Verification of the Cardinal Multiphysics Solver

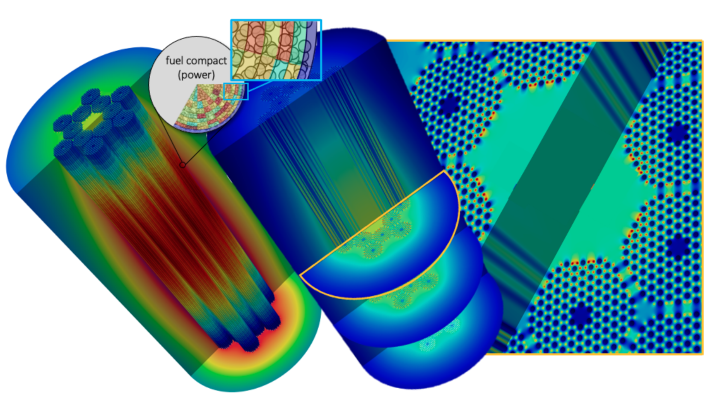
1-D Coupled Heat Transfer and Neutron Transport

Lewis Gross, April Novak, Patrick Shriwise, and Paul Wilson

August 15, 2022

Outline

Modern Multiphysics Simulation and the Importance of V&V



- Full-core multiphysics model of an HTGR using Cardinal: OpenMC power (left) and MOOSE solid temperature (right) [1].

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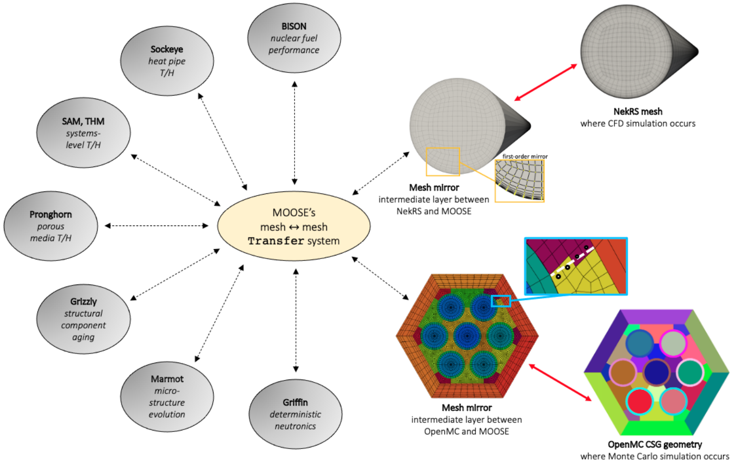
- Cardinal [1] our software choice to model this benchmark couples OpenMC [3] neutronics and NekRS [4] CFD into the MOOSE framework [5].

Cardinal's connection to the MOOSE Framework [6]



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- Using (1) and (2) gives a Doppler-broadened, macroscopic, total cross section that accounts for changes in density due to temperature as

$$\Sigma_t(x) = \frac{\rho_0\sigma_{t,0}N_A}{A}\frac{T_0}{T(x)} \equiv \Sigma_{t,0}\frac{T_0}{T(x)} \tag{4}$$

- where $\sigma_{t,0}$ is the total microscopic cross section at T_0, N_A is Avogadro's number, and A is the mass number of the medium.

Analytical Benchmark

- Based on 1-D S_2 transport, the neutron flux $\phi(x)$ is governed by

$$\frac{d}{dx}\left[\frac{1}{\Sigma_t(x)}\frac{d\phi(x)}{dx}\right]+\Sigma_t(x)\left(\lambda-1\right)\phi(x)=0 \tag{5}$$

- where $\lambda \equiv (\frac{1}{k_{eff}}\frac{\nu\Sigma_f}{\Sigma_t}+\frac{\Sigma_s}{\Sigma_t})$ is the combined in-scattering and quasi-static fission source term [2].

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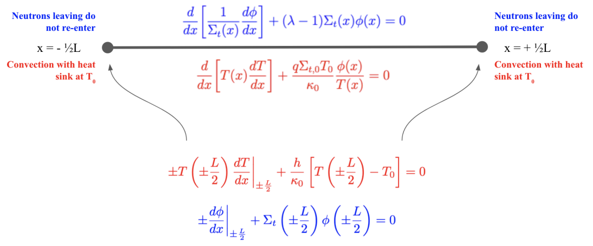
- The conduction equation and its boundary condition govern energy conservation in the slab:

$$\frac{d}{dx}\left[\kappa(T)\frac{dT(x)}{dx}\right] + q\Sigma_t(x)\phi(x) = 0 \qquad \text{AND}$$

$$-\kappa(T)\frac{dT}{dx}\bigg|_{\pm\frac{L}{2}} + h\left[T(\pm\frac{L}{2}) - T_0\right] \qquad (6)$$

- where κ is the thermal conductivity, q is the energy released **per reaction**, Σ_t is the total macroscopic cross section, and h is the heat transfer coefficient.

