It may be shown that a point $\mathbf{x}$ is an extreme point of F if and only if the corresponding point $(\mathbf{x}, \mathbf{s})'$ is a basic feasible solution to the system $A\mathbf{x} + \mathbf{s} = \mathbf{b}$, $(\mathbf{x}, \mathbf{s})' \geq \mathbf{0}$, where we have added the vector $\mathbf{s}$ of slack variables. The simplex method works only with basic feasible solutions.

Clearly a basic solution can have at most m non-zero co-ordinates. Often we meet problems such that a non-negative solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ is a basic feasible solution if and only if it has exactly m non-zero co-ordinates. Such problems are called *non-degenerate*. Now let us ignore the algebra and get started on the simplex method.

2 Simplex Method

2.1 Solving the Chemist example by the simplex method

The example (in equality form) may be written as the problem of choosing activity levels $x_j \geq 0$ in order to maximise the profit z where

$$\begin{array}{ccccccccccccc}
z & - & x_1 & - & x_2 & & & & & & & = & 0 \\
 & & 2x_1 & + & x_2 & + & x_3 & & & & & = & 11 \\
 & & x_1 & + & 3x_2 & & & + & x_4 & & & = & 18 \\
 & & x_1 & & & & & & & + & x_5 & = & 4
\end{array}$$

Step 1 Find an initial basic feasible solution. We take the obvious one $\mathbf{x} = (0,\ 0,\ 11,\ 18,\ 4)'$, with basic variables the slack variables x_3, x_4, x_5, and non-basic variables x_1, x_2 set to 0. (This corresponds to throwing away all resources, for no profit.)

Step 2 Since the profit $z = x_1 + x_2$ the current solution may be improved by increasing x_1 or x_2. Let us increase x_2, whilst leaving x_1 fixed at 0.

Step 3 When x_2 increases the basic variables x_3, x_4, x_5 must adjust in order to maintain a feasible solution, and we can continue to increase x_2 until one of these basic variables is pushed down to 0. From the first constraint we must keep $x_2 \leq 11$, and from the second we need $x_2 \leq 18/3 = 6$. The third constraint gives no upper bound on x_2 since the coefficient of x_2 is ≤ 0.

Thus x_2 can be increased up to at most 6. We now increase x_2 to 6 and obtain a new basic feasible solution $(0, 6, 5, 0, 4)'$ with z-value 6, in which the entering variable x_2 has replaced the leaving variable x_4 in the basis. In the figure we started at vertex O of the feasible region and moved along an edge (the x_2-axis) until we were stopped by the inequality corresponding to the slack variable x_4, at the new vertex P.

The above calculation was easy since
(a) The z-equation expresses the objective function in terms of the non-basic variables only. Thus when we increase x_2 (and adjust the basic variables) we know that the objective function z increases, at a rate here of 1 per unit increase in x_2.
(b) Each of the constraint equations involves exactly one basic variable, with coefficient 1.

Step 4 We now re-arrange the z-equation and constraint equations so that the conditions (a) and (b) above hold for the new basic feasible solution. The second constraint equation gave the new value 6 for the entering variable x_2 and determined the leaving variable x_4. Divide this equation by 3 to obtain the *pivot equation*

$$\frac{1}{3}x_1 + x_2 \quad + \frac{1}{3}x_4 \quad = 6,$$

and subtract multiples of the pivot equation from the z-equation and the other two constraint equations so as to eliminate x_2 from them. Expressed differently, we are using the pivot equation to substitute out for the entering variable x_2 in the other equations.

The calculations are best organised (for hand calculations) by using a 'simplex tableau'. Let us take it from the start again.

Step 1 The initial tableau corresponds to the initial equations, with basic feasible solution $(0, 0, 11, 18, 4)'$ and basic variables the slacks x_3, x_4, x_5, forming the initial basis.

		z	x_1	x_2	x_3	x_4	x_5		
(MAX)	z	1	-1	-1	0	0	0	0	R_0
current	x_3	0	2	1	1	0	0	11	R_1
basic	x_4	0	1	**3**	0	1	0	18	R_2
variables	x_5	0	1	0	0	0	1	4	R_3

Step 2 Since in the z-equation (row R_0) both x_1 and x_2 have coefficients < 0, increasing either x_1 or x_2 will increase z. We choose x_2 say as the *entering* variable.

