IDS Mathematics

Calculus Chapter 5

- Integrals -

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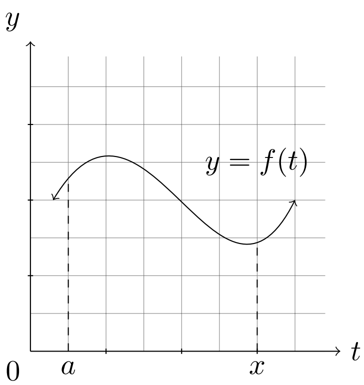
Contents

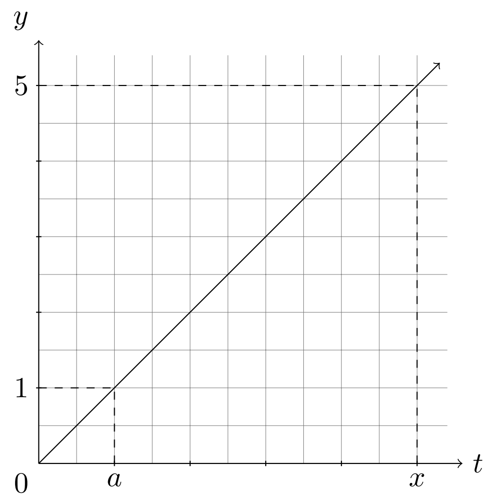
1 Introduction

1.1 Example 1

Consider:
Let $Area = g(x)$, such that: $g(x) = \int_a^x f(x)\ dt$
Considder: let $f(t) = t$, and $a = 1$,
What is the value of $g(x)$ when $x = 5$? i.e. $g(5) = ?$
As x changes, then $area$ also changes.



(a) Figure 1



(b) Figure 2

In $Figure$ 2 we know: $g(x) = \int_a^x f(t)\ dt$

$$\begin{aligned}
g(5) = \int_1^5 f(t)\ dt &= \int_0^5 f(t)\ dt - \int_0^1 f(t)\ dt \\
&= \frac{(5)(5)}{2} - \frac{(1)(1)}{2} \\
&= \frac{25}{2} - \frac{1}{2} = 12
\end{aligned}$$

GENERAL CASE:

$$g(x) = \int_a^x f(t)\ dt = \int_0^x f(t)\ dt - \int_0^a f(t)\ dt$$

Recall:

$$\begin{aligned}
\int_0^4 (x^3 - 6x)dx &= \frac{1}{4}x^4 - \frac{6}{2}x^2 \Big]_0^4 \\
&= \left(\frac{1}{4}(4)^4 - 3(4)^2\right) - \left(\frac{1}{4}(0)^4 - 3(0)^2\right) \\
&= 4^3 - 3 \cdot 4^2 = 16
\end{aligned}$$

1.2 Example 2

Let $f(x) = x^3$. Find area from $x = 0$ to $x = 1$ using the Riemann Sum definition.

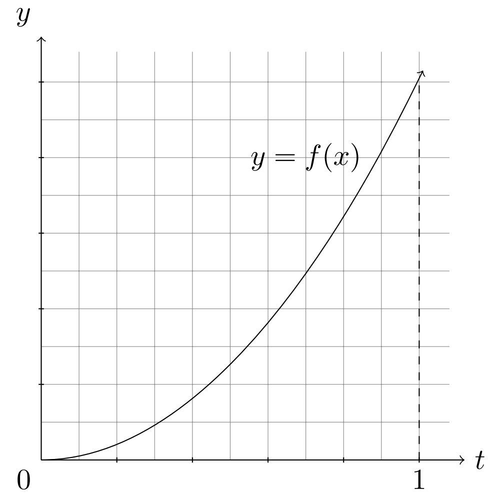


Figure 2: Example 2

$$\begin{aligned}
\text{We know: } \int_a^b f(x)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x^*)\Delta x \\
\text{Also: } \int_a^b f(x)dx &= \int_0^b f(x)dx - \int_0^a f(x)dx \\
\therefore \int_2^4 f(x)dx &= \int_0^4 f(x)dx - \int_0^2 f(x)dx \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x - \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \left(\frac{4}{n} i^3\right) \cdot \frac{4}{n} \right] - \lim_{n\to\infty} \left[\sum_{i=1}^{n} \left(\frac{2}{n} i^3\right) \cdot \frac{2}{n} \right]
\end{aligned}$$

