Libor Market Model (LFM: Lognormal Forward Rate Model) Calibration to Interest Rate Derivatives

463403 Meichen Fan
465241 Yifan Yu
463457 Ziqi Tan
463283 Shaoyuan Yang
463507 Yuwen Wang
467716 Shuang Tang
463504 Jingxuan Zou
467730 Junhao Shen

Contents

I Introduction

The main objective of our project is to calibrate the Lognormal Forward Rate Model(LFM) to interest rate derivatives — market-quoted implied cap and swaptions volatilities. In this project, we will explore four different model assumptions, find the optimal parameter set by minimizing the error measure and exam the model fit including in-sample and out-of-sample performance. The following will be our detailed procedures and results analysis. The report will be divided into two parts by different derivatives we used to calibrate.

II LFM Calibration Using Caps

The first part is to calibrate the LFM to the market caplet volatilities. We begin with considering the piecewise-constant specification for instantaneous volatilities, as shown in Table 2. The assumption assumes that the volatilities only depend on the time-to-maturity $T_k - T_{\beta(t)-1}$ of a forward rate, that is

$$\sigma_k(t) = \sigma_{k,\beta(t)} =: \eta_{k-(\beta(t)-1)} \tag{1}$$

Table 1: **Table 2 in the Textbook**

Instant. Vols	t $\in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\ldots$	$(T_{M-2}, T_{M-1}]$
Fwd Rate: $F_1(t)$	η_1	Dead	Dead	$\ldots$	Dead
$F_2(t)$	η_2	η_1	Dead	$\ldots$	Dead
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$F_{M(t)}$	η_M	η_{M-1}	η_{M-2}	$\ldots$	η_1

And then the parameters to be estimated reduce to M unknown η' s. Using this volatility structure, we have the following relation between market caplet volatilities and the unkown parameters:

$$V^2_{\text{Caplets } T_{i-1}} = \frac{1}{T_{i-1}} \sum_{j=1}^{i} \tau_{j-2,j-1} \eta^2_{i-j+1} \mathrm{d}t \tag{2}$$

Given M market caplet implied volatilities, we can solve the M η' s exactly by the algebraic procedure using MatLab

$$\eta_i^2 = i \times V_i - (i-1) \times V_{i-1} \tag{3}$$

The results of all η 's are shown in the following table.

Table 2: Calibration Result of **Table 2**

Instant. Vols	t $\in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	$(T_3, T_4]$	$(T_4, T_5]$	$(T_5, T_6]$	$(T_6, T_7]$	$(T_7, T_8]$	$(T_8, T_9]$
$F_1(t)$	0.8612	Dead	Dead	Dead	Dead	Dead	Dead	Dead	Dead	Dead
$F_2(t)$	0.4897	0.8612	Dead	Dead	Dead	Dead	Dead	Dead	Dead	Dead
$F_3(t)$	-0.5547	0.4897	0.8612	Dead	Dead	Dead	Dead	Dead	Dead	Dead
$F_4(t)$	-0.0329	-0.5547	0.4897	0.8612	Dead	Dead	Dead	Dead	Dead	Dead
$F_5(t)$	-0.3329	-0.0329	-0.5547	0.4897	0.8612	Dead	Dead	Dead	Dead	Dead
$F_6(t)$	0.0522	-0.3329	-0.0329	-0.5547	0.4897	0.8612	Dead	Dead	Dead	Dead
$F_7(t)$	0.0502	0.0522	-0.3329	-0.0329	-0.5547	0.4897	0.8612	Dead	Dead	Dead
$F_8(t)$	0.0223	0.0502	0.0522	-0.3329	-0.0329	-0.5547	0.4897	0.8612	Dead	Dead
$F_9(t)$	0.0115	0.0223	0.0502	0.0522	-0.3329	-0.0329	-0.5547	0.4897	0.8612	Dead
$F_{10}(t)$	-0.006	0.0115	0.0223	0.0502	0.0522	-0.3329	-0.0329	-0.5547	0.4897	0.8612