Step 3 Now for each constraint row R_1, R_2, R_3 in turn, we compute the ratio of the value in the last column (right hand side) to the value in the entering variable column if this latter is > 0. Here we obtain 11, 6, -. Since the minimum is attained in row R_2, the corresponding basic variable x_4 is the *leaving* variable.

Step 4 *Pivot* on the entry (value 3) in the entering variable column (x_2) and leaving variable row (R_2). We divide row R_2 by 3 and from each other row we subtract the multiple of this *pivot row* R^* that yields a zero in the x_2 column.

z	1	$-\frac{2}{3}$	0	0	$\frac{1}{3}$	0	6	$R_0' = R_0 + R^*$
x_3	0	$\mathbf{\frac{5}{3}}$	0	1	$-\frac{1}{3}$	0	5	$R_1' = R_1 - R^*$
x_2	0	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	6	R^*
x_5	0	1	0	0	0	1	4	$R_3' = R_3$

This tableau corresponds to the basic feasible solution $(0, 6, 5, 0, 4)$ with basic variables x_3, x_2, x_5 and objective function value 6. As we noted earlier, in the figure we have moved along an edge of the feasible region from vertex O to vertex P. We are now ready for the next iteration, and return to step 2.

Consider the first row R_0 of the new tableau. It shows that for all feasible solutions the objective function value z is expressed in terms of the current non-basic variables x_1 and x_4 by

$$z - \frac{2}{3}x_1 \quad + \frac{1}{3}x_4 \quad = 6.$$

Call minus the coefficients of the non-basic variables their *reduced profits*, since they give the increase of the profit z per unit increase in that non-basic variable (with the other non-basic variables held at 0). Thus here x_1 has reduced profit $\frac{2}{3}$ and for x_4 it is $-\frac{1}{3}$. Since x_1 has reduced profit > 0 we make x_1 the entering variable.

We next find the minimum of the ratios $5/\frac{5}{3}$, $6/\frac{1}{3}$, $4/1$ and note that it is attained in row R_1. The corresponding basic variable is x_3 and so this is the leaving variable. We now pivot on the entry (value $\frac{5}{3}$) in row R_1 and column x_1.

$$\begin{array}{c|c|cccccc|c|l}
\text{z} & 1 & 0 & 0 & \frac{2}{5} & \frac{1}{5} & 0 & 8 & R_0' = R_0 + \frac{2}{3}R^* \\
\hline
x_1 & 0 & 1 & 0 & \frac{3}{5} & -\frac{1}{5} & 0 & 3 & R^* \\
x_2 & 0 & 0 & 1 & -\frac{1}{5} & \frac{2}{5} & 0 & 5 & R_2' = R_2 - \frac{1}{3}R^* \\
x_5 & 0 & 0 & 0 & -\frac{3}{5} & \frac{1}{5} & 1 & 1 & R_3' = R_3 - R^*
\end{array}$$

This tableau corresponds to the basic feasible solution $(3, 5, 0, 0, 1)$ with basic variables x_1, x_2, x_5 and z-value 8. We have moved from vertex P to vertex Q in the figure. Now we know that our current solution is optimal; for row R_0 gives

$$z \quad + \frac{2}{5}x_3 + \frac{1}{5}x_4 \quad = 8,$$

each reduced profit is ≤ 0, and every feasible solution has value at most 8 (since we must have $x_3, x_4 \geq 0$) . Indeed our solution is the unique optimal solution since each reduced profit is actually < 0.

Note that the reduced profit of x_3 here is $-\frac{2}{5}$. This suggests that the marginal value to the manufacturer of the corresponding resource (ingredient R) is $\frac{2}{5}$ (measured in £1000 per ton). More on this later.

2.2 Simplex Method Rules

1 Obtain Initial Tableau

2 Optimality Condition
If each reduced profit ≤ 0 then the current tableau is optimal, so stop. If not, choose a variable x_s with reduced profit > 0 to be the *entering variable.*
(This is for a max problem. For a min problem call minus the coefficient of x_j in row R_0 its *reduced cost.* The solution is optimal if each reduced cost is ≥ 0 and if not we take a variable with reduced cost < 0 to be the entering variable.)

