Applied Category Theory Problems

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October 4, 2020

Chapter 1

Generative Effects

Exercise 1.1

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be

- *order-preserving* if $x \leq y$ implies $f(x) \leq f(y)$, for all $x, y \in \mathbb{R}$
- *metric-preserving* if $\mid x - y \mid = \mid f(x) - f(y) \mid$
- *addition-preserving* if $f(x + y) = f(x) + f(y)$

For each of the three properties defined above—call it *foo*—find an f that is *foo*-preserving and an example of an f that is not *foo*-preserving.

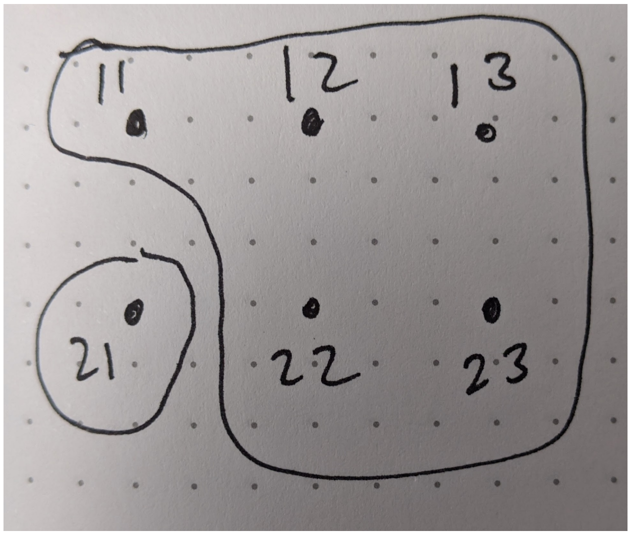
Solution

$f(x) = x$ is order-, metric-, and addition-preserving. $f(x) = x^2$ is none of these.

Exercise 1.4

See book.

Solution



Exercise 1.6, 1.7, 1.10

Discussed in person

Exercise 1.11

Let $A = \{h, 1\}$ and $B = \{1, 2, 3\}$.

Solution

1. The subsets of B are $\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$.
2. For example, $\{1, 2\} \cup \{2\} = \{1, 2\}$.
3. $A \times B = \{(h, 1), (h, 2), (h, 3), (1, 1), (1, 2), (1, 3)\}$
4. $A \sqcup B = \{h_A, 1_A, 1_B, 2_B, 3_B\}$
5. $A \cup B = \{h, 1, 2, 3, 4\}$

Exercise 1.16

Suppose that A is a set and $\{A_p\}_{p \in P}$ and $\{A'_{p'}\}_{p' \in P'}$ are two partitions of A such that for each $p \in P$ there exists a $p' \in P'$ with $A_p = A'_{p'}$.

1. Show that for each $p \in P$ there is at most one $p' \in P'$ such that $A_p = A'_{p'}$.
2. Show that for each $p' \in P'$ there is a $p \in P$ such that $A_p = A'_{p'}$.

Solution

1. If there are distinct $p_1', p_2' \in P'$ such that $A_p = A'_{p_1'}$ and $A_p = A'_{p_2'}$, then by transitivity $A'_{p_1'} = A'_{p_2'}$ which means that $p_1' = p_2'$ as P' is a partition.

2. Let $a \in A'_{p_1'}$. Then we must have that $a \in A_p$ for some p. We will show that this A_p is the desired one. There must exist p_2' such that $A_p = A'_{p_2'}$. So $a \in A'_{p_2'}$ and $a \in A'_{p_1'}$. So $A'_{p_1'} = A'_{p_2'} = A_p$ as we wanted.

Exercise 1.17

See book.

Solution

$(11, 11), (12, 12), (13, 13), (21, 21), (22, 22), (23, 23), (11, 12), (12, 11), (22, 23), (23, 22)$

Exercise 1.20

Suppose that $\sim$ is an equivalence relation on a set A, and let P be the set of $(\sim)$-closed and $(\sim)$-connected subsets $\{A_p\}_{p \in P}$.

1. Show that each part A_p is nonempty.

2. Show that if $p \neq q$ i.e. $A_p \neq A_q$, then $A_p \cap A_q = \varnothing$.

3. Show that $A = \bigcup_{p \in P} A_p$.

Solution

1. Since each A_p is $(\sim)$-connected, by definition A_p must be nonempty.

2. Suppose $a \in A_p \cap A_q$. Then for any $b \in A_p$, we know $b \sim a$, hence since A_q is $(\sim)$-closed we know $b \in A_q$. Similarly for any $b \in A_q$. Hence $A_p = A_q$.

3. Clearly $\bigcup A_p \subseteq A$. Suppose $a \in A$. Then let $X = \{b \mid a \sim b, b \in A\}$. X is closed as the equivalence relation is reflexive and connected as it is transitive, so X a set in $\{A_p\}$ and $A \subseteq \bigcup A_p$.

Exercise 1.24

Discussed in person

Exercise 1.25

Suppose that A is a set and $f : A \to \varnothing$ is a function to the empty set. Show that A is empty.

Solution

Suppose there is some $a \in A$. Then we must have $f(a) \in \varnothing$ which is clearly not possible.

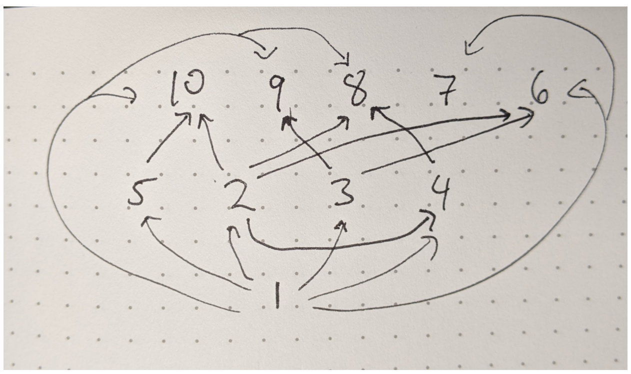
Exercise 1.27, 1.38, 1.40, 1.41, 1.42, 1.44

Discussed in person

Exercise 1.46

Write down the numbers $1, 2, \ldots, 10$ and draw an arrow $a \to b$ if a divides perfectly into b. Is it a total order?

Solution



This isn't a total order, as for example we neither have $2 \mid 7$ or $7 \mid 2$.

Exercise 1.48

Is the usual $\leq$ ordering on the set $\mathbb{R}$ of real numbers a total order?

Solution

Yes: for any $x, y \in \mathbb{R}$, we have $x \leq y$ or $y \leq x$.

Exercise 1.51

Discussed in person

Exercise 1.53

For any set S there is a coarsest partition, having just one part. What surjective function does it correspond to? There is also a finest partition, where everything is in its own partition. What surjective function does it correspond to?

