Bayesian Algorithm - NMOR

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1 Bayesian Inference Protocol

1.1 Signal Model (Dispersive Lorentzian)

For the likelihood function $\mathbb{P}(\alpha_i(t_j)|\cdot)$ in Eq. 2, the expected signal α_{theory} is modeled as a dispersive Lorentzian function of the total magnetic field, $B_{\text{total}} = B_{\text{unknown}} + B_{\text{bias},i}$. This is the well-established lineshape for the NMOR resonance near its center. (Will update the exact expression soon)

$$\alpha_{\text{theory}}(B_{\text{total}}) \propto \frac{B_{\text{total}}}{B_{\text{total}}^2 + \Gamma_B^2} \tag{1}$$

1.2 Parameter Definitions

- $\mathbb{P}(\cdot|\cdot)$: Conditional probability distribution. The term $\mathbb{P}(B_{\text{unknown}}|\{\alpha_{i-1}(t_j)\})$ represents the prior belief before the i-th cycle.
- $\{\alpha_i(t_j)\}$: The set of optical rotation measurements taken during the i-th experimental cycle.
- $\mathbb{P}(\alpha_i(t_j)|\cdot)$: The likelihood for a single measurement, typically a Gaussian distribution centered on α_{theory}.
- Γ_B: The half-width of the resonance in magnetic field units.

1.3 Sequential Update Scheme

The estimation protocol is implemented as a sequence of experimental cycles, indexed by $i = 1, 2, ..., N_{\text{exp}}$. Within each cycle i, a fixed bias field $B_{\text{bias},i}$ is applied. The system's response, the optical rotation angle α_i, is recorded as a time series over an integration time T_i, yielding a dataset for that cycle, $\{\alpha_i(t_j)\}_{j=1}^{M_i}$, where t_j are the time instances.

The data from each cycle is used to sequentially update the posterior probability distribution for the unknown magnetic field, B_{unknown}. The posterior from cycle $i-1$ serves as the prior for cycle i. The update rule is given by Bayes' theorem, where the likelihood for the entire cycle is the product of the likelihoods for each time sample:

$$\mathbb{P}_i(B_{\text{unknown}}|\{\alpha_i(t_j)\}) \propto \left[\prod_{j=1}^{M_i} \mathbb{P}(\alpha_i(t_j)|B_{\text{unknown}}, B_{\text{bias},i})\right] \cdot \mathbb{P}_{i-1}(B_{\text{unknown}}|\{\alpha_{i-1}(t_j)\}) \tag{2}$$

This procedure yields the posterior distribution $\mathbb{P}_i(B_{\text{unknown}})$ after the i-th experiment. An expected information gain measure, calculated from this posterior, is then used to select an optimized bias field, $B_{\text{bias},i+1}$, for the subsequent experimental cycle. As the posterior for B_{unknown} becomes more localized, the integration time T_{i+1} can potentially be increased to achieve higher precision on the residual field being measured.

1.4 Algorithm

1. **Initialization:** Set the experimental cycle index $i = 1$. Define an initial, non-informative prior probability distribution for the unknown magnetic field, $\mathbb{P}_0(B_{\text{unknown}})$. This is typically a uniform or broad Gaussian distribution over a plausible range.

2. **Optimization of Control Parameters:** Based on the current posterior distribution $\mathbb{P}_{i-1}(B_{\text{unknown}})$, determine the optimal control parameters for the current cycle i. This includes the bias field $B_{\text{bias},i}$ (and potentially the integration time T_i). The parameters are chosen to minimize the expected entropy of the next posterior distribution:

$$B_{\text{bias},i} = \underset{B'_{\text{bias}}}{\arg\min} \left(\mathbb{E} \left[H \left(\mathbb{P}_i(B_{\text{unknown}}) \right) \right] \right) \tag{3}$$

where the expectation $\mathbb{E}[\cdot]$ is taken over all possible measurement outcomes for a given set of control parameters, and H is an entropy measure. (to be updated and worked out rigorously, we can replace the entropy with an information measure as well trivially) (**Additional Note:** we can extend this notion to update for an optimal integration time for the subsequent measurement. It may become important because our signal grows closer to zero tending to the noise floor.)

3. **Measurement:** Apply the optimized bias field $B_{\text{bias},i}$ to the atomic ensemble. Measure the optical rotation signal $\alpha_i(t)$ for the duration of the integration time T_i, acquiring the dataset for the cycle, $\{\alpha_i(t_j)\}$.

