Causality Detection using Convergent Cross Mapping (CCM)

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1 Introduction

Granger uses the nature of predictability to distinguish causal relations between time-series variables from correlations. That is, variable X is said to cause Y in Granger's definition if given all the information in the universe, the predictability of Y decreases when X is removed from the universe. However, Granger's causality test is expected to fail when the time-series variables are from deterministic systems due to its assumption of separability. In stochastic systems, new information is injected by the noise in the cause variable X so the cause variable X and the effect variable Y are separable. In deterministic systems, information about the cause X will be redundantly present in the history of the effect Y itself and cannot formally be removed from the universe.

Alternatively, Sugihara proposes a new definition of causality in deterministic settings. That is, time series variables are causally related if they are from the same dynamic systems (they share a common attractor manifold and each variable can identify the state of the other). To detect such causality in deterministic dynamic systems, Sugihara introduces the convergent cross mapping (CCM) method. Given two time series X and Y, the idea of the method is to see whether the time indices of near-by points on the X manifold can be used to identify the near-by points on the Y manifold. The method generates a cross-mapped estimate of Y from the nearest neighbors in the constructed X shadow manifold. The convergence of the cross-mapped estimates is a necessary condition for causation.
In this report, we use the CCM test to detect causality in various dynamic systems and perform sensitivity analysis to the delay time and the dimension of the manifold embedding. Also, we study examples of some CCM test failures.

2 Numerical implementation of the Lorenz system and asymmetric coupled system

We first consider the time-series variables from the Lorenz attractor:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - \beta z\end{aligned}$$

where $\sigma = 10, r = 26$ and $\beta = 8/3$. We implement a second order Runge-Kutta ODE solver to obtain the the time series X, Y, and Z (see pseudo-code in appendix). The numerical results are shown in the blue curve of Fig 1. Visual examination on Fig 1 shows that the numerical implementation produces the butterfly shape as expected. Also, the implementation is numerically convergent since refining the grid leads to graphically identical results.
Next, we consider the following coupled difference equations

$$\begin{aligned}X(t+1) =& X(t)[r_x - r_x X(t) - \beta_{x,y} Y(t)]\\ Y(t+1) =& Y(t)[r_y - r_y Y(t) - \beta_{y,x} X(t)],\end{aligned}$$

where $\beta_{x,y} = 0$. So, Y has no influence on X and there is a unidirectional causality from X to Y.

3 Convergent cross mapping test

3.1 The Lorenz system

We construct the shadow manifolds for two time series, X and Y, obtained from the Lorenz system:

$$\dot{x} = 10(y - x)$$
$$\dot{y} = 26x - y - xz$$
$$\dot{z} = xy - (8/3)z$$

The results are shown in Figure (1). The time lag used to reconstruct shadow manifolds is 0.1 (in scale of real world time) and dimension of shadow manifold, E, is 3.

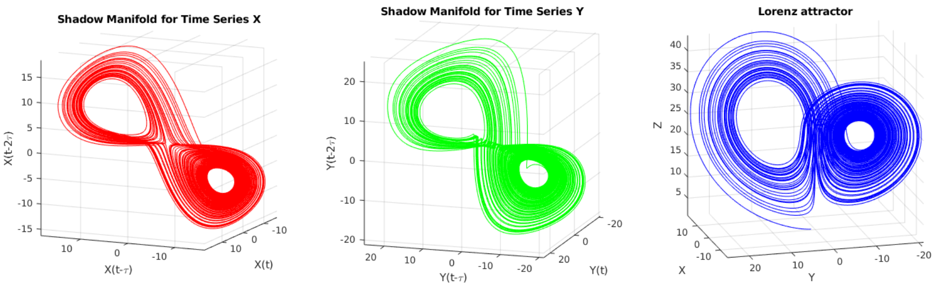

Figure 1: Comparison between Original Manifold and Shadow Manifolds (Lorenz System)

First, we compare the original manifold and two shadow manifolds reconstructed from X and Y. We can see that both reconstructed shadow manifolds have a similar butterfly shape to the original manifold.

