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How production network structure influences the transmission of microeconomic shocks into macroeconomic output changes - An analysis of German input-output data

Masterthesis

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Y	aggregate output
y_i	output of sector i
$dlny$	change of sectoral log-output (first order change)
$\widetilde{dlny}$	total change of sectoral log-output (first + higher order changes)
n	number of sectors
ϵ	vector of shocks
I	identity matrix
W	input share matrix
A	technical coefficient matrix
α	labour share
T_i	technology level (productivity) of sector i
l_i	labour input to sector i
x_{ij}	intermediate input from sector j to sector i
p_i	output price of sector i
$\tilde{v}$	influence vector with heterogeneous output share
v	influence vector with output share $1/n$
$\parallel v_n \parallel$	euclidean norm of v
k	out-degree
k_{2nd}	2nd order out-degree
δ_i	weight of sector i
σ_ϵ^2	variance of shocks
$f \in \Theta(g)$	f grows as fast as g (O-notation)
ρ	sectoral dominance
τ_n	threshold marking the beginning of the tail
$\phi()$	normal distribution
f_i	factor i

1 Introduction

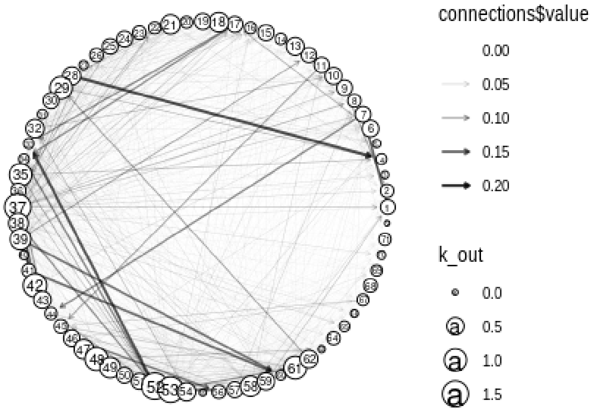


Figure 1: Circular layout of the 2015 German production network based on the technical coefficient matrix A.

It has been shown that structural heterogeneity of the production system can carry idiosyncratic fluctuations (microeconomic shocks or granular shocks) on to fluctuations in aggregate output to a larger extent than the law of large numbers (LLN) suggests. This is the case for heterogeneously sized sectors (Gabaix and Ibragimov 2011) as well as heterogeneous network structure between sectors (Acemoglu, Carvalho, et al. 2012). I focus on network structure in this thesis. The mechanism of shock volatility carrying on to aggregate output volatility can best be illustrated with a small example. It means that in an interconnected economy consisting of 10 equally sized industries with input-output links, an output change of 5 percent to one industry, does not simply translate into a proportional $5\%\frac{1}{10} = 0.5\%$ aggregate output change ($\frac{1}{10}$ is the size of one industry in relation to the whole economy). 5% would be the direct or first order effect on aggregate output. Rather in a production network, due to the interconnections the microeconomic shock also reduces the output of other industries. These higher-order effects are due to the diffusion of the first-order effect through the network and the resulting total aggregate output change can be much larger than the first order effect would suggest. Through a prominent network position, a sector can have such a large influence on aggregate output, that this changes the intensity with which sectoral shocks cancel out in aggregate output, leading to a level of aggregate volatility that contradicts the LLN (Acemoglu, Carvalho, et al. 2012). Network heterogeneity can be measured by the out-degree distribution and theory suggests that a heavy-tailed out-degree distribution is the condition for aggregate fluctuations beyond the LLN. I have analysed the degree distribution of the German industry level production network and find that the distribution for the German production network is heavy-tailed, but can not be approximated by a power law like the tails of other countries out-degree distributions. Still theory derived statistics and a simulation show that the network heterogeneity of the German production network is sufficient to lead to higher volatility in aggregate output than in a theoretical economy without a network structure. Figure 1 shows the German production network. Node sizes are the out-degree of a sector and edge strength is the value of the technical coefficient between two sectors. The plot already shows the heterogeneity in the German production network.

This thesis is structured as follows. Section 2 reviews the literature on the production network model. This also includes putting the production network literature in the context of the two macroeconomic fields business cycle analysis and economic growth.

Section 3 presents the production network model. The focus in the first part is on the interpretation of the model as a system of nested production functions. In the second part it is explained why out-degrees are the essential determinant for the influence of sectoral shocks on aggregate volatility and macroeconomic tail risk.

In section 4 I use German input-output data to analyse the structure of the German production network. This involves estimation of the first order out-degree distribution and the calculation of output elasticities for each sector in the network. These output elasticities are compared to the sectoral output shares (the first order output elasticities). Section 4 also shows the result of a simulation of shock propagation.

In Section 5 I draw conclusions. This involves pointing out some directions for future research.

2 Literature review

2.1 The production network model

Models of production networks are used to understand a variety of different economic phenomena. The literature on production networks can be divided into different parts. One part uses a model of the production network that allows to analyse the production systems reaction to shocks (Acemoglu, Carvalho, et al. (2012), Klimek et al. (2019), Pichler, Pangallo, et al. (2020)). This part can be ascribed to the business cycle literature and is the main focus of this thesis. Another part of the literature uses models of the production network to find determinants of economic growth (tightly intertwined with technological change) in the structure of the production system (Hidalgo and Hausmann (2009), McNerney et al. (2022), Acemoglu and Azar (2020)). This part of the literature is naturally more concerned with dynamics of the production network structure, while the business cycle part deals with a fixed production network structure on which shocks are applied.

The production network model concerned with the reaction of a fixed production network to disaggregate shocks is based on the mathematical concept of a diffusion process on a network. Exogenous shocks are applied to each sector or firm in the network independently and the shocks diffuse through the network until a new equilibrium is reached. *Two main insights* have been gained so far. The *first insight* is that when multiple independent shocks are applied to different industries and propagate through a network structure with heavy-tailed out-degrees, they do not cancel out to the extent that the law of large numbers suggests for an economy without a network structure. This implies that a significant amount of volatility in aggregate output can stem from idiosyncratic shocks to sectors or firms. Acemoglu, Carvalho, et al. (2012) first[1] derive this statement analytically and show that the structure of the US production network on the sectoral level theoretically allows that some amount of aggregate volatility comes from idiosyncratic shocks. Torres-González and Yang (2019) confirm that the out-degree distribution of the US production network is fat-tailed[2]. For Australia Anufriev et al. (2016) and Lithuania Constantinescu and Barauskaite (2018) also find fat-tailed out-degrees, while for Ecuador Romero et al. (2018) find much less heavy-tailed out-degrees but still enough network heterogeneity to attribute aggregate volatility to idiosyncratic shocks. Empirical studies using data on shocks and aggregate output (instead of inferring the theoretical possibility of the mechanism from the network structure, like the literature cited above) confirm that idiosyncratic shocks contribute to a significant part of aggregate volatility (Acemoglu, Akcigit, et al. (2016) (network heterogeneity), Blanco Arroyo and Alfarano (2017) (firm size heterogeneity)).

The *second insight* is that for aggregate output changes to have fat tails, the production network "needs to exhibit a sufficient level of sectoral dominance" (Acemoglu, Ozdaglar, et al. 2017, p. 56) and microeconomic shocks need to have slightly fatter tails than the normal distribution. Although the mechanism behind both insights appears similar, they are distinct, because Acemoglu, Ozdaglar, et al. (2017, p. 57) show "[...] that structural changes in an economy can simultaneously reduce aggregate volatility while increasing macroeconomic tail risk[...]". One possible source

1 They build on the article of Gabaix (2011) who derives the analytical statement that when the firm size distribution is fat-tailed, shocks on the firm level do not cancel out as much as suggested by the law of large numbers. Gabaix and Ibragimov (2011) make the statement that idiosyncratic shocks to firms account for one third of US aggregate volatility.

2 A fat-tailed distribution is a subclass of heavy-tailed distributions. Fat-tailed means the tail follows a power law. Heavy tails are all tails which go to zero slower than an exponential function (Taleb and Cirillo 2019, p. 6).

of heavy-tailed shocks could be natural disasters. By analysing data on economic damages resulting from climate stressors over the last half century Coronese et al. (2019) find that the distribution of economic damages due to natural disasters has become more rightward skewed and the tails have fattened. While the location of the damage distribution (median, mean) has not changed much, upper percentiles have moved to the right, meaning extreme damages have become more frequent the authors state. But are such shocks idiosyncratic, like the literature stresses? The term idiosyncratic shock implies that they originate from the internal processes within a firm or sector such that they are completely independent of shocks to other sectors. Hence Dosi et al. (2019, p. 69) state that "[...] the theory is agnostic about the type of shocks [...]", but in the production network literature concerned with fluctuations they are mainly understood as internal productivity shocks to a firm or sector. But if exogenous shocks such as the ones coming from natural disasters are independent for each sector, they work just the same as idiosyncratic shocks.

The part of the literature which deals with granular shocks could be called the production network literature concerned with business cycles. It is the literature originating from and staying close to Acemoglu, Carvalho, et al. (2012). It derives analytically tractable statements based on data of the structure of the economy (mainly input-output data) in connection with linear behaviour of firms and sectors. This literature then tries to empirically verify the statements with data on shocks and aggregate output. This rigid analytical framework has been applied among others to shocks due to an earthquake (Carvalho et al. 2021) and sectoral productivity shocks as well as import shocks (Acemoglu, Akcigit, et al. 2016). The level of aggregation of shocks is also a criterion which divides the literature. Another part of the literature deals with an aggregate shock to the production network. This is for example the literature investigating the effect of a COVID-19 lockdown on the production system (e.g. Pichler, Pangallo, et al. (2020)). In an empirical analysis of price propagation finds that the propagation of aggregate and idiosyncratic price shocks can differ: Belgian firms are "[...] on average completely passing through common shocks, but much less idiosyncratic shocks." (Duprez and Magerman 2018, p. 1).

By relaxing the analytical model, a few other model parameters have been identified as crucial influences on the propagation of shocks, in addition to the network structure of the economy: These are the substitutability of inputs (Inoue and Todo (2019), Pichler, Pangallo, et al. (2020)) and sectors or firms inventories which make them able to absorb shocks (Luu et al. (2018)). When the behaviour of sectors or firms then is a nonlinear function or heuristic, the analytical framework has to give way for simulations with agent-based-models. Examples are Pichler, Pangallo, et al. (2020) who simulate the response of the British economy to a COVID-19 lockdown with sectors partially having Leontief production functions and inventories, and Inoue (2016) where firms have inventories. Inventories add the capacity to absorb shocks, whereby the propagation mechanism becomes nonlinear (Luu et al. 2018) and thus requires simulation. Where no derivations of analytical expressions are possible, statistics e.g. for the "vulnerability" of sectors can be computed from simulations (Luu et al. 2018, p. 13). A limitation of the production network model "[...] is that prices play no role in the model. Firms in real economies can respond to shocks by adjusting produced quantities as well as prices. It therefore remains to be seen how prices can be incorporated [...]" Klimek et al. (2019, p. 7). I address how prices are handled in the production network model in section 3.2.3.

While the main focus of the production network literature and this thesis is on downstream propagation of shocks, from suppliers to customers, also shocks that propagate upstream can be modelled with the mathematical structure of the production network model. Such shocks are for example changes in government spendings such as a car-scrapping bonus. In fact upstream propagation of demand side shocks have been the subject of classical input-output models for a long time Rose and Miernyk (1989). Modelling the diffusion of shocks in both directions, upstream and downstream through the network at the same time imposes a hurdle to an analytical expression of such a model, but the quantitative effects on outputs can be computed with simulations (Luu et al. 2018) or an optimization procedure (Pichler and Farmer 2021).

To sum up, the production network model shows the possibility that also independent disaggregate shocks can result in sizeable aggregate fluctuations, a view that has long been rejected by economists, because it has been thought that they cancel out to a larger extent.

2.2 Business cycles

In his critique of modern macroeconomics Stiglitz (2018, p. 76) names, apart from a few other points, the representative agent paradigm together with "excessive aggregation" and the strong role of exogenous shocks as two main reasons why the discipline can not explain deep downturns in macroeconomic variables. But to explain them is what he calls "the most important challenge facing any macro-model" Stiglitz (2018, p. 70). When taking the perspective of an objective observer, aggregate output, the most prominent macroeconomic variable, is not only growing but also fluctuating. Although the term fluctuations does not quite sound like it includes deep downturns and the perception is often that crises are something extraordinary and therefore different from the "normal" fluctuations, deep downturns can be thought of as tail events in the distribution of output fluctuations. As the quote of Stiglitz indicates, the source of these fluctuations is less clear and explanations range from purely exogenous and aggregate shocks to the economy that drive aggregate output away from its steady state growth path to endogenous dynamics driven by the interactions of microeconomic agents.

(1) shocks $\longrightarrow$ (2) propagation mechanism $\longrightarrow$ (3) aggregate output

Figure 2: Business cycle models split in 3 stages.

In modern macroeconomics, models with a high level of aggregation and exogenous shocks as the source of fluctuations are prevalent. In these so-called exogenous business cycle models, exogenous shocks (1) are passed through a propagation mechanism (2) and the outcome is a fluctuating time series (3), see figure 2. Ascari et al. (2015) use the analogy of a filter to describe how exogenous shocks are passed through such a model. In a sense the propagation mechanism and therefore the economy it represents acts as a filter which transforms exogenous fluctuations (e.g. Gaussian noise) into the fluctuations observed in empirical macroeconomic time series. A "good model" would succeed to match real world data in all three of the stages into which I have split them up: (1) the shocks can be observed in the real world, (2) the propagation mechanism is an image of actually observed real world structures and processes and (3) the aggregate fluctuations which come out in the model have the same properties as real world fluctuations of aggregate output. Unfortunately this is not the case for the current state of exogenous business cycle models. Exogenous business cycle models see the economy as a stable system which is continuously driven away from its steady state by shocks. Many authors have very well described the long term state of exogenous business cycle models in a few words, for example: "The debate [over the time series properties of macroeconomics variables] until recently has been exclusively based on the foundation that the economy's dynamic behaviour is well approximated by (Gaussian) random impulses being propagated over time by an invariant linear structure." (Potter 2000, p. 1425). This quote reflects that exogenous business cycle models are characterised by features in each of the three stages which do not fulfil the criteria of a "good model": (1) Aggregate Gaussian shocks (they have not been witnessed in the size necessary to drive the observed aggregate fluctuations[3]), (2) a linear invariant propagation mechanism that represents the aggregate structure of the economy (economic structure is not invariant and likely not linear when approximated by an aggregate model, more on this below). In regard of stage (1) the production network model has the potential to bring the exogenous business cycle theory closer to reality as shocks on the sectoral or firm level (granular level) can be clearly identified events in contrast to "mysterious aggregate [...] shocks" (Gabaix 2016, p. 1-2). In the production network literature

[3] See Gallegatti and Mignacca (1994) for an analysis of the early idea that solar activity is such an aggregate shock that drives the business cycle.

the idiosyncratic shocks that are assumed to drive the business cycle fluctuations are internal productivity shocks to sectors or firms, though Dosi et al. (2019) challenge this view. In their opinion, the upstream propagation of demand side shocks in the form of microeconomic investment growth shocks is the larger channel for aggregate fluctuations (another model in this fashion is Citera et al. (2021)). In addition to aggregate shocks, exogenous business cycle models (and this time also the production network model) are based on the premise "[...] that market economies are inherently stable and that observed booms and busts are mainly due to persistent outside disturbances." (Beaudry et al. 2020, p. 42). As a result of unrealistic assumptions in (1) and (2) the exogenous business cycle models often also fail stage (3). In most exogenous business cycle models the propagation mechanism is very aggregate, typically a univariate difference equation but it could also be very disaggregate consisting of multiple agents such as a production network. Also models without exogenous shocks, purely relying on endogenous dynamics to produce fluctuations can be cast in this 3-stage structure. In the case of endogenous business cycle models, shocks (1) are removed or play a minor role and the model would still produce fluctuations.

Shortly assessing which classes of models are "good models" in the above sense will make it easier to put the production network model into the context of business cycle models. Because all models have to fulfil the criterion of stage (3), I provide an overview which model features (exogenous vs endogenous dynamics, aggregate versus disaggregate structure) allow to match the stylized facts of aggregate output changes. Aggregate output changes possess at least **four** stylized facts[4]. The first stylized fact (SF1) is that aggregate output changes are fat-tailed[5] (Fagiolo et al. (2008), Williams et al. (2017)). Lera and Sornette (2018, p. 2) call this "a broadly accepted stylized fact" but there seem to be also studies that do not find clear evidence of fat tails (Franke 2015). With Gaussian shocks, exogenous business cycle models (Ascari et al. 2015) and also the production network model (Acemoglu, Ozdaglar, et al. 2017) (as a disaggregated exogenous business cycle model) do not produce fat-tailed output changes from thin tailed shocks. But in contrast to some exogenous business cycle models which make thin tailed fluctuations out of heavy-tailed shocks (Ascari et al. 2015, p. 467), the production network as a propagation mechanism is able to preserve the heavy tails from shocks to aggregate fluctuations. That means either models have to be developed to endogenously produce fat tails

[4] I call them stylized facts but there isn't the strong evidence for every of them, that one would want to call something a fact. Stylized facts is a term that is often used for hypothesis with some evidence in economics and I lack a better term.

[5] In the literature they are called fat-tailed. Earlier I defined fat-tails equal power laws. The fat tails of the output changes here are "tails much fatter than Gaussian ones (but with finite moments of any order)" (Fagiolo et al. 2008, p. 666). Power laws can also have infinite variance and thus the definition here does not include all fat tails. But "much fatter than Gaussian" also excludes the part, which is closer to thin tails in the spectrum of heavy tails. Thus I adopt the term fat tails here.

from Gaussian shocks or it might be possible that fat tails are not the product of a propagation mechanism but simply the exogenous shocks are already fat-tailed.

The second stylized fact (SF2) is a bimodal distribution where the left peak stands for small positive growth rates and the right peak for large positive growth rates Lera, Sornette, et al. (2017). Less time is spent between the higher and the lower growth rate. This can be interpreted as a switching between two regimes. "The two peaks of the bimodal distribution can be rationalized by the fundamental out-of-equilibrium nature of the economy that is switching between boom and bust states, i.e., business cycles." (Lera and Sornette 2018, p. 1). Different regimes for booms and a busts in aggregate output would be an argument for endogenous dynamics. But stochastic regime switching models which are often used to estimate time series of aggregate output (Morley, Piger, and Tien (2013), Kim et al. (2008)) are also able to model bimodality and are exogenous business cycle models, because the signal for a regime switch is an exogenous shock. The production network model has no regime switching behaviour nor endogenous dynamics and thus can not produce a bimodal distribution of aggregate output changes. But Lera and Sornette (2018) produce a bimodal distribution with a non-linear difference equation as an aggregate representation of a production network: "In our model, the microscopic firm network is represented by an effective 'economic potential function'" (Lera and Sornette 2018, p. 5).

