Example 1. *We take as objects all sets and as morphisms all binary relations between two sets.*

If A and B are sets then a binary relation is a subset $f \subset A \times B$. So if $(a, b) \in R$, then a 'relates' to b.

In general, such a relation is not a function. It could be that $(a, b) \in f$ and $(a, c) \in f$ for $b, c \in B$ with $b \neq c$.

We define composition between morphisms as follows:
If $f : A \to B$ and $g : B \to C$ then

$$(a, c) \in g \circ f \;\; :\leftrightarrow \;\; \exists b \in B \; (a, b) \in f \wedge (b, c) \in g$$

In words, the composition relates a to c if and only if there is a 'bridge' b such that f relates a to b and g relates b to c.

For any object A we define

$$id_A = \{(a, a) \mid a \in A\}$$

One immediately sees that this serves correctly as identity morphism.

It remains to proof associativity. Assume we are given $f : A \to B$, $g : B \to C$ and $h : C \to D$.

$$(a, d) \in (h \circ g) \circ f$$

means, there exists $b \in B$ such that

$$(a, b) \in f \tag{1}$$

and

$$(b, d) \in h \circ g$$

Moreover, the latter implies there exists $c \in C$ such that

$$(b, c) \in g \tag{2}$$

and

$$(c, d) \in h \tag{3}$$

All this follows directly from the definition of $\circ$*. By putting equations (1), (2) and (3) together, we see immediately that* $(a, d) \in h \circ (g \circ f)$*. This shows* $(h \circ g) \circ f = h \circ (g \circ f)$*.*

The next example introducing a very often appearing structure in category theory.

Example 2. ***Monoid***
A monoid is a set S *together with a binary relation* $\cdot : S \times S \to S$ *fulfilling the following:*

(1) it is associative, that is $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
(2) it has a neutral element e*, which satisfies* $e \cdot x = x = x \cdot e$ *for all* $x \in S$*.*

A moinid can be seen as a category with only one object and morphisms taken to be all elements in S*. Moreover, composition is defined to be the binary relation* $\cdot$*.*

One easily verifies that all axioms are fulfilled. Note, since the category has only one element, say A*, any morphism* x *has* $dom(x) = A = cod(x)$*. Therefore, composition works between all morphisms.*

Examples for monoids are N, Q, R*, that is, the natural numbers, the rationals and the reals. The binary relation can be taken to be the addition or multiplication.*

This last example shows very nicely how much morphisms can deviate from being functions between sets. The power of category theory is that everything what holds in Set and is based only on the axioms of categories resp. morphisms, likewise holds in all monoids.

Example 3. ***The category*** Mon

From monoids we can build a category Mon which objects are all monoids and morphism functions which preserve the monoid structure in the following sense.

If $f : X \to Y$ is a morphism then
(1) for any $x, y \in X$

$$f(x \cdot y) = f(x) \cdot f(y)$$

(2) $f(e_X) = e_Y$ where e_X, e_Y are the corresponding neutral elements of X resp. Y

One easily checks that this still holds when composing two morphisms $f \circ g$. Therefore the composition of functions provides a composition between morphisms which fulfills all axioms of a category.