ESAM 495 - Data-Driven Methods for Dynamical Systems Individual Project: Comparing Granger Causality Test and Cross Convergent Mapping Test

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1 Introduction

To identify causality, Granger proposes a method that uses predictability to distinguish causation from correlation between two time series variables. Given two time series variables X and Y, X is said to be Granger causal of Y if given all the information in the universe, the predictability of Y decreases when X is removed from the universe. One of the key assumption that Granger makes is the separability of the time series. In stochastic systems, new information is injected due to the noises, which gives separable information. However, in a deterministic setup, the assumption of separability is violated due to the functional relationship between the time series variables.
Unlike the Granger causality test, the cross convergent mapping test defines that two time series variables are causally related if they are from the same dynamic systems (they share a common manifold and each variable can identify the state of the other). The new definition does not require separability and thus can detect causality in deterministic systems. In this report, we study the performance of the Granger causality test and the CCM test on time series data from stochastic systems, deterministic systems, and deterministic systems with observational noises.

2 Auto-regressive time series

2.1 Linear time series

We consider the following linear auto-regressive model:

$$X_t = \epsilon_t, \quad Y_t = X_{t-1} + \eta_t,$$

where ϵ_t and η_t are independent white noises, i.e., $\epsilon, \eta \sim \mathcal{N}(0, \sigma^2)$.

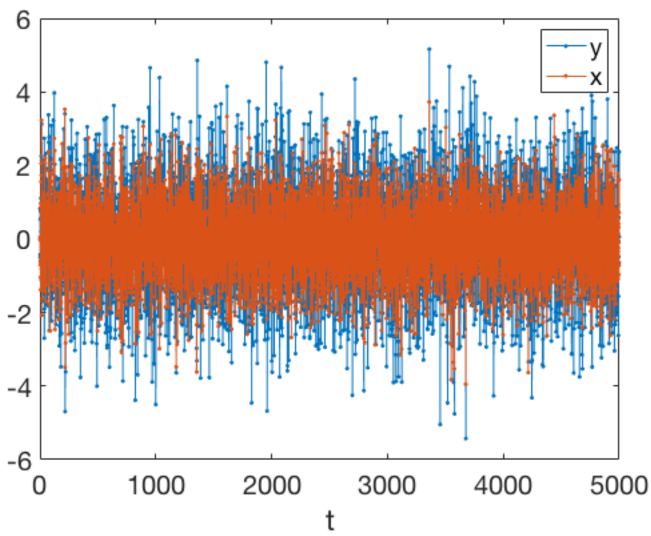

Figure 1: Linear auto-regressive time series

The data sampled from the above linear model is shown in Figure 1. For the white noises ϵ and η, their variance is set to be 1. From the time series construction, it is obvious that X causes Y and Y does not cause X. To detect the causal relationship between X and Y, we first use the Granger causality test. We regress Y on the past values of Y and find that the variance $\sigma^2(Y|Y) = 2.060$; we regress Y on the past values of X and Y and find that $\sigma^2(Y|X,Y) = 0.998$. The result $\sigma^2(Y|X,Y) \ll \sigma^2(Y|Y)$ correctly indicates that X is Granger causal of Y. Next, we use the CCM test to detect causality between X and Y. Since we know X is causal of Y, we expect the cross mapping of Y using the shadow manifold M_X to succeed and the cross mapping of X using the shadow manifold M_Y to fail. The CCM test results are shown in Figure 2. The correlation coefficients fail to converge to 1 in both directions and the CCM test fails to identify causality between X and Y. This is expected since the CCM test assumes the data are sampled from purely deterministic models and the random process in the AR system violates the assumption.

