

To solve the system

$$\begin{aligned}v'(t) &= F(t, x(t), v(t)) \\x'(t) &= v(t) \\x(0) &= x_0 \\v(0) &= v_0\end{aligned}$$

we use the Euler scheme to discretize in time. We could choose a more accurate scheme, but for showing the main idea it suffices. For each grid-point x_A we ask from where the volume arriving here has departed. This volume must exist since it solves the backward-in-time equivalent to above system. Formally this reads

$$\tilde{v}(t^{k+1}, x_A) = \tilde{v}(t^k, x_D) + \Delta t \cdot F(t^k, x_D, \tilde{v}(t^k, x_D)) \quad (1)$$

Note, by using $\tilde{v}(t, x)$ we make explicit that this is the approximate velocity of the volume currently located at x . Since we don't know x_D , equation (1) cannot be used alone in each step. The idea is to discretize $x'(t) = v(t)$ as a part of above system and to solve

$$x_A = x_D + \Delta t \cdot v(t^k, x_D) \quad (2)$$

for x_D . This is done by iterating the equation

$$x_D = x_A - \Delta t \cdot v(t^k, x_D)$$

which may be started with $x_D^0 := x_A$. For Δt being sufficiently small, the r.h.s is a contracting map and so Banach's fix-point theorem ensures a unique solution. After having found x_D , we would like to use it in (1) in order to approximate the velocity at x_A . But since x_D is certainly not matching a grid-point we have to interpolate $\tilde{v}(t^k, x_D)$ as well as $F(t^k, x_D, \tilde{v}(t^k, x_D))$ from known values.