§. Graphs:

Definition: A graph G is a pair (V, E), where $V = \{v_1, v_2, v_3, \ldots\}$ is a non-empty set whose elements are called vertices (or nodes) and $E = \{e_1, e_2, \ldots\}$ is a set such that each element e_k of E is identified with an unordered pair (v_i, v_j) of vertices. The elements of E are called edges of G. The vertices v_i, v_j associated with edge e_k are called the end vertices of e_k and the edge e_k is then denoted by $e_k = (v_i, v_j)$

Representation of a Graph:

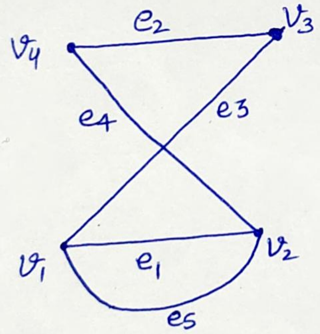
Graphs are usually represented by diagrams in which a vertex is represented by a dot or a small circle and an edge is represented by a line segment joining its end points.

Example 1 Let $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$ be such that

$$e_1 = (v_1, v_2),\ e_2 = (v_4, v_3),\ e_3 = (v_1, v_3),\ e_4 = (v_2, v_4) \text{ and } e_5 = (v_1, v_2)$$

Then $G = (V, E)$ is a graph

This graph is represented as:



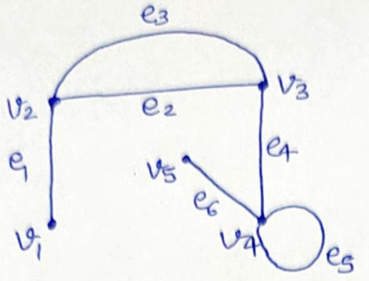
Parallel edges in a Graph:

We observe from the definition of a graph $G = (V, E)$ that while the elements of E are distinct, more than one edge in E may have same pair of end vertices. All edges having the same pair of end vertices are called parallel edges.

For example, the edge e_1 and e_5 are parallel edges. (example 1).

§. Self-loop in a graph:

We note that the definition of a graph allows an edge to be of the form (v_i, v_i). Such an edge having the same vertex as both its end vertices is called a self-loop.



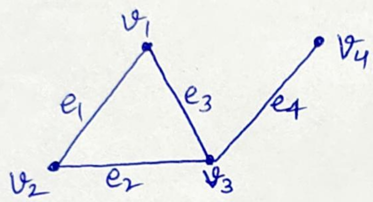
(Graph with five vertices and six edges)

(Not simple)

Edge e_5 in the above graph is a self-loop.

§. Simple Graph

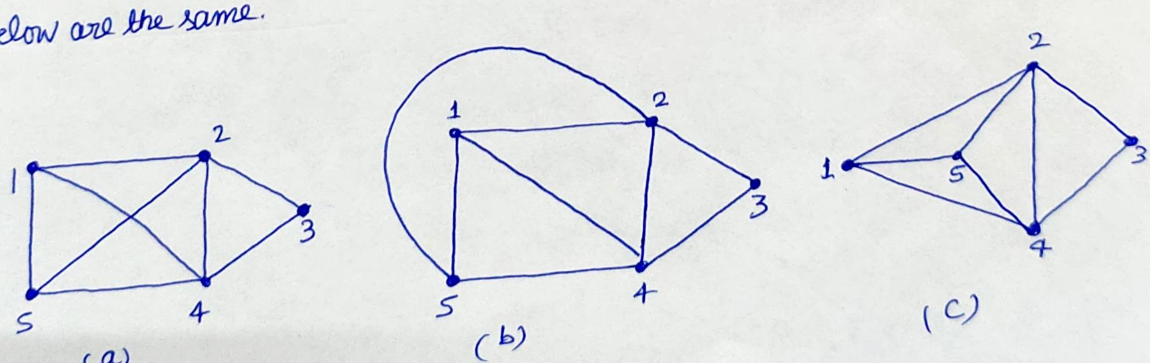
A graph that has neither self-loops nor parallel edges is called a simple graph.



Remarks:

(1) Some authors use the term graph to mean what we have called a simple graph and use the term multigraph to mean a graph with parallel edges and self-loops.

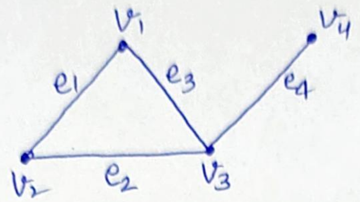
(2) It should be noted that, in drawing a graph, it is immaterial whether the edges are drawn straight or curved, long or short. What is important is how the vertices are joined up. For example, the three graphs graphs drawn in the figure given below are the same.



(Same graph drawn differently)

§. Incidence & Degree

Let e_k be an edge joining two vertices v_i and v_j of a graph G. Then the edge e_k is said to be incident on each of its end vertices v_i and v_j.



For example: in the above graph edge e_3 is incident on vertices v_1 and v_3.

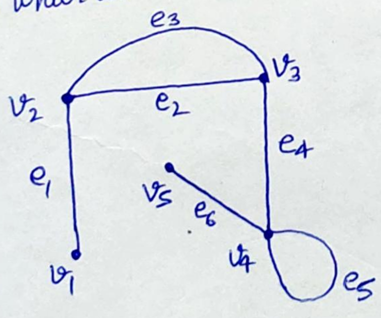
✓ Two vertices in a graph are said to be adjacent if there exists an edge joining the vertices. In the above graph v_1 and v_2 are adjacent while v_1 and v_4 are not adjacent.

✓ Two non-parallel edges are said to be adjacent if they are incident on a common vertex.

For example, edges e_1 and e_2 in the above graph are adjacent because they are incident on a common vertex v_2 while edges e_1 and e_4 are not adjacent.

Degree of a vertex

The degree of a vertex v in a graph G, written as $d(v)$ or $deg_G(v)$, is equal to the number of edges which are incident on v with self-loop counted twice.



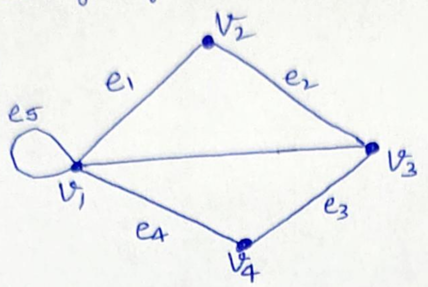
$d(v_1)=1$, $d(v_2)=3$, $d(v_3)=3$, $d(v_4)=4$, $d(v_5)=1$

✓ The degree of a vertex is sometimes also referred to as its valency

*** degree of sequence: ?

If $v_1, v_2, v_3, \ldots, v_n$ are n vertices of G, then the sequence $(d_1, d_2, d_3, \ldots, d_n)$ where $d_i = \deg(v_i)$ is the degree of sequence of G. In general, we order the vertices so that the degree sequence is monotonically increasing i.e. $d_1 \le d_2 \le d_3 \le \ldots \le d_n$.

For example, the degree of sequence of the graph



is $(2, 2, 3, 5)$.

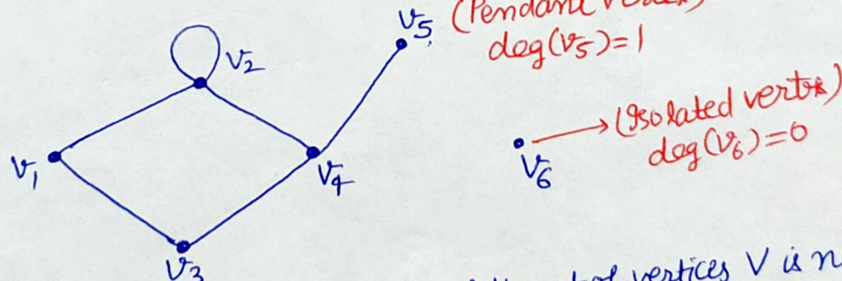
§. Finite and Infinite Graphs

A graph $G(V, E)$ is said to be <u>finite</u> if both sets V and E are finite otherwise it is called infinite graph.

