Probability Handout 1: Introduction to the Algebra of Sets

MATH 530-630

Sets

Collection of **elements** or **numbers** or **outcomes** or **events**.

Sets usually denoted by capital letters, A, B, C, etc.
Elements of sets denoted by a, b, c, etc.

If a belongs to A we write

$a \in A$

If a does not belong to A we write

$a \notin A$

Example 1: V = {x | x is a vowel} (read "|" as "such that")

= {a, e, i, o, u}

This set can be **rostered**, i.e. listed - A *finite* set.

Example 2: D = {x | x are points on a die}

= {1, 2, 3, 4, 5 , 6}

This set also can be rostered.
These sets are **discrete**, **but not all discrete sets can be rostered**.

Example 3: P = {x | x is a prime number}

This is a **denumerably infinite** set.

Example 4: T = {x | x is a triangle in the plane}

This is a nondenumerable, infinite set.

Subsets

If every element of A also belongs to B, we call A a subset of B.

This is denoted

$A \in B$

If $A \in B$ and $B \in A$, then A = B. [$A \subseteq B$]

If $A \in B$ and $A \neq B$, then A is a **proper subset** of B. [$A \subset B$]

Every set is a subset of itself.

Example 1: Tossing a die where outcomes are even constitutes a subset of the set of possible outcomes.

$$\{2, 4, 6\} \subset \{1, 2, 3, 4, 5, 6\}.$$

All sets are subsets of some particular set called the **Universal Set ($\boldsymbol{U}$)**. This is also called in probability theory the **Sample Space ($\boldsymbol{S}$)**, the set of all possible simple events or outcomes of an experiment.

Example 2: The set of all possible outcomes for the toss of a die is

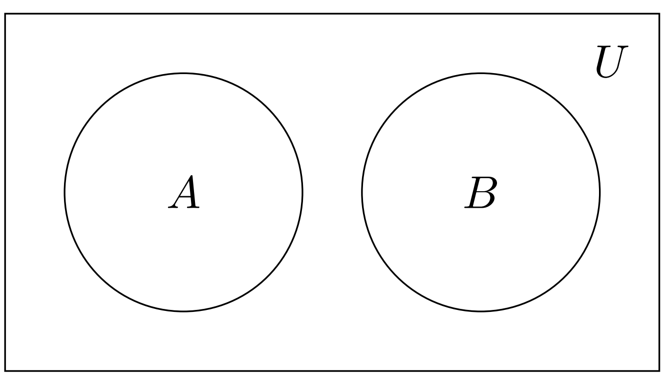
$$\text{S} = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$$

A set having no elements is called the null or empty set, denoted $\varnothing$.

Venn Diagrams

[British mathematician John Venn, 1834-1923; though Euler had a similar idea much earlier.]

Venn Diagrams provide geometric intuition about relations among sets, but they do **not** represent or substitute for formal proofs. Moreover, they are not practical when the number of sets exceeds three or so.



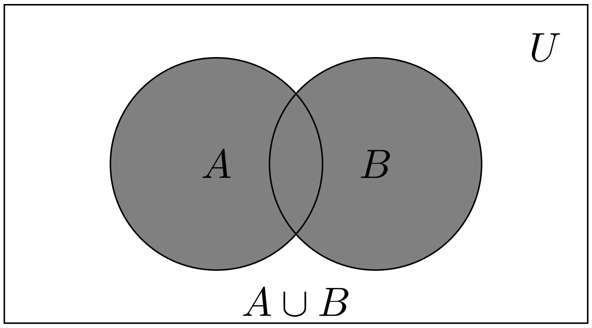
Set Operations

1. Union

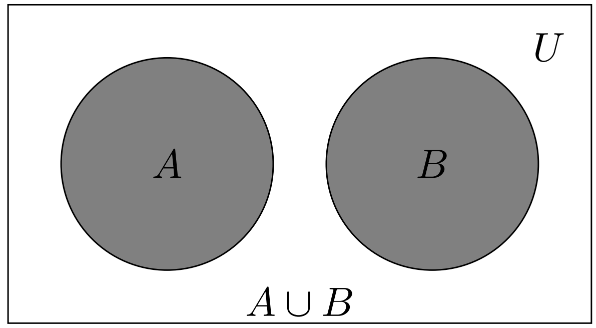
The union of sets A and B, denoted $A \cup B$, consists of all elements belonging to A **or** B [or both A and B].

$$x \in A \cup V \Rightarrow x \in A \textbf{ or } x \in B \textbf{ or } x \in \text{both A and B.}$$

(read "$\Rightarrow$" as "implies").



