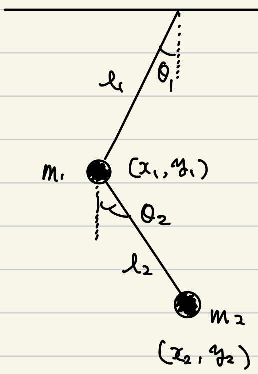

$$\begin{cases} x_1 = l_1 \sin\theta_1 \\ y_1 = -l_1 \cos\theta_1 \end{cases}$$

$$\begin{cases} x_2 = x_1 + l_2 \sin\theta_2 \\ y_2 = y_1 - l_2 \cos\theta_2 \end{cases}$$

$$L = \frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 - mg\,y_1 - mg\,y_2$$

$$= \frac{m_1}{2}\left[l_1^2 \cos^2\theta_1 \cdot \dot\theta_1^2 + l_1^2 \sin^2\theta_1^2 \cdot \dot\theta_1^2\right] + \frac{m_2}{2}\left[(l_1\cos\theta_1 \cdot \dot\theta_1 + l_2 \cos\theta_2 \cdot \dot\theta_2)^2 + (l_1 \sin\theta_1 \cdot \dot\theta_1 + l_2 \sin\theta_2 \cdot \dot\theta_2)^2\right]$$

$$+ mg\,l_1 \cos\theta_1 + m_2 g\,(l_1\cos\theta_1 + l_2 \cos\theta_2)$$

$$= \frac{m_1}{2} l_1^2 \dot\theta_1^2 + \frac{m_2}{2}\left(l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 (\cos\theta_1\cos\theta_2 + \sin\theta_1 \sin\theta_2)\right)$$

$$+ mg\,l_1 \cos\theta_1 + m_2 g\,(l_1\cos\theta_1 + l_2 \cos\theta_2)$$

$$= \frac{1}{2}(m_1+m_2) l_1^2 \dot\theta_1^2 + \frac{m_2}{2}\left(l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2)\right)$$

$$+ mg\,l_1 \cos\theta_1 + m_2 g\,(l_1\cos\theta_1 + l_2 \cos\theta_2)$$

$$= \frac{1}{2}\dot\theta \cdot \underbrace{\begin{pmatrix} (m_1+m_2) l_1^2 & m_2 l_1 l_2 \cos(\theta_1-\theta_2) \\ m_2 l_1 l_2 \cos(\theta_1-\theta_2) & m_2 l_2^2 \end{pmatrix}}_{=R} \dot\theta + mg\,l_1\cos\theta_1 + m_2 g\,(l_1\cos\theta_1 + l_2\cos\theta_2)$$

① w.r.t. θ_1

$$\frac{\partial L}{\partial \dot\theta_1} = (m_1+m_2) l_1^2 \dot\theta_1 + m_2 l_1 l_2 \dot\theta_2 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_1 g l_1 \sin\theta_1 - m_2 g\, l_1 \sin\theta_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\Leftrightarrow (m_1+m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot\theta_2 (\cancel{\dot\theta_1} - \dot\theta_2) \sin(\theta_1 - \theta_2)$$

$$\cancel{+ m_2 l_1 l_2 \dot\theta_1 \dot\theta_2} \sin(\theta_1 - \theta_2) + (m_1+m_2) g l_1 \sin\theta_1 = 0$$

⑥ w.r.t. θ_2

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\Leftrightarrow m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \cancel{\dot{\theta}_2}) \sin(\theta_1 - \theta_2)$$

$$\cancel{- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)} + m_2 g l_2 \sin\theta_2 = 0$$

$$R \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} - m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin\theta_1 \\ m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2 \end{pmatrix}$$

⑥ momentum

$$P_1 = \frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$P_2 = \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \\ m_2 l_1 l_2 \cos(\theta_1 - \theta_2) & m_2 l_2^2 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = R \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

$$H = \mathbb{P} \cdot \dot{\Theta}(\mathbb{P}) - L = \dot{\Theta}(\mathbb{P}) \cdot R \dot{\Theta}(\mathbb{P}) - \left[\frac{1}{2} \dot{\Theta}(\mathbb{P}) \cdot R \dot{\Theta}(\mathbb{P}) + m g l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)\right]$$

$$= \frac{1}{2} \dot{\Theta}(\mathbb{P}) \cdot R(\Theta) \dot{\Theta}(\mathbb{P}) - m g l_1 \cos\theta_1 - m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

$$\dot{\Theta} = \frac{\partial H}{\partial \mathbb{P}} = \frac{\partial}{\partial \mathbb{P}} \left(\frac{1}{2} \mathbb{P} \cdot R^{-1} \mathbb{P}\right) = R^{-1} \mathbb{P}$$ → consistent with the definition of $\mathbb{P}$

$$\dot{\mathbb{P}} = -\frac{\partial H}{\partial \Theta} = -\frac{1}{2} \underbrace{\mathbb{P} \cdot \frac{\partial R^{-1}(\Theta)}{\partial \Theta} \mathbb{P}}_{= \mathbb{P} \cdot R^{-1} \frac{\partial R(\Theta)}{\partial \Theta} R^{-1} \mathbb{P} \;=\; \Theta \frac{\partial R}{\partial \Theta} \Theta \quad (\because R = R^T)} - \begin{pmatrix} (m_1 + m_2) g l_1 \sin\theta_1 \\ m_2 g l_2 \cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} - m_2 l_1 l_2 \sin(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 \sin(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 \end{pmatrix} - \begin{pmatrix} (m_1 + m_2) g l_1 \sin\theta_1 \\ m_2 g l_2 \cos\theta_2 \end{pmatrix}$$