# Some new combinatorial identities related to representations of affine Lie algebras

Mirko Primc
University of Zagreb
-
joint work with Tomislav Šikić

Discretaly: A Workshop in Discrete Mathematics
Rome, Italy, February 1–2, 2018

# Funding

This work was supported by the QuantiXLie Centre of Excellence, a project Co-financed by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004).

Partially supported by the *Croatian Science Foundation* under the project 2634.

# Notation for affine Kac-Moody Lie algebras

- The classification of complex simple Lie algebras:
  $A_\ell, B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4, G_2$.
- The classification of irreducible finite dimensional representations: $L(k_1, \ldots, k_\ell), \quad k_1, \ldots, k_\ell \in \mathbb{N}_0$.

- The classification of affine Kac-Moody Lie algebras:
  $A_1^{(1)}, A_\ell^{(1)}, B_\ell^{(1)}, C_\ell^{(1)}, D_\ell^{(1)}, \ldots; A_2^{(2)}, A_{2\ell}^{(2)}, A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, E_6^{(2)}; D_4^{(3)}$.
- The standard representations of affine Lie algebras:
  $L(k_0, k_1, \ldots, k_\ell), \quad k_0, k_1, \ldots, k_\ell \in \mathbb{N}_0, \quad$ level $k$.

- rank $\ell$ is the "size" of Lie algebra, the "smallest" are $A_1^{(1)}, A_2^{(2)}$
  level $k$ s the "size" of standard module $L(k_0, k_1, \ldots, k_\ell)$,
  the "smallest" are level one $L(1, 0, \ldots, 0)$.

# Rogers-Ramanujan type combinatorial identities

- **[Rogers-Ramanujan 1894, Schur 1917]**

$$\prod_{m\geq 0}\frac{1}{(1-q^{5m+1})(1-q^{5m+4})}=\sum_{m\geq 0}\frac{q^{m^2}}{(1-q)(1-q^2)\cdots(1-q^m)}$$

- A partition of positive integer $n$
$n=\sum_{j\geq 1} f_j\cdot j=\underbrace{1+\cdots+1}_{f_1 \text{ times}}+\underbrace{2+\cdots+2}_{f_2 \text{ times}}+\ldots$
- **Combinatorially:** The number of partitions of $n$ such that

$$f_j+f_{j+1}\leq 1$$

equals the number of partitions of $n$ into parts $\not\equiv 0,\pm 2$ mod 5.
- **[Gordon 1961]** The number of partitions of $n$ such that

$$f_j+f_{j+1}\leq k$$

equals the number of partitions of $n$ into parts $\not\equiv 0,\pm(k+1)$ mod $(2k+3)$.

# Lepowsky-Wilson's approach

- **[Andrews 1967, Bressoud 1979, 1980]** generalizations of Rogers-Ramanujan identities for all moduli.
- **[Lepowsky-Milne 1978]** The product side in Gordon, Andrews and Bressoud identities is given by Weyl-Kac character formula for standard $A_1^{(1)}$-modules.
- **[Lepowsky-Wilson 1981]** a Lie-theoretic interpretation of difference conditions in Gordon, Andrews and Bressoud identities is given by the vertex operator construction of combinatorial bases of standard $A_1^{(1)}$-modules.
- **[Mandia 1987]** $B_\ell^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$ level 1.
- **[Misra 1991]** $A_\ell^{(1)}$ level 2.
- **[Capparelli, Andrews, 1992]** Two new combinatorial identities for $A_2^{(2)}$ at level 3.

# Lepowsky-Wilson's approach

- **[Tamba 1994]** $D_{\ell+1}^{(2)}$ level 2.
- **[Bos-Misra 1994]** $A_7^{(2)}$ level 2.
- **[Meurman-P 1999]** $A_1^{(1)}$ all levels.
- **[Meurman-P 2001]** $A_2^{(1)}$ level 1.
- **[Siladić 2002]** $A_2^{(2)}$ and $B_2^{(1)}$ level 1.
- **[Kanade-Russell 2014]** $D_4^{(3)}$ level 3.
- **[Nandi 2014]** $A_2^{(2)}$ level 4.
- **[Šikić-P 2016]** $C_\ell^{(1)}$ level 1; conjecture for all levels.
- **[Kanade 2017]** $A_5^{(2)}$ level 2.
- **[Kanade-Russell 2017]** $A_9^{(2)}$ level 2.

