Basel Problem

Shiwei Chen

December 2023

1 Introduction

The **Basel problem** asks for the precise summation of the reciprocals of the squares of the natural numbers, i.e, the closed form, numerical value of the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... \tag{1}$$

1.1 History

The Basel problem was initially proposed by Mathematician Pietro Mengoli in 1644, who proved the divergence of harmonic series, and the validity of Wallis' product for π, which we covered in Long Question 4. He developed the results in limits and sums which laid the groundwork for Newton and Leibniz.

Many leading mathematicians attempted to solve Mengoli's challenge. John Wallis challenged the problem in 1655, but was only able to find the sum to three decimal places. Next up was Jakob Bernouli, who proved that the series converged to a finite limit less than 2 in 1721. Christian Goldbach also attempted this problem, concluding that the number is between 41/35 and 5/3.

The problem itself is named after the town of Basel, the birthplace of Leonhard Euler, who grew to prominence after solving it in 1734. Euler found the exact sum to be $\frac{\pi^2}{6}$ and announced his discovery a year later in the Saint Petersburg Academy of Sciences.

Euler would generalize the Basel problem considerably and his ideas were used by Bernhard Riemann to define the *Riemann zeta function* $\zeta(s)$ and prove its basic properties.

Many short proofs of the Basel Problem have emerged since it was first solved by Euler, but most rely on additional knowledge beyond the Calculus level. One common approach is based on the pointwise convergence of Fourier Series. Another approach is to use the Euler-MacLaurin summation formula. Euler's original proof uses the product formula for $\sin x$, which was justified by the Weierstrass factorization theorem. Others rely on double integrals or complex analysis, such as in Cauchy's proof which uses identities derived from DeMoivre's formula.

Today, we will explore Daniel Daners' proof of the Basel Problem, which was designed to be enjoyed and appreciated by anyone with knowledge of first-year Calculus. It first appeared in Mathematics Magazine in December, 2012.

2 Daners' Solution

Let $A_n = \int_0^{\frac{\pi}{2}} \cos^{2n}(x)dx$ and $B_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n}(x)dx$

2.1 Lemma 1: $2nA_n = (2n-1)A_{n-1}$

From the definition of A_n

$$A_n = \int_0^{\frac{\pi}{2}} \cos^{2n}(x)dx = \int_0^{\frac{\pi}{2}} \cos(x)\cos^{2n-1}(x) \tag{2}$$

Take $dv = \cos(x)dx$ and $u = \cos^{2n-1}(x)$. This implies

$$du = -(2n-1)\cos^{2n-2}(x)\sin(x)dx, \tag{3}$$

and

$$v = \sin(x) \tag{4}$$

Applying (3) and (4) to (2), we get the equation

$$[\sin(x)\cos^{2n-1}(x)]_0^{\frac{\pi}{2}} + (2n-1)\int_0^{\frac{\pi}{2}} \sin^2(x)\cos^{2n-2}(x)dx \tag{5}$$

$$= (2n-1)(\int_0^{\frac{\pi}{2}} \cos^{2n-2}(x)dx - \int_0^{\frac{\pi}{2}} \cos^{2n}(x)dx) = (2n-1)(A_{n-1} - A_n) \tag{6}$$

So we have

$$\begin{aligned} &A_n = (2n-1)(A_{n-1} - A_n) \\ &\Rightarrow A_n = (2n-1)A_{n-1} - 2nA_n + A_n \\ &\Rightarrow 2nA_n = (2n-1)A_n\square \end{aligned} \tag{7}$$

2.2 Lemma 2: $A_n = (2n-1)nB_{n-1} - 2n^2 B_n$

From the definition of A_n

$$A_n = \int_0^{\frac{\pi}{2}} \cos^{2n}(x)dx \tag{8}$$

Let $u = cos^{2n}(x)$ and $dv = dx$

This implies $du = -2n\cos^{2n-1}(x)\sin(x)dx$ and $v = x$
Applying our substitutions, we now have

$$2n\int_0^{\frac{\pi}{2}} x\sin(x)\cos^{2n-1}(x)dx \tag{9}$$

Take $dv = xdx$ and $u = \sin(x)\cos^{2n-1}(x)$, which implies

$$A_n = 2n\int_0^{\frac{\pi}{2}} vdu \tag{10}$$

Note $\sin^2(x) = 1 - cos^2(x)$, so we now have

$$\begin{aligned} &du = (\cos^{2n}(x) - (2n-1)\sin^2(x)cos^{2n-2}(x))dx \\ &= (\cos^{2n}(x) - (2n-1))(\cos^{2n-2}(x) - cos^{2n}))dx \\ &= (2n\cos^{2n}(x) - (2n-1)\cos^{2n-2}(x))dx \end{aligned} \tag{11}$$

Since $dv = xdx$ this implies $v = \frac{1}{2}x^2$
By (10), we get

$$\begin{aligned} &A_n = -n\int_0^{\frac{\pi}{2}} 2nx^2\cos^{2n}(x) - (2n-1)x^2\cos^{2n-2}(x))dx \\ &= -2n^2B_n + (2n-1)nB_{n-1} \end{aligned} \tag{12}$$