Solution

Let $f : S \to \{\bullet\}$ be the unique function that sends every element of S to $\bullet$. This is a surjection corresponding to the coarsest partition, i.e. where every element is in the same set.

Let $f : S \to S$ be the identity function. This is a surjection corresponding to the finest partition, i.e. where every element is in its own set.

Exercise 1.55

Prove that the preorder of upper sets on a discrete preorder on a set X is simply the power set $P(X)$.

Solution

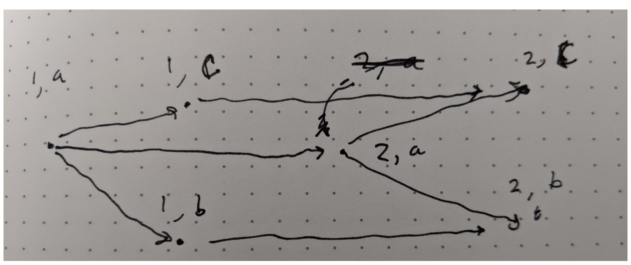
Clearly the set of upper sets $U(X)$ is a subset of the power set $P(X)$.

Let $Y \subseteq X$. We know $\varnothing$ is an upper set, so let $y \in Y$. Then since the preorder is discrete, the only element in X greater than y is y itself, which is in Y. This holds for any $y \in Y$ so Y is an upper set. Note that the ordering on both $U(X)$ and $P(X)$ is the same, i.e. $\subseteq$.

Exercise 1.57

See book.

Solution



Exercise 1.63

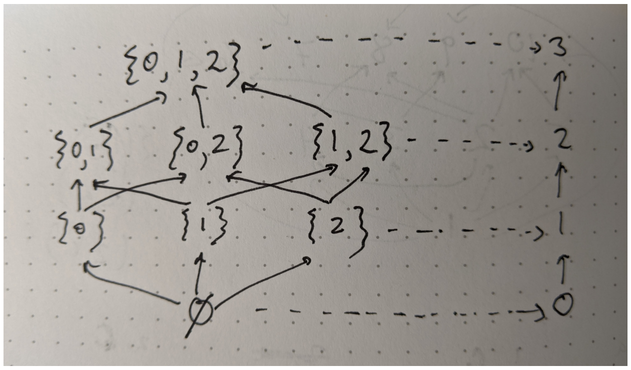
Let $X = \{0, 1, 2\}$.

1. Draw the Hasse diagram for $P(X)$.

2. Draw the Hasse diagram for the preorder $0 \leq 1 \leq 2 \leq 3$.

3. Draw the cardinality map $|\cdot|$ as dashed lines between them

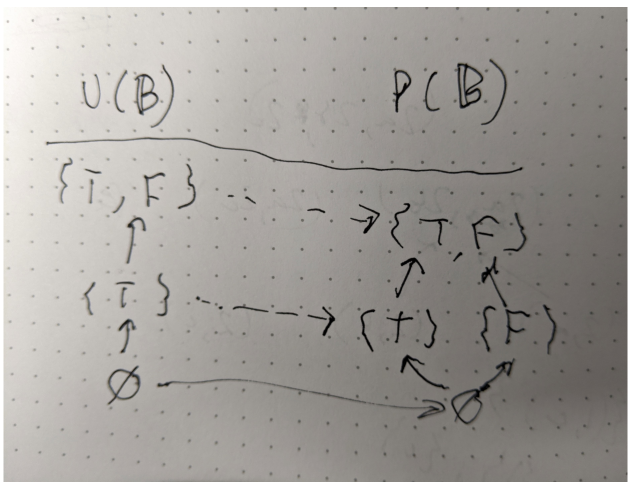
Solution



Exercise 1.65

Draw the monotone map between $U(\mathbb{B})$ and $P(\mathbb{B})$ as described in the text.

Solution



Exercise 1.66

Let $(P, \leq)$ be a preorder.

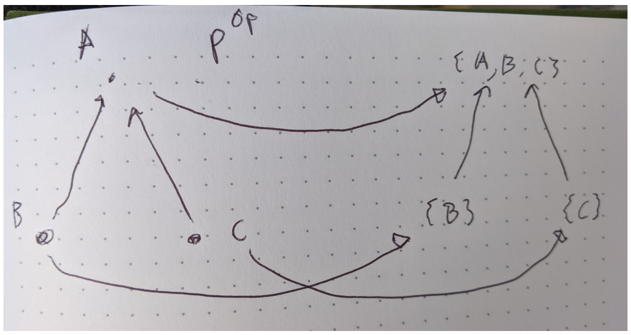
1. Show that the set $\uparrow p = \{p' \in P \mid p \leq p;\ \}$ is an upper set for any $p \in P$.
2. Show that this defines a monotone map $\uparrow\colon P^{op} \to U(P)$.
3. Show that $p \leq p'$ iff $\uparrow(p') \subseteq \uparrow(p)$.
4. Draw a picture of the map $\uparrow$ in the case where P is the preorder $(b \geq a \leq c)$.

Solution

1. Suppose $q \in\uparrow p$, then any $q' \geq q$ is transitively greater than p and hence $q' \in\uparrow p$.

2. Suppose $p \geq q$ (i.e. p is less than q in P^{op}), we want to show that $\uparrow p \subseteq\uparrow q$. So let $p' \in\uparrow p$. We know $q \leq p \leq p'$ and hence $p' \in\uparrow q$.

3. We showed the first direction in part 2, so assume $\uparrow (p') \subseteq\uparrow (p)$. This means $p \in\uparrow (p')$ and hence $p \leq p'$.

4. 

Exercise 1.67

Show that when $(P, \leq_P)$ is a discrete preorder, then every function $f : P \to Q$ is monotone regardless of the order $\leq_Q$.

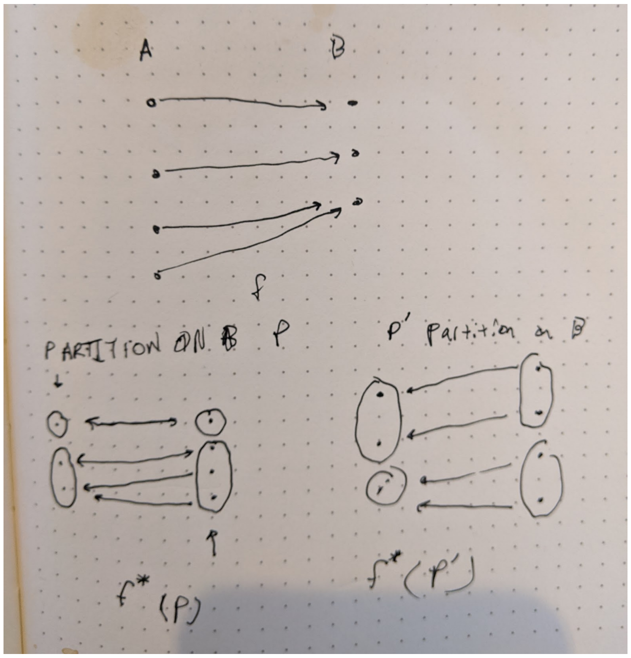
Solution

We need to show that for any $x, y \in P$ where $x \leq_P y$, we have $f(x) \leq_Q f(y)$. But the only x and y satisfying this are $x \leq_P x$, for which we have $f(x) \leq_Q f(x)$ regardless of $\leq_Q$ by the definition of a preorder.

