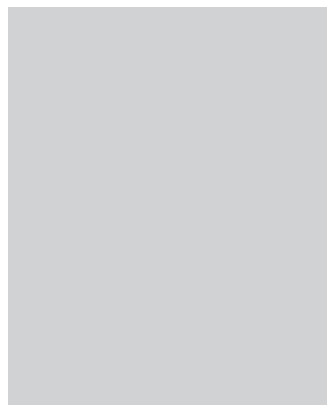


- 9.1 Systems of Linear Inequalities
- 9.2 Linear Programming Involving Two Variables
- 9.3 The Simplex Method: Maximization
- 9.4 The Simplex Method: Minimization
- 9.5 The Simplex Method: Mixed Constraints

# 9 LINEAR PROGRAMMING



*John  
von  
Neumann*

1 9 0 3 – 1 9 5 7

*J*ohn von Neumann was born in Budapest, Hungary, where his father was a successful banker. John's genius was recognized at an early age. By the age of ten, his mathematical knowledge was so great that instead of attending regular classes he studied privately under the direction of leading Hungarian mathematicians.

At the age of twenty-one, he acquired two degrees, one in chemical engineering at Zurich, and the other a Ph.D. in mathematics from the University of Budapest. He spent some time teaching at the University of Berlin, and then, in 1930, accepted a visiting professorship at Princeton University. In 1933, John von Neumann and Albert Einstein were among the first full professors to be appointed to the newly organized Institute for Advanced Study at Princeton.

During World War II, von Neumann was a

consultant at Los Alamos, and his research helped in the development of the atomic bomb. In 1954, President Eisenhower appointed him to the Atomic Energy Commission.

von Neumann is considered to be the father of modern game theory—a branch of mathematics that deals with strategies and decision making. Much of his results concerning game theory were published in a lengthy paper in 1944 titled *Theory of Games and Economic Behavior*, written with Oskar Morgenstern.

In 1955, John von Neumann was diagnosed with cancer—he died in 1957 at the age of 53. Many stories are told of his mental abilities. Even during the final months of his life, as his brother read to him in German from Goethe's *Faust*, each time a page was turned John would recite from memory the continuation of the passage on the following page.

## 9.1 SYSTEMS OF LINEAR INEQUALITIES

The following statements are inequalities in two variables.

$$3x - 2y < 6 \quad \text{and} \quad x + y \geq 6$$

An ordered pair  $(a, b)$  is a **solution of an inequality** in  $x$  and  $y$  if the inequality is true when  $a$  and  $b$  are substituted for  $x$  and  $y$ , respectively. For instance,  $(1, 1)$  is a solution of the inequality  $3x - 2y < 6$  because  $3(1) - 2(1) = 1 < 6$ . The **graph** of an inequality is the collection of all solutions of the inequality. To sketch the graph of an inequality such as

$$3x - 2y < 6$$

we begin by sketching the graph of the *corresponding equation*

$$3x - 2y = 6.$$

The graph of the equation will normally separate the plane into two or more regions. In each such region, one of the following must be true. (1) All points in the region are solutions of the inequality. (2) No points in the region are solutions of the inequality. Thus, we can determine whether the points in an entire region satisfy the inequality by simply testing *one* point in the region.

### Sketching the Graph of an Inequality in Two Variables

1. Replace the inequality sign by an equal sign, and sketch the graph of the resulting equation. (We use a dashed line for  $<$  or  $>$  and a solid line for  $\leq$  or  $\geq$ .)
2. Test one point in each of the regions formed by the graph in Step 1. If the point satisfies the inequality, then shade the entire region to denote that every point in the region satisfies the inequality.

In this section, we will work with **linear inequalities** of the form

$$ax + by < c \quad ax + by \leq c$$

$$ax + by > c \quad ax + by \geq c.$$

The graph of each of these linear inequalities is a half-plane lying on one side of the line  $ax + by = c$ . The simplest linear inequalities are those corresponding to horizontal or vertical lines, as shown in Example 1.

#### EXAMPLE 1 Sketching the Graph of a Linear Inequality

Sketch the graphs of (a)  $x > -2$  and (b)  $y \leq 3$ .

