

# Applied Category Theory Problems

April Shen & Alex Root

September 13, 2020

# Chapter 1

## Generative Effects

### Exercise 1.1

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is said to be

- *order-preserving* if  $x \leq y$  implies  $f(x) \leq f(y)$ , for all  $x, y \in \mathbb{R}$
- *metric-preserving* if  $|x - y| = |f(x) - f(y)|$
- *addition-preserving* if  $f(x + y) = f(x) + f(y)$

For each of the three properties defined above—call it *foo*—find an  $f$  that is *foo*-preserving and an example of an  $f$  that is not *foo*-preserving.

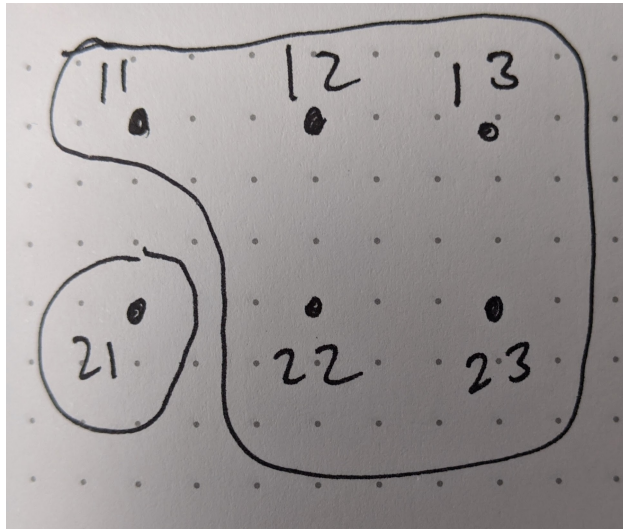
### Solution

$f(x) = x$  is order-, metric-, and addition-preserving.  $f(x) = x^2$  is none of these.

### Exercise 1.4

See book.

### Solution



### Exercise 1.6, 1.7, 1.10

Discussed in person

### Exercise 1.11

Let  $A = \{h, 1\}$  and  $B = \{1, 2, 3\}$ .

### Solution

1. The subsets of  $B$  are  $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$ .
2. For example,  $\{1, 2\} \cup \{2\} = \{1, 2\}$ .
3.  $A \times B = \{(h, 1), (h, 2), (h, 3), (1, 1), (1, 2), (1, 3)\}$
4.  $A \sqcup B = \{h_A, 1_A, 1_B, 2_B, 3_B\}$
5.  $A \cup B = \{h, 1, 2, 3, 4\}$

### Exercise 1.16

Suppose that  $A$  is a set and  $\{A_p\}_{p \in P}$  and  $\{A'_{p'}\}_{p' \in P'}$  are two partitions of  $A$  such that for each  $p \in P$  there exists a  $p' \in P'$  with  $A_p = A'_{p'}$ .

1. Show that for each  $p \in P$  there is at most one  $p' \in P'$  such that  $A_p = A'_{p'}$ .
2. Show that for each  $p' \in P'$  there is a  $p \in P$  such that  $A_p = A'_{p'}$ .

**Solution**

1. If there are distinct  $p'_1, p'_2 \in P'$  such that  $A_p = A'_{p'_1}$  and  $A_p = A'_{p'_2}$ , then by transitivity  $A'_{p'_1} = A'_{p'_2}$  which means that  $p'_1 = p'_2$  as  $P'$  is a partition.
2. Let  $a \in A'_{p'_1}$ . Then we must have that  $a \in A_p$  for some  $p$ . We will show that this  $A_p$  is the desired one. There must exist  $p'_2$  such that  $A_p = A'_{p'_2}$ . So  $a \in A'_{p'_2}$  and  $a \in A'_{p'_1}$ . So  $A'_{p'_1} = A'_{p'_2} = A_p$  as we wanted.

**Exercise 1.17**

See book.

**Solution**

(11, 11), (12, 12), (13, 13), (21, 21), (22, 22), (23, 23), (11, 12), (12, 11), (22, 23), (23, 22)

**Exercise 1.20**

Suppose that  $\sim$  is an equivalence relation on a set  $A$ , and let  $P$  be the set of  $(\sim)$ -closed and  $(\sim)$ -connected subsets  $\{A_p\}_{p \in P}$ .

