Checking whether G is 1- or 2-colorable is easy. For $k = 3, 4, \ldots, w + 1$, we test whether G is k-colorable by dynamic programming. We use notation similar to what we used in the Maximum-Weight Independent Set problem for graphs of bounded tree-width. Let $(T, \{V_t : t \in T\})$ be a tree decomposition of G. For the subtree rooted at t, and every coloring χ of V_t using the color set $\{1, 2, \ldots, k\}$, we have a predicate $q_t(\chi)$ that says whether there is a k-coloring of G_t that is equal to χ when restricted to V_t. This requires us to maintain $k^{(w+1)} \leq (w + 1)^{(w+1)}$ values for each piece of the tree decomposition.

We compute the values $q_t(\chi)$ when t is a leaf by simply trying all possible colorings of G_t. In general, suppose t has children $t_1, \ldots, t_d$, and we know the values of $q_{t_i}(\chi)$ for each choice of t_i and χ. Then there is a coloring of G_t consistent with χ on V_t if and only if there are colorings of the subgraphs $G_{t_1}, \ldots, G_{t_d}$ that are consistent with χ on the parts of V_{t_i} that intersect with V_t. Thus we set $q_t(\chi)$ equal to *true* if and only if there are colorings χ_i of V_{t_i} such that $q_{t_i}(\chi_i) = true$ and χ_i is the same as χ when restricted to $V_t \cap V_{t_i}$.

[1] ex897.854.812