Then, we apply CCM test with length set to 1 in real world time (1000 simulation time steps). To detect causality, we plot predicted time series $\hat{X}(t)$ and $\hat{Y}(t)$ against the original time series. The results are shown in the first two subplots in Figure 2, which represent a relatively accurate estimation. This result indicates that there exists bi-directional causality between $X(t)$ and $Y(t)$. Furthermore, in order to see the convergence of the predictions, we plot the correlation coefficients between X and $\hat{X}$ and between Y and $\hat{Y}$ with CCM test length from 200 to 1000, as shown in the third subplot in Figure 2. From this plot, we can learn the correlation coefficient quickly converge to 1 with the CCM test length increasing.
Note that in the Lorenz system, the cross mapping of X using the shadow manifold $M_Y$and the cross mapping of Y using the shadow manifold M_X are expected to succeed because both shadow manifolds are diffeomorphic to the Lorenz attractor. Sugihara commented that the coordinate z does not produce a valid shadow manifold since the two lobes in M are symmetric with respect to z and the shadow manifold M_Z only contains only one fixed point. The topological difference makes it impossible to reconstruct the full dynamics of the Lorenz system. However, we perform the CCM test using M_x to predict Z and using M_z to predict X and observe the convergence in both directions (see Figure 3). One possible explanation for the unexpected test success is that despite the topological difference, the local mapping between M_X and M_Z is still preserved and thus the predictive power remains.