1. Show that each part  $A_p$  is nonempty.
2. Show that if  $p \neq q$  i.e.  $A_p \neq A_q$ , then  $A_p \cap A_q = \emptyset$ .
3. Show that  $A = \bigcup_{p \in P} A_p$ .

**Solution**

1. Since each  $A_p$  is  $(\sim)$ -connected, by definition  $A_p$  must be nonempty.
2. Suppose  $a \in A_p \cap A_q$ . Then for any  $b \in A_p$ , we know  $b \sim a$ , hence since  $A_q$  is  $(\sim)$ -closed we know  $b \in A_q$ . Similarly for any  $b \in A_q$ . Hence  $A_p = A_q$ .
3. Clearly  $\bigcup A_p \subseteq A$ . Suppose  $a \in A$ . Then let  $X = \{b \mid a \sim b, b \in A\}$ .  $X$  is closed as the equivalence relation is reflexive and connected as it is transitive, so  $X$  a set in  $\{A_p\}$  and  $A \subseteq \bigcup A_p$ .

**Exercise 1.24**

Discussed in person

**Exercise 1.25**

Suppose that  $A$  is a set and  $f : A \rightarrow \emptyset$  is a function to the empty set. Show that  $A$  is empty.

### Solution

Suppose there is some  $a \in A$ . Then we must have  $f(a) \in \emptyset$  which is clearly not possible.

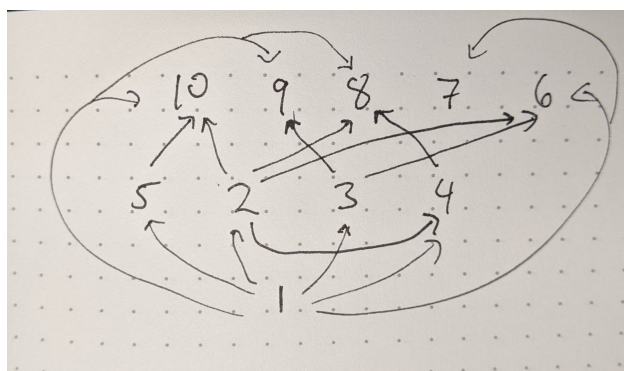
### Exercise 1.27, 1.38, 1.40, 1.41, 1.42, 1.44

Discussed in person

### Exercise 1.46

Write down the numbers  $1, 2, \dots, 10$  and draw an arrow  $a \rightarrow b$  if  $a$  divides perfectly into  $b$ . Is it a total order?

### Solution



This isn't a total order, as for example we neither have  $2 \mid 7$  or  $7 \mid 2$ .

### Exercise 1.48

Is the usual  $\leq$  ordering on the set  $\mathbb{R}$  of real numbers a total order?

### Solution

Yes: for any  $x, y \in \mathbb{R}$ , we have  $x \leq y$  or  $y \leq x$ .

### Exercise 1.51

Discussed in person

### Exercise 1.53

For any set  $S$  there is a coarsest partition, having just one part. What surjective function does it correspond to? There is also a finest partition, where everything is in its own partition. What surjective function does it correspond to?

### Solution

Let  $f : S \rightarrow \{\bullet\}$  be the unique function that sends every element of  $S$  to  $\bullet$ . This is a surjection corresponding to the coarsest partition, i.e. where every element is in the same set.

Let  $f : S \rightarrow S$  be the identity function. This is a surjection corresponding to the finest partition, i.e. where every element is in its own set.

### Exercise 1.55

Prove that the preorder of upper sets on a discrete preorder on a set  $X$  is simply the power set  $P(X)$ .

### Solution

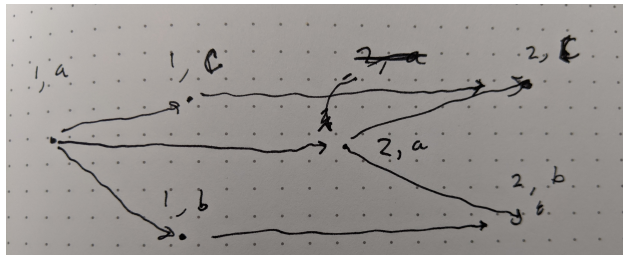
Clearly the set of upper sets  $U(X)$  is a subset of the power set  $P(X)$ .

