→ Competitive Ratio

A = online algo
OPT = optimal offline algo

$T_A(m)$ = time taken by A for m oper^n
$T_{OPT}(m)$ = " " " OPT " same seq of m oper^n

Time Efficiency

A is said to be α-competitive if there is a constant K st

$$T_A(m) \leq \alpha\, T_{OPT}(m) + K$$

for every sequence of len m, and for every m.

Good Quality

$Q_A(m)$ = size of matching comp. by A for a seq of edges.
$Q_{OPT}(m)$ = size of matching comp by OPT for same seq.

A is α competitive if $\exists$ K st

$$Q_A(m) \geq \alpha\, Q_{OPT}(m) + K$$

for all seq of len m, for every m.

→ Move to Front Algo

online seq of m search operⁿ ⇒ min time algo.

access only thru Head, and only adj ele swaps allowed.

Whenever we search any ele, bring it to the front of the list after search.

$r(e)$ = rank of ele e in list.

Search (e) = starting from head ptr, scan linearly till we find e, bring the node storing e to the front of list by a seq of swaps.

→ MTF v/s OPT

We don't know about OPT. let Search(x) be i^{th} query operⁿ.

MTF → rank$(x) = r(x)$
OPT → rank$(x) = r^*(x)$.

Actual cost of Search(x) in MTF $= 2r(x)-1$
" " " " in OPT $= r^*(x) + t_i$

no. of swaps by OPT ≥ 0.

We want to show amortised cost in MTF < amort cost in OPT ⇒ bounded by $r^*(x)$ and t_i.

$\Delta\phi \to$ cancel $(2r(x))$ term

$\phi(i) = 2 \cdot (\#\text{ inversions})$

$\phi(0) = 0$ (lists identical in beginning)

$\phi(i) \geq 0$

i) $r(x)-1$ elements preceding x now follow x.

ii) if this creates a new inversion (x,e)
$\Rightarrow$ e preceds x in OPT.

iii) if this destroys an existing inversion (x,e)
$\Rightarrow$ e follows x in OPT

$\Delta\phi = 2(\#\text{ inv created} - \#\text{ inv destroy})$

MTF $\left\{ \begin{array}{l} \#\text{ new inv created} \leq r^*(x)-1 \\ \#\text{ old inv destroyed} \geq r(x) - r^*(x) \end{array} \right.$

OPT $\left\{ \begin{array}{l} \#\text{ new inv created} \leq t_i \\ \#\text{ old destroyed} \geq 0 \end{array} \right.$

$$\Delta\phi \leq 2(r^*(x)-1+t_i-r(x)+r^*(x))$$
$$= 2(2r^*(x)+t_i-r(x)-1)$$
$$= 4r^*(x)+2t_i-2r(x)-2$$

Act MTF $= 2r(x)-1$

Amort MTF $\leq 4r^*(x)+2t_i-3$
$\leq 4r^*(x)+2t_i$
$\leq 4r^*(x)+4t_i$
$= 4(r^*(x)+t_i)$

≤ 4 (act OPT)

$\Rightarrow$ For i^{th} query oper^n, amort cost of MTF $\leq$ 4 act cost of OPT

$$T_{MTF}(m) \leq \text{Amort cost of } m \text{ oper MTF}$$
$$\leq 4 \text{ act cost of OPT}$$
$$\leq 4\, T_{OPT}(m)$$

$\Rightarrow$ MTF is 4 competitive