

# Applied Category Theory Problems

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October 4, 2020

# Chapter 1

## Generative Effects

### Exercise 1.1

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is said to be

- *order-preserving* if  $x \leq y$  implies  $f(x) \leq f(y)$ , for all  $x, y \in \mathbb{R}$
- *metric-preserving* if  $|x - y| = |f(x) - f(y)|$
- *addition-preserving* if  $f(x + y) = f(x) + f(y)$

For each of the three properties defined above—call it *foo*—find an  $f$  that is *foo*-preserving and an example of an  $f$  that is not *foo*-preserving.

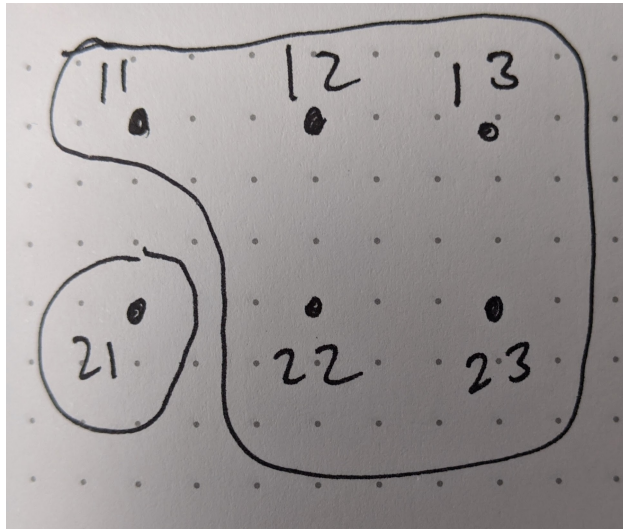
### Solution

$f(x) = x$  is order-, metric-, and addition-preserving.  $f(x) = x^2$  is none of these.

### Exercise 1.4

See book.

### Solution



### Exercise 1.6, 1.7, 1.10

Discussed in person

### Exercise 1.11

Let  $A = \{h, 1\}$  and  $B = \{1, 2, 3\}$ .

### Solution

1. The subsets of  $B$  are  $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$ .
2. For example,  $\{1, 2\} \cup \{2\} = \{1, 2\}$ .
3.  $A \times B = \{(h, 1), (h, 2), (h, 3), (1, 1), (1, 2), (1, 3)\}$
4.  $A \sqcup B = \{h_A, 1_A, 1_B, 2_B, 3_B\}$
5.  $A \cup B = \{h, 1, 2, 3, 4\}$

### Exercise 1.16

Suppose that  $A$  is a set and  $\{A_p\}_{p \in P}$  and  $\{A'_{p'}\}_{p' \in P'}$  are two partitions of  $A$  such that for each  $p \in P$  there exists a  $p' \in P'$  with  $A_p = A'_{p'}$ .

1. Show that for each  $p \in P$  there is at most one  $p' \in P'$  such that  $A_p = A'_{p'}$ .
2. Show that for each  $p' \in P'$  there is a  $p \in P$  such that  $A_p = A'_{p'}$ .

**Solution**

1. If there are distinct  $p'_1, p'_2 \in P'$  such that  $A_p = A'_{p'_1}$  and  $A_p = A'_{p'_2}$ , then by transitivity  $A'_{p'_1} = A'_{p'_2}$  which means that  $p'_1 = p'_2$  as  $P'$  is a partition.
2. Let  $a \in A'_{p'_1}$ . Then we must have that  $a \in A_p$  for some  $p$ . We will show that this  $A_p$  is the desired one. There must exist  $p'_2$  such that  $A_p = A'_{p'_2}$ . So  $a \in A'_{p'_2}$  and  $a \in A'_{p'_1}$ . So  $A'_{p'_1} = A'_{p'_2} = A_p$  as we wanted.

**Exercise 1.17**

See book.

**Solution**

(11, 11), (12, 12), (13, 13), (21, 21), (22, 22), (23, 23), (11, 12), (12, 11), (22, 23), (23, 22)

**Exercise 1.20**

Suppose that  $\sim$  is an equivalence relation on a set  $A$ , and let  $P$  be the set of  $(\sim)$ -closed and  $(\sim)$ -connected subsets  $\{A_p\}_{p \in P}$ .

1. Show that each part  $A_p$  is nonempty.
2. Show that if  $p \neq q$  i.e.  $A_p \neq A_q$ , then  $A_p \cap A_q = \emptyset$ .
3. Show that  $A = \bigcup_{p \in P} A_p$ .

**Solution**

1. Since each  $A_p$  is  $(\sim)$ -connected, by definition  $A_p$  must be nonempty.
2. Suppose  $a \in A_p \cap A_q$ . Then for any  $b \in A_p$ , we know  $b \sim a$ , hence since  $A_q$  is  $(\sim)$ -closed we know  $b \in A_q$ . Similarly for any  $b \in A_q$ . Hence  $A_p = A_q$ .
3. Clearly  $\bigcup A_p \subseteq A$ . Suppose  $a \in A$ . Then let  $X = \{b \mid a \sim b, b \in A\}$ .  $X$  is closed as the equivalence relation is reflexive and connected as it is transitive, so  $X$  a set in  $\{A_p\}$  and  $A \subseteq \bigcup A_p$ .

**Exercise 1.24**

Discussed in person

**Exercise 1.25**

Suppose that  $A$  is a set and  $f : A \rightarrow \emptyset$  is a function to the empty set. Show that  $A$  is empty.

### Solution

Suppose there is some  $a \in A$ . Then we must have  $f(a) \in \emptyset$  which is clearly not possible.

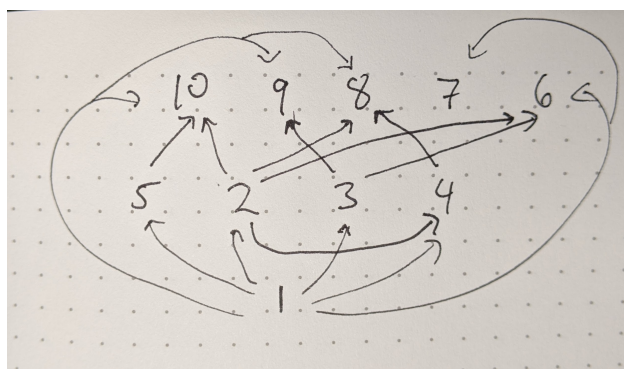
### Exercise 1.27, 1.38, 1.40, 1.41, 1.42, 1.44

Discussed in person

### Exercise 1.46

Write down the numbers  $1, 2, \dots, 10$  and draw an arrow  $a \rightarrow b$  if  $a$  divides perfectly into  $b$ . Is it a total order?

### Solution



This isn't a total order, as for example we neither have  $2 \mid 7$  or  $7 \mid 2$ .

### Exercise 1.48

Is the usual  $\leq$  ordering on the set  $\mathbb{R}$  of real numbers a total order?

### Solution

Yes: for any  $x, y \in \mathbb{R}$ , we have  $x \leq y$  or  $y \leq x$ .

### Exercise 1.51

Discussed in person

### Exercise 1.53

For any set  $S$  there is a coarsest partition, having just one part. What surjective function does it correspond to? There is also a finest partition, where everything is in its own partition. What surjective function does it correspond to?

### Solution

Let  $f : S \rightarrow \{\bullet\}$  be the unique function that sends every element of  $S$  to  $\bullet$ . This is a surjection corresponding to the coarsest partition, i.e. where every element is in the same set.

