Finite Element Method for Poisson Equations with Dirichlet Boundary Conditions

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Contents

1 Introduction

Scientific and engineering models mostly take the form in differential equations, and since finding the exact analytical solutions is usually expensive, scientists rely on numerical methods to simulate the models. The finite element method is one of the most commonly employed methods for finding numerical solutions to differential equations. It is applied to solve problems in various scientific areas including structural analysis, heat transfer, fluid flow, etc. Specifically, the finite element method reformulates a given differential equation as an equivalent variation problem, which can then be discretized and yield a system of algebraic equations.

In this report, we study the finite element method for 2D Poisson equations. The Poisson equation plays an important role in modeling problems in theoretical physics and mechanical engineering. For example, the solution to the Poisson equation can represent the equilibrium temperature subject to a density of heat source in a two-dimensional domain; it can also describe the displacement of an elastic membrane under an applied load or electric potential given charge density distribution. In practice, there are two main numerical approaches to solving 2D Poisson equations: the finite difference method and the finite element method. The finite difference method is widely employed due to its simple implementation. Nevertheless, the finite element method is more effective when it comes to complicated geometry. Besides the geometry flexibility, the finite element method is also good at solving problems with low regularity because of the variational formulation.

Suppose we have an open, bounded and connected domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$ and consider the boundary value problem

$$-\nabla^2 u = f \text{ in } \Omega \tag{1}$$

$$u = 0 \text{ on } \partial\Omega \tag{2}$$

Multiplying equation (1) by test function v and integrating both sides over Ω, by integration by parts, we obtain the variational form

$$\int_\Omega \nabla u \cdot \nabla v \; \mathrm{d}A = \int_\Omega f v \; \mathrm{d}A, \tag{3}$$

where v is a sufficiently regular function that satisfies $v = 0$ on $\partial\Omega$. Let $a(u,v)$ denote $\int_\Omega \nabla u \cdot \nabla v \; \mathrm{d}A$ and let $F(v)$ denote $\int_\Omega f v \; \mathrm{d}A$. By defining function space $V = \{v : \Omega \to \mathbb{R} \mid a(v,v) < \infty, F(v) \in \infty, \text{ and } v = 0 \text{ on } \partial\Omega\}$ for u and v, we reformulate the Poisson equation as the following variational problem

$$\text{Find } u \in V \text{ such that } a(u,v) = F(v), \text{ for all } v \in V. \tag{4}$$

Note that the regularity requirement has been reduced in variational formulation by shifting one of the derivatives from u to v. Instead of solving the Poisson equation directly which requires $u \in C^2(\Omega)$, now we solve the variational problem that only requires ∇u to be square-integrable.

2 The Lax-Milgram theorem

To investigate the well-posedness of variational problem (4), we first introduce a few notions and definitions in functional analysis.

Definition 1 (Weak derivative [1, p28]). *Let $\Omega \subset \mathbb{R}^n$ and $f \in L^1_{loc}(\Omega)$, where $L^1_{loc}(\Omega) = \{f \mid f \in L^1(K) \text{ for all compact } K \subset \text{ interior } \Omega\}$. Then, $g = D^\alpha$ is the a weak derivative if*

$$\int_\Omega g\phi \; dX = (-1)^{|\alpha|} \int_\Omega f\phi^{(\alpha)} \; dX, \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Note that $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index with length $|\alpha| = \sum_{i=1}^n \alpha_i$, and for $v \in C^\infty(\Omega)$, $D^\alpha v$ is defined as

$$D^\alpha v = \partial_x^\alpha v = v^{(\alpha)} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} v.$$

Definition 2 (Sobolev norm and Sobolev space [1, p29]). *Let $k \in \mathbb{Z}^+$ and $f \in L^1_{loc}(\Omega)$. Suppose that the weak derivative $D^\alpha f$ exists for all $\alpha \leq k$, then, for $p \in [1, \infty]$, the Sobolev norm is defined as*

$$||f||_{W_p^k(\Omega)} = \left(\sum_{|\alpha|<k} ||D^\alpha f||_{L^p(\Omega)} \right)^{1/p},$$

and the Sobolev space $W_p^k(\Omega)$ is defined as

$$W_p^k(\Omega) = \{f \in L^1_{loc}(\Omega) \mid ||f||_{W_p^k}(\Omega) < \infty\}.$$