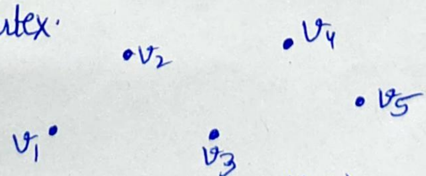
§. Isolated Vertex, Pendant Vertex and Null Graphs

A vertex having no edge incident on it is called an <u>isolated vertex</u>. In other words, a vertex v is said to be isolated vertex if degree $\deg(v)$ or $d(v)$ of v is zero.

A vertex is said to be pendant vertex if its degree is one.



A graph $G = (V, E)$ is said to be <u>null graph</u> if the set of vertices V is non-empty but the set of edges E is empty. A null graph is thus a graph in which every vertex is an isolated vertex.



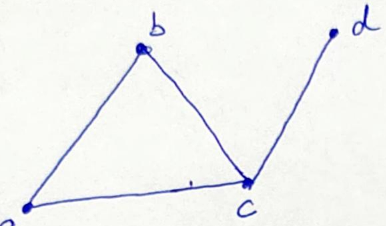
(Null graph with five vertices)

§. Undirected and Directed Graph

An undirected graph G consists of set V of vertices and a set E of edges such that each edge $e \in E$ is associated with an unordered pair of vertices.

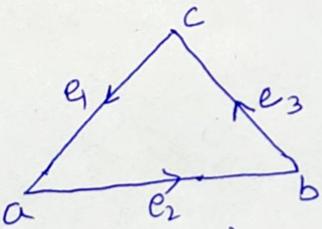
In an undirected graph we can refer to an edge joining the vertex pair i and j as either (i,j) or (j,i)

Example:



(Undirected Graph)

A directed graph (or digraph) G consists of a set V of vertices and a set E of edges such that $e \in E$ is associated with an ordered pair of vertices. In other words, if each edge of the graph G has a direction then the graph is called directed graph. In the diagram of a directed graph, each edge $e = (u, v)$ is represented by an arrow or directed curve from initial point u of e to the terminal point v.



(Directed graph)

§ Representation of Graphs

Matrix representation:

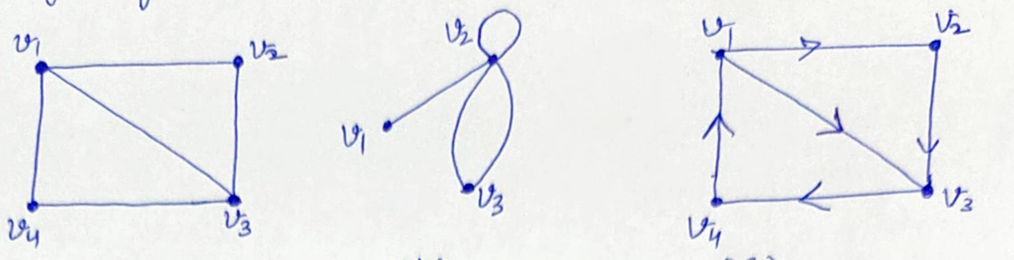
§. Adjacency Matrix:

(a) Representation of Undirected Graph: The adjacency matrix of a graph G with n vertices and no parallel edges is an $n \times n$ matrix $A = \{a_{ij}\}$ whose elements are given by
(self loops are allowed)

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge between } i^{th} \text{ and } j^{th} \text{ vertices} \\ 0, & \text{if there is no edge between them} \end{cases}$$

Note: Two graphs G_1 and G_2 are isomorphic iff the adjacency matrix of one can be obtained from the adjacency matrix of the other by interchanging some of the rows and the corresponding columns. (or Two graphs are isomorphic if and only if their vertices can be labeled in such a way that corresponding adjacency matrices are equal)

Q:1 Use adjacency matrix to represent the graphs shown in the figure below:



(a) (b) (c)

[§. Represent of Directed graph:
The adjacency matrix of a digraph D, with n vertices is the matrix $A = \{a_{ij}\}_{n \times n}$
in which

$$a_{ij} = \begin{cases} 1, & \text{if arc } (v_i, v_j) \text{ is in } D \\ 0, & \text{otherwise} \end{cases}$$

Sol: 1(a) We order the vertices as v_1, v_2, v_3, v_4. Since there are four vertices, the adjacency matrix representing the graph will ~~the~~ be a square matrix of order four. The required adjacency matrix A is

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4 \times 4} \end{array}$$

(b) $$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}_{3 \times 3} \end{array}$$

*****
Note: We can extend the idea of matrix representation to graphs having parallel edges. If G is a graph (Self loops and parallel edges are also allowed). Then we define adjacency matrix for such graphs as follows:

Define: $$a_{ij} = \begin{cases} K, & \text{if there are } K \text{ edges between vertices } v_i \text{ and } v_j \\ 0, & \text{if there is no edge between vertices } v_i \text{ and } v_j \end{cases}$$

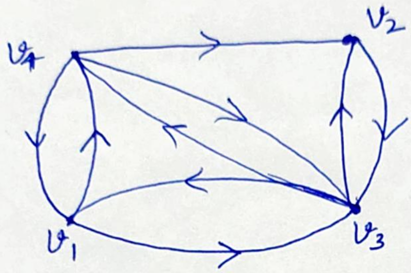
1(C)

$$A = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_2 & 0 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 0 & 1 \\ v_4 & 1 & 0 & 0 & 0 \end{array}$$

Q: Draw the diagraph G corresponding to adjacency matrix:

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{4\times 4}$$

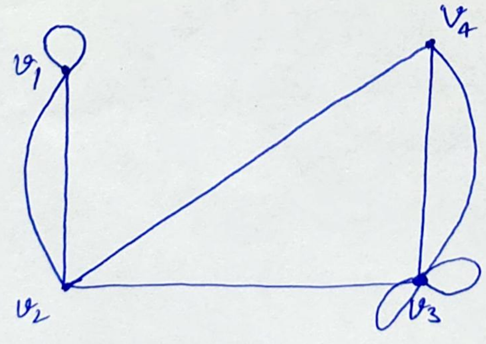
Sol Since the given matrix is a square matrix of order four, the graph G has 4 vertices v_1, v_2, v_3 and v_4.



Q: Draw the undirected graph G corresponding to adjacency matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Sol Since the given adjacency matrix is a square matrix of order 4, G has four vertices v_1, v_2, v_3 and v_4.



Representation of a Graph (Continued)

§1 Incidence Matrix (Undirected graph)

Let G be a graph with n vertices $v_1, v_2, \dots, v_n$ and m edges $e_1, e_2, \dots e_m$ and no self-loops.

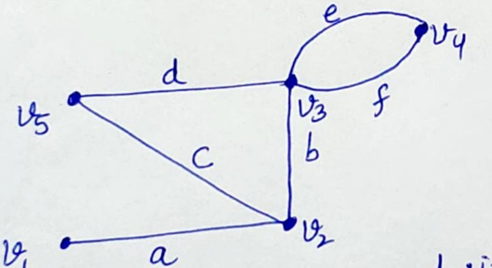
We define an $n \times m$ matrix $A = [a_{ij}]$ whose n rows correspond to the n vertices and m columns correspond to the m edges as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ edge } e_j \text{ is incident on } i^{th} \text{ vertex } v_i \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

The matrix $A = [a_{ij}]$ is called incidence matrix. Sometimes incidence matrix A of a graph G is also written as $A(G)$. Any element of the incidence matrix is either 0 or 1. That is why it is also called (0,1) matrix or a bit matrix.