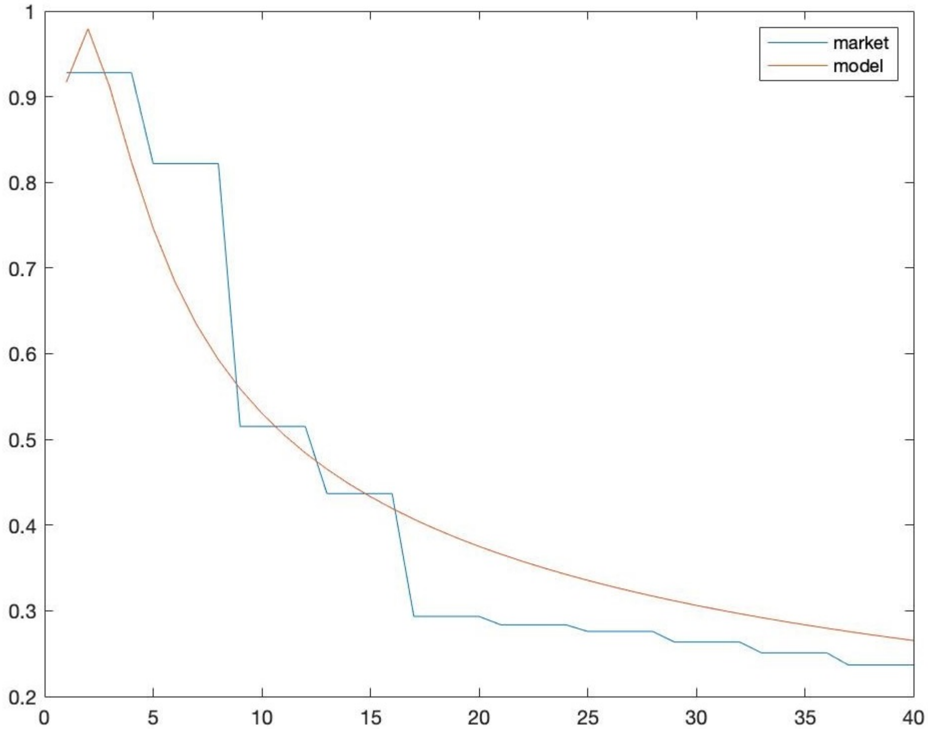
Then we consider the parametric form (**Formulation 6**) for the same quantities:

$$\sigma_i(t) = \psi\left(T_{i-1} - t; a, b, c, d\right) := \left[a\left(T_{i-1} - t\right) + d\right] e^{-b(T_{i-1}-t)} + c \tag{4}$$

$$v_i^2 = \int_0^{T_{i-1}} \left(\left[a\left(T_{i-1} - t\right) + d\right] e^{-b(T_{i-1}-t)} + c\right)^2 dt =: I^2\left(T_{i-1}; a, b, c, d\right) \tag{5}$$

This form allows a humped shape in the graph of the instantaneous volatility of the generic forward rate F_i as a function of time to maturity, $T_{i-1} - t \mapsto \sigma_i(t)$. From the equation 5, we can see that there are 4 unknown parameters [**a, b, c, d**]. First, we set the initial value of the parameter set as [1, 1, 1, 1] and calculated the predicted value of volatility. Second, using the square of difference of the predicted volatility and the market caplet volatility as the error measure, we call the FMINCON function in MatLab to find the optimal parameter set to minimize the error measure. Our optimization results are [a, b, c, d] = [**-8.1465, 3.1756, 0.0071, -0.4224**] with the minimized in-sample error at **0.2175**. Figure 1 plots term of structure of market implied volatility and that of our model implied volatility calculated with optimal parameters.

Figure 1: Term Structure of Market Volatility and Model Volatility under **Formulation 6**



III LFM Calibration Using Swaptions

The second part is to jointly calibrate the LFM to the market swaption volatilities. We also firstly use the piecewise-constant model for instantaneous volatilities, as shown in **Table 5**. The model assumes that the specification of the instantaneous volatility in a more general case

$$\sigma_k(t) = \sigma_{k,\beta(t)} := \Phi_k \psi_{k-(\beta(t)-1)} \tag{6}$$

Table 3: **Table 5** in the Textbook

Instant. Vols	t $\in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\ldots$	$(T_{M-2}, T_{M-1}]$
Fwd Rate: $F_1(t)$	$\Phi_1\psi_1$	Dead	Dead	$\ldots$	Dead
$F_2(t)$	$\Phi_2\psi_2$	$\Phi_2\psi_1$	Dead	$\ldots$	Dead
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$F_{M(t)}$	$\Phi_M\psi_M$	$\Phi_M\psi_{M-1}$	$\Phi_{M-1}\psi_{M-2}$	$\ldots$	$\Phi_M\psi_1$