A Hilbert space is defined as a complete inner product space and Sobolev spaces with $p = 2$ are Hilbert spaces and can be denoted by

$$H^k(\Omega) = W_2^k(\Omega).$$

Moreover, we define the dual of a Hilbert space H^* as the space of all bounded linear functionals on H [1, p51]. Then, the corresponding norm for H^* is defined as

$$||j||_{H^*} = \sup_{||u||_H = 1} |j(u)|.$$

Next, we introduce the definitions of continuity and coercivity of a bilinear form.

Definition 3 (Continuity of a bilinear form [1, p57]). *Let H be a Hilbert space with norm $||\cdot||$. Then, a bilinear form $a : H \times H \longrightarrow \mathbb{R}$ is continuous if there exits $C \in (0, \infty)$ such that*

$$|a(v, w) \leq ||v||_H ||w||_h, \quad \textit{for all } v, w \in H.$$

Note that a linear form is continuous if and only if it is bounded.

Definition 4 (Coercive of a bilinear form [1, p57]). *A bilinear form $a : H \times H \longrightarrow \mathbb{R}$ is coercive on $V \subset H$ if there exists $\alpha > 0$ such that*

$$a(v, w) \geq \alpha ||v||_H^2, \quad \textit{for all } v \in V$$

Following, we state Lax-Milgram Theorem, a fundamental theorem in variational analysis that ensures the existence and uniqueness of the solution to a variational problem.

Theorem 1 (Lax- Milgram theorem [1, p62]). *Let V be a closed subspace of a Hilbert space H, let $a : H \times H \longrightarrow \mathbb{R}$ be a continuous, coercive bilinear form and $F : H \longrightarrow \mathbb{R}$ be a bounded linear functional on V. Then, the variational problem:*

$$\textit{find } u \in V \textit{ such that } a(u, v) = F(v), \quad \textit{for all } v \in V$$

has a unique stable solution.

2.1 Example: 2D Poisson equation with homogeneous Dirichlet boundary condition

Consider boundary value problem

$$-\nabla^2 u = f \quad \text{in } \Omega \tag{5}$$

$$u = 0 \text{ on } \partial\Omega, \tag{6}$$

where we assume $\Omega \subset \mathbb{R}^2$ is open, bounded, connected subset with Lipschitz boundary $\partial\Omega$.

We will prove that the variational formulation of the problem is well-posed by using Lax-Milgram theorem. We start by defining Hilbert space $H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\}$. Then, multiplying equation (5) by test function $v \in H_0^1(\Omega)$, integrating over domain Ω and applying integration by parts, we obtain

$$-\int_\Omega v\nabla^2 u \, \mathrm{d}A = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}A = \int_\Omega fv \, \mathrm{d}A, \text{ for all } v \in H_0^1(\Omega). \tag{7}$$

The boundary terms produced by integration by parts vanish because we strongly impose that $v = 0$ on $\partial\Omega$. Thus , the variational problem is

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a(u,v) = F(v), \text{ for all } v \in H_0^1(\Omega), \tag{8}$$

where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}A$ and $F(v) = \int_\Omega fv \,\mathrm{d}A$.

To show well-posedness, we first prove the continuity of the bilinear form $a(\cdot,\cdot)$ on $H^1(\Omega)$. Since

$$\begin{aligned} |a(u,v)| = |(\nabla u, \nabla v)_{L^2(\Omega)}| \le\ & \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ \le\ & [\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}]^{1/2} \ [\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}]^{1/2} \\ \le\ & \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \end{aligned}$$

by definition, the bilinear form is continuous with continuity constant $C = 1$. Second, we follow Poincaré's inequality to prove the coercivity of $a(\cdot,\cdot)$.

Proposition 1 (Poincaré's inequality [1, p135])**.** *Let Ω be a bounded, Lipschitz domain. Then there exists a constant $C \in \infty$ such that*

$$\|v\|_{W_p^1(\Omega)} \le |v|_{W_p^1(\Omega)}, \quad \textit{for all } v \in W_p^1(\Omega).$$

Applying Proposition (1) with $p = 2$ gives

$$a(v,v) \ge \alpha \|v\|_{H^1(\Omega)}, \quad \text{with } \alpha = 1/C^2. \tag{9}$$

The proof for the linearity and boundedness of $F(v)$ follows from the boundedness of the Dirichlet trace operator [1, p134-135]. Therefore, by Lax-Milgram theorem, the solution to the variational problem exists and is unique.

