Key Definitions

① **Decision Prob**: one bit output: Y/N.

② **Alphabet**: finite set (Σ)

③ **Strings over Σ**: finite length sequence of elements of Σ. ($|x|$ = length)

④ **Epsilon (ε)**: the null string (len = 0)

⑤ a^n = string of a's of length n.

$a^0 = \varepsilon$ $\quad a^{n+1} = a^n a$

⑥ **Σ^***: set of all possible strings over Σ

$\{a,b\}^* = \{\varepsilon, a, b, aa, ab, baab, abb, \ldots\}$

$\{a\}^* = \{\varepsilon, a, aa, \ldots\} = \{a^n \mid n \geq 0\}$

$\{\varepsilon\}^* = \{\varepsilon\}$ $\qquad \phi^* = \{\varepsilon\}$

$\phi^0 = \varepsilon$ (any set $S^0 = \varepsilon$ and rest powers become all ϕ)

$\{\varepsilon\}$: singleton set (one element ✓) NOT empty set.

⑦ **Concatenation**:

2 strings x and y $\Rightarrow$ new string xy.

i) associative $\rightarrow (xy)z = x(yz)$

ii) $\varepsilon x = x\varepsilon = x$ (identity = ε)

iii) $|xy| = |x| + |y|$

⑧ **Prefix**:

If $x = yz$ then y is a prefix of x.

ε : prefix of all x : prefix of x

9) Operations on Sets :

i) Union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

ii) Intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

iii) Complement $\overline{A} = \{x \mid x \in \Sigma^* \text{ and } x \notin A\}$

iv) Concatenation $AB = \{xy \mid x \in A \ \& \ y \in B\}$
(finite size)

$\{\varepsilon\}A = A\{\varepsilon\} = A$

$A^{n+1} = AA^n \qquad A^0 = \{\varepsilon\}$

$\{a, b\}^n = \{x \in \{a,b\}^* \mid |x| = n\}$

v) Kleene star $A^* = \bigcup_{n \geq 0} A^n = A^0 \cup A^1 \cup A^2 \dots$

vi) $A^+ = AA^* = \bigcup_{n \geq 1} A^n$

Properties :

i) $(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$
$(AB)C = A(BC)$
Associative

ii) $A \cup B = B \cup A \qquad A \cap B = B \cap A$
$AB \neq BA$ (concatenation)
Comm

DFA

$M = (Q, \Sigma, \delta, s, F)$
Q ↓ finite, Σ ↓ finite, s ↓ start

$\delta : Q \times \Sigma \to Q$

$\delta(q, a) = q' \in Q$

if $q \in F \Rightarrow M$ accepts $x \qquad (q = \hat{\delta}(s, x))$

M rejects x

$q \notin F \Rightarrow M$ rejects x $\delta : Q \times \Sigma^* \Rightarrow Q$

① $\hat{\delta}$ defn : $\hat{\delta}(q, \varepsilon) = q$

$x = ya$ $\hat{\delta}(q, x) = \delta(\hat{\delta}(q, y), a)$

• $\hat{\delta}(q, a) = \delta(q, a)$

x is accepted by $M \Rightarrow \hat{\delta}(s, x) \in F$

x is rejected " " $\Rightarrow \hat{\delta}(s, x) \notin F$

② $L(M) = \{x \in \Sigma^* \mid \hat{\delta}(s, x) \in F\}$

③ Regular : $A \subseteq \Sigma^*$ if $A = L(M)$ for some M.

To prove something is regular, we

i) Define and construct its machine.

Define $M' = (Q', \Sigma, \delta', s', F')$

and $Q' =$ ____ , Σ , $s' =$ ____ , $F' =$ ____

and also $\delta' =$ ___ $\delta, \hat{\delta}$ ___ (imp)

ii) State what to prove.

iii) Lemma : $\hat{\delta}' \longrightarrow (\hat{\delta})$ $M : \delta, \hat{\delta}$ $M' : \delta', \hat{\delta}'$

iv) $x \in L(M) \iff x \in L(M')$

NFA

① $\delta : Q \times \Sigma \longrightarrow 2^Q$

② For any string $x \in A$, there

② Acceptance: for any string $x \in A$, there is a run of N that leads to a final state.

③ Rejection: for any string $y \notin A$, there is no run that leads to a final state.