1.3

Theorem 1.
$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Proof. We know:

$$\begin{aligned}
\int_a^b (f(x)+g(x)) &= \lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i^*)+g(x_i^*))\Delta x \\
&= \lim_{n\to\infty}\left\{\Delta x\cdot\sum_{i=1}^{n}[f(x_i^*)+g(x_i^*)]\right\} \\
&= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\Delta x+\sum_{i=1}^{n} g(x_i^*)\Delta x\right] \\
&= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\Delta x\right]+\lim_{n\to\infty}\left[\sum_{i=i}^{n} g(x_i^*)\Delta x\right] \\
&= \int_a^b f(x)dx + \int_a^b g(x)dx
\end{aligned}$$

□

1.3.1 Example 2

Solve: $\int_0^3 (x^3-6x)dx$ (using properties of integral).

$$\begin{aligned}
\int_0^3 (x^3-6x)dx &= \int_0^3 x^3\ dx - 6\int_0^3 x\ dx \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}(x_i)^3\cdot\frac{3}{n}-6\cdot\frac{9}{2} \\
&= -\frac{27}{4}
\end{aligned}$$

1.3.2 Example 6

Use the properties of integrals to evaluate $\int_0^1 (4+3x^2)\ dx$.

$$\begin{aligned}
\text{We know: } \int_0^1 x^2\ dx &= \lim_{n\to\infty}\sum_{i=1}^{n}(x^*)^2\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\frac{1}{n}=\frac{1}{3} \\
\text{And: } \int_0^1 (4+3x^2)\ dx &= \int_0^1 4\ dx + \int_0^1 x^2\ dx \\
&= 4(1-0)+3(\frac{1}{3})
\end{aligned}$$

1.3.3 Example 8

It is know that $\int_0^{10} f(x)dx = 17$ and $\int_0^{8} f(x)dx = 12$. Find $\int_8^{10} f(x)dx$.

$$\begin{aligned}\int_8^{10} f(x)dx &= \int_0^{10} f(x)dx - \int_0^{8} f(x)dx \\ &= 17 - 12 = 5\end{aligned}$$

2 Anti-derivatives

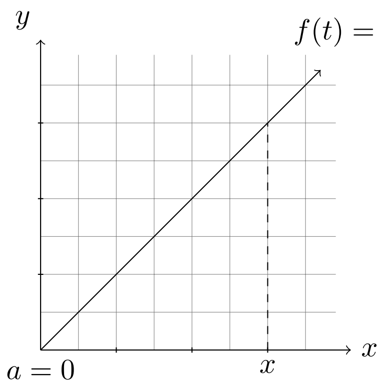
2.1 Introduction

Consider the function $f(t) = t$:

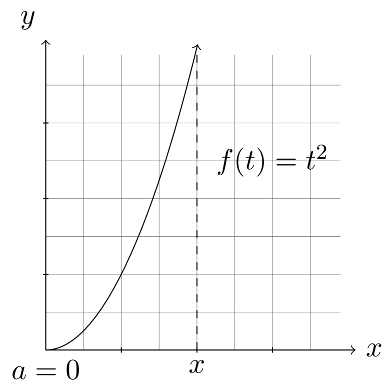
1. $g(x) = \int_a^x f(t)dt$
2. $g'(x) = \dfrac{d}{dx}\int_a^x t\ dt = x$

Supports that **differentiate** is the **inverse proceedure** to **integration** (connects to **anti-derivative**).

Now... Suppose $f(t) = t$. let $a = 0$. Find $g(x)$.



(a) $f(t) = t$



(b) $f(t) = t^2$

In Figure (a), $g(x) = \int_0^x t\ dt = \frac{1}{2}x^2$ $(*)$, from geometry.

