Finanical Portfolio Optimization Report

1. Problem Statement and Motivation

The goal of this project is to construct an optimal financial portfolio using the Markowitz Mean–Variance framework. Given historical stock data, we estimate expected returns and the covariance matrix, then solve a constrained optimization problem to determine the optimal asset weights.

We aim to:

- Minimize portfolio risk (variance)
- Achieve a target expected return
- Satisfy real-world constraints (budget, no short-selling)

This problem is essential in modern portfolio theory and demonstrates how convex optimization tools (OSQP, ECOS, SCS) solve the KKT system numerically to find optimal allocations.

2. Mathematical Formulation

Consider n assets with random return vector:

$$\mathbf{R} = (R_1, R_2, \ldots, R_n)^\top$$

Mean return vector:

$$\boldsymbol{\mu} = \mathbb{E}[\mathbf{R}] = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$$

Covariance matrix:

$$\Sigma = \mathrm{Cov}(\mathbf{R}) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}$$

Portfolio weights:

$$\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$$

Objective Function

We minimize variance with optional L2 regularization:

$$\min_{\mathbf{w}} \left(\mathbf{w}^\top \Sigma \mathbf{w} + \lambda \|\mathbf{w}\|_2^2 \right)$$

where λ is the L2-regularization strength.

Constraints

$$\mathbf{1}^\top \mathbf{w} = 1 \qquad \text{(budget constraint)} \tag{1}$$

$$\boldsymbol{\mu}^\top \mathbf{w} \geq R_{\text{target}} \qquad \text{(target return)} \tag{2}$$

$$w_i \geq 0 \quad \forall i \qquad \text{(no short-selling)} \tag{3}$$

3. Lagrangian Formulation

We convert inequalities to the standard form $g(\mathbf{w}) \leq 0$:

$$g_1(\mathbf{w}) = R_{\text{target}} - \boldsymbol{\mu}^\top \mathbf{w} \leq 0$$
$$g_{2,i}(\mathbf{w}) = -w_i \leq 0$$

Let:

$$\nu \in \mathbb{R} \quad \text{(equality multiplier)}$$
$$\gamma \geq 0 \quad \text{(return constraint multiplier)}$$
$$s_i \geq 0 \quad \text{(no-short multipliers)}$$

Lagrangian

$$\mathcal{L}(\mathbf{w}, \nu, \gamma, \mathbf{s}) = \mathbf{w}^\top (\Sigma + \lambda I) \mathbf{w} + \nu(\mathbf{1}^\top \mathbf{w} - 1) + \gamma(R_{\text{target}} - \boldsymbol{\mu}^\top \mathbf{w}) - \mathbf{s}^\top \mathbf{w}.$$

4. KKT Conditions

The optimal portfolio must satisfy the following KKT conditions.

1. Stationarity

$$\nabla_{\mathbf{w}} \mathcal{L} = 2(\Sigma + \lambda I)\mathbf{w} + \nu \mathbf{1} - \gamma \boldsymbol{\mu} - \mathbf{s} = \mathbf{0}$$

2. Primal Feasibility

$$\mathbf{1}^\top \mathbf{w} = 1$$
$$R_{\text{target}} - \boldsymbol{\mu}^\top \mathbf{w} \leq 0$$
$$w_i \geq 0 \quad \forall i$$

3. Dual Feasibility

$$\gamma \geq 0, \qquad s_i \geq 0 \; \forall i$$

4. Complementary Slackness

$$\gamma \left(R_{\text{target}} - \boldsymbol{\mu}^\top \mathbf{w} \right) = 0$$
$$s_i w_i = 0 \qquad \forall i$$

This implies:

- If $w_i > 0$ then $s_i = 0$
- If $s_i > 0$ then $w_i = 0$

5. Solver Methodology

We use three convex optimization solvers:

- **OSQP** – A fast and stable solver specifically designed for quadratic programs. Ideal for portfolio optimization because it handles covariance matrices and linear constraints very reliably.
- **ECOS** – A general conic solver (SOCP) that provides accurate solutions for many convex problems. Works well but may sometimes return "inaccurate solution" warnings on ill-conditioned data.
- **SCS** –A first–order iterative solver that is extremely scalable and can solve very large problems. It is fast but provides lower-precision, approximate solutions.