Let  $Y \subseteq X$ . We know  $\emptyset$  is an upper set, so let  $y \in Y$ . Then since the preorder is discrete, the only element in  $X$  greater than  $y$  is  $y$  itself, which is in  $Y$ . This holds for any  $y \in Y$  so  $Y$  is an upper set. Note that the ordering on both  $U(X)$  and  $P(X)$  is the same, i.e.  $\subseteq$ .

### Exercise 1.57

See book.

### Solution

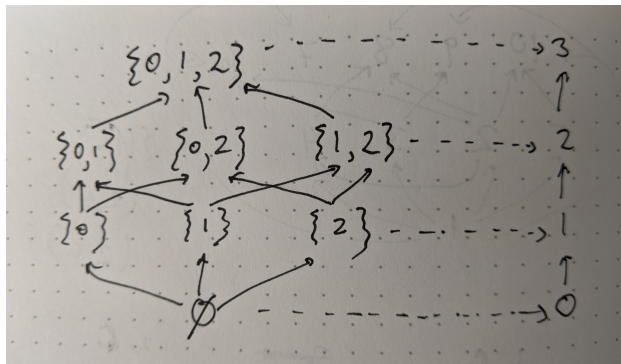


### Exercise 1.63

Let  $X = \{0, 1, 2\}$ .

1. Draw the Hasse diagram for  $P(X)$ .
2. Draw the Hasse diagram for the preorder  $0 \leq 1 \leq 2 \leq 3$ .
3. Draw the cardinality map  $|\cdot|$  as dashed lines between them

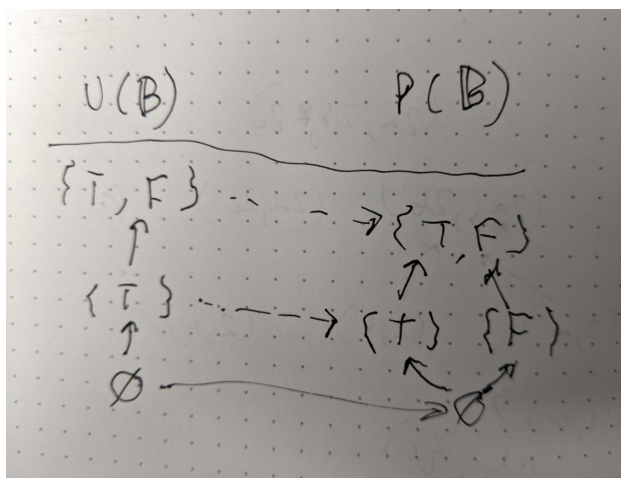
## Solution



## Exercise 1.65

Draw the monotone map between  $U(\mathbb{B})$  and  $P(\mathbb{B})$  as described in the text.

## Solution



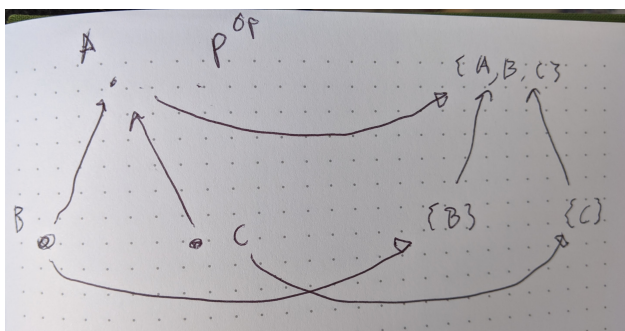
## Exercise 1.66

Let  $(P, \leq)$  be a preorder.

1. Show that the set  $\uparrow p = \{p' \in P \mid p \leq p'\}$  is an upper set for any  $p \in P$ .
2. Show that this defines a monotone map  $\uparrow: P^{op} \rightarrow U(P)$ .
3. Show that  $p \leq p'$  iff  $\uparrow(p') \subseteq \uparrow(p)$ .
4. Draw a picture of the map  $\uparrow$  in the case where  $P$  is the preorder  $(b \geq a \leq c)$ .

