We assume the same Lipschitz-condition for f:

$$|f(t,x) - f(t,y)| \leq L|x-y| \tag{1}$$

for some $L > 0$. From the scheme we have

$$y_{n+1} = y_n + hf\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n)\right) \tag{2}$$

and the local error estimated above gives

$$y(t_{n+1}) = y(t_n) + hy'\left(t_n + \frac{h}{2}\right) + o(h^3)$$

Further using $y'(t) = f(t, y(t))$ this yields

$$y(t_{n+1}) = y(t_n) + hf\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right) + o(h^3) \tag{3}$$

We will replace $o(h^3)$ by h^3R where R is some estimate of the remainder.

By introducing $e_k := y_k - y(t_k)$ and subtracting (2) from (3) we obtain

$$\begin{aligned} e_{n+1} = e_n &+ hf\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right) \\ &- hf\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n)\right) + h^3R \end{aligned}$$

Further by the triangle inequality and (1)

$$|e_{n+1}| \leq |e_n| + Lh\left|y\left(t_n + \frac{h}{2}\right) - y_n - \frac{h}{2}f(t_n, y_n)\right| + h^3R \qquad (4)$$

On the r.h.s we use the second local error to write:

$$\left|y\left(t_n + \frac{h}{2}\right) - y_n - \frac{h}{2}f(t_n, y_n)\right| =$$
$$\left|y(t_n) + \frac{h}{2}f(t_n, y(t_n)) + o(h^2) - y_n - \frac{h}{2}f(t_n, y_n)\right|$$

which again by using (1) can be estimated like

$$\left|y(t_n) + \frac{h}{2}f(t_n, y(t_n)) + o(h^2) - y_n - \frac{h}{2}f(t_n, y_n)\right| \leq$$
$$|e_n| + \frac{hL}{2}|e_n| + o(h^2)$$

Using this result in (4) yields

$$e_{n+1} \leq |e_n| + hL|e_n| + \frac{h^2L^2}{2}|e_n| + h^3R$$

By reformulating, this can be written as

$$e_{n+1} \leq \left(1 + hL\left(1 + \frac{hL}{2}\right)\right)\left(|e_n| + \frac{h^2R}{L\left(1 + \frac{hL}{2}\right)}\right) - \frac{h^2R}{L\left(1 + \frac{hL}{2}\right)}$$

which gives

$$|e_{n+1}| + \frac{h^2R}{L\left(1+\frac{hL}{2}\right)} \leq \left(1 + hL\left(1+\frac{hL}{2}\right)\right)\left(|e_n| + \frac{h^2R}{L\left(1+\frac{hL}{2}\right)}\right)$$

Now we use the estimate

$$1 + Lh + \frac{h^2L^2}{2} \leq exp(Lh)$$

to get

$$|e_{n+1}| + \frac{h^2R}{L\left(1+\frac{hL}{2}\right)} \leq exp(Lh)\left(|e_n| + \frac{h^2R}{L\left(1+\frac{hL}{2}\right)}\right)$$

From here we can do the same steps like for global error of Euler method and recursively apply this inequality to finally arrive at

Theorem.

$$|e_{n+1}| \leq exp(L(b-a))\left(|e_0| + \frac{h^2R}{L\left(1+\frac{hL}{2}\right)}\right) - \frac{h^2R}{L\left(1+\frac{hL}{2}\right)}$$

Also here one can weaken the assumption of a global Lipschitz-condition for f to

Condition. *There exists $L, M > 0$ such that*

$$|f(t,x) - f(t,y)| \leq L|x-y|$$

for all $|x-y| \leq M$.