These solvers internally attempt to satisfy the KKT optimality conditions using numerical iterative methods.

6. Results and Analysis

Weight Distributions

After solving the optimization problem, we obtain the optimal weight vector

$$\mathbf{w}^* = (w_1^*,\, w_2^*, \ldots, w_n^*)^\top,$$

which specifies how capital should be allocated across the selected assets to achieve the best possible risk–return trade-off. In other words, $\mathbf{w}^*$ tells us how to distribute our investment so that we obtain the maximum return for the minimum achievable risk under the given constraints.

Efficient Frontier

For each look-back window (3y, 5y, 10y) we compute the efficient frontier by solving:

$$\min_{\mathbf{w}}\ \mathbf{w}^\top \Sigma \mathbf{w} \quad \text{s.t. } \mathbf{1}^\top \mathbf{w} = 1,\ \boldsymbol{\mu}^\top \mathbf{w} \geq R_{\text{target}},\ w_i \geq 0.$$

This produces a set of $(\sigma_p,\ r_p)$ pairs where:

$$r_p = \boldsymbol{\mu}^\top \mathbf{w} \quad \text{and} \quad \sigma_p = \sqrt{\mathbf{w}^\top \Sigma \mathbf{w}}.$$

Best-Sharpe Portfolio

We identify the portfolio with maximum Sharpe ratio:

$$\text{Sharpe} \ = \ \frac{r_p - r_f}{\sigma_p}$$

where r_f is the risk-free rate. For each unit of risk we take, how much excess return do we get above the risk-free asset?

AUC – Area Under the Frontier

We compute the area under the risk–return curve as:

$$\text{AUC} \ = \ \int_{\sigma_{\min}}^{\sigma_{\max}} r_p(\sigma)\, \mathrm{d}\sigma \approx \sum_i (r_{p,i})\, \Delta\sigma_i$$

A larger AUC indicates a frontier that offers higher returns for given risk levels — a stronger return-for-risk trade-off.

Condition Number of the Covariance Matrix

The stability of the portfolio optimization depends strongly on the conditioning of the covariance matrix Σ. Its condition number is defined as:

$$\kappa(\Sigma) = \frac{\lambda_{\max}}{\lambda_{\min}},$$

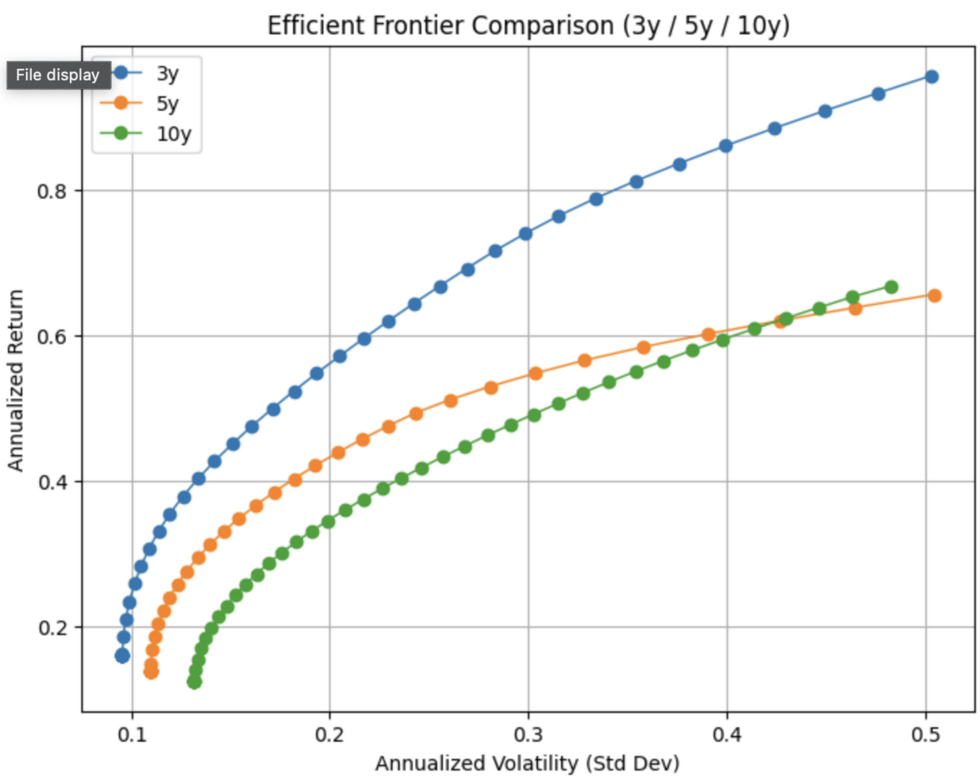
where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of Σ, respectively.