For a given graph $G = (V, E)$, an incidence matrix depends upon the ordering of the vertices of G and that of edges of G. For different ordering of vertices and edges we get different incidence matrix of the same graph G. However, any one of the incidence matrix of a graph G can be obtained from another incidence matrix of the same graph by interchanging some of the rows and columns of the matrix.

Q:1 Write the incidence matrix of the graph given below:



Sol The given graph has five vertices and six edges. The incidence matrix is given below:

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

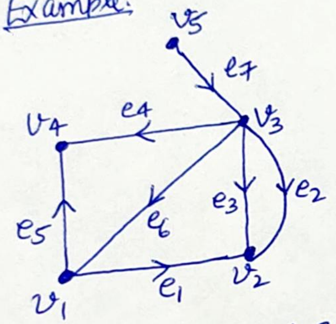
Note: Each column of incidence matrix A of a graph without self loops has exactly two 1's because each edge is incident on exactly two vertices.

2. Incidence Matrix (directed graph)

Let G be a digraph with n vertices, m edges and having no self-loops. The incidence matrix $A = [a_{ij}]$ of G has n rows, one for each vertex and m columns, one for each edge. The elements a_{ij} of A is defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if the } j^{th} \text{ edge is } \underline{\text{incident out}} \text{ of } i^{th} \text{ vertex} \\ -1, & \text{if the } j^{th} \text{ edge is } \underline{\text{incident into}} \; i^{th} \text{ vertex} \\ 0, & \text{if } j^{th} \text{ edge is not incident on } i^{th} \text{ vertex} \end{cases}$$

<u>Example:</u>



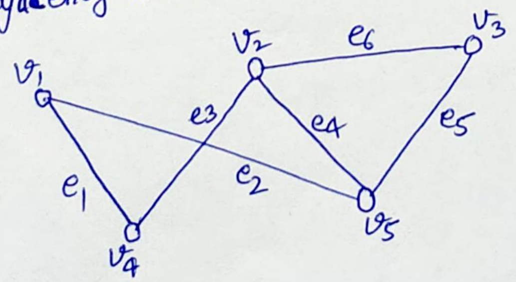
(a) diagraph of G

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(b) (incidence Matrix)

<u>Note:</u> It is clear from the definition of incidence matrix that each column of A contains exactly two non-zero entries, 1 and -1. Thus the sum of each column is zero.

<u>Q:</u> Find adjacency matrix and incidence matrix of the following graph:



Theorem: 1 A simple graph with at least two vertices has at least two vertices of same degree.

Proof: Let G be a simple graph with $n \geq 2$ vertices. The graph G has no loop and parallel edges. Hence the degree of each vertices is $\leq n-1$. Suppose all the vertices of G are of different degrees. Hence the following degrees are

$$0, 1, 2, 3, \ldots, (n-1)$$

are possible for n vertices of G. Let u be the vertex with degree 0. Then u is an isolated vertex. Let v be the vertex with degree $n-1$, then v has $n-1$ adjacent vertices. Since v is not an adjacent vertex to itself, therefore every vertex of G other than u is an adjacent vertex of G. Hence u cannot be an isolated vertex, this contradiction proves that a simple graph contains two vertices of same degree.

Another proof:- Let $v_1, v_2, v_3, \ldots v_n$ be the vertices of the graph G with degree sequence $d_1, d_2, \ldots, d_n$. If possible let $d_1 < d_2 < d_3 < \ldots < d_n$. The maximum degree of a vertex is $n-1$. Thus maximum value of d_n is $n-1$. Let $d_n = n-1, \ldots\ldots, d_2 = 1, d_1 = 0$. Since ~~degree of~~ d_n, the degree of v_n is $n-1$, so v_n is adjacent to all remaining vertices $v_1, v_2, \ldots v_{n-1}$. But it is not possible, since $d_1 = 0$. Thus the degrees of the vertices of a simple graph can not be distinct.

Thus, the degrees of at least two vertices of a simple graph must be equal.

Theorem: Show that the maximum degree of any vertex in a simple graph with n vertices is $(n-1)$.

Proof. Let G be a graph with n vertices. Since the graph is simple i.e. it has no self loops or parallel edges, any vertex v of G can be adjacent to at most remaining $(n-1)$ vertices and thus the degree of v becomes $n-1$. Hence the result.

Since the minimum degree of a vertex can be 0, thus $0 \leq deg(v) \leq n-1$ for all vertices of a simple graph.

Theorem: (Handshaking theorem)

The sum of the degrees of all vertices in a graph G is equal to twice the number of edges in G.

i.e. $$\sum_{i=1}^{n} d(v_i) = 2e$$

Proof. Let G be a graph with ~~a~~ e edges and n vertices $v_1, v_2, v_3, \ldots, v_n$. Since each edge is incident on two vertices, it contributes 2 to the sum of the degrees of the graph. Thus the sum of degrees of all the vertices in G is twice the number of edges in the graph G.

i.e. $$\boxed{\sum_{i=1}^{n} d(v_i) = 2e}$$

Note: A vertex is said to be even or odd vertex according as its degree is an even or an odd number

Theorem: The number of odd vertices (ie vertices of odd degree) in a graph is always even.

Proof: Let the number of vertices in a graph G be n. Without loss of generality, we can assume that the degree of the first r ~~vertic~~ vertices $v_1, v_2, \ldots v_r$ be even and those of the remaining $(n-r)$ vertices be odd. Then

$$\sum_{i=1}^{n} d(v_i) = \sum_{i=1}^{n} d(v_i) + \sum_{i=r+1}^{n} d(v_i) \quad \text{— (1)}$$

By Handshaking theorem, we know that the sum on the left-hand side of (1) is even. The first sum on the right hand side is also even, because each term in ~~this~~ this sum is even. Hence the seccond ~~term~~ sum on the right hand side must also be even:

$$\sum_{i=r+1}^{n} d(v_i) = \text{an even number} \quad \text{—(2)}$$

Since each term $d(v_i)$ in (2) is odd, the total number of terms in the sum must be even (to make the sum an even number). In other words, $n-r$, the number of vertices of odd degree, should be even. Hence the theorem.

Q: Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$ i.e. $^nC_2$.

Sol By Handshaking theorem, we know that

$$\sum_{i=1}^{n} d(v_i) = 2e, \text{ where } e \text{ is the number of edges with } n \text{ vertices in } G.$$

$$\Rightarrow d(v_1) + d(v_2) + \cdots\cdots + d(v_n) = 2e \quad —(1)$$

Since we know that the maximum degree of each vertex in the graph G can be $(n-1)$, therefore equation (1) reduces to

$$\underbrace{(n-1) + (n-1) + \cdots\cdots + (n-1)}_{n \text{ terms}} = 2e$$

$$\Rightarrow \quad n(n-1) = 2e$$

$$\Rightarrow \quad \boxed{e = \frac{(n-1)n}{2} = \frac{n(n-1)}{2}}$$

Hence the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$ ie $^nC_2$.

§. Isomorphic Graphs

Two Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a one-to-one correspondence between their vertex sets (V_1 and V_2) and a one-to-one correspondence between their edge sets (E_1 and E_2) so that corresponding edges of G_1 and G_2 are incident on the corresponding vertices of G_1 and G_2.

In other words, if vertices v_1 and v_2 in G_1 correspond respectively to vertices v_1' and v_2' in G_2, then an edge in G_1 with end vertices v_1 and v_2 must correspond to an edge in G_2 with v_1' and v_2' as end vertices and vice-versa.

