Applied Category Theory Problems

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September 3, 2020

Chapter 1

Generative Effects

Exercise 1.1

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be

- *order-preserving* if $x \leq y$ implies $f(x) \leq f(y)$, for all $x, y \in \mathbb{R}$
- *metric-preserving* if $\mid x - y \mid = \mid f(x) - f(y) \mid$
- *addition-preserving* if $f(x + y) = f(x) + f(y)$

For each of the three properties defined above—call it *foo*—find an f that is *foo*-preserving and an example of an f that is not *foo*-preserving.

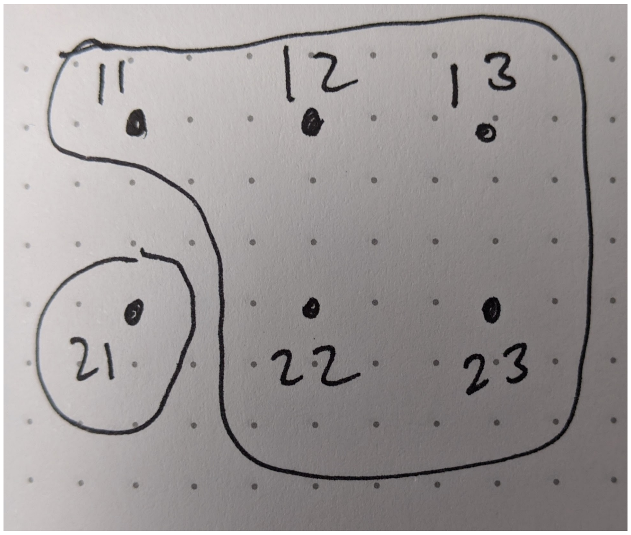
Solution

$f(x) = x$ is order-, metric-, and addition-preserving. $f(x) = x^2$ is none of these.

Exercise 1.4

See book.

Solution



Exercise 1.6, 1.7, 1.10

Discussed in person

Exercise 1.11

Let $A = \{h, 1\}$ and $B = \{1, 2, 3\}$.

Solution

1. The subsets of B are $\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$.
2. For example, $\{1, 2\} \cup \{2\} = \{1, 2\}$.
3. $A \times B = \{(h, 1), (h, 2), (h, 3), (1, 1), (1, 2), (1, 3)\}$
4. $A \sqcup B = \{h_A, 1_A, 1_B, 2_B, 3_B\}$
5. $A \cup B = \{h, 1, 2, 3, 4\}$

Exercise 1.16

Suppose that A is a set and $\{A_p\}_{p \in P}$ and $\{A'_{p'}\}_{p' \in P'}$ are two partitions of A such that for each $p \in P$ there exists a $p' \in P'$ with $A_p = A'_{p'}$.

1. Show that for each $p \in P$ there is at most one $p' \in P'$ such that $A_p = A'_{p'}$.
2. Show that for each $p' \in P'$ there is a $p \in P$ such that $A_p = A'_{p'}$.

Solution

1. If there are distinct $p_1', p_2' \in P'$ such that $A_p = A'_{p_1'}$ and $A_p = A'_{p_2'}$, then by transitivity $A'_{p_1'} = A'_{p_2'}$ which means that $p_1' = p_2'$ as P' is a partition.

2. Let $a \in A'_{p_1'}$. Then we must have that $a \in A_p$ for some p. We will show that this A_p is the desired one. There must exist p_2' such that $A_p = A'_{p_2'}$. So $a \in A'_{p_2'}$ and $a \in A'_{p_1'}$. So $A'_{p_1'} = A'_{p_2'} = A_p$ as we wanted.

Exercise 1.17

See book.

Solution

$(11, 11), (12, 12), (13, 13), (21, 21), (22, 22), (23, 23), (11, 12), (12, 11), (22, 23), (23, 22)$

Exercise 1.20

Suppose that $\sim$ is an equivalence relation on a set A, and let P be the set of $(\sim)$-closed and $(\sim)$-connected subsets $\{A_p\}_{p \in P}$.