A high condition number indicates that Σ is close to singular, meaning the assets are highly correlated and the optimization becomes sensitive to small changes in the data. A low condition number indicates a well-conditioned and numerically stable covariance matrix.

7. Visualization

In this section we present the graphical output of our Markowitz portfolio analysis

Efficient Frontier Comparison Across Windows

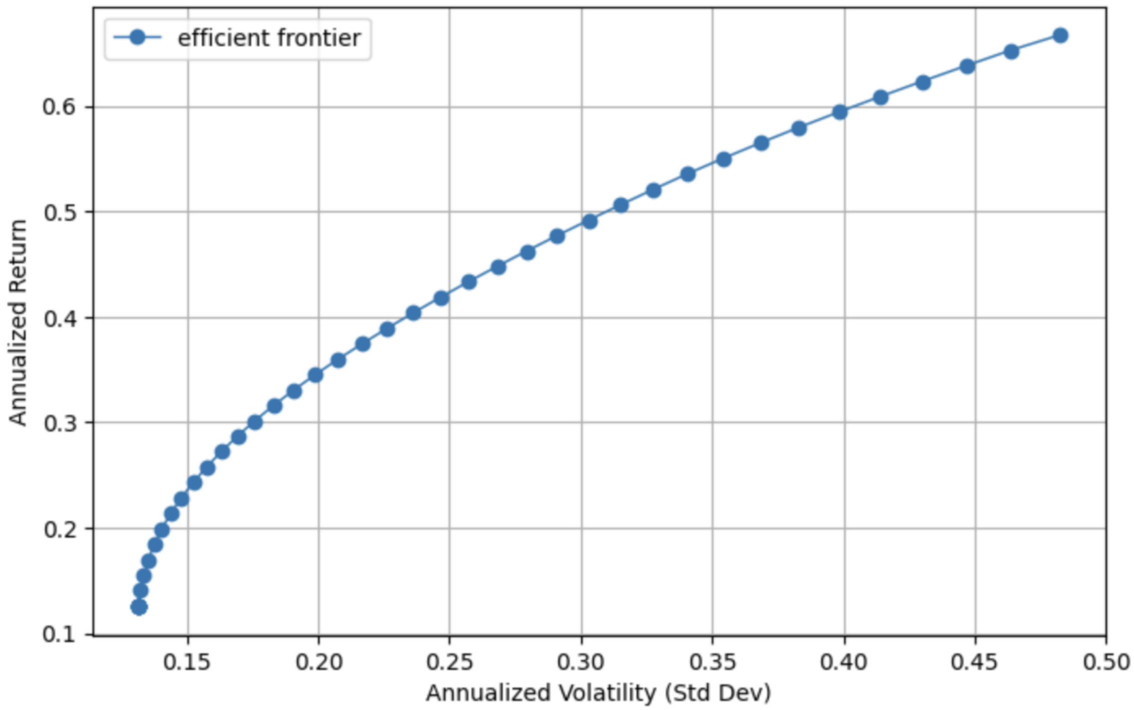


Each curve represents the set of optimal portfolios for a given historical window, plotted in *risk–return space*. For each volatility σ_p (horizontal axis), the solver finds the portfolio that maximizes expected return r_p. Mathematically, each point satisfies:

$$(\sigma_p, r_p) = \left(\sqrt{\mathbf{w}^\top \Sigma \mathbf{w}},\ \boldsymbol{\mu}^\top \mathbf{w}\right).$$

