} n(\tau) p(\tau - mT) d\tau$$

- N_m is a random variable whose distribution depends on the statistics of the *random process* $n(t)$.

You will learn about random processes and their characterisation in 3F1 & 3F4, but this is outside the scope of this course. For now, we will directly specify the distribution of N_m and analyse the detection problem.

$$Y_m = X_m + N_m, \quad m = 0, 1, 2, \ldots$$

Modelling $n(t)$ as a Gaussian process leads to the following **important** characterisation of N_m:

- For each m, N_m is a *Gaussian random variable* with zero mean, and variance σ^2 that can be estimated empirically
- Further $N_1, N_2, \ldots$ are *independent*
- Thus the sequence of random variables $\{N_m\}, m = 0, 1, \ldots$ are independent and identically distributed as $\mathcal{N}(0, \sigma^2)$.

Detection

- At the Rx, how do we detect the information symbol X_m from Y_m for $m = 0, 1, \ldots$?
- Remember that each X_m belongs to the *symbol constellation*

Detection for Binary PAM

Let's start with a simple binary constellation, then generalise.

Consider a constellation where each $X_m \in \{-A, A\}$. This is called *binary* PAM or BPSK ('Binary Phase Shift Keying')

$$Y = X + N$$

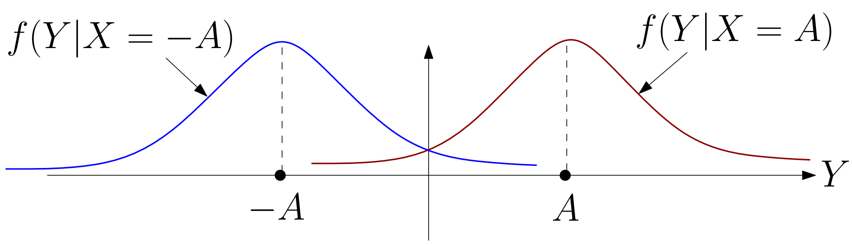
The detection problem is now:
Given Y, how to decide whether $X = A$ or $X = -A$?

Observe that:

$$Y = A + N \text{ if } X = A \quad \text{and} \quad Y = -A + N \text{ if } X = -A$$

- N is distributed as $\mathcal{N}(0, \sigma^2)$
- Therefore the pdf $f(Y|X = A)$ is Gaussian with mean A and variance σ^2
- Similarly the pdf $f(Y|X = -A)$ is Gaussian with mean $-A$ and variance σ^2

Note: Adding a constant to a random variable just shifts the mean, does not change the shape of the distribution

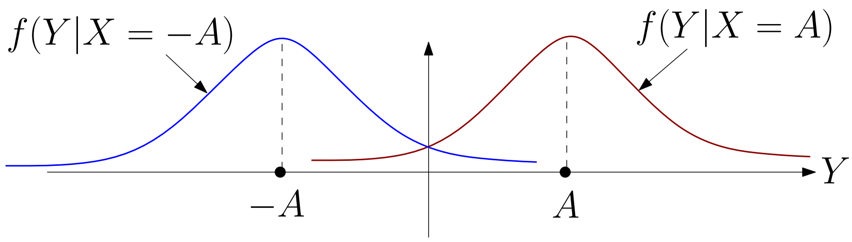


Let $\hat{X}$ denote the decoded symbol. When the symbols A and $-A$ are *a priori* equally likely, the optimal detection rule is:

$$\hat{X} = A \ \text{ if } f(Y \mid X = A) \geq f(Y \mid X = -A)$$
$$\hat{X} = -A \ \text{ if } f(Y \mid X = -A) > f(Y \mid X = A)$$

"*Choose the symbol from which Y is most likely to have occurred*"

- This decoder is called the **maximum-likelihood decoder**
- This decoder is intuitive and seems sensible, and is in fact, the optimal detection rule when all the constellation symbols are equally likely (we will not prove this here)
- It is then a special case of the Maximum a Posteriori (MAP) detection rule, which minimises the probability of detection error (discussed in 3F4)



The detection rule can be compactly written as

$$\hat{X} = \underset{x \in \{A, -A\}}{\arg\max} \; f(Y|X=x)$$

$$\begin{aligned}\hat{X} &= \underset{x \in \{A, -A\}}{\arg\max} \; f(Y|X=x) \\ &= \underset{x \in \{A, -A\}}{\arg\max} \; \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(Y-x)^2/2\sigma^2} \quad = \quad \underset{x \in \{A, -A\}}{\arg\min} \; (Y-x)^2\end{aligned}$$