Suppose we want $g'(x)$. From $(*)$, $g'(x) = \dfrac{d}{dx}\int_0^x t\ dt = \dfrac{d}{dx}\left(\frac{1}{2}x^2\right)$.

$\therefore g'(x) = \dfrac{d}{dx}\int_0^x t\ dt = x$.

Consider Figure (b):

$$\begin{aligned}\int_0^x t^2\ dx &= \lim_{n\to\infty}\Big[\sum_{i=1}^{n}\left(\frac{xi}{n}\right)^2\cdot\frac{x}{n}\Big]\\ &= \lim_{n\to\infty}\frac{x}{n}\sum_{i=1}^{n}\left(\frac{xi}{n}\right)^2\\ &= \lim_{n\to\infty}\left(\frac{x}{n}\right)^3\sum_{i=1}^{n} i^2\\ &= \lim_{n\to\infty}\left(\frac{x}{n}\right)^3\cdot\frac{n(n+1)(2n+1)}{6}\\ &= x^3\cdot\lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2}\\ &= \frac{1}{3}x^3\end{aligned}$$

Therefore: $g(x) = \int_0^x t^2\ dx = \frac{1}{3}x^3$ (*)

Area $\Rightarrow g(x) = \int_0^x t^2\ dt$ From (*): $g'(x) = \frac{d}{dx}\int_0^x t^2\ dx = x^2$.

3 The Foundamental Theorem of Calculus Part 1

Theorem 2. *If f is continuous on $[a,b]$, then the function g defined by: $g(x) = \int_a^x f(t)\ dt,\ a \leq x \leq b$, is continuous on $[a,b]$, and differentiable on (a,b), and $g'(x) = f(x)$.*

3.1 Example 2

Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2}\ dt$.

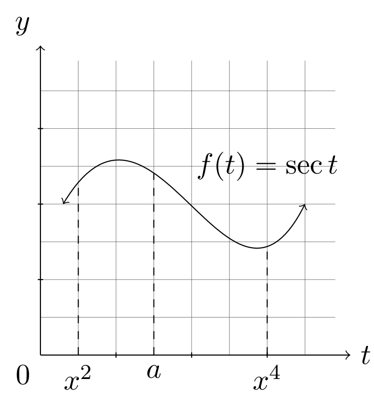
$$g'(x) = \frac{d}{dx}\int_0^x \sqrt{1+t^2}\ dt \underset{\substack{\uparrow\\ \text{FTC Part 1}}}{=} \sqrt{1+t^2}$$

3.2 Example 4 - p.g 384

Find $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t\ dt$.

$$\begin{aligned}
\text{let } g(x) &= \int_1^{x^4} \sec t\ dt \\
g'(x) &= \frac{d}{dx}\int_1^{x^4} \sec t\ dt \underset{\substack{\uparrow \\ \text{let } u = x^4}}{=} \frac{d}{dx}\int_1^{u} \sec t\ dt \\
&\underset{\substack{\uparrow \\ \text{chain rule}}}{=} \frac{d}{du}\left(\int_1^{u} \sec t\ dt\right)\frac{du}{dx} \\
&= \sec u \cdot \frac{d}{dx}(u) = \sec(x^4)\cdot 4x^3 \\
g'(x) &= \frac{d}{dx}\int_1^{x^4} = \sec(x)^4 \cdot (4x^3) \\
&= 4x^3 \cdot \sec(x^4)
\end{aligned}$$

Suppose: $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^4} \sec t\ dt$.



$$\begin{aligned}
\text{We know: } \int_{x^2}^{x^4} \sec t\ dt &= \int_{x^2}^{a} \sec t\ dt + \int_{a}^{x^4} \sec t\ dt \\
\text{And: } \int_{x^2}^{a} \sec t\ dt &= -\int_{a}^{x^2} \sec t\ dt \\
\therefore \int_{x^2}^{x^4} \sec t\ dt &= \int_{a}^{x^4} \sec t\ dt - \int_{a}^{x^2} \sec t\ dt
\end{aligned}$$

