ME 280a: HW 1

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September 9, 2016

1 Introduction and Objectives

The purpose of this study is to solve a simple finite element (FE) problem and perform a convergence study to determine the number of elements needed to reach a specific tolerance. The Galerkin FE method is used, which for certain classes of problems possesses the "best approximation property," a characteristic that signifies that the FE solution obtained is the best possible solution for a given mesh refinement and choice of shape functions. The mathematical procedure is described in Section 2.

2 Procedure

This section describes the mathematical process used to solve the following problem:

$$\frac{d}{dx}\left(E(x)\frac{du}{dx}\right)=k^2\sin\left(\frac{2\pi kx}{L}\right) \tag{1}$$

where E is the modulus of elasticity, u is the solution, k is a known constant, L is the problem domain length, and x is the spatial variable. The Galerkin FE method (FEM) achieves the best approximation property by expanding both u and a test function ψ in the same set of basis functions:

$$\begin{gathered}u\approx u^N=\sum_{j=1}^{N}a_j\phi_j\\ \psi=\sum_{i=1}^{N}b_i\phi_i\end{gathered} \tag{2}$$

where u^N is the approximate solution. Galerkin's method is stated as:

$$r^N\cdot u^N=0 \tag{3}$$

where r^N is the residual. Hence, to formulate the weak form to Eq. (1), multiply Eq. (1) through by ψ and integrate over all space, $d\Omega$.

$$\int_\Omega\frac{d}{dx}\left(E(x)\frac{du}{dx}\right)\psi d\Omega-\int_\Omega k^2\sin\left(\frac{2\pi kx}{L}\right)\psi d\Omega=0 \tag{4}$$

Applying integration by parts to the first term:

$$-\int_\Omega E(x)\frac{du}{dx}\frac{d\psi}{dx}d\Omega+\int_{\partial\Omega}E(x)\frac{du}{dx}\psi d(\partial\Omega)-\int_\Omega k^2\sin\left(\frac{2\pi kx}{L}\right)\psi d\Omega=0 \tag{5}$$

where $\partial\Omega$ refers to one dimension lower than Ω, which for this case refers to evaluation at the endpoints of the domain. Hence, for this particular 1-D problem, the above reduces to:

$$
\begin{aligned}
-\int_0^L E(x)\frac{du}{dx}\frac{d\psi}{dx}dx + E(x)\frac{du}{dx}\psi\bigg|_0^L - \int_0^L k^2 \sin\left(\frac{2\pi kx}{L}\right)\psi dx = 0 \\
\int_0^L E(x)\frac{du}{dx}\frac{d\psi}{dx}dx = \int_0^L k^2 \sin\left(\frac{2\pi kx}{L}\right)\psi dx - E(x)\frac{du}{dx}\psi\bigg|_0^L
\end{aligned} \tag{6}
$$

Inserting the approximation in Eq. (2):

$$
\int_0^L E(x)\frac{d\left(\sum_{j=1}^N a_j\phi_j\right)}{dx}\frac{d\left(\sum_{i=1}^N b_i\phi_i\right)}{dx}dx = \int_0^L k^2 \sin\left(\frac{2\pi kx}{L}\right)\left(\sum_{i=1}^N b_i\phi_i\right)dx - E(x)\frac{d\left(\sum_{j=1}^N a_j\phi_j\right)}{dx}\left(\sum_{i=1}^N b_i\phi_i\right)\Bigg|_0^L \tag{7}
$$

Recognizing that b_i appears in each term, the sum of the remaining terms must also equal zero (i.e. basically cancel b_i from each term).

$$
\int_0^L E(x)\frac{d\left(\sum_{j=1}^N a_j\phi_j\right)}{dx}\frac{d\phi_i}{dx}dx = \int_0^L k^2 \sin\left(\frac{2\pi kx}{L}\right)\phi_i dx - E(x)\frac{d\left(\sum_{j=1}^N a_j\phi_i\right)}{dx}\left(\phi_j\right)\bigg|_0^L \tag{8}
$$

This equation can be satisfied for each choice of j, and hence can be reduced to:

$$
\int_0^L E(x)\frac{d\left(a_j\phi_j\right)}{dx}\frac{d\phi_i}{dx}dx = \int_0^L k^2 \sin\left(\frac{2\pi kx}{L}\right)\phi_i dx - E(x)\frac{d\left(a_j\phi_j\right)}{dx}\left(\phi_i\right)\bigg|_0^L \tag{9}
$$

This produces a system of matrix equations of the form:

$$
\mathbf{K}\vec{x} = \vec{b} \tag{10}
$$

where:

$$
\begin{aligned}
K_{ij} &= \int_0^L E(x)\frac{d\phi_j}{dx}\frac{d\phi_j}{dx}dx \\
x_j &= a_j \\
b_i &= \int_0^L k^2 \sin\left(\frac{2\pi kx}{L}\right)\phi_i dx
\end{aligned} \tag{11}
$$