1. Show that each part A_p is nonempty.

2. Show that if $p \neq q$ i.e. $A_p \neq A_q$, then $A_p \cap A_q = \varnothing$.

3. Show that $A = \bigcup_{p \in P} A_p$.

Solution

1. Since each A_p is $(\sim)$-connected, by definition A_p must be nonempty.

2. Suppose $a \in A_p \cap A_q$. Then for any $b \in A_p$, we know $b \sim a$, hence since A_q is $(\sim)$-closed we know $b \in A_q$. Similarly for any $b \in A_q$. Hence $A_p = A_q$.

3. Clearly $\bigcup A_p \subseteq A$. Suppose $a \in A$. Then let $X = \{b \mid a \sim b, b \in A\}$. X is closed as the equivalence relation is reflexive and connected as it is transitive, so X a set in $\{A_p\}$ and $A \subseteq \bigcup A_p$.

Exercise 1.24

Discussed in person

Exercise 1.25

Suppose that A is a set and $f : A \to \varnothing$ is a function to the empty set. Show that A is empty.

Solution

Suppose there is some $a \in A$. Then we must have $f(a) \in \varnothing$ which is clearly not possible.

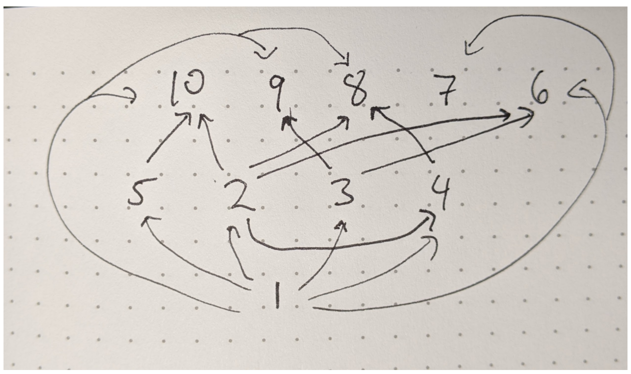
Exercise 1.27, 1.38, 1.40, 1.41, 1.42, 1.44

Discussed in person

Exercise 1.46

Write down the numbers $1, 2, \ldots, 10$ and draw an arrow $a \to b$ if a divides perfectly into b. Is it a total order?

Solution



This isn't a total order, as for example we neither have $2 \mid 7$ or $7 \mid 2$.

Exercise 1.48

Is the usual $\leq$ ordering on the set $\mathbb{R}$ of real numbers a total order?

Solution

Yes: for any $x, y \in \mathbb{R}$, we have $x \leq y$ or $y \leq x$.

Exercise 1.51

Discussed in person

Exercise 1.53

For any set S there is a coarsest partition, having just one part. What surjective function does it correspond to? There is also a finest partition, where everything is in its own partition. What surjective function does it correspond to?

Solution

Let $f : S \to \{\bullet\}$ be the unique function that sends every element of S to $\bullet$. This is a surjection corresponding to the coarsest partition, i.e. where every element is in the same set.

Let $f : S \to S$ be the identity function. This is a surjection corresponding to the finest partition, i.e. where every element is in its own set.

Exercise 1.55

Prove that the preorder of upper sets on a discrete preorder on a set X is simply the power set $P(X)$.

Solution

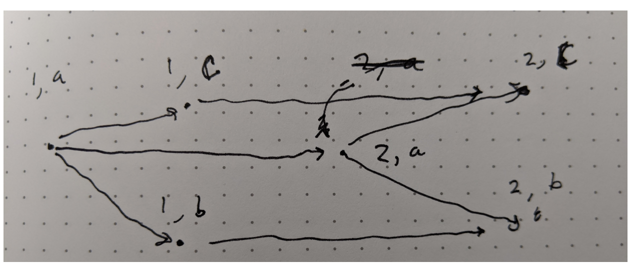
Clearly the set of upper sets $U(X)$ is a subset of the power set $P(X)$.