4 The Foundamental Theorem of Calculus Part 2

Theorem 3. *If f is continuous on $[a, b]$, then $\int_a^b f(x)\ dx = F(b) - F(a)$, where F is any antiderivative of f, that is, a function such that $F' = f$.*

4.1 Indefinite Integral

4.1.1 Example 1

Find $\int (10x^4 - 2\sec^2 x)\ dx$.

$$\begin{aligned}\int (10x^4 - 2\sec^2 x)\ dx &= \frac{10}{4+1}x^{4+1} - s\tan x + C \\ &= 2x^5 - 2\tan x + C\end{aligned}$$

4.1.2 Example 2

Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\ d\theta$.

$$\begin{aligned}\int \frac{\cos\theta}{\sin^2\theta}\ d\theta &= \int \left(\frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}\right)\ d\theta \\ &= \int \cot\theta \cdot \csc\theta \cdot d\theta \\ \therefore \int \frac{\cos\theta}{\sin^2\theta} d\theta &= -\csc\theta + C\end{aligned}$$

4.1.3 Example 3

Evaluate $\int_0^3 (x^3 - 6x)\ dx$.

$$\begin{aligned}\int_0^3 (x^3 - 6x)\ dx &= \frac{1}{4}x^4 - 3x^2\Big]_0^3 \\ &= \left[\frac{1}{4}(x)^4 - 3(3)^2\right] - 0 \\ &= -6.75\end{aligned}$$

4.1.4 Example 4

Evaluate $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)\ dx$.

$$\begin{aligned}\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)\ dx &= \frac{2}{4}x^4 - \frac{6}{2}x^2 + 3\arctan x\Big]_0^2 \\ &= \frac{1}{2}(x)^4 - 3(x)^2 + 3\arctan 2 - 0 \\ &= -4 + 3\arctan 2\end{aligned}$$

4.1.5 Example 5

Evaluate $\displaystyle\int_0^2 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\ dt.$

$$
\begin{aligned}
\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\ dt &= \int_1^9 (2 + t^{1/2} - t^{-2})\ dt \\
&= 2t + \frac{t^{3/2}}{\frac{3}{2} - \frac{t^{-1}}{-1}}\Bigg]_1^9 \\
&= \left(2 \cdot 9 + \frac{2}{3} \cdot 9^{3/2} + \frac{1}{9}\right) - \left(2 \cdot 1 + \frac{2}{3} \cdot 1^{3/2} + 1\right) \\
&= 32\frac{4}{9}
\end{aligned}
$$

4.1.6 Example 6

A particle moves alopng a line so that its velocity at time t is $v(t) = t^2 - t - 6$.

a) Find the displacement of the particle during the time period $1 \le t \le 4$.

b) Find the distance traveled during this time period.

a)

$$
\begin{aligned}
S(4) - S(1) &= \int_1^4 v(t)\ dt = \int_1^4 (t^2 - t - 6)\ dt \\
&= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2}
\end{aligned}
$$

b) Note that $v(t) = t^2 - t - 6 = (t-3)(t+2)$ and so $v(t) \le 0$ on the interval $[1, 3]$ and $v(t) \ge 0$ on $[3, 4]$. Thus, the distance traveled is

$$
\begin{aligned}
\int_1^4 |v(t)| &= \int_1^3 [-v(t)]dt + \int_3^4 v(t)dt \\
&= \int_1^3 (-t^2 + t + 6)dt + \int_3^4 (t^2 - t - 6)dt \\
&= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 \\
&= \frac{61}{6}
\end{aligned}
$$

5 The Substitution Rule

Theorem 4. *If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then*

$$\int f(g(x))g'(x)dx = \int f(u)du$$

5.1 Example 1

Find $\displaystyle\int [x^3 \cos(x^4 + 2)]dx$.