## Solution

1. Suppose  $q \in \uparrow p$ , then any  $q' \geq q$  is transitively greater than  $p$  and hence  $q' \in \uparrow p$ .
2. Suppose  $p \geq q$  (i.e.  $p$  is less than  $q$  in  $P^{op}$ ), we want to show that  $\uparrow p \subseteq \uparrow q$ . So let  $p' \in \uparrow p$ . We know  $q \leq p \leq p'$  and hence  $p' \in \uparrow q$ .
3. We showed the first direction in part 2, so assume  $\uparrow(p') \subseteq \uparrow(p)$ . This means  $p \in \uparrow(p')$  and hence  $p \leq p'$ .



4.

## Exercise 1.67

Show that when  $(P, \leq_P)$  is a discrete preorder, then every function  $f : P \rightarrow Q$  is monotone regardless of the order  $\leq_Q$ .

## Solution

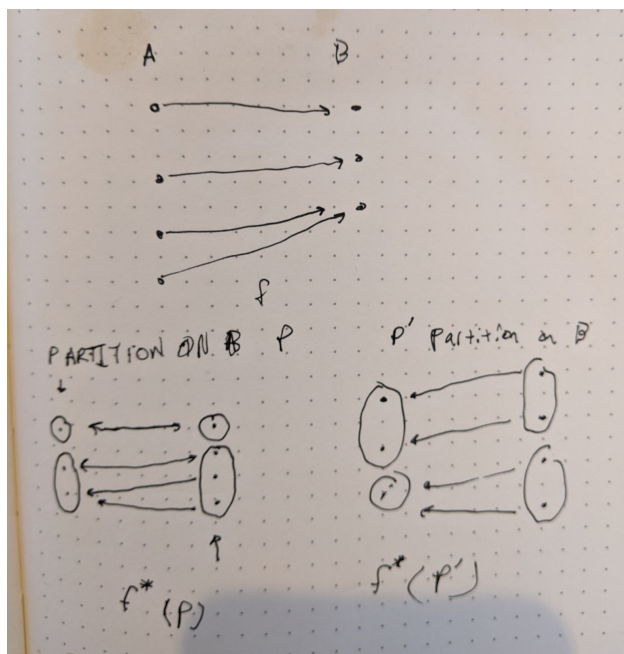
We need to show that for any  $x, y \in P$  where  $x \leq_P y$ , we have  $f(x) \leq_Q f(y)$ . But the only  $x$  and  $y$  satisfying this are  $x \leq_P x$ , for which we have  $f(x) \leq_Q f(x)$  regardless of  $\leq_Q$  by the definition of a preorder.

## Exercise 1.69

Choose two sets  $X$  and  $Y$  with at least three elements each and choose a surjective, non-identity function  $f : X \rightarrow Y$ . Write down two different partitions  $P$  and  $Q$  of  $Y$ , and find  $f^*(P)$  and  $f^*(Q)$ .



## Solution



## Exercise 1.71

Prove Proposition 1.70:

1. For any preorder  $(P, \leq_P)$ , the identity function is monotone.
2. If  $(Q, \leq_Q)$  and  $(R, \leq_R)$  are preorders and  $f: P \rightarrow Q$  and  $g: Q \rightarrow R$  are monotone, then  $(f \circ g): P \rightarrow R$  is also monotone.

## Solution

1. If  $a \leq_P b$  then clearly  $a = f(a) \leq_P f(b) = b$  if  $f$  is the identity function.
2. Suppose  $a \leq_P b$ , then  $f(a) \leq_Q f(b)$  as  $f$  is monotone, and hence  $g(f(a)) \leq_R g(f(b))$  as  $g$  is also monotone.

## Exercise 1.73

Show that a skeletal dagger preorder is just a discrete preorder, and hence can be identified with a set.

## Solution

Let  $(P, \leq)$  be a skeletal dagger preorder. We need to show that for any  $x \in P$ , the only thing comparable to  $x$  is  $x$  itself. So suppose  $x \leq y$ , then as  $P$  is a dagger preorder we know

that  $y \leq x$ . Hence as  $P$  is skeletal, we have that  $x = y$ . This implies that  $P$  is a discrete preorder.

### Exercise 1.77

Show that the map  $\Phi$  from Section 1.1.1 ('Is  $\bullet$  connected to  $\star$ ?)' is the monotone map  $\text{Prt}(\{\star, \bullet, \circ\}) \rightarrow \mathbb{B}$ .