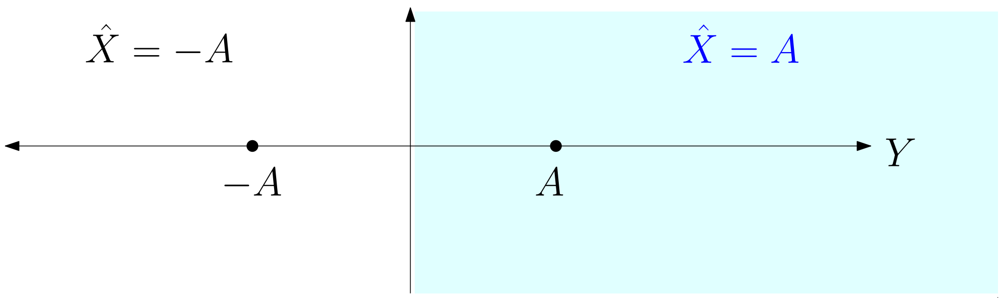
Thus the detection rule is just: $\hat{X} = A$ if $Y \geq 0$, $\hat{X} = -A$ if $Y < 0$

"Choose the constellation symbol closest to the output Y"

Decision Regions

The detection rule partitions the space of Y (the real line) into **decision regions**.

For binary PAM, we just derived the following decision regions:



Q: When does the detector make an error?

A: When $X = A$ and $Y < 0$, or When $X = -A$ and $Y > 0$

We will calculate the probability of error shortly, but let's first find the detection rule for general PAM constellations

Detection for General PAM Constellations

The detection rule can easily be extended to a general constellation $\mathcal{C}$

- E.g., $\mathcal{C}$ may be the 3-ary constellation $\{-2A, 0, 2A\}$ or a 4-ary constellation $\{-3A, -A, A, 3A\}$
- The maximum-likelihood principle is the same: "*Choose the constellation symbol from which y is most likely to have occurred*"

$$
\begin{aligned}
\hat{X} &= \arg\max_{x \in \mathcal{C}} f(Y|X = x) \\
&= \arg\max_{x \in \mathcal{C}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(Y-x)^2/2\sigma^2} = \arg\min_{x \in \mathcal{C}} (Y - x)^2
\end{aligned}
$$

Thus, the detection rule for *any* PAM constellation boils down to:
"*Choose the constellation symbol closest to the output Y*"

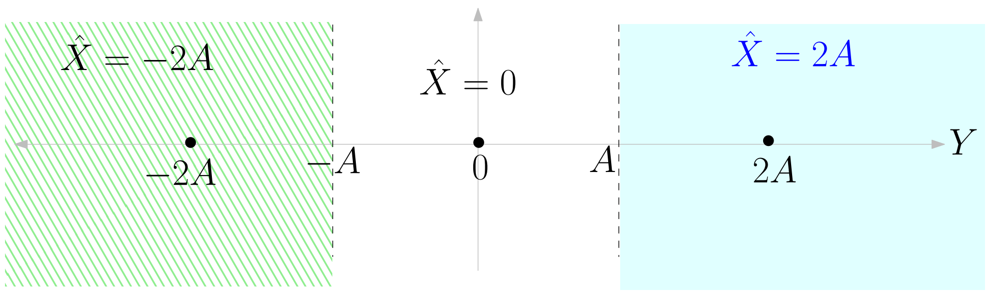
Example: 3-ary PAM

$$
Y = X + N, \quad N \sim \mathcal{N}(0, \sigma^2)
$$

What is the optimal detection rule and the associated decision regions if X belongs to the 3-ary constellation $\{-2A, 0, 2A\}$?

The "nearest symbol to Y" decoding rule yields

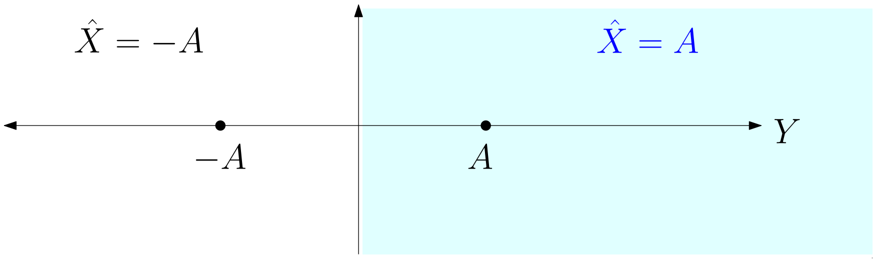
$$
\hat{X} = \begin{cases} -2A & \text{if } Y < -A \\ 0 & \text{if } -A \leq Y < A \\ 2A & \text{if } Y > A \end{cases}
$$



