Given a proposed solution to domain decomposition, we can test each domain in turn to see whether every node has a path to every other node. (This can be done very efficiently by testing for strong connectivity.) Thus *Domain Decomposition* is in NP.

We now show that *Three-Dimensional Matching* $\leq_P$ *Domain Decomposition*. To do this, we start with an instance of *Three-Dimensional Matching*, with sets X, Y, and Z of size n each, and a collection C of t ordered triples.

We construct the following instance of *Domain Decomposition*. We construct a graph $G = (V, E)$, where V consists of a node x_i' for each $x_i \in X$, y_j' for each $y_j \in Y$, and z_k' for each $z_k \in Z$. For each triple A_m in C, we will also define three nodes v_m^x, v_m^y, and v_m^z. Let U denote all nodes of the form x_i', y_j', or z_k'. We now define the following edges in G. For each triple of nodes v_m^x, v_m^y, and v_m^z, we construct a directed triangle via edges $(v_m^x, v_m^y), (v_m^y, v_m^z), (v_m^z, v_m^x)$. For each node x_i', and each node v_m^x for which x_i appears in the triple A_m, we define edges (x_i', v_m^x) and (v_m^x, x_i'). We do the analogous thing for each node y_j' and z_k'.

So the idea is to create a directed triangle for each triple, and a pair of bi-directional edges between each element and each triple that it belongs to. We want to encode the existence of a perfect tripartite matching as follows. For each triple $A_m = (x_i, y_j, z_k)$ in the matching, we will construct three 2-element domains consisting of the nodes x_i', y_j', z_k' together with the nodes v_m^x, v_m^y, and v_m^z respectively. For each triple A_m that is *not* in the matching, we will simply construct the 3-element domain on v_m^x, v_m^y, and v_m^z.

Thus, we claim that G has a decomposition into at least $3n+t-n = 2n+t$ domains if and only there is a perfect tripartite matching in C. If there is a perfect tripartite matching, then the construction of the previous paragraph produces a partition of V into $2n+t$ domains. So let us prove the other direction; suppose there is a partition of V into $2n+t$ domains. Let p denote the number of domains containing elements from U. Note that $p \leq 3n$, and $p = 3n$ if and only if each element of U appears in a 2-element domain. Let q denote the number of domains not containing elements from U. Each such domain must consist of a single triangle; since at least n triangles are involved in domains with elements of U, we have $q \leq t-n$, and $q = t-n$ if and only if the domains involving U intersect only n triangles. Now, the total number of domains is $p+q$, and so this number is $2n+t$ if and only the domains consist of $t-n$ triangles, together with $3n$ two-element domains involving elements of U. In this case, the triangles that are *not* used in the domain decomposition correspond to triples in the *Three-Dimensional Matching* instance that are all disjoint.

Thus, by deciding whether G has a decomposition into at least $2n+t$ domains, we can decide whether our original instance of *Three-Dimensional Matching* has a solution.

[1]ex742.89.672