J. Appl. Math. & Informatics Vol. **26**(2008), No. 1 - 2, pp. 349 - 353
Website: http://www.kcam.biz

ABS ALGORITHM FOR SOLVING A CLASS OF LINEAR DIOPHANTINE INEQUALITIES AND INTEGER LP PROBLEMS

CHENG-ZHI GAO AND YU-LIN DONG*

Abstract. Using the recently developed ABS algorithm for solving linear Diophantine equations we introduce an algorithm for solving a system of m linear integer inequalities in n variables, $m \leq n$, with full rank coefficient matrix. We apply this result to solve linear integer programming problems with $m \leq n$ inequalities.

AMS Mathematics Subject Classification : 11D75, 90C10.
Key words and phrases : ABS Algorithm; Diophantine inequalities; Integer points.

1. Introduction

Let $A \in Z^{m,n}$, $m \leq n$, be a full row rank matrix and let $b \in Z^m$, $x \in Z^n$. Consider the system of linear inequalities (a special case of the general linear inequality system, where m can be greater than n)

$$Ax \leq b, \qquad x \in Z^n. \tag{1}$$

The ABS methods [1] are a class of direct iteration type methods that, in a finite number of iterations, either determine the general solution of a system of linear equations or establish its inconsistency. Basic results on ABS methods for solving a linear system of equations can be found in [2], see also the paper by Xia and Zhang [7]. For solving a linear Diophantine system, using the ABS methods, see [4], where a generalization is given of a method proposed by Egervary [3] for the case of homogeneous system.

Here we extend the approach in [4] to determine all integer points of the Diophantine inequality system (1). This result is used to solve a class of integer linear programming problems.

Received December 20, 2005. Revised September 26, 2007. *Corresponding author.

Some early developments (using the Huang algorithm within the ABS class) for solving certain linear inequality systems can be found [8,9].

Spedicato and Abaffy [6] have discussed the solution of the linear programming problem with constraints $Ax \leq b$. Their method is based on the LU implicit algorithm in the ABS class. Our approach is independent of the selection of a particular algorithm in the ABS class.

The following section is devoted to introducing an ABS algorithm to solve the linear Diophantine inequalities, and dealing with inequalities ILP via the same method as in ILP.

2. Solving certain linear Diophantine inequality system and integer programming problem

Consider the linear Diophantine inequalities system

$$Ax \leq b \tag{2}$$

where $A = (a_1, \ldots, a_m)^T \in Z^{m,n}$, $m \leq n$, and $b \in Z^m$. Suppose $rank(A) = m$. Our aim is to determine the general solution of (2) using the ABS algorithm. First we give the algorithm (see [10]) for solving

$$Ax \leq b, \qquad x \in R^n. \tag{3}$$

The ABS algorithm for linear inequalities: let H_1 be arbitrary nonsingular in $R^{n,n}$, x_1 be an arbitrary vector in R^n. For $i = 1, 2, \cdots, m$, compute

(1) $p_i = H_i^T z_i$, where $z_i \in R^n$ is an arbitrary vector satisfying

$$a_i^T H_i^T z_i = a_i^T p_i > 0,$$

(2) $x_{i+1} = x_i - t_i p_i$, where

$$t_i \geq (a_i^T x_i - b_i)/a_i^T p_i$$

(3) $H_{i+1} = H_i - \dfrac{H_i a_i w_i^T H_i}{w_i^T H_i a_i}$, where $w_i \in R^n$ is an arbitrary vector satisfying

$$w_i^T H_i a_i \neq 0.$$

Then x_{m+1} is a solution of the problem (3). This algorithm retains all properties of ABS methods which do not depend on the vector b.
From the above algorithm, we observe that the key point is to preserve $a_i^T p_i > 0$, which provides us with an important clue to solve (2). Because the Rosser's theorem (see [4]) can preserve $a_i^T p_i = s_i^T z_i = \delta_i > 0$, which is also the key point to the ABS algorithm for linear Diophantine inequalities.

