ON COMPUTATIONAL POISSON GEOMETRY I: SYMBOLIC FOUNDATIONS *

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Abstract. We present a computational toolkit for (local) Poisson-Nijenhuis calculus on manifolds. Our python module PoissonGeometry implements our algorithms, and accompanies this paper. We include two examples of how our methods can be used, one for gauge transformations of Poisson bivectors in dimension 3, and a second one that determines parametric Poisson bivector fields in dimension 4.

Key words. Poisson structures, Poisson-Nijenhuis calculus, Symbolic computation, Python.

AMS subject classifications. 68W30, 97N80, 53D17

1. Introduction. The origin of the concepts in this paper is the analysis of mechanical systems of Siméon Denis Poisson in 1809 [28]. A Poisson manifold is a pair (M,Π) , with M a smooth manifold and Π a contravariant 2-tensor field (bivector field) on M satisfying the equation

$$[\![\Pi,\Pi]\!] = 0,$$

with respect to the Schouten-Nijenhuis bracket $[\![],]\!]$ for multivector fields [26, 10]. Suppose $m = \dim M$, fix a local coordinate system $x = (U; x^1, \ldots, x^m)$ on M. Then Π has the following coordinate representation [22, 32]:

(1.2)
$$\Pi = \frac{1}{2} \Pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} = \sum_{1 \le i < j \le m} \Pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

Here, the functions $\Pi^{ij} = \Pi^{ij}(x) \in C_U^{\infty}$ are called the coefficients of Π , and $\{\partial/\partial x^i\}$ is the canonical basis for vector fields on $U \subseteq M$.

The Poisson bivector, and its associated bracket, are essential elements in the comprehension of Hamiltonian dynamics [23, 10]. We recommend interested readers consult the available surveys of this field [34, 18].

Table 1 below compiles the functions in our Python module PoissonGeometry¹ their corresponding algorithm, and examples where such objects are used in the references. We describe all of our algorithms in section 2. In section 3 we present two applications that illustrate the usefulness of our computational methods. These are, a new result about gauge transformations of Poisson bivector fields in dimension 3 (Proposition 3.1), and a description of parametric families of Poisson bivectors in dimension 4 (Lemma 3.2).

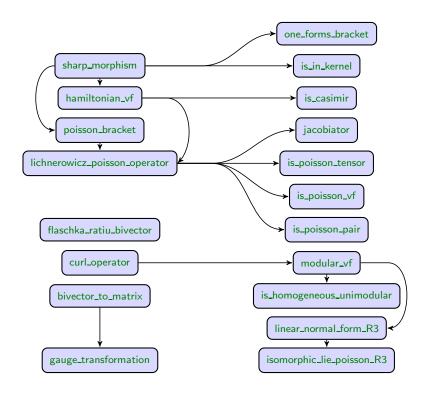
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Our code repository is found at: https://github.com/appliedgeometry/poissongeometry.

Function	Algorithm	Examples
sharp_morphism	2.1	[10, 23, 6]
poisson_bracket	2.2	[23, 6]
hamiltonian_vf	2.3	[6, 31]
lichnerowicz_poisson_operator	2.4	[27, 2]
curl_operator	2.5	[9, 2]
bivector_to_matrix	2.6	[10, 23, 6]
jacobiator	2.7	[10, 23, 6]
modular_vf	2.8	[1, 16, 2]
is_homogeneous_unimodular	2.9	[9, 23, 2, 6]
one_forms_bracket	2.10	[12, 18]
gauge_transformation	2.11	[7]
linear_normal_form_R3	2.12	[27, 6]
isomorphic_lie_poisson_R3	2.13	[27, 6]
flaschka_ratiu_bivector	2.14	[9, 14, 30, 11]
is_poisson_tensor	2.15	[14, 30, 11]
is_in_kernel	2.16	[10, 23, 2, 6]
is_casimir	2.17	[9, 14, 30, 11]
is_poisson_vf	2.18	[27, 3]
is_poisson_pair	2.19	[4, 2]

Table 1: Functions, corresponding algorithms, and examples where each particular method can be, or has been, applied. The following diagram illustrates functional dependencies in PoissonGeometry.



- **2. Implementation of Functions in PoissonGeometry.** In this section we describe the implementation of all functions of the module PoissonGeometry.
- **2.1. Key Functions.** This subsection contains functions that serve as a basis for the implementation of almost all functions of PoissonGeometry.
- **2.1.1. Sharp Morphism.** The function sharp_morphism computes the image of a differential 1-form under the vector bundle morphism $\Pi^{\natural}: \mathsf{T}^*M \to \mathsf{T}M$ induced by a bivector field Π on M and defined by

(2.1)
$$\langle \beta, \Pi^{\natural}(\alpha) \rangle := \Pi(\alpha, \beta),$$

for any $\alpha, \beta \in \mathsf{T}^*M$ [10, 23]. Here, \langle, \rangle is the natural pairing for differential 1-forms and vector fields. Equivalently, $\Pi^{\natural}(\alpha) = \mathbf{i}_{\alpha}\Pi$, with \mathbf{i}_{\bullet} the interior product of multivector fields and differential forms defined by the rule $\mathbf{i}_{\alpha \wedge \beta} := \mathbf{i}_{\alpha} \circ \mathbf{i}_{\beta}$ [19]. Analogously for vector fields. In local coordinates, if $\alpha = \alpha_j \, \mathrm{d} x^j$, $j = 1, \ldots, m$, then

(2.2)
$$\Pi^{\natural}(\alpha) = \sum_{1 \le i < j \le m} \alpha_i \Pi^{ij} \frac{\partial}{\partial x^j} - \alpha_j \Pi^{ij} \frac{\partial}{\partial x^i}.$$

Algorithm 2.1 sharp_morphism(bivector, one_form)

Input: a bivector field and a differential 1-form

Output: a vector field which is the image of the differential 1-form under the vector bundle morphism (2.1) induced by the bivector field

```
1: procedure
         m \leftarrow \text{dimension of the manifold}
                                                                ▷ Given by an instance of PoissonGeometry
         bivector \leftarrow a dictionary \{(1,2): \Pi^{12}, ..., (m-1,m): \Pi^{m-1m}\} that represents
    a bivector field according to (3.1)
         one_form \leftarrow a dictionary \{(1): \alpha_1, ..., (m): \alpha_m\} that represents a differential
     1-form according to (3.1)
         Convert each value in bivector and in one_form to symbolic expression
 5:
         sharp_dict \leftarrow the dictionary \{(1): 0, ..., (m): 0\}
 6:
 7:
         for each 1 \le i < j \le m do
                                                                               ▷ Compute the sum in (2.1)
             \operatorname{sharp\_dict}[(i)] \leftarrow \operatorname{sharp\_dict}[(i)] - \alpha_j * \Pi^{ij}
 8:
             \operatorname{sharp\_dict}[(j)] \leftarrow \operatorname{sharp\_dict}[(j)] + \alpha_i * \Pi^{ij}
 9:
10:
         if all values in sharp_dict are equal to zero then
11:
             return {0: 0}
12:
                                                                   ▶ A dictionary with zero key and value
         else
13:
             return sharp_dict
14:
         end if
15:
16: end procedure
```

Observe that the morphism (2.1) is defined, in particular, for Poisson bivector fields. So the function sharp_morphism can be applied on this class of bivector fields.