Let $Y \subseteq X$. We know $\varnothing$ is an upper set, so let $y \in Y$. Then since the preorder is discrete, the only element in X greater than y is y itself, which is in Y. This holds for any $y \in Y$ so Y is an upper set. Note that the ordering on both $U(X)$ and $P(X)$ is the same, i.e. $\subseteq$.

Exercise 1.57

See book.

Solution

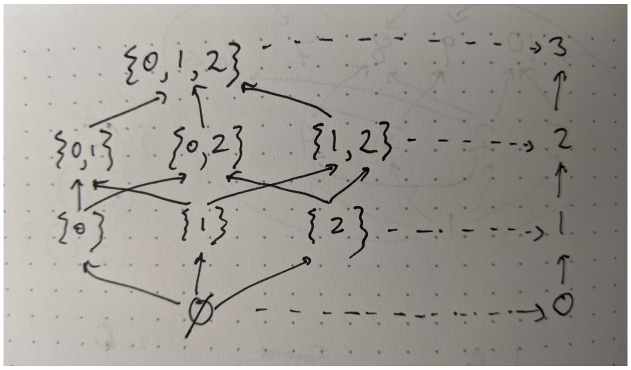


Exercise 1.63

Let $X = \{0, 1, 2\}$.

1. Draw the Hasse diagram for $P(X)$.
2. Draw the Hasse diagram for the preorder $0 \leq 1 \leq 2 \leq 3$.
3. Draw the cardinality map $|\cdot|$ as dashed lines between them

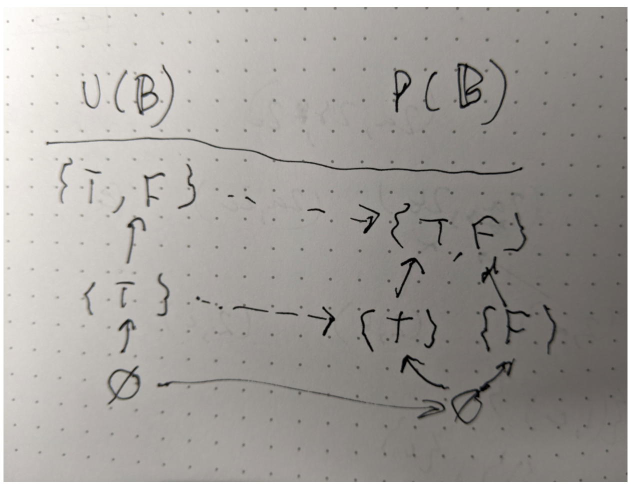
Solution



Exercise 1.65

Draw the monotone map between $U(\mathbb{B})$ and $P(\mathbb{B})$ as described in the text.

Solution



Exercise 1.66

Let $(P, \leq)$ be a preorder.

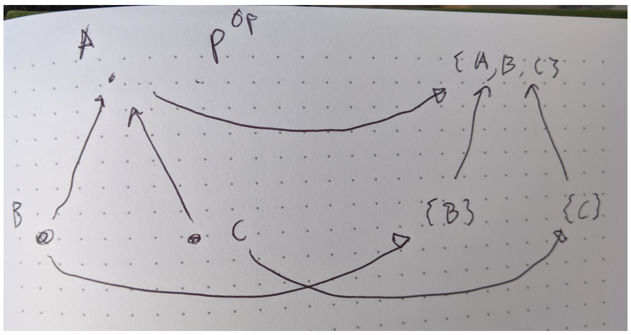
1. Show that the set $\uparrow p = \{p' \in P \mid p \leq p;\ \}$ is an upper set for any $p \in P$.
2. Show that this defines a monotone map $\uparrow\colon P^{op} \to U(P)$.
3. Show that $p \leq p'$ iff $\uparrow(p') \subseteq \uparrow(p)$.
4. Draw a picture of the map $\uparrow$ in the case where P is the preorder $(b \geq a \leq c)$.

Solution

1. Suppose $q \in \uparrow p$, then any $q' \geq q$ is transitively greater than p and hence $q' \in \uparrow p$.

2. Suppose $p \geq q$ (i.e. p is less than q in P^{op}), we want to show that $\uparrow p \subseteq \uparrow q$. So let $p' \in \uparrow p$. We know $q \leq p \leq p'$ and hence $p' \in \uparrow q$.

