Distributed Least-Squares Estimation using Active Exploration

Ian Abraham

***Abstract*—Abstract**

I. Introduction

The main contribution of this paper is the

Adapted the controller to work in a distributed manner. Formulated a consensus network to share ergodic coverage, formation control, and distribution estimates. Method for estimating distributions.

II. Ergodic Control

For a control-affine dynamical systems of the form

$$\dot{x} = f(x, u) = g(x) + h(x)u, \tag{1}$$

we formulate ergodic control as the control solution that minimizes the ergodic metric given as

$$\mathcal{E}(x(t)) = \sum_{k\in\mathbb{Z}} (\phi_k - c_k)^2 \tag{2}$$

where

$$\phi_k = \int_X \Phi(x) F_k(x) dx \tag{3}$$

and

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} F_k(x(t))dt. \tag{4}$$

Here, $\Phi(x)$ is the underlying spatial distribution and $F_k(x)$ is the basis function. In this work, the cosine basis function is used, however, any choice of basis function F_k can be used. We define the objective of the ergodic control problem as

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \mathcal{E}(x(t)) \\ \text{subject to} \quad & \dot{x}(t) = f(x(t), u(t)) \end{aligned}$$

Instead of minimizing directly over control u, the first order sensitivity to control application time is found by taking the derivative of $\mathcal{E}$ with respect to infinitesimal control application time λ defining the mode insertion gradient as

$$\begin{aligned} & \frac{d\mathcal{E}}{d\lambda} = \rho(s)^T(f_2(s,s) - f_1(s)) \\ \text{subject to} \quad & \dot{\rho} = -\sum_{k\in\mathcal{Z}} (\phi_k - c_k)\frac{\partial F_k(x)}{\partial x} - \frac{\partial f}{\partial x}^T \rho. \end{aligned} \tag{5}$$

where

$$f_1(t) = f(x(t), u_1(t)) \tag{6}$$

and

$$f_2(t,\tau) = f(x(t), u_2(\tau)) \tag{7}$$

are the systems' state f subject to the default control u_1 and optimal control u_2. Following the formulation of sequential action control (SAC), we define an auxiliary objective function J_2 as

$$J_2 = \frac{d\mathcal{E}}{d\lambda} + \frac{1}{2}\|u_2 - u_1\|_R^2 \tag{8}$$

whose minimum occurs when the mode insertion gradient is most negative, regularized by the change in the default control $u1$. Since J_2 is convex in u_2, taking a first-order optimality approach to solve for an optimal u_2 gives the closed form solution

$$u^*(t) = -R^{-1}h(x)^T\rho(t) + u_1(t). \tag{9}$$

The time of control application λ that significantly reduces the ergodic cost is obtained via

$$\lambda^* = \underset{\lambda}{\text{argmin}}\ \frac{d\mathcal{E}(\lambda)}{d\lambda}. \tag{10}$$

A. Centralized Ergodic Control

Given a network of robot agents $x_i(t)$ subject to dynamical constraints of the form

$$\dot{x}_i(t) = f_i(x_i, u_i) = g_i(x_i) + h_i(x_i)u_i, \tag{11}$$

we can define the ergodic metric as before in a centralized manner as

$$\mathcal{E}(x(t)) = \sum_{k\in\mathbb{Z}} (\phi_k - c_k(x(t))^2 \tag{12}$$

where

$$x = \begin{bmatrix} x_i \\ \vdots \\ x_N \end{bmatrix} \tag{13}$$

is the stacked centralized representation of all robot agent $x_i(t) \forall i \in \{1, \ldots, N\}$. For a network of agents, the spatial statistical representation of the collective robot network, c_k, is defined as