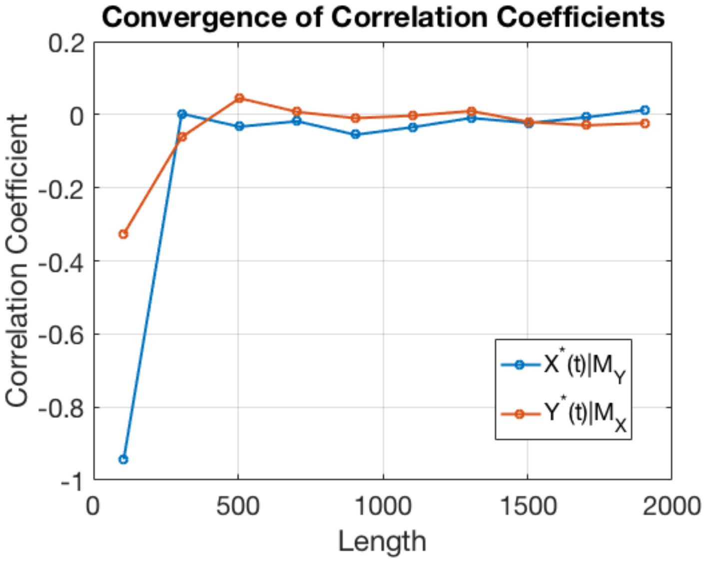

Figure 2: CCM test results

2.2 Nonlinear time series

Consider the following nonlinear auto-regressive model

$$X_t = \epsilon_t, \quad Y_t = X_{t-1}^2 + \eta_t,$$

where ϵ and η are again independent white noises.

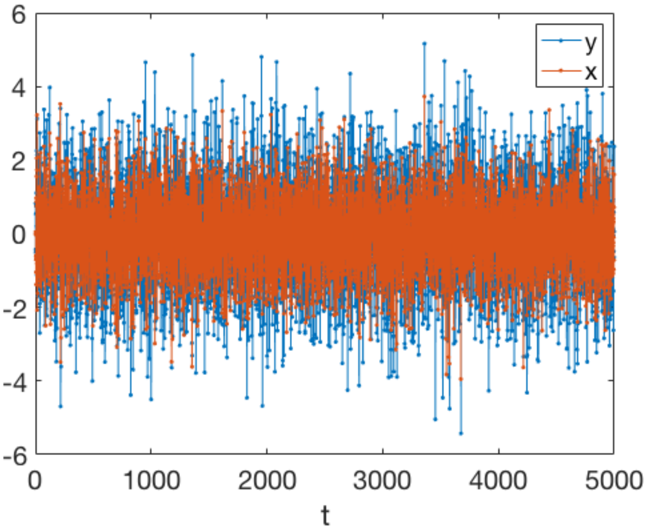

Figure 3: Nonlinear auto-regressive time series

We use Granger causality test on the data sampled from the nonlinear model. When Y is regressed on the past values of Y, we obtain that $\sigma^2(Y|Y) = 2.860$; when Y is regressed on the past values of X and Y, we obtain that $\sigma^2(Y|X,Y) = 2.856$. The result $\sigma^2(Y|X,Y) < \sigma^2(Y|Y)$ correctly indicates that X is causal of Y. However, the difference between the two variances is much smaller compared to the linear case, implying that the sensitivity and accuracy of the Granger causality test are reduced when the model is nonlinear. This is expected because the nonlinearity violates the linear assumption of the test. Next, we use the CCM test to detect causality in the nonlinear model. As shown in Figure 4, the correlation coefficient of the cross mapping of Y using M_X and the cross mapping of X using M_Y both fail to converge to 1, which is expected. We reduced the variance of the white noises in the nonlinear auto-regressive model from 1 to 0.05 and perform the CCM test. As shown in Figure 5, causality is not detected and the reduced variance does not improve the CCM test performance.

