

We consider all elements x that fulfill the following relation

$$x = x$$

Trivially, all sets fulfill this relation. The question arises, if there exists a set that contains all these elements. In other words:

Is $\{x : x = x\}$ a set?

If yes, then this set would contain all sets, since each set fulfills $x = x$.

Let us assume it is a set and denote it by X . Define

$$y := \{z \in X : z \notin z\}$$

By axioms of set-theory, this is a set and moreover either $y \in y$ or $y \notin y$.

If $y \in y$, then since $y \in X$ the definition of y implies $y \notin y$. So it must be $y \notin y$. But then, the definition of y implies $y \in y$.

All in all, this yields a contradiction and thus X cannot be a set.

Although it is not a set, it still makes perfectly sense, in terms of propositional logic, to consider expressions of the form

$$\{x : \phi(x)\}$$

where ϕ is any formula. Though, we must take in mind that the expression does not always refer to a set. In such cases they are called (proper) classes.