System Domain, Differential Equations, and Boundary Conditions



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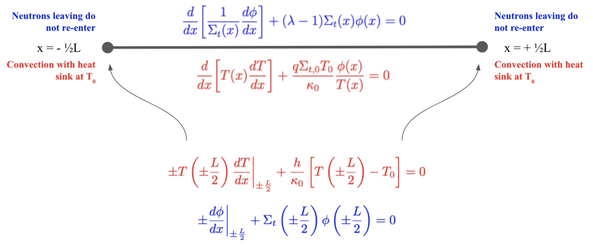

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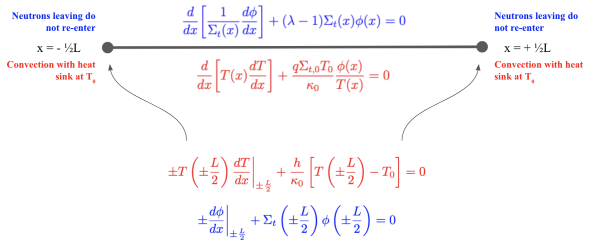

- The fundamental assumption (ansatz) of [2]: $T(x) = f\phi(x)$.
- This imposes two constraints that determine the system heat transfer coefficient h and the total microscopic cross section $\sigma_{t,0}$. The solution that satisfies the above is given by an elliptical flux shape

$$\phi(x) = \phi(0)\sqrt{1-\frac{(\lambda-1)P^2x^2}{L^2q^2\phi^2(0)}} \tag{7}$$

- where P is the slab power and L is the slab equilibrium length.

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- S_2 patch to restrict particle birth direction and scattering direction to only $\pm$x.
- Tallies flux, kappa-fission heating rate, and k-eigenvalue.

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- Convective boundary conditions at end points with heat sink at $T_0 = 293$ K.
- Jacobi Free Newton Krylov solver: 10^{-7} absolute tolerance and 10^{-9} relative tolerance.

Coupling, Data Mapping [6], and Convergence Criteria

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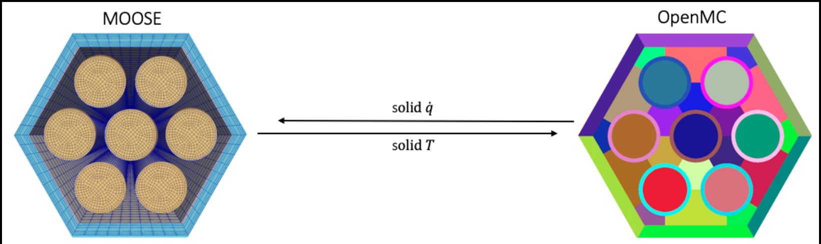

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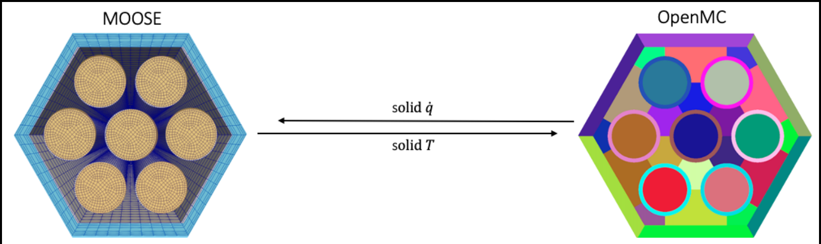

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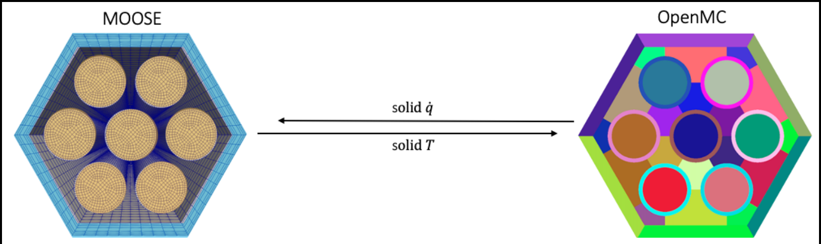

