Dynamic Mode Decomposition Discussion

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1. Explain equations (and the relationship between equations) 20.1.5, 20.1.6, and 20.1.10. What do these equations assume about the data and what do they capture?

$$\mathbf{x}_{j+1} = \mathbf{A}\mathbf{x}_j, \tag{20.1.5}$$

where $\mathbf{A}$ is linear, time-independent Koopman operator, x_j and x_{j+1} are vectors of data collected at times t_j and t_{j+1}. The matrix

$$\mathbf{X}_1^{M-1} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3 \quad \cdots \quad \mathbf{x}_{M-1}] \tag{20.1.6}$$

is constructed the vector data collected at different times ($\mathbf{x}_j$ for $j = 1 : M-1$). With the assumption of equation (20.1.5), we can deduce the equation (20.1.6) to the equation (20.1.10). Actually equation (20.1.10) gives us a recursion formula for the next time step.

$$\mathbf{A}\mathbf{X}_1^{M-1} = \mathbf{X}_2^{M} \tag{20.1.10}$$

2. What is the relationship between 20.1.7, 20.1.8, and 20.1.11?

Making use of equation (20.1.5), we are able to rewrite $\mathbf{X}_1^{M-1}$ as follows:

$$\mathbf{X}_1^{M-1} = [\mathbf{x}_1 \ \mathbf{A}\mathbf{x}_1 \ \mathbf{A}^2\mathbf{x}_1 \ ... \ \mathbf{A}^{M-2}\mathbf{x}_1]. \tag{20.1.7}$$

We try to represent the final data point $\mathbf{x}_M$ as a linear combination of the previous $M-1$ data points, as best as possible (in an L2 sense):

$$\mathbf{x_M} = \sum_{m=1}^{M-1} b_m x_m + r, \tag{20.1.8}$$

where b_m are the coefficients of the Krylov space vectors and $\mathbf{r}$ is the residual term (which is orthogonal to the Krylov space). Making use of equation (20.1.8), equation (20.1.10) can be rewritten as

$$\mathbf{X}_2^M = \mathbf{X}_1^{M-1}\mathbf{S} + \mathbf{r}e_{M-1}^* \tag{20.1.11}$$

where e_{M-1}^* is the $(M-1)$th unit vector and $\mathbf{S}$ is the coefficient matrix.

3. If x_k is a snapshot of state-space at time k and N spatial points, what are the sizes of $\mathbf{X}_1^{M-1}$, $\mathbf{X}_2^M$, r, and S?

$\mathbf{X}_1^{M-1}$ and $\mathbf{X}_2^M$ are $N \times (M-1)$ matrices, $\mathbf{r}$ is a $N \times 1$ vector and $\mathbf{S}$ is a $(M-1) \times (M-1)$ matrix.

4. What is the relationship between S and A?

Based on the linearity assumption and Krylov space method, we rewrite the relation between $\mathbf{X}_2^M$ and $\mathbf{X}_1^{M-1}$ as equation (20.1.11). In this way, the eigenvalues of $\mathbf{S}$ approximate some of the eigenvalues of the unknown Koopman operator $\mathbf{A}$.

Comparing 20.1.10 and 20.1.11 we see that $\mathbf{A}$ is being right multiplied while $\mathbf{S}$ is being left multiplied therefore $\mathbf{A}$ is $N \times N$ and $\mathbf{S}$ is a $(M-1) \times (M-1)$. In fluids applications the data is of much higher spatial resolution than temporal resolution, $N >> M$. Thus, $\mathbf{S}$ will be smaller than $\mathbf{A}$.

5. What are the characteristics of the SVD decomposition in 20.1.9? (i.e. what are the structure, dimensions, and orthogonal properties of the matrices U, Σ, and V?

$$\mathbf{X}_1^{M-1} = \mathbf{U\Sigma V}^* \tag{20.1.9}$$

U is an $N \times K$ matrix, and V is an $M-1 \times K$ matrix, Σ is an $K \times K$ rectangular diagonal matrix with non-negative real numbers on the diagonal. The diagonal entries of Σ are known as the singular values of X_1^{M-1}. The columns of U and V are called the left-singular vector and right-singular vector of X_1^{M-1}, respectively.

6. The textbook skips some steps in deriving 20.1.13. Show that you can get to this approximation of $\tilde{S} = U^*SU$ and the SVD to solve 20.1.10 for the least-squares solution.

From equations (20.1.10) and (20.1.9), we have that

$$\begin{aligned}\mathbf{A} =&\mathbf{X}_2^M(\mathbf{X}_1^{M-1})^{-1}\\ =&\mathbf{X}_2^M(\mathbf{U\Sigma V}^*)^{-1}\\ =&\mathbf{X}_2^M\mathbf{V\Sigma}^{-1}\mathbf{U}^*.\end{aligned}$$

Let $\tilde{\mathbf{S}} = \mathbf{U}^*\mathbf{AU}$ be the similarity transformation of $\mathbf{A}$. Then, A can be approximated by

$$\begin{aligned}\tilde{\mathbf{S}} =&\mathbf{U}^*\mathbf{AU}\\ =&\mathbf{U}^*(\mathbf{X}_2^M\mathbf{V\Sigma}^{-1}\mathbf{U}^*)\mathbf{U}\\ =&\mathbf{U}^*\mathbf{X}_2^M\mathbf{V\Sigma}^{-1}.\end{aligned} \tag{20.1.13}$$

Eigenvlaues and eigenvectors are preserved under a similarity transform.

7. How are eigenvectors and eigenvalues used to estimate the DMD solution?

We solve the eigenvalue problem:

$$\tilde{\mathbf{S}}\mathbf{y}_k = \mu_k\mathbf{y}_k \quad \text{for} \quad k = 1, 2, ..., K \tag{20.1.14}$$

Where K is the rank of the approximation we make ($K < rank(\mathbf{X}^M)$). We then use the eigenvalues μ_k to construct the DMD modes

$$\psi_k = \mathbf{U}\mathbf{y}_k. \tag{20.1.15}$$

Finally the approximate solution at all future times is given by

$$\mathbf{x}_{DMD}(t) = \sum_{k=1}^{K} b_k(0)\psi_k(\mathbf{x})\exp(\omega t) = \mathbf{\Psi}\ \mathrm{diag}(\exp(\omega t))\mathbf{b} \tag{20.1.16}$$

where $\omega_k = \ln(\mu_k)/\Delta t$, $b_k(0)$ is the initial amplitude of each mode, $\mathbf{\Phi}$ is the matrix whose columns are the eigenvectors ϕ, diag(ωt) is a diagonal matrix whose entries are the eigenvalues $\exp(\omega_k t)$

8. How do you calculate the initial condition for the solution?

Evaluating the approximate solution at the first time point gives $x_1 = \Psi b$. Since Ψ contains a truncated set of the eigenvectors of A, it will not be of full rank. To determine b, we use a psuedo-inverse: $b = \Psi^\dagger x_1$.

9. What makes the solution unstable and when does it become unstable?

Since approximate solution takes the form

$$\mathbf{x}_{DMD}(t) = \sum_{k=1}^{K} b_k(0)\psi_k(\mathbf{x}) \exp(\omega_k t),$$

for the solution to be stable, we need $\omega_k < 0$ for all k. And since $\omega_k = \ln|\mu_k|/\Delta t$, this means $|\mu_k| < 1$.