- Solution**
- (a) The graph of the corresponding equation  $x = -2$  is a vertical line. The points that satisfy the inequality  $x > -2$  are those lying to the right of this line, as shown in Figure 9.1.
  - (b) The graph of the corresponding equation  $y = 3$  is a horizontal line. The points that satisfy the inequality  $y \leq 3$  are those lying below (or on) this line, as shown in Figure 9.2.

Figure 9.1

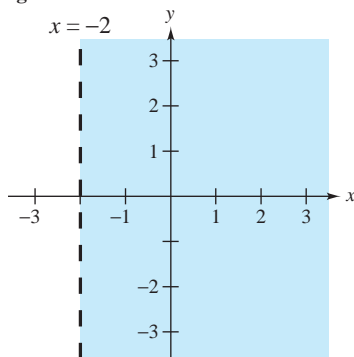
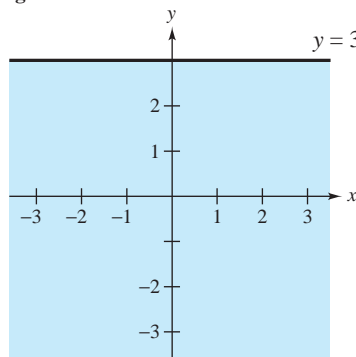


Figure 9.2

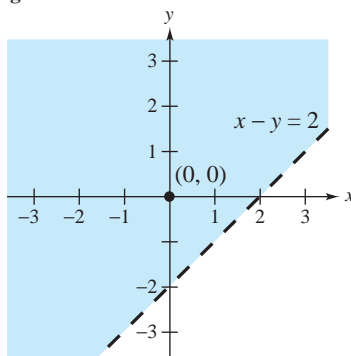


**EXAMPLE 2** *Sketching the Graph of a Linear Inequality*

Sketch the graph of  $x - y < 2$ .

**Solution** The graph of the corresponding equation  $x - y = 2$  is a line, as shown in Figure 9.3. Since the origin  $(0, 0)$  satisfies the inequality, the graph consists of the half-plane lying above the line. (Try checking a point below the line. Regardless of which point you choose, you will see that it does not satisfy the inequality.)

**Figure 9.3**



For a linear inequality in two variables, we can sometimes simplify the graphing procedure by writing the inequality in *slope-intercept* form. For instance, by writing  $x - y < 2$  in the form

$$y > x - 2$$

we can see that the solution points lie *above* the line  $y = x - 2$ , as shown in Figure 9.3. Similarly, by writing the inequality  $3x - 2y > 5$  in the form

$$y < \frac{3}{2}x - \frac{5}{2}$$

we see that the solutions lie *below* the line  $y = \frac{3}{2}x - \frac{5}{2}$ .

## Systems of Inequalities

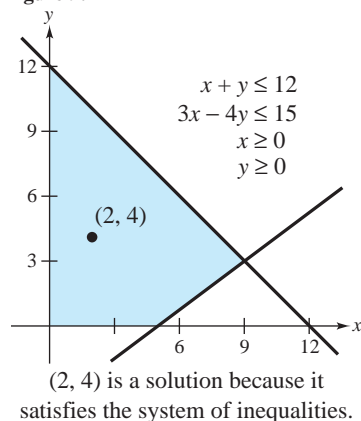
Many practical problems in business, science, and engineering involve systems of linear inequalities. Here is an example of such a system.

$$\begin{aligned}x + y &\leq 12 \\3x - 4y &\leq 15 \\x &\geq 0 \\y &\geq 0\end{aligned}$$

A **solution** of a system of inequalities in  $x$  and  $y$  is a point  $(x, y)$  that satisfies each inequality in the system. For instance,  $(2, 4)$  is a solution of this system because  $x = 2$  and  $y = 4$  satisfy each of the four inequalities in the system. The **graph** of a system of inequalities in two variables is the collection of all points that are solutions of the system. For instance, the graph of the system above is the region shown in Figure 9.4. Note that the point  $(2, 4)$  lies in the region because it is a solution of the system of inequalities.

To sketch the graph of a system of inequalities in two variables, we first sketch the graph of each individual inequality (on the same coordinate system) and then find the region that is *common* to every graph in the system. For systems of linear inequalities, it is helpful to find the *vertices* of the solution region, as shown in the following example.