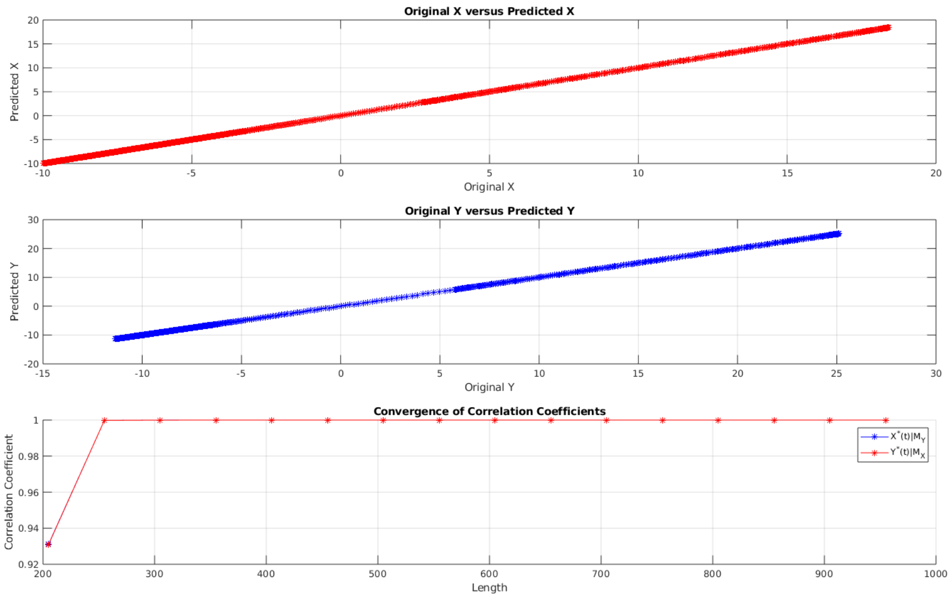

Figure 2: CCM Test Results of Lorenz System

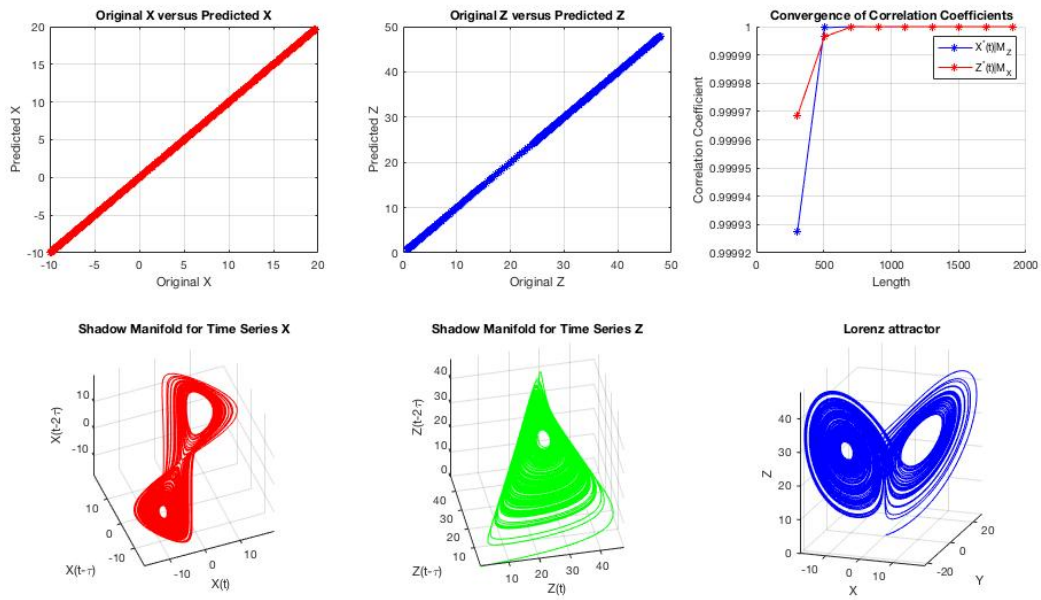

Figure 3: CCM test results of the Lorenz system with time-series variables X and Z

3.2 Asymmetric coupled system

For the asymmetric model, we modified the parameters of the equation 1 in the origin CCM paper, thus time series $Y(t)$ has no effect on $X(t)$:

$$
\begin{aligned}
X(t+1) &= X(t)\,[3.8 - 3.8X(t)] \\
Y(t+1) &= Y(t)\,[3.5 - 3.5Y(t) - 0.1X(t)]
\end{aligned}
$$

The system is simulated with initial data being $[0.8, 0.8]$ and simulation length being 10000.

When detecting causality using CCM test on this asymmetric model, we set shadow manifold dimension to 2, CCM test length to 5000 and time lag to 1 (both are in simulation time). The results are shown in Figure 4.

As shown in the results, when plotting predicted $X^*(t)$ against original $X(t)$, the points are distributed near a straight line, while in the plot for $Y(t)$ there is no obvious corresponding pattern, which implies that the causality only goes from Y to X. The results match the asymmetric system in which information only flows from X to Y. This result could also be verified by the plot of correlation coefficients, where the correlation coefficient of $(X, \hat{X})$ converges to 1 but that of $(Y, \hat{Y})$ stays close to 0.

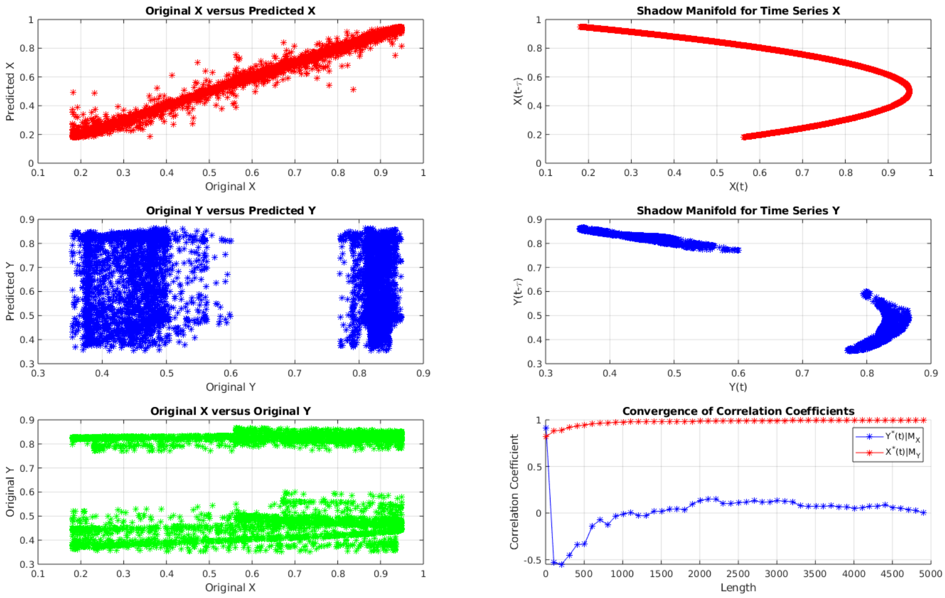

Figure 4: CCM Test Results of Asymmetric System

3.3 Sensitivity analysis

To see the effect of time lag, we vary the time lag from 0.1 to 0.4 and simulation time from 1 to 2 in real world time scale. The results are shown in Figure 5. As shown in the results, even though the shadow manifolds do not have a similar shape to the original manifold, the estimation and convergence results are still relatively good. Furthermore, we decrease the time lag to 0.01, the results are shown in Figure 6. In this case, the shadow manifolds shrink, and we could see the estimation and convergence are influenced by this change, but the overall estimation skill is still not bad.