2.1.2. Poisson Brackets. A Poisson bracket on M is a Lie bracket structure $\{,\}$ on the space of smooth functions C_M^{∞} which is compatible with the pointwise product by the Leibniz rule [10, 23]. Explicitly, the Poisson bracket induced by a Poisson bivector field Π on M is given by the formula

$$(2.3) {f,g}_{\Pi} = \left\langle \mathrm{d}g, \Pi^{\sharp}(\mathrm{d}f) \right\rangle = \left(\Pi^{\sharp}\mathrm{d}f\right)^{i} \frac{\partial g}{\partial x^{i}}, \forall f, g \in C_{M}^{\infty};$$

for i = 1, ..., m. The function poisson_bracket computes the poisson bracket, induced by a Poisson bivector field, of two scalar functions.

Algorithm 2.2 poisson_bracket(bivector, function_1, function_2)

Input: a Poisson bivector field and two scalar functions

Output: the Poisson bracket of the two scalar functions induced by the Poisson bivector field

```
1: procedure
        m \leftarrow \text{dimension of the manifold}
 2:
                                                             \triangleright Given by an instance of PoissonGeometry
        bivector \leftarrow a dictionary that represents a bivector field according to (3.1)
 3:
        function_1, function_2 \leftarrow string expressions
 4:
                                                                         ▶ Represents scalar functions
        if function_1 == function_2 then
 5:
 6:
            return 0
                                                                   \triangleright If f = g in (2.3), then \{f, g\} = 0
 7:
        else
            Convert function_1 and function_2 to symbolic expressions
 8:
            gradient\_function\_1 \leftarrow a dictionary that represents the gradient vector field
 9:
    of function_1 according to (3.1)
            sharp\_function\_1 \leftarrow sharp\_morphism(bivector, gradient\_function\_1)
10:
                                                                                  ▷ See Algorithm 2.1
            bracket \leftarrow 0
11:
            for i = 1 to m do
12:
                bracket \leftarrow bracket + sharp_function_1[(i)] * \partial(function_2)/\partial xi
13:
                                                                                           ▶ See (2.3)
14:
            end for
15:
            return bracket
        end if
16:
17: end procedure
```

2.1.3. Hamiltonian Vector Fields. The function hamiltonian_vf computes the Hamiltonian vector field

$$(2.4) X_h := \Pi^{\natural}(\mathrm{d}h).$$

of a function $h \in C_M^{\infty}$ respect to a Poisson bivector field Π on M [10, 23].

Algorithm 2.3 hamiltonian_vf(bivector, hamiltonian_function)

Input: a Poisson bivector field and a scalar function

Output: the Hamiltonian vector field of the scalar function relative to the Poisson bivector field

```
1: procedure
2: m \leftarrow \text{dimension of the manifold} \qquad \triangleright \text{Given by an instance of PoissonGeometry}
3: \text{bivector} \leftarrow \text{a dictionary that represents a bivector field according to (3.1)}
4: \text{hamiltonian\_function} \leftarrow \text{a string expression} \qquad \triangleright \text{Represents a scalar function}
```

- 5: Convert hamiltonian_function to symbolic expression
- 6: gradient_hamiltonian ← a dictionary that represents the gradient vector field of hamiltonian_function according to (3.1)
- 7: **return** sharp_morphism(bivector, gradient_hamiltonian)

▷ See Algorithm 2.1 and formula (2.4)

8: end procedure

2.1.4. Coboundary Operator. The adjoint operator of a Poisson bivector field Π on M with respect to the Schouten-Nijenhuis bracket gives rise to a cochain complex $(\Gamma \wedge \mathsf{T}M, \delta_{\pi})$, called the *Lichnerowicz-Poisson* complex of (M, Π) [22, 10, 23]. Here $\delta_{\Pi} : \Gamma \wedge^{\bullet} \mathsf{T}M \to \Gamma \wedge^{\bullet+1} \mathsf{T}M$ is the coboundary operator $(\delta_{\Pi}^2 = 0)$ defined by

(2.5)
$$\delta_{\Pi}(A) := \llbracket \Pi, A \rrbracket, \quad \forall A \in \Gamma \wedge \mathsf{T} M.$$

Here, $\Gamma \wedge \mathsf{T} M$ denotes the C_M^∞ -module of multivector fields on M. Explicitly, if $a = \deg A$, then for any $f_1, \ldots, f_{a+1} \in C_M^\infty$:

Throughout this paper the symbol $\hat{}$ will denote the absence of the corresponding factor. In particular, if $f_1 = x^1, \ldots, f_{a+1} = x^{a+1}$ are local coordinates on M, we have that for $1 \le i_1 < \cdots < i_{a+1} \le m$:

(2.6)
$$[\![\Pi, A]\!]^{i_1 \cdots i_{a+1}} = \sum_{k=1}^{a+1} (-1)^{k+1} \left\{ x^{i_k}, A^{i_1 \cdots \widehat{i_k} \cdots i_{a+1}} \right\}_{\Pi}$$

$$+\sum_{1 \le k < l \le a+1} (-1)^{k+l} \frac{\partial \Pi^{i_k i_l}}{\partial x^s} A^{s i_1 \cdots \widehat{i_k} \cdots \widehat{i_l} \cdots i_{a+1}}$$

Here $[\![\Pi,A]\!]^{i_1\cdots i_{a+1}}$, $A^{i_1\cdots i_a}$ and $\Pi^{i_ki_l}$ are the coefficients of the coordinate expressions of $[\![\Pi,A]\!]$, A and Π , in that order. The function lichnerowicz_poisson_operator computes the image of a multivector field under the coboundary operator induced by a Poisson bivector field.