Let  $f : S \rightarrow S$  be the identity function. This is a surjection corresponding to the finest partition, i.e. where every element is in its own set.

### Exercise 1.55

Prove that the preorder of upper sets on a discrete preorder on a set  $X$  is simply the power set  $P(X)$ .

### Solution

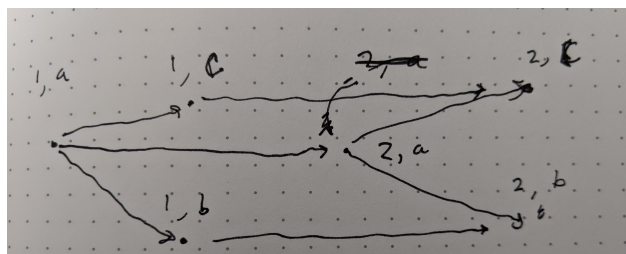
Clearly the set of upper sets  $U(X)$  is a subset of the power set  $P(X)$ .

Let  $Y \subseteq X$ . We know  $\emptyset$  is an upper set, so let  $y \in Y$ . Then since the preorder is discrete, the only element in  $X$  greater than  $y$  is  $y$  itself, which is in  $Y$ . This holds for any  $y \in Y$  so  $Y$  is an upper set. Note that the ordering on both  $U(X)$  and  $P(X)$  is the same, i.e.  $\subseteq$ .

### Exercise 1.57

See book.

### Solution

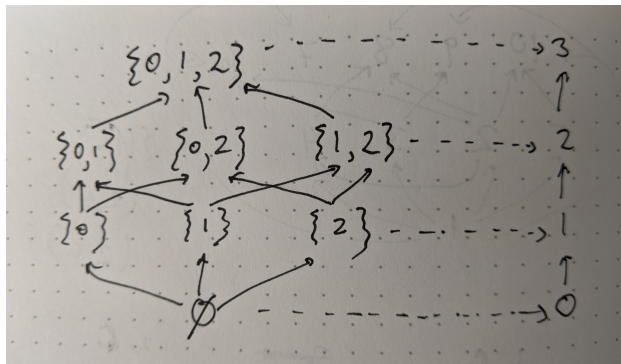


### Exercise 1.63

Let  $X = \{0, 1, 2\}$ .

1. Draw the Hasse diagram for  $P(X)$ .
2. Draw the Hasse diagram for the preorder  $0 \leq 1 \leq 2 \leq 3$ .
3. Draw the cardinality map  $|\cdot|$  as dashed lines between them

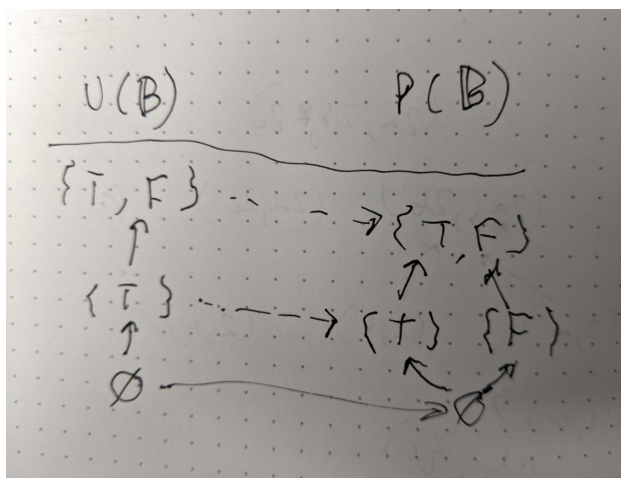
## Solution



## Exercise 1.65

Draw the monotone map between  $U(\mathbb{B})$  and  $P(\mathbb{B})$  as described in the text.

## Solution



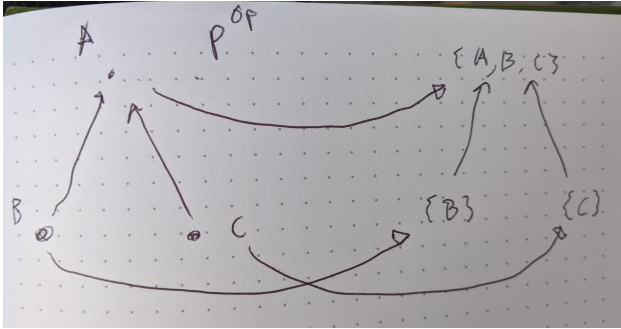
## Exercise 1.66

Let  $(P, \leq)$  be a preorder.

1. Show that the set  $\uparrow p = \{p' \in P \mid p \leq p'\}$  is an upper set for any  $p \in P$ .
2. Show that this defines a monotone map  $\uparrow: P^{op} \rightarrow U(P)$ .
3. Show that  $p \leq p'$  iff  $\uparrow(p') \subseteq \uparrow(p)$ .
4. Draw a picture of the map  $\uparrow$  in the case where  $P$  is the preorder  $(b \geq a \leq c)$ .

### Solution

1. Suppose  $q \in \uparrow p$ , then any  $q' \geq q$  is transitively greater than  $p$  and hence  $q' \in \uparrow p$ .
2. Suppose  $p \geq q$  (i.e.  $p$  is less than  $q$  in  $P^{op}$ ), we want to show that  $\uparrow p \subseteq \uparrow q$ . So let  $p' \in \uparrow p$ . We know  $q \leq p \leq p'$  and hence  $p' \in \uparrow q$ .
3. We showed the first direction in part 2, so assume  $\uparrow(p') \subseteq \uparrow(p)$ . This means  $p \in \uparrow(p')$  and hence  $p \leq p'$ .



4.

### Exercise 1.67

Show that when  $(P, \leq_P)$  is a discrete preorder, then every function  $f : P \rightarrow Q$  is monotone regardless of the order  $\leq_Q$ .

### Solution

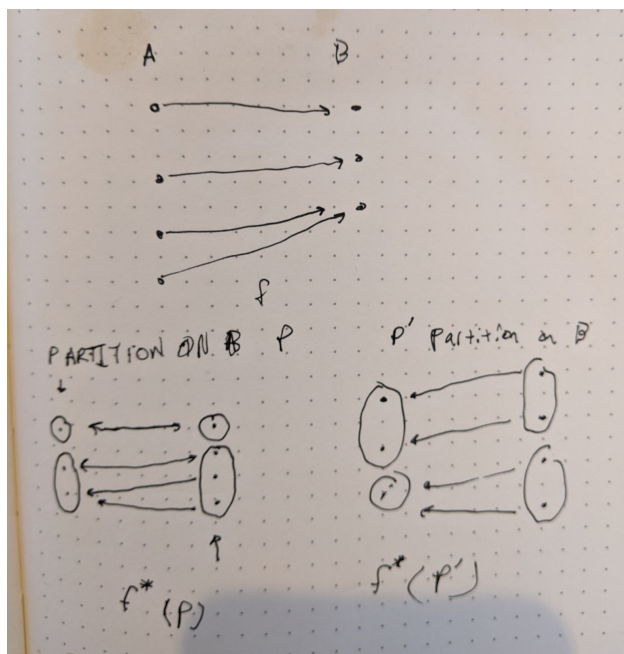
We need to show that for any  $x, y \in P$  where  $x \leq_P y$ , we have  $f(x) \leq_Q f(y)$ . But the only  $x$  and  $y$  satisfying this are  $x \leq_P x$ , for which we have  $f(x) \leq_Q f(x)$  regardless of  $\leq_Q$  by the definition of a preorder.