Exercise 1.69

Choose two sets X and Y with at least three elements each and choose a surjective, non-identity function $f : X \to Y$. Write down two different partitions P and Q of Y, and find $f^*(P)$ and $f^*(Q)$.

Solution



Exercise 1.71

Prove Proposition 1.70:

1. For any preorder $(P, \leq_P)$, the identity function is monotone.
2. If $(Q, \leq_Q)$ and $(R, \leq_R)$ are preorders and $f : P \to Q$ and $g : Q \to R$ are monotone, then $(f \mathbin{⨟} g) : P \to R$ is also monotone.

Solution

1. If $a \leq_P b$ then clearly $a = f(a) \leq_P f(b) = b$ if f is the identity function.
2. Suppose $a \leq_P b$, then $f(a) \leq_Q f(b)$ as f is monotone, and hence $g(f(a)) \leq_R g(f(b))$ as g is also monotone.

Exercise 1.73

Show that a skeletal dagger preorder is just a discrete preorder, and hence can be identified with a set.

Solution

Let $(P, \leq)$ be a skeletal dagger preorder. We need to show that for any $x \in P$, the only thing comparable to x is x itself. So suppose $x \leq y$, then as P is a dagger preorder we know that $y \leq x$. Hence as P is skeletal, we have that $x = y$. This implies that P is a discrete preorder.

Exercise 1.77

Show that the map Φ from Section 1.1.1 ('Is $\bullet$ connected to $\star$?') is the monotone map $Prt(\{\star, \bullet, \circ\}) \to \mathbb{B}$.

Solution

Let P and P' be partitions where $P \leq P'$. If $\Phi(P) = \texttt{false}$ then clearly $\Phi(P) \leq \Phi(P')$, so assume $\Phi(P) = \texttt{true}$. This means for some set X in the partition P, we know that both $\bullet, \star \in X$. As $P \leq P'$ this means there is some Y in the partition P' with $X \subseteq Y$, which implies that $\bullet, \star \in Y$. Hence $\Phi(P') = \texttt{true}$ and $\Phi(P) \leq \Phi(P')$.

Exercise 1.79

Let P and Q be preorders and $f : P \to Q$ a monotone map. Show that the pullback $f^* : U(Q) \to U(P)$ can be defined by taking $u : Q \to \mathbb{B}$ to $(f \mathbin{⨾} u) : P \to \mathbb{B}$.

Solution

Call ϕ_Q the function that takes upper sets in Q to monotone maps as defined in Proposition 1.78, and similarly ϕ_P. Let $U \in U(Q)$. We want to show $\phi_P(f^{-1}(U)) = f \mathbin{⨾} (\phi_Q(U))$.

Let $x \in P$. If $x \in f^{-1}(U)$, then we know $\phi_P(f^{-1}(U))(x) = \texttt{true}$ by definition. But we also know $f(x) \in U$ and hence $\phi_Q(U)(f(x)) = \texttt{true}$. Conversely if $x \notin f^{-1}(U)$, we will have both $\phi_P(f^{-1}(U))(x) = \texttt{false}$, as well as $f(x) \notin U$ and $\phi_Q(U)(f(x)) = \texttt{false}$. This shows that these maps are equal.

Exercise 1.80

Why is 0 a greatest lower bound for $\left\{\frac{1}{n+1} \mid n \in \mathbb{N}\right\} \subseteq \mathbb{R}$?

Solution

Assume that $\varepsilon > 0$ is a lower bound. Let $n = \lceil 1/\varepsilon \rceil$. Then

$$\tfrac{1}{n+1} \leq \tfrac{1}{1/\varepsilon + 1} \leq \tfrac{1}{1/\varepsilon} = \varepsilon.$$

Hence no such ε is a lower bound.

Exercise 1.85

Let $(P, \leq)$ be a preorder and $p \in P$, consider the set $A = \{p\}$.

1. Show that $\bigwedge A \cong p$.
2. Show that if P is a partial order, then $\bigwedge A = p$.
3. Are the analogous facts true when $\bigwedge$ is replaced by $\bigvee$?

Solution

1. Clearly $p \leq p$, so by definition $\bigwedge A \leq p$ (as a lower bound) and $\bigwedge A \geq p$ (as a greatest lower bound).
2. If the previous is true in a partial order, then we have $\bigwedge A = p$.
3. The analogous facts are true with $\bigvee$.

Exercise 1.90

In the $n \mid m$ ordering on $\mathbb{N}$, what are the meet and the join?

Solution

The meet is the greatest common divisor and the join is the least common multiple.

Exercise 1.94

Prove that for any monotone map $f : P \to Q$, if $a, b \in P$ have a join and $f(a), f(b) \in Q$ have a join, then $f(a) \vee f(b) \leq f(a \vee b)$.

Solution

We know $a, b \leq a \vee b$, so since f is monotone we have $f(a), f(b) \leq f(a \vee b)$. Hence $f(a \vee b)$ is an upper bound for $\{f(a), f(b)\}$, so by definition of the join we must have $f(a) \vee f(b) \leq f(a \vee b)$.

Exercise 1.98

Find a right adjoint for the monotone map $(3 \times -) : \mathbb{Z} \to \mathbb{R}$, and show it is correct.

Solution

Let $g(y) = \lfloor y/3 \rfloor$. Then we have $3x \leq y \Leftrightarrow x \leq y/3 \Leftrightarrow x \leq \lfloor y/3 \rfloor$, hence g is a right adjoint for $3x$.

Exercise 1.99

See book.

Solution

1. In this case f is left adjoint to g.
2. In this case f is not left adjoint to g, as $g(1) = 2 \geq 2$ but $f(2) = 2 \not\leq 1$.

Exercise 1.101

1. Does $\lceil -/3 \rceil$ have a left adjoint $L : \mathbb{Z} \to \mathbb{R}$?
2. If not, why? If so, does its left adjoint have a left adjoint?

Solution

Let $g : \mathbb{R} \to \mathbb{Z}$ be defined by $g(x) = \lceil x/3 \rceil$. We will show by contradiction that g does not have a left adjoint.

For a left adjoint $f : \mathbb{Z} \to \mathbb{R}$, we must have $f(1) \leq 0 \Leftrightarrow 1 \leq g(0) = \lceil 0/3 \rceil = 0$. Clearly the second part does not hold, so we know $f(1) \not\leq 0$.