Figure 9.4



### EXAMPLE 3 Solving a System of Inequalities

Sketch the graph (and label the vertices) of the solution set of the following system.

$$x - y < 2$$

$$x > -2$$

$$y \leq 3$$

**Solution** We have already sketched the graph of each inequality in Examples 1 and 2. The triangular region common to all three graphs can be found by superimposing the graphs on the same coordinate plane, as shown in Figure 9.5. To find the vertices of the region, we find the points of intersection of the boundaries of the region.

Vertex A:  $(-2, -4)$

Obtained by finding  
the point of  
intersection of  
 $x - y = 2$   
 $x = -2$ .

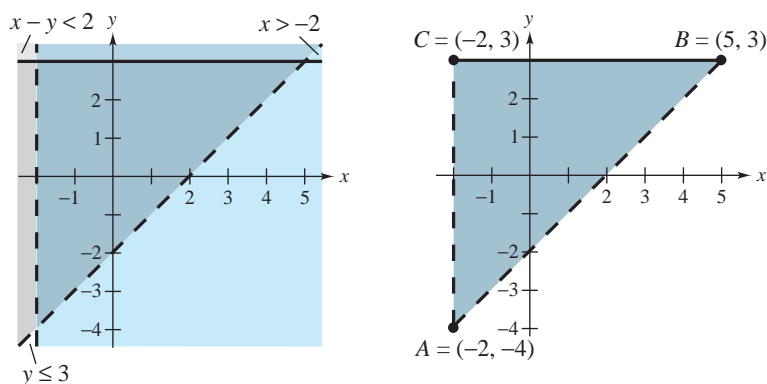
Vertex B:  $(5, 3)$

Obtained by finding  
the point of  
intersection of  
 $x - y = 2$   
 $y = 3$ .

Vertex C:  $(-2, 3)$

Obtained by finding  
the point of  
intersection of  
 $x = -2$   
 $y = 3$ .

Figure 9.5



For the triangular region shown in Figure 9.5, each point of intersection of a pair of boundary lines corresponds to a vertex. With more complicated regions, two border lines can sometimes intersect at a point that is not a vertex of the region, as shown in Figure 9.6. In order to keep track of which points of intersection are actually vertices of the region, we suggest that you make a careful sketch of the region and refer to your sketch as you find each point of intersection.

When solving a system of inequalities, you should be aware that the system might have no solution. For instance, the system

$$x + y > 3$$

$$x + y < -1$$

has no solution points because the quantity  $(x + y)$  cannot be both less than  $-1$  and greater than  $3$ , as shown in Figure 9.7.

Figure 9.6

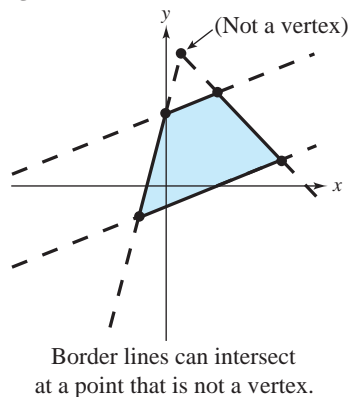
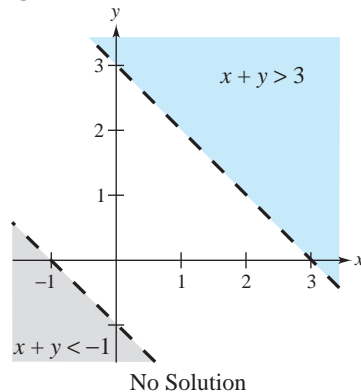


Figure 9.7



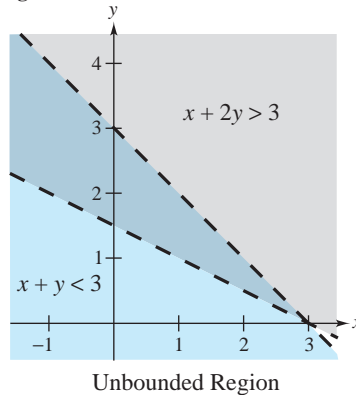
Another possibility is that the solution set of a system of inequalities can be unbounded. For instance, the solution set of

$$x + y < 3$$

$$x + 2y > 3$$

forms an *infinite wedge*, as shown in Figure 9.8.