The third stylized fact (SF3) is that the business cycle is asymmetric. That means the distribution of GDP changes is asymmetric in the way that there is "relatively small amplitude during mature expansions and substantial variation during and immediately following recessions" (Morley and Piger 2012, p. 208). Regime-switching models as exogenous business cycle models are also able to produce asymmetry in the distribution of aggregate output (Morley and Piger 2012). Again the production network model does not produce asymmetric output changes. While exogenous business cycle models in principle are able to reproduce the preceding 3 stylized facts, the exogenous nature of cycles in these models is in conflict with the fourth stylized fact (SF4), which says that aggregate fluctuations display regularity and are not just random fluctuations. In the current mainstream literature the question and explanation of cyclical fluctuations has faded into the background. This has led Beaudry et al. (2020) to call their article where they present some evidence for the regularity of fluctuations: "Putting the Cycle Back into Business Cycle Analysis". Endogenous cycles would be the result of an unstable dynamic system while the mainstream literature on business cycles views the economy as a stable system with exogenously driven fluctuations. There seems to be consensus in macroeconomics that aggregate output is a "near unit root process" (Cuestas and Garratt 2011) and therefore a stable system. More specific the estimation of linear difference equations as a model of aggregate output dynamics shows roots close to, but below 1 (because this finding is so prominent in the literature, one could call this a stylized fact as well). But Beaudry et al. (2017) show that the estimation of a nonlinear system, which the economy might actually be, with linear models can lead to the false conclusion that the system is stable. The problem is that leaving nonlinear terms out already from the beginning can lead to biased estimates of the process. Beaudry et al. (2017) show this for a nonlinear model which can generate limit cycles around an unstable fixed point. When excluding the nonlinear term a priori, the estimated behaviour of the fixed point turns from unstable to stable, which is what might have happened in the previous literature. Exogenous business cycle models clearly are not able to produce cyclical fluctuations and so is the production network model. One explanation for aggregate output as a nonlinear cyclical process is the emergence from the interaction of individuals. Letting a business cycle emerge in a disaggregated model from the interactions of heterogeneous agents is a relatively novel direction in research, but has already spawned some interesting model experiments. One very recent example is Asano et al. (2021) in whose model cycles emerge from the interactions of heterogeneous households. The cyclical behaviour comes from households taking into account neighbours' decisions when updating their states. About the role of exogenous shocks in their model they write: "Although our model has two random inputs, they are small and very different in character from the shocks that drive the dynamics of standard models." Asano et al. (2021, p. 6). Also Beaudry et al. (2020) propose a fine-grained interpretation for their nonlinear cyclical process. Their explanation is cast in a mean-field model, meaning the aggregate process is an emergent result of individual agents decisions interacting via an aggregate (global) variable (and not locally via direct network links between individuals). Positive feedback between individual agents states and their aggregate state [6] let individual decisions follow the direction of the aggregate decision but some kind of saturation reverts this process into the opposite direction at some point (Beaudry et al. 2020, p. 18-23). Also Aoki and Yoshikawa (2006) use a mean-field approach where fluctuations arise endogenously from imperfect quantity adjustments. Despite the mean-field approach where agents interact via an aggregate variable and an approach with local interactions such as Asano et al. (2021) differ, they both might be valuable on a way to explain cyclical dynamics with a heterogeneous disaggregated economy. Although also the production network model is built on heterogeneous interacting agents, no endogenous dynamics emerge after the shocks drive the production network out of its initial equilibrium. Rather it converges to a new equilibrium. But building more - but realistic - imperfections into the behaviour of sectors or firms (e.g. stochastic het-

[6] Beaudry et al. (2020) use the term "(strategic) complementarity" instead of positive feedback. One could also say increasing returns.

erogeneous adjustment time) could also lead to the emergence of endogenous cycles in the production network model. The "good model" probably lies somewhere between the exogenous and endogenous business cycle models. A very relevant question in this context is what happens when exogenous shocks interact with endogenous dynamics. This is interesting because it is well known that a nonlinear systems reaction to the same perturbation can be different in different points of the systems trajectory. A collection of articles on the reaction of nonlinear systems to perturbations (nonlinear impulse response) emphasize that only nonlinear dynamic systems can produce asymmetric reactions to symmetric shocks (Potter (1995), Dijk et al. (2007)). When negative exogenous shocks hit the recession phase of an endogenous business cycle, this could lead to a deeper downturn than in other phases. The production network model currently does not model this possible interaction of endogenous dynamics and exogenous shocks.

2.3 Economic growth (and decline)

Some recent literature tries to find structural determinants for economic growth encoded in production networks. Hidalgo and Hausmann (2009) find that the complexity of the production network is a predictor for future growth, McNerney et al. (2022) find that the output multiplier of an industry predicts future changes in prices, and that the average output multiplier of a country predicts future economic growth and Acemoglu and Azar (2020) suggest that the endogenous establishment of new links in the production network may be a driver of economic growth. This is arguably somewhat distinct from the production network literature concerned with shocks hitting the production network, because it focuses on how the current network structure determines the future development of the production network and therefore the future production capabilities while the latter is concerned with shocks hitting a fixed production network and investigating its reaction. I want to take a perspective on economic growth and decline without looking at network dynamics, by looking only at a fixed production network. In a nutshell, I try to get an idea about whether network effects are one reason why theory derived aggregate production functions are not empirically justified.

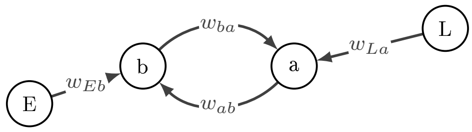


Figure 3: Stylized production network with factors energy and labour.

There are two pathways for increasing aggregate output: first, we can increase the efficiency of resource use (including the efficiency of intermediate input use). This is more the approach of the literature on economic growth and production networks described in the beginning. Second, we can use more resources. This is what I focus on in this section. Resources are the inputs which can not be produced by the production system and such inputs that cannot be produced are called factors in economics. Figure 3 shows how the factors energy and labour are linked to the production network. An aggregate production function of this stylized economy would include only energy and labour as inputs. A change in aggregate output can be expressed as the sum of factor changes multiplied with their output elasticities:

$$dY = \frac{\delta Y}{\delta L} dL + \frac{\delta Y}{\delta E} dE \tag{1}$$

In mainstream economic growth capital and labour are the only factors and every increase in aggregate output which can not be devoted to those factors is defined as an increase in their productivity. This is because the production factors are weighted with their so-called cost-share (Ayres, Bergh, et al. 2013). The cost-share is the expenditure on the factor as a percentage of GDP. So in mainstream economics, the output elasticities of the aggregate production function are assumed to equal cost-shares. Empirical cost-shares for capital and labour are typically very high and the cost share for energy very low. But there are subdisciplines of economics (ecological economics, exergy economics (Brockway et al. 2018)) which stress that the productivity that neoclassical economics[7] measures is so large because it actually includes neglected factors from the biosphere such as energy. The reason for neglecting energy in neoclassical production functions, is that the cost-share of energy is very low Ayres, Bergh, et al. (2013). Weighting the primary inputs by their cost share is an assumption derived from theory. Empirical studies which work without this assumption and derive their input weights with a high weight for energy input however have a good fit to the data with a small residual (Kümmel, Lindenberger, and Weiser 2015). When comparing the good fit of production functions without cost share weights to the bad fit of production functions with cost share weights, it appears that a large part of what neoclassical economists perceive as unobservable technological change is (1) increasing energy input and (2) increasing energy conversion efficiency (Ayres and Warr (2005), Sakai et al. (2018)).

Two *theoretical explanations* for why the cost-share theorem does not hold empirically can be found in the literature. One is that higher order effects in the production network amplify output changes above the direct effect Ayres, Bergh, et al. (2013, p.81-82). The identity output elasticity = cost share holds in a "two-factor single-good bakery" economy, Ayres, Bergh, et al. (2013) argue, but not in the real economy. In figure 3, cost-shares are basically the direct link weights of the factors (w_{La} and wEb), but when a factor change propagates through the network, the output elastic-

[7] Neoclassical economics here, because aggregate production functions with capital, labour and productivity are typical for neoclassical economics. Other subdisciplines of economics use also other factors.

ity is not equal to the input weight, because it changes due to the network structure of the whole production system. Ayres, Bergh, et al. (2013) make the example that a 50% cut in petroleum input whose output makes up 4% of the US GDP would only result in a 2% decrease in GDP without considering the influence on other sectors. But "the overall impact on the economy [...] would be far greater than 2%" Ayres, Bergh, et al. (2013, p. 82). The second reason is that constraints on input factor combinations change the input shares. This makes the cost share theorem "not even true for a single sector economy" Ayres, Bergh, et al. (2013, p. 82). This has led to the development of aggregate production functions which represent stronger physical constraints on the substitution of primary inputs than a Cobb-Douglas production function (e.g. Lindenberger et al. (2017), Keen et al. (2019)). Constraints on factor combination aside, in a similar way the cost share theorem might not even hold for an economy with only two factors (e.g. energy and labour in figure 3), when the production system has a network structure. Unfortunately usual input-output data does not include sectors representing factors. For factors coming from the biosphere, this limitation can be overcome by *environmental extensions* of input-output tables. They add additional sectors accounting for the primary energy input or the material resource input to the industries in input-output tables. In this way, the biosphere gets its own sector in input-output data. Why is it important to know the correct input share of energy in aggregate production functions? Recall again Stiglitz expression, that explaining deep downturns in macroeconomic variables is "the most important challenge facing any macro-model". It is quite obvious that a high output elasticity of the factor energy is qualified to dampen or reverse economic growth when available energy decreases due to peak oil or climate policy (see Ayres, Bergh, et al. (2013, p. 79-80) for this argument). To sum up, there are hints that macroeconomic growth models assume the wrong output elasticities, because they neglect constraints on factor combinations and network effects. Using data of an economy's production network might be a way to measure more accurate output elasticities.

3 The production network model

3.1 General mechanism

The production network model basically consists of two mechanisms: interaction between sectoral output changes (described in subsection 3.2) and aggregation of sectoral sectoral output changes (described in subsection 3.3). The interaction between sectors is essentially modelled as a linear diffusion process on a network. The production network initially is in an equilibrium in the sense that the quantities produced by each industry do not change. An output change somewhere in the network (a shock to one or multiple industries) then triggers a diffusion process: The shock first changes the output of direct customers and then propagates further through the network[8]. Because it is assumed that there are decreasing returns in single input changes, the process eventually comes to rest at a new equilibrium. The process is linear it can therefore be captured in a system of linear difference equations. System (3) shows how output changes influence the output of direct neighbours. $(1-\alpha)W$ is the diffusion matrix, and in this thesis I refer to it as the direct input share matrix, because its entries are input shares. Matrix $(1-\alpha)W$ is essentially a matrix which contains output elasticities (equation (2)). An element in $(1-\alpha)W$ captures how much the output of sector i changes in reaction to a change in supplier j's output.

$$(1-\alpha)W_{ij} = \frac{\delta y_i}{\delta y_j} \tag{2}$$

$$\frac{dy_{t+1}}{y_0} = (1-\alpha)W\frac{dy_t}{y_0} \tag{3}$$

When the actual dynamics between sectors are non-linear, system (3) can be seen as the linear approximation of a system of non-linear difference equations. Decreasing returns in single input changes guarantee that all eigenvalues of $(1-\alpha)W$ are smaller than 1 and the process (3) therefore converges to an equilibrium. System (4) shows the total change after shocks ϵ until reaching the new equilibrium. A tilde over the dy should indicate a total output change from one equilibrium to the other. Intermediate changes in-between both equilibria (equation (**??**)) are denoted without a tilde. Matrix $[I-(1-\alpha)W]^{-1}$ can be expanded to the series in (5). This shows that the total output change between the perturbed equilibrium and the new equilibrium can be written as the sum of the direct output changes due to shocks and of all higher order output changes.

$$\frac{\widetilde{dy}}{y_0} = [I-(1-\alpha)W]^{-1}\epsilon \tag{4}$$

$$[I-(1-\alpha)W]^{-1} = I + (1-\alpha)W + [(1-\alpha)W]^2 + ... = I + \sum_{k=0}^{\infty}[(1-\alpha)W]^k \tag{5}$$

These total sectoral changes can be aggregated to total aggregate output changes after a shock by multiplying a vector 1^T and the size of each sector in aggregate output. This equation relates aggregate output changes to sectoral shocks and is crucial for determining how features of the shock distribution carry over to the aggregate level.

$$\frac{\widetilde{dY}}{Y_0} = 1^T\frac{y_0}{Y_0}\frac{\widetilde{dy}}{y_0} = 1^T\frac{y_0}{Y_0}[I-(1-\alpha)W]^{-1}\epsilon \tag{6}$$

[8] In equations (3) - (6) I use relative changes, because the production network model with linear and log-linear production functions is interpreted in relative changes.

Matrix $[I-(1-\alpha)W]^{-1}$ is the matrix of *total* input shares (often matrices in this form are called Leontief inverse). I also refer to this matrix as the output multiplier It is the tool which captures the network structure for the total diffusion process: "This matrix [...] accounts for all possible direct and indirect effects of interactions between any pair of agents" (Acemoglu, Ozdaglar, et al. 2016). In contrast, as written above, the *direct* input shares matrix accounts for all possible direct effects between sectors. In the next section I will show how the equations for shock propagation between sectors can be derived simply from the assumption that a production network is a network of sectors with (log-)linear production functions. Section 3.3 then shows the derivation of network statistics which tell something about the relation between the shock distribution and the distribution of aggregate output changes.

3.2 Interaction between sectors

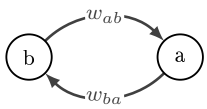
3.2.1 Linear and log-linear production functions

In a production network the relationship between the inputs a sector receives and its output is defined by a production function. These micro production functions (as opposed to its macroeconomic pendant, the aggregate production function) capture the direct effects between sectors. Figure 4 shows a minimal example network consisting of two sectors. Sector a) receives input from sector b) but also supplies sector b) with input. Thus, sector a) and b) have the production functions $y_a = f(y_b)$ and $y_b = g(y_a)$. If there is a relative change[9] in the output of sector b) (dy_b/y_{b0})[10], also the output of sector a) changes $(dy_a/y_{a0} = f(dy_b/y_{b0}))$[11], this is shown on the right side of figure 4. Because sector a) is also supplier to sector b), the output change in sector a) will change the output of sector b) another time (indicated by the subscript 2 in d_2y_b), resulting in the nested production function $d_2y_b = g(f(dy_b))$. This output change is again propagated to sector a) as figure 4 shows. To get the total output change until the new equilibrium (if an equilibrium is reached at all), all output changes for each sector have to be added. The total output change for sector a) is then $\tilde{dy}_a = f(dy_b) + f(g(f(dy_b))) + ...$, and for sector b) $\tilde{y}_b = dy_b + g(f(dy_b)) + ...$ The dots indicate that the process goes on for an infinite number of steps. A new equilibrium is only reached when the sums converge to a constant value. What we see is that higher order output changes can be calculated with nested production functions with the first order (initial) output change as the only input. This reminds of the series expansion (5). Thus if the production functions are linear, the described

9 I work with relative changes, because as explained in more detail below in this section a 3% change in output of one sector implies a 3% change in input to another sector when we assume that sectors serve customers proportional to the input quantity prior to the shock.

10 y_{b0} is the initial output of sector b).

11 This depends of course on the type of the production function. The output of sector a) can only increase with a production function that allows substitution among inputs. With a Leontief production function (fixed input shares), the output of sector a) will not increase when only one input increases.



$$dy_a = f(dy_b) = w_{ab} dy_b$$
$$\downarrow$$
$$d_2 y_b = g(dy_a) = g(f(dy_b)) = w_{ba} w_{ab} dy_b$$
$$\downarrow$$
$$d_2 y_a = f(d_2 y_b) = f(g(f(dy_b))) = w_{ab} w_{ba} w_{ab} dy_b$$
$$\downarrow$$
$$\ldots$$

Figure 4: Simple 2-sector network with linear production functions

process of nesting the production functions can be cast in the form of equation (4). The edge weights in figure 4 are the input shares. Assuming it is linear, the production function of sector a therefore becomes $y_a = f(y_b) = w_{ab} y_b$ and it is assumed that the change in output of sector a due to an output change in sector b can be written as the production function in relative differences $dy_a/y_{a0} = w_{ab} dy_b/y_{b0}$. The right side of figure 4 also shows the shock propagation for the linear production function case. At this point the diffusion equation (4) come into play. Because of the linearity of the production functions the input shares can be cast into matrix W_{simple} and the diffusion process is then sufficiently defined by a repeated multiplication of matrix W_{simple} with the vector of previous output changes. Given all eigenvalues of W_{simple} are smaller than 1 (lie inside the unit circle), the vector of total relative output changes $\tilde{dy}/y_0$ can be calculated analogous to equation (4) with the matrix 7.

$$W_{simple} = \begin{pmatrix} 0 & w_{ab} \\ w_{ba} & 0 \end{pmatrix} \tag{7}$$

Of course linear production functions are unrealistic, because they imply that output can be produced with an arbitrary amount of each input alone (no constraints on input substitution). I therefore turn now to log-linear production functions (more specifically Cobb-Douglas production functions), because they have decreasing returns in single inputs and are the standard in the production network literature[12].

So one way of deriving equation (4), and the way that best builds intuition for the model in my opinion, is by plugging production functions into each other infinitely many times. The second way is more similar to how the classic input-output model usually is derived[13]. Again we start from the production function as our model of behaviour when firms or sectors face changes in their input quantities. The Cobb-Douglas production function multiplies inputs. The w_{ij} are the input shares of the

[12] Literature originating from Acemoglu, Carvalho, et al. (2012) mainly use Cobb-Douglas production functions. For an overview of production functions used in modelling the effect of COVID-19 lockdowns on the production network see Pichler, Pangallo, et al. (2020, p. 3)

[13] To see where the differences lie to the derivation of the classical input-output model, check appendix 6.4.

industrial inputs adding up to 1. They are multiplied with $(1-\alpha)$ because α is the input share of labour input and as in Cobb-Douglas production functions all input shares have to add up to 1, the remaining part is allocated to the industrial inputs. T_i is the sector specific level of productivity and l_i the labour input.

$$y_i = f(x_j, l_i) \tag{8}$$

$$y_i = T_i l_i^{\alpha} \prod x_{ij}^{(1-\alpha)w_{ij}} \tag{9}$$

Taking the natural logarithm leads to a linear production function (10). All output quantities are measured in logarithmic terms now. Then this equation is differentiated.

$$ln(y_i) = ln(T_i) + \alpha ln(l_i) + \sum_j (1-\alpha) w_{ij} ln(x_{ij}) \tag{10}$$

$$dln(y_i) = dln(T_i) + \alpha dln(l_i) + \sum_j (1-\alpha) w_{ij} dln(x_{ij}) \tag{11}$$

Changes of logarithmic values don't appear to be interpretable very well at first, but they have a useful feature. For small differences $x_2 - x_1$, the change in natural logarithmic values approximates relative changes[14]:

$$ln(x_2) - ln(x_1) \approx \frac{x_2 - x_1}{x_1} \tag{12}$$

This is different to the linear production function, where changes have to be divided explicitly by the initial value to result in relative changes. In both cases, relative changes do the "trick" that the output change of sector i depends on output changes of its suppliers $dln(y_j)$ instead of input changes $dln(x_{ij})$. A relative change in input x_{ij} (e.g. the input increases by 5%) is equal to the relative change in output of this supplier y_j (output of industry j has increased by 5%) because it is assumed that an industry allocates additional output to its customers in proportion to what these industries already receive[15]. This implies $\frac{dx_{ij}}{x_j} = \frac{dy_j}{y_j}$. (11) can therefore be rewritten as

$$dln(y_i) = dln(T_i) + \alpha dln(l_i) + \sum_j (1-\alpha) w_{ij} dln(y_j) \tag{13}$$

[14] Throughout their article Acemoglu, Akcigit, et al. (2016) do not explicitly mention this, but consistently interpret logarithmic changes as relative changes. For a reference to equation (12), see https://stats.stackexchange.com/questions/244199/why-is-it-that-natural-log-changes-are-percentage-changes-what-is-about-logs-th

[15] This is an assumption which must not be realistic for every application field of the production network model. Pichler, Pangallo, et al. (2020) for example analyse how their models results change for different allocations, when firms face negative output shocks and have to decide which customer to serve in which amount when their demand hasn't changed.