And using this volatility structure, we have the following relation between market caplet volatilities and the unkown parameters:

$$v_i^2 = \Phi_i^2 \sum_{j=1}^{i} \tau_{j-2,j-1} \psi_{i-j+1}^2 \tag{7}$$

Thus far the parameters to be estimated reduces to M unknown Φ_k's and M unknown ψ_k's. The total number of parameters to be estimated is 2M, which is more than the number of caplets (M), so we have to utilize the swaption data to calibrate the model. The first step is to represent Φ_k 's by the combination of ψ_k's s and market caplet volatilities:

$$\Phi_i^2 = \frac{\left(v_i^{\mathrm{MKT}}\right)^2}{\sum_{j=1}^{i} \tau_{j-2,j-1} \psi_{i-j+1}^2} \tag{8}$$

Then go back to the original σ. We only have M unknown ψ_k 's which can be calibrated by matching the market swaption volatilities and the predicted swaption volatilities from the equation below:

$$\left(v_{\alpha,\beta}^{LFM}\right)^2 = \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0) w_j(o) F_i(0) F_j(0) \rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t) \sigma_j(t) dt \tag{9}$$

The error measure is also the difference of squares of the predicted volatility and the market swaption volatilities. In this equation, we input every data by:

$$S_{\alpha,\beta}(t) \approx \sum_{i=\alpha+1}^{\beta} w_i(0) F_i(t) \tag{10}$$

$$w_i(t) = \frac{P\left(t, T_i\right) \tau_i}{\sum_{k=\alpha+1}^{\beta} P\left(t, T_k\right) \tau_k} \tag{11}$$

$$F(t; T, S) := \frac{1}{\tau(T, S)} \left(\frac{P(t, T)}{P(t, S)} - 1 \right) \tag{12}$$

$$\rho_{i,j} = e^{-\beta|i-j|} \tag{13}$$

We assume β in the last item equal to 1 throughout the process to simplify the optimization procedure in reference of the textbook. We also replace the integral process of the second part of LFM swaption volatility with the summation of discrete-time periods to facilitate the optimization. Using the error measure as the minimized objective, we obtained the optimal ψ 's as follows and calculated Φ's accordingly, shown in Table 4. Below are the results of the calibration of Table 5. The minimized error measure based on swaption volatility is **0.2327**.

Table 4: Calibrated Results of ψ and Φ

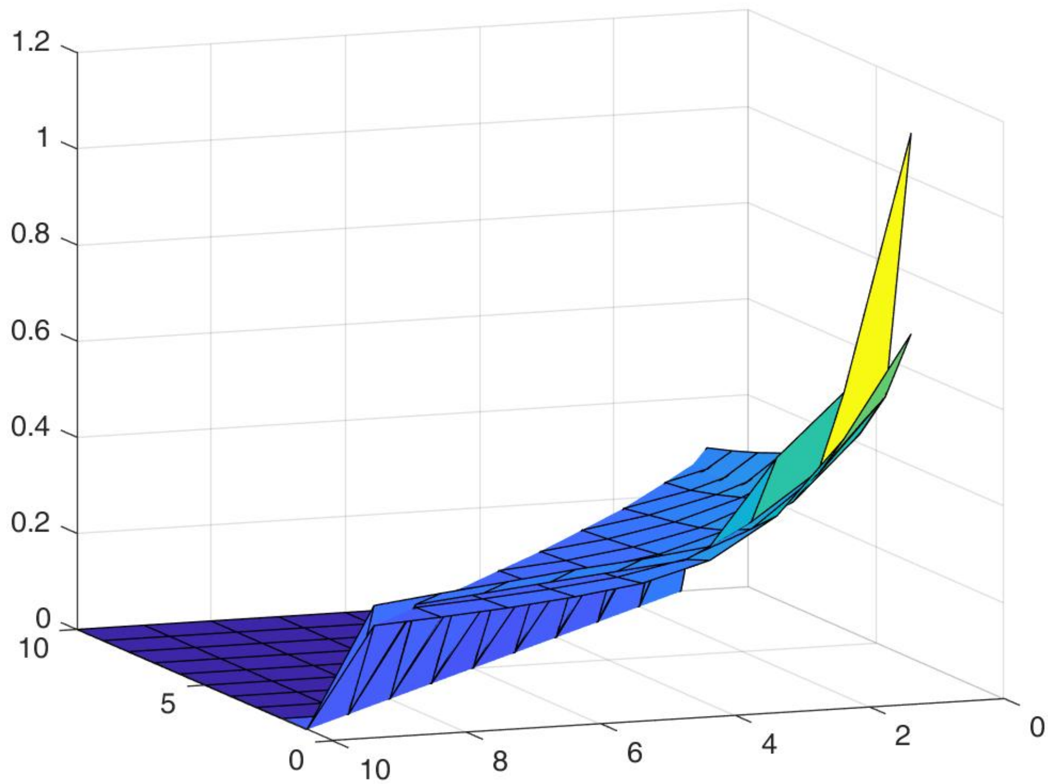
Index	ψ	ϕ
1	0.0801	11.584
2	0.0465	12.549
3	0.0253	9.2941
4	0.1093	6.0055
5	0.2474	2.2862
6	0.3707	1.4821
7	0.5211	1.0414
8	0.7627	0.7193
9	1.1429	0.488
10	1.7556	0.3204