We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\ dx$, which occurs in the integral. Thus using $x^3\ dx = dy/4$ and the Substitution Rule, we have

$$\begin{aligned}\int x^3 \cos(x^4+2)dx &= \int \cos u \cdot \frac{1}{4}\ du = \frac{1}{4}\int \cos u\ du \\ &= \frac{1}{4}\sin u + C \\ &= \frac{1}{4}\sin(x^4+2) + C\end{aligned}$$

5.2 Example 2

Evaluate $\displaystyle\int \sqrt{2x+1}\ dx$.

Let $u = 2x + 1$. Then $du = 2\ dx$, so $dx = du/2$. Thus the Substitution Rule gives

$$\begin{aligned}\int \sqrt{2x+1}\ dx &= \int \sqrt{u}\ \frac{du}{2} = \frac{1}{2}\int u^{1/2}du \\ &= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C = \frac{1}{3}u^{3/2} + C \\ &= \frac{1}{3}(2x+1)^{3/2} + C\end{aligned}$$

5.3 Example 3

Evaluate $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\ dx$.

Let $u = 1 - 4x^2$. THne $du = -8x\ dx$, so $x\ dx = -\frac{1}{8}du$ and

$$\begin{aligned}\int \frac{x}{\sqrt{1-rx^2}}\ dx &= -\frac{1}{8}\int \frac{1}{\sqrt{u}}\ du = -\frac{1}{8}\int u^{1/2}\ du \\ &= -\frac{1}{8}(x\sqrt{u}) + C = -\frac{1}{4}\sqrt{1-4x^2} + C\end{aligned}$$

5.4 Example 4

Evaluate $\int e^{5x}\ dx$.

Let $u = 5x$, then $du = 5\ dx$, so $dx = \frac{1}{5}du$. Therefore

$$\int e^{5x} = \frac{1}{5}\int e^u\ du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C$$

5.5 Example 5

Evaluate $\int \sqrt{1+x^2}\ x^5\ dx$.

An Appropriate substitution ecomes more obvious if we factor x^5 as $x^4 \cdot x$. Let $u = 1 + x^2$. Then $du = 2x\ dx$, so $x\ dx = du/2$. Also $x^2 = u - 1$, so $x^4 = (u-1)^2$:

$$\begin{aligned}
\int \sqrt{1+x^2}x^5\ dx &= \int \sqrt{1+x^2}x^4 \cdot x\ dx \\
&= \int \sqrt{u}(u-1)^2\ \frac{du}{2} = \frac{1}{2}\int \sqrt{u}(u^2 - 2u + 1)\ du \\
&= \frac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\ du \\
&= \frac{1}{2}(\frac{2}{7}u^{7/2} - 2 \cdot \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}) + C \\
&= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C
\end{aligned}$$

5.6 Example 6

Evaluate $\int \tan x\ dx$.

First we write tangent in terms of since and cosine:

$$\int \tan x\ dx = \int \frac{\sin x}{\cos x}\ dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\ dx$ and so $\sin x\ dx = -du$:

$$\begin{aligned}
\int \tan x\ dx &= \int \frac{\sin x}{\cos x}\ dx = -\int \frac{du}{u} \\
&= -\ln|u| + C = -\ln|\cos x| + C
\end{aligned}$$

5.7 Definite Integrals

Theorem 5 (**The Substitution Rule For Definite Integrals**). *If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

Consider: $\int_0^4 \sqrt{2x+1}\ dx$.

Method 1: For Now, consider the indefinite integral for $\int \sqrt{2x+1}$:

Let $u = 2x + 1$. Then $du = 2\ dx$ and $dx = \frac{du}{2}$. Therefore

$$\begin{aligned}
\int \sqrt{2x+1}\ dx &= \int \sqrt{u} \cdot \frac{du}{2} \\
&= \frac{1}{2} \int u^{1/2}\ du \\
&= \frac{1}{2} \left[\left(\frac{1}{1/2} + 1 \right)\ u^{1/2+1} \right] + C \\
&= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C \\
&= \frac{1}{3} (2x+1)^{3/2} + C
\end{aligned}$$