Leaving labour aside, as there are no shocks to labour in this thesis and also leaving productivity aside first, the production function expressed in relative changes becomes:

$$dln(y_i) = \sum_j (1-\alpha) w_{ij} dln(y_j) \tag{14}$$

Writing the resulting system of equations for all production functions i in vector-matrix notation, this is diffusion equation (3):

$$dln(y) = (1-\alpha) W dln(y) \tag{15}$$

Because such a diffusion process is a process happening between the initial and the new equilibrium, we can think of the right hand side output change as the previous periods output change and the left-hand-side output change as the current period output change. We assume the economy is initially in an equilibrium, that is characterised by previous and therefore current period output change being both zero. That means all sectoral outputs in the economy are constant over time.

$$dln(y_{t+1}) = (1-\alpha) W dln(y_t); \quad dln(y_{t+1}) = dln(y_t) = 0 \qquad \text{initial equilibrium} \tag{16}$$

Now a vector of shocks can be added to perturb the production network from its equilibrium. Due to the identity (12) the shock size as well can be interpreted as relative changes. So a shock is always a x% shock to a sectors initial output. As the initial output change $dln(y_0)$ is zero, the first round of output changes reduces to the vector of shocks. $dlny_0$ is still zero but due to the added shocks $dlny_1$ can't be zero any more. Due to the perturbation of the equilibrium, a diffusion process starts and the outputs y change from period to period. Increasing an input, which contributes to 20% to output, by 5% leads to an output increase by 20

$$dln(y_1) = (1-\alpha) W dln(y_0) + \epsilon = \epsilon \qquad \text{disturbed equilibrium} \tag{17}$$

The second round of output changes is the vector of shocks multiplied with the matrix of input shares. The third round is then the vector of shocks multiplied with the squared matrix of input shares and so on.

$$dln(y_2) = (1-\alpha) W dln(y_1) = (1-\alpha) W \epsilon \tag{18}$$

Output changes diffuse through the network until a new equilibrium is reached. (16) is a system of difference equations and therefore the diffusion process converges when all eigenvalues of $(1-\alpha)W$ are strictly smaller than 1. As matrix W is shrinked by $1-\alpha$, we assume that this is always the case. Equation (19) holds for the total relative output change $\widetilde{dln(y)}$ at which the diffusion process after a perturbation by shocks comes to rest. By rearranging the equation, it becomes clear that the total relative output change equals the vector of shocks multiplied with the inverse matrix $[I - (1-\alpha)W]^{-1}$. As this matrix captures all direct and indirect effects from the perturbation, the vector $\widetilde{dln(y)}$ captures the total relative change in sectors outputs.

This process carries on until a new equilibrium is reached. The change from the initial equilibrium to a new equilibrium after the shock is defined as the point where outputs don't change any more:

$$\widetilde{dln(y)} = (1-\alpha)W\widetilde{dln(y)} + \epsilon; \qquad \text{new equilibrium} \tag{19}$$

$$\widetilde{dln(y)} = [I - (1-\alpha)W]^{-1}\epsilon \tag{20}$$

Shocks ϵ can be shocks to the initial output level such as a forced cut of production by a government command (e.g. a COVID-19 lockdown) or also a shock to the productivity of the sector. Shocks from natural disasters or internal changes within the firm or sector can probably be better understood as productivity changes. For the mathematical framework this difference does not matter, I could have just continued to include the productivity changes from (13) on and then ϵ would become T. The matrix of input shares $(1-\alpha)W$ contains the input shares of the intermediate inputs. W is the matrix of inter-industry input shares only. The input shares of each sector in W sum up to one. They are calculated from the matrix of inter-industry transactions Z_{ij} (input-output matrix), where sales flow from j to i, by dividing the sales from j to i by the total sales to sector i:

$$W_{ij} = \frac{Z_{ij}}{\sum_j Z_{ij}} \tag{21}$$

The inter-industry input shares to one sector therefore add up to 1. The sum of input shares in the actual matrix of input shares $(1-\alpha)W$ is smaller than 1, because the sector also receives input from the factor labour with input share α. $(1-\alpha)$ is therefore the remaining input share for all industrial sectors. From a mathematical perspective it is necessary to shrink the row-stochastic matrix W with a factor $1-\alpha$, otherwise the power series does not converge and no Leontief inverse is found.

$$\alpha_i + \sum_j (1-\alpha)w_{ij} = 1 \tag{22}$$

Computing matrix W with equation (21) is equivalent to calculating the direct requirements matrix (or technical coefficient matrix) A (used in classical input-output analysis) and normalising it by its row sums:

$$A_{ij} = \frac{Z_{ij}}{y_i} \tag{23}$$

$$W_{ij} = \frac{A_{ij}}{\sum_j A_{ij}} \tag{24}$$

I emphasize this relationship because Acemoglu, Carvalho, et al. (2012) compute their empirical matrix W by normalising the empirical matrix A. Moreover, this highlights the difference to classical input-output models, where matrix A is the diffusion matrix for upstream propagation of shocks. Both matrices W and A encode different information and are therefore used for different purposes. Matrix W contains the direct input shares of each sector while A contains the direct requirements a sector needs to produce one unit of output. Section 3.2.2 shows that matrix A is also used for downstream propagation if production functions are Leontief. I also want to highlight that a meaningful diffusion matrix must not be restricted to the matrix of input shares $(1-\alpha)W$ calculated from input-output data. Klimek et al. (2019) calculate a diffusion matrix by using information about past industry output adjustments. According to the authors, their matrix can be used to study how large shocks drive the production network from its initial equilibrium into a nonequilibrium stationary state. This view is different from the major part of the production network literature where it is assumed that the production network returns to a new equilibrium after facing small shocks. The difference is that they model demand shocks and therefore upstream propagation. Concerning their models performance, they state that "out-of-sample predictions from the LRT model consistently outperform standard econometric forecasting methods, such as different types of ARIMA model" (Klimek et al. 2019, p.6).

3.2.2 Nonlinear production functions

Matrix multiplication only allows to interpret the production functions as being linear (including log-linear). Nonlinear production functions cannot be represented by such a linear equation system. If other production functions appear to be more realistic, other mechanisms than matrix multiplication have to be used. This also means that the analytical statements for aggregate volatility and macroeconomic tail risk in section 3.3 cannot be applied. Nevertheless, the Leontief production function as a special case of nonlinear production functions has a crucial feature for modelling (at least) short term shocks to the production system like the COVID-19 lockdowns: There are strong constraints on the substitution of inputs. Although using log-linear production functions Mandel and Veetil (2020) explain the implications of substitutability on point: "The crucial determinant of supply chain impacts is the elasticity of substitution, or more precisely whether inputs are complements ($\sigma < 1$) or substitutes ($\sigma \leq 1$). If inputs are strong complements, supply chains are completely disorganised during the lockdown. This leads to a massive amplification — more than two folds — of the direct impact of lockdown through supply chain effects". Also Pichler and Farmer (2021) write: "Our analysis makes it clear that bottlenecks in supply chains can strongly suppress aggregate economic output. The extent to which this is true depends on the production function. These effects are extremely strong with the Leontief production function, are much weaker with a linear production function (which allows unrealistically strong substitutions) and have an intermediate effect with our modified Leontief function." A model of downstream propagation with Leontief production functions makes use of the direct requirements matrix A (calculated with equation (23)) instead of the matrix of direct input shares W. To compute the feasible output of sector i with given input quantities x, the inputs x are divided by the direct requirement A_{ij} of sector i for that input j. This gives an output quantity that can be reached with each input. Minimising over this set gives the maximum output quantity that can be reached (equivalent to the output that can be reached with the most limiting input).

$$y_i = min_j \left\{ \frac{x_j}{A_{ij}} \right\} \tag{25}$$

Although it would be interesting to compare the results, the analytical production network model is not qualified to model propagation with Leontief production functions. The model of Pichler and Farmer (2021) expressed in computer code not only uses Leontief production functions for downstream propagation but moreover a mix of Leontief (for critical inputs) and linear (no limits on other outputs) production functions. Non-linear interaction functions between sectors must not only include the production function. Also when a sector has inventories and therefore the capability to absorb shocks this can impose non-linearity in the interaction function (see Luu et al. (2018)). At this point it might be worthwhile to note that in classical input-output models of upstream propagation, production functions are Leontief (see Appendix 6.4 for a demonstration).

3.2.3 Prices

Another simplification of the model is the absence of prices as a control variable of the sectors. Klimek et al. (2019, p. 7) emphasize: "A limitation of the Leontief IO model that extends to our work is that prices play no role in the model. Firms in real economies can respond to shocks by adjusting produced quantities as well as prices." A good reason for ignoring prices is the short run. Constant prices in the short run might not be an unrealistic assumption if we have in mind that suppliers and customers in the supply chain often have long term contracts with fixed prices. For example, concerning shock propagation due to supplier bottlenecks during the COVID-19 pandemic lockdowns Pichler, Pangallo, et al. (2020) write: "[...] we do not model prices, as we argue that price changes during the lockdown are relatively small". Following Acemoglu, Akcigit, et al. (2016) some theoretical statements about prices in the production network model can be derived when one assumes profit maximization and cost minimization as additional rules of behaviour of sectors. A sector with production function $f()$ can maximise its revenue by solving:

$$max_{x_k} f(x)p_i - \sum_j p_j x_j \tag{26}$$

This leads to the first order condition:

$$p_i \frac{\delta f(x)}{\delta x_{ik}} - p_k = 0 \tag{27}$$

In case of Cobb-Douglas (equation 9), the derivative of the production function is

$$\frac{\delta f(x)}{\delta x_{ik}} = \frac{(1-\alpha)w_{ik} x_{ik}^{(-\alpha w_{ik})} T_i l_i^{\alpha} \prod_{j \neq k} x_{ij}^{(} 1-\alpha) w_{ij}}{p_i} \tag{28}$$

$$= (1-\alpha)w_{ik} \frac{T_i l_i^{\alpha} \prod_j x_{ij}^{(} 1-\alpha) w_{ij}}{x_{ik}} \tag{29}$$

$$= (1-\alpha)w_{ik} \frac{y_i}{x_{ik}} \tag{30}$$

Inserting this into the first order condition yields

$$p_i (1-\alpha) w_{ik} \frac{y_i}{x_{ik}} = p_j \tag{31}$$

$$(1-\alpha)w_{ik} = \frac{p_k x_{ik}}{p_i y_i} \tag{32}$$

Acemoglu, Carvalho, et al. (2012) as well as Torres-González and Yang (2019) find that the technical coefficients a_{ij} in A, where $(1-\alpha)w_{ik}$ is derived from, are quite stable over time. Therefore an increase in input must be levelled by a proportional increase in output. Otherwise either the input price must increase or the output price decrease. If the coefficients where not constant this would influence the propagation of shocks.

$$0 = dln(1-\alpha)w_{ik} = dlnp_k + dlnx_{ik} - dlnp_i - dlny_i \tag{33}$$

The production network model in this thesis works with shocks to sectoral output and propagation via input quantities. Acemoglu, Carvalho, et al. (2012) note that negative shocks to a sector would usually change its output quantity as well as output price: "In general, sectoral shocks also affect upstream production through a price and a quantity effect. For instance, with a negative shock to a sector, (i) its output price increases, raising its demand for inputs; and (ii) its production decreases, reducing its demand for inputs. With Cobb-Douglas production technologies, however, these two effects cancel out.[...]". And Acemoglu, Akcigit, et al. (2016, p. 283) use productivity shocks and argue that downstream propagation of productivity shocks works also via prices and not only quantity changes: "Downstream propagation, on the other hand, is a consequence of the fact that an adverse productivity shock to a sector leads to an increase in the price of that sector's output, encouraging its customer industries to use this input less intensively and thus reduce their own production." Figure 5 depicts how such a propagation process must work given the constraint (33). Following a decrease in productivity T_i the output price p_i increases, this decreases the input x_{ji} (which means also y_i decreases and this counters the increase of p_i in (33)). Due to the decrease in x_{ji} also the output of sector j must decrease. Then because y_j decreases, p_j must decrease, so (33) holds. Imagining this is the propagation process also through the rest of the production network, one can really speak of propagation of output changes via prices here. Previously I have just assumed that a productivity shock T_i directly reduces output y_i, which then reduces input x_{ji}, then output y_j and so forth. Admittedly the first output change y_i then is not countered by a change in inputs to i, input or output price and hence would change the coefficients $(1-\alpha)w_{ik}$ (with k the suppliers of i) if (32) holds.

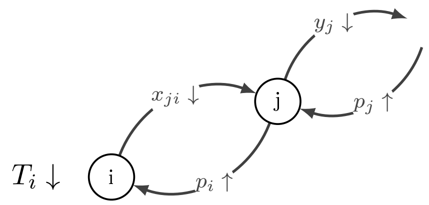


Figure 5: Price and quantity adjustment in a stylized production network.

But how does it come that the productivity change leads to a change in price p_i? When one assumes perfect competition within each sector, then the rule of behaviour of firms concerning prices is: price equals marginal cost. This is the pendant to the production function, which is the rule of behaviour concerning output changes. Let's assume the unit cost function C_i is the product of each input price times the input share. Input price is weighted with the input share, because input price is for one unit of input, but one unit of output requires only a fraction of one unit of input, namely the input share. From (36) it clearly follows that a decrease in productivity T_i leads to an increase in price p_i.

$$p_i = C_i(p) \tag{34}$$

$$C_i(p) = \frac{1}{T_i} \prod_j p_j^{\omega_{ij}} \tag{35}$$

Applying the natural logarithm and taking the total derivative

$$dlnp_i = dln\frac{1}{T_i} + \sum_j \omega_{ij} dlnp_j \tag{36}$$

In matrix notation, can then be written as:

$$dlnp = Wdlnp + ln\frac{1}{T} \tag{37}$$

This captures the first order effects, reaction of sector i's output price to price changes in its direct neighbourhood. The total effect is given by (38). Again, there has to be some friction $(1 - \alpha)$, otherwise price propagation will go on forever and no matrix $(I - W)^{-1}$ exists.

$$\widetilde{dlnp} = (I - (1 - \alpha)W)^{-1} dln\frac{1}{a} \tag{38}$$

Instead of one equation modelling quantity and price propagation we now have two separate equations for both cases. Price and quantity propagation appear to be separated in the production network models currently, despite the reasoning of Acemoglu, Akcigit, et al. (2016) that the channel of output changes after productivity shocks is also through prices. I have explained this reasoning with the help of figure 5. Price propagation there is included implicitly in the model of quantity propagation. Judged from its mathematical structure (38), also the model of price propagation seems not to interfere with quantities. The assumption of perfect competition within a sector (output price equals unit costs) leads to full pass-through of costs. This means that when a sectors marginal cost of production increase, this will be completely taken up by an increase in its output price. Full cost pass-through is a common assumption in the input-output literature (Cahen-Fourot et al. 2020). Network propagation of prices also starts to become a topic in the literature. For example, Duprez and Magerman (2018) estimate a model of how firms in a production network change their output prices in response to price changes in their network environment with empirical data. They discard the hypothesis of full pass-through of cost shocks in the Belgian production network. Firms also adjust their markups (profit). In the end such empirical analyses have to show whether and which case there are price and quantity effects in a real economy and in which direction (downstream or upstream) they propagate.

3.3 Aggregation of sectoral changes

3.3.1 Network structure and aggregate volatility

So far, it has been shown how shocks to one sector can lead to changes in other sectors output. But a main focus of macroeconomics is on the properties of *aggregate* output. Acemoglu, Carvalho, et al. (2012) derive the statement that in an economy with a degree distribution following a power law in the tail, aggregate volatility vanishes with a rate less than $\sqrt{n}$, when the production network gets more disaggregated. I will provide an explanation for this insight which is not as mathematically rigorous as in Acemoglu, Carvalho, et al. (2012) and focuses more on building intuition. I start with the observation that a change in relative aggregate output is the sector size weighted sum of relative sectoral output changes:

$$\frac{dY}{Y_0} = \sum_i \frac{y_{i0}}{Y_0}\frac{dy_i}{y_{i0}} \tag{39}$$

Again because relative changes are approximated by changes in logarithmic output in the log-linear case, this can be written as

$$dln(Y) = \sum_i \frac{y_{i0}}{Y_0} dln(y_i) \tag{40}$$

For the total sectoral changes $\tilde{dy_i}$ after a shock, I plug in equation (19). Because all output changes are total output changes from the perturbed equilibrium to the new equilibrium from now on, I drop the tilde here. The influence due to the size of the sector and the network importance of the sector can be summed up in the influence vector $\tilde{v}$.

$$dln(Y) = 1^T \frac{y_0}{Y_0} dln(y) = 1^T \frac{y_0}{Y_0}[I-(1-\alpha)W]^{-1}\epsilon = \tilde{v}\epsilon \tag{41}$$

The influence vector $\tilde{v}$ is:

$$\tilde{v} = 1^T \frac{y_0}{Y_0}[I-(1-\alpha)W]^{-1} \tag{42}$$

The tilde over the v indicates that this is the influence vector with heterogeneous sector sizes y_0/Y_0. As the focus is on the network effects, I follow Acemoglu, Carvalho, et al. (2012) in assuming that all sectors have uniform size. Therefore, the output share becomes $y_0/Y_0 = 1/n$ in the sections remainder[16]. In this way the influence vector v captures the influence of shocks on aggregate output for uniformly sized sectors.

[16] In Acemoglu, Carvalho, et al. (2012) the sectoral changes are weighted by α/n. The α appears there because their aggregate output equals GDP. In this thesis I don't define aggregate output as GDP because this involves dealing with consumption. In this thesis aggregate output is simply the sum of sectoral outputs.

$$v = 1^T \frac{1}{n}[I - (1-\alpha)W]^{-1} \tag{43}$$

The influence vector determines the relationship between the shock properties and the properties of aggregate output changes. It contains the output multiplier $[I - (1-\alpha)W]^{-1}$, which can be expanded to the following series:

$$v = 1^T \frac{1}{n} \left(I + \sum_{s=1}^{\infty} W^s \right) \tag{44}$$

$$v = \frac{1}{n} \left(I + \sum_{i=1}^{n} W_{ij} + \sum_{j=1}^{n} \sum_{k=1}^{n} W_{ij} W_{ki} + ... \right) \tag{45}$$

For my choice of matrix $(1-\alpha)W$, the series converges to a constant: "[...] [T]he fact that all eigenvalues of $(1-\alpha)W_n$ lie strictly inside the unit circle means that v_n can be expressed in terms of the convergent power series [...]" (Acemoglu, Carvalho, et al. 2012, p. 2009). An important finding is, that the first sum in (45) is the out-degree of the production network as defined in Newman (2018, p. 130):

$$k = \sum_{i=1}^{n} W_{ij} \tag{46}$$

The second sum is the second order out-degree as Anufriev et al. (2016, p. 7) defines it:

$$k_{2nd} = \sum_{j=1}^{n} \sum_{k=1}^{n} W_{ij} W_{ki} \tag{47}$$

Next, I will show how to arrive at an expression linking aggregate volatility - the standard deviation of aggregate output changes - to the out-degree distribution and shock distribution. I will start with the aggregate volatility for an economy without sectoral interactions and then carry this result over to the networked economy. In an economy without interconnections, the change in aggregate output is simply the sum of sectoral shocks weighted with the size of each sector. δ_i should be the relative size of sector i and this is equal to $1/n$ if each sector has the same size. Applying the variance operator to (48), it can then be shown that the variance of aggregate output equals (49). The proof is a standard procedure in statistics and left to the Appendix 6.1.

$$dln(Y) = \delta_1 \epsilon_1 + \delta_2 \epsilon_2 + ... + \delta_n \epsilon_n = \sum_{i=1}^{n} \delta_i \epsilon_i \tag{48}$$

$$var(dln(Y)) = \sum_{i=1}^{n} \delta_i^2 \sigma_{\epsilon_i}^2 \tag{49}$$

For $\delta_i = 1/n$, equation (49) becomes

$$var(dln(Y)) = \sum_{i=1}^{n} \frac{1}{n^2} \sigma_{\epsilon_i}^2 \tag{50}$$

Assuming all shocks have equal variance, the sum of variances becomes $n\sigma_\epsilon^2$ and thus

$$var(dln(Y)) = \frac{1}{n} \sigma_\epsilon^2 \tag{51}$$

Using the standard deviation instead of the variance,

$$var(dln(Y_n))^{1/2} = \frac{1}{\sqrt{n}} \sigma_\epsilon \tag{52}$$

As the standard deviation of shocks σ_ϵ does not change with n, the standard deviation of aggregate output changes grows with $1/\sqrt{n}$ for $n \to \infty$ (so it shrinks). This can be expressed in O-notation:

$$var(dln(Y_n))^{1/2} \in \Theta\left(\frac{1}{\sqrt{n}}\right) \tag{53}$$

The subscript n in Y_n indicates the level of aggregation of the economy. For small n the economy is disaggregated in few sectors and for large n the economy is disaggregated in many sectors. When going more disaggregated $n \to \infty$ the variance in (52) decreases with rate $\sqrt{n}$. This is exactly the theoretical argument that has long been common sense in economics. Microeconomic shocks on a high level of disaggregation n wash out to a large extent in the aggregate. In a connected economy in contrast, an output change in one sector is not just equal to the own shock to this sector. Inserting the influence vector v for δ, (49) becomes:

$$var(dln(Y_n)) = \sum_{i=1}^{n} v_{in}^2 \sigma(\epsilon_{in})^2 \tag{54}$$

Assuming the variance of shocks is identical across sectors and independent of the level of disaggregation n, this can be rewritten

$$var(dln(Y_n))^{1/2} = \sqrt{\sum_{i=1}^{n} v_{in}^2} \sqrt{\sigma(\epsilon)^2} \tag{55}$$

