

Here is a greedy algorithm for this problem. Start at the western end of the road and begin moving east until the first moment when there's a house  $h$  exactly four miles to the west. We place a base station at this point (if we went any farther east without placing a base station, we wouldn't cover  $h$ ). We then delete all the houses covered by this base station, and iterate this process on the remaining houses.

Here's another way to view this algorithm. For any point on the road, define its *position* to be the number of miles it is from the western end. We place the first base station at the easternmost (i.e. largest) position  $s_1$  with the property that all houses between 0 and  $s_1$  will be covered by  $s_1$ . In general, having placed  $\{s_1, \dots, s_i\}$ , we place base station  $i + 1$  at the largest position  $s_{i+1}$  with the property that all houses between  $s_i$  and  $s_{i+1}$  will be covered by  $s_i$  and  $s_{i+1}$ .

Let  $S = \{s_1, \dots, s_k\}$  denote the full set of base station positions that our greedy algorithm places, and let  $T = \{t_1, \dots, t_m\}$  denote the set of base station positions in an optimal solution, sorted in increasing order (i.e. from west to east). We must show that  $k = m$ .

We do this by showing a sense in which our greedy solution  $S$  “stays ahead” of the optimal solution  $T$ . Specifically, we claim that  $s_i \geq t_i$  for each  $i$ , and prove this by induction. The claim is true for  $i = 1$ , since we go as far as possible to the east before placing the first base station. Assume now it is true for some value  $i \geq 1$ ; this means that our algorithm's first  $i$  centers  $\{s_1, \dots, s_i\}$  cover all the houses covered by the first  $i$  centers  $\{t_1, \dots, t_i\}$ . As a result, if we add  $t_{i+1}$  to  $\{s_1, \dots, s_i\}$ , we will not leave any house between  $s_i$  and  $t_{i+1}$  uncovered. But the  $(i + 1)^{\text{st}}$  step of the greedy algorithm chooses  $s_{i+1}$  to be *as large as possible* subject to the condition of covering all houses between  $s_i$  and  $s_{i+1}$ ; so we have  $s_{i+1} \geq t_{i+1}$ . This proves the claim by induction.

Finally, if  $k > m$ , then  $\{s_1, \dots, s_m\}$  fails to cover all houses. But  $s_m \geq t_m$ , and so  $\{t_1, \dots, t_m\} = T$  also fails to cover all houses, a contradiction.

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<sup>1</sup>ex198.453.676