2.2 Example: 2D Poisson equation with inhomogeneous Dirichlet boundary condition

Now, we consider the case with inhomogeneous boundary condition

$$-\nabla^2 u = f \quad \text{in } \Omega \tag{10}$$

$$u = g \text{ on } \partial\Omega, \tag{11}$$

where we make the same assumptions about Ω as in the previous example. Instead of proving well-posedness directly, we pose an equivalent problem by defining $\hat{u} = u - g$, where we suppose $g \in H^1(\Omega)$. Then, we are able to rewrite the problem as

$$-\nabla^2 \hat{u} = f + \nabla^2 g \quad \text{in } \Omega \tag{12}$$

$$\hat{u} = 0 \text{ on } \partial\Omega, \tag{13}$$

where $\nabla^2 g$ is weakly defined [4, p36]. By the same proof in the homogeneous case, the equivalent equation with homogeneous Dirichlet boundary condition is well-posed and $\hat{u}$ satisfies variational equation

$$a(\hat{u}, v) = F(v) - a(g, v), \text{ for all } v \in H_0^1(\Omega), \tag{14}$$

3 Galerkin discretization

For any linear variational problem

$$\text{find } u \in V \text{ such that } a(u, v) = F(v) \text{ for all } v \in V, \tag{15}$$

we define its Galerkin approximation over a closed subspace $V_h \subset V$ as

$$\text{find } u_h \in V_h \text{ such that } a(u_h, v_h) = F(v_h) \text{ for all } v_h \in V_h. \tag{16}$$

If a variational problem is well-posed by Lax-Milgram theorem, then it can be shown that the corresponding Galerkin approximation is well-posed. For an N-dimensional subspace V_h, by introducing a basis $\{\phi_i\}$ for V_h, we can rewrite the Galerkin approximation as a linear system: $A\mathbf{u} = \mathbf{b}$, where $A_{ij} = a(\phi_i, \phi_j)$, $b_i = F(\phi_i)$, and u_i are the coefficients for the basis functions, i.e.,

$$u \approx u_h = \sum_{i=1}^{N} u_i \phi_i.$$

The structural properties of an well-posed variational problem can be passed onto the linear system. For example, coercive of the bilinear form $a(\cdot, \cdot)$ implies the positive-definiteness of A, from which, we can show that the linear system is also well-posed.

3.1 Finite Element Space

Definition 5 (Finite element [1, p69]). *(K,P,N) is a finite element if*

1. *$K \subseteq \mathbb{R}^n$ is a bounded closed set with nonempty interior and piecewise smooth boundary;*

2. *P is a finite-dimensional function space on K;*

3. *$N = \{N_1, N_2, \cdots, N_M\}$ is a basis for P^*.*

For an finite element (K, P, N), the basis $\{\phi_1, \phi_2, \cdots, \phi_M\}$ of P dual to N that satisfies $N_i(\phi_j) = \delta_{ij}$ is called the nodal basis for P. For example, let $K \subset \mathbb{R}^2$ be a triangle with vertices $(0,0), (0,1)$, and $(1,0)$. Let P be the set of all linear functions on K, and let $N = \{N_1, N_2, N_3\}$ where $N_1(v) = v(0,0), N_2(v) = v(0,1)$, and $N_2(v) = v(1,0)$, for all $v \in P$. Then, the nodal basis for the finite element (K, P, N) consists of $\phi_1(x,y) = 1 - x - y$, $\phi_2(x,y) = x$, and $\phi_3(x,y) = y$.

4 2D Poisson Equations

$$-\nabla^2 u = f \text{ in } \Omega \tag{17}$$

$$u = g \text{ on } \partial\Omega \tag{18}$$

To deal with the inhomogeneous boundary condition, we follow the same procedure as in example (2.2) and set $\hat{u} = u - g$ so that $\hat{u} = 0$ on $\partial\Omega$. Then, the variational problem can be formulated as follows:

$$\text{Find } \hat{u} \in H_0^1(\Omega) \text{ such that } a(\hat{u}, v) = F(v) - a(g, v), \text{ for all } v \in H_0^1(\Omega), \tag{19}$$

which is proven to be well-posed.

