Applied Category Theory Problems

April Shen & Alex Root

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Chapter 1

Generative Effects

Exercise 1.1

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be

- *order-preserving* if $x \leq y$ implies $f(x) \leq f(y)$, for all $x, y \in \mathbb{R}$
- *metric-preserving* if $\mid x - y \mid = \mid f(x) - f(y) \mid$
- *addition-preserving* if $f(x + y) = f(x) + f(y)$

For each of the three properties defined above—call it *foo*—find an f that is *foo*-preserving and an example of an f that is not *foo*-preserving.

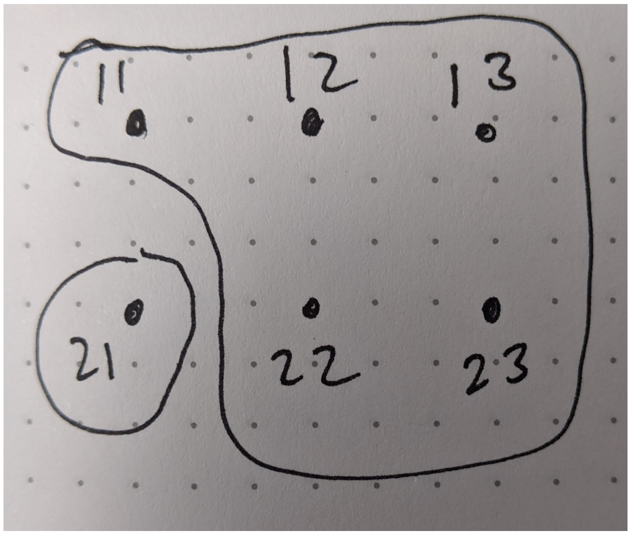
Solution

$f(x) = x$ is order-, metric-, and addition-preserving. $f(x) = x^2$ is none of these.

Exercise 1.4

See book.

Solution



Exercise 1.6, 1.7, 1.10

Discussed in person

Exercise 1.11

Let $A = \{h, 1\}$ and $B = \{1, 2, 3\}$.

Solution

1. The subsets of B are $\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$.

2. For example, $\{1, 2\} \cup \{2\} = \{1, 2\}$.

3. $A \times B = \{(h, 1), (h, 2), (h, 3), (1, 1), (1, 2), (1, 3)\}$

4. $A \sqcup B = \{h_A, 1_A, 1_B, 2_B, 3_B\}$

5. $A \cup B = \{h, 1, 2, 3, 4\}$

Exercise 1.16

Suppose that A is a set and $\{A_p\}_{p \in P}$ and $\{A'_{p'}\}_{p' \in P'}$ are two partitions of A such that for each $p \in P$ there exists a $p' \in P'$ with $A_p = A'_{p'}$.

1. Show that for each $p \in P$ there is at most one $p' \in P'$ such that $A_p = A'_{p'}$.

2. Show that for each $p' \in P'$ there is a $p \in P$ such that $A_p = A'_{p'}$.

Solution

1. If there are distinct $p'_1, p'_2 \in P'$ such that $A_p = A'_{p'_1}$ and $A_p = A'_{p'_2}$, then by transitivity $A'_{p'_1} = A'_{p'_2}$ which means that $p'_1 = p'_2$ as P' is a partition.

2. Let $a \in A'_{p'_1}$. Then we must have that $a \in A_p$ for some p. We will show that this A_p is the desired one. There must exist p'_2 such that $A_p = A'_{p'_2}$. So $a \in A'_{p'_2}$ and $a \in A'_{p'_1}$. So $A'_{p'_1} = A'_{p'_2} = A_p$ as we wanted.

Exercise 1.17

See book.

Solution

$(11, 11), (12, 12), (13, 13), (21, 21), (22, 22), (23, 23), (11, 12), (12, 11), (22, 23), (23, 22)$

Exercise 1.20

Suppose that $\sim$ is an equivalence relation on a set A, and let P be the set of $(\sim)$-closed and $(\sim)$-connected subsets $\{A_p\}_{p \in P}$.