On the other hand, we know that $f(1) \leq \inf A$ where $A = \{x \mid g(x) \geq 1, x \in \mathbb{R}\}$. However $1/n \in A$ for $n \in \mathbb{Z}^+$, as $\lceil 1/n \rceil = 1$ for all such n. But $\inf\{1/n \mid n \in \mathbb{Z}^+\} = 0$, which implies that $f(1) \leq 0$. This is a contradiction, so g must not have a left adjoint.

Exercise 1.103

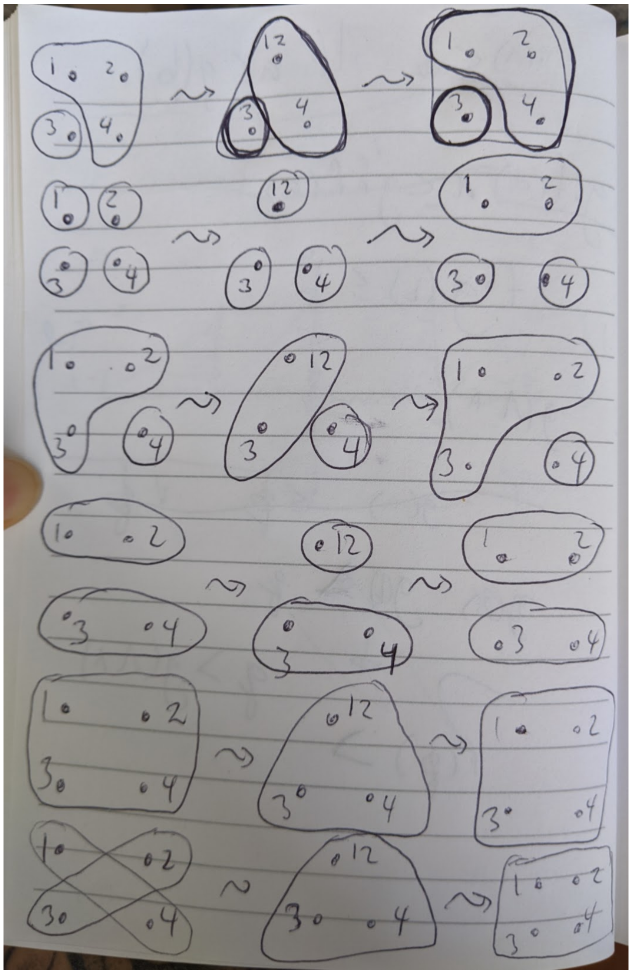
Choose 6 different partitions on the set S and for each, call it c, find $g_!(c)$ where S, T, and $g : S \to T$ are the same as they were in Example 1.102.

Exercise 1.105

Using the same S, T, and $g : S \to T$ as in Example 1.102, find the partition $g^*(c)$ for each of the 5 partitions c on the set T.

Solution

For both 1.103 and 1.105.



Exercise 1.106 (revised)

Prove that $g_!$ is left adjoint to g^*, as defined in the text.

Solution

Let S, T be sets, and let $g : S \to T$. Define $g_!, g^*$ as in the text.

We first show g^* is monotone. Let $A, B \in Prt(T)$ such that $A \leq B$. Then for each set $A_i \in A$, $A_i \subseteq B_j$ for some $B_j \in B$, and as a result $g^{-1}(A_i) \subseteq g^{-1}(B_j)$ for each $A_i \in A$. As the image of a partition under g^* is the collection of preimages of that partition via g, we have $g^*(A) \leq g^*(B)$.

Next we show $g_!$ is monotone. Let $A, B \in Prt(S)$ such that $A \leq B$. As before we know that for each set $A_i \in A$, $A_i \subseteq B_j$ for some $B_j \in B$. We consider $A, B \in Rel(S)$ i.e. subsets of $S \times S$. Note that $g_!(C)$ is the transitive closure of the relation $\{(g(x), g(y) \mid (x, y) \in C\}$. As $A \leq B$, $\{(g(x), g(y) \mid (x, y) \in A\} \subseteq \{(g(x), g(y) \mid (x, y) \in B\}$. Using the fact that the function taking a relation to its transitive closure is monotone on the set of relations ordered by inclusion, we can conclude that $g_!(A) \leq g_!(B)$.

For the next part of the proof, we use proposition 1.107, and derive our result by showing that for each $A \in Prt(S)$ and for each $B \in Prt(T)$ that both $A \leq g^* \circ g_!(A)$, and $g_! \circ g^*(B) \leq B$.

We start with showing $A \leq g^* \circ g_!(A)$. First we consider two additional functions, first $\bar{g} : Prt(S) \to Rel(T)$, where $g(r) = \{(g(x), g(y)) \mid (x, y) \in r\}$. Secondly an extension of g^* to all relations, $\bar{g^*} : Rel(T) \to Rel(S)$, so for a relation r, we have $\bar{g^*}(r) = \{(x, y) \mid (g(x), g(y)) \in r\}$ (g^* is the restriction of $\bar{g^*}$ to equivalence relations). Both $\bar{g}$ and $\bar{g^*}$ are monotone, which can be seen in proofs similar to our proofs for g^* and $g_!$. Additionally let the transitive closure of a set Q be denoted $\hat{Q}$. We now note two things, one $A = \bar{g^*} \circ \bar{g}(A)$, and two $g^* \circ g_!(A) = \bar{g^*}(\widehat{\bar{g}(A)})$. As $\bar{g^*}$ is monotone and $\bar{g}(A) \leq \widehat{\bar{g}(A)}$ we have $A \leq g^* \circ g_!(A)$.

Exercise 1.109

Complete the proof of Proposition 1.107 by showing that (for monotone $f : P \to Q$ and $g : Q \to P$)

1. if f is left adjoint to g then for any $q \in Q$ we have $f(g(q)) \leq q$, and
2. if $p \leq g(f(p))$ and $f(g(q)) \leq q$, then $p \leq g(p)$ iff $f(p) \leq q$ holds, for all $p \in P$ and $q \in Q$.

Solution

Assume f is left adjoint to g. Let $q \in Q$ and $p = g(q)$. Then we know $p \leq g(q)$, so by definition of the left adjoint $f(p) \leq q$. As we defined p to be $g(q)$ this implies $f(g(q)) \leq q$.

Next assume $p \leq g(f(p))$ and $f(g(q)) \leq q$ for any $p \in P, q \in Q$. We need to show that $p \leq g(q)$ implies $f(p) \leq q$. But $p \leq g(q)$ implies $f(p) \leq f(g(q))$ by the monotonicity of f, and $f(g(q)) \leq q$ by assumption, so $f(p) \leq q$.

Exercise 1.110

1. Show that if $f : P \to Q$ has a right adjoint g, then it is unique up to isomorphism. That is, for any other right adjoint g', we have $g(q) \cong g'(q)$ for all $q \in Q$.
2. Is the same true for left adjoints? That is, if $h : P \to Q$ has a left adjoint, is it necessarily unique up to isomorphism?

