Expedition Project (Round 2)

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1 What is a magnetometer?

A magnetometer is a device that measures the local magnetic field strength. They are used in consumer devices to measure magnetic heading (relative to the magnetic North Pole), and when combined with position, can be used to calculate absolute heading (relative to the actual North Pole). They cannot work if perpendicular to the Earth's magnetic field, so Inertial Measurement Units(IMUs) use 3 orthogonal (i.e. perpendicular) magnetometers.

2 How does interference affect magnetometers?

A major issue of using magnetometers to calculate magnetic heading is that they are susceptible to interference from magnetic fields. The various types of interference are shown below:

- Hard iron errors (magnetic fields that add or subtract to the earth's magnetic field)
  - Permanent magnets
    - Eg. Ferromagnetic material, DC motors, minerals, etc.
  - Power cables
    - Eg. ESC wires in drones
- Soft iron errors (changes the Earth's local magnetic field experiences when near magnetic materials)
  - Paramagnetic materials
    - Eg. Carbon fiber, steel, nickel, etc.

An ideal magnetometer in a perfect environment, when rotated along each plane (XY, XZ, YZ), would generate a perfect circle (if each sampled coordinate was plotted on a 2D graph) with a radius defined by the Earth's local magnetic field strength. However, various types of interference can lead to magnetometer data appearing more like this:

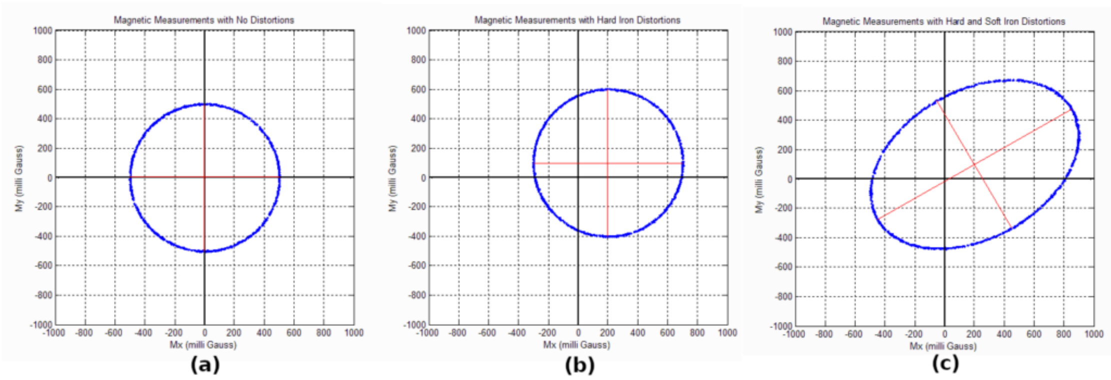

Figure 1: Magnetometer measurements affected by hard iron and soft iron distortions.

As seen, hard iron distortions cause the magnetic measurements to be off-center and soft iron distortions cause the magnetic measurements to be distorted (stretched/squashed/rotated). To address this issue, we can calculate a matrix M that when multiplied by each point x, will yield a unit sphere. Since the magnetometer measurements will never form a perfect ellipsoid due to noise, we can use a least-squares constraint-based fitting algorithm to generate an ellipsoid that most closely fits the points.

3 How do we model an ellipsoid and fit our points to one?

A quadric surface (ellipsoid, hyperboloid, etc.) can be represented by the below second degree equation of three variables.

$$ax^2 + bx^2 + cx^2 + 2fyz + 2gxz + 2hxy + 2px + 2qy + 2rz + d = 0 \text{ —- } (1)$$

Let

$$I = a + b + c, \; J = ab + bc + ac + f^2 + g^2 + h^2 \text{ —- } (2)$$

when $4J - I^2 > 0$, equation (1) must represent an ellipsoid.

$$X_i = \begin{bmatrix} x_i^2 & y_i^2 & z_i^2 & 2y_iz_i & 2x_iz_i & 2x_iy_i & 2x_i & 2y_i & 2z_i & 1 \end{bmatrix}$$

$$v = \begin{bmatrix} a & b & c & f & g & h & p & q & r & d \end{bmatrix}^T \text{ is a 10x1 matrix}$$

$$D = \begin{bmatrix} X_1 \\ X_2 \\ \dots \\ X_i \\ .. \\ X_n \end{bmatrix} \text{ D is a Nx10 matrix}$$