Table 5: Calibration Result of**Table 5**

Instant. Vols	t $\in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	$(T_3, T_4]$	$(T_4, T_5]$	$(T_5, T_6]$	$(T_6, T_7]$	$(T_7, T_8]$	$(T_8, T_9]$
$F_1(t)$	0.9280	0	0	0	0	0	0	0	0	0
$F_2(t)$	0.5832	0.5387	0	0	0	0	0	0	0	0
$F_3(t)$	0.2350	0.3173	0.2929	0	0	0	0	0	0	0
$F_4(t)$	0.6564	1.0158	1.3716	1.2662	0	0	0	0	0	0
$F_5(t)$	0.5655	1.4855	2.2990	3.1041	2.8653	0	0	0	0	0
$F_6(t)$	0.5493	0.84738	2.2259	3.4449	4.6514	4.2935	0	0	0	0
$F_7(t)$	0.5427	0.7724	1.1914	3.1297	4.8436	6.5400	6.0368	0	0	0
$F_8(t)$	0.5486	0.79432	1.1304	1.7438	4.5806	7.089	9.5717	8.8354	0	0
$F_9(t)$	0.5578	0.8221	1.1903	1.6939	2.6129	6.8638	10.6225	14.3428	13.2393	0
$F_{10}(t)$	0.5625	0.85676	1.2628	1.8283	2.6019	4.0136	10.5431	16.3165	22.0310	20.3361

Thus far, we could compute swaptions prices according to eqution 9 utilizing the parameters we obtain. We also compare the market swaptions volatility surface and model swaption volatility surface, suggested in Figure 2. As the fitted graph demonstrated below, the two surfaces suggested a significant fit, which implies the model is able to calibrate the market data well.