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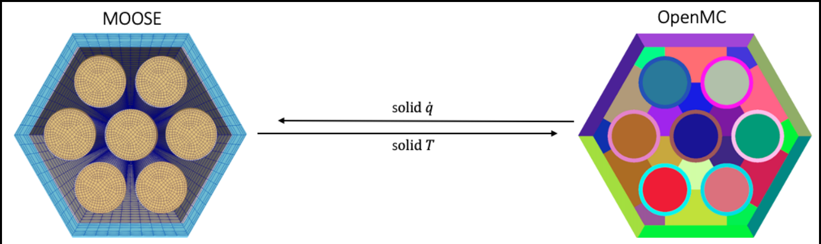

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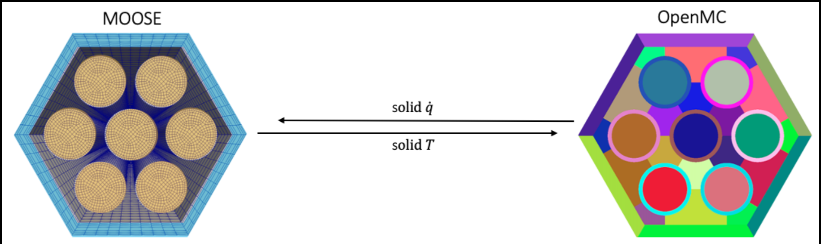

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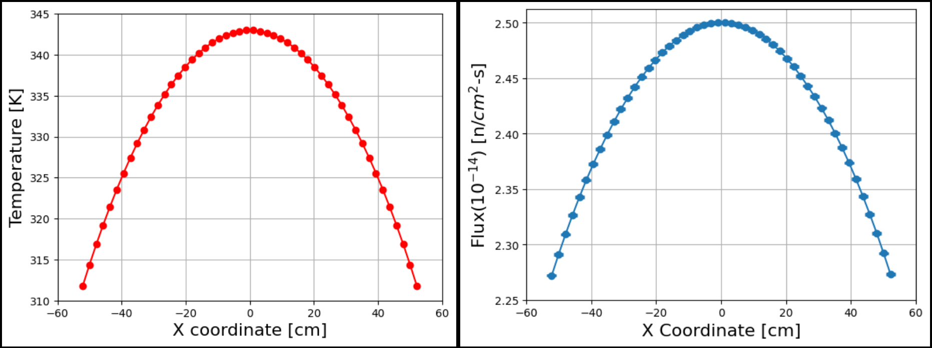
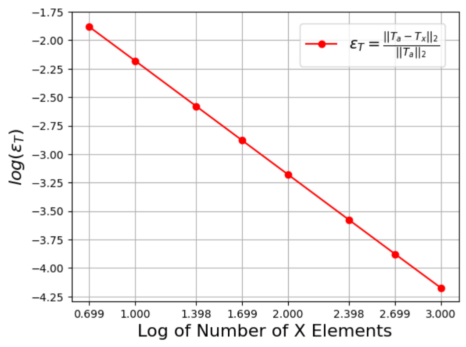
- k-eigenvalue simulation used 50,000 particles per batch, 50 inactive batches and 100 active batches.
  - Shannon Entropy study confirmed this sufficient following criteria from [9].
- Final transport solve with converged temperature used 250,000 particles per batch.

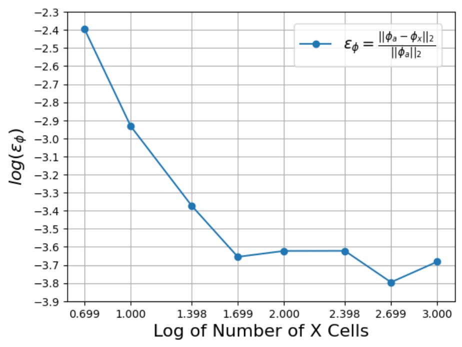
Outputs and Comparisons



- Numerical solutions for 50 mesh elements. On the right, error bars show the relative error of the flux, which are nearly smaller than the circular marker sizes.

Solution L_2 Error Norms





- Error norms as a function of heat conduction mesh element count and OpenMC cell count, respectively.