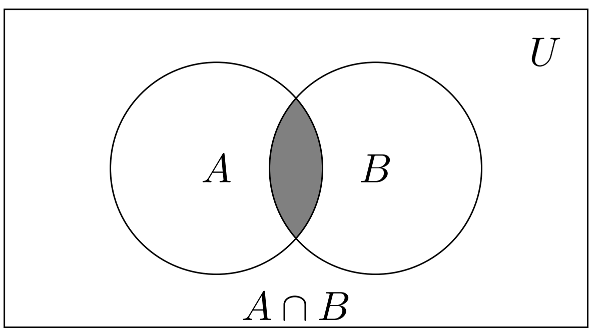
$A \cup B$



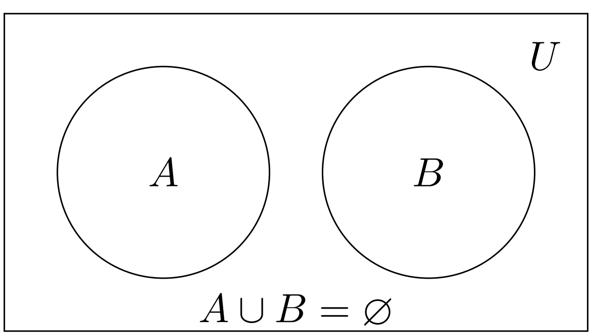
$A \cup B$

2. Intersection

The set of all elements belonging to **both** A **and** B, denoted by $A \cap B$ (or sometimes AB).



$x \in A \cap B \Rightarrow x \in A$ **and** $x \in B$

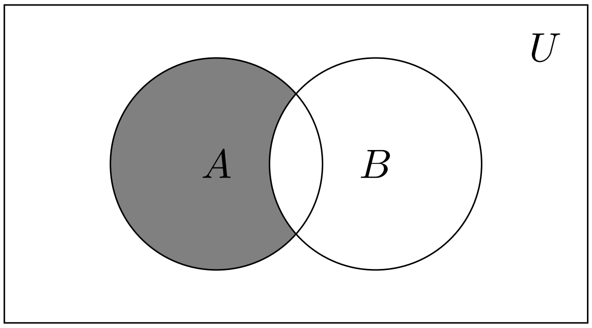


$A \cup B = \varnothing$ [Sets are mutually exclusive]

If $x \in A$, $x \notin B$ and vice versa.

3. Difference

The set of elements in A that **do not** belong to B is called the difference of A and B and denoted A – B (sometimes A/B).



$x \in A - B \Rightarrow \quad x \in A$ and $x \notin B$

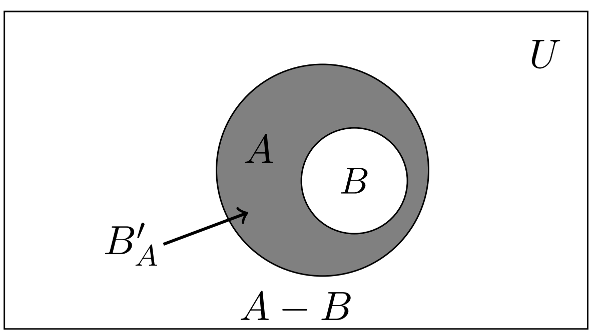
4. Complement

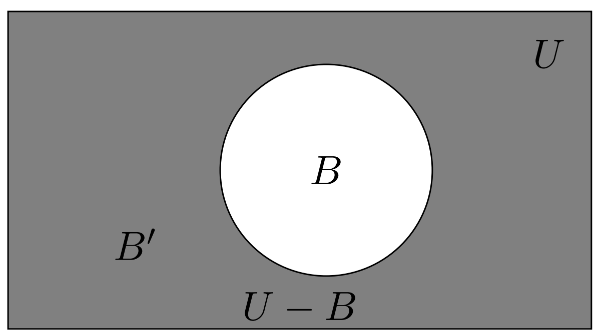
If $B \subset A$, then A – B is calld the complement of B relative to A, denoted

B'_A $\left[\text{or } \bar{B}_A \text{ or } -B_A\right]$

If A=U, U-B is called simply the complement of B denoted

B' $\left[\text{or } \bar{B} \text{ or } \sim B\right]$





Some Theorems Involving Sets

You should be able to **prove** the following:

Theorem 1-1. If $A \subset B$ and $B \subset C$, then $A \subset C$.