$N = (Q, \Sigma, \Delta, S, F)$ → set of finish states

Q, Σ: finite, finite; S → set of start states

$\Delta : Q \times \Sigma \longrightarrow 2^{Q}$ where $2^{Q} = \{A \mid A \subseteq Q\}$

$q \xrightarrow{a} p$ if $p \in \Delta(q, a)$

$\hat{\Delta} : 2^{Q} \times \Sigma^{*} \longrightarrow 2^{Q}$

- $\hat{\Delta}(A, \varepsilon) = A$
- $\hat{\Delta}(A, a) = \bigcup_{p \in A} \Delta(p, a)$
- $\hat{\Delta}(A, xa) = \bigcup_{q \in \hat{\Delta}(A, x)} \Delta(q, a)$

Acceptance: if $\hat{\Delta}(S, x) \cap F \neq \phi$

④ DFA to NFA.

by singleton sets. Proof done.

⑤ NFA to DFA

$N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$ st $L(M) = L(N)$

$M = (Q_M, \Sigma, \delta_M, s_M, F_M)$

$Q_M = 2^{Q_N}$ $\quad s_M = S_N$ $\quad F_M = \{A \subseteq Q_N \mid A \cap F_N \neq \phi\}$

$\delta_M(A, a) = \hat{\Delta}_N(A, a)$

Lemma: $\hat{\delta}_M(A, x) = \hat{\Delta}_N(A, x)$

⑥ DFA to NFA

⑦ For every εNFA $\rightarrow \exists$ DFA.

⑧ Concatenation:

$N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$ $\quad L(N_1) = A$

$N_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$ $\quad L(N_2) = B$

$N = (Q, \Sigma, \Delta, S, F)$ $\quad L(N) = AB$

$Q = Q_1 \cup Q_2 \quad S = S_1 \quad F = F_2$

$$\Delta(q, a) = \begin{cases} \Delta_1(q, a) & q \in Q_1,\ q \notin F_1 \\ \Delta_1(q, a) & q \in F_1,\ a \neq \varepsilon \\ \Delta_1(q, a) \cup S_2 & q \in F_1,\ a = \varepsilon \\ \Delta_2(q, a) & q \in Q_2 \end{cases}$$

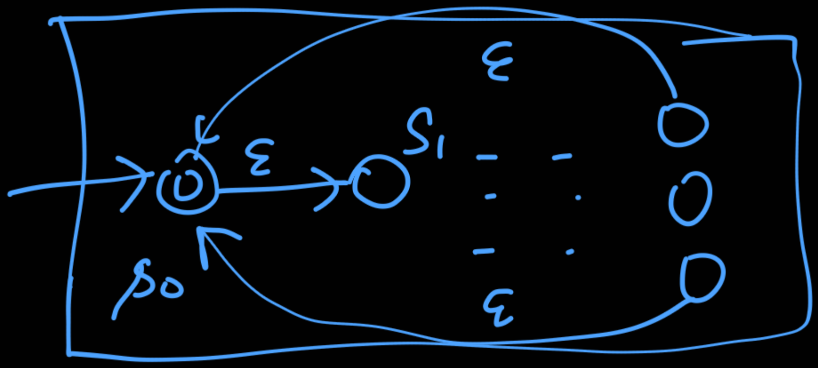
⑨ Kleene Star

$N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$ $\quad N = (Q, \Sigma, \Delta, S, F)$

$Q = Q_1 \cup \{s_0\} \quad S = \{s_0\} \quad F = \{s_0\}$

$$\Delta(q, a) = \begin{cases} \Delta_1(q, a) & q \notin F_1,\ q \in Q_1 \\ \Delta_1(q, a) & q \in F_1,\ a \neq \varepsilon \\ \Delta_1(q, a) \cup \{s_0\} & q \in F_1,\ a = \varepsilon \end{cases}$$

$$\delta(q,a) = \begin{cases} \ldots & \ldots \\ S_1 & q = s_0,\ a = \varepsilon \\ \phi & q = s_0,\ a \neq \varepsilon \end{cases}$$



Patterns

Pattern : any string characterisation.

$L(\gamma) = \{x \mid x \text{ matches } \gamma\}$ ⇒ γ is a pattern, and all strings x that match γ come in the language $L(\gamma)$.

