Note that in case when all sets B_i have exactly 2 elements (i.e. $b = 2$), the Hitting Set problem is equivalent to the Vertex Cover problem (two-element sets B_i correspond to edges). In the chapter we saw two approximation algorithm for Vertex Cover; here we generalize the one based on linear programming to arbitrary b.

Consider the following problem for Linear Programming:

$$\begin{array}{lll} \text{Min} & \sum\limits_{i=1}^{n} w_i x_i & \\ \text{s.t.} & 0 \le x_i \le 1 & \text{for all } i = 1, \ldots, n \\ & \sum\limits_{i:a_i \in B_j} x_i \ge 1 & \text{for all } j = 1, \ldots, m \text{ (all sets are hit)} \end{array}$$

Let x be the solution of this problem, and w_{LP} is a value of this solution (i.e. $w_{LP} = \sum_{i=1}^{n} w_i x_i$).

Now define the set S to be all those elements where $x_i \ge 1/b$:

$$S = \{a_i \mid x_i \ge 1/b\}$$

(1) *S is a hitting set.*

Proof. We want to prove that any set B_j intersects with S. We know that the sum of all x_i where $a_i \in B_j$ is at least 1. The set B_j contains at most b elements. Therefore some $x_i \ge 1/b$, for some $a_i \in B_j$. By definition of S, this element $a_i \in S$. So, B_j intersects with S by a_i. ∎

(2 *The total weight of all elements in S is at most $b \cdot w_{LP}$.*

Proof. For each $a_i \in S$ we know that $x_i \ge 1/b$, i.e., $1 \le bx_i$. Therefore

$$w(S) = \sum_{a_i \in S} w_i \le \sum_{a_i \in S} w_i \cdot bx_i \le b \sum_{i=1}^{n} w_i x_i = bw_{LP}$$

∎

(3 *Let S^* be the optimal hitting set. Then $w_{LP} \le w(S^*)$.*

Proof. Set $x_i = 1$ if a_i is in S^*, and $x_i = 0$ otherwise. Then the vector x satisfy constrains of our problem for Linear Programming:

$$\begin{array}{ll} 0 \le x_i \le 1 & \text{for all } i = 1, \ldots, n \\ \sum\limits_{i:a_i \in B_j} x_i \ge 1 & \text{for all } j = 1, \ldots, m \text{ (because all sets are hit)} \end{array}$$

Therefore the optimal solution is not worse that this particular one. That is,

$$w_{LP} \le \sum_{i=1}^{n} w_i x_i = \sum_{a_i \in S} w_i = w(S^*)$$

∎

Therefore we have a hitting set S, such that $w(S) \le b \cdot w(S^*)$.

[1] ex53.496.888