1. Show that each part A_p is nonempty.

2. Show that if $p \neq q$ i.e. $A_p \neq A_q$, then $A_p \cap A_q = \varnothing$.

3. Show that $A = \bigcup_{p \in P} A_p$.

Solution

1. Since each A_p is $(\sim)$-connected, by definition A_p must be nonempty.

2. Suppose $a \in A_p \cap A_q$. Then for any $b \in A_p$, we know $b \sim a$, hence since A_q is $(\sim)$-closed we know $b \in A_q$. Similarly for any $b \in A_q$. Hence $A_p = A_q$.

3. Clearly $\bigcup A_p \subseteq A$. Suppose $a \in A$. Then let $X = \{b \mid a \sim b, b \in A\}$. X is closed as the equivalence relation is reflexive and connected as it is transitive, so X a set in $\{A_p\}$ and $A \subseteq \bigcup A_p$.

Exercise 1.24

Discussed in person

Exercise 1.25

Suppose that A is a set and $f : A \to \varnothing$ is a function to the empty set. Show that A is empty.

Solution

Suppose there is some $a \in A$. Then we must have $f(a) \in \varnothing$ which is clearly not possible.

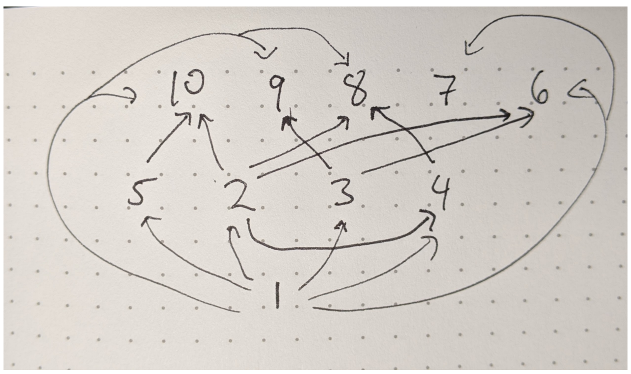
Exercise 1.27, 1.38, 1.40, 1.41, 1.42, 1.44

Discussed in person

Exercise 1.46

Write down the numbers $1, 2, \ldots, 10$ and draw an arrow $a \to b$ if a divides perfectly into b. Is it a total order?

Solution



This isn't a total order, as for example we neither have $2 \mid 7$ or $7 \mid 2$.

Exercise 1.48

Is the usual $\leq$ ordering on the set $\mathbb{R}$ of real numbers a total order?

Solution

Yes: for any $x, y \in \mathbb{R}$, we have $x \leq y$ or $y \leq x$.

Exercise 1.51

Discussed in person

Exercise 1.53

For any set S there is a coarsest partition, having just one part. What surjective function does it correspond to? There is also a finest partition, where everything is in its own partition. What surjective function does it correspond to?

Solution

Let $f : S \to \{\bullet\}$ be the unique function that sends every element of S to $\bullet$. This is a surjection corresponding to the coarsest partition, i.e. where every element is in the same set.

Let $f : S \to S$ be the identity function. This is a surjection corresponding to the finest partition, i.e. where every element is in its own set.

Exercise 1.55

Prove that the preorder of upper sets on a discrete preorder on a set X is simply the power set $P(X)$.

Solution

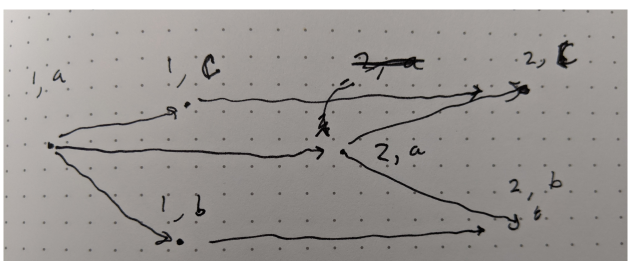
Clearly the set of upper sets $U(X)$ is a subset of the power set $P(X)$.

Let $Y \subseteq X$. We know $\varnothing$ is an upper set, so let $y \in Y$. Then since the preorder is discrete, the only element in X greater than y is y itself, which is in Y. This holds for any $y \in Y$ so Y is an upper set. Note that the ordering on both $U(X)$ and $P(X)$ is the same, i.e. $\subseteq$.