3 Feasibility Condition
For each constraint row R_i let $\hat{b}_i$ be the number in the last column and let t_{is} be the number in the entering variable (x_s) column. For each row R_i with $t_{is} > 0$ compute the ratio $\hat{b}_i/t_{is}$. If the minimum is attained in row R_r then t_{rs} is the *pivot element*, row R_r is the *pivot row*, and the basic variable corresponding to row R_r is the *leaving variable.*

4 Pivot
To pivot on t_{rs} divide the pivot row R_r by t_{rs} and subtract from each other row the multiple of the (new) pivot row which gives 0 in the x_s column. Return to the optimality condition.

2.3 Discussion of the simplex method

Three kinds of difficulties can occur in the simplex method.

Initialisation How do we find an initial tableau?
Iteration If a tableau is not optimal, can we always choose an entering variable, find a leaving variable, and construct the next tableau by pivoting?
Termination Might we go on for ever?

Initialisation In the example it was obvious how to find an initial tableau, and indeed in any activity analysis we may use the slack variables as the initial basic variables. We shall consider in the next section how to start in other cases.

Choosing an entering variable The reduced profits are minus the coefficients of the non-basic variables in the z-equation. We have already seen that if each reduced profit is ≤ 0 then we have an optimal solution, and so we stop. If the reduced profit of some non-basic variable x_j is > 0 we may choose any such x_j as entering variable. For example we may choose the variable with the largest reduced profit.

Finding a leaving variable The leaving variable is that basic variable whose non-negativity imposes the smallest upper bound on the entering variable. We impose this rule in order to ensure that the next tableau yields a non-negative solution. However, what if (a) there is no such basic variable, or (b) there are several? In case (a) the problem has an unbounded optimum - see problem 1.8. If case (b) occurs, it shows that the problem is *degenerate*, that is it has a basic feasible solution with some basic variable equal to 0 - see problem 1.9.

Pivoting It should be clear that when we pivot, the new tableau corresponds to equations with exactly the same solutions as before, and yields an expression for the objective function z which is valid for all solutions. Actually we choose our pivots carefully. Our choice of entering variable makes the corresponding objective function value increase. Our rule for the leaving variable ensures the next basic solution stays feasible (that is, non-negative).

Termination Consider a non-degenerate (maximising) problem. Then in each tableau each basic variable is > 0, and so at each iteration the objective function z will increase strictly: for the entering variable will come in at some value $t > 0$ and then z will increase by t times the reduced profit. But there are only finitely many possible tableaux, since there are only finitely many choices of basic variables, and we now see that none can be revisited. Hence the algorithm must terminate.

If the problem is degenerate then the objective function need not increase strictly at each iteration. However, we can still ensure that no tableau is revisited, that is we do not 'cycle'. One way of ensuring this is always to take as entering variable the first variable with positive reduced profit, and if there is a choice of leaving variable again take the first possible one. Another method is the perturbation or lexicographic method. (See for example Chvátal for details. The problem of possible cycling is usually ignored in practice.)

2.4 Initial basic feasible solutions

Suppose that our problem (once put into equality form) is to

$$\max \ \mathbf{c}'\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},$$

where as usual the matrix A has m rows (and rank m). By multiplying equations by -1 if necessary we may ensure that $\mathbf{b} \geq \mathbf{0}$. Now the matrix A may not contain an $(m \times m)$ identity submatrix (as conveniently happens with an activity analysis). However, we can use the simplex method to start itself.

In the first phase we may introduce m non-negative *artificial* variables $w_1, \ldots, w_m$ and solve the auxiliary problem

$$\min \ w_1 + \ldots + w_m \text{ subject to } A\mathbf{x} + \mathbf{w} = \mathbf{b}, \quad \mathbf{x}, \mathbf{w} \geq \mathbf{0}.$$

An immediately evident basic feasible solution is $\mathbf{x} = \mathbf{0}$, $\mathbf{w} = \mathbf{b}$ and we may start the simplex method as before. Note that always the objective function $z \geq 0$; and the original problem has a feasible solution if and only if the minimum value of z is 0. In the second phase we solve the original problem by starting from the optimal solution to the auxiliary problem.

The example below illustrates this 'two-phase' method. (An alternative approach is the '*big M*' method which we shall not discuss here.) One may introduce fewer than m artificial variables if the matrix A already has some unit vector columns. [With degenerate problems there is a minor complication – which we shall ignore – when an artificial variable may remain (at level 0) in an optimal basis at the end of phase I.]