4.1 Galerkin discretization

We suppose the domain is polygonal and introduce triangulation $T = \{K_m\}_{m=1}^M$, where K_m are non-overlapping triangles resulted from the geometric decomposition of Ω. We equip each triangle $K \in T$ with a Lagrange element of degree 1 and construct an finite element set $\{(K, P_K, N_K) \,|K \in T\}$. We choose P to be the linear function space on K with dimension $d = 3$, and let $N_K = \{N_1, N_2, N_3 \mid N_1(v) = v(X_1) = v(x_1, y_1), N_2(v) = v(X_2) = v(x_2, y_2)$ and $N_3(v) = v(X_3)\}$, where $X_1 = (x_1, y_1), X_2 = (x_2, y_2), X_3 = (x_3, y_3)$ denote the vertices of triangle K. Note that the local nodal basis $\{\phi_1, \phi_2, \phi_3\}$ satisfies

$N_i(\phi_j) = \delta_{ij}, i, j = 1, 2, 3$ and can be explicitly defined as

$$\phi_1(x,y) = \frac{1}{|K|}\left\|\begin{bmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}\right\|, \phi_2(x,y) = \frac{1}{|K|}\left\|\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{bmatrix}\right\|, \phi_3(x,y) = \frac{1}{|K|}\left\|\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{bmatrix}\right\|,$$

where $|K|$ denotes the area of triangle K [2, p122].

In addition, we specify a local-to-global map from $\{1, 2, 3\}$ to $\{1, \cdots, N\}$ for each $K \in T$, which matches the local degrees of freedoms to the global degrees of freedoms. Note that here we only consider degrees of freedoms in the interior of T since values of u on the boundary are given. To differentiate boundary nodes from interior nodes, we let $X_1, X_2, \cdots, X_N$ denote interior nodes of T and let $X_{N+1}, X_{N+2}, \cdots, X_{N^*}$ denote the boundary nodes of T. Then, by equipping triangulation T with corresponding finite elements, we can construct the global function space $V_h = \{v \in H_0^1(\Omega) \mid v \text{ is linear on } K, \text{for all } K \in T\}$ spanned by $\{\phi_1, \phi_2, \cdots, \phi_N\}$. In addition, we define functions $\phi_{N+1}, \phi_{N+2}, \cdots, \phi_{N^*} \in W_h = \{v \in H^1(\Omega) \mid v \text{ is linear on } K, \text{for all } K \in T\}$ such that $\phi_i(X_j) = \delta_{ij}$ for $i, j = N+1, N+2, \cdots, N^*$. Then, the Galerkin approximation of u in V_h can be written as follows:

$$u_h = \sum_{i=1}^{N} u_i\phi_i + \sum_{i+N+1}^{N^*} g_i\phi_i, \tag{20}$$

where $u_i = u(X_i) = u(x_i, y_i)$, $g_i = g(X_i) = g(x_i, y_i)$. Additionally, we define $h = \max_{K \in T} \operatorname{diam}(K)$ to be the mesh size of triangulation T, which is the longest diameter of the triangles in T.

The Galerkin approximation of the continuous variational problem can be written as

$$\sum_{i=1}^{N} a(\phi_i, \phi_j)u_j + \sum_{i=N+1}^{N^*} a(\phi_i, \phi_j)g_i = F(\phi_j), \text{ for } j = 1, 2, \cdots, N. \tag{21}$$

In matrix form, we have

$$\begin{bmatrix} a(\phi_1, \phi_1) & \cdots & a(\phi_1, \phi_N)) \\ \vdots & \ddots & \vdots \\ a(\phi_N, \phi_1) & \cdots & a(\phi_N, \phi_N)) \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} F(\phi_1) \\ \vdots \\ F(\phi_N) \end{bmatrix} - \begin{bmatrix} a(\phi_1, \phi_{N+1}) & \cdots & a(\phi_1, \phi_{N^*})) \\ \vdots & \ddots & \vdots \\ a(\phi_N, \phi_{N+1}) & \cdots & a(\phi_N, \phi_{N^*})) \end{bmatrix}\begin{bmatrix} g_{N+1} \\ \vdots \\ g_{N^*} \end{bmatrix},$$

which can be summarized as $A\mathbf{u} = \mathbf{b}$, and A is commonly referred to as the stiffness matrix.