Isomorphic Graph (Another definition)

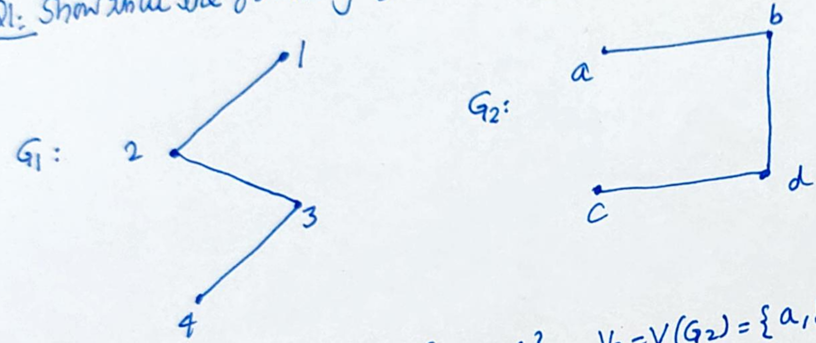
Two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are said to be isomorphic if there exists a function $f: V_1 \to V_2$ such that

(i) f is one-one and onto i.e. f is bijective

(ii) $\{a,b\}$ is an edge in E_1, if and only if $\{f(a), f(b)\}$ is an edge in E_2 for any two elements $a,b \in V_1$

The condition (ii) says that if vertices a and b are adjacent in G_1, then $f(a)$ and $f(b)$ are adjacent in G_2. In other words the function f preserves adjacency relationship and consequently the corresponding vertices in G_1 and G_2 will have the same degree. Any function f with the above properties is called an isomorphism between G_1 & G_2

Q1: Show that the following given pairs of graphs are isomorphic



Sol Here, we have $V_1 = V(G_1) = \{1,2,3,4\}$, $V_2 = V(G_2) = \{a,b,c,d\}$

$E_1 = E(G_1) = \{(1,2),(2,3),(3,4)\}$ and $E_2 = E(G_2) = \{(a,b),(b,d),(c,d)\}$

Hence $|V_1| = |V(G_1)| = 4 = |V_2|$ and $|E_1| = |E_2|$

degree of sequence in graph $G_1 = \{1,1,2,2\}$

degree of sequence in graph $G_2 = \{1,1,2,2\}$

Define a function $f: V(G_1) \to V(G_2)$ such that

$f(1)=a$, $f(2)=b$, $f(3)=d$ and $f(4)=c$

Clearly f is one-one and onto

Further

$\{1,2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a,b\} \in E(G_2)$

$\{2,3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b,d\} \in E(G_2)$

$\{3,4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d,c\} \in E(G_2)$

and $\{1,3\} \notin E(G_1)$ and $\{f(1), f(3)\} = \{a,d\} \notin E(G_2)$

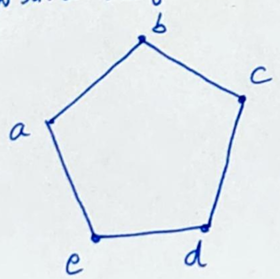
$\{1,4\} \notin E(G_1)$ and $\{f(1), f(4)\} = \{a,c\} \notin E(G_2)$

$\{2,4\} \notin E(G_1)$ and $\{f(2), f(4)\} = \{b,c\} \notin E(G_2)$

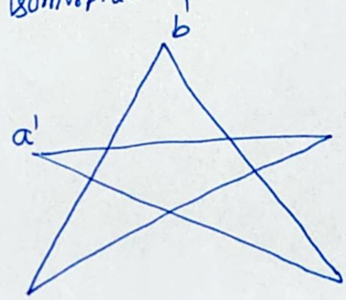
Hence f preserves adjacency as well as non-adjacency of the vertices. Therefore, G_1 and G_2 are isomorphic.

Q2: Show that the given pair of graphs are isomorphic

G_1:



G_2:



Sol Here $V(G_1) = \{a,b,c,d,e\}$, $V(G_2) = \{a',b',c',d',e'\}$

$\therefore |V(G_1)| = 5 = |V(G_2)|$

Also $|E(G_1)| = |E(G_2)|$

deg. of sequence (of G_1) $= \{2,2,2,2,2\}$

deg. of sequence of $G_2 = \{2,2,2,2,2\}$

Define a function $f: V(G_1) \to V(G_2)$ as

$f(a) = a'$, $f(b) = c'$, $f(c) = e'$, $f(d) = b'$ and $f(e) = d'$

clearly f is one-one and onto

Further

$\{a,b\} \in E(G_1)$ and $\{f(a), f(b)\} = \{a',c'\} \in E(G_2)$

$\{a,e\} \in E(G_1)$ and $\{f(a), f(e)\} = \{a',d'\} \in E(G_2)$

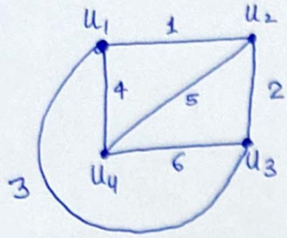
$\{b,c\} \in E(G_1)$ and $\{f(b), f(c)\} = \{c',e'\} \in E(G_2)$

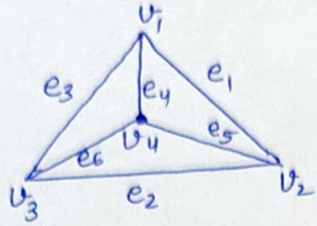
$\{c,d\} \in E(G_1)$ and $\{f(c), f(d)\} = \{e',b'\} \in E(G_2)$

$\{d,e\} \in E(G_1)$ and $\{f(d), f(e)\} = \{b',d'\} \in E(G_2)$

Hence f preserves adjacency of vertices. Therefore G_1 and G_2 are isomorphic

Q1: Show that the graphs given below are Isomorphic



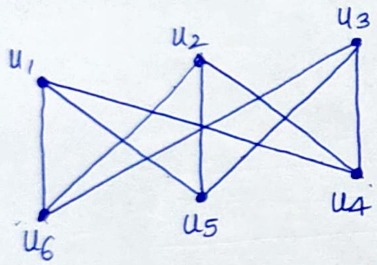


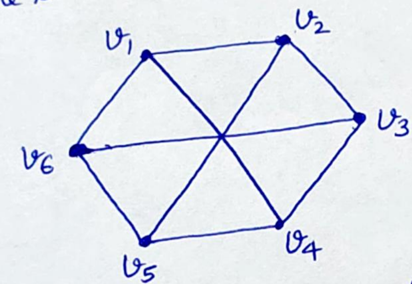
Sol: we can establish the correspondence between the two graphs as follows:

Vertex correspondence: The vertices u_1, u_2, u_3 and u_4 correspond to v_1, v_2, v_3 and v_4 respectively.

Edge correspondence: The edges 1, 2, 3, 4, 5 and 6 correspond to e_1, e_2, e_3, e_4, e_5 and e_6 respectively.

Q2: Show that the graphs given below are isomorphic:





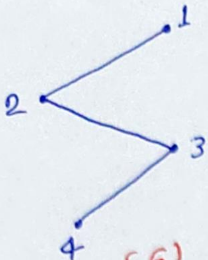
Sol: we define the correspondence between the vertex sets and edge sets as given below:

Vertex Correspondence: $u_1 \leftrightarrow v_1$, $u_2 \leftrightarrow v_3$, $u_3 \leftrightarrow v_5$, $u_4 \leftrightarrow v_2$, $u_5 \leftrightarrow v_4$ and $u_6 \leftrightarrow v_6$.