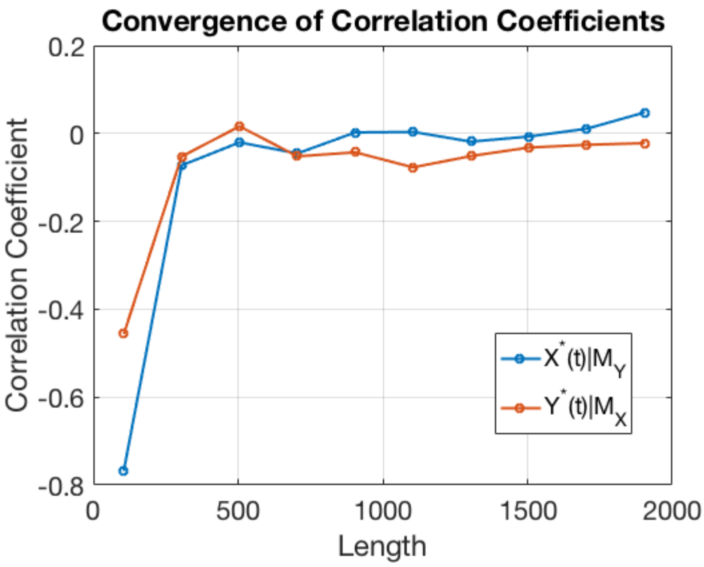

Figure 4: CCM test results

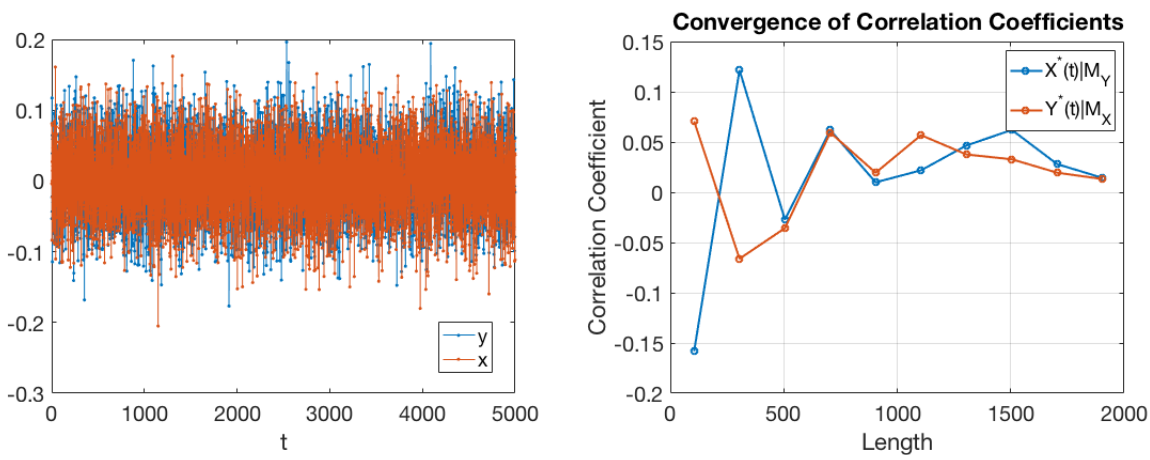

Figure 5: CCM test result with $\sigma^2 = 0.05$.

3 Deterministic systems

We consider the time-series variables from the Lorenz system:

$$\begin{aligned}\dot{x} =& \sigma(y - x)\\ \dot{y} =& rx - y - xz\\ \dot{z} =& xy - \beta z,\end{aligned}$$

where $\sigma = 10, r = 28, \beta = 8/3$, and $(x(0), y(0), z(0)) = (0, 1, 0)$. We use The MATLAB solver `ode45` to obtain the time-series X, Y, and Z. Note that X, Y, and Z are causal of each other. In this section, we use the Granger causality test and the CCM test to identify the bi-directional causal relationship between time series variables X and Y. The Lorenz attractor is plotted in Figure 6 (in xy-plane).