### Exercise 1.69

Choose two sets  $X$  and  $Y$  with at least three elements each and choose a surjective, non-identity function  $f : X \rightarrow Y$ . Write down two different partitions  $P$  and  $Q$  of  $Y$ , and find  $f^*(P)$  and  $f^*(Q)$ .



## Solution



## Exercise 1.71

Prove Proposition 1.70:

1. For any preorder  $(P, \leq_P)$ , the identity function is monotone.
2. If  $(Q, \leq_Q)$  and  $(R, \leq_R)$  are preorders and  $f : P \rightarrow Q$  and  $g : Q \rightarrow R$  are monotone, then  $(f \circ g) : P \rightarrow R$  is also monotone.

## Solution

1. If  $a \leq_P b$  then clearly  $a = f(a) \leq_P f(b) = b$  if  $f$  is the identity function.
2. Suppose  $a \leq_P b$ , then  $f(a) \leq_Q f(b)$  as  $f$  is monotone, and hence  $g(f(a)) \leq_R g(f(b))$  as  $g$  is also monotone.

## Exercise 1.73

Show that a skeletal dagger preorder is just a discrete preorder, and hence can be identified with a set.

## Solution

Let  $(P, \leq)$  be a skeletal dagger preorder. We need to show that for any  $x \in P$ , the only thing comparable to  $x$  is  $x$  itself. So suppose  $x \leq y$ , then as  $P$  is a dagger preorder we know

that  $y \leq x$ . Hence as  $P$  is skeletal, we have that  $x = y$ . This implies that  $P$  is a discrete preorder.

### Exercise 1.77

Show that the map  $\Phi$  from Section 1.1.1 ('Is  $\bullet$  connected to  $\star$ ?') is the monotone map  $\text{Prt}(\{\star, \bullet, \circ\}) \rightarrow \mathbb{B}$ .

### Solution

Let  $P$  and  $P'$  be partitions where  $P \leq P'$ . If  $\Phi(P) = \text{false}$  then clearly  $\Phi(P) \leq \Phi(P')$ , so assume  $\Phi(P) = \text{true}$ . This means for some set  $X$  in the partition  $P$ , we know that both  $\bullet, \star \in X$ . As  $P \leq P'$  this means there is some  $Y$  in the partition  $P'$  with  $X \subseteq Y$ , which implies that  $\bullet, \star \in Y$ . Hence  $\Phi(P') = \text{true}$  and  $\Phi(P) \leq \Phi(P')$ .

### Exercise 1.79

Let  $P$  and  $Q$  be preorders and  $f : P \rightarrow Q$  a monotone map. Show that the pullback  $f^* : U(Q) \rightarrow U(P)$  can be defined by taking  $u : Q \rightarrow \mathbb{B}$  to  $(f \circ u) : P \rightarrow \mathbb{B}$ .

### Solution

Call  $\phi_Q$  the function that takes upper sets in  $Q$  to monotone maps as defined in Proposition 1.78, and similarly  $\phi_P$ . Let  $U \in U(Q)$ . We want to show  $\phi_P(f^{-1}(U)) = f \circ (\phi_Q(U))$ .

Let  $x \in P$ . If  $x \in f^{-1}(U)$ , then we know  $\phi_P(f^{-1}(U))(x) = \text{true}$  by definition. But we also know  $f(x) \in U$  and hence  $\phi_Q(U)(f(x)) = \text{true}$ . Conversely if  $x \notin f^{-1}(U)$ , we will have both  $\phi_P(f^{-1}(U))(x) = \text{false}$ , as well as  $f(x) \notin U$  and  $\phi_Q(U)(f(x)) = \text{false}$ . This shows that these maps are equal.

### Exercise 1.80

Why is 0 a greatest lower bound for  $\{\frac{1}{n+1} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ ?

### Solution

Assume that  $\varepsilon > 0$  is a lower bound. Let  $n = \lceil 1/\varepsilon \rceil$ . Then

$$\frac{1}{n+1} \leq \frac{1}{1/\varepsilon+1} \leq \frac{1}{1/\varepsilon} = \varepsilon.$$

Hence no such  $\varepsilon$  is a lower bound.

**Exercise 1.85**

Let  $(P, \leq)$  be a preorder and  $p \in P$ , consider the set  $A = \{p\}$ .

1. Show that  $\bigwedge A \cong p$ .
2. Show that if  $P$  is a partial order, then  $\bigwedge A = p$ .
3. Are the analogous facts true when  $\bigwedge$  is replaced by  $\bigvee$ ?

**Solution**

1. Clearly  $p \leq p$ , so by definition  $\bigwedge A \leq p$  (as a lower bound) and  $\bigwedge A \geq p$  (as a greatest lower bound).
2. If the previous is true in a partial order, then we have  $\bigwedge A = p$ .
3. The analogous facts are true with  $\bigvee$ .

**Exercise 1.90**

In the  $n \mid m$  ordering on  $\mathbb{N}$ , what are the meet and the join?

**Solution**

The meet is the greatest common divisor and the join is the least common multiple.

**Exercise 1.94**

Prove that for any monotone map  $f : P \rightarrow Q$ , if  $a, b \in P$  have a join and  $f(a), f(b) \in Q$  have a join, then  $f(a) \vee f(b) \leq f(a \vee b)$ .

**Solution**

We know  $a, b \leq a \vee b$ , so since  $f$  is monotone we have  $f(a), f(b) \leq f(a \vee b)$ . Hence  $f(a \vee b)$  is an upper bound for  $\{f(a), f(b)\}$ , so by definition of the join we must have  $f(a) \vee f(b) \leq f(a \vee b)$ .

**Exercise 1.98**

Find a right adjoint for the monotone map  $(3 \times -) : \mathbb{Z} \rightarrow \mathbb{R}$ , and show it is correct.

**Solution**

Let  $g(y) = \lfloor y/3 \rfloor$ . Then we have  $3x \leq y \Leftrightarrow x \leq y/3 \Leftrightarrow x \leq \lfloor y/3 \rfloor$ , hence  $g$  is a right adjoint for  $3x$ .

**Exercise 1.99**

See book.

**Solution**

1. In this case  $f$  is left adjoint to  $g$ .
2. In this case  $f$  is not left adjoint to  $g$ , as  $g(1) = 2 \geq 2$  but  $f(2) = 2 \not\leq 1$ .

**Exercise 1.101**

1. Does  $\lceil -/3 \rceil$  have a left adjoint  $L : \mathbb{Z} \rightarrow \mathbb{R}$ ?
2. If not, why? If so, does its left adjoint have a left adjoint?

**Solution**

Let  $g : \mathbb{R} \rightarrow \mathbb{Z}$  be defined by  $g(x) = \lceil x/3 \rceil$ . We will show by contradiction that  $g$  does not have a left adjoint.

For a left adjoint  $f : \mathbb{Z} \rightarrow \mathbb{R}$ , we must have  $f(1) \leq 0 \Leftrightarrow 1 \leq g(0) = \lceil 0/3 \rceil = 0$ . Clearly the second part does not hold, so we know  $f(1) \not\leq 0$ .