Atomic Patterns : 5

$L(\gamma) \subseteq \Sigma^*$

Σ : alphabet set

$a \in \Sigma$	$L(a) = \{a\}$
ε	$L(\varepsilon) = \{\varepsilon\}$
ϕ	$L(\phi) = \phi$
$\#$	$L(\#) = \Sigma$
$@$	$L(@) = \Sigma^*$

Compound Patterns : 6

$\alpha + \beta$	$L(\alpha + \beta) = L(\alpha) \cup L(\beta)$
$\alpha \cap \beta$	$L(\alpha \cap \beta) = L(\alpha) \cap L(\beta)$

$\alpha \cap \beta \qquad L(\alpha \cap \beta) = L(\alpha) \cap L(\beta)$

$\alpha\beta \qquad L(\alpha\beta) = L(\alpha)L(\beta)$

$= \{xy \mid x \in L(\alpha) \text{ and } y \in L(\beta)\}$

$\alpha^* \qquad L(\alpha^*) = L(\alpha)^0 \cup L(\alpha)^1 \cup \ldots$

$= \{x_1 x_2 \ldots x_n \mid n \geq 0;\ x_i \in L(\alpha)\}$

$\alpha^+ \qquad L(\alpha^+) = L(\alpha) \cup L(\alpha)^2 \cup \ldots$

$= L(\alpha)^+$

$\neg\alpha \qquad L(\neg\alpha) = \overline{L(\alpha)} = \Sigma^* - L(\alpha)$

- $\varepsilon = \neg(\#@) \qquad @ = \#^*$

$\alpha^+ = \alpha\alpha^* \qquad \# = a_1 + a_2 + \ldots$

<u>DeMorgan</u>: $\alpha \cap \beta = \neg(\neg\alpha \cup \neg\beta)$

<u>Regular Expressions</u>

Atomic: $a \in \Sigma$, ε, ϕ

compound: $+$, $\cdot$, $*$

patterns constructed from the above

syntax $\rightarrow$ reg exp.

Precedence $\rightarrow$ $* > \cdot > +$

<u>Theorem</u>: Let $A \subseteq \Sigma^*$. Then these are equivalent:

① A is regular, then $\exists$ a finite automaton

M st $L(M) = A$.

(2) $A = L(\alpha)$ for some pattern α.

(3) $A = L(\alpha)$ for some reg expression α.

Proof in other pdf.

Homomorphism

Let Σ and Γ be finite alphabet sets.

$h: \Sigma^* \longrightarrow \Gamma^*$ st $\forall\ x, y \in \Sigma^*$

$$h(xy) = h(x)\,h(y)$$

$$h(\varepsilon) = \varepsilon.$$

Any func $h: \Sigma \to \Gamma^*$ uniquely defines $h': \Sigma^* \longrightarrow \Gamma^*$

$A \subseteq \Sigma^*$ then $h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$

$B \subseteq \Gamma^*$ then $h^{-1}(B) = \{y \mid h(y) \in B\} \subseteq \Sigma^*$

$h(A)$: image of set A under h

$h^{-1}(B)$: preimage of set B under h.

Theorem 1: Let $h: \Sigma^* \to \Gamma^*$ be a homomorphism. if $A \subseteq \Sigma^*$ is regular then $h(A)$ is regular.

Theorem 2: Let $h : \Sigma^* \to \Gamma^*$ be a homo. If $B \subseteq \Gamma^*$ is regular then $h^{-1}(B)$ is regular.

NFA with ε transitions

$M = (Q, \Sigma, \hat{\varepsilon}, \Delta, S, F)$ (ε = null string)

$\hat{\varepsilon}$: special symbol, not in Σ. (M : ε NFA)
(treated as null string)

$M_E = (Q, \Sigma \cup \{\hat{\varepsilon}\}, \Delta, S, F)$
ordinary NFA over $\Sigma \cup \{\hat{\varepsilon}\}$.

Defn: M accepts $x \in \Sigma^*$ if
$\exists\, y \in (\Sigma \cup \{\hat{\varepsilon}\})^*$ st M_E accepts y

x is obtained from y by removing all occurences of symbol $\hat{\varepsilon}$.

$x = h(y)$ where $h : (\Sigma \cup \{\hat{\varepsilon}\})^* \to \Sigma^*$

$h(a) = a$ $\quad h(\hat{\varepsilon}) = \varepsilon$ $\quad h(\varepsilon) = \varepsilon$
metasymbol $\quad$ null str

Thus $L(M) = h(L(M_E))$

To show $B = \{a^n b^n \mid n \geq 0\}$ is not regular.