Eigenvalue comparisons across each spatial discretization

Resolution	k_{eff}	(numerical - analytical) [pcm]
analytical	0.29557	-
5	0.29624 $\pm$ 0.00003	67 $\pm$ 3
10	0.29581 $\pm$ 0.00004	24 $\pm$ 4
25	0.29563 $\pm$ 0.00004	6 $\pm$ 4
50	0.29553 $\pm$ 0.00004	-4 $\pm$ 4
100	0.29557 $\pm$ 0.00003	0 $\pm$ 3
250	0.29561 $\pm$ 0.00004	4 $\pm$ 4
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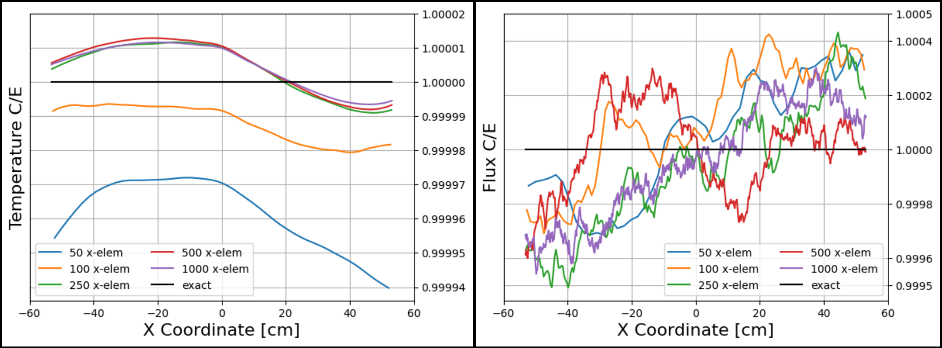
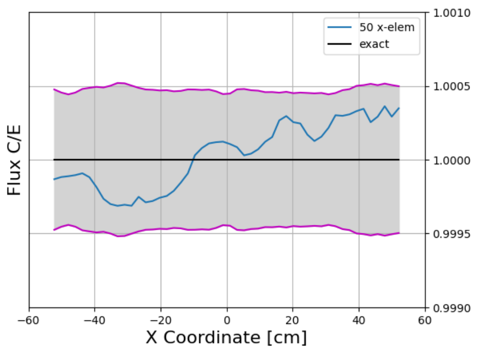
- k_{eff} is a system-wide parameter, so it converges much faster than flux and is not as dependent on number of cells.

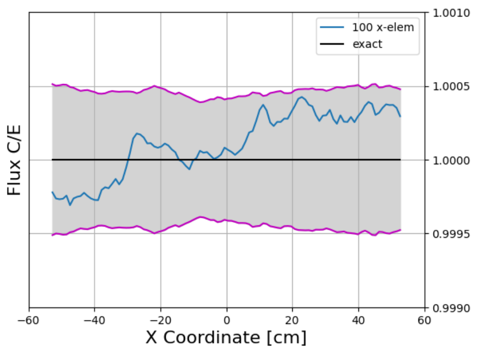
Computed to Expected Ratios



- C/E for fine cases ($N = 50, 100, 250, 500, 1000$). Note the scales of the y-axes - the temperature is everywhere being predicted to within 0.006% and flux is everywhere being predicted to within 0.05%.

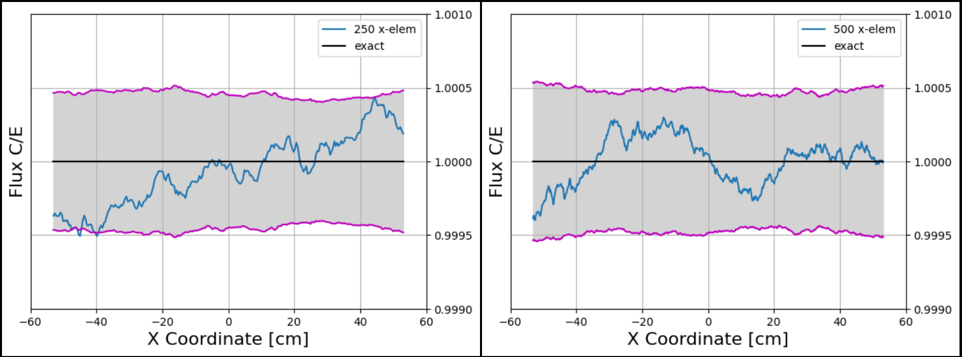
Individual Flux C/E with 2σ Error Bars for Fine Cases





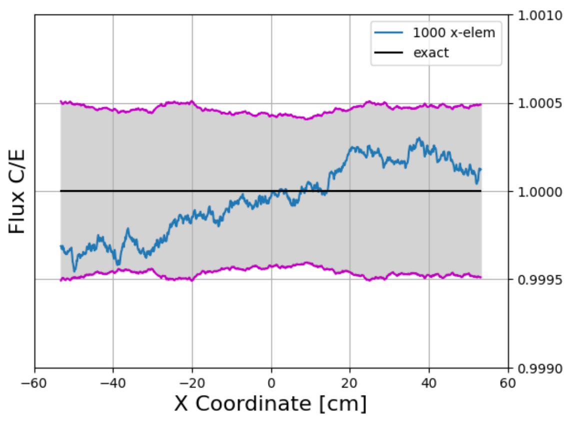
- C/E in blue with 2σ error bars (gray bounded by purple). 50 and 100 cells.