Let's find v subject to the constraint $kJ - I^2 = 1, k = 4$ (Lagrange multiplier)

$$L = ||Dv||^2 + \lambda(kJ - I^2) \text{ ——————— } (3)$$

$$\nabla \mathrm{L} = \mathrm{D}^T Dv - \lambda \mathrm{Cv} = 0 \text{ ——————— } (4)$$

where $C = \begin{bmatrix} C_1 & 0_{6x4} \\ 0_{4x6} & 0_{4x4} \end{bmatrix}$, and $C_1 = \begin{bmatrix} -1 & k/2-1 & k/2-1 & 0 & 0 & 0 \\ k/2-1 & -1 & k/2-1 & 0 & 0 & 0 \\ k/2-1 & k/2-1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & 0 \\ 0 & 0 & 0 & 0 & 0 & -k \end{bmatrix}$

$$D^T D = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22}, \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

where matrices S_{11}, S_{12}, S_{22} are of size 6x6, 6x4, 4x4 vector v_1, v_2 are of size 6 and 4.

$$(S_{11} - \lambda C_1)v_1 + S_{12}v_2 = 0 \text{ ——————— } (5)$$

$$S_{12}^T v_1 + S_{22}v_2 = 0 \text{ ——————— } (6)$$

$$v_2 = -S_{22}^{-1} S_{12}^T v_1$$

Substituting the equation for v2 in equation 6 we get the following eigen system

$$C_1^{-1}(S_{11} - S_{12}S_{22}^{-1}S_{12}^T)v_1 = \lambda v_1 \text{ ———— } (7)$$

In most cases, matrix $C_1^{-1}(S_{11} - S_{12}S_{22}^{-1}S_{12}^T)$ will be positive and eigen system (7) will have and only have one positive eigen value. Let u_1 be the eigenvector associated with the only positive eigenvalue of the general eigen system(7), and let $u_2 = -S_{22}^{-1}S_{12}^T u_1$, then $u = [u_1, u_2]^T$ will be the solution to (4). However, the matrix $(S_{11} - S_{12}S_{22}^{-1}S_{12}^T)$ can be singular in some cases. In this situations, the corresponding u_1 can be replaced with the eigenvector associated with the largest eigenvalue.

4 Transforming Ellipsoid to unit sphere

Let X be a point on the surface of a sphere centered around the origin with radius r. The sphere can be represented by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad X^T X = r^2,\ x^2 + y^2 + z^2 = r^2$$

The above sphere can be transformed to an ellipsoid by applying the following transformation.

$$X' = O + R.S.X \qquad O = \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix} R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

O represents origin of the ellipsoid. R the rotation matrix. S the scaling matrix.

Let's find the inverse mapping for X (transforms ellipsoid to sphere)

$$X = (RS)^{-1}.(X' - O) \text{ Let's define } M = (RS)^{-1}$$

$$X = M(X' - O) \text{ ———————————— } (8)$$

$$X^T X = (X' - O)^T M^T M(X' - O) = r^2$$

$$(X' - O)^T A(X' - O) = r^2 \text{ where } A = M^T M \text{ is a symmetric Matrix}$$

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

Let's expand the expression $(X - O)^T A(X - O) = r^2$

$$(X^T - O^T)A(X - O) = r^2$$

$$X^T A(X - O) - O^T A(X - O) = r^2$$

$$X^T AX - X^T AO - O^T AX + O^T AO = r^2$$

$$X^T AO = O^T AX$$

$$X^T AX - 2O^T AX + O^T AO - r^2 = 0 \text{ ——————— } (9)$$

Equation(9) represents the quadric surface in terms of matrices and it is equal to equation (1). Origin of the ellipsoid O can be extracted as below.

$$O^T A = -\begin{bmatrix} p & q & r \end{bmatrix}, AO = -\begin{bmatrix} p \\ q \\ r \end{bmatrix}, \text{ A is symmetric matrix}$$

$$O = A^{-1} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \text{ ——— } (10)$$

$$d = O^T AO - r^2$$

$$r = \sqrt{O^T AO - d} \text{ ——— } (11)$$

Let's find M

$$M^T M = A, A = PDP^T$$

P is the eigenvectors of A. D a diagonal matrix with the eigenvalues of A.

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$M = PD_{sqrt}P^T \text{ ——— } (12)$$

$$D_{sqrt} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix}$$

The below transform maps the ellipsoid to a unit sphere

$$X = \frac{1}{r} M(X' - O) \text{ ——- } (13)$$