Exercise 1.57

See book.

Solution

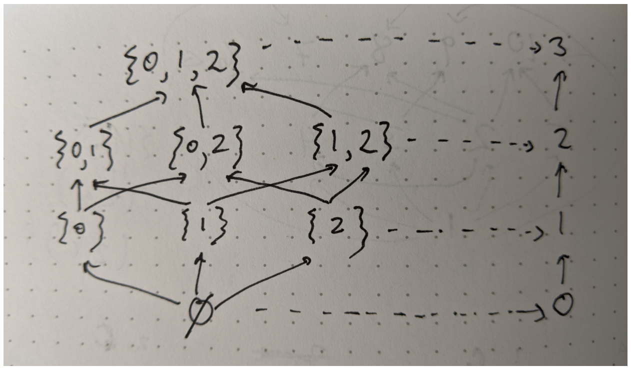


Exercise 1.63

Let $X = \{0, 1, 2\}$.

1. Draw the Hasse diagram for $P(X)$.
2. Draw the Hasse diagram for the preorder $0 \leq 1 \leq 2 \leq 3$.
3. Draw the cardinality map $|\cdot|$ as dashed lines between them

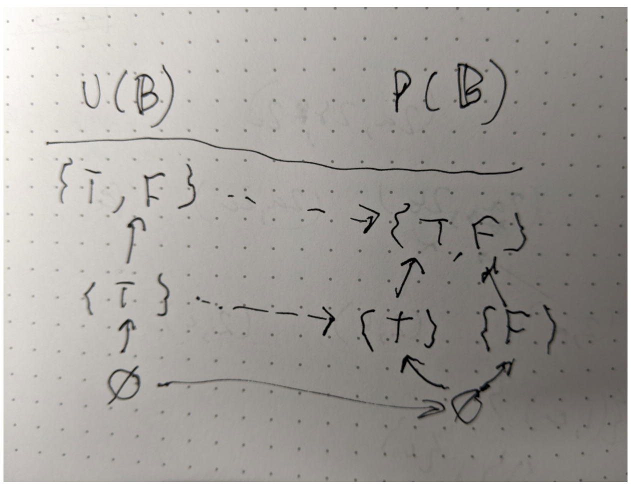
Solution



Exercise 1.65

Draw the monotone map between $U(\mathbb{B})$ and $P(\mathbb{B})$ as described in the text.

Solution



Exercise 1.66

Let $(P, \leq)$ be a preorder.

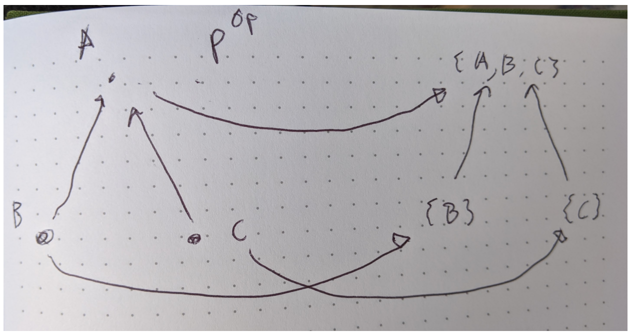
1. Show that the set $\uparrow p = \{p' \in P \ \mid \ p \leq p;\ \}$ is an upper set for any $p \in P$.
2. Show that this defines a monotone map $\uparrow : P^{op} \to U(P)$.
3. Show that $p \leq p'$ iff $\uparrow(p') \subseteq \uparrow(p)$.
4. Draw a picture of the map $\uparrow$ in the case where P is the preorder $(b \geq a \leq c)$.

Solution

1. Suppose $q \in \uparrow p$, then any $q' \geq q$ is transitively greater than p and hence $q' \in \uparrow p$.

2. Suppose $p \geq q$ (i.e. p is less than q in P^{op}), we want to show that $\uparrow p \subseteq \uparrow q$. So let $p' \in \uparrow p$. We know $q \leq p \leq p'$ and hence $p' \in \uparrow q$.