On the other hand, we know that  $f(1) \leq \inf A$  where  $A = \{x \mid g(x) \geq 1, x \in \mathbb{R}\}$ . However  $1/n \in A$  for  $n \in \mathbb{Z}^+$ , as  $\lceil 1/n \rceil = 1$  for all such  $n$ . But  $\inf\{1/n \mid n \in \mathbb{Z}^+\} = 0$ , which implies that  $f(1) \leq 0$ . This is a contradiction, so  $g$  must not have a left adjoint.

**Exercise 1.103**

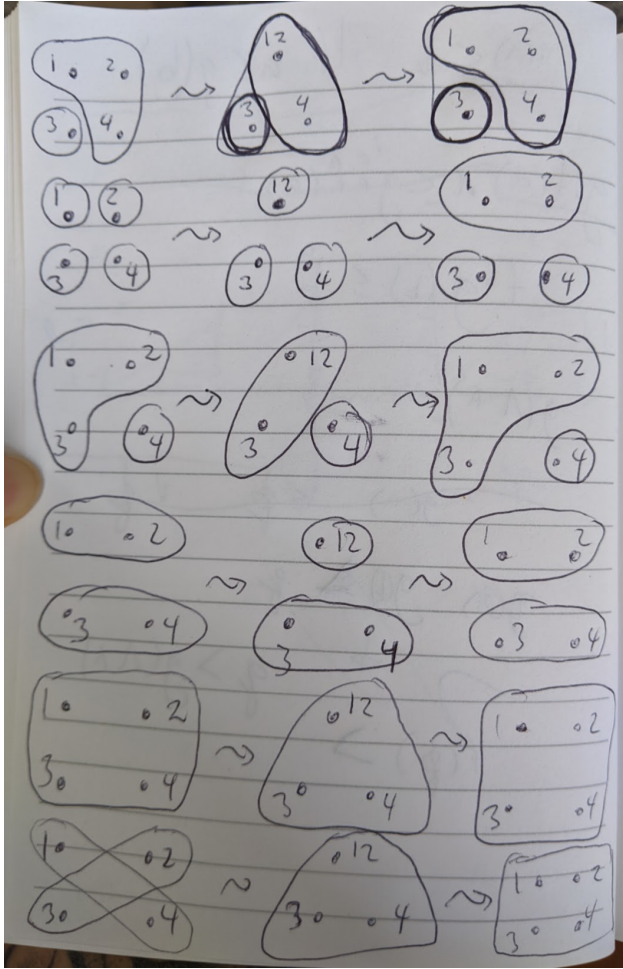
Choose 6 different partitions on the set  $S$  and for each, call it  $c$ , find  $g_!(c)$  where  $S$ ,  $T$ , and  $g : S \rightarrow T$  are the same as they were in Example 1.102.

**Exercise 1.105**

Using the same  $S$ ,  $T$ , and  $g : S \rightarrow T$  as in Example 1.102, find the partition  $g^*(c)$  for each of the 5 partitions  $c$  on the set  $T$ .

**Solution**

For both 1.103 and 1.105.



### Exercise 1.106 (revised)

Prove that  $g_!$  is left adjoint to  $g^*$ , as defined in the text.

### Solution

Let  $S, T$  be sets, and let  $g : S \rightarrow T$ . Define  $g_!, g^*$  as in the text.

We first show  $g^*$  is monotone. Let  $A, B \in \text{Prt}(T)$  such that  $A \leq B$ . Then for each set  $A_i \in A$ ,  $A_i \subseteq B_j$  for some  $B_j \in B$ , and as a result  $g^{-1}(A_i) \subseteq g^{-1}(B_j)$  for each  $A_i \in A$ . As the image of a partition under  $g^*$  is the collection of preimages of that partition via  $g$ , we have  $g^*(A) \leq g^*(B)$ .

Next we show  $g_!$  is monotone. Let  $A, B \in \text{Prt}(S)$  such that  $A \leq B$ . As before we know that for each set  $A_i \in A$ ,  $A_i \subseteq B_j$  for some  $B_j \in B$ . We consider  $A, B \in \text{Rel}(S)$  i.e. subsets of  $S \times S$ . Note that  $g_!(C)$  is the transitive closure of the relation  $\{(g(x), g(y)) \mid (x, y) \in C\}$ . As  $A \leq B$ ,  $\{(g(x), g(y)) \mid (x, y) \in A\} \subseteq \{(g(x), g(y)) \mid (x, y) \in B\}$ . Using the fact that the function taking a relation to its transitive closure is monotone on the set of relations ordered by inclusion, we can conclude that  $g_!(A) \leq g_!(B)$ .

For the next part of the proof, we use proposition 1.107, and derive our result by showing that for each  $A \in \text{Prt}(S)$  and for each  $B \in \text{Prt}(T)$  that both  $A \leq g^* \circ g_!(A)$ , and  $g_! \circ g^*(B) \leq B$ .

We start with showing  $A \leq g^* \circ g_!(A)$ . First we consider two additional functions, first  $\bar{g} : \text{Prt}(S) \rightarrow \text{Rel}(T)$ , where  $g(r) = \{(g(x), g(y)) \mid (x, y) \in r\}$ . Secondly an extension of  $g^*$  to all relations,  $\bar{g}^* : \text{Rel}(T) \rightarrow \text{Rel}(S)$ , so for a relation  $r$ , we have  $\bar{g}^*(r) = \{(x, y) \mid (g(x), g(y)) \in r\}$  ( $g^*$  is the restriction of  $\bar{g}^*$  to equivalence relations). Both  $\bar{g}$  and  $\bar{g}^*$  are monotone, which can be seen in proofs similar to our proofs for  $g^*$  and  $g_!$ . Additionally let the transitive closure of a set  $Q$  be denoted  $\hat{Q}$ . We now note two things, one  $A = \bar{g}^* \circ \bar{g}(A)$ , and two  $g^* \circ g_!(A) = \bar{g}^*(\widehat{\bar{g}(A)})$ . As  $\bar{g}^*$  is monotone and  $\bar{g}(A) \leq \widehat{\bar{g}(A)}$  we have  $A \leq g^* \circ g_!(A)$ .

### Exercise 1.109

Complete the proof of Proposition 1.107 by showing that (for monotone  $f : P \rightarrow Q$  and  $g : Q \rightarrow P$ )

1. if  $f$  is left adjoint to  $g$  then for any  $q \in Q$  we have  $f(g(q)) \leq q$ , and
2. if  $p \leq g(f(p))$  and  $f(g(q)) \leq q$ , then  $p \leq g(p)$  iff  $f(p) \leq q$  holds, for all  $p \in P$  and  $q \in Q$ .

### Solution

Assume  $f$  is left adjoint to  $g$ . Let  $q \in Q$  and  $p = g(q)$ . Then we know  $p \leq g(q)$ , so by definition of the left adjoint  $f(p) \leq q$ . As we defined  $p$  to be  $g(q)$  this implies  $f(g(q)) \leq q$ .

Next assume  $p \leq g(f(p))$  and  $f(g(q)) \leq q$  for any  $p \in P, q \in Q$ . We need to show that  $p \leq g(q)$  implies  $f(p) \leq q$ . But  $p \leq g(q)$  implies  $f(p) \leq f(g(q))$  by the monotonicity of  $f$ , and  $f(g(q)) \leq q$  by assumption, so  $f(p) \leq q$ .

