The target area is given by

$$[\xi_1, \xi_2] \times [\eta_1, \eta_2]$$

If the initial velocity in horizontal resp. vertical direction is u_0 resp. v_0, then after n steps the horizontal resp. vertical position of the particle is

$$x_n = nu_0 - \frac{1}{2}n(n-1) \tag{1}$$

resp.

$$y_n = nv_0 - \frac{1}{2}n(n-1) \tag{2}$$

For the vertical velocity v_n at step $n > 0$ we have

$$v_n = v_{n-1} - 1$$

Thus,

$$v_n = v_0 - n$$

When the particle reaches the highest point, it must necessarily fulfill

$$v_m = 0$$

Therefore the previous equation implies

$$m = v_0$$

By putting this in the equation (2) yields:

$$y_{max} = \frac{1}{2}v_0(v_0 + 1)$$

The latter shows, finding the highest possible v_0, automatically gives the highest possible vertical position y_{max}.

For given u_0 let us compute all possible steps that makes the particle horizontally ending up in $[\xi_1, \xi_2]$:
By (1) the determining relation for this is

$$\xi_1 \leq nu_0 - \frac{1}{2}n(n-1) \leq \xi_2$$

This is an quadratic relation in n. By using $\xi_1, \xi_2 \geq 0$ one easily verifies that the $n \geq 0$ that satisfy the above relation are exactly those that satisfy

$$\sqrt{\left(u_0 + \frac{1}{2}\right)^2 - 2\xi_2} + u_0 + \frac{1}{2} \leq n \leq \sqrt{\left(u_0 + \frac{1}{2}\right)^2 - 2\xi_1} + u_0 + \frac{1}{2}$$

Next, for give u_0 and given n within the above interval, let us determine the interval of possible v_0 such that the particle vertically ends up within $[\eta_1, \eta_2]$ at step n.
By use of (2), we can formulate this as

$$\eta_1 \leq n v_0 - \frac{1}{2} n(n-1) \leq \eta_2$$

This yields

$$\frac{1}{n}\left(\eta_1 + \frac{1}{2} n(n-1)\right) \leq v_0 \leq \frac{1}{n}\left(\eta_2 + \frac{1}{2} n(n-1)\right)$$

Note, when doing the corresponding iterations, one has to ensure by use of suitable Gauss-brackets, all velocities and steps to be natural numbers.