$x \in A \Rightarrow x \in B \Rightarrow x \in C$.

Then $x \in A \Rightarrow x \in C$.

Theorem 1-2. $A \cup B = B \cup A$. [Commutative law for unions].

Theorem 1-3. $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$. [Associative law for unions].

Theorem 1-4. $A \cap B = B \cap A$. [Commutative law for intersections].

Theorem 1-5. $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$. [Associative law for intersections].

Theorem 1-6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. [First distributive law].

Theorem 1-7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. [Second distributive law].

[In 1-6 and 1-7 "$\cup$" and "$\cap$" are exchanged - "Principle of Duality"]

Theorem 1-8. $A - B = A \cap B'$.

Theorem 1-9. If $A \subset B$, then $B' \subset A'$.

Theorem 1-10. $A \cup \varnothing = A$. $A \cap \varnothing = \varnothing$.

Theorem 1-11. $A \cup U = U$. $A \cap U = A$.

Theorem 1-12a. $(A \cup B)' = A' \cap B'$. [DeMorgan's First Law].

Theorem 1-12b. $(A \cap B)' = A' \cup B'$. [DeMorgan's Second Law].

Theorem 1-13. $A = (A \cap B) \cup (A \cap B')$.

Principle of Duality: Any true relation remains true if we replace '$\cup$' by '$\cap$', '$\cap$' by '$\cup$', sets by their complements and if we reverse inclusion symbols '$\subset$' and '$\supset$'.

Consider the following sets:

U = {1/2, 0, π, 5, $-\sqrt{2}$, -4}

A = {$-\sqrt{2}$, π, 0}

B = {5, 1/2, $-\sqrt{2}$, -4}

C = {1/2, -4}

Use these sets to illustrate Theorems 2-13.

a) $A \cup B = \{1/2, 0, \pi, 5, -\sqrt{2}, -4\} = B \cup A$.

b) $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C = \{1/2, 0, \pi, 5, -\sqrt{2}, -4\}$.

c) $A \cap B = \{-\sqrt{2}\} = B \cap A$.

d) $A \cap (B \cup C) = A \cap$ ({1/2, -4}) = $\{-\sqrt{2}, \pi, 0\} \cap$ {1/2, -4} = $\varnothing$

e) $A \cap (B \cap C) = \{-\sqrt{2}, \pi, 0\} \cap \{5, 1/2, -\sqrt{2}, -4\} = (A \cap B) \cup (A \cap C) = \{-\sqrt{2}\} \cup \varnothing = \{-\sqrt{2}\}$.

f) $A \cup (B \cap C) = \{-\sqrt{2}, \pi,$ 0} $\cup$ {1/2,-4} = {1/2, 0, π, 5, $-\sqrt{2}$, -4}.
$= (A \cup B) \cap (A \cup C)$ = {1/2, 0, π, 5, $-\sqrt{2}$, -4} $\cap$ {1/2, 0, π, 5, $-\sqrt{2}$, -4} = {1/2, 0, π, 5, $-\sqrt{2}$, -4}.

g) $B - A = B \cap A'$
$\{5, 1/2, -\sqrt{2}, -4\}$ - $\{-\sqrt{2}, \pi,$ 0} = {5, 1/2, -4} = $\{5, 1/2, -\sqrt{2}, -4\} \cap$ {5, 1/2, -4}.

h) $C \subset B$
{1/2, -4} $\subset \{5, 1/2, -\sqrt{2}, -4\}$.
$C' = \{0, \pi, 5, -\sqrt{2}\}$.
$B' = \{0, \pi\} \Rightarrow B' \subset A'$.

i) $A \cup U$ = {1/2, 0, π, 5, $-\sqrt{2}$, -4}.

j) $(A \cup B)' = \varnothing$. $A' \cap B'$ = {5, 1/2, -4} $\cap$ {π, 0} = $\varnothing$.

k) $(A \cap B)'$ = {1/2, 0, π, 5, -4}.
$A' \cup B'$ = {5, 1/2, -4} $\cup$ {π, 0} = {1/2, 0, π, 5, -4}.

l) (A $\cap$ B) $\cup$ (A $\cap B'$) = {1/2} $\cup$ $\{-\sqrt{2}, \pi, 0\} \cap$ {π, 0} = {$-\sqrt{2}, \pi$, 0} = A.