# Difference conditions in Gordon's identities

The difference condition

$$f_j + f_{j+1} \leq k \quad \text{for all} \quad j \in \mathbb{N}$$

in Gordon's identity can be written as

$$\sum_{a \in \mathcal{Z}} f_a \leq k \quad \text{for all downward lines} \quad \mathcal{Z}$$

in the array of natural numbers

$$\begin{array}{cccccccccccc} & 2 & & \color{blue}{4} & & 6 & & \color{red}{8} & & 10 & & \\ 1 & & \color{blue}{3} & & 5 & & 7 & & \color{red}{9} & & & \ldots \end{array}$$

# Arrays $\mathcal{A}_{1,2}$ and $\mathcal{A}_{1,1}$

$\mathcal{A}_{1,2}$ is the array[1] of natural numbers

$$\begin{array}{ccccccccccc} 2 & & 4 & & 8 & & 10 & & 14 & & \\ & \color{red}{3} & & 6 & & \color{red}{9} & & 12 & & & \cdots \\ \color{red}{1} & & \color{red}{5} & & \color{red}{7} & & \color{red}{11} & & \color{red}{13} & & \end{array}$$

$\mathcal{A}_{1,1}$ is the array of natural numbers

$$\begin{array}{ccccccccccc} 1 & & 3 & & 5 & & 7 & & 9 & & \\ & 2 & & 4 & & 6 & & 8 & & & \cdots \\ 1 & & 3 & & 5 & & 7 & & 9 & & \end{array}$$

[1] it is *"something like" the* $(1,2)$*-specialized weigted crystal of negative roots for* $A_1^{(1)}$

# Capparelli's identity

**[Capparelli 1993]** The number of partitions of $n = \sum_{a\in\mathcal{A}} f_a \cdot a$ such that

$$f_2 = 0 \quad \text{and} \quad \sum_{a\in\mathcal{Z}} f_a \leq 1 \quad \text{for all downward paths} \quad \mathcal{Z},$$

(i.e. paths $\mathcal{Z}$ going downwards, possibly in zigzag maner) in the array of natural numbers $\mathcal{A}$

$$\begin{array}{cccccccccccc}
2 & & 4 & & \color{red}{8} & & 10 & & 14 & & & \\
& 3 & & \color{red}{6} & & 9 & & 12 & & & \cdots & \\
1 & & 5 & & \color{red}{7} & & 11 & & 13 & & &
\end{array}$$

equals the number of partitions of $n$ into distinct parts $\not\equiv \pm 2$ mod 6.

# A generalization of Capparelli's identities

**[Capparelli $k=1$ 1993, Meurman-P 1999]**[2] Let $k_0, k_1 \in \mathbb{N}_0$, $k = k_0 + k_1$.

The number of partitions of $n = \sum_{a\in\mathcal{A}} f_a \cdot a$ such that

$$f_1 \leq k_0,\ f_2 \leq k_1 \quad \text{and} \quad \sum_{a\in\mathcal{Z}} f_a \leq k \quad \text{for all downward paths} \quad \mathcal{Z}$$

in the array of natural numbers $\mathcal{A}$

$$\begin{array}{cccccccccccc}
2 & & 4 & & 8 & & 10 & & 14 & & & \\
& 3 & & 6 & & 9 & & 12 & & & \cdots & \\
1 & & 5 & & 7 & & 11 & & 13 & & &
\end{array}$$

equals the number of partitions of $n$ given by the coefficient of $q^n$ in

$$\prod_{\substack{r\equiv 0,\pm(k_0+1) \mod 2(k+2) \\ s\equiv \pm 2(k_1+1) \mod 4(k+2)}} (1-q^r)(1-q^s) \Big/ \prod_{j=1}^{\infty}(1-q^j).$$