Now we give the algorithm for linear Diophantine inequalities:

(1) *Choose* $x_1 \in Z^n$ *arbitrary, and* $H_1 \in Z^{n,n}$ *arbitrary and unimodular. Let* $i = 1$, $r_1 = 0$.
(2) *Compute* $\tau_i = a_i^T x_i - b_i$ *and* $s_i = H_i a_i$.

(3) *If ($s_i = 0$ and $\tau_i \leq 0$) then let $x_{i+1} = x_i$, $H_{i+1} = H_i$, $r_{i+1} = r_i$, and go to step (7) (the ith inequality is redundant). If ($s_i = 0$ and $\tau_i > 0$)*

(4) *$\{s_i \neq 0\}$ Compute $\delta_i = gcd(s_i)$. Compute the search direction $p_i = H_i^T z_i$, where $z_i \in Z^n$, is an arbitrary integer vector satisfying $z_i^T H_i a_i = \delta_i$. Compute*

$$\alpha_i = \tau_i/\delta_i, \qquad t_i \geq \alpha_i$$

where $t_i \in Z$ and

$$x_{i+1} = x_i - t_i p_i.$$

(5) *$\{$Updating $H_i\}$ Update H_i to H_{i+1} by*

$$H_{i+1} = H_i - \frac{H_i a_i w_i^T H_i}{w_i^T H_i a_i}$$

where $w_i \in Z^n$ is an arbitrary integer vector satisfying $w_i^T s_i = \delta_i$.

(6) *Let $r_{i+1} = r_i + 1$.*

(7) *If $i = m$ then Stop (x_{m+1} is a solution) else let $i = i + 1$ and go to step (2).*

Now let us give the follow similar theorems via the combination of the algorithms given by [4] and [10].

Theorem 1. *Let A be full rank and suppose that the Diophantine system (2) is solvable. Consider the sequence of Abaffians generated by the basic ABS algorithm with the following parameter choices:*

(a) *H_1 is unimodular.*

(b) *for $i = 1, \ldots, m$, w_i is such that $w_i^T H_i a_i = \delta_i$, $\delta_i = gcd(H_i a_i)$. Then the following properties are true:*

(c) *the sequence of Abaffians generated by the algorithm is well-defined and consists of integer matrices.*

(d) *if y_{i+1} is a special integer solution of the first i equations, x can be written in the form $x = y_{i+1} + H_{i+1}^T q$ for some integer vector q.*

Proof. It is easy to prove (c) by induction, using the ABS property that $H_i a_i \neq 0$ for full rank matrices, Rosser's algorithm to define w_i and observing that H_i is updated by an integer correction.

To prove (d) we observe that, from general properties of linear systems, any solution x of the first i inequalities, $i = 1, \ldots, m$, can be written as the sum of a special integer solution y_{i+1} and some solution $x^\star$ of the homogeneous systems $Ax = 0$. Let x be any integer solution of the first i inequalities. Then $x = y_{i+1} + x^\star$ implies $x^\star$ to be integer and uniquely defined. From ABS properties any solution x of the first i inequalities can be written in the form $x = y_{i+1} + H_{i+1}^T q$, for some $q \in Z^n$. If $q \in Z^n$ and H_{i+1} have been determined according to the conditions of the Theorem, then x is an integer vector. It remains to show that our x can be obtained by selecting some $q \in Z^n$. Take $q = (H_1^T)^{-1} x^\star$ and observe that $q \in Z^n$ since H_1 is unimodular. Now we have

$$\begin{aligned} x &= y_{i+1} + H_{i+1}^T (H_1^T)^{-1} x^\star \\ &= y_{i+1} + (H_1^T - H_1^T W_i (W_i^T H_1 A_i)^{-T} A_i^T H_1^T)(H_1^T)^{-1} x^\star \\ &= y_{i+1} + x^\star \end{aligned}$$

since $A_i^T x^\star = 0$, $x^\star$ being a solution of the homogeneous equations. □