Now:

$$\begin{aligned}
\int_0^4 \sqrt{2x+1}\ dx &= \frac{1}{3}(2x+1)^{3/2} \Big]_0^4 \\
&= \frac{1}{3}[(2)(4)+1]^{3/2} - \frac{1}{3}[(2)(0)+1]^{3/2} \\
&= \frac{26}{3}
\end{aligned}$$

Method 2: For Now, consider the indefinite integral for $\int \sqrt{2x+1}$:

Let $u = 2x + 1$. When $x = 4$, $u = 9$ and when $x = 0$, $u = 1$.

$$\begin{aligned}
\int_0^4 \sqrt{2x+1} = \int_1^9 \frac{1}{2}\sqrt{u}\ du &= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big]_1^9 \\
&= \frac{1}{3}(9^{3/2} - 1^{3/2}) = \frac{26}{3}
\end{aligned}$$

5.8 Example 8

Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$.

Let $u = 3 - 5x$. Then $du = -5\ dx$, so $dx = -du/5$. When $x = 1$, $u = -2$ and when, $x = 2, u = -7$.

Thus

$$\begin{aligned}\int_1^2 \frac{dx}{(3-5x)^2} &= -\frac{1}{5}\int_{-2}^{-7}\frac{du}{u^2} \\ &= -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = \frac{1}{5u}\Bigg]_{-2}^{-7} \\ &= -\frac{1}{5}\left(-\frac{1}{7}+\frac{1}{2}\right) = \frac{1}{14}\end{aligned}$$

5.9 Example 9

Evaluate $\displaystyle\int_1^e \frac{\ln x}{x}\ dx$.

Let $u = \ln x$. Then $du = \frac{dx}{x}$. When $x = 1$, $u = |\ln x| = 0$ and when $x = e$, $u = \ln e = 1$.

$$\therefore \int_1^e \frac{\ln x}{x}\ dx = \int_0^1 u\ du = \frac{u^2}{2}\Bigg]_0^1 = \frac{1}{2}$$

Theorem 6 (**Integrals of Symmetric Functions**)**.** *Suppose f is continuous on $[a, -a]$.*

a) If f is even $[f(-x)] = f(x)$, then $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$.

b) If f is odd $[f(-x)] = -f(x)$, then $\int_{-a}^{a} f(x)\ dx = 0$.

5.10 Example 10

Evaluate $\displaystyle\int_{-2}^{2}(x^6+1)\ dx$.

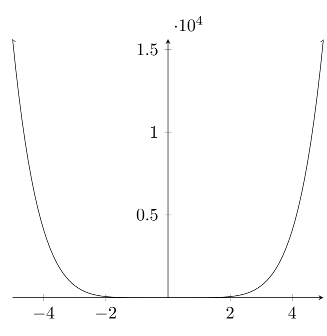


Figure 4: $f(x) = x^6 + 1$

Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$. it is even and so

$$\int_{-2}^{2} (x^6 + 1)\ dx = 2 \int_{0}^{2} (x^6 + 1)\ dx$$
$$= 2\left[\frac{1}{7}x^7 + x\right]_0^2 = 2\left(\frac{128}{7} + 2\right) = \frac{284}{7}$$

5.11 Example 11

Evaluate $\displaystyle\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\ dx$.

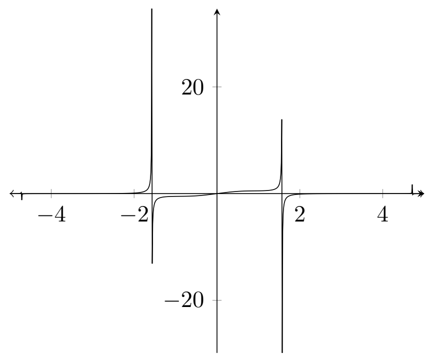


Figure 5: $f(x) = \dfrac{\tan x}{1 + x^2 + x^4}$

Since $f(x) = \dfrac{\tan x}{1 + x^2 + x^4}$ satisfies $f(-x) = -f(x)$. it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\ dx = 0$$

