**Leon 1-2a.** Describe what step-by-step procedure might be involved inside the network in making a telephone connection.
**Solution:**

1. The telephone number specifies an "address" where the receiver is located. In the North American telephone numbering system the first three digits are the area code which specifies the main geographical region of the receiver; the next three digits specify a particular telephone office in the North American network. The final four digits identify the particular location of the receiver.

2. When a telephone number is dialed equipment at the other end of the telephone line uses the sequence of dialed numbers to determine a route across the telephone network from the call-originating phone to the destination phone. A circuit is established between the originating and destination phone along the identified route.

3. A ringing tone is then applied at the destination to indicate that there is an incoming call. If the destination party (a person, an answering machine, or some other device) is ready to answer, the call is completed through the lifting of the phone set or some equivalent action. The setting up of telephone calls is discussed in Chapter 4.

**1-2b.** Now consider a personal communication service that provides a user with a personal telephone number. When the number is dialed, the network establishes a connection to wherever the user is located at the given time. What functions must the network now perform in order to implement this service?
**Solution:**

1. The key difference here is that the personal telephone number is not tied to a specific location. Instead the personal number is associated with one or more pieces of equipment, for example, a cell phone, that can request service from various points in the network. This necessitates the translation of the personal telephone number to a number that corresponds to a specific location in the network at a given time.

2. When the telephone number is dialed, a message requesting a connection setup is sent to the "home" location of the personal number.

3. The home location must somehow be able to redirect the connection setup process to the current location of the user. For example, the user may register one or more forwarding "addresses" to which calls are to be redirected. The signaling system is used to locate the user. In cellular telephony, for example, requests for connections to a given mobile telephone are broadcast over specific signaling channels.

4. Once the location of the destination is identified a connection is established.

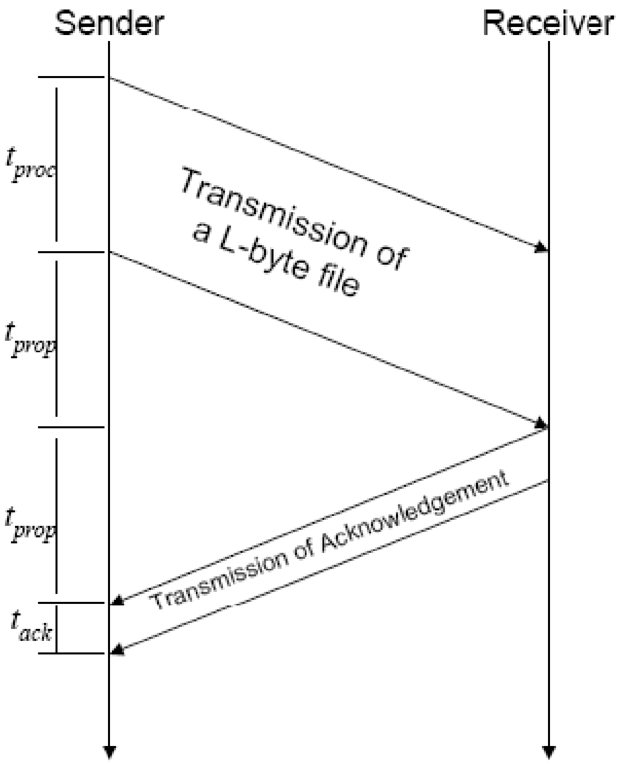
**Leon 1-5.** Suppose that network addresses are scarce, so addresses are assigned so that they are not globally unique; in particular suppose that the same block of addresses may be assigned to different organizations. How can the organizations make use of these addresses? Can users from two such organizations communicate with each other?
**Solution:**
To make the example concrete suppose that two organizations are assigned the same set of telephone numbers. Clearly, users within each organization can communicate with each other as long as they have a unique address within the organization. However, communications outside an organization poses a problem since any given address is no longer unique across multiple organizations.
A possible approach to enabling communications between users in different organizations is to use a two-step procedure as follows. Each organization has a special gateway to communicate outside the organization. Internal users contact the gateway to establish calls to other organizations. Gateways have procedures to establish connections with each other. This enables gateways to establish connections between their internal users and users in other networks.