[2] **[Feigin-Kedem-Loktev-Miwa-Mukhin 2001]**

# Some combinatorial identities for $A_1^{(1)}$

**[Meurman-P 1999]** Let $k_0, k_1 \in \mathbb{N}_0$, $k = k_0 + k_1$.

The number of colored partitions of $n = \sum_{a\in\mathcal{A}} f_a \cdot a$ such that

$$\sum_{a\in\mathcal{Z}} f_a \le k \quad \text{for all downward paths} \quad \mathcal{Z}, \quad f_1 \le k_0, \; f_{\color{red}1} \le k_1,$$

in the array of natural numbers $\mathcal{A}$

$$\begin{array}{ccccccccccc}
\color{red}1 & & \color{red}3 & & \color{red}5 & & \color{red}7 & & \color{red}9 & & \\
& 2 & & 4 & & 6 & & 8 & & & \cdots \\
1 & & 3 & & 5 & & 7 & & 9 & &
\end{array}$$

equals the number of partitions of $n$ given by the product

$$\prod_{\substack{r\equiv 0,k_0+1 \mod (k+2) \\ s\equiv k_1+1 \mod (k+2)}} (1-q^r)(1-q^s) \Big/ \prod_{i \text{ odd},\, j\in\mathbb{N}} (1-q^i)(1-q^j).$$

# New combinatorial identities for $C_\ell^{(1)}$-modules

**[Šikić-P 2016]** Let $\ell \geq 2$.

The number of colored partitions of $n = \sum_{a \in \mathcal{A}_\ell} f_a \cdot a$ such that

$$\sum_{a \in \mathcal{Z}} f_a \leq 1 \quad \text{for all downward paths } \mathcal{Z} \text{ in the array } \mathcal{A}_\ell$$

$$\begin{array}{cccccccccccc}
 & & & & 5 & & 7 & & \color{red}{9} & & & \\
 & & & 4 & & 6 & & \color{red}{8} & & 10 & & \\
 & & 3 & & 5 & & 7 & & \color{red}{9} & & & \dots \\
 & 2 & & 4 & & 6 & & \color{red}{8} & & 10 & & \\
1 & & 3 & & 5 & & \color{red}{7} & & 9 & & &
\end{array}$$

(on the figure $\ell = 2$) equals the number of partitions of $n$ into parts $\not\equiv 0, \pm 2 \mod (2\ell + 2)$.

# About the proof

The proof follows [Meurman-P 1999] for $A_1^{(1)} = C_1^{(1)}$:

Instead of constructing a basis of $L(\Lambda_0) \cong U(\widehat{\mathfrak{g}}_{<0})/J$ , we construct a Groebner basis of the maximal submodule $J$. We need

- generators $\{r\}$ of $J$,
- relations $x_1 r_1 \sim x_2 r_2$,
- a tally of generators and a tally of relations needed.

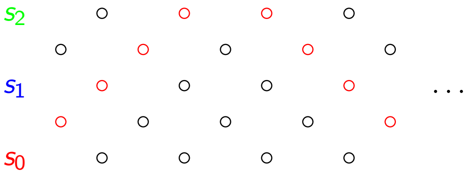
A basis of $\widehat{\mathfrak{g}}_{<0} = \mathfrak{g} \otimes t^{-1} + \mathfrak{g} \otimes t^{-2} + \dots$ is parametrized by

$$
\begin{array}{ccccccccccc}
 & & 3 & & 5 & & 7 & & 9 & & \\
 & 2 & & 4 & & 6 & & 8 & & 10 & \\
1 & & 3 & & 5 & & 7 & & 9 & &
\end{array}
$$

and the generators $r$ of $J$ are parametrized by frequences $(f_a \mid a \in \mathcal{Z})$ such that $\sum_{a\in\mathcal{Z}} f_a = 2$.

# The congruence arrays

The congruence array $\mathcal{C}_{s_0,s_1,s_2}$ for $C_2^{(1)}$ is the array of natural numbers starting on the left with $s_0, s_1, s_2$




and filled out to the right in such a way that adjacent diagonals change by adding a number $s_i$ for some $i = 0, 1, 2$. "Red places" $\circ$ change by the rule

$$\text{“along”}\ \circ \xrightarrow{s_1} \circ \xrightarrow{s_2} \circ \xrightarrow{s_2} \circ \xrightarrow{s_1} \circ\ , \qquad \text{“across”}\ \circ \xrightarrow{s_0} \circ \xrightarrow{s_0} \circ\ ,$$

and the other places on diagonals follow the same rule.

