

In the category of sets, given two set B and C , the set of all functions from C to B is often denoted by B^C . It can be constructed as a subset of $PP(C \times B)$, where P denotes the power-set operator. We now aim to generalize this concept in terms of category theory.

Definition. *In a category \mathcal{C} for two objects $B, C \in \mathcal{C}$ we call an object exponential, denoted by*

$$C^B$$

if the following is satisfied:

(1) $C^B \in \mathcal{C}$

(2) *there is a morphism $\epsilon \in \text{Hom}(C^B \times B, C)$ such that:*

For any $f \in \text{Hom}(A \times B, C)$ there exists a unique $\tilde{f} \in \text{Hom}(A, C^B)$ with $\epsilon \circ (\tilde{f} \times \text{id}_B) = f$.

At first glance this definition might look a little bewildering. First of all C^B is required to be just an object. Don't let yourself confuse by its special notation. To understand the meaning of ϵ consider in *Set* a function $f : A \times B \rightarrow C$. If we fix one element of A , say a , then we obtain a function $f_a : B \rightarrow C$. That is

$$f_a(b) = f(a, b)$$

This way

$$a \mapsto f_a$$

actually is a mapping

$$A \rightarrow C^B$$

This mapping corresponds exactly to the transpose $\tilde{f} \in \text{Hom}(A, C^B)$. On the other hand, this transpose is connected to the function f by the property that

$$\tilde{f}(a)(b) = f(a, b)$$

The r.h.s we also can write as

$$\left(\tilde{f}(a), b\right) \mapsto \tilde{f}(a)(b)$$

or by regarding as product

$$(\tilde{f} \times id_B)(a, b) \mapsto \tilde{f}(a)(b)$$

From these expressions we see that this map operates by evaluating the the first component, which is in C^B , at the point given in the second component, which is in B .

This reveals the sense of ϵ , and it is to considered like an abstract evaluation.

Like for products, the exponential must not necessarily exist. We say a category is **cartesian closed** if for any pair of objects its product and exponential exists.