Figure 2: Market Swaption Volatility Surface and Model Volatility Surface



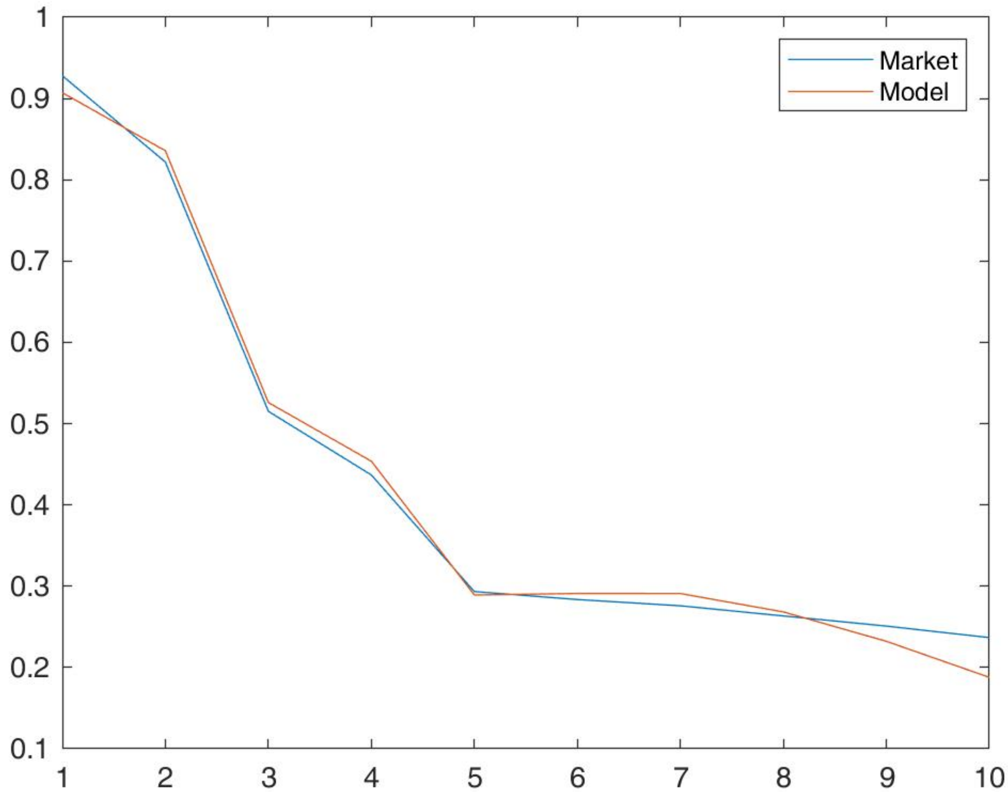
Then we moved to the parametric form (**Formulation 7**):

$$\sigma_i(t) = \Phi_i \psi \left(T_{i-1} - t; a, b, c, d\right) := \Phi_i \left(\left[a \left(T_{i-1} - t\right) + d\right] e^{-b(T_{i-1}-t)} + c\right) \tag{14}$$

$$v_i^2 = \Phi_i^2 \int_0^{T_{i-1}} \left(\left[a \left(T_{i-1} - t\right) + d\right] e^{-b(T_{i-1}-t)} + c\right)^2 dt = \Phi_i^2 I^2 \left(T_{i-1}; a, b, c, d\right) \tag{15}$$

In this form, we introduced a parameter ψ that is varied by the Φ 's for each maturity T_i. We have calculated Φ 's in the previous part. Thus, in Formulation 7 we are only left with parameters a, b, c, and d. Since the number of parameters is smaller than the number of caplets, we did not need swaption to calibrate the model. Similar to the previous steps, we used the FMINCON to minimize the error measure which is calculated as the difference of squares of market caplet volatilities and predicted caplet volatilities. We set the initial values of [a, b, c, d] at [1, 1, 1, 1] and got the optimized results [**0.7539, 0.0565, -13.0406, 13.1194**] with minimum error measure **0.0041**. The fitted graph of term structure of market and model caplet implied volatility is as follows.

Figure 3: Term Structure of Market Volatility and Model Volatility under **Formulation 7**



IV Results Analysis

Originally, we consider the assumption that the piecewise-constant instantaneous volatilities $\sigma_{i,\beta(t)}$ depend only on the time to maturity. In **Table 2**, the parameters η alone can be used to exactly fit the (square of the) market caplet volatilities (multiplied by time) v_i^2. Moving to the parametric form in **Formulation 6** with the same assumption, we obtained the minimum error measure of the cap volatility model deviation from the market at **0.2175** with the optimized parameter set.

In the second part, we further expand the assumption that the piecewise-constant instantaneous volatilities $\sigma_{i,\beta(t)}$ follow the separable structure in which we define parameters Φ and ψ , where $\sigma_{i,\beta(t)} = \Phi_i \psi_{\beta(t)}$. In **Table 5**, we can also exactly fit the market caplet volatilities by finding the optimal ψ s with the calibration of market swaption volatilities and calculating Φs by ψs. The joint calibration helps solve the problem with more unknown parameters than the number of known values. For **Formulation 7**, applying the optimized s from calibration of **Table 5**, we reached another minimum error measure **0.0041** based on the cap volatilities.

From the in-sample errors and the fitted graphs of the model and market data, we observed an improvement in LFM fitting and flexibility in the shape of the term structure of volatilities from **Formulation 6** to **Formulation 7**. By introducing parametric forms Φ and ψ, we altered parameters for each maturity T_i so that we are able to add flexibility to improve the joint calibration to caplets and swaptions.