Solution

1. Suppose g and g' are right adjoint to $f : P \to Q$. Then for any $q \in Q, p \in P$ we have

   $$f(p) \leq q \Leftrightarrow p \leq g(q) \Leftrightarrow p \leq g'(q).$$

   In particular this holds for $p = g(q)$, which means $g(q) \leq g'(q)$ as by reflexivity $g(q) \leq g(q)$. Similarly for $p = g'(q)$, we have $g'(q) \leq g(q)$ as $g'(q) \leq g'(q)$. Thus $g(q) \cong g'(q)$ for all $q \in Q$.

2. The same holds for left adjoints. To show this, suppose f and f' are left adjoint to $g : Q \to P$. Then for any $p \in P, q \in Q$ we have

$$f(p) \leq q \Leftrightarrow p \leq g(q) \Leftrightarrow f'(p) \leq q.$$

The rest of the proof follows analogously to part 1.

Exercise 1.112

Complete the proof of Proposition 1.111 by showing that left adjoints preserve joins.

Solution

Let $f : P \to Q$ and $g : Q \to P$ be monotone maps with f left adjoint to g. Let $A \subseteq P$ and let $j = \bigvee A$ be its join. Then since f is monotone $f(a) \leq f(j)$ for all $a \in A$, so $f(j)$ is an upper bound for $f(A)$.

Next we will show that $f(j)$ is a least upper bound. Suppose b is some other upper bound for $f(A)$. So for any $a \in A$, we know $f(a) \leq b$ which implies $a \leq g(b)$ by the fact that f is left adjoint. Hence $g(b)$ is an upper bound for A, and by definition of the join we know $j \leq g(b)$, which implies $f(j) \leq b$, again by definition of a Galois connection. Therefore $f(j)$ is the join of $f(A)$.

Exercise 1.114

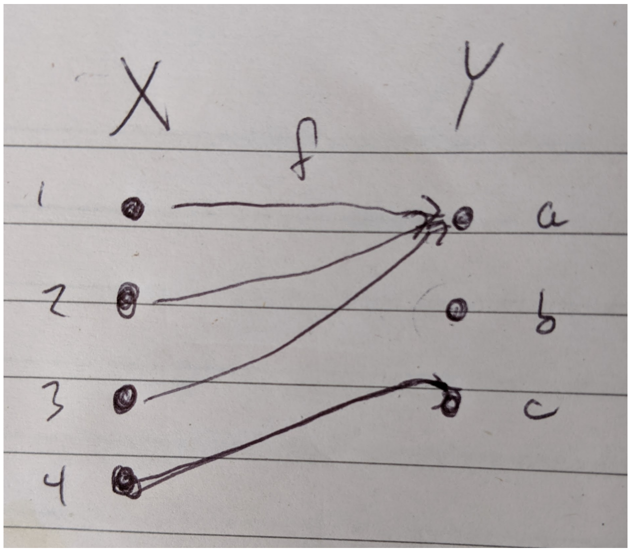
Discussed in person.

Exercise 1.118

Choose sets X and Y with 2-4 elements each, and a function $f : X \to Y$.

1. Choose $B_1, B_2 \subseteq Y$ and find $f^*(B_1)$ and $f^*(B_2)$.
2. Choose $A_1, A_2 \subseteq A$v and find $f_!(A_1)$ and $f_!(A_2)$.
3. Find $f_*(A_1)$ and $f_*(A_2)$.

Solution



1. $f^*(\{a, b\}) = \{1, 2, 3\}$ and $f^*(\{c\}) = \{4\}$
2. $f_!(\{1, 4\}) = \{a, c\}$ and $f_!(\{2, 3\}) = \{a\}$
3. $f_*(\{1, 4\}) = \{b\}$ and $f_*(\{2, 3\}) = \{a, b\}$

Exercise 1.119

Suppose f is left adjoint to g. Show that

1. $p \leq (f \fatsemi g)(p)$
2. $(f \fatsemi g \fatsemi f \fatsemi g)(p) \cong (f \fatsemi g)(p)$

Solution

1. From Proposition 1.107 we immediately have that $p \leq (f \fatsemi g)(p)$ from the fact that f is left adjoint to g.
2. If we apply Proposition 1.107 to $(f \fatsemi g)(p) \in P$, this gives us $(f \fatsemi g)(p) \leq (f \fatsemi g)((f \fatsemi g)(p))$. If we apply the other part to $f(p) \in Q$, this gives us $(g \fatsemi f)(f(p)) \leq f(p)$, and because g is monotone this implies that $(f \fatsemi g \fatsemi f \fatsemi g)(p) \leq (f \fatsemi g)(p)$. Hence $(f \fatsemi g \fatsemi f \fatsemi g)(p) \cong (f \fatsemi g)(p)$.

Exercise 1.124, 1.125

Discussed in person.

Chapter 2

Resource Theories

Exercise 2.5

Is $(\mathbb{R}, \leq, 1, *)$ a symmetric monoidal preorder?

Solution

It is not a monoidal preorder, as for example $-5 \leq -1$ and $-2 \leq -1$, but $-5*-2 \not\leq -1*-1$.

Exercise 2.8

Check that if $(M, *, e)$ is a commutative monoid then $(\mathbf{Disc}_M, =, *, e)$ is a symmetric monoidal preorder.

Solution

Unitality, associativity, and symmetry come for free from the definition of a commutative monoid, so we only need to check monotonicity. Since $x \leq y \Leftrightarrow x = y$ this is easy to check, as $x_1 \leq y_1$ and $x_2 \leq y_2$ implies $x_1 = y_1$ and $x_2 = y_2$ which in turn implies that $x_1 * x_2 = y_1 * y_2$.

Exercise 2.20

Formally prove that $t \leq v+w, w+u \leq x+z, v+x \leq y$ implies $t+u \leq y+z$. Be explicit about where reflexivity and transitivity are used, and why symmetry need not be used.

Solution

In the below, R=reflexivity, M=monotonicity, A=associativity, T=transitivity. Symmetry is not used, which you can tell from the diagram from the fact that no wires cross.

$$\begin{array}{rclr}
t \leq v+w, u \leq u & \Rightarrow & t+u \leq (v+w)+u = v+(w+u) & [\text{R, M, A, T}] \quad (2.1) \\
(2.1), w+u \leq x+z & \Rightarrow & v+(w+u) \leq v+(x+z) & [\text{M}] \quad (2.2) \\
(2.1, 2.2) & \Rightarrow & t+u \leq v+(x+z) = (v+x)+z & [\text{T, A}] \quad (2.3) \\
v+x \leq y, z \leq z & \Rightarrow & (v+x)+z \leq y+z & [\text{M}] \quad (2.4) \\
(2.3, 2.4) & \Rightarrow & t+u \leq y+z & [\text{T}] \quad (2.5)
\end{array}$$

Exercise 2.21

Skipped.