### Solution

Let  $P$  and  $P'$  be partitions where  $P \leq P'$ . If  $\Phi(P) = \mathbf{false}$  then clearly  $\Phi(P) \leq \Phi(P')$ , so assume  $\Phi(P) = \mathbf{true}$ . This means for some set  $X$  in the partition  $P$ , we know that both  $\bullet, \star \in X$ . As  $P \leq P'$  this means there is some  $Y$  in the partition  $P'$  with  $X \subseteq Y$ , which implies that  $\bullet, \star \in Y$ . Hence  $\Phi(P') = \mathbf{true}$  and  $\Phi(P) \leq \Phi(P')$ .

### Exercise 1.79

Let  $P$  and  $Q$  be preorders and  $f : P \rightarrow Q$  a monotone map. Show that the pullback  $f^* : U(Q) \rightarrow U(P)$  can be defined by taking  $u : Q \rightarrow \mathbb{B}$  to  $(f \circ u) : P \rightarrow \mathbb{B}$ .

### Solution

Call  $\phi_Q$  the function that takes upper sets in  $Q$  to monotone maps as defined in Proposition 1.78, and similarly  $\phi_P$ . Let  $U \in U(Q)$ . We want to show  $\phi_P(f^{-1}(U)) = f \circ (\phi_Q(U))$ .

Let  $x \in P$ . If  $x \in f^{-1}(U)$ , then we know  $\phi_P(f^{-1}(U))(x) = \mathbf{true}$  by definition. But we also know  $f(x) \in U$  and hence  $\phi_Q(U)(f(x)) = \mathbf{true}$ . Conversely if  $x \notin f^{-1}(U)$ , we will have both  $\phi_P(f^{-1}(U))(x) = \mathbf{false}$ , as well as  $f(x) \notin U$  and  $\phi_Q(U)(f(x)) = \mathbf{false}$ . This shows that these maps are equal.

### Exercise 1.80

Why is 0 a greatest lower bound for  $\{\frac{1}{n+1} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ ?

### Solution

Assume that  $\varepsilon > 0$  is a lower bound. Let  $n = \lceil 1/\varepsilon \rceil$ . Then

$$\frac{1}{n+1} \leq \frac{1}{1/\varepsilon + 1} \leq \frac{1}{1/\varepsilon} = \varepsilon.$$

Hence no such  $\varepsilon$  is a lower bound.

**Exercise 1.85**

Let  $(P, \leq)$  be a preorder and  $p \in P$ , consider the set  $A = \{p\}$ .

1. Show that  $\bigwedge A \cong p$ .
2. Show that if  $P$  is a partial order, then  $\bigwedge A = p$ .
3. Are the analogous facts true when  $\bigwedge$  is replaced by  $\bigvee$ ?

**Solution**

1. Clearly  $p \leq p$ , so by definition  $\bigwedge A \leq p$  (as a lower bound) and  $\bigwedge A \geq p$  (as a greatest lower bound).
2. If the previous is true in a partial order, then we have  $\bigwedge A = p$ .
3. The analogous facts are true with  $\bigvee$ .

**Exercise 1.90**

In the  $n \mid m$  ordering on  $\mathbb{N}$ , what are the meet and the join?

**Solution**

The meet is the greatest common divisor and the join is the least common multiple.

**Exercise 1.94**

Prove that for any monotone map  $f : P \rightarrow Q$ , if  $a, b \in P$  have a join and  $f(a), f(b) \in Q$  have a join, then  $f(a) \vee f(b) \leq f(a \vee b)$ .

**Solution**

We know  $a, b \leq a \vee b$ , so since  $f$  is monotone we have  $f(a), f(b) \leq f(a \vee b)$ . Hence  $f(a \vee b)$  is an upper bound for  $\{f(a), f(b)\}$ , so by definition of the join we must have  $f(a) \vee f(b) \leq f(a \vee b)$ .

**Exercise 1.98**

Find a right adjoint for the monotone map  $(3 \times -) : \mathbb{Z} \rightarrow \mathbb{R}$ , and show it is correct.