A curve with **higher returns for the same risk** is more efficient. Here, the 3-year frontier dominates the others, giving the largest AUC (area under curve).

Single-Window Efficient Frontier

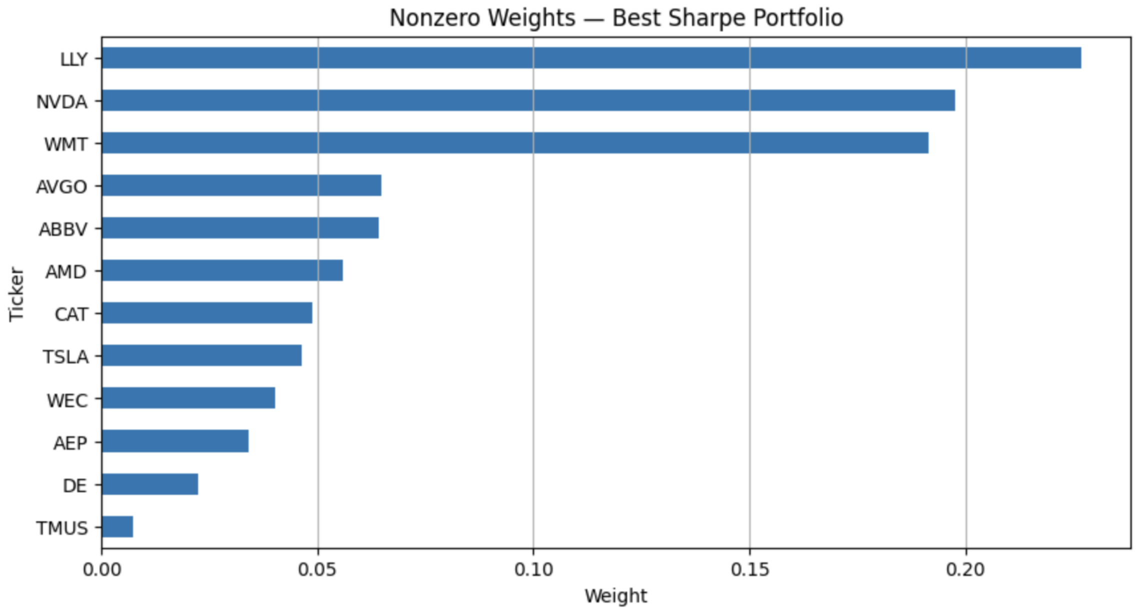


This curve illustrates the tradeoff between risk and expected return within one specific lookback window. The frontier is obtained by solving:

$$\min_{\mathbf{w}} \mathbf{w}^\top \Sigma \mathbf{w} \quad \text{s.t.} \quad \mathbf{1}^\top \mathbf{w} = 1, \quad \boldsymbol{\mu}^\top \mathbf{w} \geq R_{\text{target}}, \quad w_i \geq 0.$$

As R_{target} increases, the optimizer selects portfolios with higher expected return at the cost of higher volatility.

Best-Sharpe Portfolio Weights



This bar plot shows all assets with strictly positive weights in the final best–Sharpe portfolio.

Large bars indicate assets that significantly contribute to expected return. Small but nonzero weights indicate assets selected for diversification benefits.

9. Conclusion

This study presents a complete Markowitz framework with modern regularization, shrinkage, and solver diagnostics. The KKT conditions ensure theoretical correctness, while practical solvers enable stable computation even under noisy market data.