3. We showed the first direction in part 2, so assume $\uparrow (p') \subseteq \uparrow (p)$. This means $p \in \uparrow (p')$ and hence $p \leq p'$.

4. 

Exercise 1.67

Show that when $(P, \leq_P)$ is a discrete preorder, then every function $f : P \to Q$ is monotone regardless of the order $\leq_Q$.

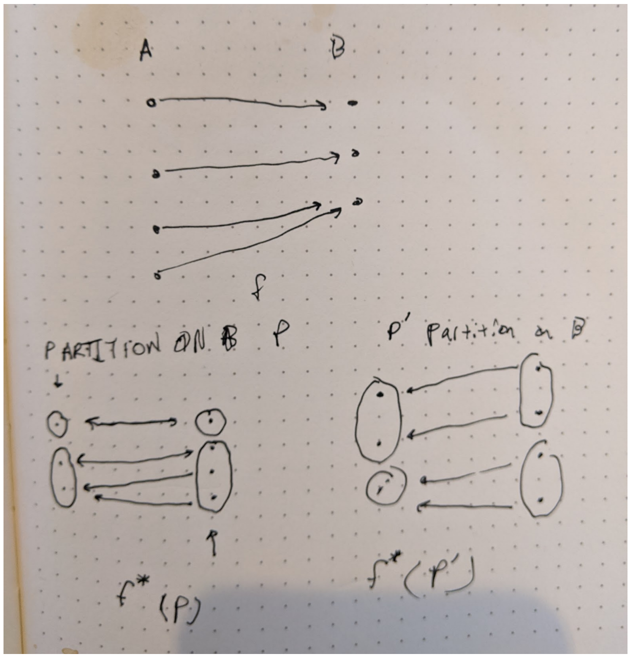
Solution

We need to show that for any $x, y \in P$ where $x \leq_P y$, we have $f(x) \leq_Q f(y)$. But the only x and y satisfying this are $x \leq_P x$, for which we have $f(x) \leq_Q f(x)$ regardless of $\leq_Q$ by the definition of a preorder.

Exercise 1.69

Choose two sets X and Y with at least three elements each and choose a surjective, non-identity function $f : X \to Y$. Write down two different partitions P and Q of Y, and find $f^*(P)$ and $f^*(Q)$.

Solution



Exercise 1.71

Prove Proposition 1.70:

1. For any preorder $(P, \leq_P)$, the identity function is monotone.
2. If $(Q, \leq_Q)$ and $(R, \leq_R)$ are preorders and $f : P \to Q$ and $g : Q \to R$ are monotone, then $(f \mathbin{⨟} g) : P \to R$ is also monotone.

Solution

1. If $a \leq_P b$ then clearly $a = f(a) \leq_P f(b) = b$ if f is the identity function.
2. Suppose $a \leq_P b$, then $f(a) \leq_Q f(b)$ as f is monotone, and hence $g(f(a)) \leq_R g(f(b))$ as g is also monotone.

Exercise 1.73

Show that a skeletal dagger preorder is just a discrete preorder, and hence can be identified with a set.

Solution

Let $(P, \leq)$ be a skeletal dagger preorder. We need to show that for any $x \in P$, the only thing comparable to x is x itself. So suppose $x \leq y$, then as P is a dagger preorder we know that $y \leq x$. Hence as P is skeletal, we have that $x = y$. This implies that P is a discrete preorder.

Exercise 1.77

Show that the map Φ from Section 1.1.1 ('Is $\bullet$ connected to $\star$?') is the monotone map $Prt(\{\star, \bullet, \circ\}) \to \mathbb{B}$.

Solution

Let P and P' be partitions where $P \leq P'$. If $\Phi(P) = \texttt{false}$ then clearly $\Phi(P) \leq \Phi(P')$, so assume $\Phi(P) = \texttt{true}$. This means for some set X in the partition P, we know that both $\bullet, \star \in X$. As $P \leq P'$ this means there is some Y in the partition P' with $X \subseteq Y$, which implies that $\bullet, \star \in Y$. Hence $\Phi(P') = \texttt{true}$ and $\Phi(P) \leq \Phi(P')$.