The first factor on the right hand side is the euclidean norm of the influence vector $\parallel v_n \parallel$.

$$var(dln(Y_n))^{1/2} = \parallel v_n \parallel \sqrt{\sigma(\epsilon)^2} \tag{56}$$

Therefore aggregate volatility in a networked economy grows with the euclidean norm of the influence vector for $n \to \infty$.

$$var(dln(Y_n))^{1/2} \in \Theta\left(\| v_n \|\right) \tag{57}$$

To assess whether the influence vector differs from the rate $1/\sqrt{n}$ at which aggregate volatility in the unconnected economy changes for $n \to \infty$, one has to determine how the influence vector v_n depends on n. For this assessment, the fact is used that v_n can be written as a convergent power series.

$$v_n = \frac{1}{n} 1^T \sum_{k=0}^{\infty} [(1-\alpha)W_n]^k \tag{58}$$

Because this series converges, every new summand is smaller than the previous one and therefore a large part of v_n is in the first two terms of the sum (k=(1,2)). Acemoglu, Carvalho, et al. (2012) use this fact to define a lower bound for the value of v_n.

$$v_n \geq \frac{1}{n} 1^T + \frac{(1-\alpha)}{n} 1^T W_n \tag{59}$$

With the right hand side of (59), we have an easy to handle approximation of v_n from below. If this lower bound suffices to increase the scaling of aggregate volatility over $1/\sqrt{n}$, then the exact v_n would increase scaling even more. The squared euclidean norm (see Appendix 6.2 for detailed algebraic expressions) of (59) is

$$\begin{aligned} \| v_n \|^2 &\geq \frac{3-2\alpha}{n} + \frac{(1-\alpha)^2}{n^2} \| W_n^T 1 \|^2 \\ &\geq \Theta\left(\frac{1}{n}\right) + \Theta\left(\frac{1}{n^2} \sum_{i=1}^{n} (k_i^n)^2\right) \end{aligned} \tag{60}$$

When taking the square root of this expression, it becomes obvious that aggregate volatility scales with more than $1/\sqrt{n}$ in an economy with network structure.

$$\| v_n \| \geq \Theta\left(\frac{1}{\sqrt{n}}\right) + \Theta\left(\frac{1}{n}\sqrt{\sum_{i=1}^{n} (k_i^n)^2}\right)$$

(61)

Moreover from (3.3.1), it can be inferred that "[...] aggregate volatility is higher in economies whose corresponding degree sequence have a 'heavier tails.'" (Acemoglu, Carvalho, et al. 2012, p. 1991). This can intuitively be seen in the following way. As shown in (62) for two distinct sequences of numbers of equal length with identical sum (and identical mean therefore), the sum of the squared sequence of numbers is higher which has higher variance. In this style, for two degree distributions with identical mean, the distributions degree sequence is higher which has the higher variance. In most cases, a heavier tail implies a higher variance and thus the heavier the tail the larger is the second term in (3.3.1)[17]. This means that an economy with a heavy tailed out-degree distribution has higher aggregate volatility from microeconomic shocks than an economy with thin tailed out-degrees or without network structure, given all of them consist of the same number of sectors n.

$$\begin{aligned} Var(x_1) &> Var(x_2) \\ E(x_1^2) - E(x_1)^2 &> E(x_2^2) - E(x_2)^2 \\ E(x_1^2) &> E(x_2^2) \\ \sum x_1^2 &> \sum x_2^2 \end{aligned} \tag{62}$$

One can also compute a lower bound of the euclidean norm of the influence vector v including second order effects and show that heavy-tailed first and second order out-degrees lead to higher aggregate volatility than out-degree distributions with thin tails. This can be looked up in Acemoglu, Carvalho, et al. (2012).

3.3.2 Network structure and heavy-tailed aggregate output changes

Volatility is only one property of the distribution of aggregate output changes that depends on the network structure. Another one is the volume of the tail of the distribution. Acemoglu, Ozdaglar, et al. (2017) show that whether heavy-tails in microeconomic shocks can also appear in the aggregate, depends on the network structure of an economy. Sectoral dominance is a statistic for the presence of heavy-tails in aggregate output. It is a function of the influence vector:

$$\rho = \frac{v_{max}}{||v||/\sqrt{n}} \tag{63}$$

To arrive at this statistic, Acemoglu, Ozdaglar, et al. (2017) define that a variable has a heavy-tail if its probability for tail events is larger than the probability would be if the variable was normally distributed. (64) compares the log probability of variable Y having realisations in the left tail below τ times the standard deviation of Y to the probability that the log standard-normal distribution has realisations left of τ times its standard deviation (σ of N(0,1) is 1, Φ is the CDF of the standard-normal distribution). Variable Y has a heavier tail below some value τ than the

[17] Although in two different distributions with identical first and second moment, one distribution can have a heavier tail (https://stats.stackexchange.com/questions/180952/t-distribution-having-heavier-tail-than-normal-distribution and https://stats.stackexchange.com/questions/86429/which-has-the-heavier-tail-lognormal-or-gamma), this intuition based explanation should apply for many combinations two degree distributions.

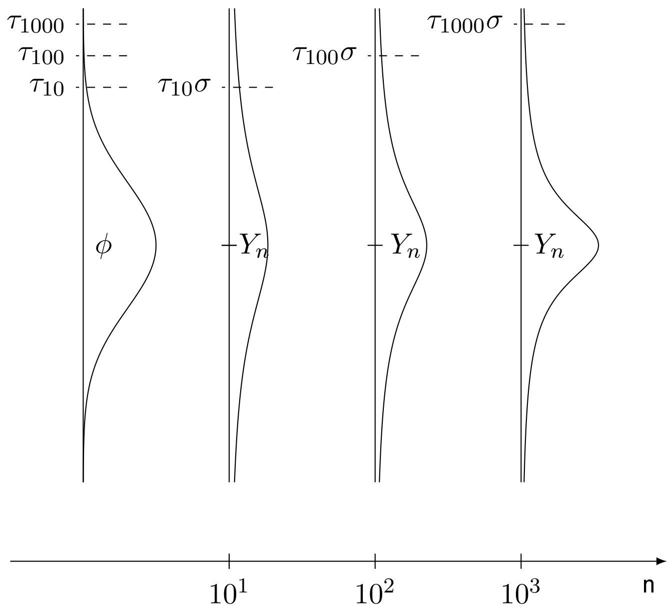


Figure 6: Tail volume for different levels of disaggregation n.

normal distribution if $Pr(Y < -\tau\sigma) > \Phi(-\tau)$. $R(\tau)$ therefore is 1 if the tails of Y (Y is the distribution of aggregate output changes) and the normal distribution are equal. Because logarithms turn the relationship around ($0.5 < 0.9$ but $|ln(0.5)| > |ln(0.9)|$), $R(\tau)$ is smaller than 1 if Y has a heavier tail. τ is the variable that determines where the tail of a distribution starts and as one goes further away from the mean, τ increases. In case of a heavy-tail, Acemoglu, Ozdaglar, et al. (2017) argue the difference between numerator and denominator in (64) should become more pronounced as further as we go away from the mean. Therefore for $\tau \to \infty$, $R(\tau)$ will converge to 0 if Y is more heavy-tailed than the normal distribution.

$$R(\tau) = \frac{logPr(Y < -\tau\sigma)}{log\Phi(-\tau)} \tag{64}$$

$$lim_{\tau\to\infty}R(\tau) = 0 \tag{65}$$

This serves as a method to assess whether aggregate output of a production network is heavy tailed. One can now substitute Y with (41), to get an expression that includes the structure of the production network and microeconomic shocks.

$$Pr(Y < -\tau\sigma) \leq 1/2Pr(v_i\epsilon_i < -\tau\sigma) \tag{66}$$

$$R(\tau) \leq \frac{log[1/2F(\frac{-\tau\sigma}{v_i})]}{log\Phi(-\tau)} \tag{67}$$

where $F()$ is the CDF of the shocks ϵ. To take the lower bound of the tail probability F for choices of v_i, the realisation of the influence vector v_i is set to the largest value of the influence vector v_{max}.

$$R(\tau) \leq \frac{log[1/2F(\frac{-\tau\sqrt{n}\sigma}{\sqrt{n}v_{max}})]}{log\Phi(-\tau)} \tag{68}$$

Expanding the argument of $F()$ by $\sqrt{n}/\sqrt{n}$ and setting $\sigma =\parallel v \parallel$ (because this is the standard deviation of aggregate output changes when the variance of shocks is set to 1, according to (56)) shows that the statistic $\rho = v_{max}\sqrt{n}/\parallel v \parallel$ is included.

$$R(\tau) \leq \frac{log[1/2F(\frac{-\tau\sqrt{n}}{\rho})]}{log\Phi(-\tau)} \tag{69}$$

For larger ρ, this upper bound of $R(\tau)$ is smaller, thus has a more heavy tailed distribution of Y. Acemoglu, Ozdaglar, et al. (2017) point out, that two distinct versions of network heterogeneity influence the size of ρ. If the economy has a very dominant sector in terms of network position v_{max}, then ρ is large. At the same time larger overall network heterogeneity $\parallel v \parallel$ decreases ρ. This result is contrary to the finding for volatility, where a larger overall network heterogeneity increases aggregate volatility. Here, more overall network heterogeneity in relation to a dominant sector decreases the heavy-tailedness of aggregate output. This result becomes important again when I analyse the tails of aggregate output changes of the German production network in section 4.4.

Acemoglu, Ozdaglar, et al. (2017) take this assessment one step further by asking the following question: How can we know at which rate the heavy-tails in aggregate output would vanish when shocks happen to a more disaggregate version of the production network? The heavy-tails would vanish with some rate proportional to the level of disaggregation n, but at which rate exactly? Acemoglu, Ozdaglar, et al. (2017) therefore index the tail threshold τ with the level of disaggregation n and set it to $\tau_n = c\sqrt{n}$, due to the following reasoning. Imagine that due to the law of large numbers, heavy-tailedness vanishes at the usual rate of $\sqrt{n}$ when the economy gets more disaggregated ($n \to \infty$). If this would be the case, then moving the threshold τ_n upwards at the same rate $\sqrt{n}$, this would keep $R(\tau_n)$ constant. As explained above, this is because the tail of Y should become heavier in relation to the normal distribution when moving τ upwards. To understand this better, have a look at figure 3.3.2. The leftmost distribution is a normal distribution, its tail starts where τ_{10} is marked. To its right is a heavy tailed distribution of Y which is the result of shocks to 10 sectors in which the economy is disaggregated. $\tau_{10}\sigma$ marks the beginning of its tail. The tail of the normal distribution starting at τ_{10} has less volume than the tail of the distribution of Y_{10}. For the threshold moving upwards, $\tau \to \infty$, the difference should become more pronounced. When Y had been created by a more disaggregated economy, then due to the law of large numbers also the volume of the tail should shrink, presumably at rate $\sqrt{n}$. In the figure it can be seen that the tails of the distributions to the right become thinner. Because moving the threshold τ upwards increases the volume of the heavy tail in relation to the normal tail, we now move the threshold upwards at rate $\sqrt{n}$. So if the volume of the tail shrinks with the same rate $\sqrt{n}$ when we move to the right, as it increases when we move the threshold upwards, then $R(\tau_n)$ does not change. But if the tail shrinks with a smaller rate than $\sqrt{n}$ when moving to the right, this means $R(\tau_n)$ shrinks. If the heavy-tails vanish with a rate slower than $\sqrt{n}$ when the economy becomes more disaggregate, Acemoglu, Carvalho, et al. (2012) define that in this case the economy has tail risk. In this spirit (71), defines tail risk as $R(\tau_n)$ becoming smaller for increasing τ_n with $\sqrt{n}$.

$$R(\tau_n) \leq \frac{log[1/2F(\frac{-\tau_n\sqrt{n}}{\rho})]}{log\Phi(-\tau_n)} \tag{70}$$

$$lim_{n\to\infty}R(\tau_n) = 0 \tag{71}$$

Acemoglu, Ozdaglar, et al. (2017) show that for a shock distribution F with exponential tail, when $n \to \infty$, then $R(\tau_n)$ converges to 0 when condition (72) is fulfilled. Hence, an economy perturbed by exponentially tailed shocks faces macroeconomic tail risk, when overall network heterogeneity $\parallel v \parallel$ increases less than $\sqrt{n}$ times as fast as the dominance of one sector (v_{max}) when going more disaggregated.

$$lim_{n\to\infty}\rho = \infty \tag{72}$$

3.3.3 Aggregate output elasticities of factors

In case the production network includes factor inputs, the output elasticities or input shares of an aggregate production function can be calculated from the influence vector. Because the influences of factors might not add up to 1, as usually required for the input shares of production functions[18], output elasticities can be calculated by dividing the influence of one factor f_i by the sum of the influences of all factors:

$$\frac{\delta Y}{\delta f_i} = \frac{v_{fi}}{\sum_i v_{fi}} \tag{73}$$

[18] This is an assumption also in growth accounting outside of mainstream economic growth theory (Kümmel and Lindenberger 2014). It implies that doubling the size of our economic system with the current technology level also doubles the output it produces.

4 Network statistics

4.1 Input-output data

To construct the German production network, I utilize input-output data on the sectoral level which is available for download on the website of the Federal Statistical Office of Germany (Destatis 2021). The input-output data is subject to revisions every few years. Thus their inter-temporal comparability is only guaranteed within a revision period. I chose the revision period 2019 which corresponded to input-output tables of the years 2015-2017 when I retrieved the data. To make the German input-output data more comparable to the US input-output data, I first report some general statistics of technical coefficient matrices. I follow Torres-González and Yang (2019), who provide a very detailed statistical description of US input-output data. The German input-output data consists of 72 sectors. The sector "Waren u. Dienstleistungen privater Haushalte o.a.S." has no edges connecting it to other sectors but produces a positive total output (in all years). Its total output makes up for around 0.12 percent of aggregate output which is the sixth smallest of output shares. I exclude this unconnected sector from matrix Z and total output X because it is irrelevant for shock propagation. The production network then consists of 71 sectors. The German input-output table is more aggregated compared to the US input-output table consisting of 400 industries Acemoglu, Carvalho, et al. (2012) or the Australian input-output tables with 114 industries Anufriev et al. (2016, p. 13).

The 2015 technical coefficient matrix A (non-normalised matrix W) has 1727 *zero entries* of 5184 entries in total, this is 33 percent. In comparison, the share of zero entries in the technical coefficient matrix of the US varies between the extremes 88.2% in 1967 and 20.6% in 1983 with a generally decreasing trend[19] Torres-González and Yang (2019). Torres-González and Yang (2019, p. 487) use only non-zero entries for their statistical analysis of pooled technical coefficients, "since the existence of zero values is highly affected by the construction procedure, which has gone through substantial changes for a few times in the past decades". This caution isn't relevant for comparing statistics of the German pooled technical coefficients over the years 2015 to 2017 because this period is subject to the same revision.

Torres-González and Yang (2019, p. 488) report large self-loops for the US data: "[A]lmost all diagonal entries in each matrix contain large technical coefficients, which indicates that intra-industry trade is an important component". In the German data, there are 67 *self-loops* across all years. Only 4 sectors do not have intra-

[19] One would think that the economy should become more connected over time and the share should therefore decrease. But the definition of industries also changes over time and new industries are added.

	2015	2016	2017
# Sectors	71	71	71
Coefficients (min, mean, max)	(0, 0.006, 0.54)	(0, 0.006, 0.52)	(0, 0.006, 0.53)
# Self-loops (mean, max)	67 (0.1, 0.54)	67 (0.1, 0.52)	67 (0.1, 0.53)
Zero coefficients	31.4 %	31.6 %	31.5 %
Indegree (min, mean, max)	(0.14, 0.41, 0.75)	(0.18, 0.41, 0.75)	(0.2, 0.42, 0.75)
Outdegree (min, mean, max)	(0.003, 0.41, 1.6)	(0.007, 0.41, 1.63)	(0.009, 0.42, 1.56)

Table 1: Summary statistics of A.

industrial trade, respectively use their own production as input. With 0.1 the mean of the self-loops is indeed high compared to the mean of pooled coefficients (0.006) and also the maximum coefficient values for each year can be found within the self-loops. The technical coefficients range from 0 to 0.54 across all years. Their mean is 0.006 across all years. Because technical coefficients can be interpreted as the input required to produce one unit of output (compare equation (25)), one could say that the production of one unit of output of a single sector requires on average 0.006 units of each input. Also the entries of the US technical coefficient matrix A lie between 0 and 1 with the majority of values very close to 0 Torres-González and Yang (2019, p. 490).

4.2 Degree distributions

4.2.1 In-degrees

The technical coefficient matrix A is also used to compute the in-degrees because the in-degrees of the input share matrix W are equal to 1. Acemoglu, Carvalho, et al. (2012, p. 1985) write: "[...] one can define an indegree for any given sector. However, in view of Assumption 1, the (weighted) indegrees of all sectors are equal to 1. We show in Section 4 that this is a good approximation to the patterns we observe in the U.S. data." This shows how Acemoglu, Carvalho, et al. (2012) understand matrix A as the empirical version of matrix W[20]. But neither their in-degrees are very close to 1 nor are their in-degrees extremely concentrated as would be the case for matrix A being a shrinked matrix $(1-\alpha)W$. Acemoglu, Carvalho, et al. (2012, p. 1998) note that "the indegrees of most sectors are concentrated around the mean: on average, 71 percent of the sectors are within one standard deviation of the mean input share." To make my thesis comparable to Acemoglu, Carvalho, et al. (2012) and Anufriev et al. (2016), I also report the in-degrees of matrix A. In-degrees are the row sums of the diffusion matrix ((74) Newman (2018, p. 130)). Acemoglu, Carvalho, et al. (2012) first obtain matrix A before they normalise it (according to (24)) to matrix W.

[20] The reason is that the calculation of the empirical matrix A (equation (23)) equals their theory derived matrix of input shares (32).

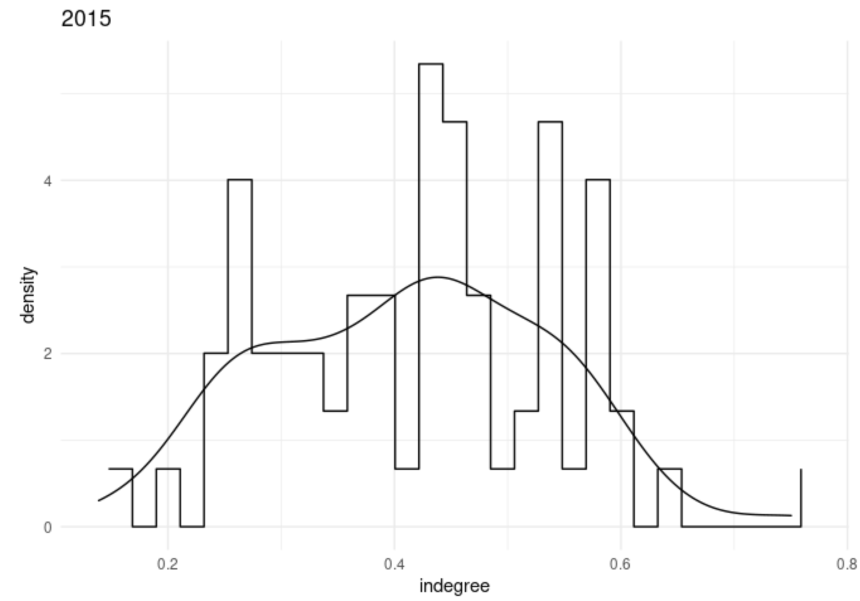


Figure 7: Plot of the PDF of German in-degrees.

