

Let us define two families of 10-tuples of non-negative natural numbers smaller or equal than k :

$$I_1 := \{(p_0, \dots, p_9) \in \{0, \dots, k\}^{10} : p_0 + \dots + p_9 = n\}$$

$$I_2 := \{(p_0, \dots, p_9) \in \{0, \dots, k\}^{10} : p_0 + \dots + p_9 = n \text{ and } p_0 < k\}$$

With this the problem can be solved by the following formula:

$$\sum_{(p_0, \dots, p_9) \in I_1} \binom{n}{p_0, p_1, \dots, p_9} - \sum_{(p_0, \dots, p_9) \in I_2} \binom{n-1}{p_0, p_1, \dots, p_9}$$

Explanation:

A multinomial coefficient of the form

$$\binom{n}{p_0, p_1, \dots, p_9}$$

presents the number of different ways n elements can be colored with 9 different colors where a color i is used exactly p_i times.

In our example colors are presented by the digits from 0 to 9 and each coloring represents a n -digit number for which the first color is not 0.

The first summand accounts for all colorings and the second for the colorings that start with 0. Thus, we obtain the number of all n -digit numbers in request by just subtracting the latter summand from the former.

Multinomial coefficients can be calculated by the following formula:

$$\binom{n}{p_1, p_2, \dots, p_m} = \frac{n!}{p_1! p_2! \dots p_m!}$$

That this indeed presents the number of ways to color n elements with m colors by using a color i exactly p_i times, can be seen as follows:

We construct recursively n trees with n levels by first taking each of

the n elements as root. Further, for each such tree we add $n - 1$ child nodes, by using all of the n elements except the one being the root for the corresponding tree. We repeat this recursively for each new node, by always adding child nodes that consist of all the elements in the current node's level except the element of the current node. So the child nodes of the node j_s with path to root

$$j_s \rightarrow j_{s-1} \rightarrow \cdots \rightarrow j_1$$

would be

$$\{1, 2, \cdots, n\} - \{j_1, j_2, \cdots, j_s\}$$

Since $p_1 + p_2 + \cdots + p_m = n$, each path from a root to a leaf can be interpreted as picking first p_1 elements, then p_2 elements, and so on. This way, two each coloring of the n elements corresponds such a path. But this correspondence is not unique! For instance, a specific selection of p_1 elements for the color 1, has exactly as many paths corresponding to this as there are permutations of p_1 elements. Note, for each permutation of the p_1 elements with color 1 we may find a corresponding path that starts with this elements. The number of these permutations exactly is $p_1!$. Using the same arguments but applied on the p_2 elements colored with 2, we have for each permutation of the p_1 elements further $p_2!$ permutations of the p_2 elements. Thus exactly $p_1! \cdot p_2!$ paths exist that represents the fixed coloring of p_1 and p_2 . Following up this way, we find exactly $p_1! \cdot p_2! \cdots p_m!$ pathes the present a fixed coloring of p_1, p_2, \cdots, p_m . The number of pathes in the tree is easily seen to be $n!$. Altogether, this gives

$$\binom{n}{p_1, p_2, \cdots, p_m} \cdot p_1! \cdot p_2! \cdots p_m! = n!$$