Figure 9.8



## Applications

Our last example in this section shows how a system of linear inequalities can arise in an applied problem.

### EXAMPLE 4 An Application of a System of Inequalities

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Set up a system of linear inequalities that describes the minimum daily requirements for calories and vitamins.

**Solution** We let

$x$  = number of cups of dietary drink X

$y$  = number of cups of dietary drink Y.

Then, to meet the minimum daily requirements, the following inequalities must be satisfied.

For calories:  $60x + 60y \geq 300$

For vitamin A:  $12x + 6y \geq 36$

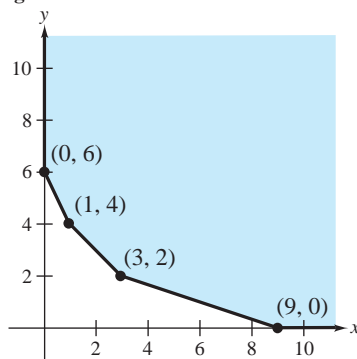
For vitamin C:  $10x + 30y \geq 90$

$$x \geq 0$$

$$y \geq 0$$

The last two inequalities are included because  $x$  and  $y$  cannot be negative. The graph of this system of linear inequalities is shown in Figure 9.9.

**Figure 9.9**

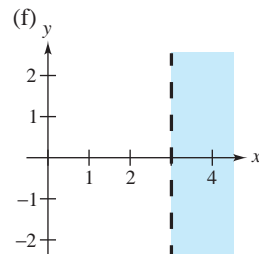
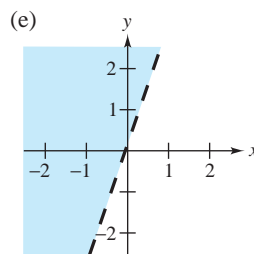
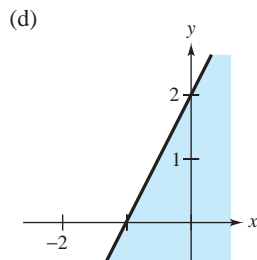
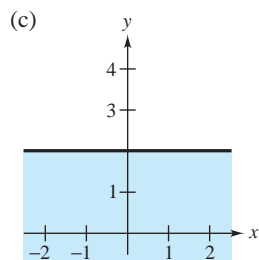
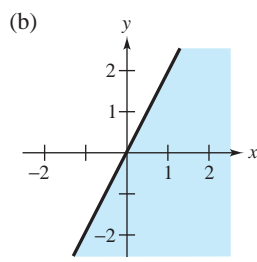
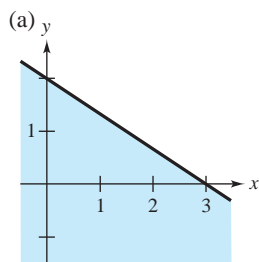


Any point inside the region shown in Figure 9.9 (or on its boundary) meets the minimum daily requirements for calories and vitamins. For instance, 3 cups of dietary drink X and 2 cups of dietary drink Y supply 300 calories, 48 units of vitamin A, and 90 units of vitamin C.

## SECTION 9.1 EXERCISES

In Exercises 1–6, match the linear inequality with its graph. [The graphs are labeled (a)–(f).]

- |                     |                         |                     |
|---------------------|-------------------------|---------------------|
| 1. $x > 3$          | 2. $y \leq 2$           | 3. $2x + 3y \leq 6$ |
| 4. $2x - y \geq -2$ | 5. $x \geq \frac{y}{2}$ | 6. $y > 3x$         |



In Exercises 7–22, sketch the graph of the given linear inequality.