	2015	2016	2017
Coefficients (min, mean, max)	(0, 0.014, 0.78)	(0, 0.014, 0.8)	(0, 0.014, 0.81)
Outdegree (min, mean, max)	(0.01, 1, 4.13)	(0.02, 1, 4.12)	(0.04, 1, 3.9)

Table 2: Summary statistics of W.

$$k_i^{in} = \sum_{j=1}^{n} A_{ij} \tag{74}$$

The mean in-degree of the German direct requirements matrix A is 0.41 to 0.42 across all three years (see table 1). The in-degrees range from 0.2 to 0.75 in 2017 which isn't quite the "good approximation" to matrix W nor to a shrinked matrix $(1-\alpha)W$. A normalised matrix $(1-\alpha)W$ with $alpha = 0.3333$ (choice for calculation of influence vector v in Acemoglu, Carvalho, et al. (2012)) would have all in-degrees equal to $(1 - 0.3333)$. Also for the German data, it depends on the perspective whether the in-degrees of A are seen as a good approximation of the in-degrees of W. On the one hand the entries of A can be seen as kind of shrinked input shares, because they lie between 0 and 1 and their sums (in-degrees) do not exceed 1. On the other hand in-degrees vary more than just a bit in my opinion, which isn't consistent with a shrinked input share matrix $(1 - \alpha)W$ where all in-degrees are smaller than 1 but all equal.

4.2.2 First order out-degrees

As shown in section 3.3.1, the out-degree distribution of a production network is the statistic which is decisive for how much volatility from the shocks is passed to

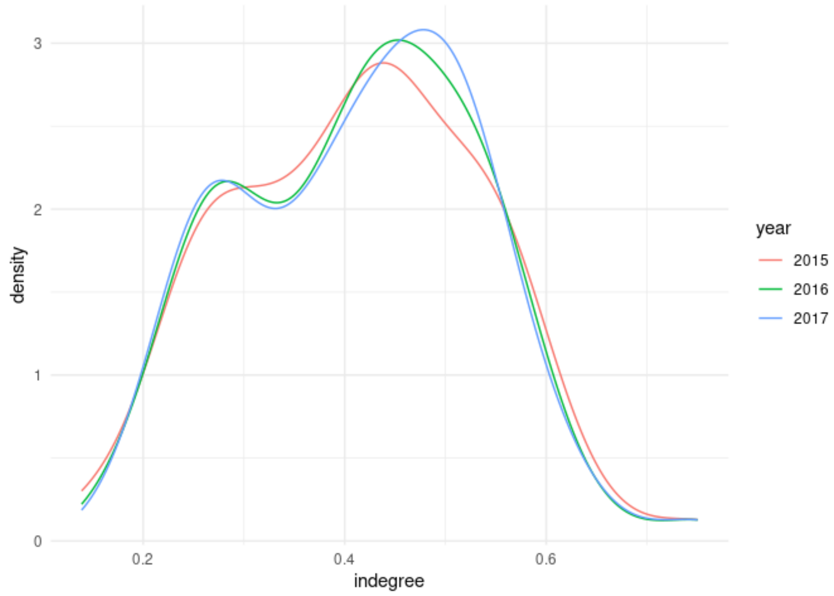


Figure 8: Plot of the PDF of German in-degrees.

aggregate output. A heavy-tailed out-degree distribution leads to higher aggregate volatility than a thin tailed out-degree distribution. To assess the tail of the out-degree distribution, I fit the tails of multiple standard distributions to the tail of the log-log counter cumulative density function (log-log CCDF) of the first and second order out-degrees (results for the second order out-degrees can be found in Appendix 6.3). Computing the CCDF involves three steps. First, out-degrees k (see equation (46) on how to get them from matrix W) are sorted in decreasing order. Second, the counter cumulative probability for each out-degree is given by $r(k_{decreasing})/n$ where $r(k_{decreasing})$ is the rank of out-degree $k_{decreasing}$ and n is the highest rank (Newman (2018)). This yields the CCDF. In a third step, one computes the logarithm of the sorted out-degrees $k_{decreasing}$ and their CCDF $r(k_{decreasing})/n$ to arrive at the log-log CCDF.

The CCDF of 2015 is plotted in figure 9. By looking at the CCDF, it is impossible to differentiate between a heavy tail and a thin tail. In the log-log plot though, it is much simpler to spot a heavy tail by eye. A tail that decays with a powerlaw, the strongest form of heavy tails, appears linear in the log-log CCDF (Newman 2018, p. 321-322). Tails decaying faster than linear (decrease superlinear) in the log-log plot are lighter than the powerlaw distribution. The tail of the CCDF (figures 10 - 16) looks quite linear. It thus makes sense to estimate the tail by a power law. Throughout the thesis I refer to two different tail lengths. For the power law fitted by a Pareto distribution later, an optimal tail length can be computed. Another possibility is to determine a tail with an exogenous cut-off value. Acemoglu, Carvalho, et al. (2012) estimate their linear model on a tail which makes up for 20% of the

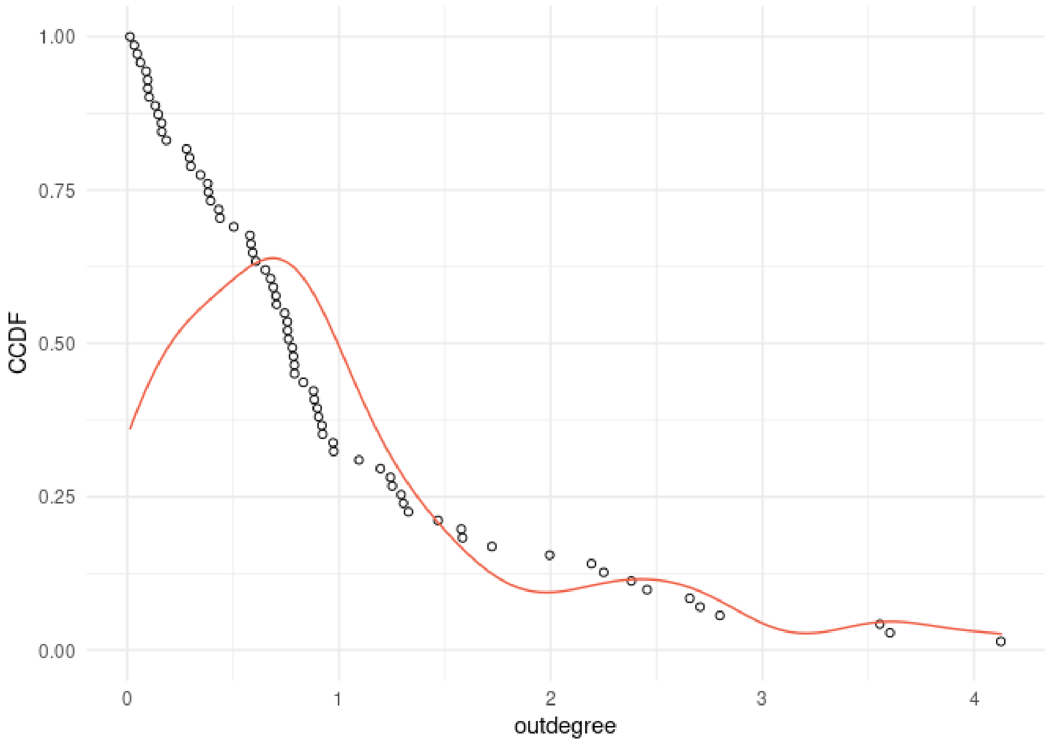


Figure 9: CCDF of the 2015 first order out-degrees with a fitted density function.

whole out-degree distribution. I will refer to this tail with exogenous cut-off value as the 20% tail.

One way to estimate the tail exponent of a distribution following a power law in the tail is by using a linear ordinary least squares (OLS) regression. The OLS regression (75) estimates the relation between the logarithmic out-degree and its logarithmic rank (Gabaix and Ibragimov 2011, p.1). The estimated parameters are reported in table 3. The tail exponent estimated with the linear regression is around 2.5 for all three years. This is plausible, tail exponents of real world networks typically lie between 2 and 3 (Clauset et al. 2009, p. 662).

$$log(rank(k)) = c - \hat{\beta} log(k) \tag{75}$$

Gabaix and Ibragimov (2011) point out that the OLS estimate is downward biased in small samples and propose a modification to regression (75). The Gabaix-Ibragimov correction subtracts 0.5 from the rank of the out-degrees, modifying (75) to (76). Also, the calculation of the standard error changes to (77). The estimates of the tail exponent by linear regression with Gabaix-Ibragimov (GI) correction is around 2.9 for all three years and thus a bit higher than the estimate without GI-correction. The standard error according to (77) is roughly 4.8 times higher than the, already high, standard error of the regression without GI-correction. The standard

	year	distribution	parameter	xmin	stderror	p-value
1	2015	lm	2.46	14.00	0.23	
2	2016	lm	2.57	14.00	0.24	
3	2017	lm	2.42	14.00	0.26	
4	2015	lm_GIcorrect	2.90	14.00	1.10	
5	2016	lm_GIcorrect	3.03	14.00	1.15	
6	2017	lm_GIcorrect	2.83	14.00	1.07	
7	2015	ksr	2.69	14.00		
8	2016	ksr	2.74	14.00		
9	2017	ksr	2.50	14.00		
10	2015	powerlaw	1.61	27.00		0.09
11	2016	powerlaw	1.63	24.00		0.02
12	2017	powerlaw	1.61	26.00		0.10
13	2015	powerlaw_exogcut	2.30	14.00		0.29
14	2016	powerlaw_exogcut	2.33	14.00		0.19
15	2017	powerlaw_exogcut	2.21	14.00		0.47

Table 3: Estimated parameters of 2015-2017 first order out-degree log-log CCDF tails.

errors are very high, likely due to the small sample size of only 14 out-degrees that make up the 20% tail. The regression lines of the OLS and GI-corrected OLS can be seen in figure 10 and 11.

$$log(rank(k) - 0.5) = c - \hat{\beta} log(k) \tag{76}$$

$$se = \sqrt{\frac{2}{n}} \hat{\beta} \tag{77}$$

Because Acemoglu, Carvalho, et al. (2012) and Anufriev et al. (2016) deploy a Nadaraya–Watson kernel regression, I also do this. I also select the bandwidth with least squares cross-validation. The resulting fitted function can be seen in figure 12. It has an average slope varying from -2.5 to -2.74 across all three years which is close to the OLS estimates without GI-correction. The kernel regression does not yield a standard error.

The R-package *poweRlaw* (Gillespie 2015) allows to directly fit a powerlaw and other heavy-tailed distributions to (the tail of) an empirical distribution. This is done via maximum likelihood estimation (ML). The estimation routine in the package is equivalent to the ML estimation routine of Clauset et al. (2009) which is used by Acemoglu, Carvalho, et al. (2012) and Anufriev et al. (2016). I fit a power law (the tail of a Pareto distribution) on two different tail lengths. One estimation uses the same tail length as the OLS estimations (20% of the distribution) to make the tail exponents of the fitted power law and the tail exponent of the linear regressions directly comparable. The other estimation uses an endogenously determined

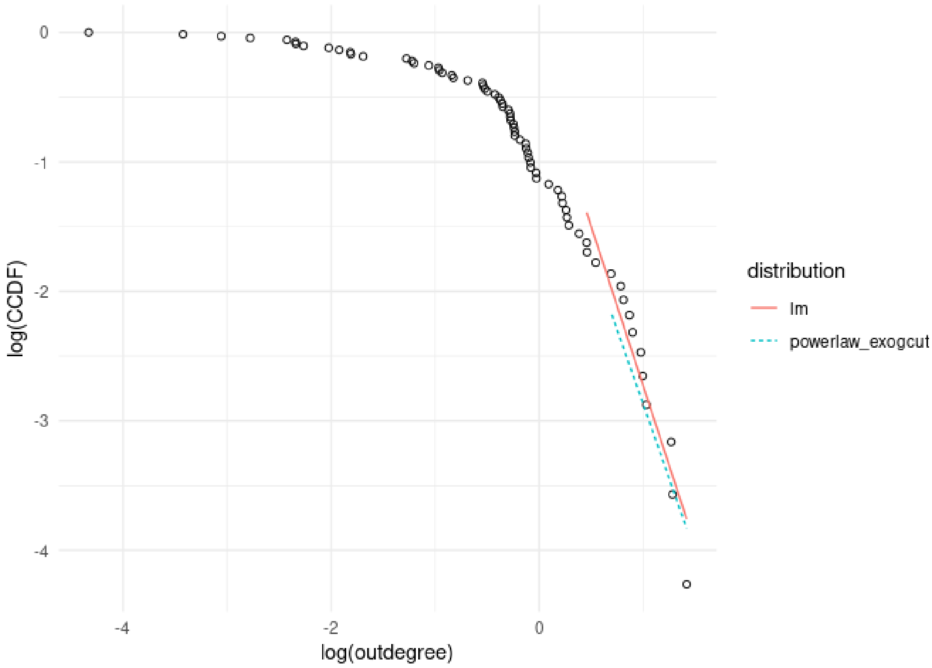


Figure 10: Log-log CCDF of the 2015 first order out-degrees with a fitted linear model versus power law on the 20% tail.

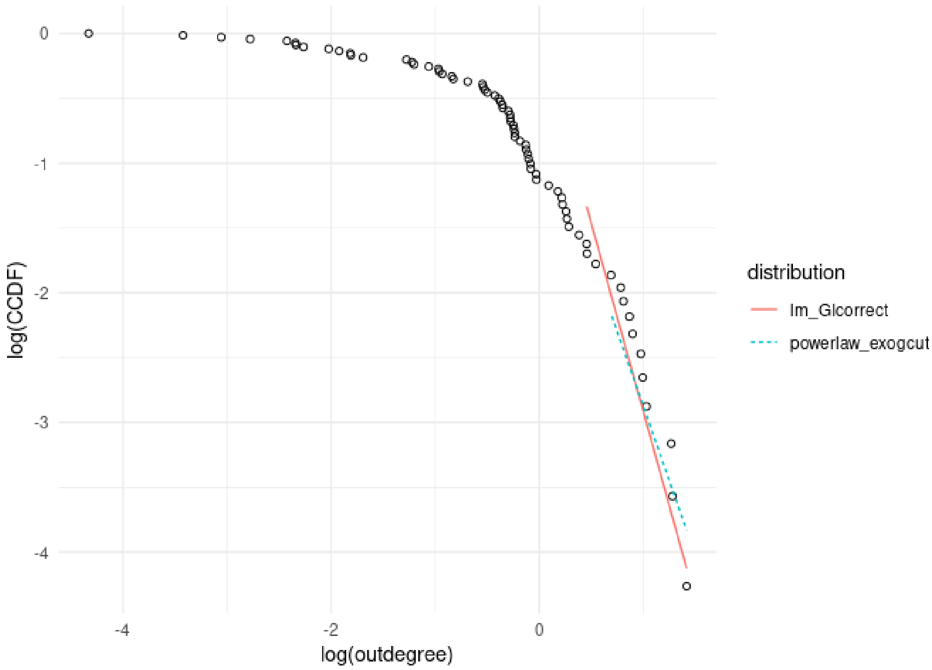


Figure 11: Log-log CCDF of the 2015 first order out-degrees with a fitted linear model with GI-correction versus power law on the 20% tail.

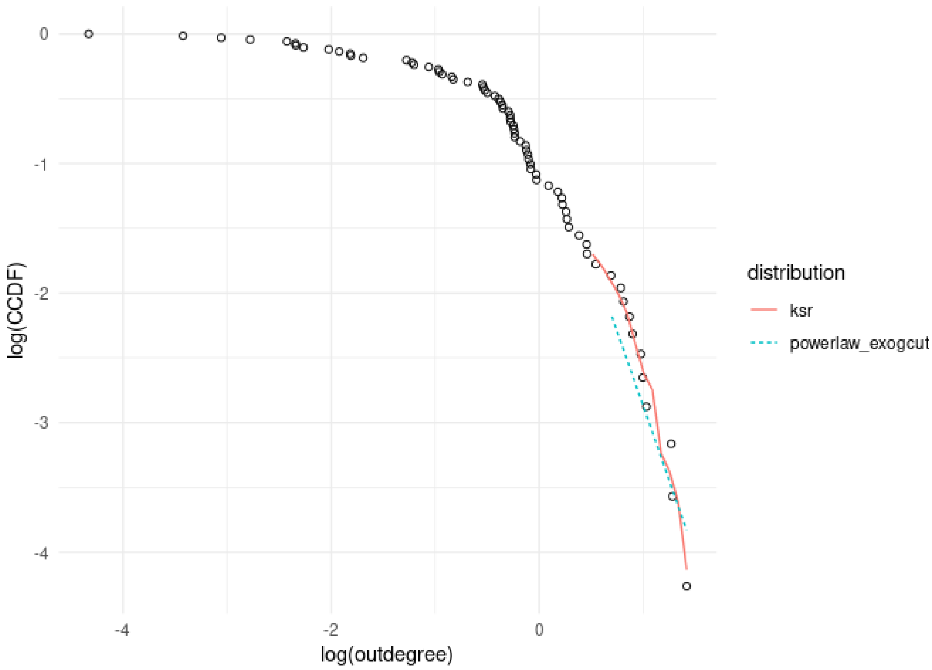


Figure 12: Log-log CCDF of the 2015 first order out-degrees with a kernel smoothing regression versus power law on the 20% tail.

tail length which is found "by minimising the Kolmogorov-Smirnoff statistic (as described in Clauset, Shalizi, Newman (2009))" Gillespie (2020a, p. 18). This statistic is simply the distance between the empirical CDF and the fitted CDF Gillespie (2015, p. 4). The ML estimation with endogenous cut-off is the same routine as Acemoglu, Carvalho, et al. (2012) use as a comparison for their OLS estimates. The estimated tail exponents for the power law ("powerlaw") and the power law fit to the 20% tail ("powerlaw_exogcut") are reported in table 3. The tail with endogenously determined tail cut-off is much longer than the 20 % tail. Also the tail exponents differ. While with 2.21 to 2.33 the tail exponent of the fitted Pareto distribution with 20% tail is smaller but still not too far away from the tail exponents estimated with OLS, the tail exponents of the endogenously fitted Pareto distribution are much smaller, around 1.6. Looking at figures 14 - 16 it becomes already clear that the power law might not be the best fit to the tail. Whether it is likely that the data really follows a power law can be tested in the poweRlaw-package with a goodness-of-fit test via a bootstrapping procedure. "If the p-value is large, than any difference between the empirical data and the model can be explained with statistical fluctuations. If $p \simeq 0$, then the model does not provide a plausible fit to the data" (Gillespie 2020b, p. 4-5). Clauset et al. (2009, p. 678) tell what size a "large" p-value would be: "If we discover that the p-value for the power law is reasonably large (say, $p > 0.1$), then the power law is not ruled out." and "If p is large (close to 1), then the difference between the empirical data and the model can be attributed to statistical fluctuations alone; if it is small, the model is not a plausible fit to the data." Clauset et al. (2009, p. 676).