### Exercise 1.110

1. Show that if  $f : P \rightarrow Q$  has a right adjoint  $g$ , then it is unique up to isomorphism. That is, for any other right adjoint  $g'$ , we have  $g(q) \cong g'(q)$  for all  $q \in Q$ .
2. Is the same true for left adjoints? That is, if  $h : P \rightarrow Q$  has a left adjoint, is it necessarily unique up to isomorphism?

### Solution

1. Suppose  $g$  and  $g'$  are right adjoint to  $f : P \rightarrow Q$ . Then for any  $q \in Q, p \in P$  we have

$$f(p) \leq q \Leftrightarrow p \leq g(q) \Leftrightarrow p \leq g'(q).$$

In particular this holds for  $p = g(q)$ , which means  $g(q) \leq g'(q)$  as by reflexivity  $g(q) \leq g(q)$ . Similarly for  $p = g'(q)$ , we have  $g'(q) \leq g(q)$  as  $g'(q) \leq g'(q)$ . Thus  $g(q) \cong g'(q)$  for all  $q \in Q$ .

2. The same holds for left adjoints. To show this, suppose  $f$  and  $f'$  are left adjoint to  $g : Q \rightarrow P$ . Then for any  $p \in P, q \in Q$  we have

$$f(p) \leq q \Leftrightarrow p \leq g(q) \Leftrightarrow f'(p) \leq q.$$

The rest of the proof follows analogously to part 1.

### Exercise 1.112

Complete the proof of Proposition 1.111 by showing that left adjoints preserve joins.

### Solution

Let  $f : P \rightarrow Q$  and  $g : Q \rightarrow P$  be monotone maps with  $f$  left adjoint to  $g$ . Let  $A \subseteq P$  and let  $j = \bigvee A$  be its join. Then since  $f$  is monotone  $f(a) \leq f(j)$  for all  $a \in A$ , so  $f(j)$  is an upper bound for  $f(A)$ .

Next we will show that  $f(j)$  is a least upper bound. Suppose  $b$  is some other upper bound for  $f(A)$ . So for any  $a \in A$ , we know  $f(a) \leq b$  which implies  $a \leq g(b)$  by the fact that  $f$  is left adjoint. Hence  $g(b)$  is an upper bound for  $A$ , and by definition of the join we know  $j \leq g(b)$ , which implies  $f(j) \leq b$ , again by definition of a Galois connection. Therefore  $f(j)$  is the join of  $f(A)$ .

### Exercise 1.114

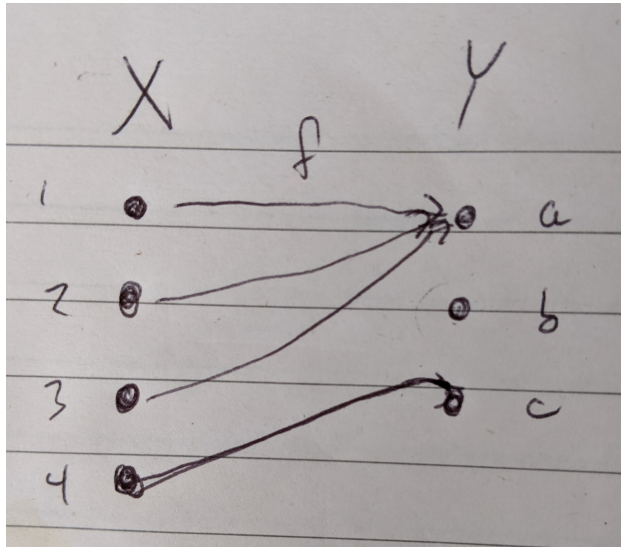
Discussed in person.

### Exercise 1.118

Choose sets  $X$  and  $Y$  with 2-4 elements each, and a function  $f : X \rightarrow Y$ .

1. Choose  $B_1, B_2 \subseteq Y$  and find  $f^*(B_1)$  and  $f^*(B_2)$ .
2. Choose  $A_1, A_2 \subseteq X$  and find  $f_!(A_1)$  and  $f_!(A_2)$ .
3. Find  $f_*(A_1)$  and  $f_*(A_2)$ .

## Solution



1.  $f^*({a, b}) = \{1, 2, 3\}$  and  $f^*({c}) = \{4\}$
2.  $f_!({1, 4}) = \{a, c\}$  and  $f_!({2, 3}) = \{a\}$
3.  $f_*({1, 4}) = \{b\}$  and  $f_*({2, 3}) = \{a, b\}$

## Exercise 1.119

Suppose  $f$  is left adjoint to  $g$ . Show that

1.  $p \leq (f \circ g)(p)$
2.  $(f \circ g \circ f \circ g)(p) \cong (f \circ g)(p)$

## Solution

1. From Proposition 1.107 we immediately have that  $p \leq (f \circ g)(p)$  from the fact that  $f$  is left adjoint to  $g$ .
2. If we apply Proposition 1.107 to  $(f \circ g)(p) \in P$ , this gives us  $(f \circ g)(p) \leq (f \circ g)((f \circ g)(p))$ . If we apply the other part to  $f(p) \in Q$ , this gives us  $(g \circ f)(f(p)) \leq f(p)$ , and because  $g$  is monotone this implies that  $(f \circ g \circ f \circ g)(p) \leq (f \circ g)(p)$ . Hence  $(f \circ g \circ f \circ g)(p) \cong (f \circ g)(p)$ .

## Exercise 1.124, 1.125

Discussed in person.



# Chapter 2

## Resource Theories

### Exercise 2.5

Is  $(\mathbb{R}, \leq, 1, *)$  a symmetric monoidal preorder?

### Solution

It is not a monoidal preorder, as for example  $-5 \leq -1$  and  $-2 \leq -1$ , but  $-5 * -2 \not\leq -1 * -1$ .

### Exercise 2.8

Check that if  $(M, *, e)$  is a commutative monoid then  $(\mathbf{Disc}_M, =, *, e)$  is a symmetric monoidal preorder.

### Solution

Unitality, associativity, and symmetry come for free from the definition of a commutative monoid, so we only need to check monotonicity. Since  $x \leq y \Leftrightarrow x = y$  this is easy to check, as  $x_1 \leq y_1$  and  $x_2 \leq y_2$  implies  $x_1 = y_1$  and  $x_2 = y_2$  which in turn implies that  $x_1 * x_2 = y_1 * y_2$ .

### Exercise 2.20

Formally prove that  $t \leq v + w, w + u \leq x + z, v + x \leq y$  implies  $t + u \leq y + z$ . Be explicit about where reflexivity and transitivity are used, and why symmetry need not be used.

#### Solution

In the below, R=reflexivity, M=monotonicity, A=associativity, T=transitivity. Symmetry is not used, which you can tell from the diagram from the fact that no wires cross.

$$\begin{array}{llll}
 t \leq v + w, u \leq u & \Rightarrow & t + u \leq (v + w) + u = v + (w + u) & [\text{R, M, A, T}] \quad (2.1) \\
 (2.1), w + u \leq x + z & \Rightarrow & v + (w + u) \leq v + (x + z) & [\text{M}] \quad (2.2) \\
 (2.1, 2.2) & \Rightarrow & t + u \leq v + (x + z) = (v + x) + z & [\text{T, A}] \quad (2.3) \\
 v + x \leq y, z \leq z & \Rightarrow & (v + x) + z \leq y + z & [\text{M}] \quad (2.4) \\
 (2.3, 2.4) & \Rightarrow & t + u \leq y + z & [\text{T}] \quad (2.5)
 \end{array}$$

### Exercise 2.21

Skipped.