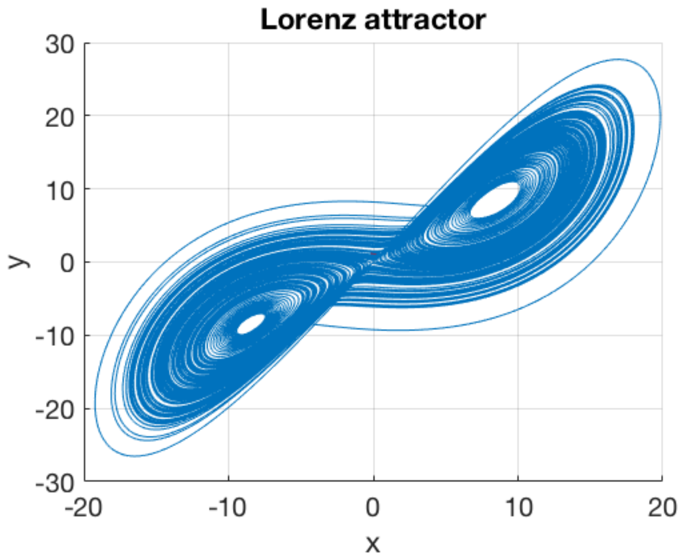

Figure 6: Lorenz attractor

First, we perform Granger causality test on the time series data X and Y. We have that $\sigma^2(Y|X, Y) = 1.248 \times 10^{-8}$ when Y is regressed on the past values of X and $\sigma^2(Y|Y) = 0.0041$ when Y is regressed on the past values of X and Y. The result $\sigma^2(Y|X, Y) \ll \sigma^2(Y|Y)$ suggests that X is Granger causal of Y. We perform the same test swapping X and Y and obtain that $\sigma^2(X|X, Y) = 3.0888 \times 10^{-13} \ll 0.0017 = \sigma^2(X|X)$, which indicates that Y is also Granger causal of X. Granger causality test successfully detects the bi-directional causal relationship between X and Y, which is unexpected since the deterministic nature of the Lorenz system violates Granger's redundancy assumption (X and Y are functionally related and thus considered to be redundant information).

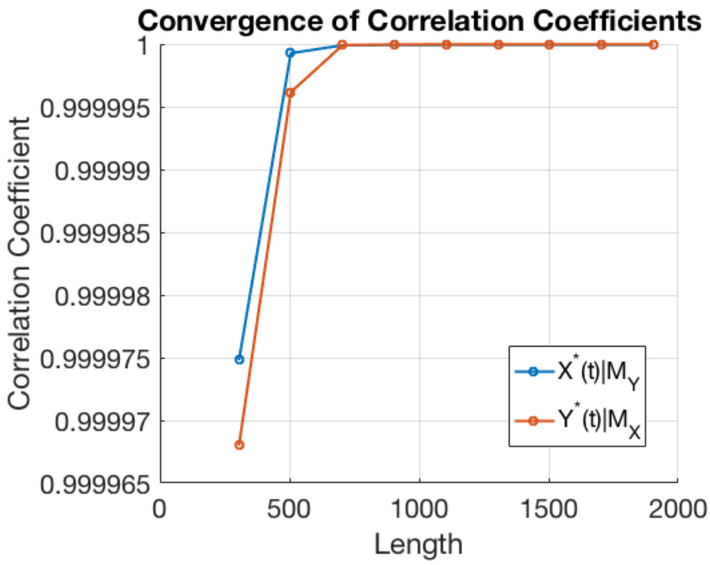

Figure 7: CCM test results

Following, we use the CCM test on the Lorenz time series X and Y. The convergence plot is shown in Figure 7, which correctly suggests bi-directional causal relationship between X and Y.

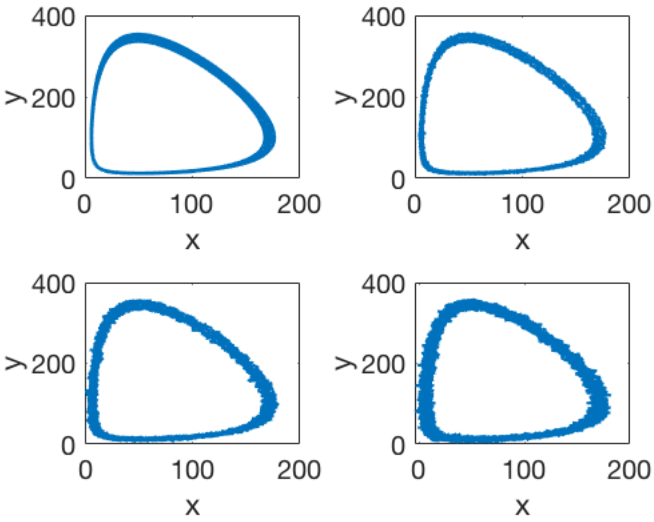

Figure 8: Noisy Lotka-Volterra limit cycles: noise variance = 0 (top left), 1 (top right),2 (bottom left), and 3 (bottom right).