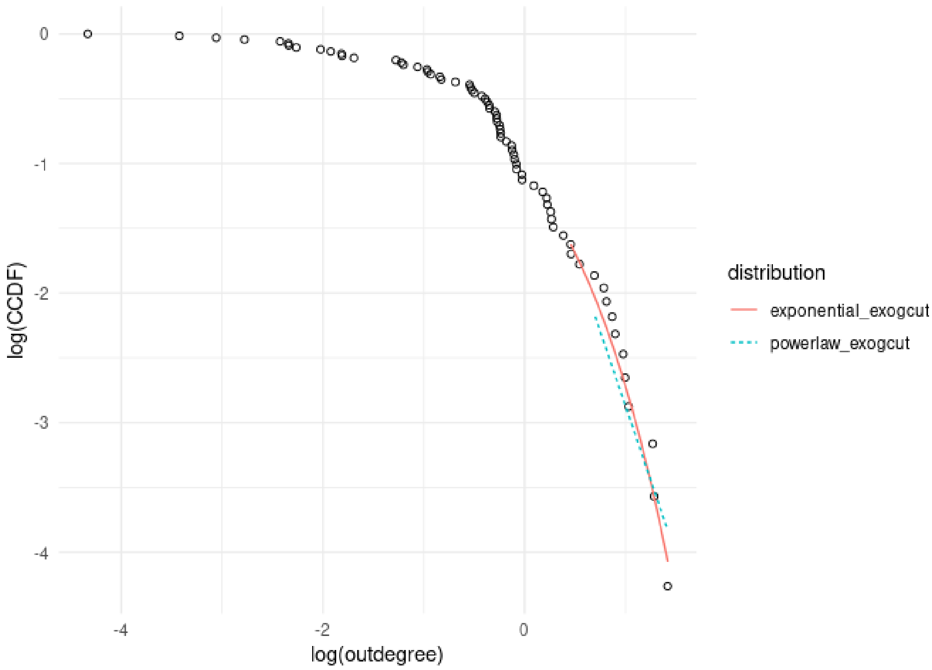


Figure 13: Log-log CCDF of the 2015 first order out-degrees with fitted exponential distribution versus power law on the 20% tail.

The p-values are reported in column "p-value" of table (3). Strangely, the p-value is larger for the power law with exogenous tail length. This is surprising since the algorithm which minimises the Kolmogorov-Smirnoff statistic has chosen the longer tail over the 20 % tail length and all other tail lengths[21]. With 0.02 to 0.1 the p-values for the powerlaw fitted to the endogenously determined tail are very low. They are below the threshold of 0.1 cited above. Also for the 20% tail, p-values are not very large, but with 0.19 to 0.47 above this threshold. Thus with these p-values, the endogenously determined tail does not follow a powerlaw, but the powerlaw is not ruled out for the 20% tail.

With the poor but existent fit of the power law to the 20% tail, one can already be confident that the tail is heavier than a normal distributions tail. But it would be valuable to know if a powerlaw is really the heavy-tailed distribution which best describes the tail. The poweRlaw package allows to fit also other heavy-tailed distributions and compare their fit to the powerlaw fit. The distributions I compare to the power law are: the exponential distribution, the log-normal distribution

[21] For the US out-degree distributions used in Acemoglu, Carvalho, et al. (2012), for which I have conducted the same statistical analysis and for which outputs can be found in Jany (2021), it is the opposite: The p-values are much larger for the endogenous tail than for the 20% tail, as one would expect it for the endogenous tail having the optimal tail length. The same R-code is applied to both German- and US-data, errors in code should therefore be ruled out. One explanation for this could be that the routine which finds the endogenous cut-off requires a minimal length for the tail, although this is not mentioned in the package's documentation (Gillespie 2015).

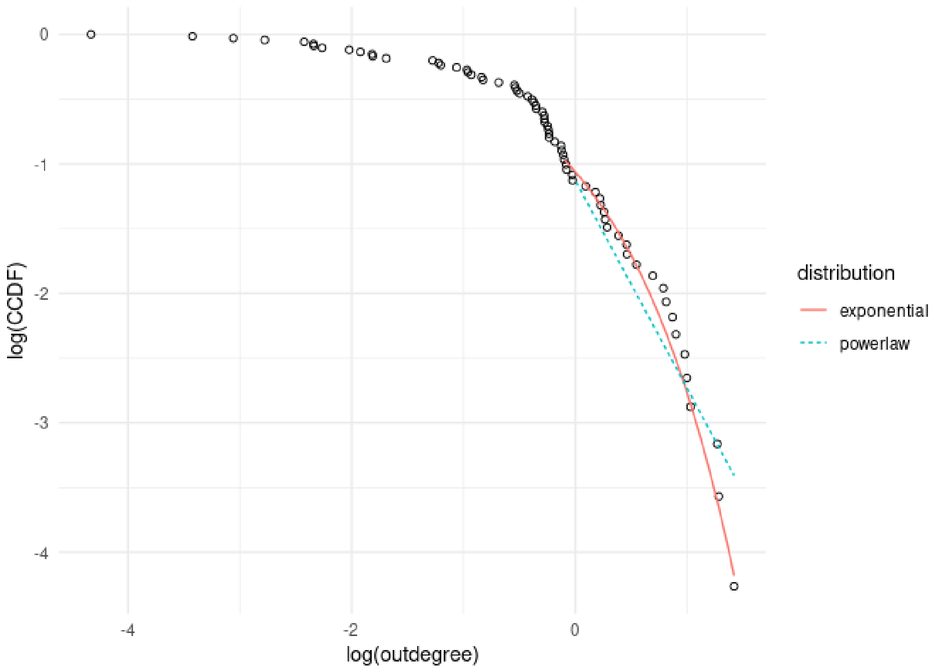


Figure 14: Log-log CCDF of the 2015 first order out-degrees with fitted exponential distribution versus power law on the power laws optimal tail length.

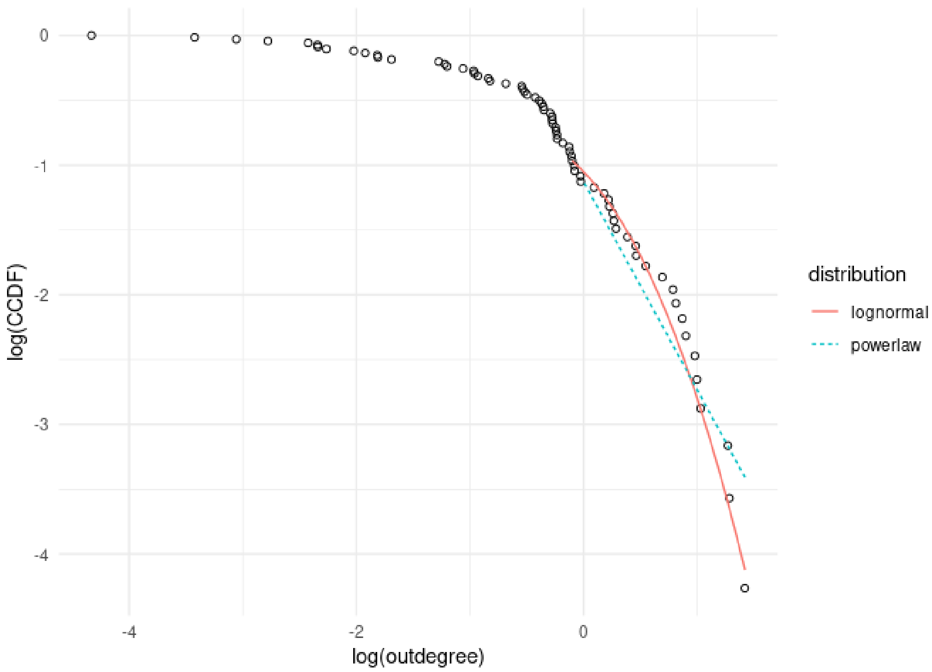


Figure 15: Log-log CCDF of the 2015 first order out-degrees with fitted lognormal distribution versus power law on power laws optimal tail length.

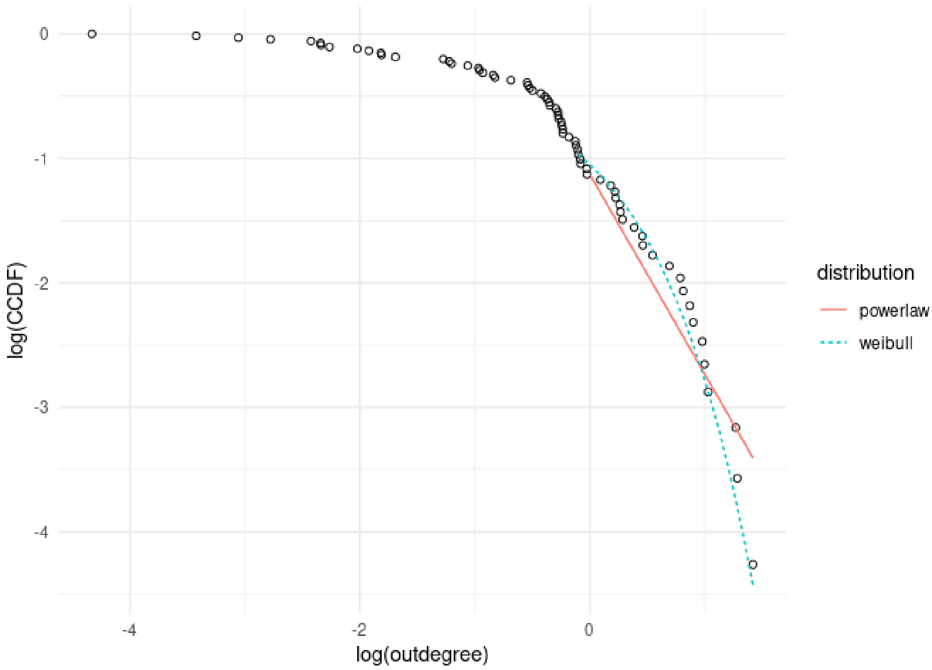


Figure 16: Log-log CCDF of the 2015 first order out-degrees with fitted Weibull distribution versus power law on power laws optimal tail length.

and the Weibull distribution. The exponential distribution is the border case between a heavy-tailed and a light-tailed distribution: "The exponential distribution is the absolute minimum alternative candidate for evaluating the heavy-tailedness of the distribution. The reason is definitional: the typical quantitative definition of a 'heavy-tail' is that it is not exponentially bounded [10]. Thus if a powerlaw is not a better fit than an exponential distribution (as in the above example) there is scarce ground for considering the distribution to be heavy-tailed at all, let alone a power law." (Alstott et al. 2014, p. 5). Plots of the three distributions versus the powerlaw can be seen in figures 14 - 16. All three distributions and the power law are fitted to the endogenously determined tail as well as to the 20% tail. The fit of two distributions can be compared within the poweRlaw-package with a likelihood-ratio test with Vuong's p-value (Clauset et al. 2009). There is a one-sided test and a two-sided test and I perform both. For the one-sided test, the p-value is the probability of having the test statistic tending to one distribution if actually the other distribution is a better fit. Thus, independent of the decision of the test statistic for one or the other distribution, a high p-value always tells that neither distribution should be considered as the better one, I therefore report only p-values and not the test statistics: "If p is large, on the other hand, the sign is not reliable and the test does not favor either model over the other. It is one of the advantages of this approach that it can tell us not only which of two hypotheses is favored, but also when the data are insufficient to favor either of them." (Clauset et al. 2009, p. 680). The p-values in table 4 are quite high for every pair of a powerlaw and another heavy-tailed distribution. Thus the test can not discriminate between a powerlaw and other heavy-tailed distributions for both tail lengths. A result which is quite surprising again, is that the p-values for the powerlaw versus exponential distribution are larger than the other p-values. For the US-data, the p-values are smallest, by far, for the powerlaw versus exponential distribution (Jany 2021). Such a behaviour seems consistent, if the out-degrees really follow a powerlaw, as the exponential distribution is the boundary case between heavy and thin-tailed. Thus the test should have more difficulty discriminating between log-normal versus powerlaw and Weibull versus powerlaw. I lack a good explanation for why this is the other way around for the German data.

	distribution	lognormal	weibull	exponential
1	powerlaw	0.79	0.82	0.87
2	powerlaw_exogcut	0.69	0.76	0.90

Table 4: P-values of the one-sided likelihood-ratio tests.

In case of the two-sided test, the p-values are the probability of getting a positive or negative test statistic if actually no distribution fits better. A high p-value here therefore indicates that the test can not discriminate between both distributions. The p-values of the two-sided test are reported in table 5. For log-normal and Weibull p-values are large, therefore the test cannot distinguish whether the power law or these distributions fit better. Also the p-value for the exponential distribution is quite high here. But this time there is the tendency that the exponential distribution has the smallest p-values, which implies that it is easier for the test to distinguish between exponential versus powerlaw than between the other heavy-tailed distributions versus powerlaw. The test statistic for powerlaw versus exponential is negative for both tail lengths, meaning it favours the exponential distribution. Still the p-value is high, thus it can not be ruled out that the power law fits better. In comparison, the p-value of the two-sided test for power law versus exponential distribution is very low for the US out-degree distribution (around 0.07) and also the p-value of the one-sided test is very low (around 0.035) (Jany 2021). As the test statistic for the US data favours the power law, this is a clear indication that the US out-degree distribution is better described by a power law than an exponential distribution. Despite the result that we can not say the German out-degree distribution follows a power law in the tail, this doesn't rule out that it is still a heavy tail, just maybe not as heavy as a power law. The tests can not clearly rule out the possibility of an exponential tail neither that of a tail following a power law. This result alone implies that the tail must be heavier than a thin tail, otherwise the exponential distribution would have been favoured more clearly by the tests. Because the powerlaw is the extreme case of a heavy tail, it might be a good idea to also compare the fit between the exponential distribution and the two other heavy-tailed distributions to get a more conclusive statement about the heavy-tailedness of the out-degree distribution.

	distribution	lognormal	weibull	exponential
1	powerlaw	0.43	0.36	0.25
2	powerlaw_exogcut	0.62	0.48	0.20

Table 5: P-values of the two-sided likelihood-ratio tests.

The estimations of the tail of the second-order out-degree distribution can be found in Appendix 6.3. The estimated tail exponents are much lower for the second-order out-degrees. Also the tail exponents of the US second-order out-degrees are lower than the tail exponents of first-order out-degrees, although their difference is not so pronounced (Acemoglu, Carvalho, et al. 2012, p. 2000). US tail exponents also are already much smaller than the German tail exponents. In contrast to the first-order out-degrees, the p-values for the powerlaw fitted to both tail lengths of second-order out-degrees lie almost all above 0.1. Thus apart from the endogenous tail of 2017, the powerlaw can not be ruled out for all other years and tail lengths. But the p-values of the likelihood-ratio test are all quite high, meaning the test can not distinguish whether the powerlaw or any of the other three distributions fits better.

4.3 Output multipliers

In section 3.3.3 I have argued that the input shares of an aggregate production function of a networked economy can be calculated by dividing the influence of each factor by the sum of influences of the factors. Unfortunately the German input-output data does not contain sectors representing factors. All sectors receive inputs. I anyway compare the influence vector to output shares (equation (78)) to show that network structure significantly changes the output elasticity of a sector. I argue that this can also be expected for the output elasticities of factors. The comparison also provides a feeling for the heterogeneity the network structure adds to the output shares.

$$costshare_i = \frac{y_i}{Y} \tag{78}$$

The left plot in figure 17 shows the output shares versus entries in influence vector v. The influence vector v solely is the influence over the network with homogeneous output shares $1/n$. The right plot shows the output shares versus the influence vector $\tilde{v}$. The influence vector is the output multiplier (network importance) multiplied with the output share (relative sector size) as in equation (42). Equation (41) tells that the influence vector multiplied with the shocks gives the aggregate output change. The output share tells that if the economy would be unconnected, a 1% change

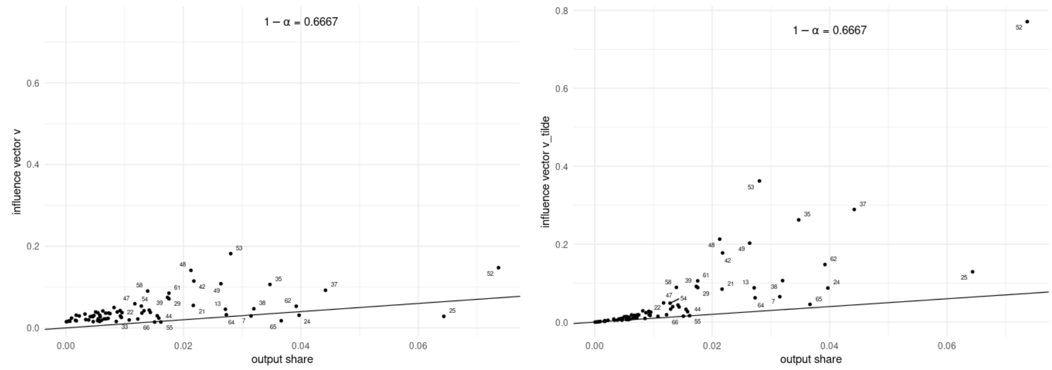


Figure 17: Output share versus influence vector v and $\tilde{v}$ of 2015. Labour share 0.3333.

in sector 24 would lead to a 0.04% change in aggregate output. In a connected economy with heterogeneous output shares (right plot), a 1% change in output of sector 24 would lead to a 0.1% change in aggregate output[22] according to the values of the influence vector in the left plot of figure 17. Sectors placed above the black diagonal in the right plot have a higher influence vector than output share. This is the majority of sectors. One could say: The production network model shows that changes of aggregate output due to sectoral changes must be higher in the networked economy than in the unconnected economy (with heterogeneous sector size). A critical parameter is the labour input share α. For a small labour input share the output multipliers and therefore influence vector v become very large. The choice of $\alpha = 0.3333$ by Acemoglu, Carvalho, et al. (2012) appears quite arbitrary. In general the performance of the theory derived input share matrix has to be verified empirically before taking the quantitative statements above at face value.

4.4 Simulation

	shockdistr	shocks	homogeneous	networked	heterogeneousnetworked
1	normal	1.00	0.12	0.46	1.07
2	laplace	1.41	0.17	0.64	1.50

Table 6: Standard deviation of shocks and the predicted standard deviation of aggregate output in the homogenous and networked economy. The predicted standard deviation is calculated with (52) and (56) from shock volatility (the values are close to the empirical volatility which is included in the time series).

I apply the quantitative illustration of Acemoglu, Ozdaglar, et al. (2017) to the German input-output data. Shocks are drawn from a distribution and applied to each sector of the production network.[23] Aggregate output is then calculated according

[22] Remember that although the production network model works with changes of logarithmic output, they approximate relative changes in non-logarithmic output, but only for small changes. See explanation around equation (12).

[23] Instead of using the empirical growth rate mean and standard deviation for the shock distribution like Acemoglu, Ozdaglar, et al. (2017), I use zero mean and standard deviation one. This is simpler and suffices the illustrative purpose because it does not change the qualitative result that aggregate output has a certain distributional shape.

to equation (41). This procedure is repeated 1000 times and as a result the time series of aggregate output and shocks can be compared. The results clearly show that network heterogeneity attenuates the cancellation of shock volatility in aggregate output. Consider especially the difference between figure 18 and 19. In both cases, an economy with network structure and an economy without any heterogeneity, the same shocks with the same volatility is used but the volatility of the networked economy is almost 4 times higher. Due to the law of large numbers, the standard deviation of the economy without heterogeneity is already very small for $n = 71$. In the networked economy the standard deviation is still much larger, although the law of large numbers also applies here. Something even more counter-intuitive happens when the actual output shares are included in addition to the network structure (when $\tilde{v}$ is the influence vector (42)). Figure 20 shows that then the standard deviation of aggregate output is larger than the standard deviation of shocks. With larger n the standard deviation of aggregate output will likely also decrease but for this small number of industries volatility is amplified. Appendix 6.5 provides the theoretical underpinning for this result in showing that the CCDF of the influence vector with network structure and actual output shares is more heavy-tailed than the influence vector with network structure and homogeneous output shares $1/n$.