### Exercise 2.29

Consider  $(\mathbb{B}, \leq)$  with monoidal product  $\vee$ . What's the monoidal unit? Does it satisfy the rest of the conditions?

#### Solution

The monoidal unit should be **false**. Discussed why in person (i.e. truth table)

### Exercise 2.31

Show that there is a monoidal structure on  $(\mathbb{N}, \leq)$  where the monoidal product is standard  $*$ . What should the monoidal unit be?

#### Solution

The monoidal unit should be 1. We will show monotonicity as the other conditions are obvious. If  $x_1 \leq y_1$  and  $x_2 \leq y_2$ , then there are  $a_1, a_2 \in \mathbb{N}$  such that  $y_1 = x_1 + a_1$  and  $y_2 = x_2 + a_2$ . Then

$$y_1 * y_2 = (x_1 + a_1) * (x_2 + a_2) = x_1 * x_2 + a_1 * x_2 + a_2 * x_1 + a_1 * a_2 \geq x_1 * x_2.$$

### Exercise 2.33

Consider the divisibility order  $(\mathbb{N}, |)$ . Does 0 as monoidal unit and  $+$  as monoidal product satisfy the conditions?

### Solution

It does not, as for example  $2 \mid 4$  and  $1 \mid 1$ , but  $(2 + 1) \nmid (4 + 1)$ , so monotonicity fails.

### Exercise 2.34

Consider the preorder **NMY** with Hasse diagram  $\mathbf{no} \rightarrow \mathbf{maybe} \rightarrow \mathbf{yes}$ , monoidal unit **yes** and “min” as the monoidal product. Define what “min” should be and check that the axioms hold.

### Solution

min	no	maybe	yes
no	no	no	no
maybe	no	no	maybe
yes	no	maybe	yes

### Exercise 2.35

Let  $S$  be a set and let  $P(S)$  be its power set, with the subset relation as order. Does  $P(S)$  with unit  $S$  and product given by set intersection satisfy the conditions of symmetric monoidal preorder?

### Solution

The other conditions are easy to show, so we will show monotonicity only. Let  $A_1, A_2, B_1, B_2 \subseteq S$  where  $A_1 \subseteq B_1$  and  $A_2 \subseteq B_2$ . Then  $A_1 \cap A_2 \subseteq A_1 \subseteq B_1$  and  $A_1 \cap A_2 \subseteq A_2 \subseteq B_2$ , which implies that  $A_1 \cap A_2 \subseteq B_1 \cap B_2$ .

### Exercise 2.36

Let  $\mathbf{Prop}^{\mathbb{N}}$  denote the set of all mathematical statements one can make about a natural number, where we consider two statements to be the same if one is true if and only if the other is true. Given  $P, Q \in \mathbf{Prop}^{\mathbb{N}}$ , we say  $P \leq Q$  if for all  $n \in \mathbb{N}$ , whenever  $P(n)$  is true, so is  $Q(n)$ . Define a monoidal unit and product on  $\mathbf{Prop}^{\mathbb{N}}$ .

### Solution

We define the monoidal unit to be the statement “ $n$  is a natural number” (i.e. a statement that’s always true) and the monoidal product to be logical AND. Note that this SMP effectively reduces to the previous example, where we consider subsets  $P = \{n \in \mathbb{N} \mid P(n)\}$ .

### Exercise 2.39

Complete the proof of Proposition 2.38.

### Solution

These conditions are inherited from the original SMP, so we have decided this problem is dumb.

### Exercise 2.40

What is  $\mathbf{Cost}^{\text{op}}$  as a preorder? What is the monoidal unit and product?

### Solution

$\mathbf{Cost}^{\text{op}}$  is the same as  $\mathbf{Cost}$  but using  $\leq$  rather than  $\geq$ . From Proposition 2.38, we know that  $\mathbf{Cost}^{\text{op}}$  can use the same monoidal unit and product as the original, i.e. 0 and +.

### Exercise 2.43

Check that the map  $g : (\mathbb{B}, \leq, \text{true}, \wedge) \rightarrow ([0, \infty], \geq, 0, +)$  with  $g(\text{false}) = \infty$  and  $g(\text{true}) = 0$  is monoidal monotone. Is  $g$  strict?

### Solution

Clearly  $g(\text{false}) \leq g(\text{true})$  as  $\infty \geq 0$ , so  $g$  is monotone. In addition  $g(\text{true}) = 0$ , so the function preserves identities exactly. Finally note that  $g(\text{false}) + g(\text{false}) = \infty + \infty = \infty = g(\text{false}) = g(\text{false} \wedge \text{false})$ , the other products are trivial as we've shown the identity is preserved. Hence  $g$  is strict monoidal monotone.

### Exercise 2.44

Let  $\mathbf{Bool}$  and  $\mathbf{Cost}$  be as above, and consider  $d, u : [0, \infty] \rightarrow \mathbb{B}$  as follows:

$$d(x) := \begin{cases} \text{false} & \text{if } x > 0 \\ \text{true} & \text{if } x = 0 \end{cases} \quad u(x) := \begin{cases} \text{false} & \text{if } x = \infty \\ \text{true} & \text{if } x < \infty \end{cases}$$

Is  $d$  monotonic, monoidal, and/or strict? Is  $u$ ?

### Solution

Both  $d$  and  $u$  are monotonic. They also both map the unit in  $\mathbf{Cost}$ , namely 0, exactly to the unit in  $\mathbf{Bool}$ , true.

Finally, they both strictly preserve the monoidal product, which can be shown by a truth table. In short, for  $d$ , anything added to a number  $x > 0$  continues to be greater than 0, while  $0 + 0 = 0$ ; for  $u$ , anything added to  $\infty$  is still  $\infty$ , while the sum of two finite numbers is still finite.

### Exercise 2.45

1. Is  $(\mathbb{N}, \leq, 1, *)$  a monoidal preorder?
2. If not, why not? If so, does there exist a monoidal monotone  $(\mathbb{N}, \leq, 0, +) \rightarrow (\mathbb{N}, \leq, 1, *)$ ? If not, why not?
3. Is  $(\mathbb{Z}, \leq, *, 1)$  a monoidal preorder?

### Solution

1. Yes,  $(\mathbb{N}, \leq, 1, *)$  is a monoidal preorder, as we showed in Exercise 2.31.
2. Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be defined by  $f(n) = 1$  for any  $n \in \mathbb{N}$ . Then clearly  $f(0) = 1$  so the unit is preserved, and for any  $n, m \in \mathbb{N}$  we have  $f(n) * f(m) = 1 * 1 = 1 = f(n + m)$ , so the product is preserved. Hence  $f$  is strict monoidal monotone.
3. No,  $(\mathbb{Z}, \leq, 1, *)$  is not, similar to what we showed in Exercise 2.5.

### Exercise 2.50

1. Show that if you start with a preorder  $(P, \leq)$ , define a **Bool**-category as in Example 2.47, and turn it back to a preorder as in Theorem 2.49, you get back the preorder you started with.
2. Similarly, show that if you start with a **Bool**-category, turn it into a preorder, then turn it back to a **Bool**-category, you get back the **Bool**-category you started with.

