Calculus Notes

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2.1 Ideas about limits

We write

$$\lim_{x \to a} f(x) = L$$

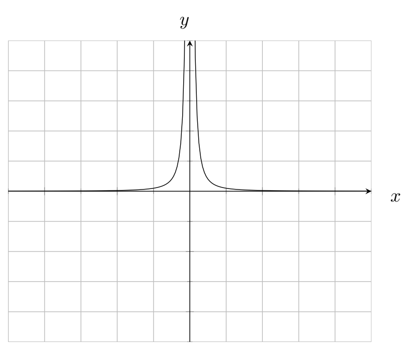
and say the limit of f(x) as x approaches a is L

1

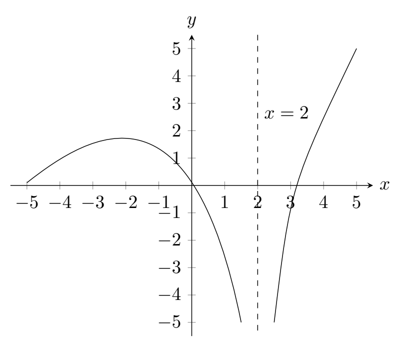
$$\lim_{x \to a} f(x) = L$$

if and only if

- $\lim_{x \to a^-} f(x) = L$
- $\lim_{x \to a^+} f(x) = L$



(a) $y = \frac{1}{x^2}$



(b) y approaches $-\infty$ as x approaches 2

As seen in figure a) and b), both functions have a limit approaching infinity. We write their limits as

$$\lim_{x \to a} f(x) = (\pm)\infty$$

Thus, we say $x = a$ are their **asymtotes**.

2.2 Finding Limits

Example 1. Show that $\lim_{x \to 0} |x| = 0$

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

By analysis,

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$
$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} x = 0$$

By theorem 1,

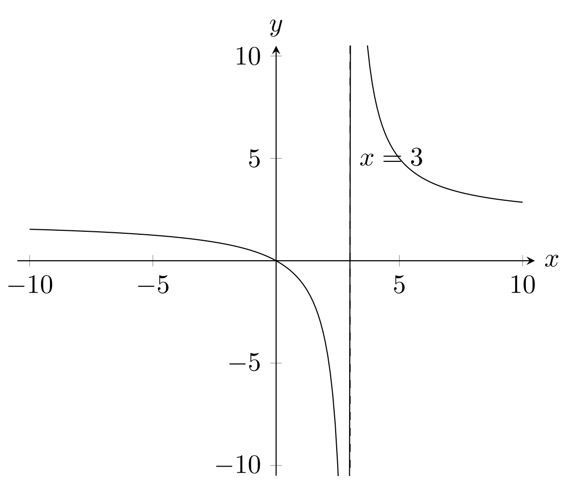
$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^+} |x|$$
$$\Downarrow$$
$$\lim_{x\to 0} |x| = 0$$

Example 2. Find $\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$

By analysis, if x approaches 3 from the right side, then $(x+3)$ is a small positive integer, then $(x+3)$ is a really small pos. integer; the result is a large pos. int.

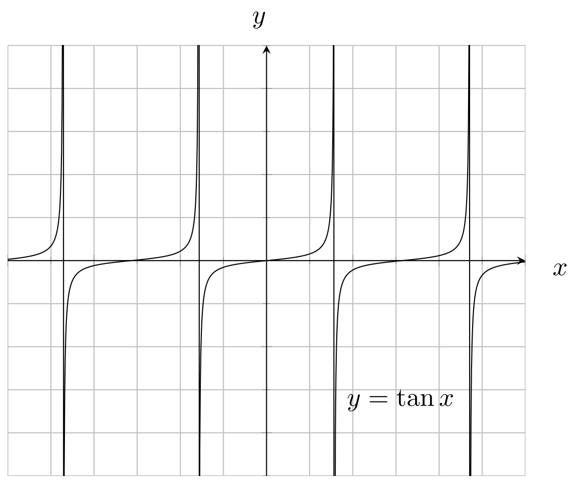
If x approaches 3 from the left side, then $(x+3)$ is a small neg. integer. The result is a large neg. int.

$$\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$$
$$\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$$



However, the lim is still non-existant.

Example 3. Limit of $y = tan(x)$



The vertical asymtotes are:

$$x = \frac{\pi}{2} + n\pi \ (n \in \mathbb{Z})$$

2.3 Asymtotes

When at least one of the following is true:

$$\lim_{x\to a} f(x) = \infty, \lim_{x\to a^-} f(x) = \infty, \lim_{x\to a^+} f(x) = \infty,$$

$$\lim_{x\to a} f(x) = -\infty, \lim_{x\to a^-} f(x) = -\infty, \lim_{x\to a^+} f(x) = -\infty,$$

$$\boxed{x = a \text{ is a vertical asymtote.}}$$

2.4 Limit Laws

*Suppose c is a constant and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists,

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \tag{1}$$

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \tag{2}$$

$$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x) \tag{3}$$

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \tag{4}$$

$$\lim_{x\to a}\frac{[f(x)]}{[g(x)]} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \; (\lim_{x\to a} g(x) \neq 0) \tag{5}$$

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$$

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

2.5 Limit Calculations

Example 1. Find $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$

$$\begin{aligned} F(h) &= \frac{(3+h)^2-9}{h} \\ &= \frac{h^2+6h+9-9}{h} \\ &= h+6 \end{aligned}$$

By evaluation, $\lim_{h\to 0} h+6 = 6$

2.5.1 Direct Substitution Property

If f is a **polynomial** or a **rational function** and a is in the domain of f, then:

$$\boxed{\lim_{x\to a} f(x) = f(a)}$$

Example 2. Find $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

$$\begin{aligned}
&= \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} \\
&= \lim_{t\to 0} \frac{t^2+9-9}{t^2(\sqrt{t^2+9}+3)} \\
&= \lim_{t\to 0} \frac{t^2}{t^2\sqrt{t^2+9}+3t^2} \\
&= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3} \\
&= \frac{1}{\sqrt{\lim_{t\to 0} t^2+9}+3} \\
&= \frac{1}{\sqrt{9}+3} = \frac{1}{6}
\end{aligned}$$

Example 3. Find $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$