Algorithm 2.4 lichnerowicz_poisson_operator(bivector, multivector)

Input: a Poisson bivector field and a multivector field

Output: the image of the multivector field under the coboundary operator (2.5) induced by the Poisson bivector field

- 1: procedure
- $m \leftarrow \text{dimension of the manifold}$
- ▷ Given by an instance of PoissonGeometry
- 3: $\mathbf{a} \leftarrow \text{degree of the multivector field}$
- 4: bivector \leftarrow a dictionary $\{(1,2): \Pi^{12}, ..., (m-1,m): \Pi^{m-1m}\}$ that represents a bivector field according to (3.1)
- 5: multivector \leftarrow a dictionary $\{(1, ..., a): A^{1 \cdots a}, ..., (m-a+1, ..., m): A^{m-a+1 \cdots m}\}$ that represents a bivector field according to (3.1) or a string expression

```
if a + 1 > m then
 6:
                                                                              \triangleright deg [\Pi, A] = deg(A) + 1 in (2.5)
              return {0: 0}
 7:
                                                                        ▶ A dictionary with zero key and value
          else if multivector is a string expression then
 8:
              return hamiltonian_vf(bivector, str(-1^*) + multivector)
 9:
                                                                                             ▷ See Algorithm 2.3
          else
10:
              Convert each value in bivector and in multivector to symbolic expression
11:
              image_multivector \leftarrow the dictionary \{(1,...,a+1): 0,...,(m-a,...,m): 0\}
12:
              for each 1 \le i_1 < \dots < i_{a+1} \le m do
13:
                   for k = 1 to a+1 do
                                                                            ▷ Compute first summation in (2.6)
14:
                       image_multivector[(i_1, ..., i_{a+1})] \leftarrow image_multivector[(i_1, ..., i_{a+1})]
15:
     + (-1)^{k+1} \left\{ \times i_k, \widetilde{A^{i_1 \cdots \widehat{i_k} \cdots i_{a+1}}} \right\}_{\Pi} \quad \triangleright \text{ See Algorithm 2.2 to compute the Poisson bracket } \{,\}_{\Pi}
16:
                   for each 1 \le k < l \le a+1 do
17:
                                                                         \triangleright Compute second summation in (2.6)
                        image\_multivector[(i_1, ..., i_{a+1})] \leftarrow image\_multivector[(i_1, ..., i_{a+1})]
18:
     + (-1)^{k+l} \frac{\partial \Pi^{ik}{}^{il}}{\partial x^{s}} \mathsf{A}^{s i_{1} \cdots \hat{i_{k}} \cdots \hat{i_{l}} \cdots i_{a+1}}
19:
20:
              end for
21:
              if all values in image_multivector are equal to zero then
                   return \{0:0\}
                                                                        ▶ A dictionary with zero key and value
22:
23:
              else
                   return image_multivector
24:
25:
              end if
26:
          end if
27: end procedure
```

2.1.5. Curl (Divergence) Operator. Fix a volume form Ω_0 on an oriented Poisson manifold (M, Π, Ω_0) . The *divergence* (relative to Ω_0) of an a-multivector field A on M is the unique (a-1)-multivector field $\mathcal{D}_{\Omega_0}(A)$ on M such that

$$\mathbf{i}_{\mathcal{D}_{\Omega_0}(A)}\Omega_0 = \mathrm{d}\mathbf{i}_A\Omega_0.$$

This induces a (well defined, Ω_0 -dependent) coboundary operator $\mathcal{D}_{\Omega_0}: A \mapsto \mathcal{D}_{\Omega_0}(A)$ on the module of multivector fields on M, called the *curl operator* [19, 23]. As any other volume form on M is a multiple $f\Omega_0$ of Ω_0 by a nowhere vanishing function $f \in C_M^{\infty}$, we have $\mathcal{D}_{f\Omega_0} = \mathcal{D}_{\Omega_0} + \frac{1}{f}\mathbf{i}_{df}$. In local coordinates, expressing Ω_0 as

$$\Omega_0 = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m,$$

then for any a-multivector field $A = A^{i_1 \cdots i_a} \partial/\partial x^{i_1} \wedge \cdots \wedge \partial/\partial x^{i_a}$ on M, with $1 \le i_1 < \cdots < i_a \le m$, the divergence of A with respect to the volume form $f\Omega_0$ is given by: (2.10)

$$\mathcal{D}_{f\Omega_0}(A) = \sum_{k=1}^m (-1)^{k+1} \left(\frac{\partial A^{i_1 \cdots i_a}}{\partial x^{i_k}} + \frac{1}{f} \frac{\partial f}{\partial x^{i_k}} A^{i_1 \cdots i_a} \right) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\widehat{\partial}}{\partial x^{i_k}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_a}}$$

Let f_0 be a nonzero scalar function. The function curl_operator computes the divergence of a multivector field respect to the volume form $f_0\Omega_0$, for Ω_0 in (2.9).

Algorithm 2.5 curl_operator(multivector, function)

Input: a multivector field and a nonzero scalar function f_0 Output: the divergence of the multivector field with respect to the volume form $f_0\Omega_0$

```
1: procedure
           m \leftarrow \text{dimension of the manifold}

    □ Given by an instance of PoissonGeometry

           \mathbf{a} \leftarrow \text{degree of the multivector field}
           multivector \leftarrow a dictionary \{(1,...,a): A^{1\cdots a}, ..., (m-a+1,...,m): A^{m-a+1\cdots m}\}
     that represents a multivector field according to (3.1) or a string expression
 5:
           function \leftarrow a string expression
                                                                                         ▶ Represents a nonzero function
           if multivector is a string expression then
 6:
                return {0: 0}
 7:
                                                                                ▶ A dictionary with zero key and value
 8:
           else
                Convert each value in multivector and function to symbolic expression
 9:
10:
                curl_multivec \leftarrow the dictionary \{(1,...,a-1): 0, ..., (m-a+2,...,m): 0\}
                for each 1 \le i_1 < \cdots < i_a \le m do
11:
                                                                                   \triangleright Compute the summation in (2.10)
                     for k = 1 to m do
12:
                      \begin{array}{ll} & \text{curl\_multivec}[(i_1,...,\widehat{\imath_k},...,i_{\mathbf{a}})] \leftarrow \text{curl\_multivec}[(i_1,...,\widehat{\imath_k},...,i_{\mathbf{a}})] \ + \\ \left(\frac{\partial \mathsf{A}^{i_1\cdots i_a}}{\partial \mathsf{x} i_k} \ + \ \frac{1}{\text{function}} \ * \ \frac{\partial (\text{function})}{\partial \mathsf{x} i_k} \ * \ \mathsf{A}^{i_1\cdots i_a}\right) \end{array}
13:
                     end for
14:
                end for
15:
                if all values in curl_multivec are equal to zero then
16:
                     return {0: 0}
                                                                                ▶ A dictionary with zero key and value
17:
18:
                     return curl_multivec
19:
                end if
20:
           end if
21:
22: end procedure
```

2.2. Matrix of a bivector field. The function bivector_to_matrix computes the (local) matrix $\left[\Pi^{ij}\right]$ of a bivector field Π on M [10, 23], the coefficients of Π in (1.2). In particular, it computes the matrix of a Poisson bivector field.