Probability of Detection Error

$$Y = X + N$$

Consider binary PAM with $X \in \{A - A\}$. The decision regions are:



The detector makes an error when $X = A$ and $Y < 0$, or when $X = -A$ and $Y > 0$

The probability of detection error is

$$\begin{aligned} P_e &= P(\hat{X} \neq X) \\ &= P(X = -A)P(\hat{X} = A | X = -A) + P(X = A)P(\hat{X} = -A | X = A) \\ &= \tfrac{1}{2}P(\hat{X} = A \mid X = -A) + \tfrac{1}{2}P(\hat{X} = -A \mid X = A) \end{aligned}$$

(The symbols are equally likely $\Rightarrow P(X = A) = P(X = -A) = \frac{1}{2}$)

Let us first examine $P(\hat{X} = A \mid X = -A)$

$$\begin{aligned} P(\hat{X} = A \mid X = -A) &= P(Y > 0 \mid X = -A) \\ &= P(-A + N > 0 \mid X = -A) \\ &= P(N > A \mid X = -A) \overset{(a)}{=} P(N > A) \end{aligned}$$

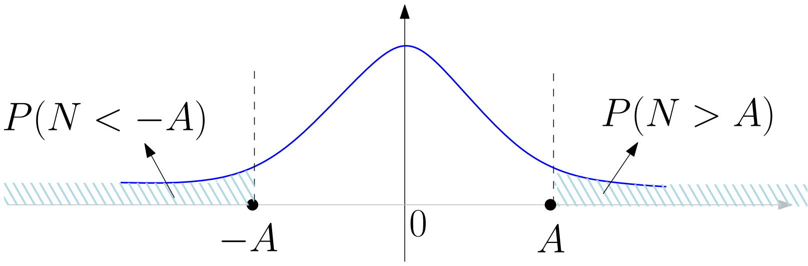
(a) is true because the noise random variable N is **independent** of the transmitted symbol X. Similarly,

$$\begin{aligned} P(\hat{X} = -A \mid X = A) &= P(Y < 0 \mid X = A) \\ &= P(A + N < 0 \mid X = A) \\ &= P(N < -A \mid X = A) = P(N < -A) \end{aligned}$$

The probability of detection error is therefore

$$\begin{aligned} P_e &= \tfrac{1}{2}P(\hat{X} = A \mid X = -A) + \tfrac{1}{2}P(\hat{X} = -A \mid X = A) \\ &= \tfrac{1}{2}P(N > A) \; + \; \tfrac{1}{2}P(N < -A) \\ &\overset{(b)}{=} P(N > A) \overset{(c)}{=} P\left(\frac{N}{\sigma} > \frac{A}{\sigma}\right) \end{aligned}$$

- (b) holds due to the symmetry of the Gaussian pdf $\mathcal{N}(0, \sigma^2)$:

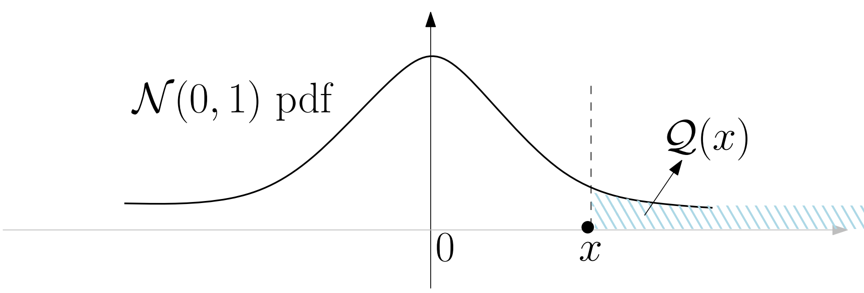


- In (c), we have expressed the probability in terms of a **standard** Gaussian random variable with distribution $\mathcal{N}(0, 1)$
- Recall from 1B Paper 7 (Probability) that if N is distributed as $\mathcal{N}(0, \sigma^2)$ then $\frac{N}{\sigma}$ is distributed as $\mathcal{N}(0, 1)$