The following theorem provides an ABS characterization of the integer solvability of the general Diophantine linear inequalities. Now, consider the ILP

$$\max\ c^T x :\ Ax \leq b, \qquad x \in Z^n. \tag{4}$$

where $c \in Z^n$, $b \in Z^m$, $A \in Z^{m,n}$, $m \leq n$. Assume that the system $Ax \leq b$ is integerly consistent. From the Theorem 3.1(d), we can get the representation of the general solution of the system (2) in the following

$$x = x_{m+1} + H_{m+1}^T q \in Z^n. \tag{5}$$

Using (5), then (4) is equivalent to the following unconstrained problem

$$\max \left(c^T x_{m+1} + c^T H_{m+1}^T q \right) :\ q \in Z^n. \tag{6}$$

If $H_{m+1} c = 0$, then x_{m+1} is an optimal solution to (4) (in fact, $x = x_{m+1} + H_{m+1}^T q$, for any $q \in Z^n$, is an optimal solution). Otherwise, (4) is integerly unbounded and hence has no solution.

From the above observation, the method we use to deal with (4) is easier to be done than the method by H. Esmaeili in [5].

3. Conclusion

In this paper we have introduced a class of ABS algorithms for linear Diophantine systems to solve linear inequalities and linear programming problems in the integer space where the number of inequalities is less than or equal to the number of variables. The possibility of extension of the ABS approach to the case of variables for either the real or integer case is evidently worthwhile investigating in future.

References

1. J. Abaffy, C.G. Broyden and E. Spedicato, *A class of direct methods for linear equations*, Numer. Math. **45** (1984) 361-176.
2. J. Abaffy and E. Spedicato, *ABS Projection Algorithms: Mathematical Techniques for Linear and Nonlinear Equations*. Ellis Horwood, Chisester, 1989.
3. E. Egervary, *On rank-diminishing operations and their applications to the solution of linear equations*, ZAMP **9** (1960) 376-386.
4. H. Esmaeili, N. Mahdavi-Amiri and E. Spedicato, *A class of ABS algorithms for Diophantine linear systems*, Numer. Math. **90** (2001) 101-115.
5. H. Esmaeili, N. Mahdavi-Amiri and E. Spedicato, *ABS solution of a class of linear integer inequality and integer LP problems*, Optimization Methodes and Software, **16** 179-192.

6. E. Spedicato and J. Abaffy, *On the use of the ABS algorithms for some linear programming problesm*, Preprint, University of Bergamo, 1987.
7. Z. Xia and Z. Zhang, *ABS algorithm for solving linearly constrained optimization problems via the active set strategy*, this issue.
8. L. Zhang (September 1991). *A method for finding a feasible point of inequalities.* In: E. Spedicato (Ed.). Proceedings of the First International Conference on ABS Algorithms. pp. 131-137. Luoyang, University of Bergamo.
9. L. Zhang, *An algorithm for the least Euclidean norm solution of a linear system of inequalities via the Huang ABS algorithm and the Goldfarb-Idnani strategy*, Report 95/2, University of Bergamo, 1995.
10. J. Zhao, *ABS algorithm for solving linear inequalities*, Report 91/21, University of Bermago, 1991.

Cheng-Zhi Gao received his postgraduate degree from Department of Applied Mathematics of Dalian University of Technology. His research interests are numerical computing and software, numerical mathematics for optimization.

NO.25, Kai-Xuan Road, Luoyang, Henan, 471009, P. R. China
e-mail: gczljhyn_81@163.com

Yu-Lin Dong is a doctor student in Department of Applied Mathematics, Dalian University of Technology. His research interests are data mining, numerical computing and software, numerical mathematics for optimization, nonsmooth optimization.

Department of Applied Mathematics, Dalian University of Technology, 116024, P. R. China
e-mail: yulindong_68@yahoo.com