Individual Flux C/E with 2σ Error Bars for Fine Cases



- C/E in blue with 2σ error bars (gray bounded by purple). 250 and 500 cells

Individual Flux C/E with 2σ Error Bars for Fine Cases



- C/E in blue with 2σ error bars (gray bounded by purple). 1000 cells.

Why do the error bars appear on the same order despite cell refinement?

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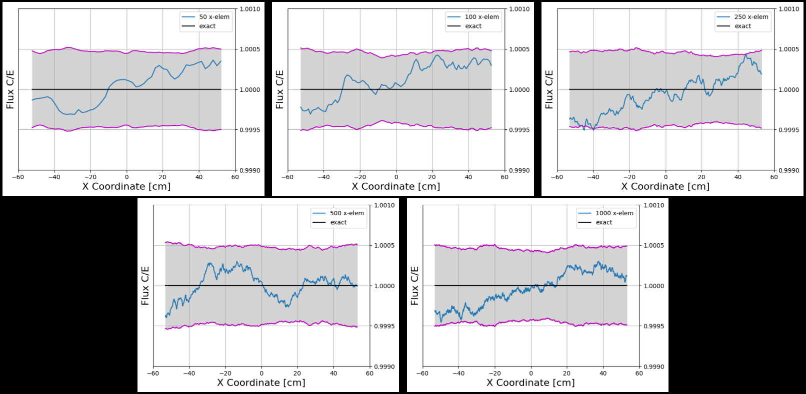

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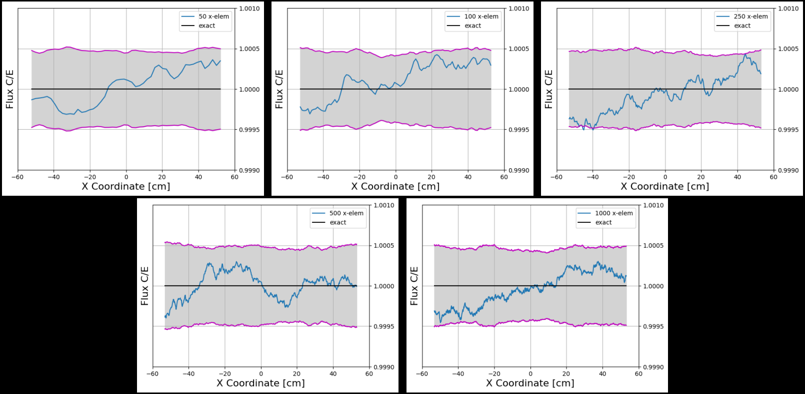

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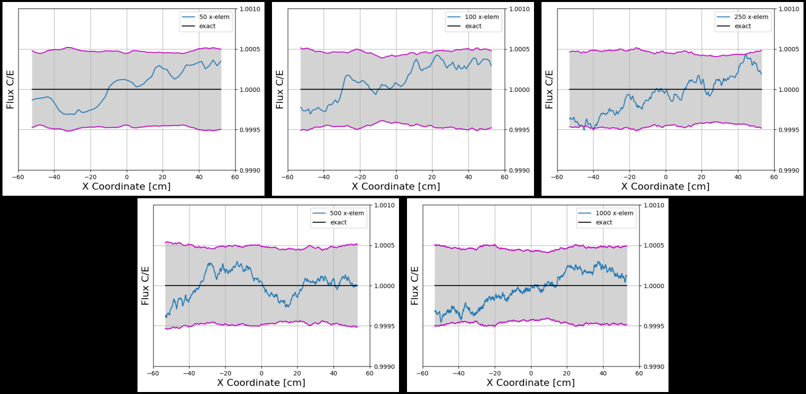

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- Nearly all points fall within 2σ (95% Confidence Interval), meaning that Cardinal is computing the correct flux within statistical uncertainty.