4.2 Local and global assembly

To efficiently compute the global stiffness matrix A, we first compute a 3×3 local stiffness matrix a^K for all triangles $K \in T$. Then, we assemble the global stiffness matrix by looping over each triangle and adding the local matrices to the global one using the local-to-global map. Consequently, we have

$$A_{ij} = \sum_{K \in T} a_{ij}^K,$$

where A_{ij} denotes the entries of the global stiffness matrix A, and a_{ij}^K denotes the entries of local stiffness matrix a^K for triangle K. Since we have the explicit formula for the local basis functions $\{\phi_1, \phi_2, \phi_3\}$ for each K, we can compute the local stiffness matrices exactly. For K with vertices $(x_j, y_j), j = 1, 2, 3$,

$$\nabla \phi_j(x, y) = \frac{1}{2|K|}(y_{j+1} - y_{j+2}, x_{j+2} - x_{j+1})$$

with index mod 3, from which we obtain

$$\begin{aligned} a_{ij}^K &= \int_K \nabla \phi_i \cdot \nabla \phi_j \, \mathrm{d}x\mathrm{d}y \\ &= \int_K \frac{|K|}{4|K|^2}(y_{i+1} - y_{i+2}, x_{i+2} - x_{i+1}) \cdot (y_{j+1} - y_{j+2}, x_{j+2} - x_{j+1}) \, \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Recall that $|K|$ denotes the area of K [2, 122].

We adopt the similar approach to compute the RHS vector $\mathbf{b}$. It is worth mentioning that we use Gaussian quadrature to evaluate vector $[F(\phi_1), F(\phi_2), F(\phi_3)]^T$. Let h_j denote $f\phi_j$, then,

$$\begin{aligned} F(\phi_j) &= \int_K f\phi_j \, \mathrm{d}x\mathrm{d}y \\ &= \frac{|K|}{6}\left[h_j\left(\frac{x_j + x_{j+1}}{2}, \frac{y_j + y_{j+1}}{2}\right) + h_j\left(\frac{x_j + x_{j+2}}{2}, \frac{y_j + y_{j+2}}{2}\right)\right], \end{aligned}$$

with index mod 3.

4.3 Numerical result

Consider the model problem

$$\nabla^2 u = -5\pi^2 \sin(\pi x)\cos(2\pi y) \text{ in } \Omega \tag{22}$$

$$u = -\sin(\pi x)\cos(2\pi y) \text{ on } \partial\Omega, \tag{23}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is the unit disk. The exact solution to the problem is $u = -\sin(\pi x)\cos(2\pi y)$. We use MATLAB's PDE Toolbox to generate a triangulation T_h of the disk Ω with $h = 0.075$ (see figure (1)). Solving the linear system $A\mathbf{u} = \mathbf{b}$ given by Galerkin discretization over T_h, we obtain the Galerkin approximation $I_h u = u_h$. Plotting u_h and the exact solution u over Ω give figure (2).