Algorithm 2.6 bivector_to_matrix(bivector)

Input: a bivector field

Output: the (local) matrix of the bivector field

```
1: procedure
        m \leftarrow \text{dimension of the manifold}
2:
                                                                 \triangleright Given by an instance of \mathsf{PoissonGeometry}
       bivector \leftarrow a dictionary \{(1,2): \Pi^{12}, ..., (m-1,m): \Pi^{m-1m}\} that represents
3:
   a bivector field according to (3.1)
4:
        matrix \leftarrow a \text{ symbolic } m \times m\text{-matrix}
        for each 1 \le i < j \le m do
5:
            Convert \Pi^{ij} to symbolic expression
6:
            \text{matrix}[i-1,j-1] \leftarrow \Pi^{ij}
7:
            matrix[i - 1, i - 1] \leftarrow (-1) * matrix[i - 1, i - 1]
8:
        end for
9:
```

```
10: return matrix11: end procedure
```

2.3. Jacobiator. The Schouten-Nijenhuis bracket of a bivector field Π with itself, $[\![\Pi,\Pi]\!]$, is computed with the jacobiator function. This 3-multivector field is called the Jacobiator of Π . The Jacobi identity (1.1) for Π follows from the vanishing of its Jacobiator [10, 23].

Algorithm 2.7 jacobiator(bivector)

Input: a bivector field

Output: the Schouten-Nijenhuis bracket of the bivector field with itself

```
1: procedure
```

- 2: bivector \leftarrow a dictionary that represents a bivector field according to (3.1)
- 3: return lichnerowicz_poisson_operator(bivector, bivector) ▷ See Algorithm 2.4
- 4: end procedure
- **2.4.** Modular Vector Field. For (M,Π,Ω) an orientable Poisson manifold, and a fixed volume form Ω on M, the map

$$(2.11) Z: h \longmapsto \mathcal{D}_{\Omega}(X_h)$$

is a derivation of C_M^{∞} . Therefore it defines a vector field on M, called the modular vector field of Π relative to Ω [33, 1, 10, 23]. Here, \mathcal{D}_{Ω} is the curl operator relative to Ω (2.8). Then, Z is a Poisson vector field of Π which is independent of the choice of a volume form, modulo Hamiltonian vector fields: $Z_{f\Omega} = Z - \frac{1}{f}X_f$. Here $Z_{f\Omega}$ is the modular vector field of Π relative to the volume form $f\Omega$ and $f \in C_M^{\infty}$ a nowhere vanishing function. In this context, the Poisson bivector field Π is said to be unimodular if Z is a Hamiltonian vector field (2.4). Equivalently, if Z is zero for some volume form on M. We can compute the modular vector field Z of Π (2.11) relative to a volume form $f\Omega$ as the (minus) divergence of Π (2.10):

(2.12)
$$Z_{f\Omega} = -\mathcal{D}_{f\Omega}(\Pi) = \mathcal{D}_{f\Omega}(-\Pi).$$

Let f_0 be a nonzero scalar function. The function modular_vectorfield computes the modular vector field of a Poisson bivector field with respect to the volume form $f_0\Omega_0$.

Algorithm 2.8 modular_vf(bivector, function)

Input: a Poisson bivector field and a nonzero scalar function f_0

Output: the modular vector field of the Poisson bivector field (2.12) relative to the volume form $f_0\Omega_0$,

```
1: procedure
```

- 2: bivector \leftarrow a dictionary $\{(1,2): \Pi^{12}, ..., (m-1,m): \Pi^{m-1m}\}$ that represents a bivector field according to (3.1)
- 3: function \leftarrow a string expression

 $\, \triangleright \,$ Represents a scalar function

4: for each $1 \le i < j \le m$ do

 \triangleright A dictionary for $-\Pi$ in (2.12)

- 5: bivector[(i, j)] $\leftarrow -\Pi^{ij}$
- 6: end for

7: return curl_operator(bivector, function) ▷ See Algorithm 2.5 and formula (2.12) 8: end procedure

2.5. Unimodularity of Homogeneous Poisson bivector fields. We can verify whether an homogeneous Poisson bivector field is unimodular or not with the is_homogeneous_unimodular function. A Poisson bivector field Π on \mathbb{R}_x^m ,

(2.13)
$$\Pi = \frac{1}{2}\Pi^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \qquad i, j = 1 \dots, m;$$

is said to be *homogeneous* if each coefficient Π^{ij} is an homogeneous polynomial [23]. To implement this function we use the following fact: an homogeneous Poisson bivector field on \mathbb{R}_x^m is unimodular on (the whole of) \mathbb{R}_x^m if and only if its modular vector field (2.11) relative to the Euclidean volume form is zero [20].

Algorithm 2.9 is_homogeneous_unimodular(bivector)

Input: a homogeneous Poisson bivector field on \mathbb{R}^m

Output: verify if the modular vector field respect to the Euclidean volume form on \mathbb{R}^m of the Poisson bivector field is zero or not

2.6. Bracket on Differential 1-Forms. The function one_forms_bracket computes the Lie bracket of two differential 1-forms $\alpha, \beta \in \Gamma \mathsf{T}^*M$ induced by a Poisson bivector field Π on M [10, 23] and defined by

$$\{\alpha,\beta\}_{\Pi} := \mathbf{i}_{\Pi^{\natural}(\alpha)} d\beta - \mathbf{i}_{\Pi^{\natural}(\beta)} d\alpha + d\langle\beta,\Pi^{\natural}(\alpha)\rangle.$$

Here, d is the exterior derivative for differential forms and $\{df, dg\}_{\Pi} = d\{f, g\}_{\Pi}$, by definition for all $f, g \in C_M^{\infty}$. The bracket on the right-hand side of this equality is the Poisson bracket for smooth functions on M induced by Π (2.3). In coordinates, if $\alpha = \alpha_k dx^k$ and $\beta = \beta_l dx^l$, for $k, l = 1, \ldots, m$:

$$(2.14) \quad \{\alpha,\beta\}_{\Pi} = \sum_{1 \leq i < j \leq m} \left[\left(\Pi^{\natural} \alpha \right)^{j} \left(\frac{\partial \beta_{i}}{\partial x^{j}} - \frac{\partial \beta_{j}}{\partial x^{i}} \right) - \left(\Pi^{\natural} \beta \right)^{j} \left(\frac{\partial \alpha_{i}}{\partial x^{j}} - \frac{\partial \alpha_{j}}{\partial x^{i}} \right) \right] dx^{i}$$

$$+ \sum_{1 \leq i \leq m} \left[\left(\Pi^{\natural} \alpha \right)^{i} \left(\frac{\partial \beta_{j}}{\partial x^{i}} - \frac{\partial \beta_{i}}{\partial x^{j}} \right) - \left(\Pi^{\natural} \beta \right)^{i} \left(\frac{\partial \alpha_{j}}{\partial x^{i}} - \frac{\partial \alpha_{i}}{\partial x^{j}} \right) \right] dx^{j} + \frac{\partial \left[\left(\Pi^{\natural} \alpha \right)^{l} \beta_{l} \right]}{\partial x^{k}} dx^{k}$$