Suppose B is regular. Then $\exists$ DFA M st $L(M) = B$. Let k = no. of states in M. Consider string $n > k$.
By pigeonhole principle, some state must be repeated in the string from first 'a' to the n^{th} 'a'.
Let that state be p.

$$\overbrace{a\,a\,a\,a\,\_\,\_\,.\,a\,.\,\_\,\_\,.\,a\,.\,a}^{n}\;\overbrace{b\,b\,\_\,\_\,.\,\_\,\_\,.\,b}^{n}$$

$$\underset{s}{\uparrow} \xleftarrow{\quad} u \xrightarrow{\quad} \underset{p}{\uparrow} \leftarrow v \rightarrow \underset{p}{\uparrow} \xleftarrow{\qquad} w \xrightarrow{\qquad} \underset{f \in F}{\uparrow}$$

$\hat{\delta}(s, u) = p \quad \hat{\delta}(p, v) = p \quad |v| > 0$
$\hat{\delta}(p, w) \in F$

Now consider run of M on $x = uw$.

$$\hat{\delta}(s, uw) = \hat{\delta}(\hat{\delta}(s, u), w)$$
$$= \hat{\delta}(p, w) \in F$$

$uw \in L(M)$ but $uw = a^{n-|v|} b^{n} \notin B$.
$|v| \geq 1$

Similarly consider uv^2w.

$$\hat{\delta}(s, uv^2w) = \hat{\delta}(\hat{\delta}(\hat{\delta}(\hat{\delta}(s,u), v), v), w)$$
$$= \hat{\delta}(\hat{\delta}(\hat{\delta}(p, v), v), w)$$
$$= \hat{\delta}(\hat{\delta}(p, v), w)$$
$$= \hat{\delta}(p, w) \in F$$

$= \delta^*(p, w) \in F$

$uv^2w = a^{n+|v|} b^n \notin B \qquad |v| \geq 1$

Pumping Lemma

Statement of Lemma:

Let A be a regular set. P holds for A:

"There exists $k \geq 0$ st for any string x, y, z with $xyz \in A$ and $|y| \geq k$, there exists strings u, v, w st $y = uvw$, $v \neq \varepsilon$ and for all $i \geq 0$, the string $xuv^iwz \in A$"

Contrapositive Statement of P:

For all $k \geq 0$, $\exists$ strings x, y, z st $xyz \in A$, $|y| \geq k$ and $\forall$ u, v, w with $y = uvw$, $v \neq \varepsilon$, $\exists$ $i \geq 0$ st $xuv^iwz \notin A$

Then A is not regular.

Example: $A = \{a^n b^m \mid n \geq m\}$

i) Consider any $k \geq 0$. Let $x = a^k$ $y = b^k$ $z = \varepsilon$.
Then $xyz \in A$

ii) Consider any split of y. $u = b^j$ $v = b^m$ $w = b^n$
st $j + m + n = k$.

Let $i = 2 \Rightarrow xuv^2wz = a^k b^j b^{2m} b^n$

$= a^k b^{k+m} \notin A$

$= a^k b \quad \notin A$

CRUX To prove using Pumping Lemma

i) Consider any $K \geq 0$ and x, y, z st $xyz \in A$. x, y, z in terms of K.

ii) Consider any uvw, $|v| \geq 1$ st $y = uvw$

iii) Take any powers of v to show, str $\notin A$

Example 2 $\quad A = \{ww \mid w \in \{a,b\}^*\}$

$x = \varepsilon \quad y = a^K \quad z = ba^K b$

$u = a^j \quad v = a^m \quad w = a^n$

$j + m + n = K$

$m > 0$

$$xuv^2wz = a^j a^{2m} a^n ba^K b$$
$$= a^K a^m b a^K b \quad \notin A$$

$x = a^n b^m$

$y = a^n b^m \quad z = \varepsilon$

$u = a^n \quad v = b^m \quad w = \varepsilon$

Use of closure properties :

i) $A = \{x \in \{a,b\}^* \mid \#a(x) = \#b(x)\}$

Consider $A' = A \cap L(a^* b^*)$

i) $L(a^* b^*)$ is regular.

ii) If A is regular and $L(a^* b^*)$ is

ii) if A is regular, and ...
then by closure under $\cap$, A' is regular.
$A' = \{a^n b^n \mid n \geq 0\} \Rightarrow$ not regular

Equivalence Relation on Q

$p \approx q$ iff $\forall x \in \Sigma^* \; (\hat{\delta}(p, x) \in F$ iff $\hat{\delta}(q, x) \in F)$
$\approx$ is an equivalence relation

$[p] = \{q \mid q \approx p\}$: set of all eq. classes.