Exercise 2.29

Consider $(\mathbb{B}, \leq)$ with monoidal product $\vee$. What's the monoidal unit? Does it satisfy the rest of the conditions?

Solution

The monoidal unit should be `false`. Discussed why in person (i.e. truth table)

Exercise 2.31

Show that there is a monoidal structure on $(\mathbb{N}, \leq)$ where the monoidal product is standard $*$. What should the monoidal unit be?

Solution

The monoidal unit should be 1. We will show monotonicity as the other conditions are obvious. If $x_1 \leq y_1$ and $x_2 \leq y_2$, then there are $a_1, a_2 \in \mathbb{N}$ such that $y_1 = x_1 + a_1$ and $y_2 = x_2 + a_2$. Then

$$y_1 * y_2 = (x_1 + a_1) * (x_2 + a_2) = x_1 * x_2 + a_1 * x_2 + a_2 * x_1 + a_1 * a_2 \geq x_1 * x_2.$$

Exercise 2.33

Consider the divisibility order $(\mathbb{N}, |)$. Does 0 as monoidal unit and $+$ as monoidal product satisfy the conditions?

Solution

It does not, as for example $2 \mid 4$ and $1 \mid 1$, but $(2+1) \nmid (4+1)$, so monotonicity fails.

Exercise 2.34

Consider the preorder **NMY** with Hasse diagram $\texttt{no} \to \texttt{maybe} \to \texttt{yes}$, monoidal unit $\texttt{yes}$ and "min" as the monoidal product. Define what "min" should be and check that the axioms hold.

Solution

min	no	maybe	yes
no	no	no	no
maybe	no	no	maybe
yes	no	maybe	yes

Exercise 2.35

Let S be a set and let $P(S)$ be its power set, with the subset relation as order. Does $P(S)$ with unit S and product given by set intersection satisfy the conditions of symmetric monoidal preorder?

Solution

The other conditions are easy to show, so we will show monotonicity only. Let $A_1, A_2, B_1, B_2 \subseteq S$ where $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$. Then $A_1 \cap A_2 \subseteq A_1 \subseteq B_1$ and $A_1 \cap A_2 \subseteq A_2 \subseteq B_2$, which implies that $A_1 \cap A_2 \subseteq B_1 \cap B_2$.

Exercise 2.36

Let $\texttt{Prop}^{\mathbb{N}}$ denote the set of all mathematical statements one can make about a natural number, where we consider two statements to be the same if one is true if and only if the other is true. Given $P, Q \in \texttt{Prop}^{\mathbb{N}}$, we say $P \leq Q$ if for all $n \in \mathbb{N}$, whenever $P(n)$ is true, so is $Q(n)$. Define a monoidal unit and product on $\texttt{Prop}^{\mathbb{N}}$.

Solution

We define the monoidal unit to be the statement "n is a natural number" (i.e. a statement that's always true) and the monoidal product to be logical AND. Note that this SMP effectively reduces to the previous example, where we consider subsets $P = \{n \in \mathbb{N} \mid P(n)\}$.

Exercise 2.39

Complete the proof of Proposition 2.38.

Solution

These conditions are inherited from the original SMP, so we have decided this problem is dumb.

Exercise 2.40

What is $\mathbf{Cost}^{\mathrm{op}}$ as a preorder? What is the monoidal unit and product?

Solution

$\mathbf{Cost}^{\mathrm{op}}$ is the same as $\mathbf{Cost}$ but using $\leq$ rather than $\geq$. From Proposition 2.38, we know that $\mathbf{Cost}^{\mathrm{op}}$ can use the same monoidal unit and product as the original, i.e. 0 and $+$.

Exercise 2.43

Check that the map $g : (\mathbb{B}, \leq, \text{true}, \wedge) \to ([0, \infty], \geq, 0, +)$ with $g(\text{false}) = \infty$ and $g(\text{true}) = 0$ is monoidal monotone. Is g strict?

Solution

Clearly $g(\text{false}) \leq g(\text{true})$ as $\infty \geq 0$, so g is monotone. In addition $g(\text{true}) = 0$, so the function preserves identities exactly. Finally note that $g(\text{false}) + g(\text{false}) = \infty + \infty = \infty = g(\text{false}) = g(\text{false} \wedge \text{false})$, the other products are trivial as we've shown the identity is preserved. Hence g is strict monoidal monotone.

Exercise 2.44

Let **Bool** and **Cost** be as above, and consider $d, u : [0, \infty] \to \mathbb{B}$ as follows:

$$d(x) := \begin{cases} \text{false} & \text{if } x > 0 \\ \text{true} & \text{if } x = 0 \end{cases} \qquad u(x) := \begin{cases} \text{false} & \text{if } x = \infty \\ \text{true} & \text{if } x < \infty \end{cases}$$

Is d monotonic, monoidal, and/or strict? Is u?

Solution

Both d and u are monotonic. They also both map the unit in **Cost**, namely 0, exactly to the unit in **Bool**, true.

Finally, they both strictly preserve the monoidal product, which can be shown by a truth table. In short, for d, anything added to a number $x > 0$ continues to be greater than 0, while $0 + 0 = 0$; for u, anything added to ∞ is still ∞, while the sum of two finite numbers is still finite.

Exercise 2.45

1. Is $(\mathbb{N}, \leq, 1, *)$ a monoidal preorder?
2. If not, why not? If so, does there exist a monoidal monotone $(\mathbb{N}, \leq, 0, +) \to (\mathbb{N}, \leq, 1, *)$? If not, why not?
3. Is $(\mathbb{Z}, \leq, *, 1)$ a monoidal preorder?

Solution

1. Yes, $(\mathbb{N}, \leq, 1, *)$ is a monoidal preorder, as we showed in Exercise 2.31.
2. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = 1$ for any $n \in \mathbb{N}$. Then clearly $f(0) = 1$ so the unit is preserved, and for any $n, m \in \mathbb{N}$ we have $f(n) * f(m) = 1 * 1 = 1 = f(n + m)$, so the product is preserved. Hence f is strict monoidal monotone.
3. No, $(\mathbb{Z}, \leq, 1, *)$ is not, similar to what we showed in Exercise 2.5.

Exercise 2.50

1. Show that if you start with a preorder $(P, \leq)$, define a **Bool**-category as in Example 2.47, and turn it back to a preorder as in Theorem 2.49, you get back the preorder you started with.
2. Similarly, show that if you start with a **Bool**-category, turn it into a preorder, then turn it back to a **Bool**-category, you get back the **Bool**-category you started with.

Solution

1. Let $(P, \leq)$ be a preorder and let $\mathcal{X}$ be the constructed **Bool**-category. So $\mathrm{Ob}(\mathcal{X}) = P$ and $\mathcal{X}(x, y) = \text{true}$ if and only if $x \leq y$. As this is an if-and-only-if, if you do the reverse construction you get back exactly the original preorder.
2. This proof is basically the same as the previous.