4. **Posterior Update:** Update the posterior distribution for the unknown magnetic field using the newly acquired data and the posterior from the previous step as the new prior, according to Bayes' theorem:

$$\mathbb{P}_i(B_{\text{unknown}}) \propto \left[\prod_{j=1}^{M_i} \mathbb{P}(\alpha_i(t_j) | B_{\text{unknown}}, B_{\text{bias},i}) \right] \cdot \mathbb{P}_{i-1}(B_{\text{unknown}}) \tag{4}$$

5. **Iteration:** Increment the cycle index, $i \leftarrow i+1$. If the desired precision has not been reached or the total experimental time has not elapsed, return to Step 2. Otherwise, the final posterior distribution $\mathbb{P}_i(B_{\text{unknown}})$ is the result of the measurement.

2 Adaptive Experimental Design

The adaptive protocol selects the bias field for the $(i+1)$-th experimental cycle, $B_{\text{bias},i+1}$, by maximizing the expected information gain. This gain is quantified using the Kullback-Leibler (KL) divergence.

2.1 Utility Function and Optimization

The utility of a chosen B_{bias} is the expected information gain, $E[U_{KL}]$, which is the KL divergence averaged over all possible measurement outcomes α. The KL divergence, U_{KL}, for a specific outcome α is:

$$U_{KL}(\alpha, B_{\text{bias}}) = \int P(B'_{\text{unknown}} | \alpha, B_{\text{bias}}) \log \left(\frac{P(B'_{\text{unknown}} | \alpha, B_{\text{bias}})}{P_i(B'_{\text{unknown}})} \right) dB'_{\text{unknown}} \tag{5}$$

The expected utility is then:

$$E[U_{KL}(B_{\text{bias}})] = \int P(\alpha | B_{\text{bias}}) \cdot U_{KL}(\alpha, B_{\text{bias}})\, d\alpha \tag{6}$$

where the marginal likelihood $P(\alpha | B_{\text{bias}})$ is given by:

$$P(\alpha | B_{\text{bias}}) = \int P(\alpha | B_{\text{unknown}}, B_{\text{bias}}) \cdot P_i(B_{\text{unknown}})\, dB_{\text{unknown}} \tag{7}$$

The optimal bias field for the next cycle is the one that maximizes this expected utility:

$$B_{\text{bias},i+1} = \underset{B_{\text{bias}}}{\arg\max} \left(E[U_{KL}(B_{\text{bias}})] \right) \tag{8}$$

3 Summary of Assumptions

The validity of the proposed Bayesian inference protocol and the underlying theoretical model rests on a set of key assumptions.

- **Low $B_{unknown}$ magnitude:** We now initially assume the $B_{unknown}$ we are trying to measure is small enough that we are within the dispersive lorentzian regime.
- **Doppler-Free Regime:** The experiment utilizes a cold-atom ensemble. It is assumed that the atomic temperature is sufficiently low such that the velocity distribution is narrow, and the resulting Doppler shift is negligible compared to the natural linewidth and all other relevant rates.
- **Perfect Control:** In the context of the adaptive protocol, the experimental control parameters, particularly the bias field B_{bias}, are assumed to be set with perfect accuracy and precision, free from noise or calibration errors.

4 Further Directives

A list of directions (not exhaustive) we have to explore going ahead are as follows:

- **Expected Information Measure and Code:** Quantify and make rigorous the expected information measure to finalize and implement the algorithm in code. The normalization of the posterior would be trivial given that we working with a $1D$ parameter estimation problem, so I assume a simple numerical integration method would do the trick.
- **Obtain actual expression for $\alpha_{\textbf{theory}}$:** Once the full expression for α_{theory} is obtained (a personal exercise), we can possibly try and account for $B_{unknown}$ outside the dispersive Lorentzian regime, and given that we have system parameters such as the saturation parameter, and relaxation time on hand, we can make estimates of maximum possible dynamic range and the sensitivity of the system.
- **Centering around the Dispersive Lorentzian Regime:** Thinking of alternative and quicker methods to set initial B_{bias} such that we are within the dispersive Lorentzian regime while initialising our bayesian algorithm.
- **Incorporate B_{bias} uncertainty:** To account for the fact that the knowledge we have about our controllable parameter B_{bias} is not absolute and to incorporate its own uncertainty in the bayes rule update.