Therefore, $A_n = (2n-1)nB_{n-1} - 2n^2B_n$ $\square$

2.3 Lemma 3: $\lim_{n\to\infty} \frac{B_n}{A_n} = 0$

We know that $\frac{\pi}{2}x <= \sin(x)$, when $x \in [0, \frac{\pi}{2}]$,
and also that $\sin^2(x) = (1 - \cos^2(x))$

Now,

$$\begin{aligned} &B_n = \int_0^{\frac{\pi}{2}} x^2\cos^{2n}(x)dx \leq \frac{\pi^2}{4}\int_0^{\frac{\pi}{2}} \sin^2(x)\cos^{2n}(x)dx \\ &= \frac{\pi^2}{4}\int_0^{\frac{\pi}{2}} (1-\cos^2(x))\cos^{2n}(x)dx \\ &= \frac{\pi^2}{4}\int_0^{\frac{\pi}{2}} \cos^{2n}(x) - \cos^{2n+2} dx \\ &= \frac{\pi^2}{4}(A_n - A_{n+1}) \end{aligned} \tag{13}$$

By Lemma 1, $2nA_n = (2n-1)A_{n-1}$ Using $n+1$ and rearranging Lemma 1,

$$\begin{aligned}
&2(n+1)A_{n+1} = (2n)A_n \\
&\Rightarrow 2nA_{n+1} + 2A_{n+1} = 2nA_n \\
&\Rightarrow 2nA_{n+1} - 2nA_n = -2A_{n+1} \\
&\Rightarrow A_{n+1} - A_n = -\frac{A_{n+1}}{n} \\
&\Rightarrow A_n - A_{n+1} = \frac{A_{n+1}}{n}
\end{aligned} \tag{14}$$

Applying (14) to (13), we get

$$\begin{aligned}
&B_n \leq \frac{\pi^2}{4}\frac{A_n}{n} \\
&\Rightarrow \frac{B_n}{A_n} \leq \frac{\pi^2}{4}\frac{A_n}{n}\frac{1}{A_n} = \frac{\pi^2}{4n} \\
&\Rightarrow 0 \leq \lim_{n\to\infty}\frac{B_n}{A_n} \leq \lim_{n\to\infty}\frac{\pi^2}{4n} = 0
\end{aligned} \tag{15}$$

$\therefore$ by the Squeeze Theorem, $\lim_{n\to\infty}\frac{B_n}{A_n} = 0$ $\square$

2.4 Assembling the Pieces

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n^2} \tag{16}$$

By Lemma 2,

$$A_n = (2n-1)nB_{n-1} - 2n^2B_n \Rightarrow \frac{1}{n^2} = (\frac{(2n-1)B_{n-1}}{nA_n} - \frac{2B_n}{A_n}) \tag{17}$$

Applying (16) to (17), we get

$$\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n^2} = (\frac{(2n-1)B_{n-1}}{nA_n} - \frac{2B_n}{A_n}) \tag{18}$$

By Lemma 1, $2nA_n = (2n-1)A_{n-1} \Rightarrow A_n = \frac{(2n-1)A_{n-1}}{2n}$
Apply this definition of A_n to (18) to get

$$\begin{aligned}
&\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n^2} = (\frac{2B_{n-1}}{A_{n-1}} - \frac{2B_n}{A_n}) \\
&= 2\lim_{N\to\infty}\sum_{n=1}^{N}(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n})
\end{aligned} \tag{19}$$

Note that the equation we obtain in (19) is a telescoping series, so we can condense the equation down to down to

$$2((\frac{B_0}{A_0}) - \lim_{N\to\infty} \frac{B_N}{A_N}) = 2(\frac{B_0}{A_0} - 0) \tag{20}$$

By Lemma 3.
So now we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= 2(\frac{B_0}{A_0}) = 2\frac{\int_0^{\frac{\pi}{2}} x^2 dx}{\int_0^{\frac{\pi}{2}} dx} \\ &= 2\frac{\frac{\pi^3}{24}}{\frac{\pi}{2}} \\ &= \frac{4\pi^2}{24} = \frac{\pi^2}{6}\square \end{aligned} \tag{21}$$

2.5 Consequences of Euler's Proof

The *Riemann zeta function* can be written as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{22}$$

From which we can see that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{2} \tag{23}$$

The function is one of the most important functions in mathematics due to its connection with the Riemann Hypothesis, a conjecture that the Riemann zeta function has its zeroes only at the negative even integers and complex numbers with real part 1/2

Many consider this hypothesis to be the most important unsolved problem in mathematics, and it is a part of the seven problems in the Millennium Prize Problems, whose proof will reward its author with 1 million dollars.

The Riemann Hypothesis builds upon the prime number theorem, and if proved, it would immediately solve many other open problems in number theory and refine our understanding of the behavior of prime numbers.