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{1}{N}\sum_i^N F_k(x_i(t)) = \frac{1}{T}\int_{t_0}^{t_0+T} \tilde{F}_k(x(t)). \tag{14}$$

where $\tilde{F}_k(x(t)) = \frac{1}{N}\sum_j^N F_k(x_i(t))$. Here, the mode insertion gradient is given as

$$\frac{d\mathcal{E}}{d\lambda} = \rho(s)(f(x(s), u_2(s)) - f(x(s), u_1(s))) \tag{15}$$

where

$$\rho(t) = \begin{bmatrix} \rho_i \\ \vdots \\ \rho_N \end{bmatrix}, f = \begin{bmatrix} f_i \\ \vdots \\ f_N \end{bmatrix} \tag{16}$$

Where the solution to the costate equation ρ is obtained by the backwards differential equation

$$\dot{\rho} = -q \sum_{k \in \mathcal{Z}} (\phi_k - c_k) \frac{\partial \tilde{F}_k(x)}{\partial x} - \frac{\partial f}{\partial x}^T \rho. \tag{17}$$

Assuming that each robot's states $x_i(t)$ are independent of one another,

$$\frac{\partial \tilde{F}_k(x)}{\partial x} = \begin{bmatrix} \frac{\partial F_k(x_i)}{\partial x_i} \\ \vdots \\ \frac{\partial F_k(x_N)}{\partial x_N} \end{bmatrix} \tag{18}$$

and

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_i}{\partial x_i} & 0 & \cdots & 0 \\ 0 & \frac{\partial f_{i+1}}{\partial x_{i+1}} & & \\ \vdots & & \ddots & \\ 0 & & & \frac{\partial f_N}{\partial x_N} \end{bmatrix} \tag{19}$$

As done previously, we define a secondary cost function

$$J_2 = \frac{d\mathcal{E}}{d\lambda} + \frac{1}{2}\|u_2 - u_1\|_R^2. \tag{20}$$

Taking the derivative of (20) with respect to control u_2 and setting the resulting expression equal to zero gives

$$u^*(t) = -R^{-1} h(x)^T \rho(t) + u_1(t). \tag{21}$$

If we expand (21) we get

$$\begin{bmatrix} u_i^*(t) \\ \vdots \\ u_N^*(t) \end{bmatrix} = -R^{-1} \begin{bmatrix} h_i(x_i) & & \\ & \ddots & \\ & & h_N(x_i) \end{bmatrix}^T \begin{bmatrix} \rho_i(t) \\ \vdots \\ \rho_N(t) \end{bmatrix} + \begin{bmatrix} u_{1,i}(t) \\ \vdots \\ u_{1,N}(t) \end{bmatrix} \tag{22}$$

Since $h(x)$ is a block diagonal matrix, the transpose of a block diagonal matrix is the transpose of each element on the diagonal. With R as a positive definite diagonal matrix whose diagonal elements R_i are unique to each robot agent, we find the solution that minimizes (20) with respect to the control u_2 is

$$\begin{bmatrix} u_i^*(t) \\ \vdots \\ u_N^*(t) \end{bmatrix} = \begin{bmatrix} -R_i^{-1} h_i(x)^T \rho_i(t) + u_{1,i}(t) \\ \vdots \\ -R_N^{-1} h_N(x)^T \rho_N(t) + u_{1,N}(t) \end{bmatrix} \tag{23}$$

which is the stacked representation of each individual robot agent. As done before, finding the time of control application that significantly reduces the ergodic metric is found via

$$\lambda^* = \underset{\lambda}{\text{argmin}} \frac{d\mathcal{E}(\lambda)}{d\lambda}. \tag{24}$$

B. Distributed Ergodic Control

Theorem 2.1: For a connected network of agents with control-affine dynamics, the solution to the decentralized ergodic control is equivalent to the centralized ergodic control problem.