**Solution**

Let  $g(y) = \lfloor y/3 \rfloor$ . Then we have  $3x \leq y \Leftrightarrow x \leq y/3 \Leftrightarrow x \leq \lfloor y/3 \rfloor$ , hence  $g$  is a right adjoint for  $3x$ .

**Exercise 1.99**

See book.

**Solution**

1. In this case  $f$  is left adjoint to  $g$ .
2. In this case  $f$  is not left adjoint to  $g$ , as  $g(1) = 2 \geq 2$  but  $f(2) = 2 \not\leq 1$ .

**Exercise 1.101**

1. Does  $\lceil -/3 \rceil$  have a left adjoint  $L : \mathbb{Z} \rightarrow \mathbb{R}$ ?
2. If not, why? If so, does its left adjoint have a left adjoint?

**Solution**

Let  $g : \mathbb{R} \rightarrow \mathbb{Z}$  be defined by  $g(x) = \lceil x/3 \rceil$ . We will show by contradiction that  $g$  does not have a left adjoint.

For a left adjoint  $f : \mathbb{Z} \rightarrow \mathbb{R}$ , we must have  $f(1) \leq 0 \Leftrightarrow 1 \leq g(0) = \lceil 0/3 \rceil = 0$ . Clearly the second part does not hold, so we know  $f(1) \not\leq 0$ .

On the other hand, we know that  $f(1) \leq \inf A$  where  $A = \{x \mid g(x) \geq 1, x \in \mathbb{R}\}$ . However  $1/n \in A$  for  $n \in \mathbb{Z}^+$ , as  $\lceil 1/n \rceil = 1$  for all such  $n$ . But  $\inf\{1/n \mid n \in \mathbb{Z}^+\} = 0$ , which implies that  $f(1) \leq 0$ . This is a contradiction, so  $g$  must not have a left adjoint.

**Exercise 1.103**

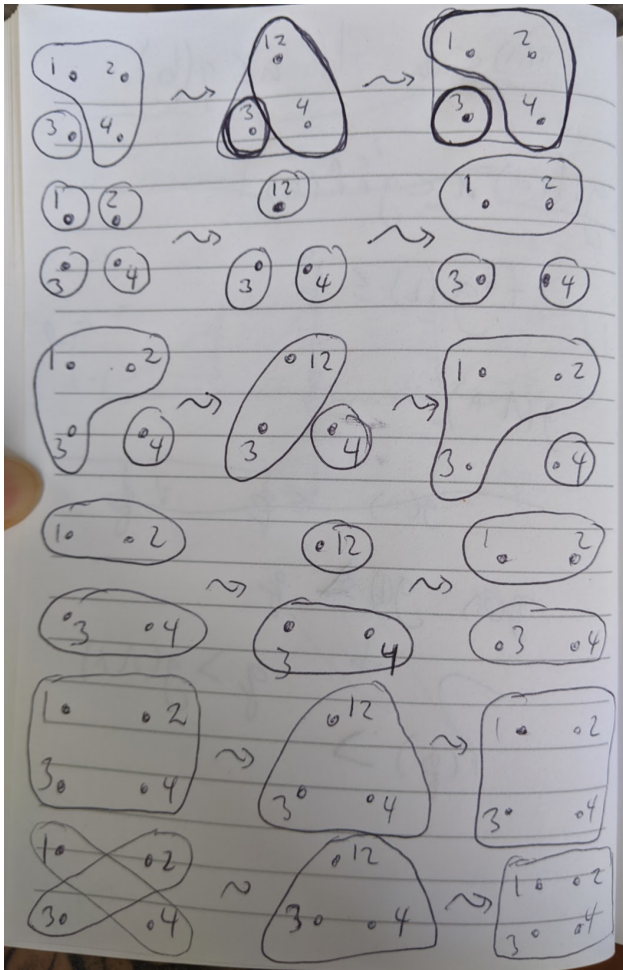
Choose 6 different partitions on the set  $S$  and for each, call it  $c$ , find  $g_!(c)$  where  $S$ ,  $T$ , and  $g : S \rightarrow T$  are the same as they were in Example 1.102.

**Exercise 1.105**

Using the same  $S$ ,  $T$ , and  $g : S \rightarrow T$  as in Example 1.102, find the partition  $g^*(c)$  for each of the 5 partitions  $c$  on the set  $T$ .

**Solution**

For both 1.103 and 1.105.