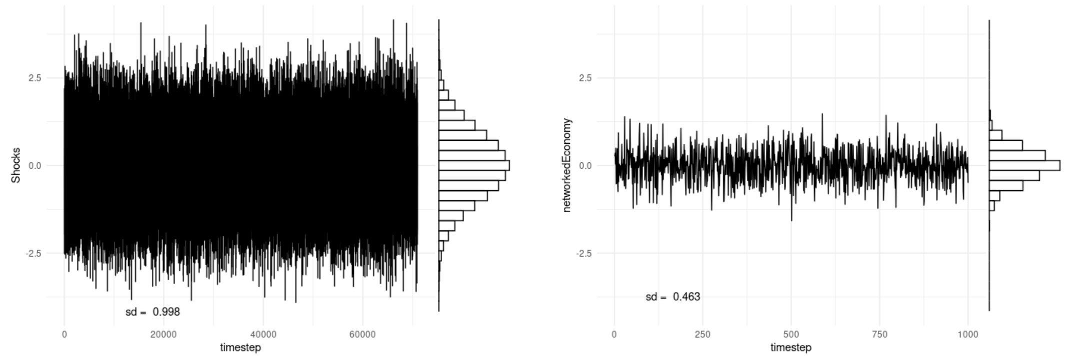


Figure 18: Shock propagation through the 2015 German production network.
1000 time steps, each time step one shock drawn from a normal distribution (standard deviation = 1) to each sector. Empirical standard deviations of the time series are shown in the lower left region of each plot. Volatility cancels out in the aggregate, but is still higher than in an economy without network connections.

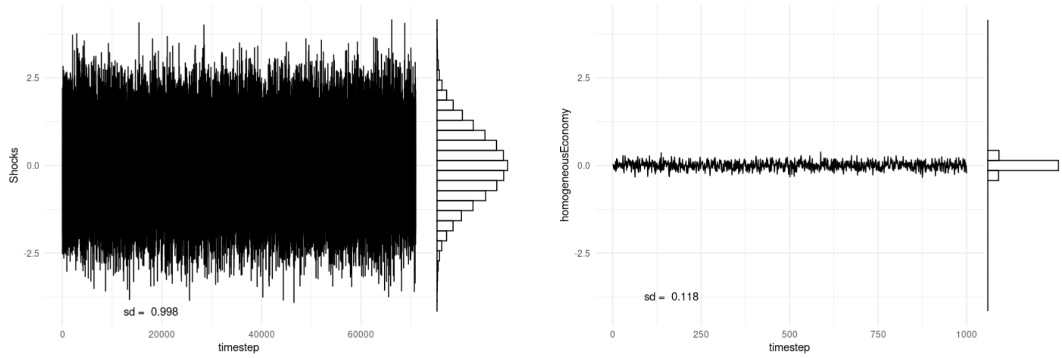


Figure 19: Shock propagation through a homogeneous 71 sector economy without a production network. Volatility cancels out very much in the aggregate.

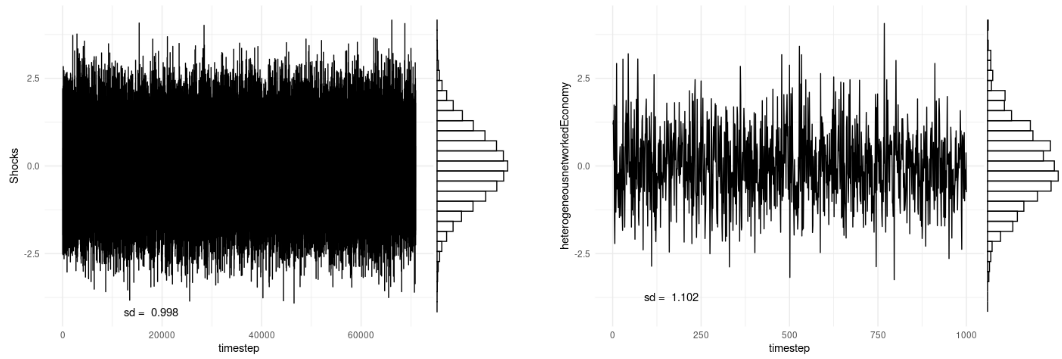


Figure 20: Shock propagation through the 2015 German production network with actual output shares. Volatility is even amplified in the aggregate.

Like Acemoglu, Ozdaglar, et al. (2017) I also draw shocks from "a doubly exponential (Laplace) distribution" to investigate whether heavy tails in shocks can carry on to the aggregate level. Quantile-quantile plots in figure 21 show whether the tails of the distributions differ from the tails of a normal distribution. Deviations from the diagonal line indicate that the tails differ from the normal. Downward sloping points away from the diagonal in the lower left corner and upward sloping points away from the diagonal in the upper right corner indicate tails heavier than normal. Clearly the Laplacian shocks are heavy-tailed. Aggregate output is not so heavy-tailed and there does not seem to be a large difference between networked and a homogeneous economy. However aggregate output in the networked economy with heterogeneous output shares clearly is more heavy-tailed. This shows that heavy tails of shocks can carry on to the aggregate level in the German production network. Section 3.3.2 has also shown that the network statistic ρ (sectoral dominance) is decisive for the extent to which heavy tails from shocks appear in the aggregate. I therefore calculate ρ for the homogeneous, the networked and the networked plus heterogeneous output share economy (the values are shown in table 7). As could be expected from the QQ-plots, the networked economy with heterogeneous output shares has the largest value. This confirms that this economy has the heaviest tail. But also the networked economy has a higher ρ and therefore heavier tails than the homogeneous economy which has not become so clear from only the QQ-plots. An interesting result is that heterogeneous output shares increase sectoral dominance by such a large amount. There must be a very dominant sector in terms of output shares in the German economy. Figure 17 in the last chapter clearly shows this. Sector 52 has a much higher $\tilde{v}_i$ than v_i.´

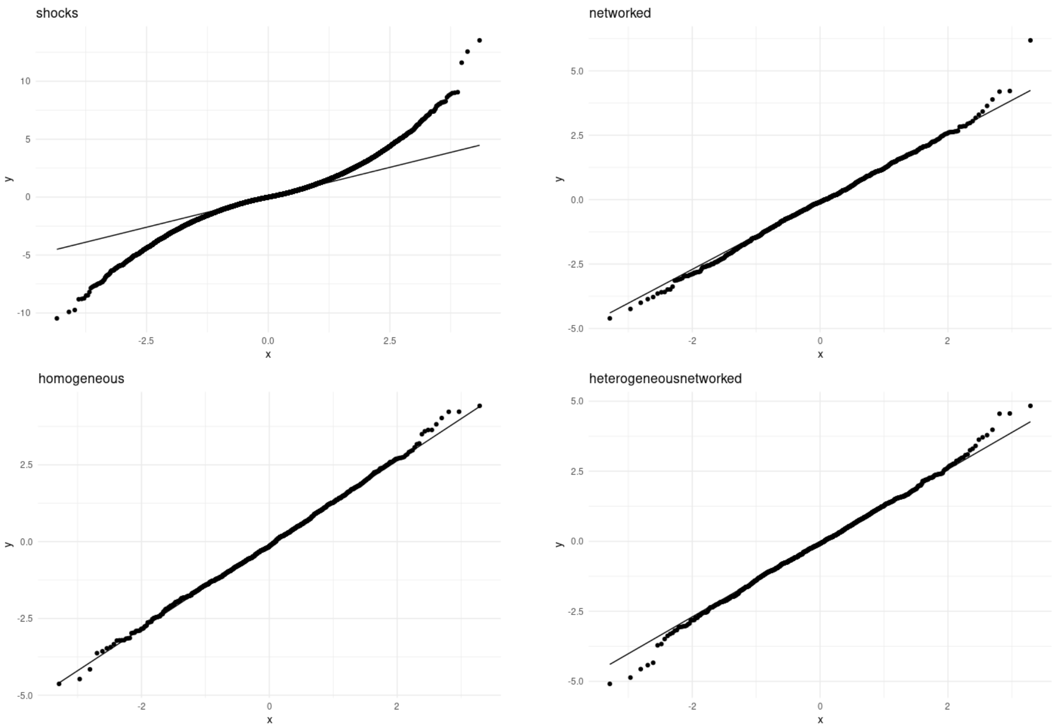


Figure 21: QQ-plots of the distributions of Laplacian shocks and aggregate output after shock propagation through the 2015 German production network.

homogeneous	networked	heterogeneousnetworked
1.00	3.37	6.10

Table 7: Values of ρ for economies of different sectoral heterogeneity at the same level of disaggregation $n = 71$ sectors.

5 Conclusion

I have analysed the structure of the German production network to find whether sectoral shocks can lead to aggregate fluctuations and macroeconomic tail risk. I showed that it is not very likely that the tails of first- and second-order out-degree follow a powerlaw like other countries' out-degrees. Still they are likely heavy-tailed for two reasons: First, the tests do not clearly decide against the hypothesis of a powerlaw and other heavy tailed distributions, which would be the expected behaviour for a thin tail. Second, the simulation in section 4.4 shows that for the same shock distribution, aggregate volatility is higher in the German production network than in an economy without network structure. This implies that aggregate fluctuations can come from sectoral shocks in the German economy, but this mechanism is less pronounced than for example in its US counterpart. Moreover the calculations of statistic ρ for sectoral dominance are larger for the German production network than for the unconnected economy. Thus heavy-tailedness of shocks is also transferred to the aggregate level. I have highlighted that the production network model makes some assumptions that are crucial determinants for quantitative and even qualitative results. Parameters shaping the distribution of aggregate output are the network structure, shape of interaction function, shape of aggregation function and shock distribution Acemoglu, Ozdaglar, et al. (2016). I have only analysed how the network structure (specifically the out-degree distribution), and to a minor part the shock distribution, influences aggregate output. I have only assumed a log-linear interaction function, although other shapes might be closer to reality. Interaction functions can be nonlinear production functions such as the Leontief production function. They can represent sectors inventories via thresholds or different forms of rationing when negative shocks reduce a suppliers output below actual demand. Despite the possibility to model a more complex production process with an agent-based-model in computer code, empirical studies have shown that the mathematical expression of linear diffusion appears to be a useful linear approximation to the actual process of shock diffusion in a real world production network. I have also tried to find out how prices are related to the production network model and found that the models assume price changes are either accounted for in the quantity changes already or absent, especially in the short run. A task for future research would be to find diffusion matrices which best capture the empirical output elasticities between sectors. This does not need to be derived solely from the combination of theory with input-output data as in this thesis, but could also be derived by observing past reactions of the production network to shocks, as Klimek et al. (2019) do. In the literature review I have also provided a classification of the production network model into the main fields of macroeconomics. The model is basically an exogenous business cycle model, but I have pointed to the possibility of making it also an endogenous business cycle model by letting dynamics emerge from the interaction of firms or sectors that do not converge to an equilibrium. I have also highlighted that seeing the production network model as a model representing the whole production system, it could be used as an alternative for the aggregate production function or even a way to derive an aggregate production function from the micro level. To wrap up, much information about economic processes in the structure of production networks has long not been recognized, because resources for computation and simulation where missing. Modelling the production system as a network has the potential to better understand the process of production on the macroeconomic level.

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6 Appendix

6.1 Proof of transformation from (48) to (49)

I start from the general case where an aggregate output change dY equals the sum of weighted sectoral shocks. The weights δ_i can be the size of each sector or in the production network model the network influence of each sector. The crucial point for this derivation of the output volatility is the independence of the sectoral shocks z_i. The derivation of the variance of a weighted sum of independent random variables is then a well known procedure in statistics[24].

$$dY = \delta_1 z_1 + \delta_2 z_2 + ... + \delta_n z_n \tag{79}$$

$$dY = \sum_{i=1}^{n} \delta_i z_i \tag{80}$$

dY is the weighted sum of independent variables z_i.

$$var(dY) = E[(dY - E(dY))^2] \tag{81}$$

Plugging in equation (80) for dY,

$$var(dY) = E\left[\left(\sum_{i=1}^{n} \delta_i z_i - \mu_{dY}\right)^2\right] \tag{82}$$

Because dY consists of n independent variables, its mean μ_{dY} can be written as the weighted sum of the independent variables means

$$\mu_{dY} = \sum_{i=1}^{n} \delta_i \mu_{z_i} \tag{83}$$

Plugging expression (83) into equation (82),

$$var(dY) = E\left[\left(\sum_{i=1}^{n} \delta_i z_i - \sum_{i=1}^{n} \delta_i \mu_{z_i}\right)^2\right] \tag{84}$$

$$var(dY) = E\left[\left(\sum_{i=1}^{n} (\delta_i z_i - \mu_{z_i})\right)^2\right] \tag{85}$$

$$var(dY) = E\left[\sum_{i=1}^{n} (\delta_i z_i - \mu_{z_i}) \sum_{j=1}^{n} \left(\delta_j z_j - \mu_{z_j}\right)\right] \tag{86}$$

[24] https://online.stat.psu.edu/stat414/lesson/24/24.3

$$var(dY) = E\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i\delta_j\left(z_i-\mu_{z_i}\right)\left(z_j-\mu_{z_j}\right)\right] \tag{87}$$

Because the variables z_i are independent, covariances are zero. Therefore one can set $i = j$. As the expectation now wraps around the last term, this is the variance of the variables z_i.

$$var(dY) = \sum_{i=1}^{n}\delta_i^2 E\left[\left(z_i-\mu_{z_i}\right)^2\right] \tag{88}$$

$$var(dY) = \sum_{i=1}^{n}\delta_i^2\sigma_{z_i}^2 \tag{89}$$

6.2 Euclidean norm of influence vector v

The euclidean norm of a vector v is defined as:

$$\parallel v \parallel = \sqrt{\sum_{i=1}^{N} v_i^2} \tag{90}$$

In vector notation this writes:

$$\parallel v \parallel = \sqrt{v^T v} \tag{91}$$

The squared euclidean norm therefore is:

$$\parallel v \parallel^2 = v^T v \tag{92}$$

Plugging in the lower bound of v_n (right hand side of (59)) in v, the euclidean norm is

$$v_n \geq \frac{1}{n}1^T + \frac{(1-\alpha)}{n}1^T W_n \tag{93}$$

$$\begin{aligned}
\parallel v_n \parallel^2 &= v_n^T v_n \\
&\geq \left(\frac{1}{n}1^T + \frac{(1-\alpha)}{n}1^T W_n\right)\left(\frac{1}{n}1 + \frac{(1-\alpha)}{n}W_n^T 1\right) \\
&\geq \frac{1}{n}1^T\frac{1}{n}1 + \frac{1}{n}1^T\frac{(1-\alpha)}{n}W_n^T 1 + \frac{(1-\alpha)}{n}1^T W_n\frac{1}{n}1 + \frac{(1-\alpha)}{n}1^T W_n\frac{(1-\alpha)}{n}W_n^T 1 \\
&\geq \frac{1}{n^2}1^T 1 + \frac{1}{n}\frac{(1-\alpha)}{n}1^T W_n^T 1 + \frac{1}{n}\frac{(1-\alpha)}{n}1^T W_n 1 + \frac{(1-\alpha)^2}{n^2}1^T W_n W_n^T 1
\end{aligned} \tag{94}$$

The two terms in the middle can be combined because $1^T W_n^T 1$ is equivalent to $1^T W_n 1$, both expressions sum up all entries of matrix W_n.

$$\| v_n \|^2 \geq \frac{1}{n^2} 1^T 1 + \frac{2(1-\alpha)}{n^2} 1^T W_n 1 + \frac{(1-\alpha)^2}{n^2} \| W_n^T 1 \|^2 \tag{95}$$

Because its a row-stochastic matrix, the sum of all elements of W_n is equal to n. Also $1^T 1$ is n. Therefore the first two summands can be combined into a single term.

$$\begin{aligned} \| v_n \|^2 &\geq \frac{3-2\alpha}{n} + \frac{(1-\alpha)^2}{n^2} \| W_n^T 1 \|^2 \\ &\geq \Theta\left(\frac{1}{n}\right) + \Theta\left(\frac{1}{n^2} \sum_{i=1}^{n} (k_i)^2\right) \end{aligned} \tag{96}$$

$W_n^T 1$ is the column sum of the direct input share matrix and therefore corresponds to the vector of out-degrees k.

6.3 Second order out-degree distribution and fitted distributions

	year	distribution	parameter	xmin	stderror	p-value
1	2015	lm	1.71	14.00	0.18	
2	2016	lm	1.73	14.00	0.18	
3	2017	lm	1.69	14.00	0.18	
4	2015	lm_GIcorrect	2.01	14.00	0.76	
5	2016	lm_GIcorrect	2.03	14.00	0.77	
6	2017	lm_GIcorrect	1.99	14.00	0.75	
7	2015	ksr	1.92	14.00		
8	2016	ksr	1.94	14.00		
9	2017	ksr	1.96	14.00		
10	2015	powerlaw	1.17	45.00		0.34
11	2016	powerlaw	1.15	42.00		0.52
12	2017	powerlaw	1.15	44.00		0.08
13	2015	powerlaw_exogcut	1.51	14.00		0.77
14	2016	powerlaw_exogcut	1.52	14.00		0.81
15	2017	powerlaw_exogcut	1.51	14.00		0.39

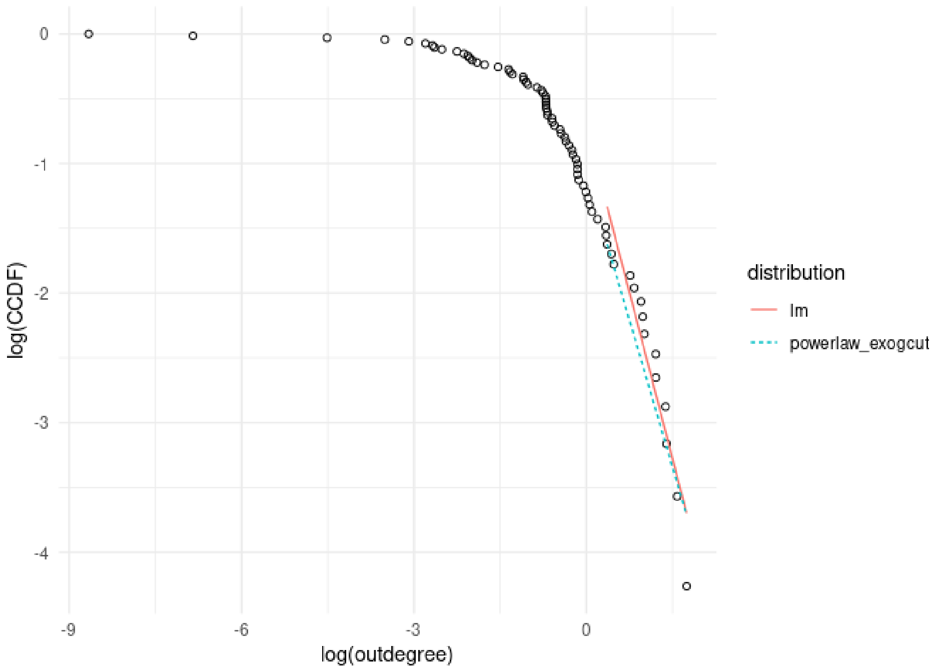

Figure 22: Log-log CCDF of the 2015 second order out-degrees with a fitted linear model versus power law on the 20% tail.

