Double Limits: A User’s Guide

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Double limits capture the notion of limits in double categories. In ordinary category theory, a limit is the best way to construct new objects from a given collection of objects related in a certain way. Double limits, extend this idea to the richer structure of double categories. For each of the limits we can think of in an ordinary category, we can ask ourselves: how do these limits look in double categories?

In ordinary category theory, many results can be extended to double categories. For instance, in an ordinary category, we can determine if it has all limits (resp. finite limits) by checking if it has all products and equalizers (resp. binary products, a terminal object, and equalizers) (see Thm 5.1.26 in [3]). In a double category, we need to introduce a new notion of limit, known as a tabulator. One of the main theorems by Paréeacute; and Grandis states that a double category has all small double limits if and only if it has small double products, double equalizers, and tabulators. Therefore, these components are sufficient to construct small double limits. To explain this concept thoroughly, we will introduce their definitions in this post. There are various definitions depending on your focus, but for the sake of simplicity, this guide aims to be accessible to anyone with a background in category theory. For an introduction to double categories, you can check here.

We give an overview of how limits behave in this two-dimensional setting, following Grandis and Paré's "Limits in double categories." In particular, we make several definitions more explicit for use in further computations.

Introduction

Recall that double categories consist of two types of morphisms, horizontal and vertical, which interact in a compatible way. Often, composition of one arrow type is weaker than the other. Therefore, we may also think of limits in two different directions. However, limits with respect to the weaker class of morphisms tend to be badly behaved. Hence, in this post, we will only focus on horizontal double limits.

Throughout this article, we will refer to the class of morphisms with strong composition as "arrows," written horizontally, with composition denoted by $\circ$. The weaker arrows will be called "proarrows," written as vertical dashed arrows, and with composition denoted by $\bullet$. Identity arrows/proarrows for an object X will be written $\mathbf{1}_X$ and $\mathbf{1}^\bullet_X$ respectively. Sometimes, we will also refer to the identity *cell* associated to an arrow $f : X \to Y$. This is obtained by taking both proarrow edges to be the respective vertical identities on objects:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\mathbf{1}^\bullet_f"] & Y \arrow[d, equal] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

There's an analogous construction for proarrows, but we won't need it in this article.

Double limits are defined for double diagrams and a double diagram is a double functor from an indexing double category $\mathbb{I}$ to an arbitrary double category $\mathbb{A}$. A limit for a given double diagram D is a universal double cone over D. This is a very high-level definition, but we will try to explain each unfamiliar term and illustrate it with examples.

The first thing we need to understand is a double diagram for which we take the limits.

Diagrams

A double diagram in $\mathbb{A}$ of shape $\mathbb{I}$ is a double functor $F : \mathbb{I} \to \mathbb{A}$ between double categories $\mathbb{I}$ and $\mathbb{A}$ and in strict double categories, a double functor is simultaneously a functor on the horizontal and vertical structures, preserving cells as well as their vertical compositions, horizontal compositions, identities. That is, for every cell $\alpha : u \to v$,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] & & FA \arrow[r, "Ff"] \arrow[d, "Fu"'] \arrow[dr, phantom, "F\alpha"] & FB \arrow[d, "Fv"] \\
C \arrow[r, "g"'] & D & \mapsto & FC \arrow[r, "Fg"'] & FD
\end{tikzcd}$$

and for every composable pair of cells $\alpha : u \to v$ and $\beta : v \to w$

1. preserve horizontal compositions of cells: $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$,
2. preserve vertical compositions of cells: $F(\gamma \bullet \alpha) = F(\gamma) \bullet F(\alpha)$,
3. preserve cell-wise horizontal identity: for each proarrow $u : A \twoheadrightarrow B$, $F(1_u) = 1_{Fu}$,
4. preserve cell-wise vertical identity: for each arrow $f : A \to A'$, $F(1^\bullet_f) = 1^\bullet_{Ff}$,

We will also need the notion of a double natural transformation. These are defined componentwise, much in the same way as ordinary natural transformations. For double functors $F, G : \mathbb{I} \to \mathbb{A}$,
a horizontal transformation $H : F \Rightarrow G$ is given by the following data:

- horizontal $\mathbb{A}$-arrows $Hi : Fi \to Gi$ for every object $i \in \mathbb{I}$
- an $\mathbb{A}$-cell Hu for every proarrow $u : i \twoheadrightarrow j$ in $\mathbb{I}$ of the shape

$$\begin{tikzcd}
Fi \arrow[r, "Hi"] \arrow[d, "Fu"'] \arrow[dr, phantom, "Hu"] & Gi \arrow[d, "Gu"] \\
Fj \arrow[r, "Hj"'] & Gj
\end{tikzcd}$$

- Identities and composition are preserved.
- For every cell $\alpha \in \mathbb{I}$ with proarrow edges u, v and arrow edges f, g, the component cells of u and v satisfy $(F\alpha | Hv) = (Hu | G\alpha)$

Vertical transformations satisfy analogous requirements with respect to vertical morphisms, given Section 1.4 of [1].

We will also use the notion of a modification to define double limits. Suppose we have double functors $F, F', G, G' : \mathbb{I} \to \mathbb{A}$, horizontal transformations $H : F \Rightarrow G, K : F' \Rightarrow G'$ and vertical transformations $U : F \Rightarrow F', V : G \Rightarrow G'$. A modification is an assignment of an $\mathbb{A}$-cell μi to each object $i \in \mathbb{I}$:

$$\begin{tikzcd}
Fi \arrow[r, "Hi"] \arrow[d, "Ui"'] \arrow[dr, phantom, "\mu i"] & Gi \arrow[d, "Vi"] \\
F'i \arrow[r, "Ki"'] & G'j
\end{tikzcd}$$

such that, for every horizontal $f : i \to j$, $(\mu i | Vf) = (Uf | \mu j)$:

$$\begin{tikzcd}
Fi \arrow[r, "Hi"] \arrow[d, "Ui"'] \arrow[dr, phantom, "\mu i"] & Gi \arrow[r, "Gf"] \arrow[d, "Vi"] \arrow[dr, phantom, "Vf"] & Gj \arrow[d, "Vj"] \\
F'i \arrow[r, "Ki"'] & G'i \arrow[r, "G'f"'] & G'j
\end{tikzcd}
=
\begin{tikzcd}
Fi \arrow[r, "Ff"] \arrow[d, "Ui"'] \arrow[dr, phantom, "Uf"] & Fj \arrow[r, "Hj"] \arrow[d, "Uj"] \arrow[dr, phantom, "\mu j"] & Gj \arrow[d, "Vj"] \\
F'i \arrow[r, "F'f"'] & F'j \arrow[r, "Kj"'] & G'j
\end{tikzcd}$$

Double limits will be defined as a universal double cone. But what are cones or double cones in double categories? You may ask.

Like ordinary categories, cones for a functor F in double categories also consist of an object X and morphisms from X to the objects Fi, for each object i of $\mathbb{I}$. Note that there two types of morphisms, those of horizontal direction or arrows and those of vertical direction or proarrows. The morphisms involved in cones are the horizontal ones but must be compatible with vertical ones. Let's dive into the definition to see how that works.

A double cone for a double functor $F : \mathbb{I} \to \mathbb{A}$ consists of an X with arrows $pi : X \to Fi$ for each object i of

$\mathbb{I}$, and cells $pu : \mathbf{1}^\bullet_X \to Fu$ for each
every proarrow $u : i \nrightarrow j$, satisfying the following axioms:

1. for each object i in $\mathbb{I}$,

$$p(\mathbf{1}^\bullet_i) = \mathbf{1}^\bullet_{pi}$$

2. for each composable pair of proarrows u and v in $\mathbb{I}$,

$$p(v \bullet u) = pv \bullet pu$$

3. for every cell $\alpha : u \to v$ in $\mathbb{I}$,

$$(pu | F\alpha) = pv$$

$$\begin{tikzcd}
 & Fi \arrow[r, "Ff"] \arrow[dd, "Fu"', "|" marking] \arrow[ddr, phantom, "F\alpha"] & Fj \arrow[dd, "Fv", "|" marking] \\
X \arrow[ur, "p_i"] \arrow[urr, "p_j"'] \arrow[dr, "p_k"'] \arrow[drr, "p_l"] & & \\
 & Fk \arrow[r, "Fg"'] & Fl
\end{tikzcd}$$