6 Area Between Curves

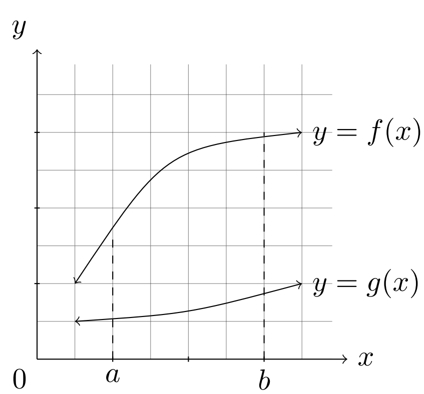


Figure 6: Area between the lines

$$
\begin{aligned}
Area &= \int_a^b f(x)\ dx - \int_a^b g(x)\ dx \\
&= \int_a^b [f(x) - g(x)]\ dx
\end{aligned}
$$

6.1 Example 1

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

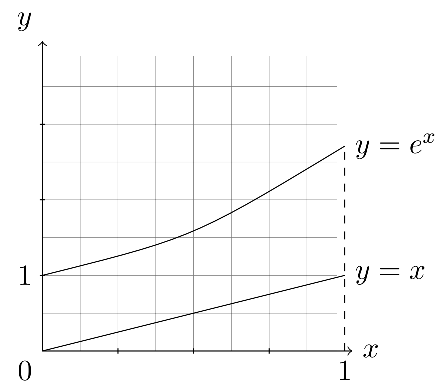


Figure 7: Caption

$$
\begin{aligned}
A = \int_0^1 (e^x - x)\ dx &= e^x - \frac{1}{2}x^2\Big]_0^1 \\
&= \left(e^1 - \frac{1}{2}1^2\right) - \left(e^0 - \frac{1}{2}0^2\right) \\
&= e^1 - \frac{1}{2} - 1 = e - \frac{3}{2}\ \text{units}^2
\end{aligned}
$$

6.2 Example 2

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Figure 8: Example 2

First, we solve the two equations and find the interesections: $(0,0)$ and $(1,1)$, so

$$\begin{aligned}
A &= \int_0^1 [(2x - x^2) - (x^2)]\ dx = \int_0^1 (2x - 2x^2)\ dx \\
&= 2\int_0^1 (x - x^2)\ dx \\
&= 2\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 \\
&= 2\left\{\left[\frac{1}{2}(1)^2 - \frac{1}{3}(1)^3\right] - \left[\frac{1}{2}(0)^2 - \frac{1}{3}(0)^3\right]\right\} \\
&= 2\Big(\frac{1}{6}\Big) = \frac{1}{3}\text{units}^2
\end{aligned}$$

6.3 Example 5

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Figure 9: Example 5

The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$. Then the required area is:

$$\begin{aligned}
A &= \int_0^{\pi/2} |\cos x - \sin x|\ dx = A_1 + A_2 \\
&= \int_0^{\pi/4} (\cos x - \sin x)\ dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\ dx \\
&= \big[\sin x - \cos x\big]_0^{\pi/4} + \big[-\cos x + \sin x\big]_{\pi/4}^{\pi/2} \\
&= \bigg(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\bigg) + \bigg(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\bigg) \\
&= 2\sqrt{2} - 2
\end{aligned}$$

6.4 Example 6 **On test do both ways

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

(a) Example 6

(b) Example 6

Method 1

$$\text{Area} = \int_{-3}^{1} \left[\sqrt{2x+6} - (-\sqrt{2x+6})\right] + \int_{-1}^{5} \left[\sqrt{2x+6} - (x-1)\right]\, dx$$

Method 2
We should look at the figure sideways, then $y^2 = 2x+6$ becomes $x = (y^2-6)/2$, $y = x-1$ becomes $x = y+1$. The intersections becomes $(-1, -2)$ and $(5, 4)$.

$$\text{Area} = \int_{-2}^{4} \left[(y+1) - \left(\frac{y^2-6}{2}\right)\right]\, dy$$