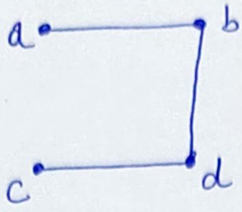
Edge correspondence: $(u_1, u_4) \leftrightarrow (v_1, v_2)$, $(u_1, u_5) \leftrightarrow (v_1, v_4)$, $(u_1, u_6) \leftrightarrow (v_1, v_6)$
$(u_2, u_4) \rightarrow (v_3, v_2)$, $(u_2, u_5) \leftrightarrow (v_3, v_4)$, $(u_2, u_6) \leftrightarrow (v_3, v_6)$
$(u_3, u_4) \leftrightarrow (v_5, v_2)$, $(u_3, u_5) \leftrightarrow (v_5, v_4)$, $(u_3, u_6) \leftrightarrow (v_5, v_6)$

Thus two graphs are isomorphic.

Q3: Show that the two graphs shown in the figure are isomorphic:



(G_1)



(G_2)

Sol **Vertex Correspondence:** $1 \leftrightarrow a,\ 2 \leftrightarrow b,\ 3 \leftrightarrow d,\ 4 \leftrightarrow c$

Edge Correspondence: $(1,2) \leftrightarrow (a,b),\ (2,3) \leftrightarrow (b,d),\ (3,4) \leftrightarrow (d,c)$

Hence the two graphs are isomorphic.

Note: We observe that $(1,2) \in E(G_1)$ and $(a,b) \in E(G_2)$

or $(1,2) \longleftrightarrow (a,b)$

$(2,3) \longleftrightarrow (b,d)$

$(3,4) \longleftrightarrow (d,c)$

(Preserves adjacency of vertices)

$(1,3) \notin E(G_1)$ and $(a,d) \notin E(G_2)$

$(1,4) \notin E(G_1)$ and $(a,c) \notin E(G_2)$

$(2,4) \notin E(G_1)$ and $(b,c) \notin E(G_2)$

(Preserves non-adjacency of vertices)

Important Note: If G_1 and G_2 are isomorphic graphs, then G_1 and G_2 have

(i) Same number of vertices

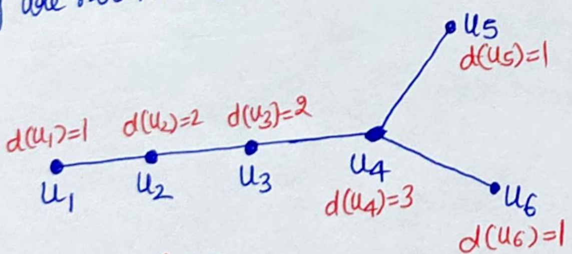
(ii) Same number of edges

(iii) Same degree sequence

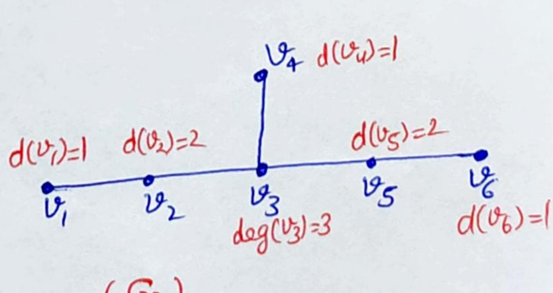
*** If any of these quantities differ in two graphs, they can not be isomorphic.

However these conditions are by no means sufficient.

**** For example: The two graphs shown in the figure satisfy three conditions, yet they are not isomorphic.



(G_1)



(G_2)

For graph (G_1)

(i) No. of vertices $= 6$

(ii) No. of edges $= 5$

(iii) degree sequence $= (1, 1, 1, 2, 2, 3)$

For graph (G_2)

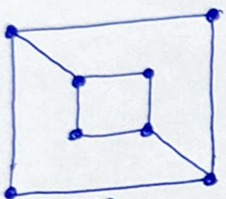
(ii) No. of vertices $= 6$

(i) No. of edges $= 5$

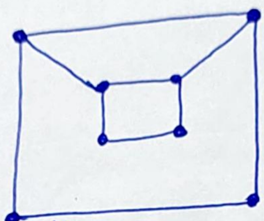
(iii) deg. of sequence $= (1, 1, 1, 2, 2, 3)$

If the two graphs G_1 and G_2 are to be isomorphic, then u_4 in G_1 must be correspondence to v_3 because there are no other vertices of degree three. Now in graph G_1 there are two pendant vertices u_5 and u_6 adjacent to u_4 while in graph (G_2) there is only one pendant vertex v_4, adjacent to v_3. Hence these two graphs are not isomorphic.

Q: Explain why the two graphs in the figure given below are not isomorphic:



(a)



(b)

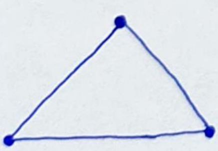
Sol

In graph (a), no vertices of degree two are adjacent while in the graph (b) vertices of degree two are adjacent. Since isomorphism preserves adjacency of vertices, the graphs are not isomorphic.

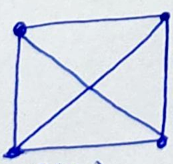
§. **Complete Graphs**: A simple graph G is said to be complete if every vertex in G is connected with every other vertex i.e. If G contains exactly one edge between each pair of distinct vertices.

If a complete graph G has n vertices, then it will be denoted by K_n. It should be noted that K_n has $\frac{n(n-1)}{2}$ edges and degree of each vertex is $(n-1)$.

Graphs K_3, K_4, K_5 and K_6 are shown below.

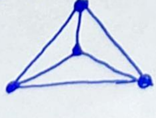


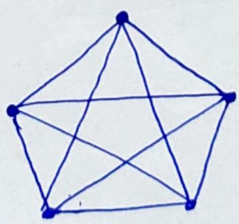
(a) Graph K_3



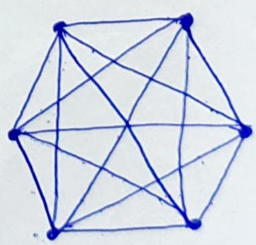
(K_4)

or,





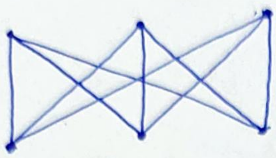
K_5



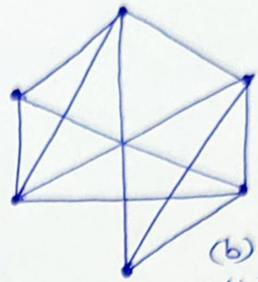
(K_6)

§. **Regular Graphs:** A graph G is called regular if each vertex in G has the same degree. If a graph G is regular with $d(v)=r$ for each vertex v in G, then G is called r-regular.

Example:

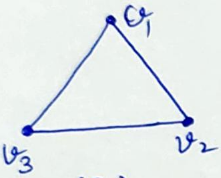


(a)
3-regular graph

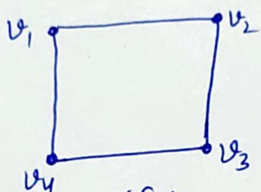


(b)
(4-regular graph)

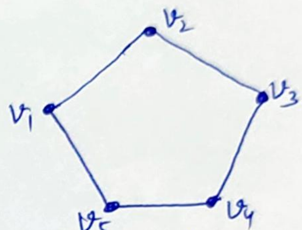
§. **Cycles:** The cycle C_n, $n \geq 3$ consists of n vertices $v_1, v_2, \ldots v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, ---- $\{v_{n-1}, v_n\}$ and $\{v_n, v_1\}$.



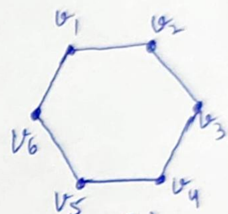
(C_3)



(C_4)



(C_5)

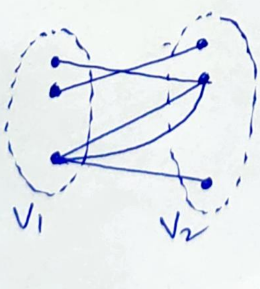


(C_6)

§. **Bipartite Graphs:**

A graph $G=(V,E)$ is called bipartite if its vertex set V can be decomposed into two disjoint subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2. In other words, a graph $G=(V,E)$ is bipartite if its vertex set V can be partitioned into two subsets V_1 and V_2 such that each edge in G has one end vertex in V_1 and another in V_2.