Exercise 1.79

Let P and Q be preorders and $f : P \to Q$ a monotone map. Show that the pullback $f^* : U(Q) \to U(P)$ can be defined by taking $u : Q \to \mathbb{B}$ to $(f \mathbin{⨟} u) : P \to \mathbb{B}$.

Solution

Call ϕ_Q the function that takes upper sets in Q to monotone maps as defined in Proposition 1.78, and similarly ϕ_P. Let $U \in U(Q)$. We want to show $\phi_P(f^{-1}(U)) = f \mathbin{⨟} (\phi_Q(U))$.

Let $x \in P$. If $x \in f^{-1}(U)$, then we know $\phi_P(f^{-1}(U))(x) = \texttt{true}$ by definition. But we also know $f(x) \in U$ and hence $\phi_Q(U)(f(x)) = \texttt{true}$. Conversely if $x \notin f^{-1}(U)$, we will have both $\phi_P(f^{-1}(U))(x) = \texttt{false}$, as well as $f(x) \notin U$ and $\phi_Q(U)(f(x)) = \texttt{false}$. This shows that these maps are equal.

Exercise 1.80

Why is 0 a greatest lower bound for $\left\{\frac{1}{n+1} \mid n \in \mathbb{N}\right\} \subseteq \mathbb{R}$?

Solution

Assume that $\varepsilon > 0$ is a lower bound. Let $n = \lceil 1/\varepsilon \rceil$. Then

$$\tfrac{1}{n+1} \leq \tfrac{1}{1/\varepsilon+1} \leq \tfrac{1}{1/\varepsilon} = \varepsilon.$$

Hence no such ε is a lower bound.

Exercise 1.85

Let $(P, \leq)$ be a preorder and $p \in P$, consider the set $A = \{p\}$.

1. Show that $\bigwedge A \cong p$.
2. Show that if P is a partial order, then $\bigwedge A = p$.
3. Are the analogous facts true when $\bigwedge$ is replaced by $\bigvee$?

Solution

1. Clearly $p \leq p$, so by definition $\bigwedge A \leq p$ (as a lower bound) and $\bigwedge A \geq p$ (as a greatest lower bound).
2. If the previous is true in a partial order, then we have $\bigwedge A = p$.
3. The analogous facts are true with $\bigvee$.

Exercise 1.90

In the $n \mid m$ ordering on $\mathbb{N}$, what are the meet and the join?

Solution

The meet is the greatest common divisor and the join is the least common multiple.

Exercise 1.94

Prove that for any monotone map $f : P \to Q$, if $a, b \in P$ have a join and $f(a), f(b) \in Q$ have a join, then $f(a) \vee f(b) \leq f(a \vee b)$.

Solution

We know $a, b \leq a \vee b$, so since f is monotone we have $f(a), f(b) \leq f(a \vee b)$. Hence $f(a \vee b)$ is an upper bound for $\{f(a), f(b)\}$, so by definition of the join we must have $f(a) \vee f(b) \leq f(a \vee b)$.

Exercise 1.98

Find a right adjoint for the monotone map $(3 \times -) : \mathbb{Z} \to \mathbb{R}$, and show it is correct.

Solution

Let $g(y) = \lfloor y/3 \rfloor$. Then we have $3x \leq y \Leftrightarrow x \leq y/3 \Leftrightarrow x \leq \lfloor y/3 \rfloor$, hence g is a right adjoint for $3x$.

Exercise 1.99

See book.

Solution

1. In this case f is left adjoint to g.
2. In this case f is not left adjoint to g, as $g(1) = 2 \geq 2$ but $f(2) = 2 \not\leq 1$.

Exercise 1.101

1. Does $\lceil -/3 \rceil$ have a left adjoint $L : \mathbb{Z} \to \mathbb{R}$?
2. If not, why? If so, does its left adjoint have a left adjoint?

Solution

Let $g : \mathbb{R} \to \mathbb{Z}$ be defined by $g(x) = \lceil x/3 \rceil$. We will show by contradiction that g does not have a left adjoint.