Note that this implies that $Ff \circ p_i = p_j$ and
$Fg \circ p_k = p_l$.
We can observe that the cells pu for every u are made of two green arrows and Fu, which is indeed a cell such that the horizontal source of pu is the identity proarrow $\mathbf{1}^\bullet_X$.

$$\begin{tikzcd} & Fi \arrow[dd, "|" marking] \\ X \arrow[ur] \arrow[dr] \arrow[r, phantom, "pu"] & {} \\ & Fk \end{tikzcd} = \begin{tikzcd} X \arrow[r, "p_i"] \arrow[d, "1^\bullet_X"', "|" marking] \arrow[dr, phantom, "pu"] & Fi \arrow[d, "Fu", "|" marking] \\ X \arrow[r, "p_k"'] & Fk \end{tikzcd}$$

For example, let's take cones for the functor F from an indexing double category which is the discrete double category (made of only two objects i and j), to an arbitrary double category, defined such that $Fi = A$ and $Fj = B$. Then, a double cone X for F is a candidate product for A and B.

Notice that the above description of a double cone satisfies the requirements of a horizontal transformation. We can consider a constant functor $DA : \mathbb{I} \to \mathbb{A}$ at an object A of $\mathbb{A}$, then the data of a double cone with vertex A is determined by a horizontal transformation $x : DA \Rightarrow F$. The componentwise definition of x unrolls to precisely the conditions specified above.

We have now all the setup needed for defining double limits, since as we mentioned above, double limits are universal double cones. That is, a double cone for an underlying functor F through which any other double cones factor.

Double Limits

Limits

Let $F : \mathbb{I} \to \mathbb{A}$ be a double functor. The (horizontal) *double limit* of F is a universal cone (A, x) for F.

Explicitly, this requires several things:

- For any other double cone (A', x'), there exists a unique arrow $c : A' \to A$ in $\mathbb{A}$ with $x \circ Dc = x'$ (where D is the constant functor at the vertex of A)

- Let $(A', x'), (A'', x'')$ be double cones with a proarrow $u : A' \nrightarrow A''$. For every collection of cell η_i where i is an object of $\mathbb{I}$, associated to components of each cone, which organize into a modification, there exists a unique $\mathbb{A}$-cell τ such that $(\tau | xi) = \eta_i$:

$$\begin{tikzcd} A' \arrow[r, "c'"] \arrow[d, "u"'] \arrow[dr, phantom, "\tau"] & A \arrow[r, "xi"] \arrow[d, equal] \arrow[dr, phantom, "1^\bullet"] & Fi \arrow[d, equal] \\ A'' \arrow[r, "c''"'] & A \arrow[r, "xi"'] & Fi \end{tikzcd} = \begin{tikzcd} A' \arrow[r, "x'i"] \arrow[d, "u"'] \arrow[dr, phantom, "\eta_i"] & Fi \arrow[d, equal] \\ A'' \arrow[r, "x''i"'] & Fi \end{tikzcd}$$

In other words, a cell built from a proarrow and the components of two cones (viewed as natural transformations) can be factored uniquely via τ and $1^\bullet$.

To get a better feel for double limits in practice, let's examine (binary) products in a double category. Just as in 1-category theory, products are constructed as the double limit of the diagram $\bullet\ \bullet$ (two discrete objects). Spelling out the universal properties of a double limit, the (double) product of objects $A, B \in \mathbb{A}$ consists of an object $A \times B$ which satisfies the usual requirements for a product with respect to horizontal morphisms (with projection maps π_A, π_B. Additionally, given cells α, β as below:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, equal] \\ D \arrow[r] & A \end{tikzcd} , \begin{tikzcd} C \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\ D \arrow[r] & B \end{tikzcd}$$