Algorithm 2.10 one_forms_bracket(bivector, one_form_1, one_form_2)

Input: a Poisson bivector field and two differential 1-forms

Output: a differential 1-form which is the Lie bracket induced by the Poisson bivector field of the two differential 1-forms

```
1: procedure
                      m \leftarrow \text{dimension of the manifold}
   2:
                                                                                                                                                          ▶ Given by an instance of PoissonGeometry
   3:
                      bivector \leftarrow a dictionary that represents a bivector field according to (3.1)
                      one_form_1 \leftarrow a dictionary \{(1): \alpha_1, ..., (m): \alpha_m\} that represents a differential
           1-form according to (3.1)
   5:
                      one_form_2 \leftarrow a dictionary \{(1): \beta_1, ..., (m): \beta_m\} that represents a differential
            1-form according to (3.1)
                      sharp_1 \leftarrow sharp\_morphism(bivector, one\_form_1)
   6:
                                                                                                                                                                                                              ▶ See Algorithm 2.1
                      sharp_2 \leftarrow sharp_morphism(bivector, one_form_2)
   7:
                      CONVERT each of one_form_1 and one_form_2 to a symbolic expression
   8:
                      forms_bracket \leftarrow the dictionary \{(1): 0, ..., (m): 0\}
   9:
 10:
                      for 1 \le i < j \le m do
                                                                                                                                               ▷ Compute the first two summations in (2.14)
                                forms_bracket[(i)] \leftarrow forms_bracket[(i)] + sharp_1[(j)] * (\partial \beta_i/\partial x_j - \beta_i)
 11:
           \partial \beta_i / \partial x_i) - sharp_2[(j)] * (\partial \alpha_i / \partial x_i - \partial \alpha_i / \partial x_i)
                                 forms\_bracket[(j)] \leftarrow forms\_bracket[(j)] + sharp\_1[(i)] * (\partial \beta_i / \partial x_i - \beta_i) + sharp\_1[(i)] + sharp\_1[(i)]
12:
           \partial \beta_i/\partial x_j) - sharp_2[(i)] * (\partial \alpha_i/\partial x_i - \partial \alpha_i/\partial x_j)
                      end for
13:
                      for k, l = 1 to m do
                                                                                                                                                                              ▷ Compute the last sum in (2.14)
14:
                                forms\_bracket[(k)] \leftarrow forms\_bracket[(k)] + \partial(sharp\_1[(l)] * \beta_l)/\partial x k
15:
16:
                      if all values in forms_bracket are equal to zero then
17:
                                return \{0:0\}
18:
                                                                                                                                                                ▶ A dictionary with zero key and value
19:
                      else
                                return forms_bracket
20:
                      end if
21:
22: end procedure
```

2.7. Gauge Transformations. Let Π be a bivector field on M. Suppose we are given a differential 2-form λ on M such that the vector bundle morphism

$$\left(\mathrm{id}_{\mathsf{T}^*M} - \lambda^\flat \circ \Pi^\natural\right) : \mathsf{T}^*M \to \mathsf{T}M \qquad \text{is invertible}.$$

Then, there exists a bivector field $\overline{\Pi}$ on M (well) defined by the skew-symmetric morphism

(2.16)
$$\overline{\Pi}^{\natural} = \Pi^{\natural} \circ \left(\operatorname{id}_{\mathsf{T}^*M} - \lambda^{\flat} \circ \Pi^{\natural} \right)^{-1}.$$

Here, $\lambda^{\flat}: TM \to T^*M$ is the vector bundle morphism given by $X \mapsto \mathbf{i}_X \lambda$. The bivector field $\overline{\Pi}$ is called the λ -gauge transformation of Π [29, 7, 8]. A pair of bivector fields Π and $\overline{\Pi}$ on M are said to be gauge equivalent if they are related by (2.16) for some differential 2-form λ on M satisfying (2.15). If Π is a Poisson bivector field, then $\overline{\Pi}$ is a Poisson bivector field if and only if λ is closed along the symplectic leaves of Π . A gauge transformation modifies only the leaf-wise symplectic form of Π by means of the pull-back of λ , preserving the characteristic foliation. Furthermore, gauge

transformations preserve unimodularity. The function gauge_transformation computes the gauge transformation of a bivector field.

Algorithm 2.11 gauge_transformation(bivector, two_form)

Input: a bivector field and a differential 2-form

Output: a bivector field which is the gauge transformation induced by the differential 2-form of the given bivector field

```
1: procedure
 2:
        m \leftarrow \text{dimension of the manifold}
                                                        bivector \leftarrow a dictionary that represents a bivector field according to (3.1)
 3:
        two_form \leftarrow a dictionary that represents a differential 2-form according to
    (3.1)
 5:
       bivector\_matrix \leftarrow bivector\_to\_matrix(bivector)
                                                                            ▶ See Algorithm 2.6
 6:
        2_{\text{form\_matrix}} \leftarrow \text{bivector\_to\_matrix}(\text{two\_form})
 7:
       identity \leftarrow the m \times m identity matrix
       if det(identity - 2\_form\_matrix * bivector\_matrix) == 0 then
 8:
           return False
                                                             ▶ Means that (2.15) is not invertible
9:
       else
10:
           gauge_matrix ← bivector * (identity - 2_form_matrix * bivector_matrix)
11:
           gauge\_bivector \leftarrow an empty dictionary dict()
12:
           for 1 \le i < j \le m do
13:
               gauge_bivector[(i, j)] \leftarrow gauge_matrix[i - 1, j - 1]
14:
15:
           return gauge_bivector, det(identity - 2_form_matrix * bivector_matrix)
16:
        end if
17:
18: end procedure
```

Observe that the function gauge_transformation can be used to compute the gauge transformation induced by a closed differential 2-form of a Poisson bivector field.

2.8. Classification of Lie-Poisson bivector fields on \mathbb{R}^3 . A Lie-Poisson bivector field is a homogeneous Poisson bivector field (2.13) for which each Π^{ij} is a linear polynomial [17, 15, 10]. A pair of homogeneous Poisson bivector fields Π and $\widetilde{\Pi}$ on \mathbb{R}^m are said to be equivalent (or isomorphic) if there exists an invertible linear operator $T: \mathbb{R}^m \to \mathbb{R}^m$ such that

$$(2.17) \qquad \qquad \widetilde{\Pi} = T^* \Pi.$$

Under this equivalence relation in the 3-dimensional case there exist 9 non-trivial equivalence classes of Lie-Poisson bivector fields [24].

The function linear_normal_form_R3 computes a normal form of a Lie-Poisson bivector field on \mathbb{R}^3 . The normal forms are based on well-known classifications of (real) 3-dimensional Lie algebra isomorphisms [24].