### Solution

1. Let  $(P, \leq)$  be a preorder and let  $\mathcal{X}$  be the constructed **Bool**-category. So  $\text{Ob}(\mathcal{X}) = P$  and  $\mathcal{X}(x, y) = \text{true}$  if and only if  $x \leq y$ . As this is an if-and-only-if, if you do the reverse construction you get back exactly the original preorder.
2. This proof is basically the same as the previous.

### Exercise 2.52

Which distance is bigger under the above description,  $d(\text{Spain}, \text{US})$  or  $d(\text{US}, \text{Spain})$ ?

### Solution

$d(\text{US}, \text{Spain})$  is bigger, since the US is larger in area than Spain.

### Exercise 2.55

Consider the SMP  $(\mathbb{R}_{\geq 0}, \geq, 0, +)$ . How would you characterise the difference between a Lawvere metric space and a  $(\mathbb{R}_{\geq 0}, \geq, 0, +)$ -category?

**Solution**

The latter is the same as a Lawvere metric space except without infinite distances.

**Exercise 2.58**

See book.

**Solution**

$d(\nearrow)$	$A$	$B$	$C$	$D$
$A$	0	6	3	11
$B$	2	0	5	5
$C$	5	3	0	8
$D$	11	9	6	0

**Exercise 2.60**

See book.

**Solution**

$\nearrow$	$A$	$B$	$C$	$D$
$A$	0	$\infty$	3	$\infty$
$B$	2	0	$\infty$	5
$C$	$\infty$	3	0	$\infty$
$D$	$\infty$	$\infty$	6	0

**Exercise 2.61**

Interpret what a **NMY**-category is (Exercise 2.34).

**Solution**

This is a bit like a Hasse diagram (where edges either exist or don't), except with the possibility of an edge being “maybe” present.

**Exercise 2.62, 2.63**

Discussed in person.

**Exercise 2.67**

Draw the Hasse diagram for the preorder corresponding to the regions US, Spain, and Boston with the “regions of the world” Lawvere metric space.

### Solution

This has a single nonidentity arrow,  $\text{Boston} \rightarrow \text{US}$ , and no others, basically telling us that Boston is in the US but there are no other such containment relations among the three regions.

### Exercise 2.68

1. Find another monoidal monotone  $g : \mathbf{Cost} \rightarrow \mathbf{Bool}$  different from the one defined in the Eq. 2.66.
2. Find a Lawvere metric space  $\mathcal{X}$  on which your monoidal monotone  $g$  and the monoidal monotone  $f$  given in Eq. 2.66 give different answers,  $\mathcal{X}_f \neq \mathcal{X}_g$ .

### Solution

Let

$$g(x) := \begin{cases} \text{false} & \text{if } x = \infty \\ \text{true} & \text{if } x < \infty \end{cases}$$

Then for the regions Lawvere metric space used above, the constructed preorder will be different. For  $\mathcal{X}_g$  every point is equivalent to every other point, as all regions are finite distances from one another, whereas there are points that are not connected for  $\mathcal{X}_f$ , e.g. Spain and US, as their distance is not zero.

### Exercise 2.73

1. Show that a skeletal dagger  $\mathbf{Cost}$ -category is an extended metric space.
2. Use Exercise 1.73 to make sense of the analogy “preorders are to sets as Lawvere metric spaces are to extended metric spaces.”

### Solution

Let  $\mathcal{X}$  be a skeletal dagger  $\mathbf{Cost}$ -category, and let  $x, y, z \in \mathcal{X}$ . We will define  $d(x, y) = \mathcal{X}(x, y)$ . From the definition of a  $\mathcal{V}$ -category we know that  $0 \geq \mathcal{X}(x, x) = d(x, x)$ , so we have property (a) from Definition 2.51. We also know that  $\mathcal{X}(x, y) + \mathcal{X}(y, z) \geq \mathcal{X}(x, z)$ , which gives us property (d).

From the fact that  $\mathcal{X}$  is dagger, we know that the identity function is a  $\mathcal{V}$ -functor from  $\mathcal{X}$  to  $\mathcal{X}^{\text{op}}$ . Hence for each  $x, y \in \mathcal{X}$ , we have  $\mathcal{X}(x, y) \geq \mathcal{X}(y, x)$  as well as  $\mathcal{X}(y, x) \geq \mathcal{X}(x, y)$ , which implies that  $\mathcal{X}(x, y) = \mathcal{X}(y, x)$ . This proves property (c) holds.

Finally, suppose that  $\mathcal{X}(x, y) = 0$ . This implies that  $0 \geq \mathcal{X}(x, y)$ , and from property (c)  $0 \geq \mathcal{X}(y, x)$ . Therefore due to the fact that  $\mathcal{X}$  is skeletal, we have  $x = y$ . This proves property (b). Hence  $\mathcal{X}$  is an extended metric space.

In Exercise 1.73 we showed that skeletal dagger preorders are sets, and analogously we now know that skeletal dagger Lawvere metric spaces are extended metric spaces.

### Exercise 2.75

Show that the  $\mathcal{V}$ -product of  $\mathcal{V}$ -categories is indeed a  $\mathcal{V}$ -category. Point out exactly where the symmetry condition is used.

#### Solution

Let  $\mathcal{X} \times \mathcal{Y}$  be the  $\mathcal{V}$ -product of  $\mathcal{V}$ -categories.

For  $(x, y) \in \text{Ob}(\mathcal{X} \times \mathcal{Y})$ , we have  $(\mathcal{X} \times \mathcal{Y})((x, y), (x, y)) = \mathcal{X}(x, x) \otimes \mathcal{Y}(y, y)$ . Since  $\mathcal{X}$  and  $\mathcal{Y}$  are  $\mathcal{V}$ -categories, we know  $I \leq \mathcal{X}(x, x)$  and  $I \leq \mathcal{Y}(y, y)$ , which implies that  $I \otimes I = I \leq \mathcal{X}(x, x) \otimes \mathcal{Y}(y, y)$ , again by the properties of the monoidal preorder  $\mathcal{V}$ . Therefore  $I \leq (\mathcal{X} \times \mathcal{Y})((x, y), (x, y))$ .

Next, let  $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \text{Ob}(\mathcal{X} \times \mathcal{Y})$ . Then

$$\begin{aligned} & (\mathcal{X} \times \mathcal{Y})((x_1, y_1), (x_2, y_2)) \otimes (\mathcal{X} \times \mathcal{Y})((x_2, y_2), (x_3, y_3)) \\ &= (\mathcal{X}(x_1, x_2) \otimes \mathcal{Y}(y_1, y_2)) \otimes (\mathcal{X}(x_2, x_3) \otimes \mathcal{Y}(y_2, y_3)) \\ &= (\mathcal{X}(x_1, x_2) \otimes \mathcal{X}(x_2, x_3)) \otimes (\mathcal{Y}(y_1, y_2) \otimes \mathcal{Y}(y_2, y_3)) \quad \text{by symmetry} \\ &\leq \mathcal{X}(x_1, x_3) \otimes \mathcal{Y}(y_1, y_3) \\ &= (\mathcal{X} \times \mathcal{Y})((x_1, y_1), (x_3, y_3)). \end{aligned}$$

(The inequality is from the definition of  $\mathcal{V}$ -category and the monoidal product in  $\mathcal{V}$ .) Therefore  $\mathcal{X} \times \mathcal{Y}$  is a  $\mathcal{V}$ -category.

### Exercise 2.78

Considering  $\mathbb{R}$  as a **Cost** category, form the **Cost**-product  $\mathbb{R} \times \mathbb{R}$ . What is the distance from  $(5, 6)$  to  $(-1, 4)$ ?