For a left adjoint $f : \mathbb{Z} \to \mathbb{R}$, we must have $f(1) \leq 0 \Leftrightarrow 1 \leq g(0) = \lceil 0/3 \rceil = 0$. Clearly the second part does not hold, so we know $f(1) \not\leq 0$.

On the other hand, we know that $f(1) \leq \inf A$ where $A = \{x \mid g(x) \geq 1, x \in \mathbb{R}\}$. However $1/n \in A$ for $n \in \mathbb{Z}^+$, as $\lceil 1/n \rceil = 1$ for all such n. But $\inf\{1/n \mid n \in \mathbb{Z}^+\} = 0$, which implies that $f(1) \leq 0$. This is a contradiction, so g must not have a left adjoint.

Exercise 1.103

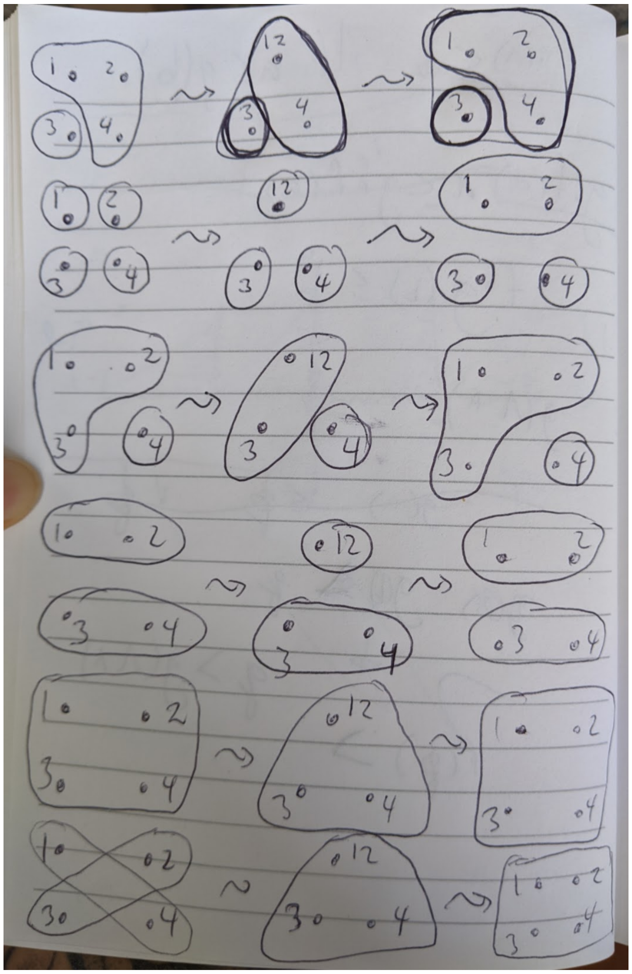
Choose 6 different partitions on the set S and for each, call it c, find $g_!(c)$ where S, T, and $g : S \to T$ are the same as they were in Example 1.102.

Exercise 1.105

Using the same S, T, and $g : S \to T$ as in Example 1.102, find the partition $g^*(c)$ for each of the 5 partitions c on the set T.

Solution

For both 1.103 and 1.105.



Exercise 1.106 (revised)

Prove that $g_!$ is left adjoint to g^*, as defined in the text.

Solution

Let S, T be sets, and let $g : S \to T$. Define $g_!, g^*$ as in the text.

We first show g^* is monotone. Let $A, B \in Prt(T)$ such that $A \leq B$. Then for each set $A_i \in A$, $A_i \subseteq B_j$ for some $B_j \in B$, and as a result $g^{-1}(A_i) \subseteq g^{-1}(B_j)$ for each $A_i \in A$. As the image of a partition under g^* is the collection of preimages of that partition via g, we have $g^*(A) \leq g^*(B)$.

Next we show $g_!$ is monotone. Let $A, B \in Prt(S)$ such that $A \leq B$. As before we know that for each set $A_i \in A$, $A_i \subseteq B_j$ for some $B_j \in B$. We consider $A, B \in Rel(S)$ i.e. subsets of $S \times S$. Note that $g_!(C)$ is the transitive closure of the relation $\{(g(x), g(y) \mid (x, y) \in C\}$. As $A \leq B$, $\{(g(x), g(y) \mid (x, y) \in A\} \subseteq \{(g(x), g(y) \mid (x, y) \in B\}$. Using the fact that the function taking a relation to its transitive closure is monotone on the set of relations ordered by inclusion, we can conclude that $g_!(A) \leq g_!(B)$.