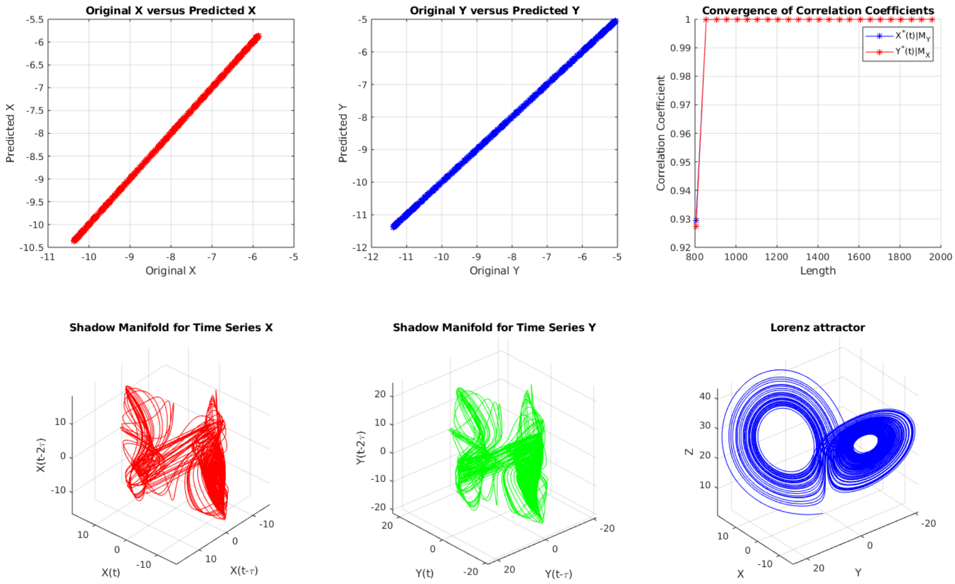

Figure 5: CCM Test Results of Lorenz System With Longer Time Lag

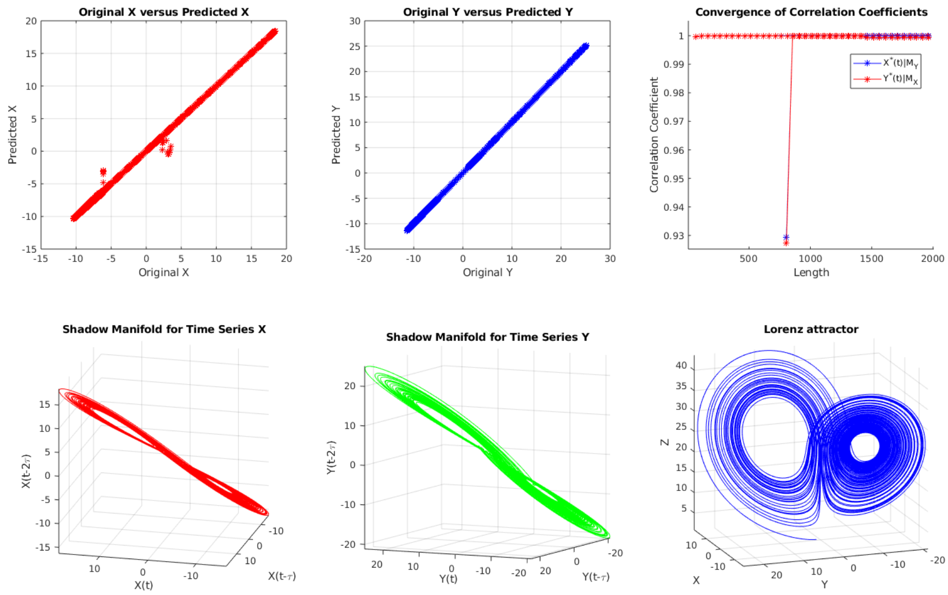

Figure 6: CCM Test Results of Lorenz System With Shorter Time Lag

To test the effect of shadow manifold dimension, we first increase the shadow manifold dimension from 2 to 5 for the asymmetric system. The results are shown in Figure 7. Then we increase the dimension to 20, the results of which are shown in Figure 8. Both tests are of length 2000. In those results, we could see that, with the shadow manifold dimension increasing, CCM gives worse predictions. When the dimension is 20, the correlation coefficient of $X(t)$ is under 0.5. Thus, this asymmetric system represents a relatively high sensitivity to shadow manifold dimension for CCM test.

