Definition. ***Functor***
A functor F between to categories C and D consists of two functions. One which maps objects from C to objects in D and one which maps morphisms from C to morphisms in D. Moreover, the following must hold:
(1) for $f \in Hom(A, B)$, $F(f) \in Hom(F(A), F(B))$
(2) $F(id_A) = id_{F(A)}$
(3) $F(g \circ f) = F(g) \circ F(f)$

Functors defined this way are structure preserving mappings between categories. Let us consider some examples.

Example 1. *Remember the natural numbers and integers, that is N resp. Z, are monoids and thus define categories. These both categories consist of only one object and their morphisms are each element in N resp. Z. Having two morphisms m and n, composition is defined by $m + n$. The identity morphism is given by the element 0 and with this associativity trivially is fulfilled.*

Next, let F by a functor between these categories. Since both only exists of one object, say x resp. y, we have $F : x \mapsto y$. So nothing interesting here.

Now, let us consider two morphisms $m, n \in N$.
By (3) we have the requirement

$$F(n + m) = F(n) + F(m)$$

And by (2)

$$F(0) = 0$$

Functions that fulfill these two requirements are exactly the linear *maps between N and Z. Therefore F must have the form $F : n \mapsto a \times n$ for some $a \in N$.*

For readers with knowledge in topology the next example will be of interest:

Example 2. *Given a topological space X we define a category C by taking as objects all its elements. The morphisms are defined as follows: For two objects $x, y \in X$, the morphism $x \to y$ is included in $Hom(C)$ if and only if*

$$\forall U \ U \in \mathcal{B}(x) \to y \in U$$

That is, if for any open set U that contains x it also contains y. The composition is defined by

$$(y \to z) \circ (x \to y) = x \to z$$

*Let us verify that indeed all axioms for a category are fufilled: Clearly we have $x \to x \in Hom(C)$. Hence for any object x, $id_x \in Hom(C)$.
Moreover, if $x \to y$ and $y \to z$ are in $Hom(C)$, then each open set U containing x must contain y. But U is also an open set for y, so it must contain z as well. This shows, the composition is well-defined. Trivially, this composition is associative.*

*Let X, Y be topological spaces and C_X, C_Y the corresponding categories of the sort like above. If $f : X \to Y$ is a continuous mapping, then f is a functor between these categories.
For this to see we verify the axioms of a functor:*

*(1) Assume $x \to y$. We have to show $f(x) \to f(y)$.
Let V be an open set containing $f(x)$. Then by continuity $f^{-1}(V)$ is open as well. Moreover, $x \in f^{-1}(V)$ and thus $y \in f^{-1}(V)$. But this*

implies $f(y) \in V$ *that shows* $(f(x) \to f(y)) \in Hom(C_Y)$.

(2) is trivially fulfilled, since $f(x \to x) = f(x) \to f(x)$.

(3) is trivial as well but to familiarize the reader with the abstract notation of category theory, let us write down all the details: Assume $x \to y$ *and* $y \to z$, *then*

$$f(y \to z \circ x \to y) = f(x \to z) = f(x) \to f(z)$$

By using $f(x) \to f(y)$ *and* $f(y) \to f(z)$ *we can rewrite the r.h.s as*

$$f(x) \to f(z) = f(y) \to f(z) \circ f(x) \to f(y)$$

This finally yields

$$f(x) \to f(y) = f(y) \to f(z) \circ f(x) \to f(y)$$

The question arises if a functor between these categories necessarily is a continuous function. The answer is no.
Just consider the function $f : [0, 1] \to \{0, 1\}$, *with* $f|[0, 1/2[= 0$ *and* $f|[1/2, 1] = 1$. *Let* $\{0, 1\}$ *carry the discrete topology and* $[0, 1]$ *having a topology generated by the following basis: all open sets for points* $\xi \in]0, 1]$ *and an open set for* 0 *which is* $\{0, 1/4\}$. *This implies* $0 \to 1/4$ *and trivially* $f(0) \to f(1/4)$. *By construction,* f *is not continuous at* $1/2$.