Quotient Automata

$M = (Q, \Sigma, \delta, s, F)$ $\quad M/\approx \; = (Q', \Sigma, \delta', s', F')$

$Q' = \{[p] \mid p \in Q\}$ (the $\approx$ equi classes are states of $M/\approx$)

$\delta'([p], a) = [\delta(p, a)]$

$s' = [s]$ $\quad F' = \{[p] \mid p \in F\}$

1. δ' is well defined.
2. $p \in F$ iff $[p] \in F'$
3. $\forall x \in \Sigma^*$ $\hat{\delta}'([p], x) = [\hat{\delta}(p, x)]$

- Two equivalent states of $M/\approx$ are equal and $\sim \subseteq Q' \times Q'$ is the identity relation

(To prove a func $f: A \to B$ is well defined PT. each element in A has exactly one image in B)

State Minimisation Algo

i) if $p \in F$ and $q \notin F$ or vice versa, mark $\{p,q\}$.

ii) Repeat until no change in marking:

- if $\{p,q\}$ unmarked st $\{\delta(p,a), \delta(q,a)\}$ is marked for some a, mark $\{p,q\}$.

iii) $p \approx q$ iff $\{p,q\}$ is not marked.

Theorem: The pair $\{p,q\}$ is marked by the Algo iff $\exists x \in \Sigma^*$ st $\hat{\delta}(p,x) \in F$ and $\hat{\delta}(q,x) \notin F$ or vice versa. That is, iff $p \not\approx q$.

Isomorphic DFAs

Two DFAs M and N are isomorphic if $\exists$ a bijection $F: Q_M \to Q_N$ st

$M = (Q_M, \Sigma, \delta_M, s^M, F^M)$

$N = (Q_N, \Sigma, \delta_N, s^N, F^N)$

$F(s^M) = s^N$

$p \in F_M$ iff $F(p) \in F_N$

$F(\delta_M(p,a)) = \delta_N(F(p),a) \; \forall p \in Q_M \; \forall a \in \Sigma$

or if $\delta_M(p,a) = q$ then $\delta_N(F(p),a) = F(q)$

Myhill Nerode Theorem

If M and N are any two DFAs with

If M and N are ... no inaccessible states st $L(M) = L(N)$ then $M/\approx$ and $N/\approx$ are isomorphic.

Corollary – The output of collapsing algo is the minimal DFA for the set and it is unique upto isomorphism.

Myhill Nerode Relation

Let $M = (Q, \Sigma, \delta, s, F)$ be a DFA st $L(M) = R$. M induces an equivalence relation $\equiv_M$ on Σ^* defined by

$x \equiv_M y$ iff $\hat{\delta}(s, x) = \hat{\delta}(s, y)$ (x, y may or may not $\in R$)

- $\equiv_M$ is an equivalence relation.

$\equiv_M$ satisfies 3 more prop:

(1) Right Congruence: $\forall x, y \in \Sigma^*, \forall a \in \Sigma$
$$x \equiv_M y \Rightarrow xa \equiv_M ya$$

(2) Refinement: $\forall x, y \in \Sigma^*$
$$x \equiv y \Rightarrow (x \in R \text{ iff } y \in R)$$

(3) Finite index: there are only finitely many equivalence classes.

Myhill Nerode Relation

Let $R \subseteq \Sigma^*$ be a regular set. An equivalence rel^n $\equiv$ on Σ^* is a Myhill Nerode rel^n for R if it satisfies the above 3 properties.

(1, 3 indep of R, 2 dep on R)

Any DFA $M = (Q, \Sigma, \delta, s, F)$ with $L(M) = R$ defines a Myhill Nerode relation $\equiv_M$.

DFA to MN Reln:

$(x \equiv_M y)$ iff $(\hat{\delta}(s, x) = \hat{\delta}(s, y))$

for DFA $M = (Q, \Sigma, \delta, s, F)$

MN Reln to DFA:

Let $R \subseteq \Sigma^*$ and $\equiv$ be any MN rel^n for R. From $\equiv$ we can construct DFA $M_\equiv$ st $L(M_\equiv) = R$.

for $x \in \Sigma^*$, let $[x] = \{y \mid y \equiv x\}$ be the equi classes of x.