The $\mathcal{Q}$-function

The error probability is usually expressed in terms of the $\mathcal{Q}$-function, which is defined as:

$$\mathcal{Q}(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$



- $\mathcal{Q}(x)$ is the probability that a **standard Gaussian** $\mathcal{N}(0, 1)$ random variable takes value greater than x
- Also note that $\mathcal{Q}(x) = 1 - \Phi(x)$, where $\Phi(.)$ is the cdf of a $\mathcal{N}(0, 1)$ random variable

P_e in terms of the signal-to-noise ratio

The probability of detection error is therefore

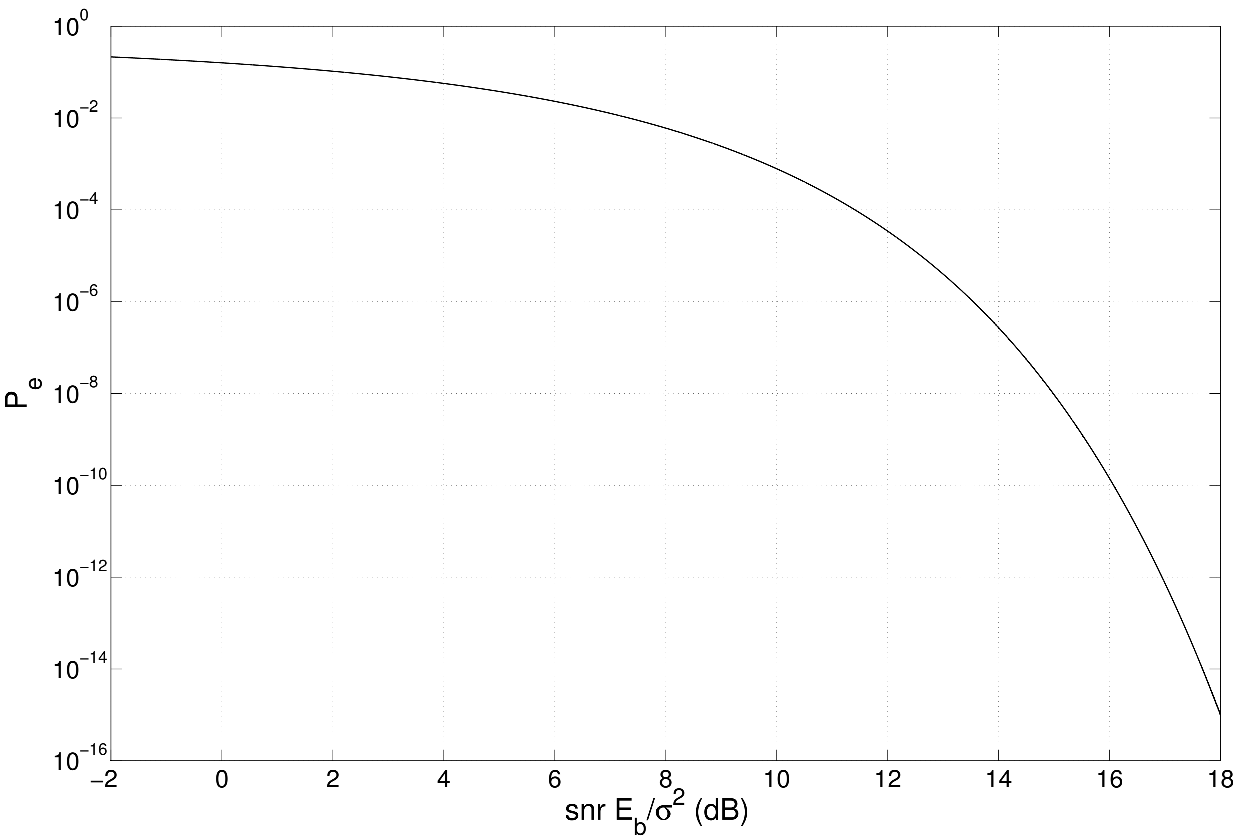
$$P_e = P(N > A) = P\left(\frac{N}{\sigma} > \frac{A}{\sigma}\right) = \mathcal{Q}\left(\frac{A}{\sigma}\right) = \mathcal{Q}\left(\sqrt{\frac{E_s}{\sigma^2}}\right)$$

where E_s is the *average energy per symbol* of the constellation:

$$E_s = \frac{1}{2}(A^2 + (-A)^2) = A^2$$

- For a binary constellation, each symbol corresponds to 1 bit. $\Rightarrow$ the *average energy per bit* E_b is also equal to A^2 in this case
- For an M-point constellation, the average energy per symbol $E_s = E_b \log_2 M$
- $\frac{E_b}{\sigma^2}$ is called the signal-to-noise ratio (snr) of the transmission scheme
- P_e can be plotted as a function of the snr $\frac{E_b}{\sigma^2}$ ...