**Leon 1-9a.** Suppose that an interactive video game is accessed over a communication network. What requirements are imposed on the network if the network is connection-oriented? connectionless?
**Solution:**
We suppose that the game involves the interaction between a player and a server across a network. To support an interactive video game over a communications network, the network, whether connection-oriented or connectionless, must provide real-time delivery of the player's commands to the server, and of the server's responses to the player. With a connection-oriented network, connections between the player and the servers transfer the sequence of commands and responses throughout the game with very little delay. In a connectionless network, user commands may be delivered to the other end with variable delay, out-of-sequence, or not at all. The user’s network software is responsible for ensuring the ordered and correct delivery of game commands. In-time delivery of commands cannot be assured.

**1-9b.** Repeat part (a) if the game involves several players located at different sites.
**Solution:**
The requirements on the network depend on how the game is implemented. In the centralized approach the players interact through a central server that processes the commands from all of the players, maintains a view of the state of the overall system, and issues appropriate responses to all the players. Alternatively, the game could be implemented in a decentralized fashion, where each player receives commands from some or all of the players, maintains a local view of the system state, and transmits responses to some or all of the players.
In the centralized approach, the network requirements are essentially the same as those in part (a). In the decentralized approach, the real-time response requirement may apply only to a subset of players when they happen to be interacting with each other. If players are located at different sites, the network could provide multicast capability so messages can be exchanged among the server and all the players.

**1-9c.** Repeat part (b) if one or more of the players is in motion, for example, kids in the back of the van during a summer trip.
**Solution:**
If one or more players is in motion, then the network must also be able to locate each mobile player and then deliver and receive information to and from such player. Additional delay and loss issues come into play when a mobile user is “handed-off” from one radio coverage area to another.

Leon 1-15 Let d be the distance between a sender and a receiver (10cm, 10m, 100m, 100km, 1000km, and 2x36,000 km), and R be the transmission speed of a sender and a receiver (10,000 bits/sec, 1 megabit/sec, 100 megabits/sec, and 10 gigabits/sec.) Overall transmission scheme is shown as following:




$t_{proc}$ : processing time for a L-byte file, $t_{proc} = L / R$,

$t_{prop}$ : propagation delay, $t_{proc} = d / (2.3 \times 10^8)$,

$t_{ack}$ : processiong time for an 1-byte acknowledgement file, $t_{ack} = 1 / R$

---

Total amount of time for completing transmission of a L-byte file:

$$
\begin{aligned}
t_{total} &= t_{proc} + 2 \times t_{prop} + t_{ack} \\
&= L / R + 2 \times d / (2.3 \times 10^8) + 1 / R \\
&= \frac{L+1}{R} + 2 \frac{d}{2.3 \times 10^8}
\end{aligned}
$$

i) if L=1,

| R(bits/sec) \ d(meter) | 0.1 | 10 | 100 | $10^5$ | $5x10^6$ | $72x10^6$ |
|---|---|---|---|---|---|---|
| 10,000 | 0.0002 | 0.0002 | 0.0002 | 0.00107 | 0.0437 | 0.626 |
| $10^6$ | 2e-6 | 2.09e-6 | 2.87e-6 | 8.7e-5 | 0.0435 | 0.626 |
| $10^8$ | 2.09e-8 | 1.07e-7 | 8.9e-7 | 8.7e-5 | 0.0435 | 0.626 |
| $10^{10}$ | 1.07e-9 | 8.7e-8 | 8.7e-7 | 8.7e-5 | 0.0435 | 0.626 |

ii) if L=1000,

| R(bits/sec) \ d(meter) | 0.1 | 10 | 100 | $10^5$ | $5x10^6$ | $72x10^6$ |
|---|---|---|---|---|---|---|
| 10,000 | 0.1 | 0.1 | 0.1 | 0.101 | 0.144 | 0.726 |
| $10^6$ | 0.001 | 0.001 | 0.001 | 0.0019 | 0.044 | 0.627 |
| $10^8$ | 1.0e-5 | 1.0e-5 | 1.09e-5 | 0.00088 | 0.0435 | 0.626 |
| $10^{10}$ | 1.0e-7 | 1.87e-7 | 9.7e-7 | 0.8e-5 | 0.0435 | 0.626 |

iii) if L=1,000,000

| R(bits/sec) \ d(meter) | 0.1 | 10 | 100 | $10^5$ | $5x10^6$ | $72x10^6$ |
|---|---|---|---|---|---|---|
| 10,000 | 100 | 100 | 100 | 100 | 100.04 | 100.63 |
| $10^6$ | 1 | 1 | 1 | 1.0009 | 1.0435 | 1.626 |
| $10^8$ | 0.01 | 0.01 | 0.01 | 0.0109 | 0.00535 | 0.636 |
| $10^{10}$ | 0.0001 | 0.0001 | 0.0001 | 0.00097 | 0.00436 | 0.626 |