V Appendix: Calibration Model applied with Bloomberg Market Data

We further extracted the Bloomberg caplets market data to apply our calibration model. We pulled the Cap at-the-money volatilities along with the corresponding strike prices, discounted factor, and swaption volatilities matrix as of 2014-09-01. The results using Libor market data retrieved from Bloomberg are provided as the following.

Table 6: **Calibration Result of Table 2**

Instant. Vols	t $\in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	$(T_3, T_4]$	$(T_4, T_5]$	$(T_5, T_6]$	$(T_6, T_7]$	$(T_7, T_8]$	$(T_8, T_9]$
$F_1(t)$	0.4027	Dead	Dead	Dead	Dead	Dead	Dead	Dead	Dead	Dead
$F_2(t)$	0.2847	0.4027	Dead	Dead	Dead	Dead	Dead	Dead	Dead	Dead
$F_3(t)$	0.0266	0.2847	0.4027	Dead	Dead	Dead	Dead	Dead	Dead	Dead
$F_4(t)$	0.0687	0.0266	0.2847	0.4027	Dead	Dead	Dead	Dead	Dead	Dead
$F_5(t)$	-0.0010	0.0687	0.0266	0.2847	0.4027	Dead	Dead	Dead	Dead	Dead
$F_6(t)$	-0.0426	-0.0010	0.0687	0.0266	0.2847	0.4027	Dead	Dead	Dead	Dead
$F_7(t)$	0.0849	-0.0426	-0.0010	0.0329	0.0266	0.2847	0.4027	Dead	Dead	Dead
$F_8(t)$	-0.0419	0.0849	-0.0426	-0.0010	0.0329	0.0266	0.2847	0.4027	Dead	Dead
$F_9(t)$	0.0149	-0.0419	0.0849	-0.0426	-0.0010	0.0329	0.0266	0.2847	0.4027	Dead
$F_{10}(t)$	0.0273	0.0149	0.0849	0.0849	-0.0426	-0.0010	0.0687	0.0266	0.4897	0.4027

Table 7: Calibration Result of **Formulation 6**

a	b	c	d	SSE
6.7694	2.4605	0.1697	-1.5783	0.012

Table 8: Calibrated Results of ψ, Φ and SSE

Index	ψ	ϕ
1	0.3499	1.8136
2	0.1550	2.1665
3	0.1954	1.9665
4	0.2134	1.8441
5	0.3131	1.5433
6	0.4658	1.1643
7	0.6028	0.9523
8	0.7455	0.7308
9	0.9002	0.5919
10	1.1243	0.4826
SSE	0.1458	

Table 9: Result of **Table 5**

Instant. Vols	t $\in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	$(T_3, T_4]$	$(T_4, T_5]$	$(T_5, T_6]$	$(T_6, T_7]$	$(T_7, T_8]$	$(T_8, T_9]$
$F_1(t)$	0.6346	0	0	0	0	0	0	0	0	0
$F_2(t)$	0.3357	0.281	0	0	0	0	0	0	0	0
$F_3(t)$	0.3842	0.4233	0.3544	0	0	0	0	0	0	0
$F_4(t)$	0.3935	0.4196	0.4623	0.3870	0	0	0	0	0	0
$F_5(t)$	0.4832	0.5774	0.6157	0.6783	0.5678	0	0	0	0	0
$F_6(t)$	0.5423	0.7188	0.8589	0.9159	1.0091	0.8447	0	0	0	0
$F_7(t)$	0.5740	0.7019	0.9303	1.1116	1.1854	1.306	1.0933	0	0	0
$F_8(t)$	0.5448	0.7100	0.8680	1.1505	1.3748	1.4660	1.6152	1.3521	0	0
$F_9(t)$	0.5328	0.6578	0.8573	1.0481	1.3892	1.6600	1.7703	1.9503	1.6326	0
$F_{10}(t)$	0.5426	0.6655	0.8216	1.0706	1.3090	1.7350	2.0732	2.2109	2.4358	2.0390