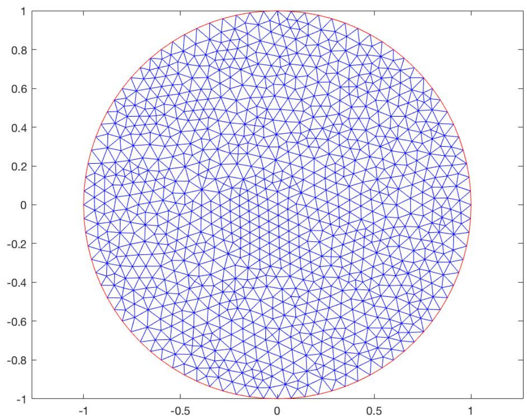
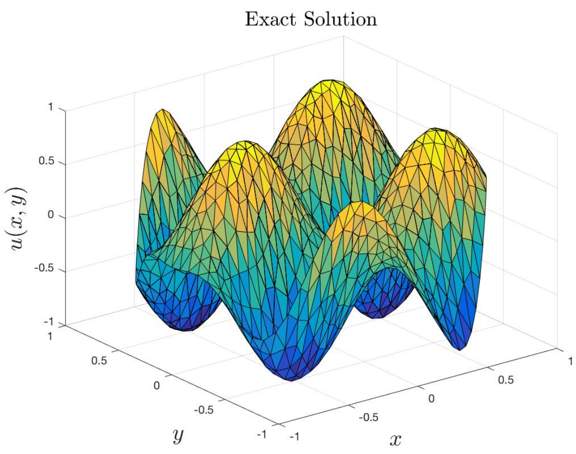
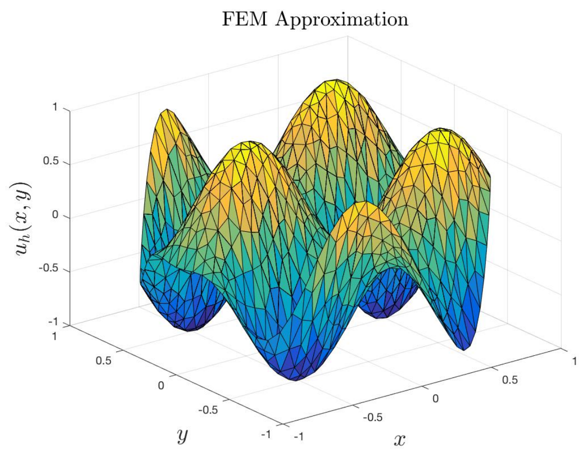


Figure 1: Triangulation T of the domain Ω with $h = 0.075$



(a) Exact solution u



(b) Galerkin approximation u_h

Figure 2: FEM approximation with $h = 0.075$

4.4 Error estimate and convergence

To prove the convergence of the finite element method, we look for error bounds of the form $||u - u_h||_{H^1(\Omega)} \leq Ch^p$, where C is a constant and p is the rate of convergence. We consider the following theorems:

Theorem 2 (Cea's theorem [1, p64])**.** *Let H be a Hilbert space and V_h be a finite dimensional subspace of H. Suppose the variational problem*

$$\textit{Find } u \in H \textit{ such that } a(u, v) = F(v), \textit{ for any } v \in H$$

satisfies the hypothesis of Lax-Milgram theorem, and u and u_h are the weak solution and the Galerkin approximation of the problem, respectively. Then,

$$||u - u_h||_{H(\Omega)} \leq C_0 \inf_{v \in V_h} ||u - v||_{H(\Omega)}. \tag{24}$$

Theorem 3 ([4, p90-91])**.** *Let V_h be a finite function space constructed with continuous Lagrange element of order p on a regular triangulation T of Ω with mesh size h. Let $u \in H^{p+1}(\Omega)$ and $I_h : H^{p+1} \longrightarrow V_h$ be the interpolation operator associated with V_h. Then, there exists a constant $C_1 \in \infty$ independent of u such that*

$$||u - \mathbf{I}_h u||_{H^1(\Omega)} \leq C_1 h^p ||u||_{H^{p+1}(\Omega)}. \tag{25}$$

Suppose that the conditions for Cea's theorem 2 and theorem 3 hold. Then, following from (24) and (25), we obtain

$$||u - u_h||_{H^1(\Omega)} \leq C_3 h^p ||u||_{H^{p+1}(\Omega)}. \tag{26}$$

Since we use linear Lagrange elements to approximate variational problem (19) in section 4, we have $||u - u_h||_{H^1(\Omega)} \leq C_3 h ||u||_{H^2(\Omega)}$, which implies $||u - u_h||_{H^1(\Omega)} = o(h)$.

