Products play an important role in applications of category theory. Its definition is like always leaned at the corresponding construct from set theory.

Definition 1. *Product*
In a category C the product of two objects A_1, A_2 is an object denoted by $A_1 \times A_2$ together with two morphisms

$$p_1 : A_1 \times A_2 \to A_1$$

and

$$p_2 : A_1 \times A_2 \to A_2$$

such that the following holds:
For any object X and morphisms $f_1 : X \to A_1$ and $f_2 : X \to A_2$ there exists a unique morphism denoted by $\langle f_1, f_2 \rangle : X \to A_1 \times A_2$ such that

$$p_1 \circ \langle f_1, f_2 \rangle = f_1$$

and

$$p_2 \circ \langle f_1, f_2 \rangle = f_2$$

The morphisms p_1, p_2 are called projections and $\langle f_1, f_2 \rangle$ the product morphism of f_1, f_2. This shall remind to the product in set theory. One easily verifies that the set theoretic product of two sets fulfills all requirements of this categorical version.
The notation of a product entails that the product itself only depends on the factors. It turns out that this is correct up to isomorphism.

Theorem 1. *The product $A_1 \times A_2$ of two objects A_1, A_2 is unique up to isomorphism.*

Proof. Let us assume there exists two objects P and Q and corresponding morphisms p_1, p_2 and q_1, q_2 such that both fulfill the requirements to be the product of A_1 and A_2. We show, there exists an isomorphism $j : P \to Q$. We apply the product properties of P onto the case $X = P$,

$f_1 = p_1$ and $f_2 = p_2$. Then by assumption there is a unique morphism $i : P \to P$ such that $p_1 \circ i = p_1$ and $p_2 \circ i = p_2$. We conclude

$$i = id_P \tag{1}$$

We repeat this application but for the case $X = Q$, $f_1 = q_1$ and $f_2 = q_2$. This yields a unique morphism

$$\langle q_1, q_2 \rangle : Q \to P$$

In analogy, by using the product properties of Q, we find from the case $X = P$, $f_1 = p_1$ and $f_2 = p_2$, the existence of

$$\langle p_1, p_2 \rangle : P \to Q$$

We show that $\langle q_1, q_2 \rangle$ and $\langle p_1, p_2 \rangle$ yields the desired isomorphism and its inverse. Using the properties of product morphism one gets:

$$p_1 \circ \langle q_1, q_2 \rangle \circ \langle p_1, p_2 \rangle = q_1 \circ \langle p_1, p_2 \rangle = p_1$$

and

$$p_2 \circ \langle q_1, q_2 \rangle \circ \langle p_1, p_2 \rangle = q_2 \circ \langle p_1, p_2 \rangle = p_2$$

Comparing this with the above morphism i and its uniqueness, (1) implies

$$\langle q_1, q_2 \rangle \circ \langle p_1, p_2 \rangle = id_P$$

By analogous steps or just symmetry, one finds

$$\langle p_1, p_2 \rangle \circ \langle q_1, q_2 \rangle = id_P$$

which yields the desired result. $\square$

The next theorem shows that the product is associative in case all factors do exist in C.

Theorem 2. *If A_1, A_2, A_3 are objects from a category C, then*

$$(A_1 \times A_2) \times A_3 = A_1 \times (A_2 \times A_3)$$

under the condition that all factors are defined.

Proof. The proof is similar to the one of theorem 1. By using several projections one yields a unique morphism $f : (A_1 \times A_2) \times A_3 \to A_1 \times (A_2 \times A_3)$ and another $g : A_1 \times (A_2 \times A_3) \to (A_1 \times A_2) \times A_3$. The like in proof of the above theorem, one compares the projections of $f \circ g$ and $g \circ f$ with the projections of $id : (A_1 \times A_2) \times A_3 \to (A_1 \times A_2) \times A_3$ resp. $id : A_1 \times (A_2 \times A_3) \to A_1 \times (A_2 \times A_3)$. $\square$

This last theorem gives sense to products of arbitrary finite factors. Moreover, if a category contains the product of any two objects, it also contains all finite products. TODO p46 spaces closed of products