Algorithm 2.12 linear_normal_form_R3(bivector)

Input: a Lie-Poisson bivector field on \mathbb{R}^3

Output: a linear normal form for the Lie-Poisson bivector field

1: procedure

```
bivector \leftarrow a dictionary \{(1,2): \Pi^{12}, (1,3): \Pi^{13}, (2,3): \Pi^{23}\} that represents
 2:
    a Lie-Poisson bivector field on \mathbb{R}^3 according to (3.1)
        Convert each value in bivector to symbolic expression
 3:
        parameter \leftarrow x1 * \Pi^{23} - x2 * \Pi^{13} + x3 * \Pi^{12}
 4:
        hessian\_parameter \leftarrow Hessian matrix of parameter
 5:
        if modular\_vf(bivector) == 0 then
 6:
                                                                           ▷ See Algorithm 2.8
 7:
           if rank(hessian\_parameter) == 0 then
               return {0: 0}
                                                          ▶ A dictionary with zero key and value
 8:
           else if rank(hessian\_parameter) == 1 then
9:
               return \{(1,2): 0, (1,3): 0, (2,3): x1\}
10:
           else if rank(hessian\_parameter) == 2 then
11:
12:
               if index(hessian\_parameter) == 2 then
                                                                     ▶ Index of quadratic forms
                   return \{(1,2): 0, (1,3): -x2, (2,3): x1\}
13:
               else
14:
                   return \{(1,2): 0, (1,3): \times 2, (2,3): \times 1\}
15:
               end if
16:
17:
           else
               if index(hessian\_parameter) == 3 then
18:
                                                                     ▶ Index of quadratic forms
                   return \{(1,2): x3, (1,3): -x2, (2,3): x1\}
19:
20:
21:
                   return \{(1,2): -x3, (1,3): -x2, (2,3): x1\}
               end if
22:
           end if
23:
        else
24:
           if rank(hessian\_parameter) == 0 then
25:
               return \{(1,2): 0, (1,3): x1, (2,3): x2\}
26:
           else if rank(hessian\_parameter) == 1 then
27:
               return \{(1,2): 0, (1,3): x1, (2,3): 4*x1 + x2\}
28:
29:
           else
               if index(hessian\_parameter) == 2 then
30:
                                                                     ▶ Index of quadratic forms
                   return \{(1,2): 0, (1,3): x1 - 4*a*x2, (2,3): 4*a*x1 + x2\}
31:
32:
                   return \{(1,2): 0, (1,3): x1 + 4*a*x2, (2,3): 4*a*x1 + x2\}
33:
               end if
34:
           end if
35:
        end if
36:
37: end procedure
```

2.9. Isomorphic Lie-Poisson Tensors on \mathbb{R}^3 . Using the function isomorphic_lie_poisson_R3 we can verify whether two Lie-Poisson bivector fields on \mathbb{R}^3 are isomorphic (2.17), or not.

Algorithm 2.13 isomorphic_lie_poisson_R3(bivector_1, bivector_2)

Input: two Lie-Poisson bivector fields

Output: verify if the Lie-Poisson bivector fields are isomorphic or not

```
1: procedure
```

2: bivector_1 \leftarrow a dictionary that represents a bivector field according to (3.1)

```
3: bivector_2 ← a dictionary that represents a bivector field according to (3.1)
4: if linear_normal_form_R3(bivector_1) == linear_normal_form_R3(bivector_2)
then ▷ See Algorithm 2.12
5: return True
6: else
7: return False
8: end if
9: end procedure
```

2.10. Flaschka-Ratiu Bivector Fields. Given m-2 functions $K_1, ..., K_{m-2} \in C_M^{\infty}$ on an oriented m-dimensional manifold (M, Ω) , with volume form Ω , we can construct a Poisson bivector field Π on M defined by

$$\mathbf{i}_{\Pi}\Omega := \mathrm{d}K_1 \wedge \cdots \wedge \mathrm{d}K_{m-2}.$$

Clearly, Π is non-trivial on the open subset of M where K_1, \ldots, K_{m-2} are (functionally) independent. Moreover, by construction, each K_l is a Casimir function of Π . These class of Poisson bivector fields are called *Flaschka-Ratiu* bivector fields [9]. In coordinates, if $\Omega = dx^1 \wedge \cdots \wedge dx^m$, then

(2.18)
$$\Pi = (-1)^{i+j} \det P_{[i,j]} \cdot \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \qquad 1 \le i < j \le m.$$

Here P denotes the $(m-2) \times m$ -matrix whose k-th row is $(\partial K_k/\partial x^1, \ldots, \partial K_k/\partial x^m)$, for $k = 1, \ldots, m-2$; and $P_{[i,j]}$ the matrix P without the columns i and j. Moreover, the symplectic form ω_S of Π on a 2-dimensional (symplectic) leaf $S \subseteq M$ is given by (2.19)

$$\omega_{S} = \frac{1}{|\Pi|^{2}} \left[(-1)^{i+j+1} \det P_{[i,j]} \cdot \mathrm{d}x^{i} \wedge \mathrm{d}x^{j} \right] \Big|_{S}, \quad |\Pi|^{2} := \sum_{1 \leq i < j \leq m} \left(\det P_{[i,j]} \right)^{2}.$$

The function flaschka_ratiu_bivector computes the Flaschka-Ratiu bivector field and the corresponding symplectic form of a 'maximal' set of scalar functions [9, 14, 30, 11].

Algorithm 2.14 flaschka_ratiu_bivector(casimir_list)

Input: m-2 scalar functions

Output: the Flaschka-Ratiu bivector field induced by the m-2 functions and the symplectic form of this Poisson bivector field

```
1: procedure
 2:
        m \leftarrow \text{dimension of the manifold}

    □ Given by an instance of PoissonGeometry

        casimir_list \leftarrow a list ['K1', ..., 'K\{m-2\}'] with m-2 string expressions
 3:
                                                ▶ Each string expression represents a scalar function
       if at least two functions in casimir_list are functionally dependent then
 4:
            return {0: 0}
 5:
                                                            ▶ A dictionary with zero key and value
        else
 6:
            matrix_gradients \leftarrow a symbolic (m-2) \times m-matrix
 7:
            for i = 1 to m - 2 do
 8:
                Convert Ki to symbolic expressions
9:
               Compute the gradient vector \nabla Ki of Ki
10:
```

```
Append \nabla Ki to matrix_gradients as its i-th row
11:
            end for
12:
13:
            flaschka\_bivector \leftarrow an empty dictionaty dict()
            sum\_bivector \leftarrow 0
14:
            for each 1 \le i < j \le m do
15:
               Remove from matrix_gradients the i-th and j-th columns
16:
               flaschka_bivector[(i, j)] \leftarrow (-1)^{i+j} * \det(\text{matrix\_gradients})
17:
               sum\_bivector \leftarrow sum\_bivector + det(matrix\_gradients)**2
18:
               APPEND to matrix_gradients the i-th and j-th removed
19:
            end for
20:
            symplectic\_form \leftarrow an empty dictionary dict()
21:
            for each 1 \le i < j \le m do
22:
               symplectic_form[(i,j)] \leftarrow (-1) * sum\_bivector * flaschka\_bivector[(i,j)]
23:
            end for
24:
            return flaschka_bivector, symplectic_form
25:
26:
        end if
27: end procedure
```