iv) if L=1,000,000,000

| R(bits/sec) \ d(meter) | 0.1 | 10 | 100 | $10^5$ | $5x10^6$ | $72x10^6$ |
|---|---|---|---|---|---|---|
| 10,000 | 1e5 | 1e5 | 1e5 | 1e5 | 1e5 | 1e5 |
| $10^6$ | 1000 | 1000 | 1000 | 1000 | 1000 | 1000.6 |
| $10^8$ | 10 | 10 | 10 | 10.001 | 10.043 | 10.626 |
| $10^{10}$ | 0.1 | 0.1 | 0.1 | 0.10087 | 0.14348 | 0.72609 |

[5] Prove that finite additivity follows from countable additivity

For a finite set of $n$ disjoint events $E_1, \ldots, E_n$, to show $P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i)$.

Define $E_i = \emptyset, \forall i \geq n+1$, then $E_i, i \geq 1$ are still disjoint. Then,

$$\begin{aligned} P(\bigcup_{i=1}^{n} E_i) &= P(\bigcup_{i \geq 1} E_i) \\ &=^{(1)} \sum_{i \geq 1} P(E_i) \\ &= \sum_{i=1}^{n} P(E_i) + \sum_{i \geq n+1} P(E_i) \\ &=^{(2)} \sum_{i=1}^{n} P(E_i) \end{aligned}$$

(1) is from countable additivity, and (2) is from the fact that $P(E_i) = P(\emptyset) = 0, \forall i \geq n+1$.

[6]

(1)

$P(A) + P(A^c) = P(A \bigcup A^c) = P(\Omega) = 1$

$\Longrightarrow P(A^c) = 1 - P(A)$.

(2)

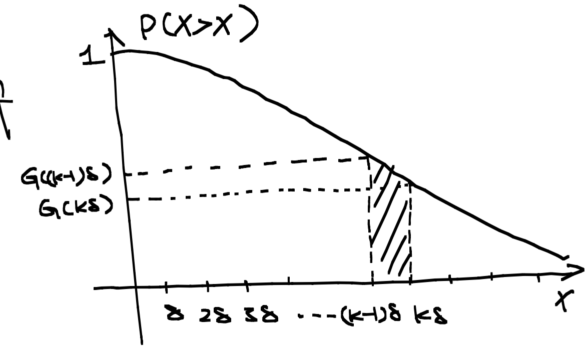
$P(B) = P(A \bigcup (A \setminus B)) = P(A) + P(A \setminus B)$

Since $P(A \setminus B) \geq 0$, then $P(A) \leq P(B)$.

(7) (Method 1)

(a)

Partition the $x$ axis into intervals of length $\delta$, as in the figure.




let $G(x) = P(X>x)$ ; $x \geq 0$

the probability of $(k-1)\delta < X \leq k\delta$ is

$Pr((k-1)\delta < X \leq k\delta) = G((k-1)\delta) - G(k\delta)$.

So

$$(k-1)\delta \cdot Pr((k-1)\delta < X \leq k\delta) \leq E[X; (k-1)\delta < X \leq k\delta]$$
$$\leq k\delta \cdot Pr((k-1)\delta < X \leq k\delta)$$

Because $E(X) = \sum_{k=1}^{\infty} E[X; (k-1)\delta < X \leq k\delta]$ holds for any $\delta > 0$.

$$\lim_{\delta\to 0} \sum_{k=1}^{\infty} (k-1)\delta \cdot Pr((k-1)\delta < X \leq k\delta) \leq E(X) \leq \lim_{\delta\to 0} \sum_{k=1}^{\infty} k\delta \cdot Pr((k-1)\delta < X \leq k\delta)$$

$$\lim_{\delta\to 0} \sum_{k=1}^{\infty} (k-1)\delta\,[G((k-1)\delta) - G(k\delta)] \leq E(X) \leq \lim_{\delta\to 0} \sum_{k=1}^{\infty} k\delta\,[G((k-1)\delta) - G(k\delta)]$$

$$\lim_{\delta\to 0} \sum_{k=1}^{\infty} \delta\,[G(k\delta)] \leq E(X) \leq \lim_{\delta\to 0} \sum_{k=1}^{\infty} \delta\,[G((k-1)\delta)]$$

$$\int_0^{\infty} G(x)\,dx \leq E(X) \leq \int_0^{\infty} G(x)\,dx$$

Hence $E(X) = \int_0^{\infty} G(x)\,dx = \int_0^{\infty} P(X>x)\,dx$.