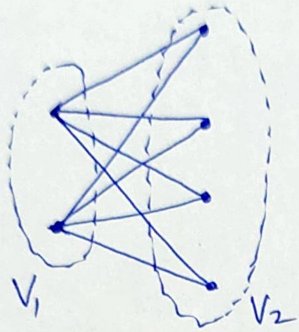
In other words, $G=(V,E)$ is bipartite if V can be written as the union of two disjoint sets V_1 and V_2 so that no two members of V_1 or V_2 are adjacent.



§. Complete Bipartite Graph

Let $G=(V,E)$ be a bipartite graph and let V_1 and V_2 be the partition of the vertex set V of G. The bipartite graph G is said to be complete bipartite if each vertex in V_1 is joined to each vertex in V_2 by just one edge. This graph is denoted by $K_{m,n}$ if V_1 has m vertices and V_2 has n vertices.

For example, the complete bipartite graph $K_{2,4}$ is shown below



(The complete bipartite Graph $K_{2,4}$)

Note that the complete bipartite graph $K_{m,n}$ has $m+n$ ~~edges~~ vertices and mn edges.

Q: Show that the maximum number of edges in a complete bipartite graph of n vertices is $\frac{n^2}{4}$.

Sol Let G be a complete bipartite graph with n vertices. Let n_1 and n_2 be the number of vertices in the partitions V_1 and V_2 of vertex set of G. Since G is complete bipartite, each vertex in V_1 is joined to each vertex in V_2 by exactly one edge. Thus G has $n_1 n_2$ edges where $n_1+n_2=n$.

But we know that the maximum value of $n_1 n_2$ subject to condition $n_1+n_2=n$ is $\frac{n^2}{4}$. Thus the number of edges in G is $\frac{n^2}{4}$.

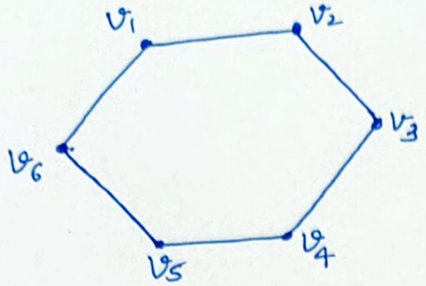
GATE 2014

Q: The maximum number of edges in a bipartite graph on 12 vertices is ___

(a) 36 (b) 48 (c) 12 (d) 24

Q: Show that C_6 is a bipartite graph.

Sol: We know that C_6 is given by



$\therefore$ Vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

The vertex set V can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.

Therefore the graph C_6 is bipartite

§. **Walk:** A walk in a graph G is a finite alternating sequence

$$v_0 - e_1 - v_1 - e_2 - v_2 - e_3 \cdots\cdots - e_n - v_n$$

of vertices and edges of the graph such that each edge e_i in the sequence joins vertices v_{i-1} and v_i, $1 \leq i \leq n$. The end vertices v_0 and v_n are the end or terminal vertices of the walk. The vertices $v_1, v_2, \ldots v_{n-1}$ are called internal vertices. The integer n, the number of edges in the walk is called in length of the walk.

✓ A walk is called open when the terminal vertices are distinct

✓ For the same end terminal vertices, it is termed as closed.

Note that a walk may repeat both vertices and edges.

Special Types of Walk:

(1) A walk is called a trail if all its edges are distinct. A trail is open or closed depends on whether its end vertices are distinct or not.

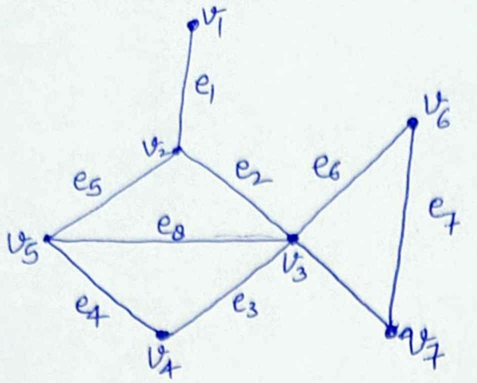
(2) A closed trail is called a circuit.

(3) A walk is called a path if all its vertices and edges are distinct.

(4) A path in which only repeated vertex is the first vertex is called a cycle to describe such a closed path.

	Repeated Edge	Repeated Vertex	Starts and Ends ~~are~~ at same points?
Walk (open)	allowed	allowed	No
Walk (Closed)	allowed	allowed	Yes
Trail	No	allowed	No
Circuit	No	allowed	Yes
Path	No	no	NO
Cycle	NO	first and last only	Yes

For example: In the graph given below



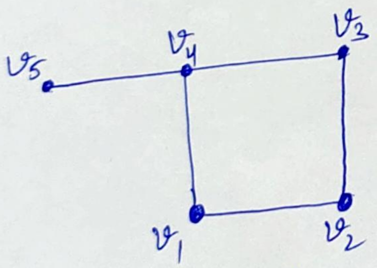
(i) the sequence $v_1-e_1-v_2-e_5-v_5-e_8-v_3-e_3-v_4-e_4-v_5-e_5-v_2-e_2-v_3-e_6-v_6$ walk of length 8. It contains repeated vertices v_2, v_3 and v_5 and repeated edge e_5.

(ii) The sequence $v_1-e_1-v_2-e_5-v_5-e_8-v_3-e_3-v_4-e_4-v_5$ is a trail. It contains repeated vertex v_5 but does not contain repeated edge.

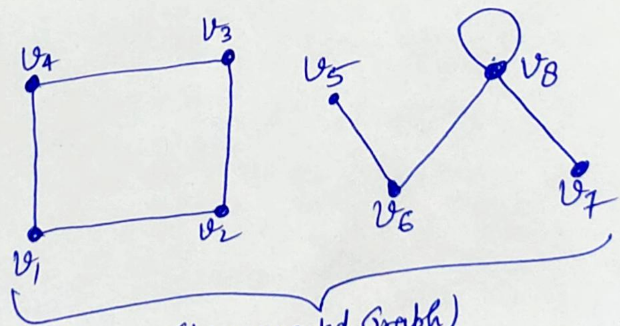
(iii) The sequence $v_1-e_1-v_2-e_5-v_5-e_8-v_3-v_4$ is a path. It does not contain repeated vertex and repeated edge.

(iv) The sequence $v_2-e_2-v_3-e_3-v_4-e_4-v_5-e_5-v_2$ is a cycle. It does not contain repeated vertex and repeated edge except the first and last vertex.

f. Connected Graph: A graph G is said to be connected if there exists a path between every pair of vertices in G. A graph which is not connected is called a disconnected graph. Thus a graph G is disconnected if we can find a pair v_i, v_j of vertices in G such that there is no path between v_i and v_j.



(Connected Graph)



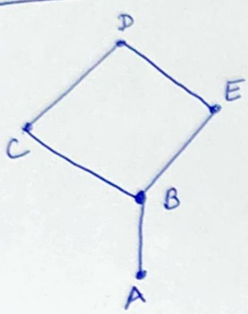
(disconnected Graph)

§. Euler Path: A path in a graph G is called Euler path if it includes every edge exactly once. Since the path contains every edge exactly once, it is also called Euler trail.