- **2.11. Test Type Functions.** In this section we describe our implementation of some useful functions in the PoissonGeometry module which allow us to verify whether a given geometric object on a Poisson manifold satisfies certain property. The algorithms for each of these functions are similar, as they are decision-making processes.
- **2.11.1.** Jacobi Identity. We can verify in PoissonGeometry if a given bivector field Π is a Poisson bivector field or not.

```
Algorithm 2.15 is_poisson_tensor(bivector)

Input: a bivector field
```

Output: verify if the bivector field is a Poisson bivector field or not

```
1: procedure
2: bivector ← a dictionary that represents a bivector field according to (3.1)
3: if lichnerowicz_poisson_operator(bivector, bivector) == {0: 0} then

▷ See Algorithm 2.4
4: return True
5: else
6: return False
7: end if
8: end procedure
```

2.11.2. Kernel of a Bivector Field. The kernel of a bivector field Π is the subspace $\ker \Pi := \{\alpha \in \mathsf{T}^*M \mid \Pi^{\natural}(\alpha) = 0\}$ of T^*M . It is defined as the kernel of its sharp morphism (2.1), and is defined likewise for Poisson bivector fields [10, 23].

Algorithm 2.16 is_in_kernel(bivector, one_form)

Input: a bivector field and a differential 1-form

Output: verify if the differential 1-form belongs to the kernel of the (Poisson) bivector field

```
1: procedure
      bivector \leftarrow a dictionary that represents a bivector field according to (3.1)
3:
      one_form \leftarrow a dictionary that represents a differential 1-form according to
   (3.1)
      if sharp\_morphism(bivector, one\_form) == \{0: 0\} then
4:
                                                                       ▶ See Algorithm 2.1
          return True
5:
      else
6:
          return False
7:
8:
      end if
9: end procedure
```

2.11.3. Casimir Functions. A function $K \in C_M^{\infty}$ is said to be a Casimir funtion of a Poisson bivector field Π if its Hamiltonian vector field (2.4) is zero. Equivalently, if its exterior derivative dK belongs to the kernel of Π [10, 9, 23].

Algorithm 2.17 is_casimir(bivector, function)

Input: a Poisson bivector field and a scalar function

Output: verify if the scalar function is a Casimir function of the Poisson bivector field

```
1: procedure
      bivector \leftarrow a dictionary that represents a bivector field according to (3.1)
3:
      function \leftarrow a string expression
                                                                 ▶ Represent a scalar function
      if hamiltonian_vf(bivector, function) == \{0: 0\} then
4:
                                                                         ▷ See Algorithm 2.3
5:
          return True
      else
6:
          return False
7:
      end if
9: end procedure
```

2.11.4. Poisson Vector Fields. A vector field W on M is said to be a Poisson vector field of a Poisson bivector field Π if it commutes with respect to the Schouten-Nijenhuis bracket, $[W,\Pi] = 0$ [10, 23].

Algorithm 2.18 is_poisson_vf(bivector, vector_field)

Input: a Poisson bivector field and a vector field

Output: verify if the vector field is a Poisson vector field of the Poisson bivector field

```
    procedure
    bivector ← a dictionary that represents a bivector field according to (3.1)
    vector_field ← a dictionary that represents a vector field according to (3.1)
```

```
4: if lichnerowicz_poisson_operator(bivector, vector_field) == {0: 0} then

▷ See Algorithm 2.4

5: return True
6: else
7: return False
8: end if
9: end procedure
```

2.11.5. Poisson Pairs. We can verify whether a couple of Poisson bivector fields Π and Ψ form a Poisson pair. That is, if the sum $\Pi + \Psi$ is again a Poisson bivector field or, equivalently, if Π and Ψ commute with respect to the Schouten-Nijenhuis bracket, $\llbracket \Pi, \Psi \rrbracket = 0$ [10, 23].

Algorithm 2.19 is_poisson_pair(bivector_1, bivector_2)

Input: two Poisson bivector fields.

Output: verify if the bivector fields commute with respect to the Schouten-Nijenhuis bracket

```
1: procedure
      bivector_1 \leftarrow a dictionary that represents a bivector field according to (3.1)
2:
      bivector 2 \leftarrow a dictionary that represents a bivector field according to (3.1)
3:
      if lichnerowicz_poisson_operator(bivector_1, bivector_2) == {0: 0} then
4:
                                                                        ▷ See Algorithm 2.4
5:
          return True
6:
      else
7:
          return False
      end if
9: end procedure
```

3. PoissonGeometry: Syntax and Applications. PoissonGeometry is our python module for local calculus on Poisson manifolds. First we define a tuple of symbolic variables that emulate local coordinates on a finite (Poisson) smooth manifold M. By default, these symbolic variables are just the juxtaposition of the symbol x and an index of the set $\{1, \ldots, m = \dim M\}$: $(x1, \ldots, xm)$.

Scalar Functions. A local representation of a scalar function in PoissonGeometry is written using *string literal expressions*. For example, the function $f = a(x^1)^2 + b(x^2)^2 + c(x^3)^2$ should be written exactly as follows: 'a * x1**2 + b * x2**2 + c * x3**2'. It is *important* to remember that all characters that are not local coordinates are treated as (symbolic) parameters: a, b and c for the previous example.

Multivector Fields and Differential forms. Both multivector fields and differential forms are written using dictionaries with tuples of integers as keys and string type values. If the coordinate expression of an a-multivector field A on M, with $a \in \mathbb{N}$, is given by,

$$A = \sum_{1 \le i_1 < i_2 < \dots < i_a \le m} A^{i_1 i_2 \dots i_a} \frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_a}}, \quad A^{i_1 \dots i_a} = A^{i_1 \dots i_a}(x),$$

then A should be written using a dictionary, as follows:

$$(3.1) \ \Big\{ (1,...,a) : \mathcal{A}^{1\cdots a}, \ ..., \ (i_1,...,i_a) : \mathcal{A}^{i_1\cdots i_a}, \ ..., \ (m-a+1,...,m) : \mathcal{A}^{m-a+1\cdots m} \Big\}.$$

Here each key (i_1, \ldots, i_a) is a tuple containing ordered indices $1 \leq i_1 < \cdots < i_a \leq m$ and the corresponding value $\mathcal{A}^{i_1 \cdots i_a}$ is the string expression of the scalar function (coefficient) $A^{i_1 \cdots i_a}$ of A.

The syntax for differential forms is the same. It is important to remark that we can only write the keys and values of *non-zero coefficients*. See the documentation for more details.

