

We can conclude that  $A$  must be a minimum-cost arborescence. Let  $r$  be the designated root vertex in  $G = (V, E)$ ; recall that a set of edges  $A \subseteq E$  forms an arborescence if and only if (i) each node other than  $r$  has in-degree 1 in  $(V, A)$ , and (ii)  $r$  has a path to every other node in  $(V, A)$ .

We claim that in a directed acyclic graph, any set of edges satisfying (i) must also satisfy (ii). (Note that this is certainly not true in an arbitrary directed graph.) For suppose that  $A$  satisfies (i) but not (ii), and let  $v$  be a node not reachable from  $r$ . Then if we repeatedly follow edges backwards starting at  $v$ , we must re-visit a node eventually, and this would be a cycle in  $G$ .

Thus, every way of choosing a single incoming edge for each  $v \neq r$  yields an arborescence. It follows that an arborescence  $A$  has minimum cost if and only, for each  $v \neq r$ , the edge in  $A$  entering  $v$  has minimum cost over all edges entering  $v$ ; and similarly, an edge  $(u, v)$  belongs to a minimum-cost arborescence if and only if it has minimum cost over all edges entering  $v$ .

Hence, if we are given an arborescence  $A \subseteq E$  with the guarantee that for every  $e \in A$ ,  $e$  belongs to *some* minimum-cost arborescence in  $G$ , then for each  $e = (u, v)$ ,  $e$  has minimum cost over all edges entering  $v$ , and hence  $A$  is a minimum-cost arborescence.

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