Acknowledgements

- Benchmark authors: David P. Greisheimer and Gabriel Kooreman
- Co-authors: April J. Novak, Patrick Shriwise, Paul P.H. Wilson
- OpenMC, Cardinal, and MOOSE teams!

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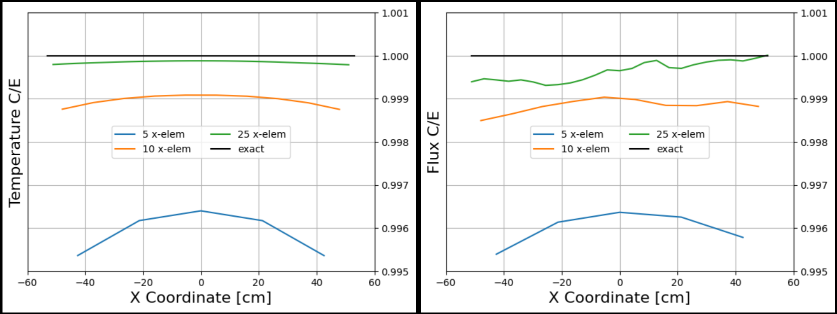
Coarse C/E results



- C/E for coarse cases ($N = 5, 10, 25$). The coarse cases' errors are a few orders of magnitude larger than the fine cases. A significant improvement in agreement can be seen between each coarse case.

Ensuring Validity of the Fundamental Ansatz

- Taking the heat conduction ODE

$$\frac{d}{dx}\left[\kappa(T(x))\frac{dT(x)}{dx}\right] + q\Sigma_t(x)\phi(x) = 0 \tag{8}$$

- and using the thermal conductivity and cross section temperature dependence

$$\frac{d}{dx}\left[\kappa_0 T(x)\frac{dT(x)}{dx}\right] + q\Sigma_{t,0}\frac{T_0}{T(x)}\phi(x) = 0 \tag{9}$$

- Taking the neutron transport ODE

$$\frac{d}{dx}\left[\frac{1}{\Sigma_t(x)}\frac{d\phi(x)}{dx}\right] + \Sigma_t(x)\left(\lambda - 1\right)\phi(x) = 0 \tag{10}$$

- and inserting cross section temperature dependence gives

$$\frac{d}{dx}\left[\frac{T(x)}{\Sigma_{t,0}T_0}\frac{d\phi(x)}{dx}\right] + \Sigma_{t,0}\frac{T_0}{T(x)}\left(\lambda - 1\right)\phi(x) = 0 \tag{11}$$

Ensuring Validity of the Fundamental Ansatz

- these two equations are very close, and after some re-arranging, they look even closer

$$\frac{d}{dx}\left[T(x)\frac{dT(x)}{dx}\right]+\frac{q\Sigma_{t,0}}{\kappa_0}\frac{T_0}{T(x)}\phi(x)=0 \qquad \text{AND}$$

$$\frac{d}{dx}\left[\frac{T(x)}{\Sigma_{t,0}T_0}\frac{d\phi(x)}{dx}\right]+\Sigma_{t,0}\frac{T_0}{T(x)}\left(\lambda-1\right)\phi(x)=0 \quad (12)$$

- Applying the ansatz $T(x)=f\phi(x)$ gives

$$\frac{d}{dx}\left[f^2\phi(x)\frac{d\phi(x)}{dx}\right]+\frac{q\Sigma_{t,0}}{\kappa_0}\frac{T_0}{f}=0 \qquad \text{AND}$$

$$\frac{d}{dx}\left[\frac{f\phi(x)}{\Sigma_{t,0}T_0}\frac{d\phi(x)}{dx}\right]+\Sigma_{t,0}\frac{T_0}{f}\left(\lambda-1\right)=0 \quad (13)$$

$$\frac{d}{dx}\left[\phi(x)\frac{d\phi(x)}{dx}\right]+\frac{q\Sigma_{t,0}}{\kappa_0}\frac{T_0}{f^3}=0 \qquad \text{AND}$$

$$\frac{d}{dx}\left[\phi(x)\frac{d\phi(x)}{dx}\right]+\left(\Sigma_{t,0}\frac{T_0}{f}\right)^2\left(\lambda-1\right)=0 \quad (14)$$