Exercise 2.52

Which distance is bigger under the above description, $d(\text{Spain, US})$ or $d(\text{US, Spain})$?

Solution

$d(\text{US, Spain})$ is bigger, since the US is larger in area than Spain.

Exercise 2.55

Consider the SMP $(\mathbb{R}_{\geq 0}, \geq, 0, +)$. How would you characterise the difference between a Lawvere metric space and a $(\mathbb{R}_{\geq 0}, \geq, 0, +)$-category?

Solution

The latter is the same as a Lawvere metric space except without infinite distances.

Exercise 2.58

See book.

Solution

$d(\nearrow)$	A	B	C	D
A	0	6	3	11
B	2	0	5	5
C	5	3	0	8
D	11	9	6	0

Exercise 2.60

See book.

Solution

$\nearrow$	A	B	C	D
A	0	∞	3	∞
B	2	0	∞	5
C	∞	3	0	∞
D	∞	∞	6	0

Exercise 2.61

Interpret what a **NMY**-category is (Exercise 2.34).

Solution

This is a bit like a Hasse diagram (where edges either exist or don't), except with the possibility of an edge being "maybe" present.

Exercise 2.62, 2.63

Discussed in person.

Exercise 2.67

Draw the Hasse diagram for the preorder corresponding to the regions US, Spain, and Boston with the "regions of the world" Lawvere metric space.

Solution

This has a single nonidentity arrow, Boston $\to$ US, and no others, basically telling us that Boston is in the US but there are no other such containment relations among the three regions.

Exercise 2.68

1. Find another monoidal monotone $g : \mathbf{Cost} \to \mathbf{Bool}$ different from the one defined in the Eq. 2.66.
2. Find a Lawvere metric space $\mathcal{X}$ on which your monoidal monotone g and the monoidal monotone f given in Eq. 2.66 give different answers, $\mathcal{X}_f \neq \mathcal{X}_g$.

Solution

Let

$$g(x) := \begin{cases} \text{false} & \text{if } x = \infty \\ \text{true} & \text{if } x < \infty \end{cases}$$

Then for the regions Lawvere metric space used above, the constructed preorder will be different. For $\mathcal{X}_g$ every point is equivalent to every other point, as all regions are finite distances from one another, whereas there are points that are not connected for $\mathcal{X}_f$, e.g. Spain and US, as their distance is not zero.

Exercise 2.73

1. Show that a skeletal dagger **Cost**-category is an extended metric space.
2. Use Exercise 1.73 to make sense of the analogy "preorders are to sets as Lawvere metric spaces are to extended metric spaces."

Solution

Let $\mathcal{X}$ be a skeletal dagger **Cost**-category, and let $x, y, z \in \mathcal{X}$. We will define $d(x,y) = \mathcal{X}(x,y)$. From the definition of a $\mathcal{V}$-category we know that $0 \geq \mathcal{X}(x,x) = d(x,x)$, so we have property (a) from Definition 2.51. We also know that $\mathcal{X}(x,y) + \mathcal{X}(y,z) \geq \mathcal{X}(x,z)$, which gives us property (d).

From the fact that $\mathcal{X}$ is dagger, we know that the identity function is a $\mathcal{V}$-functor from $\mathcal{X}$ to $\mathcal{X}^{\text{op}}$. Hence for each $x, y \in \mathcal{X}$, we have $\mathcal{X}(x,y) \geq \mathcal{X}(y,x)$ as well as $\mathcal{X}(y,x) \geq \mathcal{X}(x,y)$, which implies that $\mathcal{X}(x,y) = \mathcal{X}(y,x)$. This proves property (c) holds.

Finally, suppose that $\mathcal{X}(x,y) = 0$. This implies that $0 \geq \mathcal{X}(x,y)$, and from property (c) $0 \geq \mathcal{X}(y,x)$. Therefore due to the fact that $\mathcal{X}$ is skeletal, we have $x = y$. This proves property (b). Hence $\mathcal{X}$ is an extended metric space.

In Exercise 1.73 we showed that skeletal dagger preorders are sets, and analogously we now know that skeletal dagger Lawvere metric spaces are extended metric spaces.

Exercise 2.75

Show that the $\mathcal{V}$-product of $\mathcal{V}$-categories is indeed a $\mathcal{V}$-category. Point out exactly where the symmetry condition is used.

Solution

Let $\mathcal{X} \times \mathcal{Y}$ be the $\mathcal{V}$-product of $\mathcal{V}$-categories.

For $(x, y) \in \mathrm{Ob}(\mathcal{X} \times \mathcal{Y})$, we have $(\mathcal{X} \times \mathcal{Y})((x, y), (x, y)) = \mathcal{X}(x, x) \otimes \mathcal{Y}(y, y)$. Since $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{V}$-categories, we know $I \leq \mathcal{X}(x, x)$ and $I \leq \mathcal{Y}(y, y)$, which implies that $I \otimes I = I \leq \mathcal{X}(x, x) \otimes \mathcal{Y}(y, y)$, again by the properties of the monoidal preorder $\mathcal{V}$. Therefore $I \leq (\mathcal{X} \times \mathcal{Y})((x, y), (x, y))$.

Next, let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathrm{Ob}(\mathcal{X} \times \mathcal{Y})$. Then

$$\begin{aligned}
&(\mathcal{X} \times \mathcal{Y})((x_1, y_1), (x_2, y_2)) \otimes (\mathcal{X} \times \mathcal{Y})((x_2, y_2), (x_3, y_3)) \\
=\ &(\mathcal{X}(x_1, x_2) \otimes \mathcal{Y}(y_1, y_2)) \otimes (\mathcal{X}(x_2, x_3) \otimes \mathcal{Y}(y_2, y_3)) \\
=\ &(\mathcal{X}(x_1, x_2) \otimes \mathcal{X}(x_2, x_3)) \otimes (\mathcal{Y}(y_1, y_2) \otimes \mathcal{Y}(y_2, y_3)) \quad \text{by symmetry} \\
\leq\ &\qquad\qquad \mathcal{X}(x_1, x_3) \qquad\qquad \otimes \qquad\qquad \mathcal{Y}(y_1, y_3) \\
=\ &(\mathcal{X} \times \mathcal{Y})((x_1, y_1), (x_3, y_3).
\end{aligned}$$

(The inequality is from the definition of $\mathcal{V}$-category and the monoidal product in $\mathcal{V}$.) Therefore $\mathcal{X} \times \mathcal{Y}$ is a $\mathcal{V}$-category.

Exercise 2.78

Considering $\mathbb{R}$ as a **Cost** category, form the **Cost**-product $\mathbb{R} \times \mathbb{R}$. What is the distance from $(5, 6)$ to $(-1, 4)$?