(b) Similar to (a).

$$(k-1)^n\delta^n \cdot \Pr((k-1)\delta < X \le k\delta) \le E[X^n; (k-1)\delta < X \le k\delta]$$

$$\le k^n\delta^n \Pr((k-1)\delta < X \le k\delta)$$

$$\lim_{\delta\to 0}\sum_{k=1}^{\infty}(k-1)^n\delta^n[G((k-1)\delta)-G(k\delta)] \le E(X) \le \lim_{\delta\to 0}\sum_{k=1}^{\infty}k^n\delta^n[G((k-1)\delta)-G(k\delta)]$$

$$\lim_{\delta\to 0}\sum_{k=1}^{\infty}[(k\delta)^n-((k-1)\delta)^n][G(k\delta)] \le E(X) \le \lim_{\delta\to 0}\sum_{k=1}^{\infty}[(k\delta)^n-((k-1)\delta)^n]G((k-1)\delta))$$

$$\int_0^\infty G(x)\,dx^n \le E(X) \le \int_0^\infty G(x)\,dx^n$$

$$E(X) = \int_0^\infty nx^{n-1}\,P(X>x)\,dx$$

$\square$

A more elegant method is given in the next page using the concept of indicator function.

An indicator function takes a value of 1 or 0 & is usually very useful in proving results in probability.

(a) Define $\underset{\substack{\uparrow \\ \text{Indicator function}}}{I_\alpha}(X) = \begin{cases} 1 & \text{if } X \geq \alpha \text{ st } \alpha \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$\int P\{X > \alpha\} \, d\alpha = \int_0^\infty E\left(I_\alpha(X)\right) d\alpha$$

$$= E \int_0^\infty I_\alpha(X) \, d\alpha = E \int_0^X 1 \, d\alpha$$

$$= E(X).$$

(b) Again Define $I_\alpha(X) = \begin{cases} 1 & X \geq \alpha, \ \alpha \geq 0 \\ 0 & \text{else.} \end{cases}$

$$\int_0^\infty n x^{n-1} P\{X > \alpha\} = \int_0^\infty n \alpha^{n-1} P\{X > \alpha\} \, d\alpha$$

$$= \int_0^\infty n \alpha^{n-1} E\left(I_\alpha(X)\right) d\alpha$$

$$= E \int_0^\infty n \alpha^{n-1} I_\alpha(X) \, d\alpha$$

$$= E\left(\int_0^X n \alpha^{n-1} d\alpha\right) = E(X^n)$$

[8]

(a) $E\left( E(X|Y)\right)$

$$= \int_{-\infty}^{\infty} E(X|Y)\, dF_Y(y)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \; dF_X(x|y) \quad dF_Y(y)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot \frac{dF_{XY}(x,y)}{dF_Y(y)} \; dF_Y(y)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot \; dF_{XY}(x,y)$$

$$= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} dF_{XY}(x,y)$$

$$= \int_{-\infty}^{\infty} x \cdot \; dF_X(x) \; = \; E(X) \;//.$$

(b) If $P\{X \le x, Y \le y\} = P\{X \le x\}\, P\{Y \le y\}$

$$\Rightarrow E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, dF_{XY}(x,y)$$

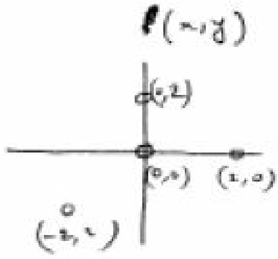
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, dF_X(x)\, dF_Y(y)$$

$$= \int_{-\infty}^{\infty} x \, dF_X(x) \int_{-\infty}^{\infty} y \, dF_Y(y)$$

$$= E(X)\, E(Y) \quad //.$$

The reverse is not true.