(By P_3, there are only finitely many equi classes)

$M_\equiv = (Q, \Sigma, \delta, s, F)$

$Q = \{[x] \mid x \in \Sigma^*\}$ $\quad s = [\varepsilon]$

$F = \{[x] \mid x \in R\}$ $\quad \delta([x], a) = [xa]$

(P1) $x \in R$ iff $[x] \in F$

(P2) $\hat{\delta}([x], y) = [xy]$

$\equiv \rightarrow M_{\equiv} \rightarrow \equiv_{M_{\equiv}}$ (identical)

$M \rightarrow \equiv_M \rightarrow M_{\equiv_M}$ (isomorphic)

<u>Theorem</u>: Let Σ be a finite alphabet. upto isomorphism of automata, there is a one-to-one correspondence between deterministic finite automata over Σ (with no inaccessible states) accepting $R \subseteq \Sigma^*$ and Myhill Nerode rel^n for R on Σ.

<u>Refinement</u>

A rel^n $\equiv_1$ refines another reln $\equiv_2$ if

$$\equiv_1 \subseteq \equiv_2 \qquad \text{(Partial order)}$$

i.e. $\forall x, y$ if $x \equiv_1 y$ then $x \equiv_2 y$.

$\forall x \quad [x]_1 \subseteq [x]_2$

reflexive, transitive, anti-symmetric,

$\equiv_1 \subseteq \equiv_2 \implies \equiv_1$: finer

$\equiv_2$: coarser

finest ⇒ every state into a diff eq class
coarsest ⇒ all states into one class

zyada eq. classes, but no. of elements in each kam V/S kam equivalence classes, but each me zyada no. of element.

Coarsest : Equivalence - R

Let $R \subseteq \Sigma^*$ (NOT nece. Regular)
Eq. reln $\equiv_R$ on Σ^* as $x \equiv_R y$
iff $\forall z \in \Sigma^* \; (xz \in R$ iff $yz \in R)$

- $\equiv_R$ is a right congruence.
- $\equiv_R$ refines R
- coarsest relation.

Myhill Nerode Theorem

Let $R \subseteq \Sigma^*$. The following are equi:

i) R is regular
ii) $\exists$ a Myhill-Nerode reln for R
iii) $\equiv_R$ is of finite index.

Example: $A = \{a^n b^n \mid n \geq 0\}$
for each a^k, $k \geq 0$ there is an $\equiv_A$-class
since $a^k b^k \in A$, $a^m b^k \notin A$, if $k \neq m$.
i.e. $k \neq m \Rightarrow a^k \not\equiv_A a^m$
$\therefore$ ∞-many eq classes $\Rightarrow$ A is not regular.

Context Free Grammar

$G = (N, \Sigma, P, S)$

N = non-terminal symbols (finite)

Σ = terminal symbols (finite) $(N \cap \Sigma = \phi)$

P = productions : $N \times (N \cup \Sigma)^*$

(non terminal to non or terminal)

$S \in N$ = start symbol

Notations: $A, B, C \Rightarrow$ non terminals

$a, b, c \Rightarrow$ terminals

$\alpha, \beta, \gamma \Rightarrow$ strings over $(N \cup \Sigma)^*$

A language Z is called context free language, if $\exists$ a grammar G st $Z = L(G)$

One Step Derivable:

Let $\alpha, \beta \in (N \cup \Sigma)^*$. Then β is derivable from α in one step $\alpha \xrightarrow{1} \beta$ if β can be obtained from α by replacing some occurence of a non-terminal A in α with γ where $A \to \gamma \in P$.

if $\exists\ \alpha_1, \alpha_2 \in (N \cup \Sigma)^*$ st $\alpha = \alpha_1 A \alpha_2$ and $\exists\ A \to \gamma \in P$ then $\alpha \xrightarrow[G]{1} \beta = \alpha_1 \gamma \alpha_2$.

$B \subseteq \Sigma^*$ is a CFL if $B = L(G)$ for some CFG G.

$X = \{ w \in \{0,1\}^* \mid w$ has equal no. of 0's &

and 1's

$S = 0S1S \mid 1S0S \mid \varepsilon$

$X = \{w \in \{0,1\}^* \mid w = rev(w)\}$ (even + odd)

$S \to \varepsilon \mid \underbrace{0 \mid 1}_{\text{odd}} \mid 0S0 \mid 1S1$

$X = \{w \in \{0,1\}^* \mid w$ is odd & middle symbol $= 0\}$

$S \to 0 \mid 0S1 \mid 1S0 \mid 0S0 \mid 1S1$

$X = \{w \in \{0,1\}^* \mid w$ contains atleast 3 1's$\}$

$S \to A1A1A1A$

$A \to 0A \mid 1A \mid \varepsilon$