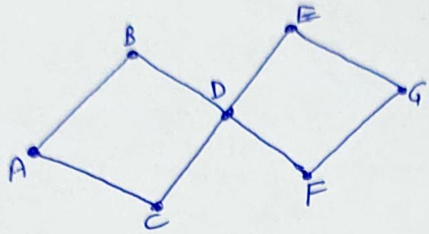
§. Euler circuit: An Euler path that is circuit is called Euler Circuit.
(A closed Eulerian trail that uses each edge exactly once is called Eulerian circuit)

§. Eulerian Graph: A graph which has an Eulerian circuit is called an Eulerian graph.

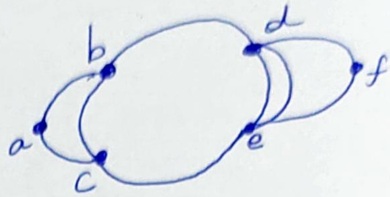
Examples:



(a)



(b)



(C)

The graph of Figure (a) has an Euler path but no Euler circuit. Note that two vertices A and B are of degree 1 and 3 respectively. This means that AB can be used to either arrive at vertex A or leave vertex A but not for both. Thus, an Euler path can be found if we start either from vertex A or from B. ABCDEB and BCDEBA are two Euler paths. Starting from any vertex no Euler Circuit can be found.

The graph of figure (b) has both Euler circuit and Euler path. ABDEGFDCA is an Euler path and circuit. Note that all vertices are of even degree.

No Euler path and circuit is possible in figure (C). Note that all vertices are not even degree and more than two vertices are of odd degree.

The existence of Euler path and circuit depends on the degree of vertices.

Result :(i) A nonempty connected graph G is Eulerian if and only if its vertices are all of even degree.

(ii) Euler graphs are connected.

Note: To determine whether a graph G (Connected) has an Euler circuit; we note the following points:

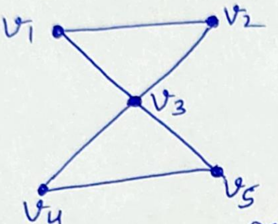
(1) List the degree of all vertices in the graph.

(2). If any value is zero, the graph is not connected and hence it can not have Euler path or Euler circuit

(3) If all the degrees are even, then G has both Euler path and Euler circuit

(4) If exactly two vertices are odd degree, then G has Euler path but no Euler Circuit

Example 1 Let G be a graph as shown below, Verify that G has an Eulerian circuit



Sol We observe that G is connected and all the vertices are having even degree.

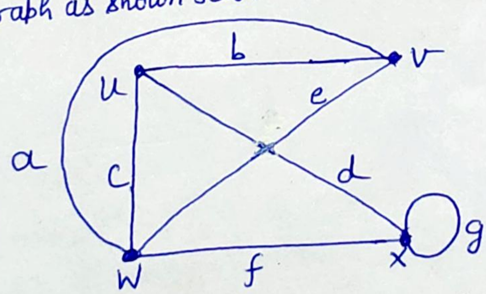
$d(v_1)=2$, $d(v_2)=2$, $d(v_3)=4$, $d(v_4)=2$, $d(v_5)=2$

Thus G has an Eulerian Circuit. By inspection, we find the Eulerian circuit

$$v_1 - v_3 - v_5 - v_4 \rightarrow v_3 - v_2 - v_1$$

Example 2: Show that the graph as shown below has no Eulerian circuit but has an Eulerian trail.

Sol



Here, $deg(u)=deg(v)=3$ and $deg(w)=4$, $deg(x)=4$. Since u and v ~~have~~ are only two vertices of odd degree; therefore the graph does not contain Euler circuit but the path $b-a-c-d-g-f-e$ is an Eulerian trail.

§. Hamiltonian Graphs

1. Hamiltonian path: A Hamiltonian path is a simple path that contains all vertices of G where the end points may be distinct.

2. Hamiltonian Circuit: A circuit in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice is known as Hamiltonian circuit

3. Hamiltonian Graph: A graph G is called a Hamiltonian Graph if it contain a Hamiltonian circuit.

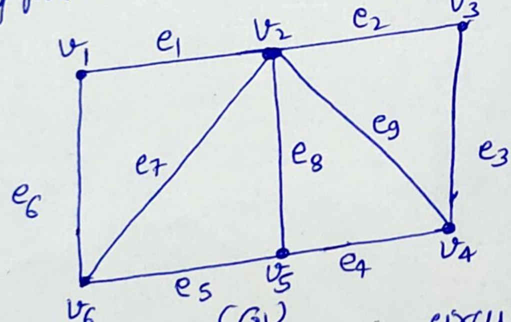
Example: The complete graph K_n $(n \geq 3)$ is Hamiltonian.

Dirac's theorem: Let G be a graph with $n \geq 3$ vertices. If $\deg(v) \geq n/2$ for all vertices v of G, then G is Hamiltonian.

Give an example of a graph which is Hamiltonian but not Eulerian

****Q: Given an example of a graph which is Hamiltonian but non-Eulerian and vice-versa.

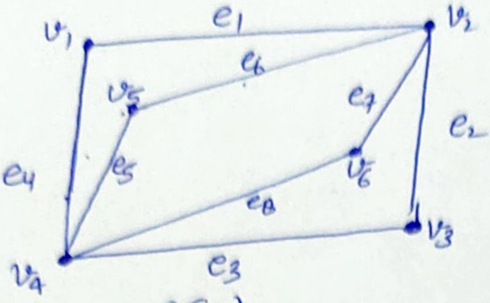
Sol The following graph G_1 as shown below is Hamiltonian but non-Eulerian



(G_1)

The graph contains a Hamiltonian circuit $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_6 e_6 v_1$. Since the degree of each vertex is not even the graph is non-Eulerian

The graph G_2 shown below is Eulerian but not Hamiltonian



(G_2)

The graph is Eulerian since the degree of each vertex is even. It does not contains Hamiltonian circuit. This can be seen by noting that any circuit containing every vertex must contain a vertex twice except starting vertex and ending vertex.

Result: (1) If G is a simple graph with n vertices and e edges with $n \geq 3$ and

$$e \geq \frac{1}{2}(n^2-3n+6)$$

then G is Hamiltonian.

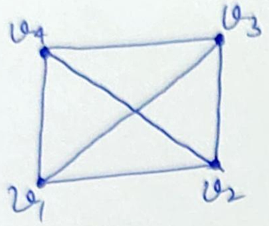
(2) If G is a bipartite graph with odd number of vertices then G is not Hamiltonian.

(3) A graph with n vertices and with no-loops and parallel edges which has at least $\frac{1}{2}(n-1)(n-2)+2$ edges, is Hamiltonian.

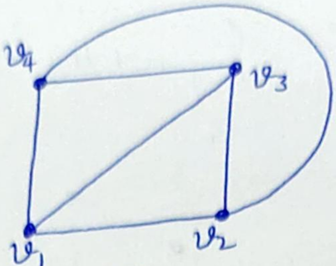
§. Planar Graphs:

A graph G is said to be planar if the graph can be drawn in a plane so that no edges cross except at vertices.