Counterexample: $P\{X=0, Y=0\} = \frac{1}{4}$
$P\{X=0, Y=2\} = \frac{1}{4}$
$P\{X=2, Y=0\} = \frac{1}{4}$
$P\{X=-2, Y=-2\} = \frac{1}{4}$




$E(X) = -2\cdot\frac{1}{4} + 2\cdot\frac{1}{4} + 0 = 0$
$E(Y) = -2\cdot\frac{1}{4} + 2\cdot\frac{1}{4} + 0 = 0$

$E(XY) = 0 \qquad P\{X \le 0, Y \le 0\} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
$P\{X \le 0\} = 3/4; \; P\{Y \le 0\} = 3/4$

$\Rightarrow P\{X \le x, Y \le y\} \neq P\{X \le x\} P\{Y \le y\}$

even though $E(XY) = E(X)\, E(Y)$

(c) $X$ & $Y$ are independent.

$$\psi_{X+Y}(t) = E\left[e^{t(X+Y)}\right] = E\left(e^{tX} \cdot e^{tY}\right)$$

$$= E\left(e^{tX}\right) E\left(e^{tY}\right)$$

$$= \psi_X(t)\, \psi_Y(t).$$

$\psi_X(t) = \left(\dfrac{\lambda}{\lambda - t}\right)^n$ Note: $\psi'_X(t) = E\left(e^{tX} \cdot X\right)$

$$\psi'_X(0) = E\left(e^{0X} X\right)$$

$$= E(X).$$

$$\psi'_X(t)\Big|_{t=0} = E(X) = \frac{d}{dt}\left(\frac{\lambda}{\lambda - t}\right)^n\Big|_{t=0}$$

$$= (\lambda - t)^{-n-1} \cdot \lambda^n \cdot n \Big|_{t=0} = \frac{n\lambda^n}{\lambda^{n+1}} = \frac{n}{\lambda}.$$

Similarly: $E(X^2) = \dfrac{d^2}{dt^2} \psi_X(t)\Big|_{t=0} = \dfrac{n(n+1)}{(\lambda - t)^{n+2}} \lambda^n\Big|_{t=0} = \dfrac{n(n+1)}{\lambda^2}$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{n^2}{\lambda^2} - \frac{n(n+1)}{\lambda^2} = \frac{n}{\lambda^2}$$

(d) To show:

$$\text{Var}(X) = E(\text{Var}(X|Y)) + \text{Var}(E(X|Y))$$

$$E(\text{Var}(X|Y)) + \text{Var}(E(X|Y))$$

$$= E\left(E\left([X - E(X|Y)]^2 \mid Y\right)\right) + \text{Var}(E(X|Y))$$

$$= E\left[(X - E(X|Y))^2\right] + \text{Var}(E(X|Y))$$

$$= E(X^2) - 2E[X\,E(X|Y)] + E(E^2(X|Y))$$
$$+E(E^2(X|Y)) - E[E(X|Y)]^2$$

$$= E(X^2) - 2E(\cancel{E(X|Y)}\,E(X|Y))$$

$$+ 2E(\cancel{E^2(X|Y)}) - E^2(X)$$

$$= E(X^2) - E^2(X) = \text{Var}(X).$$

[10]

$$Y = \sum_{i=1}^{N} X_i$$ ; $X_i$ are iid

$X_i$ are independent of $N$.

(a) Find $\psi_Y(t) = E\left(e^{t\sum_{i=1}^{N} X_i}\right)$

First $E\left(e^{t\sum_{i=1}^{N} X_i} \mid N = n\right)$

$$= E\left(e^{t\sum_{i=1}^{n} X_i} \mid N = n\right) = E\left(e^{t\sum_{i=1}^{n} X_i}\right)$$

$$= E\left(e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}\right)$$

$$= E\left(e^{tX_1}\right) E\left(e^{tX_2}\right) \cdots E\left(e^{tX_n}\right)$$

$$= \left[\psi_X(t)\right]^n.$$

Because $X_i$ is independent of $N$ & $X_i$ are independent of each other.