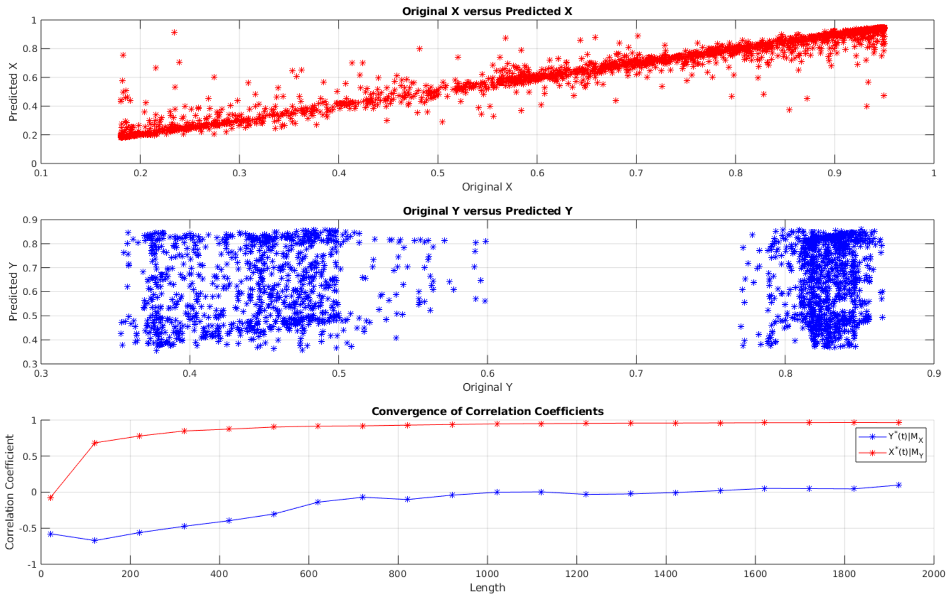

Figure 7: CCM Test Results of Asymmetric System With Shadow Manifold Dimension 5

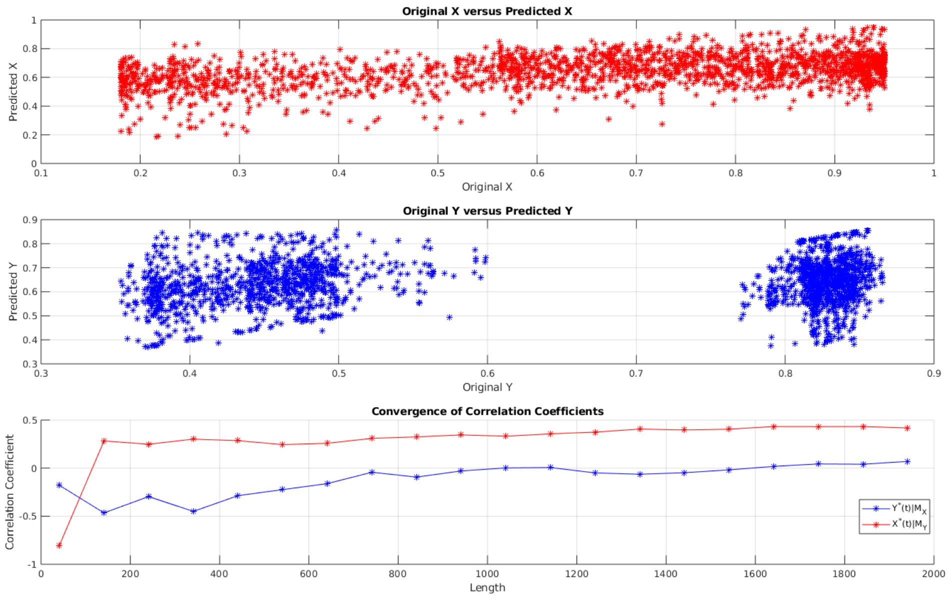

Figure 8: CCM Test Results of Asymmetric System With Shadow Manifold Dimension 20

4 CCM test failures

4.1 Non-coupled system with external force

We simulate a non-coupled system with external force, in which we expect the CCM test to succeed even though there is no causality between the variables. The system equations are:

$$\dot{x} = x(z - x)$$
$$\dot{y} = y(z - y)$$
$$\dot{z} = z^2(x - z)$$

In this system, information flows between X and Z and from Z to Y. We simulated this system with length being 50 and simulation time step being 0.005, then implement CCM test with time lag being 0.002 and shadow manifold length(L) being 0.2 (all are in real world time scale). As shown by the results in Figure 9, CCM test fails in this case, since it falsely detects causality between X and Y, which actually does not exist.