P_e vs snr for binary PAM



To get P_e of 10^{-3}, we need *snr* $E_b/\sigma^2 \approx 9$ dB

Error Probability vs Transmit Power

The probability of error for binary PAM decays rapidly as *snr* $\uparrow$:

- $\mathcal{Q}(x) \approx e^{-x^2/2}$ for large $x > 0 \quad \Rightarrow \; P_e \approx e^{-\mathsf{snr}/2}$

Can we set the snr $\frac{E_b}{\sigma^2}$ to be as high as we want, by increasing E_b? (i.e., by increasing A since $E_b = E_s = A^2$ for binary PAM)

- The problem is that transmitted power also increases!
- Intuition: 1 symbol transmitted every T seconds with average energy $E_s \Rightarrow$ transmit power is E_s/T
- Thus as you increase the snr, you battery drains faster!

Power of PAM signal

$$x_b(t) = \sum_k X_k \, p(t - kT)$$

With any constellation the power of the baseband PAM signal $x_b(t)$ is

$$\frac{E_s}{T} = \frac{E_b \log_2 M}{T},$$

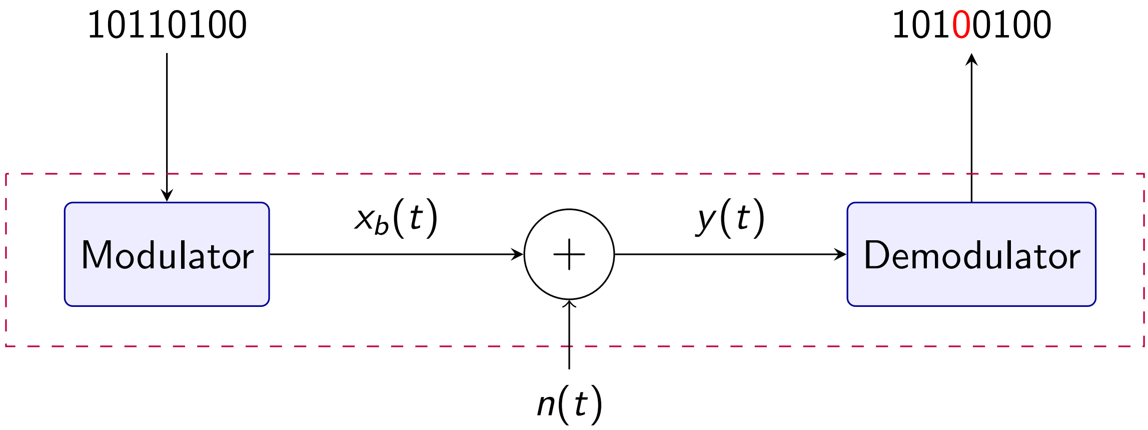
where

- E_s is the average symbol energy of the constellation.
- E_b is the average energy per bit

Intuition:

- In each symbol period of length T, a symbol with average energy E_s modulates a unit energy pulse
- A rigorous calculation of power has to take into account the fact that the transmitted symbols $X_1, X_2, \ldots$ are drawn *randomly* from the constellation (done in 3F4)

Pulse Amplitude Modulation - The Key Points



PAM is a way to map a sequence of information bits to a *continuous-time baseband* waveform

1. Pick a constellation, map the information bits to symbols $X_1, X_2, \ldots$ in the constellation
2. These symbols then modulate the amplitude of a *pulse shape* $p(t)$ to generate the baseband waveform $x_b(t)$

$$x_b(t) = \sum_k X_k p(t - kT)$$

Desirable properties of the pulse shape $p(t)$:

- $p(t)$ should decay quickly in time; its bandwidth W shouldn't be too large
- *Orthonormal* shifts property for simple and reliable decoding

At the receiver, first **demodulate** then **detect**:

- The demodulator is a *matched filter* with IR $h(t) = p(-t)$
- Matched filter output is *sampled* at times $\ldots, 0, T, 2T, \ldots$. At time mT, the output is

$$Y_m = X_m + N_m$$

  N_m is Gaussian noise with zero mean and variance σ^2 that can be empirically estimated
- Detection rule: $\hat{X}_m =$ the constellation symbol closest to Y_m

Probability of detection error can be calculated:

- Decays exponentially with snr E_s/σ^2
- E_s is average energy/symbol of the constellation; power of PAM waveform $x_b(t)$ is E_s/T