3.1. Applications. We will now describe two applications. One of gauge_transformation, used here to derive a characterization of gauge transformations on \mathbb{R}^3 (see, Subsection 2.7), and a second one of jacobiator, used here to construct a family of Poisson bivector fields on \mathbb{R}^4 (1.1).

Gauge Transformations on \mathbb{R}^3 . For an arbitrary bivector field on \mathbb{R}^3_x ,

$$(3.2) \Pi = \Pi^{12} \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + \Pi^{13} \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + \Pi^{23} \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3},$$

and an arbitrary differential 2-form

(3.3)
$$\lambda = \lambda_{12} dx^{1} \wedge dx^{2} + \lambda_{13} dx^{1} \wedge dx^{3} + \lambda_{23} dx^{2} \wedge dx^{3},$$

we compute:

```
>>> pg3 = PoissonGeometry(3)
>>> P = {(1,2): 'P12', (1,3): 'P13', (2,3): 'P23'}
>>> lambda = {(1,2): 'L12', (1,3): 'L13', (2,3): 'L23'}
>>> (gauge_bivector, determinant) = pg3.gauge_transformation(P, lambda)
>>> print(gauge_bivector)
>>> print(determinant)

{(1,2): P12/(L12*P12 + L13*P13 + L23*P23 + 1), (1,3): P13/(L12*P12 + L13*P13 + L23*P23 + 1), (2,3): P23/(L12*P12 + L13*P13 + L23*P23 + 1)}
(L12*P12 + L13*P13 + L23*P23 + 1)**2
```

The symbols P12, P13, P23, and L12, L13, L23 stand for the coefficients of Π and λ , in that order. Then, (see (2.15)):

(3.4)
$$\det \left(\mathrm{Id} - \lambda^{\flat} \circ \Pi^{\natural} \right) = \left(\lambda_{12} \Pi^{12} + \lambda_{13} \Pi^{13} + \lambda_{23} \Pi^{23} + 1 \right)^{2}$$

So, for $1 \le i < j \le 3$, the λ -gauge transformation $\overline{\Pi}$ of Π is given by:

$$(3.5) \ \overline{\Pi} = \frac{\Pi^{12}}{\lambda_{ij}\Pi^{ij} + 1} \frac{\partial}{\partial x^{1}} \wedge \frac{\partial}{\partial x^{2}} + \frac{\Pi^{13}}{\lambda_{ij}\Pi^{ij} + 1} \frac{\partial}{\partial x^{1}} \wedge \frac{\partial}{\partial x^{3}} + \frac{\Pi^{23}}{\lambda_{ij}\Pi^{ij} + 1} \frac{\partial}{\partial x^{2}} \wedge \frac{\partial}{\partial x^{3}}$$

With these ingredients, we can now show:

PROPOSITION 3.1. Let Π be a bivector field on a 3-dimensional smooth manifold M. Then, given a differential 2-form λ on M, the λ -gauge transformation $\overline{\Pi}$ (2.16) of Π is well defined on the open subset,

$$(3.6) {F := \langle \lambda, \Pi \rangle + 1 \neq 0} \subseteq M.$$

Moreover, $\overline{\Pi}$ is given by

$$(3.7) \overline{\Pi} = \frac{1}{E} \Pi.$$

If Π is Poisson and λ is closed along the leaves of Π , then $\overline{\Pi}$ is also Poisson.

Proof. Suppose (3.2) and (3.3) are coordinate expressions of Π and λ on a chart $(U; x^1, x^2, x^3)$ of M. Observe that the pairing of Π and λ is given by

$$\langle \lambda, \Pi \rangle = \lambda_{12} \Pi^{12} + \lambda_{13} \Pi^{13} + \lambda_{23} \Pi^{23}.$$

Hence (3.4) yields $\det(\mathrm{Id} - \lambda^{\flat} \circ \Pi^{\natural}) = (\langle \lambda, \Pi \rangle + 1)^2$. This implies that the morphism $\mathrm{Id} - \lambda^{\flat} \circ \Pi^{\natural}$ is invertible on the open subset in (3.6) and, in consequence, the λ -gauge transformation of Π (2.15). Finally, formula (3.7) follows from (3.5) as

$$\overline{\Pi} = \frac{\Pi^{12}}{F} \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + \frac{\Pi^{13}}{F} \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + \frac{\Pi^{23}}{F} \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3},$$

for
$$F$$
 in (3.6.

Parametrized Poisson Bivector Fields. Poisson bivector fields also play an important role in the theory of deformation quantization, which is linked to quantum mechanics [5]. They appear in star products, that is, in general deformations of the associative algebra of smooth functions of a symplectic manifold [13]. Our module PoissonGeometry can be used to study particular problems around deformations of Poisson bivector fields and star products.

For example, we can modify the following 4-parametric bivector field on \mathbb{R}^4

$$\Pi = a_1 x^2 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + a_2 x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + a_3 x^4 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^4} + a_4 x^1 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3},$$

using the jacobiator function to construct a family of Poisson bivector fields on \mathbb{R}^4 :

>>> pg4 = PoissonGeometry(4)
>>> P =
$$\{(1,2): 'a1*x2', (1,3): 'a2*x3', (1,4): 'a3*x4', (2,3): 'a4*x1'\}$$

>>> pg4.jacobiator(P)
 $\{(1,2,3): -2*a4*x1*(a1 + a2), (2,3,4): -2*a3*a4*x4\}$

Therefore

$$\llbracket \Pi, \Pi \rrbracket = -2a_4(a_1 + a_2) x^1 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} - 2a_3 a_4 x^4 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4}$$

Hence, we have two cases, explained in the following:

LEMMA 3.2. If $a_4 = 0$, then Π determines a 3-parametric family of Poisson bivector fields on \mathbb{R}^4_x :

$$(3.8) \Pi = a_1 x^2 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + a_2 x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + a_3 x^4 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^4}.$$

If $a_2 = -a_1$ and $a_3 = 0$, then Π determines a 2-parametric family of Poisson bivector fields on \mathbb{R}^4_π :

(3.9)
$$\Pi = a_1 x^2 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} - a_1 x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + a_4 x^1 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3}.$$

Related Work. Notable contributions in similar directions include; computations of normal forms in Hamiltonian dynamics (in Maxima) [31], symbolic tests of the Jacobi identity for generalized Poisson brackets and their relation to hydrodynamics [21], and an implementation of the Schouten-Bracket for multivector fields (in Sage ²).

Our work here is, to the best of our knowledge, the first comprehensive implementation of routine computations used in Poisson geometry, and in Python (based on SymPy[25]).

Future Directions. With the algorithms in this paper, numerical extensions for the same methods can be developed. Explicit computations of Poisson cohomology can also be explored. These are the subjects of ongoing, and forthcoming work.

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