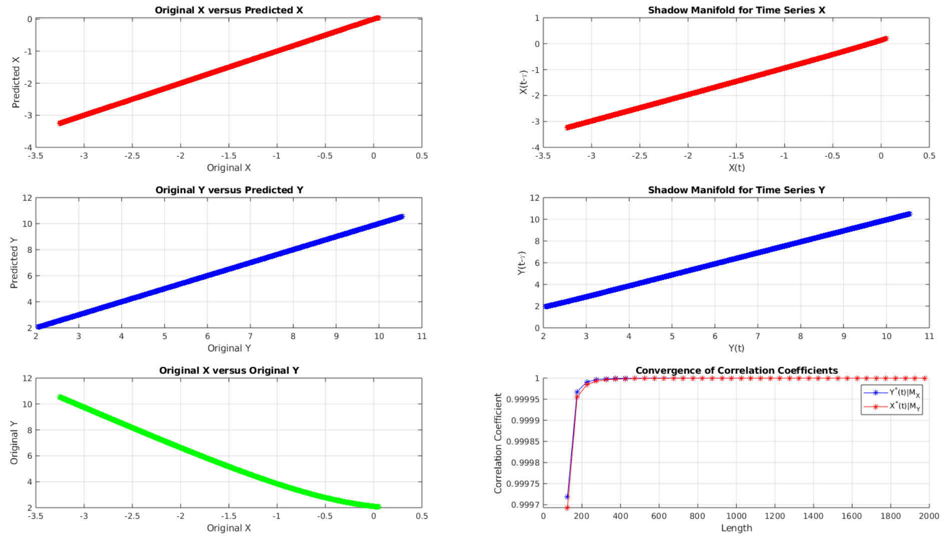

Figure 9: CCM Test Failed for Non-Coupled System with External Force

4.2 Lotka-Volterra model

We use CCM to test bidirectional causality between the two time series variables X and Y from a limit cycle in Lotka-Volterra model

$$\begin{aligned}\dot{x} =& x - 0.01xy \\ \dot{y} =& -y + 0.02xy,\end{aligned}$$

and the correlation coefficient converges to 1. Since observational error is very common in real-life data collection, we add normally distributed random noise to the variables and perform the same test. The cross-mapped results are displayed in Figure 10. The CCM test fails to detect causal relations in both directions, which is expected since in practice, observational errors and process noise will limit the level of convergence of the CCM estimates, as suggested by Sugihara. When the noise is relatively small, the CCM test can still identify causality via predictability that increases with length L; when the noise becomes too large, the cross-mapped estimates become unreliable and Granger causality test becomes the better method.