3. We showed the first direction in part 2, so assume $\uparrow (p') \subseteq \uparrow (p)$. This means $p \in \uparrow (p')$ and hence $p \leq p'$.

4. 

Exercise 1.67

Show that when $(P, \leq_P)$ is a discrete preorder, then every function $f : P \to Q$ is monotone regardless of the order $\leq_Q$.

Solution

We need to show that for any $x, y \in P$ where $x \leq_P y$, we have $f(x) \leq_Q f(y)$. But the only x and y satisfying this are $x \leq_P x$, for which we have $f(x) \leq_Q f(x)$ regardless of $\leq_Q$ by the definition of a preorder.

Exercise 1.69

Choose two sets X and Y with at least three elements each and choose a surjective, non-identity function $f : X \to Y$. Write down two different partitions P and Q of Y, and find $f^*(P)$ and $f^*(Q)$.

Solution

Exercise 1.71

Prove Proposition 1.70:

1. For any preorder $(P, \leq_P)$, the identity function is monotone.

2. If $(Q, \leq_Q)$ and $(R, \leq_R)$ are preorders and $f : P \to Q$ and $g : Q \to R$ are monotone, then $(f \mathbin{⨾} g) : P \to R$ is also monotone.

Solution

1. If $a \leq_P b$ then clearly $a = f(a) \leq_P f(b) = b$ if f is the identity function.
2. Suppose $a \leq_P b$, then $f(a) \leq_Q f(b)$ as f is monotone, and hence $g(f(a)) \leq_R g(f(b))$ as g is also monotone.

Exercise 1.73

Show that a skeletal dagger preorder is just a discrete preorder, and hence can be identified with a set.

Solution

Let $(P, \leq)$ be a skeletal dagger preorder. We need to show that for any $x \in P$, the only thing comparable to x is x itself. So suppose $x \leq y$, then as P is a dagger preorder we know that $y \leq x$. Hence as P is skeletal, we have that $x = y$. This implies that P is a discrete preorder.

Exercise 1.77

Show that the map Φ from Section 1.1.1 ('Is $\bullet$ connected to $\star$?') is the monotone map $Prt(\{\star, \bullet, \circ\}) \to \mathbb{B}$.

Solution

Let P and P' be partitions where $P \leq P'$. If $\Phi(P) = \texttt{false}$ then clearly $\Phi(P) \leq \Phi(P')$, so assume $\Phi(P) = \texttt{true}$. This means for some set X in the partition P, we know that both $\bullet, \star \in X$. As $P \leq P'$ this means there is some Y in the partition P' with $X \subseteq Y$, which implies that $\bullet, \star \in Y$. Hence $\Phi(P') = \texttt{true}$ and $\Phi(P) \leq \Phi(P')$.

Exercise 1.79

Let P and Q be preorders and $f : P \to Q$ a monotone map. Show that the pullback $f^* : U(Q) \to U(P)$ can be defined by taking $u : Q \to \mathbb{B}$ to $(f \mathbin{⨟} u) : P \to \mathbb{B}$.

Solution

Call ϕ_Q the function that takes upper sets in Q to monotone maps as defined in Proposition 1.78, and similarly ϕ_P. Let $U \in U(Q)$. We want to show $\phi_P(f^{-1}(U)) = f \mathbin{⨟} (\phi_Q(U))$.

Let $x \in P$. If $x \in f^{-1}(U)$, then we know $\phi_P(f^{-1}(U))(x) = \texttt{true}$ by definition. But we also know $f(x) \in U$ and hence $\phi_Q(U)(f(x)) = \texttt{true}$. Conversely if $x \notin f^{-1}(U)$, we will have both $\phi_P(f^{-1}(U))(x) = \texttt{false}$, as well as $f(x) \notin U$ and $\phi_Q(U)(f(x)) = \texttt{false}$. This shows that these maps are equal.