We consider a velocity field in a fluid given by the function

$$u : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$$

Moreover, we regard a small volume within this fluid which location shall be tracked by $x : [0, T] \to \mathbb{R}^n$. So if we want to know the velocity of this volume at time t we must consider $u(t, x(t))$. This actually we can regard as a function in time only $v : [0, T] \to \mathbb{R}^n$ defined by

$$v(t) := u(t, x(t))$$

By Newton's law, the change of velocity is proportional to forces acting on the volume. Thus,

$$v'(t) = F(t, x(t), v(t)) \tag{1}$$

Here $F(t, x, u)$ collects all these forces. Without going into more details, the appearance of u in that function is due to friction forces or sometimes referred as momentum diffusion and proportional to $\nabla \cdot \nabla u$.

By replacing $v(t)$ with $u(t, x(t))$ and applying the chain rule we compute

$$v'(t) = u'(t, x(t)) = \frac{\partial u}{\partial t} + \left(\frac{dx(t)}{dt} \cdot \nabla \right) u$$

Since velocity of the volume is given by

$$v(t) = \frac{dx(t)}{dt}$$

we find

$$\begin{aligned} &\frac{\partial u}{\partial t} + \left(\frac{dx(t)}{dt} \cdot \nabla \right) u = \\ &\frac{\partial u}{\partial t} + \left(v(t) \cdot \nabla \right) u = \frac{\partial u}{\partial t} + \left(u \cdot \nabla \right) u \end{aligned}$$

By replacing this result at the l.h.s of (1) we arrive at

$$\frac{\partial u}{\partial t} + \left(u \cdot \nabla \right) u = F(t, x, u)$$