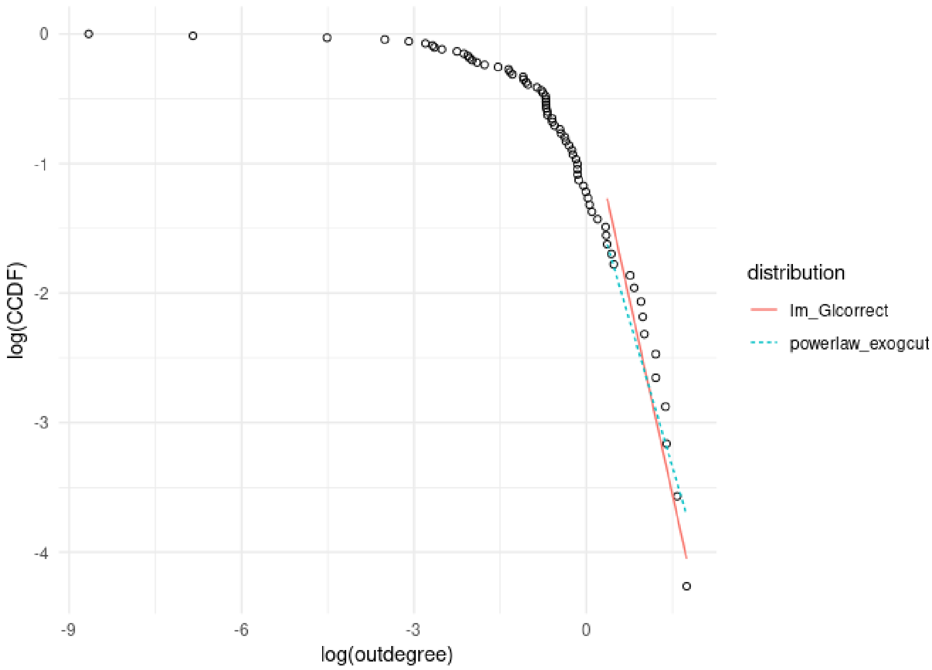

Figure 23: Log-log CCDF of the 2015 second order out-degrees with a fitted linear model with GI-correction versus power law on the 20% tail.

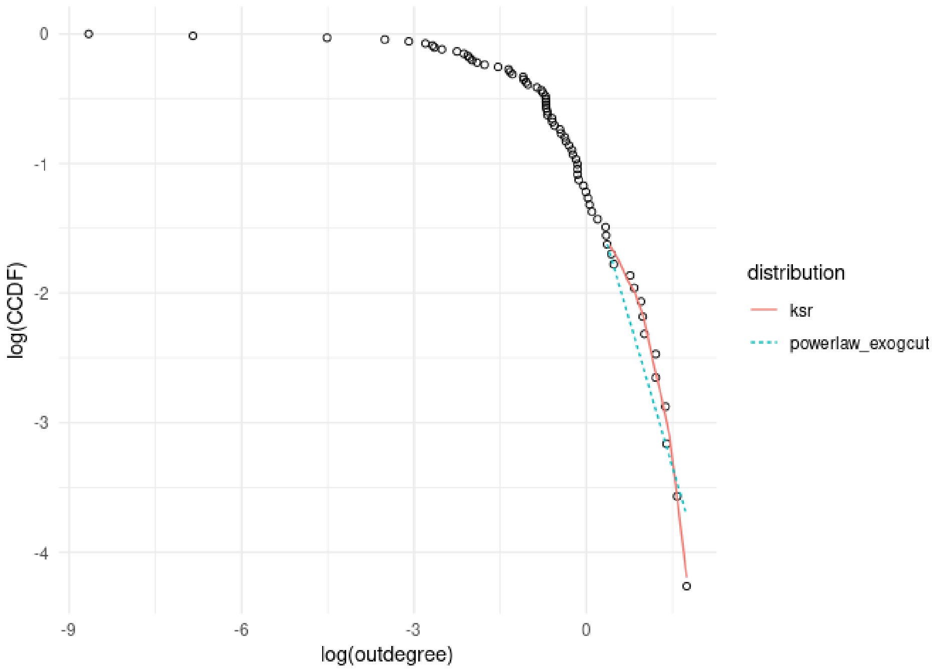

Figure 24: Log-log CCDF of the 2015 second order out-degrees with a kernel smoothing regression versus power law on the 20% tail.

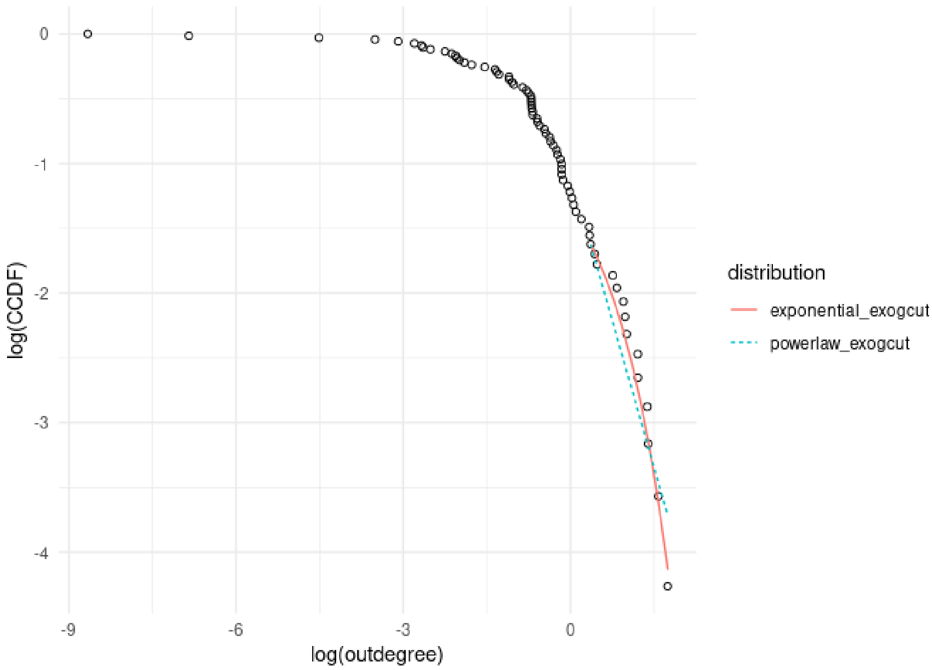

Figure 25: Log-log CCDF of the 2015 second order out-degrees with fitted exponential distribution versus power law on the 20% tail.

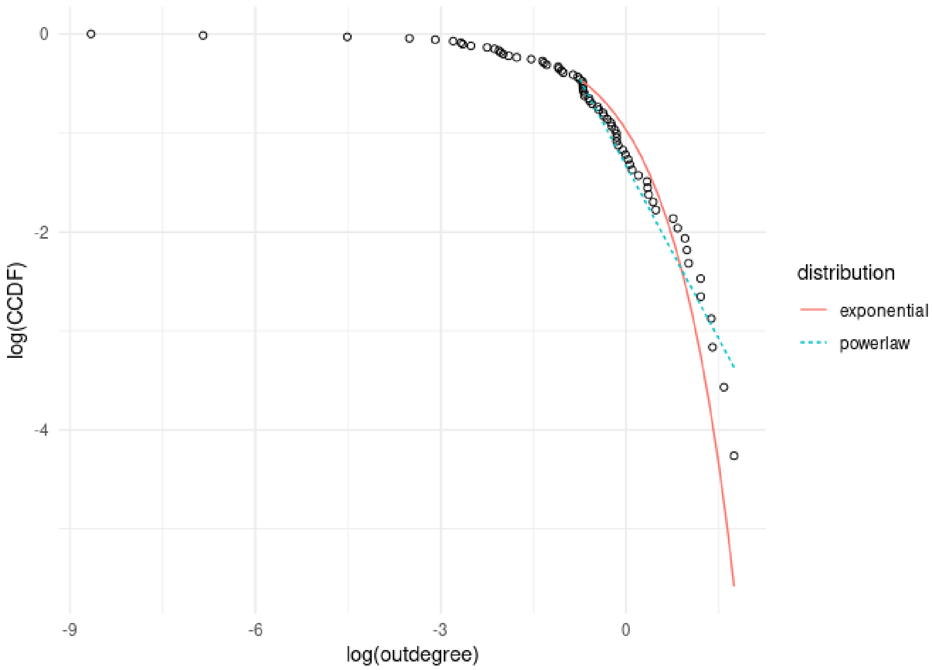

Figure 26: Log-log CCDF of the 2015 second order out-degrees with fitted exponential distribution versus power law on the power laws optimal tail length.

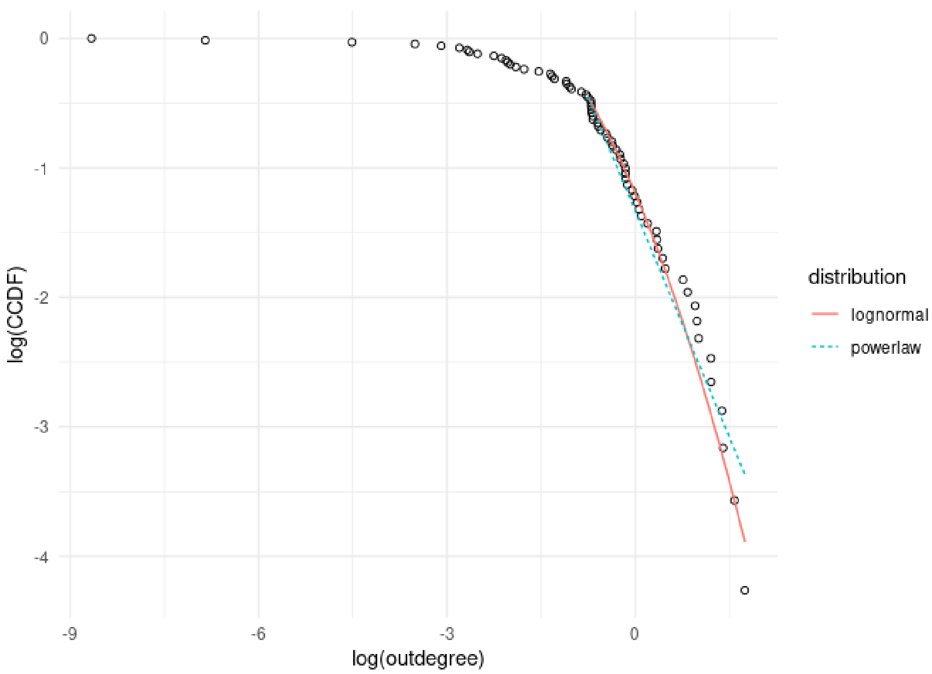

Figure 27: Log-log CCDF of the 2015 second order out-degrees with fitted lognormal distribution versus power law on power laws optimal tail length.

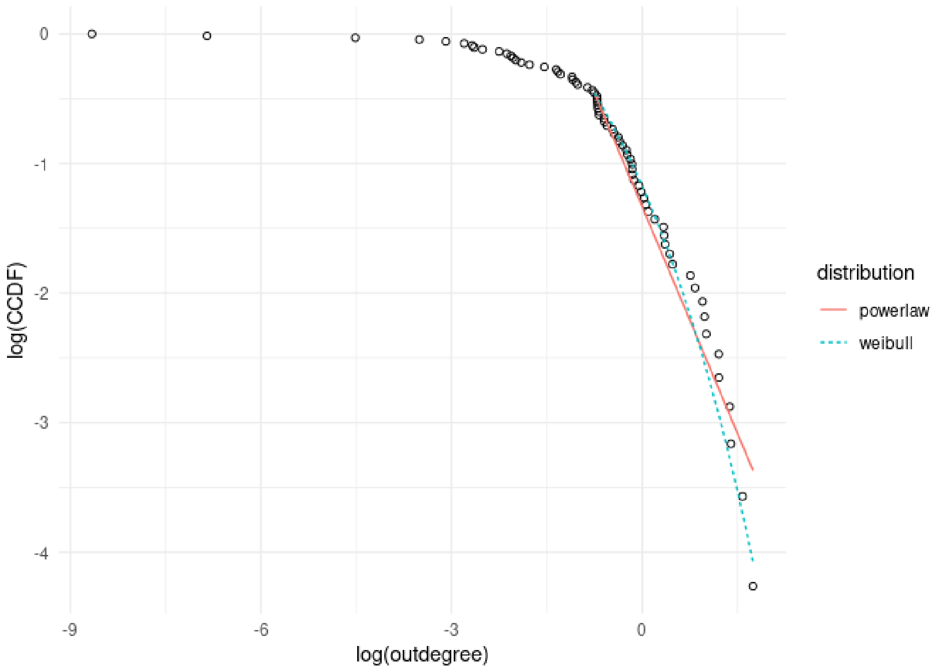

Figure 28: Log-log CCDF of the 2015 second order out-degrees with fitted Weibull distribution versus power law on power laws optimal tail length.

	distribution	lognormal	weibull	exponential
1	powerlaw	0.75	0.77	0.26
2	powerlaw_exogcut	0.72	0.80	0.93

Table 8: P-values of the one-sided likelihood-ratio test.

	distribution	lognormal	weibull	exponential
1	powerlaw	0.51	0.46	0.52
2	powerlaw_exogcut	0.56	0.41	0.15

Table 9: P-values of the two-sided likelihood-ratio test.

6.4 Differences of the production network model and the classical input-output model.

While the mathematical structure of the production network model and the classical input-output model is equivalent, their difference boils down to the production functions they assume and thus to the form their shocks and output values have (relative differences versus absolute values). To derive the upstream propagation of the classical input-output model, one starts from the accounting identity of input-output tables ((97) or (98)), where x is the vector of total sectoral outputs, Z the input-output matrix showing the intermediate inputs and c the vector of final consumption. It is important to note that the sales in Z_{ij} still run from j to i as in (21) for calculating the downstream propagation matrix W. But matrix Z^T is the transpose where transactions run from rows to columns. This also results in the transposed direct requirements matrix A^T. In the transposed matrices, shocks diffuse upstream instead of downstream. Figure **??** shows the example network used here.

$$x_i = \sum_i Z_{ij} + c_i \tag{97}$$

$$x = Z^T 1 + c \tag{98}$$

$$x = A^T x + c \tag{99}$$

$$x = (I - A^T)^{-1} c \tag{100}$$

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix} \qquad x = \begin{pmatrix} 10 \\ 5 \\ 6 \end{pmatrix} \tag{101}$$

$$A^T = \begin{pmatrix} 0 & \frac{3}{5} & \frac{5}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad c = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} \tag{102}$$

When the output of sector 2 doubles to 10 and the output of sector 3 increases by 1.5 to 9 (because final demand here doubles and increases by 1.5) this requires the supplying sector 1 to increase its production to 15.5 instead of 10. This results from multiplying $(I - A^T)^{-1}$ with c_{new}. The new outputs can be seen in x_{new}. By using the direct requirements matrix A as the diffusion matrix, the classical input-output model uses Leontief production functions.

$$(I - A^T)^{-1} = \begin{pmatrix} 1 & \frac{3}{5} & \frac{5}{6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad c_{new} = \begin{pmatrix} 2 \\ 10 \\ 9 \end{pmatrix} \qquad x_{new} = \begin{pmatrix} 2 + 6 + 7.5 = 15.5 \\ 10 \\ 9 \end{pmatrix} \tag{103}$$

Multiplying final consumption with the technical coefficients works perfectly well for interpreting upstream propagation with Leontief production functions. The classical input output model has also been adapted to model of downstream propagation, the Gosh model, where interpretation becomes more difficult. The diffusion matrix of the Gosh-model is a matrix E, which includes the shares of a sectors input in its total output.

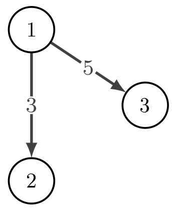

Figure 29: Network of flow matrix Z

$$E_{ij} = \frac{Z_{ij}}{x_i} \tag{104}$$

$$x = (I - E)^{-1} v \tag{105}$$

In the Gosh-model as well, v and x are vectors of absolute output. But in framework with absolute values a model of downstream propagation is implausible (there is a large literature on the implausibility of the Gosh-model, see e.g. Oosterhaven (1988), Oosterhaven (1989)). The crucial point is that multiplying input shares of one industry with the absolute outputs of other industries is not what production functions do. They multiply input shares with the amount that an industry 2 gives to industry 1 as input. That's just a fraction of the output of industry 2. So the production function has to fulfil another task here. Additionally to calculating how much output is produced from which input, the production function has to serve as a allocator of a sectors output to its customers. But production functions are defined for inputs being a fraction of other industries output. They are not concerned with allocation. A model of downstream propagation with linear production functions makes only sense if the production functions work with relative input changes and relative output changes as I have been shown in section 3.2.1. The production network model circumvents this with changes of log-linear production functions, which makes the model work coherently as well.

An example of downstream propagation with the Gosh-model is shown in (106). If one assumes linear production functions of the form $y_2 = e_{21} y_1$, then doubling the output of sector 1 (20 instead of 10) leads to an increase in output of sector 2 by $12/5 = 2.4$ and in sector 3 by $16.66/6 = 2.8$. Assuming linear production functions with relative changes $dy_2/y_2 = e_{21} dy_1/y_1$, doubling the output of sector 1 ($dy_1/y_1 = 100\%$) leads only to an increase in output of sector 2 by $100\%(3/5) = 60\%$ and in $100\%(5/6) = 83.33\%$ in sector 3. This appears more realistic as production functions usually are not assumed to have increasing returns to scale, where doubling the input more than doubles the output.

$$E = \begin{pmatrix} 0 & 0 & 0 \\ \frac{3}{5} & 0 & 0 \\ \frac{5}{6} & 0 & 0 \end{pmatrix} \qquad (I-E)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{5} & 1 & 0 \\ \frac{5}{6} & 0 & 1 \end{pmatrix} \qquad v = \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix} \qquad x = \begin{pmatrix} 20 \\ 12 \\ 16.66 \end{pmatrix} \tag{106}$$

To sum up, the classical input-output model is a model of upstream propagation with Leontief production functions while the production network model is a model of downstream propagation with log-linear production functions. The essential difference between the production network model of downstream propagation to the Gosh-model is its use of output changes instead of absolute output. This allows to interpret the diffusion matrix as a network of nested production functions.

6.5 CCDF of influence vectors

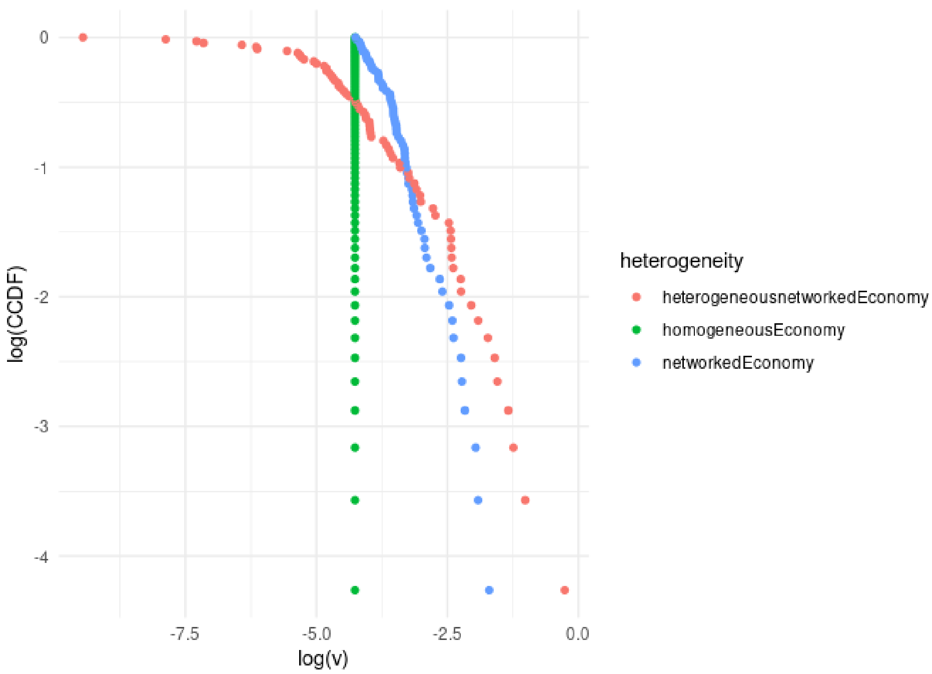

Figure 30: Log-log CCDFs of different influence vectors of the 2015 German production network.

Affirmation

I hereby declare that I have composed my Master's thesis *"How production network structure influences the transmission of microeconomic shocks into macroeconomic output changes - An analysis of German input-output data"* independently using only those resources mentioned, and that I have as such identified all passages which I have taken from publications verbatim or in substance. I agree that the work will be reviewed using plagiarism testing software. Neither this paper, nor any extract of it, has been previously submitted to an examining authority, in this or a similar form.

I have ensured that the written version of this thesis is identical to the version saved on the enclosed storage medium.

January 25, 2022

(Signature)