there exists a *unique* cell $\alpha \times \beta$ such that

$$
\begin{tikzcd}
C \arrow[r, dashed] \arrow[d, "u"', "\shortmid" marking] \arrow[dr, phantom, "\alpha \times \beta"] & A \times B \arrow[r, "\pi_A"] \arrow[d, equal] \arrow[dr, phantom, "\mathbf{1}^\bullet_{\pi_A}"] & A \arrow[d, equal] \\
D \arrow[r, dashed] & A \times B \arrow[r, "\pi_A"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[r] \arrow[d, "u"', "\shortmid" marking] \arrow[dr, phantom, "\alpha"] & A \arrow[d, equal] \\
D \arrow[r] & A
\end{tikzcd}
$$

An identical condition must also hold for B and π_B.

Equalizers can be extended to the double setting in a similar manner. Taking the double limit of the diagram $\bullet \rightrightarrows \bullet$ yields *double equalizers*. For horizontal $f, g : A \rightrightarrows B$ in $\mathbb{A}$, the double equalizer of f and g consists of an object $Eq(f, g)$ equipped with a horizontal arrow $e : Eq(f, g) \to A$, which is the equalizer of f, g in the ordinary sense with respect to horizontal arrows. Additionally, for every cell η with $(\eta | \mathbf{1}^\bullet_f) = (\eta | \mathbf{1}^\bullet_g)$, there exists a unique τ such that $(\tau | \mathbf{1}^\bullet) = \eta$:

$$
\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, "u"', "\shortmid" marking] \arrow[dr, phantom, "\tau"] & Eq(f,g) \arrow[r, "e"] \arrow[d, equal] \arrow[dr, phantom, "1^\bullet"] & A \arrow[d, equal] \\
Y \arrow[r, dashed] & Eq(f,g) \arrow[r, "e"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "u"', "\shortmid" marking] \arrow[dr, phantom, "\eta"] & A \arrow[d, equal] \\
Y \arrow[r, "y"'] & A
\end{tikzcd}
$$

Tabulators

Until now, we have considered examples of double limits of diagrams built from horizontal morphisms. Tabulators bring proarrows into the mix. They are an interesting case obtained as the limit over the diagram consisting of a single proarrow: $\bullet \nrightarrow \bullet$.

Suppose that $u : A \nrightarrow B$ is a proarrow. The tabulator of u is the double limit of the diagram consisting of just u. Unrolling the limit, this amounts to an object Tu along with a cell τ:

$$\begin{tikzcd} Tu \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\tau"] & A \arrow[d, "u", "|" marking] \\ Tu \arrow[r] & B \end{tikzcd}$$

such that, for any cell η of the following shape,

$$\begin{tikzcd} C \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[d, "u", "|" marking] \\ C \arrow[r] & B \end{tikzcd}$$

there exists a unique horizontal morphism $f : C \to T$ such that $(1^{\bullet}_{f} | \tau) = \eta$:

$$\begin{tikzcd} C \arrow[r, dashed, "f"] \arrow[d, equal] \arrow[dr, phantom, "1^{\bullet}_{f}"] & Tu \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\tau"] & A \arrow[d, "u", "|" marking] \\ C \arrow[r, dashed, "f"'] & Tu \arrow[r] & B \end{tikzcd} = \begin{tikzcd} C \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[d, "u", "|" marking] \\ C \arrow[r] & B \end{tikzcd}$$

Additionally, any proarrow $v : C \nrightarrow D$ with horizontal morphisms to A and B forming a tetrahedron can be uniquely factored through Tu:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "v"', "|" marking] \arrow[dr] & A \arrow[d, "u", "|" marking] \\ D \arrow[r] \arrow[ur] & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} C \arrow[dd, "v"', "|" marking] \arrow[dr] & & A \arrow[dd, "u", "|" marking] \\ & Tu \arrow[ur] \arrow[dr] & \\ D \arrow[ur] & & B \end{tikzcd}$$

In an ordinary category, the existence of all finite products and equalizers is enough to guarantee the existence of all limits. However, in the double setting, we need something extra: tabulators. The following result gives us a similar condition for limits in double categories.

