Approximate analytic connection of the asymptotic ratio to signal density

Signal density is defined as:

$$\text{Signal density} = \frac{1}{N} \cdot \frac{\left(\sum_k (\Delta\mu_k)^2\right)^2}{\sum_k (\Delta\mu_k)^4} =: \frac{1}{N} \cdot \frac{M_2^2}{M_4}$$

I am using M_2 and M_4 as short notation for the sums of squares or fourth-powers of the elements of $\Delta\mu$ over neurons.

The asymptotic ratio is the ratio of the asymptotic SNR along $\Delta\mu$ to the shuffled asymptotic SNR along $\Delta\mu$. When I checked those two variables separately (not as a ratio), there was no correlation to the asymptotic SNR, but there was a correlation to the shuffled asymptotic SNR. Let's inspect more closely the shuffled asymptotic SNR.

Shuffled asymptotic SNR

The shuffled asymptotic SNR is the ratio of the rate of change of the signal, which is

$$\text{Signal slope} = \frac{1}{N} \cdot \sum_k (\Delta\mu_k)^2 = \frac{1}{N} \cdot M_2$$

to the rate of change of the noise, which is harder to determine because it appears to converge to a constant value or nearly constant value (it depends on the properties of the distribution of signal and noise among neurons).

The noise along $\Delta\mu$ in the shuffled case (idealized to mean that there are no noise correlations at all) is:

$$\text{Noise} = \frac{\sum_k (\Delta\mu_k)^2 \sigma_k^2}{\sum_k (\Delta\mu_k)^2} = \frac{\sum_k (\Delta\mu_k)^2 \sigma_k^2}{M_2}$$

In other words, it is the weighted average of the noise variances σ_k^2 based on the signal quantities $(\Delta\mu_k)^2$. Since it is a weighted average, it is bounded above by the maximum value of the random variable $\sigma_k^2 \sim \text{Dist}$, where k is a random neuron and Dist is the distribution from which the noise variance of a random neuron comes from. If we reasonably assume that individual neurons will carry only up to a maximal amount of noise then the noise in the signal direction will necessarily converge to a constant. If we measure the slope in the asymptotic regime it must be 0. The way we are calculating the slope of the noise is by fitting a line to all noise values from ensembles larger than 100 neurons. Of course, the asymptotic regime can never be exactly reached, so the values of the slopes of the noise in the shuffled case would be a nonzero finite slope value that captures the rate of convergence to the asymptotic value. It is reasonable to assume that this quantity is proportional to the value of the asymptotic noise itself, if the noise in the signal direction of all sessions converges at a similar rate per additional neuron in the ensemble. Let us denote the constant of proportionality as γ. Now the estimated rate of change of noise is:

$$\text{Noise slope} = \gamma \frac{\sum_k (\Delta\mu_k)^2 \sigma_k^2}{M_2}$$

We take the ratio of the signal slope to the noise slope to get the estimated asymptotic SNR:

$$\text{Shuffled asymp. SNR} = \frac{\text{Signal slope}}{\text{Noise slope}} = \frac{M_2/N}{\gamma \sum_k (\Delta\mu_k)^2 \sigma_k^2 / M_2} = \frac{1}{\gamma N} \cdot \frac{M_2^2}{\sum_k (\Delta\mu_k)^2 \sigma_k^2}$$

This quantity will become proportional to signal density if we include another assumption, that $\sigma_k^2 \propto (\Delta\mu_k)^2$. This assumption is almost certainly false, since the noise density is approximately twice the signal density, whereas proportionality would imply that they should be equal. However, a reasonable assumption is that $\sigma_k^2 \sim (\Delta\mu_k)^2$, meaning that they are correlated. If that is the case, then the term in the denominator of the shuffled asymptotic SNR, $\sum_k (\Delta\mu_k)^2 \sigma_k^2 \sim M_4$ becomes correlated with M_4, which in turn means that shuffled asymptotic SNR is correlated with signal density.

$$\text{Shuffled asymp. SNR} \sim \frac{1}{\gamma N} \cdot \frac{M_2^2}{M_4} \sim \text{Signal density}$$

Since the signal density did not correlate with the non-shuffled asymptotic SNR, it would still be correlated to the ratio of the real asymptotic SNR to the shuffled asymptotic SNR, just inversely, which is the direction of the correlation that we observed empirically.