Ensuring Validity of the Fundamental Ansatz

- In order to make the ansatz hold, this implies that

$$\frac{q\Sigma_{t,0}}{\kappa_0}\frac{T_0}{f^3} = \left(\Sigma_{t,0}\frac{T_0}{f}\right)^2(\lambda - 1) \qquad \text{OR} \qquad \Sigma_{t,0} = \frac{q}{(\lambda - 1)\kappa_0 T_0 f} \tag{15}$$

  which is a condition for the total cross section based on system parameters.
- A similar process of matching coefficients must be applied to the boundary conditions to gain a condition for the heat transfer coefficient. Though, at this point, realize that

$$\frac{d}{dx}\left[\phi(x)\frac{d\phi(x)}{dx}\right] + \left(\Sigma_{t,0}\frac{T_0}{f}\right)^2(\lambda - 1) = 0 \tag{16}$$

- is a separable ODE that can be solved for an analytical solution. The result for the heat transfer coefficient is given by

$$h\left(\sqrt{\frac{L(\lambda - 1)}{\kappa_0 P}} - \frac{2T_0}{P}\right) = 1 \tag{17}$$

Deriving the Benchmark ODE for $\phi(x)$

- Going from the steady-state, mono-energetic, 1-D neutron transport equation to the ODE that describes neutron transport for this benchmark:

$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\Sigma_t(x)\psi(x,\mu)=\int_{-1}^{1}\frac{1}{2}\big[\Sigma_s(x)+\frac{\nu\Sigma_f(x)}{k_{eff}}\big]\psi(x,\mu')d\mu' \quad (18)$$

For now, lump the fission term into the scattering cross section to get

$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\Sigma_t(x)\psi(x,\mu)=\int_{-1}^{1}\frac{1}{2}\Sigma_s(x)\psi(x,\mu')d\mu' \quad (19)$$

Define the scalar flux and the magnitude of current

$$\phi(x)=\int_{-1}^{1}\psi(x,\mu)d\mu \qquad \text{AND} \qquad J(x)=\int_{-1}^{1}\mu\psi(x,\mu)d\mu. \quad (20)$$

Considering S_2 transport means restricting the angular cosine to $\mu=\pm1$:

$$\psi(x,\mu)=\psi(x,-1)\delta(\mu+1)+\psi(x,1)\delta(\mu-1) \quad (21)$$

(sometimes denoted $\psi^+\equiv\psi(x,1)\delta(\mu-1)$ and $\psi^-\equiv\psi(x,-1)\delta(\mu+1)$)

Deriving the Benchmark ODE for $\phi(x)$

Now carrying out the integral definitions with S_2 quantities gives

$$\phi(x) = \int_{-1}^{1} \left[\psi(x,-1)\delta(\mu+1) + \psi(x,1)\delta(\mu-1)\right] d\mu = \psi(x,-1) + \psi(x,1) \tag{22}$$

and

$$J(x) = \int_{-1}^{1} \mu \left[\psi(x,-1)\delta(\mu+1) + \psi(x,1)\delta(\mu-1)\right] d\mu = \psi(x,1) - \psi(x,-1) \tag{23}$$

Evaluating (19) at $\mu = \pm 1$ gives

$$-\frac{\partial\psi(x,-1)}{\partial x} + \Sigma_t(x)\psi(x,-1) = \frac{1}{2}\Sigma_s(x)\phi(x); \tag{24}$$

$$\frac{\partial\psi(x,1)}{\partial x} + \Sigma_t(x)\psi(x,1) = \frac{1}{2}\Sigma_s(x)\phi(x). \tag{25}$$

Adding (24) and (25) gives

$$-\frac{\partial\psi(x,-1)}{\partial x} + \frac{\partial\psi(x,1)}{\partial x} + \Sigma_t(x)\left(\psi(x,-1) + \psi(x,1)\right) = \Sigma_s(x)\phi(x) \tag{26}$$

Deriving the Benchmark ODE for $\phi(x)$

The results for $\phi(x)$ and $J(x)$ can simplify (26)

$$\frac{dJ(x)}{dx} + \Sigma_t(x)\phi(x) = \Sigma_s(x)\phi(x) \tag{27}$$

Subtracting (24) and (25) gives

$$-\frac{\partial\psi(x,-1)}{dx} - \frac{\partial\psi(x,1)}{dx} + \Sigma_t(x)\left(\psi(x,-1) - \Sigma_t(x)\psi(x,1)\right) = 0 \tag{28}$$

Which can be transformed with similar tricks to

$$\frac{d\phi(x)}{dx} + \Sigma_t(x)J(x) = 0 \qquad \text{OR} \qquad J(x) = -\frac{1}{\Sigma_t(x)}\frac{d\phi(x)}{dx} \tag{29}$$