To numerically investigate the convergence of FEM, we solve the model problem in section 4.3 multiple times, while refining the triangulation each time by halving the mesh size h. The error $||u - u_h||_{H^1(\Omega)}$ can be computed using Gaussian quadrature:

$$||u - u_h||_{H^1(\Omega)} = \left[\sum_{K \in T} \int_K |u - u_h|^2 + |\nabla u - \nabla u_h|^2 \mathrm{d}x\mathrm{d}y \right]^{1/2} \tag{27}$$

$$= \left[\sum_{K \in T} |K| (|u(\xi, \eta) - u_h(\xi, \eta)|^2 + |\nabla u(\xi, \eta) - \nabla u_h|^2) \right]^{1/2} \tag{28}$$

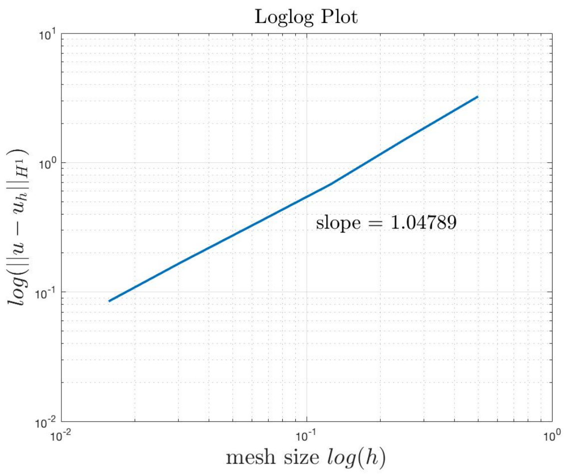


Figure 3: Log-log plot

where $\xi = (x_1 + x_2 + x_3)/3$ and $\eta = (y_1 + y_2 + y_3)/3$. Note that since u_h is linear on each element domain $K \in T$, the term ∇u_h in (28) is independent of (x, y) and can be computed exactly. Let $e_h = ||u - u_h||_{H^1(\Omega)}$, then we can calculate the rate of convergence using formula:

$$p = \log_2 \left| \frac{e_h}{e_{h/2}} \right|. \tag{29}$$

The numerical implementation suggests that the Galerkin approximation u_h converges to solution u at a linear rate (see table 1). Moreover, we plot error e_h against mesh size h in logarithmic scale (see figure 3) and find that the slope of the line is approximately equal to 1, which also suggests $e_h = o(h)$. The numerical result is consistent with the previous convergence analysis.

Table 1: Table of Rates of Convergence

Mesh size h	$\lVert u - u_h \rVert_{H^1(\Omega)}$	Rate of Convergence
0.5	3.2432	–
0.25	1.5000	1.1125
0.125	0.67719	1.1473
0.0625	0.34013	0.99345
0.03125	0.17177	0.98559
0.015625	0.084593	1.0219

5 Conclusion and future works

To conclude, in this report, we studied the mathematical framework of finite element methods. Most importantly, we discussed the Lax-Milgram theorem and applied it to show the well-posedness of the variational formulation of Poisson equations with Dirichlet boundary conditions.

Applying Galerkin discretization, we created a finite element solver for Poisson equations with inhomogeneous Dirichlet boundary conditions. To investigate the convergence of the finite element method, we used Cea's theorem to find error bounds and show linear convergence in H_1-norm. Using the FEM solver to obtain the numerical solutions to a model problem, we computed the numerical errors in H_1-norm and confirmed the rate of convergence is 1.

For future works, we can develop FEM solvers for Poisson equations with Neumann boundary conditions or other types of PDEs, i.e. heat equations and wave equations.

Appendices

A Compute local stiffness matrices

In our code, we adopt Long Chen's algorithm in "Programming of Finite Element Methods in MATLAB" to compute local stiffness matrices [3, p6]:

$$
\begin{aligned}
a_{ij}^K =& \int_K \nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}x\mathrm{d}y \\
=& \frac{1}{2}|\det(B)|(B^{-1}G_i)\cdot(B^{-1}G_j),
\end{aligned}
$$

where

$$
B = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}, G_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } G_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.
$$