- |                       |                        |                     |
|-----------------------|------------------------|---------------------|
| 7. $x \geq 2$         | 8. $x \leq 4$          | 9. $y \geq -1$      |
| 10. $y \leq 3$        | 11. $y < 2 - x$        | 12. $y > 2x - 4$    |
| 13. $2y - x \geq 4$   | 14. $5x + 3y \geq -15$ |                     |
| 15. $y \leq x$        | 16. $3x > y$           | 17. $y \geq 4 - 2x$ |
| 18. $y \leq 3 + x$    | 19. $3y + 4 \geq x$    | 20. $6 - 2y < x$    |
| 21. $4x - 2y \leq 12$ | 22. $y + 3x > 6$       |                     |

In Exercises 23–32, sketch the graph of the solution of the given system of linear inequalities.

- |                |                 |                    |
|----------------|-----------------|--------------------|
| 23. $x \geq 0$ | 24. $x \geq -1$ | 25. $x + y \leq 1$ |
| $y \geq 0$     | $y \geq -1$     | $-x + y \leq 1$    |
| $x \leq 2$     | $x \leq 1$      | $y \geq 0$         |
| $y \leq 4$     | $y \leq 2$      |                    |

26.  $3x + 2y < 6$   
 $x > 0$   
 $y > 0$
27.  $x + y \leq 5$   
 $x \geq 2$   
 $y \geq 0$
28.  $2x + y \geq 2$   
 $x \leq 2$   
 $y \leq 1$
29.  $-3x + 2y < 6$   
 $x + 4y > -2$   
 $2x + y < 3$
30.  $x - 7y > -36$   
 $5x + 2y > 5$   
 $6x - 5y > 6$
31.  $x \geq 1$   
 $x - 2y \leq 3$   
 $3x + 2y \geq 9$   
 $x + y \leq 6$
32.  $x + y < 10$   
 $2x + y > 10$   
 $x - y < 2$

In Exercises 33–36, derive a set of inequalities to describe the given region.

33. Rectangular region with vertices at (2, 1), (5, 1), (5, 7), and (2, 7).
34. Parallelogram with vertices at (0, 0), (4, 0), (1, 4), and (5, 4).
35. Triangular region with vertices at (0, 0), (5, 0), and (2, 3).
36. Triangular region with vertices at (−1, 0), (1, 0), and (0, 1).
37. A furniture company can sell all the tables and chairs it produces. Each table requires 1 hour in the assembly center and  $1\frac{1}{3}$  hours in the finishing center. Each chair requires  $1\frac{1}{2}$  hours in the assembly center and  $1\frac{1}{2}$  hours in the finishing center. The company's assembly center is available 12 hours per day, and its finishing center is available 15 hours per day. If  $x$  is the number of tables produced per day and  $y$  is the number of chairs, find a system of inequalities describing all possible production levels. Sketch the graph of the system.
38. A store sells two models of a certain brand of computer. Because of the demand, it is necessary to stock at least twice as many units of model A as units of model B. The cost to the store for the two models is \$800 and \$1200, respectively. The management does not want more than \$20,000 in computer inventory at any one time, and it wants at least four model A computers and two model B computers in inventory at all times. Devise a system of inequalities describing all possible inventory levels, and sketch the graph of the system.

39. A person plans to invest no more than \$20,000 in two different interest-bearing accounts. Each account is to contain at least \$5000. Moreover, one account should have at least twice the amount that is in the other account. Find a system of inequalities to describe the various amounts that can be deposited in each account, and sketch the graph of the system.
40. Two types of tickets are to be sold for a concert. One type costs \$15 per ticket and the other type costs \$25 per ticket. The promoter of the concert must sell at least 15,000 tickets including 8000 of the \$15 tickets and 4000 of the \$25 tickets. Moreover, the gross receipts must total at least \$275,000 in order for the concert to be held. Find a system of inequalities describing the different numbers of tickets that can be sold, and sketch the graph of the system.
41. A dietitian is asked to design a special diet supplement using two different foods. Each ounce of food X contains 20 units of calcium, 15 units of iron, and 10 units of vitamin B. Each ounce of food Y contains 10 units of calcium, 10 units of iron, and 20 units of vitamin B. The minimum daily requirements in the diet are 300 units of calcium, 150 units of iron, and 200 units of vitamin B. Find a system of inequalities describing the different amounts of food X and food Y that can be used in the diet, and sketch the graph of the system.
42. Rework Exercise 41 using minimum daily requirements of 280 units of calcium, 160 units of iron, and 180 units of vitamin B.