For the next part of the proof, we use proposition 1.107, and derive our result by showing that for each $A \in Prt(S)$ and for each $B \in Prt(T)$ that both $A \leq g^* \circ g_!(A)$, and $g_! \circ g^*(B) \leq B$.

We start with showing $A \leq g^* \circ g_!(A)$. First we consider two additional functions, first $\bar{g} : Prt(S) \to Rel(T)$, where $g(r) = \{(g(x), g(y)) \mid (x, y) \in r\}$. Secondly an extension of g^* to all relations, $\bar{g^*} : Rel(T) \to Rel(S)$, so for a relation r, we have $\bar{g^*}(r) = \{(x, y) \mid (g(x), g(y)) \in r\}$ (g^* is the restriction of $\bar{g^*}$ to equivalence relations). Both $\bar{g}$ and $\bar{g^*}$ are monotone, which can be seen in proofs similar to our proofs for g^* and $g_!$. Additionally let the transitive closure of a set Q be denoted $\hat{Q}$. We now note two things, one $A = \bar{g^*} \circ \bar{g}(A)$, and two $g^* \circ g_!(A) = \bar{g^*}(\widehat{\bar{g}(A)})$. As $\bar{g^*}$ is monotone and $\bar{g}(A) \leq \widehat{\bar{g}(A)}$ we have $A \leq g^* \circ g_!(A)$.

Exercise 1.109

Complete the proof of Proposition 1.107 by showing that (for monotone $f : P \to Q$ and $g : Q \to P$)

1. if f is left adjoint to g then for any $q \in Q$ we have $f(g(q)) \leq q$, and
2. if $p \leq g(f(p))$ and $f(g(q)) \leq q$, then $p \leq g(p)$ iff $f(p) \leq q$ holds, for all $p \in P$ and $q \in Q$.

Solution

Assume f is left adjoint to g. Let $q \in Q$ and $p = g(q)$. Then we know $p \leq g(q)$, so by definition of the left adjoint $f(p) \leq q$. As we defined p to be $g(q)$ this implies $f(g(q)) \leq q$.

Next assume $p \leq g(f(p))$ and $f(g(q)) \leq q$ for any $p \in P, q \in Q$. We need to show that $p \leq g(q)$ implies $f(p) \leq q$. But $p \leq g(q)$ implies $f(p) \leq f(g(q))$ by the monotonicity of f, and $f(g(q)) \leq q$ by assumption, so $f(p) \leq q$.

Exercise 1.110

1. Show that if $f : P \to Q$ has a right adjoint g, then it is unique up to isomorphism. That is, for any other right adjoint g', we have $g(q) \cong g'(q)$ for all $q \in Q$.
2. Is the same true for left adjoints? That is, if $h : P \to Q$ has a left adjoint, is it necessarily unique up to isomorphism?

Solution

1. Suppose g and g' are right adjoint to $f : P \to Q$. Then for any $q \in Q, p \in P$ we have

   $$f(p) \leq q \Leftrightarrow p \leq g(q) \Leftrightarrow p \leq g'(q).$$

   In particular this holds for $p = g(q)$, which means $g(q) \leq g'(q)$ as by reflexivity $g(q) \leq g(q)$. Similarly for $p = g'(q)$, we have $g'(q) \leq g(q)$ as $g'(q) \leq g'(q)$. Thus $g(q) \cong g'(q)$ for all $q \in Q$.

2. The same holds for left adjoints. To show this, suppose f and f' are left adjoint to $g : Q \to P$. Then for any $p \in P, q \in Q$ we have

$$f(p) \leq q \Leftrightarrow p \leq g(q) \Leftrightarrow f'(p) \leq q.$$

The rest of the proof follows analogously to part 1.

Exercise 1.112

Complete the proof of Proposition 1.111 by showing that left adjoints preserve joins.

Solution