4 Deterministic systems with noises

Since the Granger causality test is designed to detect causal relationships in stochastic systems that satisfy the requirement of separability, and the CCM test is designed for non-separable dynamical models (deterministic models), in this section, we consider a deterministic system with observation noises, which does not satisfy both Granger and CCM's model assumptions. The time series data in consideration are sampled from the limit cycle in Lotka-Volterra model:

$$\begin{aligned}\dot{x} =& x - 0.01xy\\ \dot{y} =& -y + 0.02xy.\end{aligned}$$

Since observational error is common in real-life data collection, normally distributed noises with zero mean and variable $\sigma^2 = 0, 1, 2, 3$ are added to the variables (see Figure 8). Note that in the above system, there exists bi-directional causal relationship between X and Y.

We first use Granger causality test on X and Y and the results are shown in Table 1. When the system is completely deterministic (without noise), causation is detected in both directions; as the variance of the added noise increases, bi-directional causation is detected but the differences between the residual variances become very small, indicating reduced sensibility. Next, we apply the CCM test to the data sampled from the Lotka-Volterra system with added observational noises. The convergence results are shown in Figure 9. In all four cases, the correlation coefficient converges to 1 and bi-directional causation is detected, which means the added noises do not have much effect on the performance of the CCM test. However, as we further increase the variance of the noise, the CCM test performance does worsen.

variance	0	1	2	3
$\sigma^2(Y\|X,Y)$	7.18e-21	2.1145	7.9981	18.3882
$\sigma^2(Y\|Y)$	0.266	2.2822	8.1651	18.5519
$\sigma^2(X\|X,Y)$	4.20e-21	2.0153	7.8580	17.8043
$\sigma^2(X\|X)$	0.06781	2.0595	7.9011	17.8474