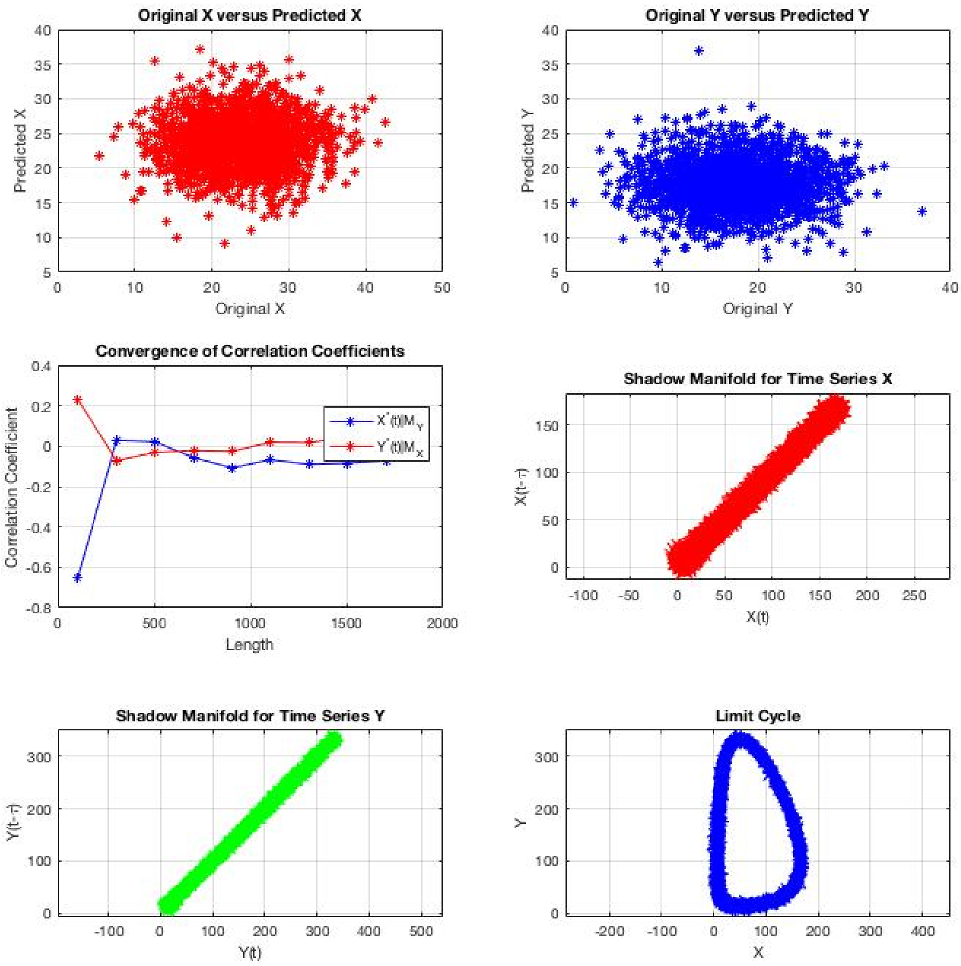

Figure 10: CCM Test Failed for Lotka-Volterra Model

5 Conclusion

In conclusion, in the Lorenz system, the CCM test is successful in detecting causality among all three times-series variables X, Y and Z. The convergence of the cross-mapped estimates for X and Y is expected due to the diffeomorphism between the X and Y shadow manifolds and the Lorenz attractor M. The test works surprisingly well even when there the Z shadow manifold is topologically different from M_X and M_Y shadow manifolds and not diffeomorphic with M.

In the sensitivity analysis, we conclude that the CCM test is fairly robust when it comes to time lag change, despite that the structures of the shadow manifolds are affected. However, the test exhibits a high sensitivity to the embedding dimension. The cross-mapped estimation skills decrease as we increase the dimension.

Note that the CCM test can falsely detect causality between two non-coupled time-series variables in a system with external force. Also, the test fails to converge when the time-series data is very noisy.

A Pseudocode

procedure GENERATING TIME SERIES FROM LORENZ SYSTEM
  initialize $T, steps, \sigma, r, \beta, X(0), Y(0), Z(0)$
  $T \leftarrow$ the final time
  $steps \leftarrow$ number of steps taken to T
  $\sigma, r, \beta \leftarrow$ manually set parameters of Lorenz System
  $X(0), Y(0), Z(0) \leftarrow$ initials of the system
  for do$k = 1 : steps$
    Use Runge-Kutta(2) to solve the ODEs numerically
  end for
end procedure

Algorithm 1: Lorenz System Generator

procedure GENERATING TIME SERIES FROM ASYMMETRIC SYSTEM
  initialize $T, X(0), Y(0)$
  $T \leftarrow$ the final time
  $X(0), Y(0) \leftarrow$ initials of the system
  for do$k = 1 : T$
    $X(k) = X(k-1)[3.8 - 3.8X(k-1)]$;
    $Y(k) = Y(k-1)[3.5 - 3.5Y(k-1) - 0.1X(k-1)]$
  end for
end procedure

Algorithm 2: Asymmetric System Generator

procedure CONSTRUCT CCM FROM TWO TIME SERIES X AND Y
  data $X, Y \leftarrow$ two time series from observations
  initialization τ, E, L
  $\tau \leftarrow$ time lags
  $E \leftarrow$ dimension of the shadow manifold
  $L \leftarrow$ length for the data of interest
  $Xdata, Ydata \leftarrow X(0 : L), Y(0 : L)$
  Construct the Shadow Manifolds M_x, M_y
  for k=1:length(M_x) **do**
    Use `knnsearch` to find the k-nearest neighbors of $M_x(k)$
    Use the neighbors to calculate the weight and $\hat{Y}(t)|M_x$
    Use `knnsearch` to find the k-nearest neighbors of $M_y(k)$
    Use the neighbors to calculate the weight and $\hat{X}(t)|M_y$
  end for
  Calculate the residuals and correlations
end procedure

Algorithm 3: Convergent Cross Map