Since $\frac{dJ(x)}{dx}$ appears in (27), we can take the derivative of both sides of (29) and substitute it in

$$\frac{dJ(x)}{dx} = -\frac{d}{dx}\left[\frac{1}{\Sigma_t(x)}\frac{d\phi(x)}{dx}\right] \tag{30}$$

Now the equation that only depends on $\phi(x)$ is given by

$$-\frac{d}{dx}\left[\frac{1}{\Sigma_t(x)}\frac{d\phi(x)}{dx}\right] + \Sigma_t(x)\phi(x) = \Sigma_s(x)\phi(x) \tag{31}$$

Deriving the Benchmark ODE for $\phi(x)$

At this point, we "un-lump" the scattering cross section to write out the fission term

$$-\frac{d}{dx}\left[\frac{1}{\Sigma_t(x)}\frac{d\phi(x)}{dx}\right]+\Sigma_t(x)\phi(x)=\left[\Sigma_s(x)+\frac{\nu\Sigma_f}{k_{eff}}\right]\phi(x) \tag{32}$$

$$-\frac{d}{dx}\left[\frac{1}{\Sigma_t(x)}\frac{d\phi(x)}{dx}\right]+\Sigma_t(x)\left[1-\frac{\Sigma_s(x)+\frac{\nu\Sigma_f}{k_{eff}}}{\Sigma_t}\right]\phi(x)=0 \tag{33}$$

$$\frac{d}{dx}\left[\frac{1}{\Sigma_t(x)}\frac{d\phi(x)}{dx}\right]+\Sigma_t(x)\left[\frac{\Sigma_s(x)+\frac{\nu\Sigma_f}{k_{eff}}}{\Sigma_t}-1\right]\phi(x)=0 \tag{34}$$

Now define

$$\lambda\equiv\frac{\Sigma_s(x)+\frac{\nu\Sigma_f}{k_{eff}}}{\Sigma_t} \tag{35}$$

giving the final result:

$$\frac{d}{dx}\left[\frac{1}{\Sigma_t(x)}\frac{d\phi(x)}{dx}\right]+\Sigma_t(x)\left(\lambda-1\right)\phi(x)=0 \tag{36}$$

Deriving the Benchmark ODE for $\phi(x)$

The next task is to apply boundary conditions so that $\phi(x)$ can be specified. In discrete ordinates with $\mu = \pm 1$ (S_2), we use the vacuum boundary condition. The angular flux for positive angular cosines is zero at the left boundary and is zero for negative angular cosines at the right boundary. Using the previous results for $\phi(x)$ and $J(x)$ at the boundaries gives

$$\phi(x = \frac{L}{2}) = \psi(x = \frac{L}{2}, \mu = -1) + \psi(x = \frac{L}{2}, \mu = 1) \qquad \text{AND}$$
$$\phi(x = -\frac{L}{2}) = \psi(x = -\frac{L}{2}, \mu = -1) + \psi(x = -\frac{L}{2}, \mu = 1) \quad (37)$$

and

$$J(x = \frac{L}{2}) = -\psi(x = \frac{L}{2}, -1) + \psi(x = \frac{L}{2}, 1) \qquad \text{AND}$$
$$J(x = -\frac{L}{2}) = -\psi(x = -\frac{L}{2}, -1) + \psi(x = -\frac{L}{2}, 1) \quad (38)$$

Deriving the Benchmark ODE for $\phi(x)$

Now, terms can be crossed out due to vacuum boundaries. This gives that

$$\phi(x = \frac{L}{2}) = \psi(x = \frac{L}{2}, \mu = 1) \qquad \text{AND} \qquad \phi(x = -\frac{L}{2}) = \psi(x = -\frac{L}{2}, \mu = -1) \tag{39}$$

$$J(x = \frac{L}{2}) = \psi(x = \frac{L}{2}, \mu = 1) \qquad \text{AND} \qquad J(x = -\frac{L}{2}) = -\psi(x = -\frac{L}{2}, \mu = -1) \tag{40}$$

Using this with (29) gives the desired boundary conditions

$$J(x = \pm\frac{L}{2}) = \pm\phi(x = \pm\frac{L}{2}) \tag{41}$$

And the boundary conditions of interest are now

$$\left.\frac{d\phi}{dx}\right|_{x=\pm\frac{L}{2}} \pm \Sigma_t(x = \pm\frac{L}{2})\phi(x = \pm\frac{L}{2}) = 0 \tag{42}$$