Solution

$$(\mathbb{R} \times \mathbb{R})((5, 6), (-1, 4)) = \mathbb{R}(5, -1) + \mathbb{R}(6, 4) = 6 + 2 = 8$$

Exercise 2.82

Prove that a monoidal preorder $(V, \leq, I, \otimes)$ is monoidal closed iff, given any $v \in V$, the map $(- \otimes v) : V \to V$ given by multiplying with v has a right adjoint, written $(v \multimap -) : V \to V$.

Solution

Let $w, w' \in V$. Then as $v \leq v$, we know $w \leq w' \Rightarrow w \otimes v \leq w' \otimes v$ by the definition of a monoidal preorder. So the map $(- \otimes v)$ is monotone.

Next we suppose $\mathcal{V}$ is closed. Then as $(v \multimap w) \leq (v \multimap w)$, by the definition of monoidal closure we have $((v \multimap w) \otimes v) \leq w$.

Finally if $w \leq w'$, then we want to show $(v \multimap w) \leq (v \multimap w')$. But from the previous step we have $((v \multimap w) \otimes v) \leq w \leq w'$, which again by definition of closure gives us $(v \multimap w) \leq (v \multimap w')$. Thus the map $(v \multimap -)$ is monotone.

As the definition of a hom-element gives us the rest of the conditions of a Galois connection, this shows that a SMP is closed if and only if $(- \otimes v)$ has a right adjoint, namely $(v \multimap -)$.

Exercise 2.84

Show that $\mathbf{Bool} = (\mathbb{B}, \leq, \text{true}, \wedge)$ is monoidal closed.

Solution

We show that $\multimap$ is the $\Rightarrow$ operator. Let $a, v, w \in \mathbb{B}$. Then the usual rules of logic tell us

$$a \wedge v \leq w \quad \text{iff} \quad a \leq (v \Rightarrow w).$$

Exercise 2.92

1. What is $\bigvee \varnothing$ in the case of (a) **Bool** and (b) **Cost**?
2. What is $x \vee y$ in the same cases?

Solution

1. In **Bool**, $\bigvee \varnothing = \text{false}$. In **Cost**, it is ∞.
2. In **Bool**, $x \vee y$ is logical-or. In **Cost**, it is *min*.

Exercise 2.93

Show that $\mathbf{Bool} = (\mathbb{B}, \leq, \text{true}, \wedge)$ is a quantale.

Solution

We've already showed that **Bool** is closed. As the join is logical-or, we can clearly "or" together as many booleans as we like and get a boolean value, hence **Bool** is a quantale.

Exercise 2.94

Let S be a set. Is the power set monoidal preorder $(P(S), \subseteq, S, \cap)$ a quantale?

Solution

$P(S)$ clearly contains all joins, i.e. set unions, so to show that it is a quantale we will show that it is monoidal closed. We define the hom-element to be $A \multimap B := \overline{B} \cup C$ (where $\overline{B}$ is the complement of B in S).

Then for any $A, B, C \subseteq S$, we have $A \cap B \subseteq C \Rightarrow A \subseteq \overline{B} \cup C$, as A consists of the part of A not in B, i.e. within $\overline{B}$, and the part of A within B, which we've assumed is a subset of C.

Finally if $A \subseteq \overline{B} \cup C$, then

$$A \cap B \subseteq (\overline{B} \cup C) \cap B = (\overline{B} \cap B) \cup (C \cap B) = B \cap C \subseteq C.$$

Thus we have shown $A \cap B \subseteq C \Leftrightarrow A \subseteq \overline{B} \cup C$, which proves that $(P(S), \subseteq, S, \cap)$ is a quantale.

Exercise 2.103

Write down the 2 $\times$2 -identity matrix for each of the quantales $(\mathbb{N}, \leq, 1, *)$, **Bool** $= (\mathbb{B}, \leq,$ true, $\wedge)$, and **Cost** $= ([0, \infty], \geq, 0, +)$.

Solution

(Note the first example only has finite joins, which is okay for matrix multiplication.) For $\mathbb{N}$ the standard identity matrix works.

For **Bool**, the matrix with true on the diagonal and false everywhere else is the identity matrix.

For **Cost**, the matrix with 0 on the diagonal and ∞ on the off diagonals is the identity.

Exercise 2.104

Let $\mathcal{V} = (V, \leq, I, \otimes, \multimap)$ be a quantale. Prove the identity and associative laws for $\mathcal{V}$-matrix multiplication.

Solution

Let A and B be sets and $M : A \times B \to V$ a $\mathcal{V}$-matrix. Then

$$(I_A * M)(a, b) = \bigvee_{a' \in A} I_A(a, a') \otimes M(a', b).$$

These entries are nonzero only when $a = a'$ by definition of I_A, so this gives us

$$= \left[\bigvee_{a' \neq a} I_A(a, a') \otimes M(a', b) \right] \vee I_A(a, a) \otimes M(a, b)$$

$$= \left[\bigvee_{a' \neq a} \left(\bigvee \varnothing \otimes M(a', b)\right)\right] \vee I \otimes M(a, b)$$
$$= \left[\bigvee_{a' \neq a} \left(\bigvee_{x \in \varnothing} (x \otimes M(a', b))\right)\right] \vee M(a, b)$$

by application of Proposition 2.87. As the join over the empty set is 0, we have

$$= \left[\bigvee_{a' \neq a} 0\right] \vee M(a, b) = M(a, b).$$

This proves $I_A * M = M$.

Next, let $M : A \times B \to V$, $N : B \times C \to V$, and $P : C \times D \to V$. Then

$$\begin{aligned}((M * N) * P)(a, d) &= \left[\bigvee_{b \in B} M(a, b) \otimes N(b, c)\right] * P \\ &= \bigvee_{c \in C} \left(\left[\bigvee_{b \in B} M(a, b) \otimes N(b, c)\right] \otimes P(c, d)\right) \\ &= \bigvee_{c \in C} \bigvee_{b \in B} (M(a, b) \otimes N(b, c)) \otimes P(c, d),\end{aligned}$$

again using Proposition 2.87. By the associativity of the monoidal product, this gives us

$$= \bigvee_{c \in C} \bigvee_{b \in B} M(a, b) \otimes (N(b, c) \otimes P(c, d)).$$

It is clear that we can switch the order of the joins, giving us

$$\begin{aligned}&= \bigvee_{b \in B} \bigvee_{c \in C} M(a, b) \otimes (N(b, c) \otimes P(c, d)) \\ &= \bigvee_{b \in B} M(a, b) \otimes \left[\bigvee_{c \in C} N(b, c) \otimes P(c, d)\right] \\ &= M * \left[\bigvee_{c \in C} N(b, c) \otimes P(c, d)\right] \\ &= (M * (N * P))(a, d).\end{aligned}$$

Therefore $(M * N) * P = M * (N * P)$.