#### Solution

$$(\mathbb{R} \times \mathbb{R})((5, 6), (-1, 4)) = \mathbb{R}(5, -1) + \mathbb{R}(6, 4) = 6 + 2 = 8$$

### Exercise 2.82

Prove that a monoidal preorder  $(V, \leq, I, \otimes)$  is monoidal closed iff, given any  $v \in V$ , the map  $(- \otimes v) : V \rightarrow V$  given by multiplying with  $v$  has a right adjoint, written  $(v \multimap -) : V \rightarrow V$ .

#### Solution

Let  $w, w' \in V$ . Then as  $v \leq v$ , we know  $w \leq w' \Rightarrow w \otimes v \leq w' \otimes v$  by the definition of a monoidal preorder. So the map  $(- \otimes v)$  is monotone.

Next we suppose  $\mathcal{V}$  is closed. Then as  $(v \multimap w) \leq (v \multimap w)$ , by the definition of monoidal closure we have  $((v \multimap w) \otimes v) \leq w$ .



Finally if  $w \leq w'$ , then we want to show  $(v \multimap w) \leq (v \multimap w')$ . But from the previous step we have  $((v \multimap w) \otimes v) \leq w \leq w'$ , which again by definition of closure gives us  $(v \multimap w) \leq (v \multimap w')$ . Thus the map  $(v \multimap -)$  is monotone.

As the definition of a hom-element gives us the rest of the conditions of a Galois connection, this shows that a SMP is closed if and only if  $(- \otimes v)$  has a right adjoint, namely  $(v \multimap -)$ .

### Exercise 2.84

Show that **Bool** =  $(\mathbb{B}, \leq, \text{true}, \wedge)$  is monoidal closed.

### Solution

We show that  $\multimap$  is the  $\Rightarrow$  operator. Let  $a, v, w \in \mathbb{B}$ . Then the usual rules of logic tell us

$$a \wedge v \leq w \quad \text{iff} \quad a \leq (v \Rightarrow w).$$

### Exercise 2.92

1. What is  $\bigvee \emptyset$  in the case of (a) **Bool** and (b) **Cost**?
2. What is  $x \vee y$  in the same cases?

### Solution

1. In **Bool**,  $\bigvee \emptyset = \text{false}$ . In **Cost**, it is  $\infty$ .
2. In **Bool**,  $x \vee y$  is logical-or. In **Cost**, it is  $\min$ .

### Exercise 2.93

Show that **Bool** =  $(\mathbb{B}, \leq, \text{true}, \wedge)$  is a quantale.

### Solution

We've already showed that **Bool** is closed. As the join is logical-or, we can clearly “or” together as many booleans as we like and get a boolean value, hence **Bool** is a quantale.

### Exercise 2.94

Let  $S$  be a set. Is the power set monoidal preorder  $(P(S), \subseteq, S, \cap)$  a quantale?

### Solution

$P(S)$  clearly contains all joins, i.e. set unions, so to show that it is a quantale we will show that it is monoidal closed. We define the hom-element to be  $A \multimap B := \overline{B} \cup C$  (where  $\overline{B}$  is the complement of  $B$  in  $S$ ).

Then for any  $A, B, C \subseteq S$ , we have  $A \cap B \subseteq C \Rightarrow A \subseteq \overline{B} \cup C$ , as  $A$  consists of the part of  $A$  not in  $B$ , i.e. within  $\overline{B}$ , and the part of  $A$  within  $B$ , which we've assumed is a subset of  $C$ .

Finally if  $A \subseteq \overline{B} \cup C$ , then

$$A \cap B \subseteq (\overline{B} \cup C) \cap B = (\overline{B} \cap B) \cup (C \cap B) = B \cap C \subseteq C.$$

Thus we have shown  $A \cap B \subseteq C \Leftrightarrow A \subseteq \overline{B} \cup C$ , which proves that  $(P(S), \subseteq, S, \cap)$  is a quantale.

### Exercise 2.103

Write down the  $2 \times 2$  -identity matrix for each of the quantales  $(\mathbb{N}, \leq, 1, *)$ , **Bool** =  $(\mathbb{B}, \leq, \text{true}, \wedge)$ , and **Cost** =  $([0, \infty], \geq, 0, +)$ .

### Solution

(Note the first example only has finite joins, which is okay for matrix multiplication.) For  $\mathbb{N}$  the standard identity matrix works.

For **Bool**, the matrix with true on the diagonal and false everywhere else is the identity.

For **Cost**, the matrix with 0 on the diagonal and  $\infty$  on the off diagonals is the identity.

### Exercise 2.104

Let  $\mathcal{V} = (V, \leq, I, \otimes, \multimap)$  be a quantale. Prove the identity and associative laws for  $\mathcal{V}$ -matrix multiplication.

### Solution

Let  $A$  and  $B$  be sets and  $M : A \times B \rightarrow V$  a  $\mathcal{V}$ -matrix. Then

$$(I_A * M)(a, b) = \bigvee_{a' \in A} I_A(a, a') \otimes M(a', b).$$

These entries are nonzero only when  $a = a'$  by definition of  $I_A$ , so this gives us

$$\begin{aligned} &= \left[ \bigvee_{a' \neq a} I_A(a, a') \otimes M(a', b) \right] \vee I_A(a, a) \otimes M(a, b) \\ &= \left[ \bigvee_{a' \neq a} \left( \bigvee \emptyset \otimes M(a', b) \right) \right] \vee I \otimes M(a, b) \end{aligned}$$

$$= \left[ \bigvee_{a' \neq a} \left( \bigvee_{x \in \emptyset} (x \otimes M(a', b)) \right) \right] \vee M(a, b)$$

by application of Proposition 2.87. As the join over the empty set is 0, we have

$$= \left[ \bigvee_{a' \neq a} 0 \right] \vee M(a, b) = M(a, b).$$

This proves  $I_A * M = M$ .

Next, let  $M : A \times B \rightarrow V$ ,  $N : B \times C \rightarrow V$ , and  $P : C \times D \rightarrow V$ . Then

$$\begin{aligned} ((M * N) * P)(a, d) &= \left[ \bigvee_{b \in B} M(a, b) \otimes N(b, c) \right] * P \\ &= \bigvee_{c \in C} \left( \left[ \bigvee_{b \in B} M(a, b) \otimes N(b, c) \right] \otimes P(c, d) \right) \\ &= \bigvee_{c \in C} \bigvee_{b \in B} (M(a, b) \otimes N(b, c)) \otimes P(c, d), \end{aligned}$$

again using Proposition 2.87. By the associativity of the monoidal product, this gives us

$$= \bigvee_{c \in C} \bigvee_{b \in B} M(a, b) \otimes (N(b, c) \otimes P(c, d)).$$

It is clear that we can switch the order of the joins, giving us

$$\begin{aligned} &= \bigvee_{b \in B} \bigvee_{c \in C} M(a, b) \otimes (N(b, c) \otimes P(c, d)) \\ &= \bigvee_{b \in B} M(a, b) \otimes \left[ \bigvee_{c \in C} N(b, c) \otimes P(c, d) \right] \\ &= M * \left[ \bigvee_{c \in C} N(b, c) \otimes P(c, d) \right] \\ &= (M * (N * P))(a, d). \end{aligned}$$

Therefore  $(M * N) * P = M * (N * P)$ .