Let us call this problem Trade. Trade is in NP, since a pair of subsets  $A_i$  and  $A_j$  can serve as a certificate. We can verify that

- $\sum_{a \in A_i} v_i(a) > \sum_{a \in A_i} v_i(a)$
- $\sum_{a \in A_i} v_j(a) > \sum_{a \in A_j} v_j(a)$

by summing up the valuation of each person, which is clearly polynomial time doable.

We will prove *Trade* is NP-complete by reduction from *Subset Sum* problem. Suppose we have an instance of *Subset Sum*, that is, n integers  $w_1, w_2, \ldots, w_n$ , and another integer W. The goal is to find a subset  $S \subseteq w_1, w_2, \ldots, w_n$ , s.t.  $\sum_{w_k \in S} w_k = W$ .

Construct an instance of Trade as follows: there are n+1 objects  $a_1, a_2, \ldots, a_{n+1}, p_i$  possesses objects  $a_1, a_2, \ldots, a_n$ , and  $p_j$  possesses  $a_{n+1}$ . The first n objects are corresponding to the n numbers in  $Subset\ Sum$  problem, and the valuations for them are  $v_i(a_k) = v_j(a_k) = w_k$   $(k = 1, 2, \ldots, n)$ . The valuations of the last object are  $v_i(a_{n+1}) = W + 1$ , and  $v_j(a_{n+1}) = W - 1$ .

Since  $p_j$  only possesses a single object  $a_{n+1}$ , so if there exist  $A_i$  and  $A_j$ , then  $A_j$  must be  $\{a_{n+1}\}$ .  $A_i$  will be a subset of  $\{a_1, a_2, \ldots, a_n\}$ .

Now we will prove that if there is a subset of numbers that sums up to W, if and only if there is a subset  $A_i \subseteq \{a_1, a_2, \ldots, a_n\}$ , together with  $A_j = \{a_{n+1}\}$ , satisfying the two inequalities stated before.

If there is a subset S for Subset Sum problem, s.t.  $\sum_{k \in S} w_k = W$ , then we let

$$A_i = \{a_k : w_k \in S\}$$

According to our construction:

$$\sum_{a \in A_j} v_i(a) = v_i(a_{n+1}) = W + 1 > W = \sum_{a \in A_i} v_i(a)$$

and

$$\sum_{a \in A_i} v_j(a) = W > W - 1 = v_j(a_{n_1}) = \sum_{a \in A_j} v_j(a)$$

so  $A_i$  and  $A_j$  satisfy the requirement of Trade problem.

If there is a subset  $A_i \subseteq \{a_1, a_2, \dots, a_n\}$ , and  $A_j = \{a_{n+1}\}$  satisfying the two inequalities, then we will let

$$S = \{w_k : a_k \in A_i\}$$

According to our construction,  $\sum_{a \in A_i} v_j(a) > W-1$ ,  $\sum_{a \in A_i} v_i(a) < W+1$ , and  $\sum_{w_k \in S} w_k = \sum_{a \in A_i} v_i(a) = \sum_{a \in A_i} v_j(a)$ , so  $\sum_{w_k \in S} w_k = W$ .

 $<sup>^{1}</sup>$ ex508.142.105