Figure 4: Market Swaption Volatility Surface and Model Volatility Surface

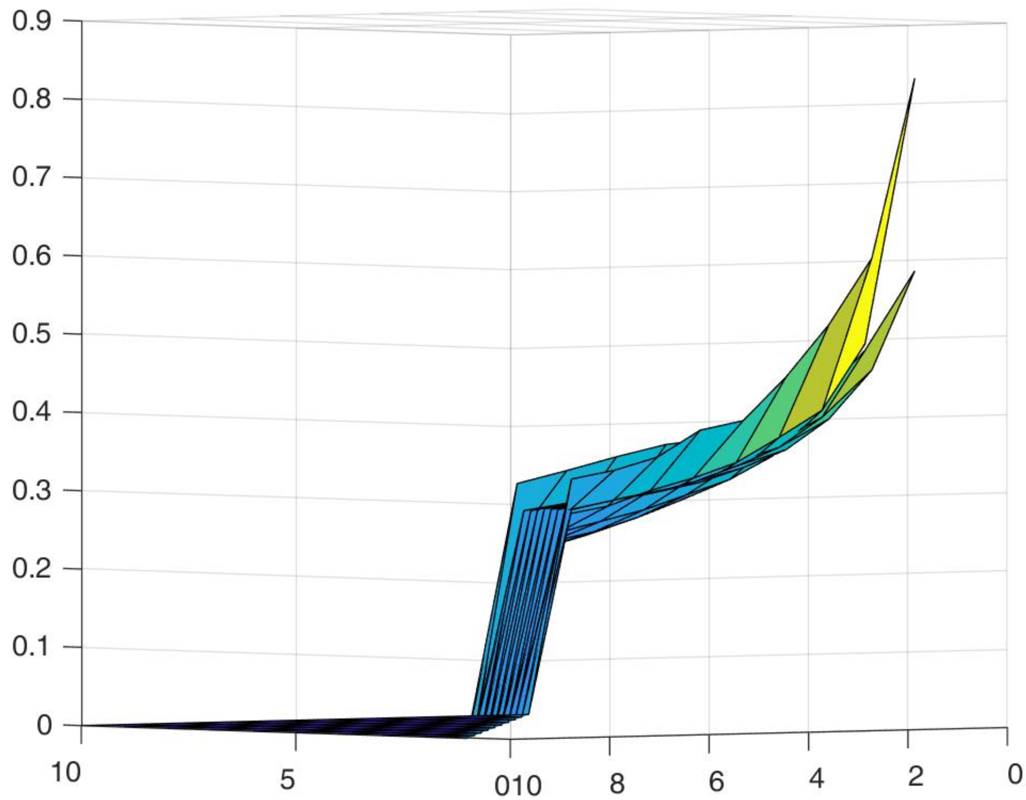


Figure 5: Term Structure of Market Volatility and Model Volatility under **Formulation 6**

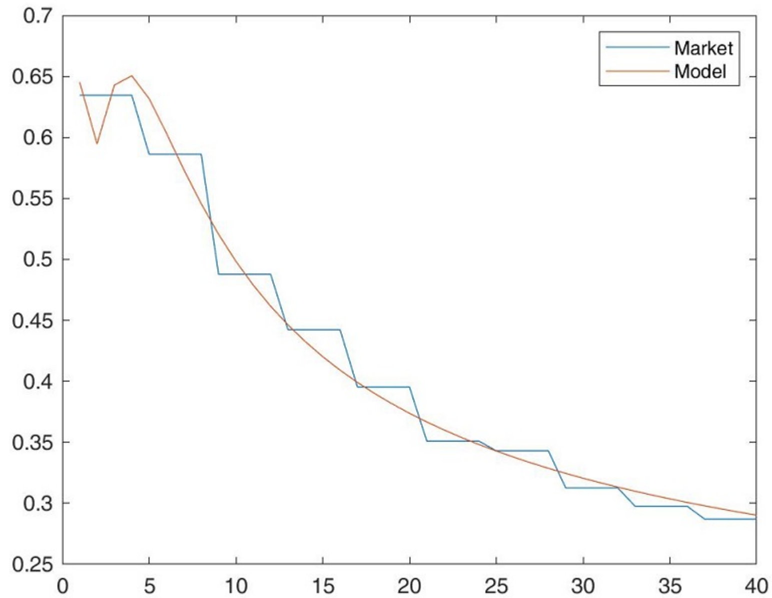


Figure 6: Term Structure of Market Volatility and Model Volatility under **Formulation 7**