In order to ease notations, let us assume for a moment the function f fulfills a Lipschitz-condition of the form

$$|f(t,x) - f(t,y)| \leq L|x-y| \tag{1}$$

for some $L > 0$. From our scheme we have

$$y_{n+1} = y_n + f(t_n, y_n) \tag{2}$$

If y denotes the true solution, then by Taylor we may write

$$y(t_{n+1}) = y(t_n) + y'(t_n)h + o(h^2)$$

Further using $y'(t) = f(t, y(t))$ this yields

$$y(t_{n+1}) = y(t_n) + f(t_n, y(t_n))h + h^2 R \tag{3}$$

In the last step we have in addition replaced $o(h^2)$ by some estimate R of the remainder. Our aim is to compare y_{n+1} with $y(t_{n+1})$, therefore let us subtract (3) from (2):

$$y_{n+1} - y(t_{n+1}) = y_n - y(t_n) + f(t_n, y_n) - f(t_n, y(t_n))h - h^2 R$$

By introducing $e_k := y_k - y(t_k)$ we further can estimate

$$|e_{n+1}| \leq |e_n| + Lh|e_n| + h^2 R$$

Note, we have used (1).
We reformulate appropriately to obtain

$$|e_{n+1}| \leq (1 + Lh)(|e_n| + hR/L) - hR/L$$

which gives

$$|e_{n+1}| + hR/L \leq (1 + Lh)(|e_n| + hR/L)$$

In order to ease computation we make the 'pessimistic' estimate of $1 + Lh \leq exp(Lh)$. Thus we have

$$|e_{n+1}| + hR/L \leq exp(Lh)(|e_n| + hR/L) \tag{4}$$

This holds for all indexes and so also

$$|e_n| + hR/L \leq exp(Lh)(|e_{n-1}| + hR/L)$$

Inserting the later into (4) gives

$$|e_{n+1}| + hR/L \leq exp(2Lh)(|e_{n-1}| + hR/L)$$

By proceeding this way, we find

$$|e_{n+1}| + hR/L \leq exp((n+1)Lh)(|e_0| + hR/L)$$

Altogether this shows

Theorem.

$$|e_{n+1}| \leq exp(L(b-a))(|e_0| + hR/L) - hR/L$$

One can finally weaken the assumption of a global Lipschitz-condition for f. Without giving all the details here, the idea is to replace it by:

Condition. *There exists $L, M > 0$ such that*

$$|f(t,x) - f(t,y)| \leq L|x-y|$$

for all $|x-y| \leq M$.

This is like a local version of (1). By carefully repeating above proof, one can see that it is always possible to choose h small enough in order to have $|y_n - y(t_n)| \leq M$. The rest of the argumentation remains the same.