Example of two-phase method

$$\min \quad x_1 + x_2 + 3x_3 + x_4$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 \quad + 6x_4 &\geq 14 \\ 3x_1 + x_2 + 2x_3 - 7x_4 &= -11 \\ x_1, \ldots, x_4 &\geq 0 \end{aligned}$$

In appropriate equality form this is

$$\min \quad x_1 + x_2 + 3x_3 + x_4$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 \quad + 6x_4 - x_5 &= 14 \\ -3x_1 - x_2 - 2x_3 + 7x_4 \quad &= 11 \\ x_1, \ldots, x_5 &\geq 0. \end{aligned}$$

In phase I we solve the auxiliary program

$$\min \quad w_1 + w_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 \quad + 6x_4 - x_5 + w_1 \quad &= 14 \\ -3x_1 - x_2 - 2x_3 + 7x_4 \quad + w_2 &= 11 \\ x_1, \ldots, x_5 \geq 0, \;\; w_1, w_2 &\geq 0. \end{aligned}$$

In the first tableau w_1 and w_2 will be the basic variables. We must express the objective function z in terms of the non-basic variables. We do this by using the equations to substitute out for the current basic variables. In the tableau below the row R_0 is obtained by starting with the row

$$1 \mid 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad -1 \mid 0$$

which corresponds to an equation for the objective function, and adding on the two constraint rows.

z	1	-1	2	-2	13	-1	0	0	25
w_1	0	2	3	0	6	-1	1	0	14
w_2	0	-3	-1	-2	**7**	0	0	1	11

z	1	$\frac{32}{7}$	$\frac{27}{7}$	$\frac{12}{7}$	0	-1	0	$-\frac{13}{7}$	$\frac{32}{7}$
w_1	0	$\boldsymbol{\frac{32}{7}}$	$\frac{27}{7}$	$\frac{12}{7}$	0	-1	1	$-\frac{6}{7}$	$\frac{32}{7}$
x_4	0	$-\frac{3}{7}$	$-\frac{1}{7}$	$-\frac{2}{7}$	1	0	0	$\frac{1}{7}$	$\frac{11}{7}$

z	1	0	0	0	0	0	-1	-1	0
x_1	0	1	$\frac{27}{32}$	$\frac{3}{8}$	0	$-\frac{7}{32}$	$\frac{7}{32}$	$-\frac{3}{16}$	1
x_4	0	0	$\frac{7}{32}$	$-\frac{1}{8}$	1	$-\frac{3}{32}$	$\frac{3}{32}$	$\frac{1}{16}$	2

The third iteration completes phase I. We now drop the artificial variables. We know that the original problem is feasible, and an initial basic feasible solution is $\mathbf{x} = (1, 0, 0, 2, 0)'$, with basic variables x_1, x_4. To start phase II, we must express the original objective function z in terms of the current non-basic variables. Proceeding as above, to the equation

$$z - x_1 - x_2 - 3x_3 - x_4 = 0$$

we add the appropriate multiple ($c_1 = 1$) of the row R_1 equation to remove the term in x_1 and the appropriate multiple ($c_4 = 1$) of the row R_2 equation to remove the term in x_4. We obtain

$$z + \frac{1}{16}x_2 - \frac{11}{4}x_3 - \frac{5}{16}x_5 = 3.$$

We now complete phase II.

z	1	0	$\frac{1}{16}$	$-\frac{11}{14}$	0	$-\frac{5}{16}$	3
x_1	0	1	$\boldsymbol{\frac{27}{32}}$	$\frac{3}{8}$	0	$-\frac{7}{32}$	1
x_4	0	0	$\frac{7}{32}$	$-\frac{1}{8}$	1	$-\frac{7}{32}$	2

z	1	$-\frac{2}{27}$	0	$-\frac{25}{9}$	0	$-\frac{8}{27}$	$\frac{79}{27}$
x_2	0	$\frac{32}{27}$	1	$\frac{4}{9}$	0	$-\frac{7}{27}$	$\frac{32}{27}$
x_4	0		0		1		$\frac{47}{27}$

This tableau corresponds to the bfs $\mathbf{x}^* = (0, \frac{32}{27}, 0, \frac{47}{27}, 0),'$ with value $\frac{79}{27}$; and this solution is optimal since all the relevant reduced costs are ≥ 0, that is all the coefficients in row R_0 are ≤ 0.