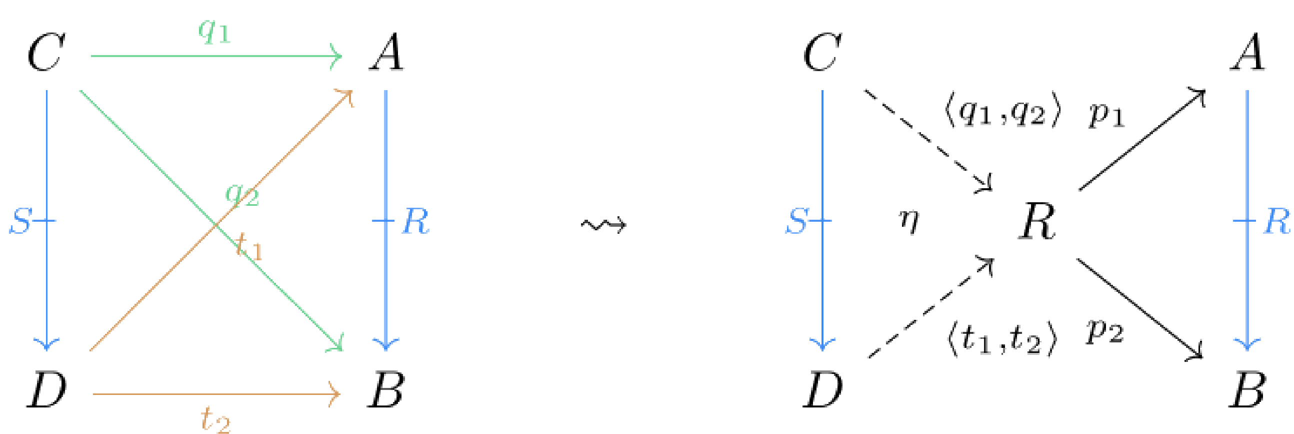
Theorem (5.5 in [1]): A double category $\mathbb{A}$ has all small double limits if and only if it has small double products, equalizers, and tabulators.

Examples in $\mathbb{R}\mathrm{elset}$

In this section, we consider the double category $\mathbb{R}\mathrm{elset}$ of sets with functions as horizontal morphisms and relations as vertical morphisms, for more information see [add reference].

Tabulators

A tabulator for a proarrow or relation $R \subseteq A \times B$ is R itself with the projection maps $p_1 : R \to A$ and $p_2 : R \to B$. For every other double cone $(C, q) = (C, q_1, q_2)$ of R, there exists a unique function or arrow $h = \langle q_1, q_2 \rangle : C \to TR$ ($TR = R$), such that $q_i = p_i \circ h$; and for every relation $S \subseteq C \times D$ and such that $(D, t) = (D, t_1, t_2)$ is also a double cone for R, there exists a unique cell $\eta = (SR) : S \to \mathbf{1}^{\bullet}_{R}$, such that $(\eta | pR) = q_1 \nrightarrow t_2$.



Product

The double product of two sets A and B is the cartesian product with the usual projection maps and we also have the following:

$$
\begin{tikzcd}
C \arrow[r, "q_1"] \arrow[d, "S"', "|" marking] \arrow[dr, "q_2" near start] & A \\
D \arrow[ur, "t_1"' near end] \arrow[r, "t_2"'] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
C \arrow[dr, dashed, "\langle q_1, q_2 \rangle"] \arrow[dd, "S"', "|" marking] & & A \\
\arrow[r, phantom, "\eta" near start] & A \times B \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\
D \arrow[ur, dashed, "\langle t_1, t_2 \rangle"'] & & B
\end{tikzcd}
$$

References

[1] Grandis, Marco, and Robert Paré. "Limits in double categories." Cahiers de topologie et géométrie différentielle catégoriques 40.3 (1999): 162-220.

[2] Patterson, Evan. "Products in double categories, revisited." arXiv preprint arXiv:2401.08990 (2024).

[3] Leinster, Tom. "Basic category theory." *arXiv preprint arXiv:1612.09375* (2016).