Proof: Assume that the robot agents are connected such that it can be represented via a graph Laplacian L. A consensus matrix P can be generated from the graph Laplacian such that P is both row and column stochastic. In the distributed case, the ergodic metric is defined as

$$\mathcal{E}_i(x_i(t)) = \sum_{k \in \mathbb{Z}} (\phi_k - \sum_j P_{ij} c_{k,j})^2 \tag{25}$$

for each agent i with neighbors j. Because of the row and column stochasticity of P, the operation $\sum_j P_{ij} c_{k,j}$ takes an average of the local c_k values for each neighboring agent j. Thus, we can rewrite the consensus on c_k as

$$\sum_j P_{ij} c_{k,j} = \frac{1}{N} \sum_j \frac{1}{T} \int_{t_0}^{t_0+T} F_k(x_j(t)) dt. \tag{26}$$

The summation over all neighboring nodes can be brought into the integral to give

$$\begin{aligned} \sum_j P_{ij} c_{k,j} &= \frac{1}{T} \int_{t_0}^{t_0+T} \frac{1}{N} \sum_j F_k(x_j(t)) dt \\ &= \frac{1}{T} \int_{t_0}^{t_0+T} \tilde{F}_k(x(t)) dt \end{aligned} \tag{27}$$

which for a fully connected network is equivalent to (14). The mode insertion gradient for an individual agent is then,

$$\frac{d\mathcal{E}_i}{d\lambda} = \rho_i(s)(f_i(x_i(s), u_{2,i}(s)) - f_i(x_i(s), u_{1,i}(s))) \tag{28}$$

with the costate equation obtained from the backwards-differential equation

$$\dot{\rho}_i = -\sum_{k \in \mathcal{Z}} (\phi_k - \sum_j P_{ij} c_{k,j}) \frac{\partial F_{k,i}(x_i)}{\partial x_i} - \frac{\partial f_i}{\partial x_i}^T \rho_i. \tag{29}$$

The solution to the secondary cost function for an individual agent

$$J_{2,i} = \frac{d\mathcal{E}_i}{d\lambda} + \frac{1}{2}\|u_{2,i} - u_{1,i}\|_{R_i}^2 \tag{30}$$

is

$$u_i^*(t) = -R_i^{-1} h_i(x_i)^T \rho_i(t) + u_{1,i}(t) \tag{31}$$

which is the same as the individual agent's control solution as the stacked representation of the centralized case. Individual time of control application λ can be found with

$$\lambda^* = \underset{\lambda}{\text{argmin}} \frac{d\mathcal{E}_i(\lambda)}{d\lambda}. \tag{32}$$

However, the time of action of one agent may interfere with another agent, impeding on reduction of the ergodic metric. This can be solved by having a consensus on the first order sensitivity of control application and finding the time of application that achieves the largest decrease in the ergodic metric. This can be written as

$$\lambda^* = \underset{\lambda}{\text{argmin}}\ N \sum_j P_{ij} \frac{d\mathcal{E}_j(\lambda)}{d\lambda} \tag{33}$$

and is equivalent to finding the time of control application in the centralized ergodic control formulation.

III. Network and Consensus

IV. Estimation using DEC

The least-squares problem is formulated as a minimization of the sum of square errors of a set of data y_i taken at $x_i \in \mathbb{R}^n$ and a prediction of the underlying function that is likely to represent the data $\tilde{y}(x_i)$. Formally, the least-squares problem is described as

$$\min_c \sum_i^N (y_i - c^T \phi(x_i))^2 \tag{34}$$

where $\tilde{y}(x) = c^T \phi(x)$ is linear in c and $\phi(x)$ is the basis function. Given enough samples N taken across the independent data space x, the least-squares estimation of c should provide a reasonable estimate of the underlying data. However, in this application, we seek to estimate spatial properties that span a larger range than the available singular sensor is able to sense. As a result, we turn to utilizing a distribution of sensors that together span the spatial independent data set.

The solution to the minimization problem in eq (34) is typically written in closed-form as

$$c^\star = (X^T X)^{-1} X^T y, \tag{35}$$

where $c^\star$ is the optimal parameter set, $X_{ij} = \phi_j(x_i)$ and y is a vector of the sensor measurements. However, this closed-form solution assumes an existing large diffuse data set. The aim of this work is to estimate global spatial properties as data is acquired in real-time. This prevents the need for trajectory planning ahead of time while

we present three methods for minimizing the least squares. Kalman filter, stochastic gradient descent, recursive least squares

V. Results

VI. Conclusion