<u>Now</u> $E\left(e^{t\sum_{i=1}^{N} X_i} \mid N\right) = \left[\psi_X(t)\right]^N$

$$\psi_Y(t) = E\left[E\left(e^{t\sum_{i=1}^{N} X_i} \mid N\right)\right]$$

$$= E\left(\psi_X(t)\right)^N$$

(b) To compute the mean & variance of $Y$, differentiate $\psi_Y(t)$ as follows:

$$\psi_Y'(t) = E[N(\psi_X(t))^{N-1}\psi_X'(t)],$$

$$\psi_Y''(t) = E[N(N-1)(\psi(t))^{N-2}(\psi_X'(t))^2 + N(\psi_X(t))^{N-1}\psi_X''(t)].$$

Then,

$$E[Y] = \psi_Y'(t)|_{t=0}$$
$$= E[N\,E(X)]$$
$$= E[N]E[X]$$

$$E[Y^2] = \psi_Y''(t)|_{t=0}$$
$$= E[N(N-1)E^2(X) + N\,E(X^2)]$$

Hence,
$$Var(Y) = E[Y^2] - E^2[Y]$$
$$= E[N]Var[X] + E^2[X]\,Var(N).$$

[11] (a) $P(\text{all five cards are sevens}) = 0.$

(b) $P(\text{at least one is a seven})$
$= 1 - P(\text{none of them is seven})$
$= 1 - \dfrac{\binom{48}{5}}{\binom{52}{5}} \doteq 0.341$

(c) $P(\text{none of them is seven}) = \dfrac{\binom{48}{5}}{\binom{52}{5}} \doteq 0.659$

(d) $P(\text{two out of five are seven}) = \dfrac{\binom{48}{3}\binom{4}{2}}{\binom{52}{5}} \doteq 0.04$

[12] $P\{X=k\} = \binom{n}{k} p^k (1-p)^{n-k}$

$$= \frac{n!}{k!\,(n-k)!}\, p^k\, (1-p)^n\, (1-p)^{-k}$$

$$= \frac{\overbrace{n(n-1)(n-2)\cdots(n-k+1)}^{k \text{ terms}}}{k!}\, p^k\, (1-p)^n\, (1-p)^{-k}$$

$$= \frac{np\,(n-1)p\,(n-2)p\cdots(n-k+1)p}{k!}\, (1-p)^n\, (1-p)^{-k} \longrightarrow (1).$$

Now, $\lim\limits_{\substack{n\to\infty \\ p\to 0 \\ np\to\lambda}} P\{X=k\}$

$$= \frac{\overbrace{\lambda\cdot\lambda\cdot\lambda\cdots\lambda}^{k \text{ terms}}}{k!}\, e^{-\lambda}\cdot 1.$$

$$\boxed{P\{X=k\} = \frac{\lambda^k}{k!}\, e^{-\lambda} \Rightarrow \text{Poisson distribution.}}$$

[13]

Carefully show that definitions I and II given in the notes for a Poisson process are equivalent.
Hint: In deriving II from I, set up a differential equation in the following manner. First define $P_0(t+h) := P(\{N(t+h) = 0\})$. Then

$$
\begin{aligned}
P_0(t+h) &= P(\{N(t) = 0, N(t+h) - N(t) = 0\}) \\
&= P(\{N(t) = 0\})P(\{N(t+h) - N(t) = 0\}) \\
&= P_0(t)(1 - \lambda h + o(h)), \\
\text{hence } \frac{P_0(t+h) - P_0(t)}{h} &= -\lambda P_0(t) + \frac{o(h)}{h}.
\end{aligned}
$$

Now let $h \to 0$ and set up and solve the appropriate differential equation. After solving for $P_0(t)$ proceed to solve for $P_n(t) := P(\{N(t) = n\})$ in much the same manner.

Let us first derive II from I

Following the hint, we define

$$
\begin{aligned}
P_0(t+h) &\triangleq P\{N(t+h) = 0\} \\
&= P\{N(t) = 0,\ N(t+h) - N(t) = 0\} \\
&= P\{N(t) = 0\}\, P\{N(t+h) - N(t) = 0\} \quad \text{(Independent Increments)} \\
&= P_0(t)\,(1 - \lambda h + o(h)) \quad \text{(Stationary Increments)}.
\end{aligned}
$$

$$\Rightarrow \frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0 + \frac{o(h)}{h}$$