Table 1: Granger test results

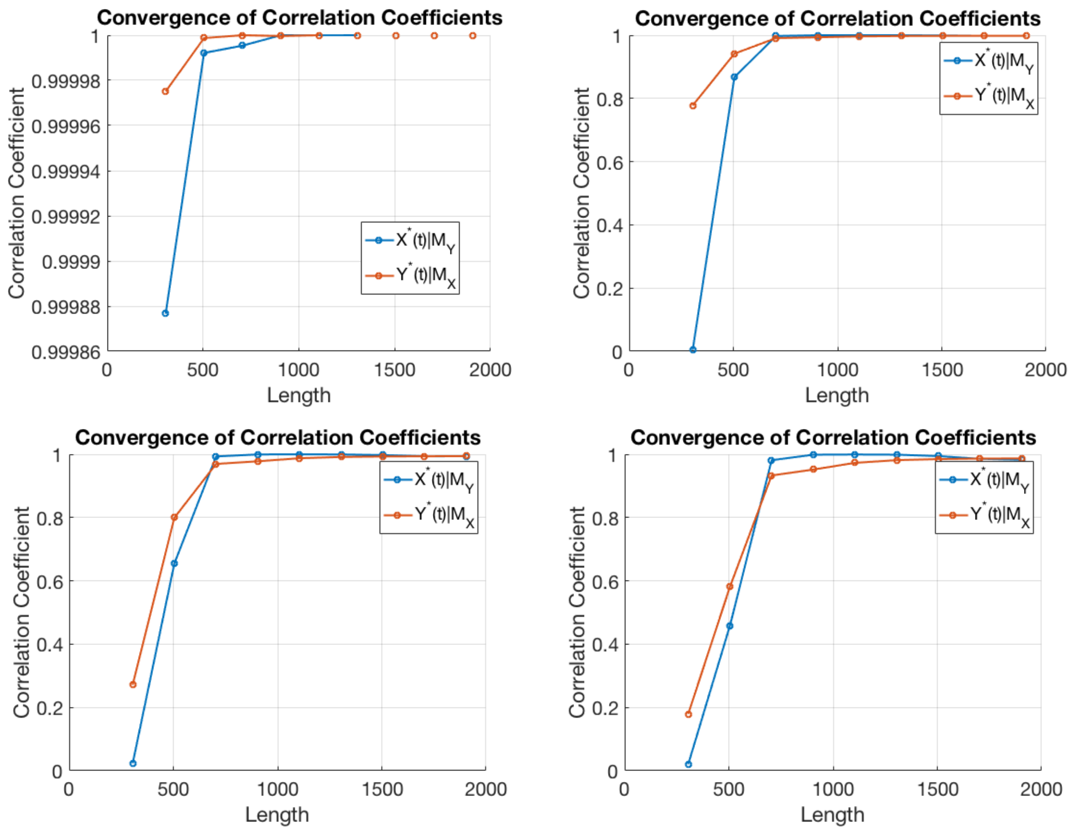

Figure 9: CCM test results with noise variance = 0 (top left), 1 (top right,), 2 (bottom left), and 3 (bottom right).

5 Discussion of results

The Granger causality test performs well on data sampled from the linear auto-regressive model, which is expected; the performance declines when the model is nonlinear. Since the linear and nonlinear models are purely stochastic and the time series do not share the same manifold, the CCM test fails in both examples, also as expected. When the data are sampled from Lorenz attractor, which is chaotic and deterministic, both the Granger causality test and the CCM test successfully identify causal relationships. When the time series data are sampled from a deterministic system with observational errors, when the noise is small, the Granger causality test and the CCM test both perform well. As the noise becomes larger, the Granger test has reduced sensitivity and the CCM test has slower convergence. With added noises, The CCM test in general performs better than the Granger causality test. In conclusion, the Granger causality test works for both linear auto-regressive time series and time series from deterministic systems, and the CCM test works for deterministic systems with or without noises.

A Granger causality test algorithm

Data: Time series $[X_t]_{t=1}^N$ and $[Y_t]_{t=1}^N$, integers p and q ($p \geq q$)
if q=0 **then**
Construct matrix

$$\mathbf{X} = \begin{bmatrix} 1 & Y_p & \cdots & Y_1 \\ \vdots & & & \\ 1 & Y_{N-1} & \cdots, & Y_{N-p} \end{bmatrix}$$

;
else
Construct matrix

$$\mathbf{X} = \begin{bmatrix} 1 & Y_p & \cdots & Y_1 & X_{p+1} & \cdots & X_{p+1-q} \\ \vdots & \dots & \dots & \vdots & \vdots & \dots & \vdots \\ 1 & Y_{N-1} & \cdots, & Y_{N-p} & X_N & \cdots & X_{N-q} \end{bmatrix}$$

;
end
Solve the linear system

$$\mathbf{X}^T \mathbf{X} \vec{b} = \mathbf{X}^T \vec{y},$$

for $\vec{b}$, where $\vec{y} = [Y_{p+1}, Y_{p+2}, \cdots, Y_N]^T$, $\vec{b} = [\beta_0, \beta_1, \cdots, \beta_p, \alpha_0, \cdots, \alpha_q]^T$.
Compute the variance of the residual vector $\sigma^2 = \mathrm{Var}(\vec{r}) = \mathrm{Var}(\vec{y} - \mathbf{X}\vec{b})$.
Result: Variance σ^2 .

B CCM algorithm

```
procedure CONSTRUCT CCM FROM TWO TIME SERIES X AND Y
    data X, Y ← two time series from observations
    initialization τ, E, L
    τ ← time lags
    E ← dimension of the shadow manifold
    L ← length for the data of interest
    Xdata, Ydata ← X(0 : L), Y(0 : L)
    Construct the Shadow Manifolds M_x, M_y
     for k=1:length(M_x) do
     end
    Use knnsearch to find the k-nearest neighbors of M_x(k)
    Use the neighbors to calculate the weight and Ŷ(t)|M_x
    Use knnsearch to find the k-nearest neighbors of M_y(k)
    Use the neighbors to calculate the weight and X̂(t)|M_y

    Calculate the residuals and correlations
end procedure
```