B MATLAB code for solving 2D Poisson equation with Dirichlet boundary condition

```
function [ ] = FEM_poisson( f,g, Hmax)
% f(X) - RHS function , X = [x,y];
% g(x,y) -  derichlet boundary condition;
% Hmax - meshsize
%% generate mesh
model = createpde(1);
geometryFromEdges(model,@circleg);
MESH=generateMesh(model,'Hmax',Hmax); % use PDE toolbox to generate triangu
pdeplot(model)
axis equal

[p,e,t] = meshToPet(model.Mesh); % export mesh data
node = p'; % nodes
elem = t(1:3,:)'; % elements
bdy = e(1,:); % boundary nodes
bdy(2,:) = 0; % input homogeneous dirichlet boundary condition
for j = 1:length(bdy(1,:))
    i = bdy(1,j);
```

```
        bdy(2,j)=g(node(i,1),node(i,2));
    end
    M = size(elem,1); % number of finite elements
    N = size(node,1); % numbe of nodes

    %% construct global stiffness matrix
    % construct lobal stiffness matrix
    locS = zeros(3*M,3);locB = zeros(3,M);
    for l = 1:M
        p = [node(elem(l,1),1),node(elem(l,2),1),node(elem(l,3),1);
            node(elem(l,1),2),node(elem(l,2),2),node(elem(l,3),2)];
            % p = [x1,x2,x3;y1,y2,y3]; (xi,yi), i = 1,2,3, are the coordinates
            % three vertices of the triangle l

        % CONSTRUCT LOCAL STIFFNESS
        % we adopt Long Chen's algorithm to compute local stiffness matrices
        Bnode = [p(1,1)-p(1,3),p(2,1)-p(2,3); p(1,2)-p(1,3),p(2,2)-p(2,3)];
        G = [[1,0]',[0,1]',[-1,-1]'];
        area = 0.5*abs(det(Bnode));
        s = zeros(3,3);
        for i = 1:3
            for j = 1:3
                s(i,j) = area*((Bnode\G(:,i))'*(Bnode\G(:,j)));
            end
        end
        % store local stiffness matrices
        locS(3*(l-1)+1:3*(l-1)+3,:) = s;

        % we use Gaussian quadrature to compute local load vectors
        mid1 = ( node(elem(l,2),:)+node(elem(l,3),:) )/2;
        mid2 = ( node(elem(l,3),:)+node(elem(l,1),:) )/2;
        mid3 = ( node(elem(l,1),:)+node(elem(l,2),:) )/2;
        b1 = area.* (f(mid2)+f(mid3))/6;
        b2 = area.* (f(mid3)+f(mid1))/6;
        b3 = area.* (f(mid1)+f(mid2))/6;
        % store local stiffness matrices
```

```
    locB(:,l) = [b1;b2;b3];
end
% assemble global stiffness matrix
S= sparse(N,N); B =zeros(N,1);
for l = 1:M
    for i = 1:3
        for j = 1:3
            S(elem(l,i),elem(l,j)) = S(elem(l,i),elem(l,j)) + locS((l-1)*3+
            % global assembly of global stiffness matrix
        end
    end
    for k = 1:3
            B(elem(l,k)) = B(elem(l,k)) + locB(k,l);
            % global assembly of global load vector
    end
end

%% implement dirichlet boundary condition
Xbdy = bdy(1,:);
Xint = setdiff(1:N,Xbdy); %separate boundary and interior nodes
Uh = zeros(N,1);
Uh(Xbdy) = bdy(2,:)';
B = B - S*Uh; % insert nonzero dirichlet BCs
S2 = S(Xint,Xint);
B2 = B(Xint);
Uh(Xint) = S2\B2; % use backslash operator to solve sparse linear system

%% plot solution & FEM approximation
figure (1)
trisurf(elem,node(:,1)',node(:,2)',Uh); % plot numerical solution
title('FEM Approximation')
xlabel('x'); ylabel('y'); zlabel('u_h(x,y)')
axis equal
end
```

References

[1] Brenner, Susanne C., and L. Ridgway. Scott. *The Mathematical Theory of Finite Element Methods*. 3rd ed., Springer, 2008.

[2] Carstensen, Carsten, et al. "Remarks around 50 Lines of Matlab: Short Finite Element Implementation." *Numerical Algorithms*, vol. 20, no. 2-3, Aug. 1999, pp. 117–137., doi:https://doi.org/10.1023/A:1019155918070.

[3] Chen, Long. "Programming of Finite Element Methods in MATLAB." Cornell University Library,*Cornell University*, 14 Apr. 2018, www.cornell.edu.

[4] Braess, Dietrich. *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*. 3rd ed., Cambridge University Press, 2007.