### Exercise 1.106 (revised)

Prove that  $g_!$  is left adjoint to  $g^*$ , as defined in the text.

### Solution

Let  $S, T$  be sets, and let  $g : S \rightarrow T$ . Define  $g_!, g^*$  as in the text.

We first show  $g^*$  is monotone. Let  $A, B \in \text{Prt}(T)$  such that  $A \leq B$ . Then for each set  $A_i \in A$ ,  $A_i \subseteq B_j$  for some  $B_j \in B$ , and as a result  $g^{-1}(A_i) \subseteq g^{-1}(B_j)$  for each  $A_i \in A$ . As the image of a partition under  $g^*$  is the collection of preimages of that partition via  $g$ , we have  $g^*(A) \leq g^*(B)$ .

Next we show  $g_!$  is monotone. Let  $A, B \in \text{Prt}(S)$  such that  $A \leq B$ . As before we know that for each set  $A_i \in A$ ,  $A_i \subseteq B_j$  for some  $B_j \in B$ . We consider  $A, B \in \text{Rel}(S)$  i.e. subsets of  $S \times S$ . Note that  $g_!(C)$  is the transitive closure of the relation  $\{(g(x), g(y)) \mid (x, y) \in C\}$ . As  $A \leq B$ ,  $\{(g(x), g(y)) \mid (x, y) \in A\} \subseteq \{(g(x), g(y)) \mid (x, y) \in B\}$ . Using the fact that the function taking a relation to its transitive closure is monotone on the set of relations ordered by inclusion, we can conclude that  $g_!(A) \leq g_!(B)$ .

For the next part of the proof, we use proposition 1.107, and derive our result by showing that for each  $A \in \text{Prt}(S)$  and for each  $B \in \text{Prt}(T)$  that both  $A \leq g^* \circ g_!(A)$ , and  $g_! \circ g^*(B) \leq B$ .

We start with showing  $A \leq g^* \circ g_!(A)$ . First we consider two additional functions, first  $\bar{g} : \text{Prt}(S) \rightarrow \text{Rel}(T)$ , where  $g(r) = \{(g(x), g(y)) \mid (x, y) \in r\}$ . Secondly an extension of  $g^*$  to all relations,  $\bar{g}^* : \text{Rel}(T) \rightarrow \text{Rel}(S)$ , so for a relation  $r$ , we have  $\bar{g}^*(r) = \{(x, y) \mid (g(x), g(y)) \in r\}$  ( $g^*$  is the restriction of  $\bar{g}^*$  to equivalence relations). Both  $\bar{g}$  and  $\bar{g}^*$  are monotone, which can be seen in proofs similar to our proofs for  $g^*$  and  $g_!$ . Additionally let the transitive closure of a set  $Q$  be denoted  $\hat{Q}$ . We now note two things, one  $A = \bar{g}^* \circ \bar{g}(A)$ , and two  $g^* \circ g_!(A) = \bar{g}^*(\widehat{\bar{g}(A)})$ . As  $\bar{g}^*$  is monotone and  $\bar{g}(A) \leq \widehat{\bar{g}(A)}$  we have  $A \leq g^* \circ g_!(A)$ .

### Exercise 1.109

Complete the proof of Proposition 1.107 by showing that (for monotone  $f : P \rightarrow Q$  and  $g : Q \rightarrow P$ )

1. if  $f$  is left adjoint to  $g$  then for any  $q \in Q$  we have  $f(g(q)) \leq q$ , and
2. if  $p \leq g(f(p))$  and  $f(g(q)) \leq q$ , then  $p \leq g(p)$  iff  $f(p) \leq q$  holds, for all  $p \in P$  and  $q \in Q$ .

### Solution

### Exercise 1.110

1. Show that if  $f : P \rightarrow Q$  has a right adjoint  $g$ , then it is unique up to isomorphism. That is, for any other right adjoint  $g'$ , we have  $g(q) \cong g'(q)$  for all  $q \in Q$ .
2. Is the same true for left adjoints? That is, if  $h : P \rightarrow Q$  has a left adjoint, is it necessarily unique up to isomorphism?

### Solution

### Exercise 1.112

Complete the proof of Proposition 1.111 by showing that left adjoints preserve joins.

### Solution