Example. *Let us consider the category of all partial ordered sets. This in general is denoted by* **Pos**.

Remember a partial ordered set consists of a binary relation such that for any elements we have:
(1) if $a \leq b$ *and* $b \leq c$ *then* $a \leq c$
(2) if $a \leq b$ *and* $b \leq a$ *then* $a = b$
(3) $a \leq a$

The category **Pos** *is now formed by taking as objects all partially ordered sets. Morphisms between any two objects* A, B *in* Pos*, are taken to be all monotone functions, that is,* $f \in Hom(A, B)$ *iff from*

$$a \leq b$$

follows

$$f(a) \leq f(b)$$

Exercise*:*
Do verify that all axioms of a category are fulfilled by this definitions and hence Pos *really is a category.*

Next we are going to show that this category contains all products for any pair of objects:
Given two objects A, B *in* Pos*, we just take the Cartesian product* $A \times B$ *as underlying set and define the following binary relation:*

$$(a_1, b_1) \leq (a_2, b_2)$$

iff

$$a_1 \leq a_2$$

and

$$b_1 \leq b_2$$

Exercise:
Do verify that the above axioms (1), ... (3) are fulfilled for this binary relation and thus making $A \times B$ belonging to Pos.

As ***projections*** *we take the usual projections of Cartesian products, that is,*

$$p_A : (a, b) \mapsto a$$

$$p_B : (a, b) \mapsto b$$

Exercise:
Do verify that p_A, p_B indeed are morphisms in Pos (show that they are monotone).

Remember the definition of product in category theory. We have to show for two morphisms $f : X \to A$ and $g : X \to B$ the existence of a unique morphism $\langle f, g \rangle$ such the $p_A \circ \langle f, g \rangle = f$ resp. $pr_B \circ \langle f, g \rangle = g$.

The function $\langle f, g \rangle$ is given by

$$x \mapsto (f(x), g(x))$$

The uniqueness can by readable verified together with the definition of the projection maps.

Exercise:
Show that $\langle f, g \rangle$ defined as above is a morphism in Pos (show that it is monotone).

Next, we show that the category Pos contains the exponential for any pair of objects. This together with the above result implies the Pos is Cartesian closed.

We define

$$C^B := Hom(B, C)$$

That is, C^B is just the set of all monotone functions from B to C. The first thing we have to verify is that C^B is in Pos. For this we can make $Hom(B, C)$ a partial ordered set by introducing the following binary relation:

$$g \leq h$$

iff for all $b \in B$

$$g(b) \leq h(b)$$

So, $g \leq h$ iff it is the case evaluated on all elements of B.

***Exercise**:*
Do verify that the above axioms (1), ... (3) are fulfilled for this binary relation and thus making C^B belonging to Pos.

Now, we define $\epsilon \in Hom(C^B \times B, C)$ by just being the canonical evaluation, that is,

$$\epsilon : (g, b) \mapsto g(b)$$

***Exercise**:*
Show that ϵ is a morphism in Pos (show it is monotone).

Finally, we define the transpose $\tilde{f}$ of a given morphism $f : A \times B \to C$ by

$$\tilde{f} : a \mapsto f(a, \cdot)$$

So, $\tilde{f}$ at an element $a \in A$ is obtained from f by fixing the first parameter of f with a.
Again, it remains to show that $\tilde{f}$ is a morphism in Pos. For this assume $a_1, a_2 \in A$ with $a_1 \leq a_2$. Then

$$\tilde{f}(a_1) = f(a_1, \cdot)$$

and

$$\tilde{f}(a_2) = f(a_2, \cdot)$$

But since for any $b \in B$

$$f(a_1, b) \leq f(a_2, b)$$

it follows by definition of the binary relation on C^B

$$\tilde{f}(a_1) \leq \tilde{f}(a_2)$$

Thus $\tilde{f}$ is monotone and hence a morphism in Pos.

From the above definitions it follows

$$\epsilon \circ (\tilde{f} \times id_B) = f$$

We can verify this point-wise:

$$\epsilon \circ (\tilde{f} \times id_B)(a, b) = \epsilon(f(a, \cdot), b) = f(a, b)$$

***Exercise**:*
Verify the $\tilde{f}$ is uniquely determined by requiring $\epsilon \circ (\tilde{f} \times id_B) = f$.

Altogether we have shown that Pos is a Cartesian close category.