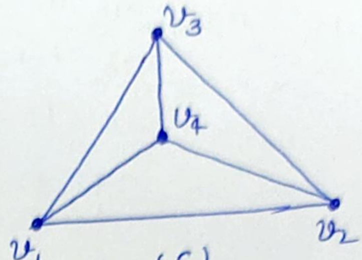
For example, the complete graph with four vertices is usually represented by as in Fig (A). But this graph can also be drawn with non crossing of edges as in Fig (B) or Fig (C)



(A)



(B)



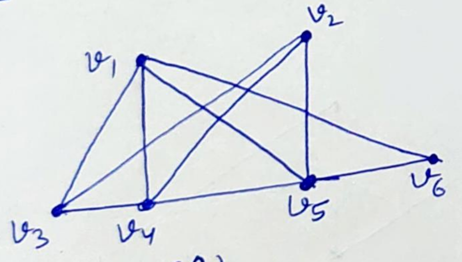
(C)

Hence the complete graph Graph with four verties is planar

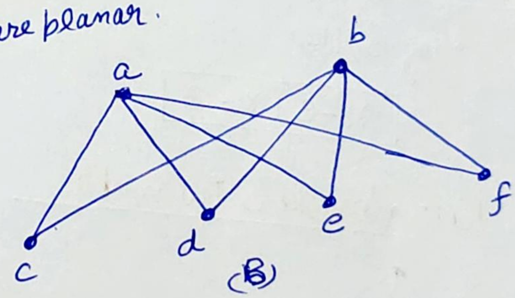
A graph G that cannot be drawn on a plane without crossings of its edges is called non-planar. A drawing of a planar graph G on a ~~paper~~ plane or surface such that no edges intersect is called embedding of G.

In order to show that a graph is non-planar, we have to show that it is not possible to draw the graph on a plane without crossings of edges.

Q:1 Show that the graphs given below are planar.

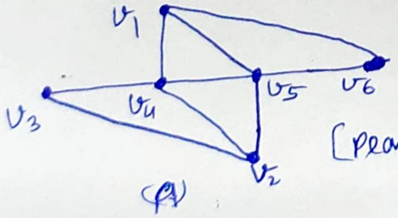


(A)



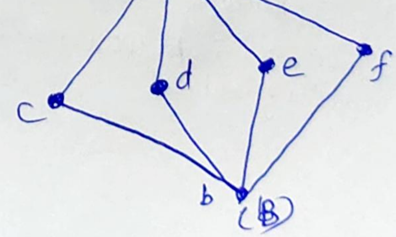
(B)

Sol The graphs (A) & (B) are planar because we can draw these graphs in a plane without crossing of edges as shown below:



[Planar embadding of (A)]

(A)

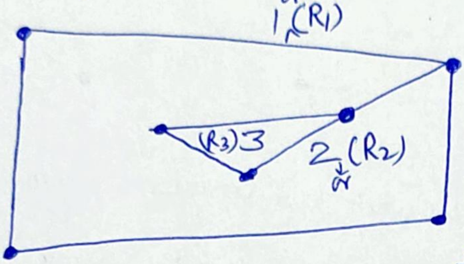


(B)

§. Region of a Graph:

Any planar graph partitions the plane into number of disjoint regions called one of which is infinite. A region is characterised by the cycle that forms its boundary. If the area of the region is finite then the region is called a finite region. If the region is infinite it is called infinite region. A planar graph has only one infinite region

We can define the degree of a region of a planar graph to be the number of encounters with edges in a walk round the boundary of the region.

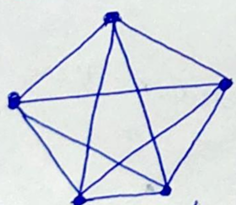


The above figure has 3 regions, two are finite and one is infinite. The infinite region 1 has degree 4, the region 3 has degree 3 and region 2 has degree 9 (Nine). (Note that each edge is encountered twice, once on each side)

§. Euler's Formula If a connected planar graph G has n vertices, e edges and r regions, then

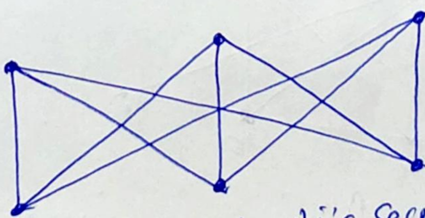
$$\boxed{n - e + r = 2}$$

Kuratowski's Two Graphs



(A) K_5, Kuratowski's first Graph

(Non-Planar Graph with smallest number of vertices)



(B) $K_{3,3}$ (Kuratowski's Second graph)

(Non-Planar Graph with smallest number of edges)

Results:

(i) If G is a connected simple planar graph with $n\ (\geq 3)$ vertices and e edges, then

$$e \leq 3n-6$$

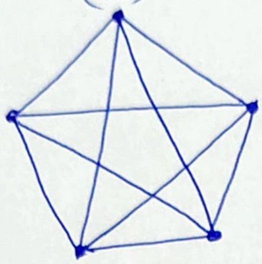
Here

(ii) If G is a connected simple planar graph with $n \geq 3$ vertices and e edges and no circuits of length 3, then $e \leq 2n-4$. (Only a necessary condition not sufficient)

Q:1 Show that the graph K_5 is not coplanar. (non-planar)

Sol Since K_5 is a simple graph, the smallest possible length for any cycle of K_5

(K_5)



is three. We shall suppose that the graph is planar. The graph has 5 vertices and 10 edges so that $n=5$, $e=10$.

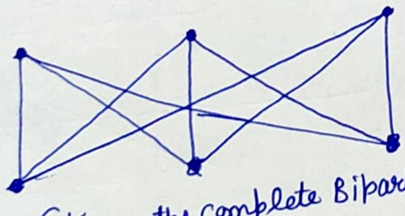
Now we have

$$3n-6 = 15-6 = 9 < 10$$

Thus the graph violates the inequality $e \leq 3n-6$ and hence it is not coplanar

Q2 Show that the graph $K_{3,3}$ is not coplanar

Sol



($K_{3,3}$, the complete Bipartite Graph)

Since $K_{3,3}$ has no circuits of length 3 (it is bipartite) and has 6 vertices and 9 edges i.e. $n=6$ and $e=9$ so that $2n-4 = 12-4 = 8$

Hence the inequality $e \leq 2n-4$ does not satisfy and the graph is not coplanar

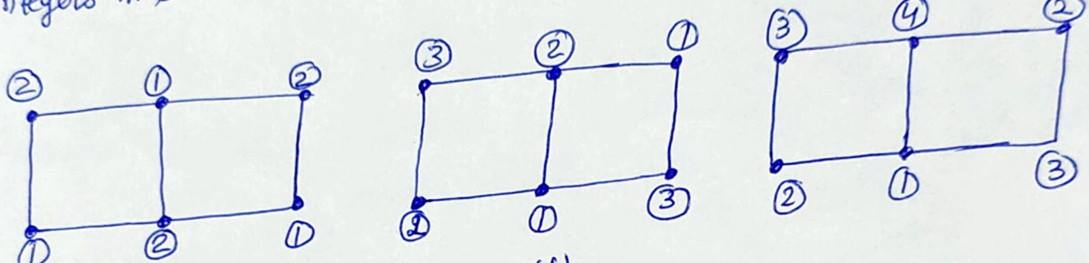
§. Graph Coloring:

The assign of colors to the verties of G, one color to each vertex, so that adjacent vertices are assigned different colors is called the proper coloring of G or simply vertex coloring. The n-coloring of G is a coloring of G using n-colors. If G has n-coloring, then G is said to be n-colorable.

Chromatic Number:

The chromatic number of a graph G is the minimum number of colors to color the vertices of the graph G and is denoted by $\chi(G)$. Thus, a graph G is n-colorable if $\chi(G) < n$.

The graph of Figure below of n coloring for $n = 2, 3, 4$ are displayed, with positive integers in small circles designating the colors.



(a)

(b)

There is no easy way of finding $\chi(G)$ of a graph G. The following rules may be helpful in finding $\chi(G)$.

(1) A graph consisting of only isolated vertices is 1-chromatic.

(2) $\chi(G) \leq |V|$, where $|V|$ is the number of vertices of G.

(3) If a subgraph of G requires m colors then $\chi(G) \geqslant m$.

(4) If the degree of a vertex of G is d, then atmost d colors are required to color vertices adjacent to it.

(5) Every K-chromatic graph has at least K vertices v such that degree$(v) \geqslant K-1$