Taking limit on both sides we have

$$\lim_{h\to 0} \frac{P_0(t+h) - P_0(t)}{h} = \lim_{h\to 0} -\lambda P_0 + \frac{o(h)}{h}$$

$$\Rightarrow \frac{dP_0(t)}{dt} = -\lambda P_0(t) \Rightarrow \frac{dP_0(t)}{P_0} = -\lambda\, dt$$

$$\Rightarrow P_0(t) = e^{-\lambda t + c}$$

But $P_0(0) = P\{N(0) = 0\} = 1 = e^{-\lambda t} \cdot e^{c}$

$$\Rightarrow \boxed{P_0(t) = e^{-\lambda t}}$$

Similarly, for $n \geq 1$

$$P_n(t+h) = P\{N(t+h) = n\}$$

$$= P\{N(t) = n,\ N(t+h) - N(t) = 0\}$$

$$+ P\{N(t) = n-1,\ N(t+h) - N(t) = 1\}$$

? $\Rightarrow$ $+ P\{N(t+h) = n,\ N(t+h) - N(t) \geq 2\}$

~~?~~ correct! up to $t+h$ there is still $n$ but in the interval $N(t+h)$ to $N(t)$ it is $\geq$.

$$= P_n(t)\, P_0(h) + P_{n-1}(t)\, P_1(h) + o(h)$$

$$= (1-\lambda h)\, P_n(t) + \lambda h\, P_{n-1}(t) + o(h)$$

$1 - P_0(h) = 1 - (1-\lambda h) = \lambda h$

$$\Rightarrow \frac{P_n(t+h) - P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(h)}{h}$$

Letting $h \to 0$, we have

$$\frac{dP_n(t)}{dt} = +\lambda P_{n-1}(t) - \lambda P_n(t)$$

Multiplying by $e^{\lambda t}$ on both sides,

$$e^{\lambda t}\left[\frac{dP_n(t)}{dt} + \lambda P_n(t)\right] = \left(+\lambda P_{n-1}(t)\right) \cdot e^{\lambda t}$$

$$= \frac{d}{dt}\left(P_n(t) \cdot e^{\lambda t}\right) = \left(+\lambda P_{n-1}(t)\right) e^{\lambda t} \longrightarrow 1$$

Now <u>when $n = 1$.</u>

$$\Rightarrow \frac{d}{dt}\left(P_1(t)\, e^{\lambda t}\right) = \lambda P_0(t)\, e^{\lambda t} = \lambda$$

But $P_n(0) = P\{N(0) = n\} = 0$

$$\Rightarrow \boxed{P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}}$$

Now let us derive (I) from (II).

First we will show that

$P\{N(h) = 1\} = \lambda h + o(h)$

$P\{N(h) = 1\} = e^{-\lambda h} \frac{(\lambda h)^1}{1!}$ from (II).

$$\Rightarrow \cancel{\lim_{h \to 0}} = \lambda h \left( 1 - \lambda h + \frac{(\lambda h)^2}{2!} + \cdots \right)$$

$$= \lambda h + o(h)$$

Now we will show that

$P\{N(h) \geq 2\} = o(h)$

$$P\{N(h) \geq 2\} = \sum_{n=2}^{\infty} e^{-\lambda h} \frac{(\lambda h)^n}{n!}$$

$$= e^{-\lambda h} \left( \frac{\lambda h^2}{2!} + \frac{\lambda h^3}{3!} + \cdots \right)$$

$$= o(h).$$

[14]

We need to make an assumption here that bits are corrupted "independently" with probability $p$.

$\Rightarrow P\{$A packet arrives successfully | packet length $= L\}$

$$= (1-p)^L$$

Let $\alpha \triangleq P\{$A packet arrives successfully$\}$

$$= \sum_{L=0}^{\infty} P\{\text{A packet arrives successfully} \mid \text{Packet length} = L\} \cdot P\{\text{Packet length} = L\}$$

$$= \sum_{L=0}^{\infty} (1-p)^L e^{-\mu} \frac{\mu^L}{L!}$$

$$= \sum_{L=0}^{\infty} e^{-\mu} \frac{[(1-p)\mu]^L}{L!} = e^{-\mu p} \underbrace{\sum_{L=0}^{\infty} e^{-\mu(1-p)} \frac{[(1-p)\mu]^L}{L!}}_{1}$$

$$\Rightarrow \alpha = e^{-\mu p}.$$

$$\Rightarrow \boxed{\text{Rate of successful packets} = e^{-\mu p} \cdot \lambda.}$$

Note: Here we assumed that packet length was Poisson distributed with rate $\mu$.

$\Rightarrow P\{L=0\} = e^{-\mu}$

$\Rightarrow$ a non-zero probability exists of a packet having a length of zero!

We could easily eliminate this issue by using a shifted Poisson process

For example:

$$P\{L = k\} = e^{-\mu} \frac{\mu^{k-x}}{(k-x)!}$$

& renormalize the LHS so that the probabilities sum to 1.