Similarly we define the congruence array $\mathcal{C}_{s_0,s_1,\ldots,s_\ell}$ for $C_\ell^{(1)}$.

# Example: the congruence array $\mathcal{C}_{3,1,1}$

$$
\begin{array}{ccccccccccc}
1 & & 3 & & 7 & & 13 & & 17 & & \\
& 2 & & 6 & & 10 & & 14 & & 18 & \\
1 & & 5 & & 9 & & 11 & & 15 & & \dots \\
& 4 & & 8 & & 10 & & 12 & & 16 & \\
3 & & 7 & & 9 & & 11 & & 13 & &
\end{array}
$$

Diagonals change by the rule:

$$\circ \xrightarrow{1} \circ \xrightarrow{1} \circ \xrightarrow{1} \circ \xrightarrow{1} \circ \qquad \circ \xrightarrow{3} \circ \xrightarrow{3} \circ$$

$\mathcal{C}_{3,1,1}$ is periodic (up to flipping upside down every second triangle)
$\mod 2(k+\ell+1) = \mod (3+7) = \mod 10$,
so it is sufficient to fill out only the upper-left triangular part and get that all the elements in $\mathcal{C}_{3,1,1}$ are numbers

$$0, 0, \pm 1, \pm 1, \pm 2, \pm 3, 3, 4, 5, 6, 7 \mod 10.$$

# Lepowsky's product formula

Lepowsky's product formula for the principally specialized character of a standard $C_\ell^{(1)}$-module is

$$ch_q\, L(k_0, k_1, \ldots, k_\ell) = \prod_{a \in \mathcal{C}_{k_0+1,k_1+1,\ldots,k_\ell+1}} (1-q^a) \Big/ \prod_{r \in \mathcal{C}_{1,1,\ldots,1}} (1-q^r)$$

Roughly speaking, Lepowsky's product formula is a generating function for number of colored partitions on array $\mathcal{C}_{1,\ldots,1}$ satisfying "congruence condition" that no part is in $\mathcal{C}_{k_0+1,\ldots,k_\ell+1}$

Our example gives the principally specialized character $ch_q L(2,0,0)$

$$\prod_{a \equiv 0,0,\pm1,\pm1,\pm2,\pm3,3,4,5,6,7 \mod 10} (1-q^a) \Big/ \prod_{i \text{ odd},\ j \in \mathbb{N}} (1-q^i)(1-q^j)^2.$$

**Conjecture [Šikić-P 2016]** Let $k \geq 2$.

The number of colored partitions of $n = \sum_{a \in \mathcal{A}_2} f_a \cdot a$ such that

$$\sum_{a \in \mathcal{Z}} f_a \leq k \quad \text{for all downward paths } \mathcal{Z} \text{ in the array } \mathcal{A}_2:$$

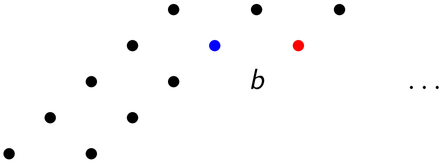
$$\begin{array}{cccccccccccc}
 & & & & 5 & & 7 & & 9 & & & \\
 & & & 4 & & 6 & & 8 & & 10 & & \\
 & & 3 & & 5 & & 7 & & 9 & & & \dots \\
 & 2 & & 4 & & 6 & & 8 & & 10 & & \\
1 & & 3 & & 5 & & 7 & & 9 & & &
\end{array}$$

equals the number of colored partitions of $n$ given by the coefficient of $q^n$ in

$$\prod_{\substack{i \text{ odd} \\ r \not\equiv 0, \pm 1, \pm 2, \pm 3 \mod (2k+6) \\ s \not\equiv 0, \pm 1, \pm(k+1), \pm(k+2), k+3 \mod (2k+6)}} (1-q^i)^{-1}(1-q^r)^{-1}(1-q^s)^{-1}$$

# A program for counting colored partitions

We can construct a partition $n = \sum_{a\in\mathcal{A}_2} f_a\, a$ satisfying difference condition by adding rows of frequences in steps: assume we completed a list of frequences $f_a$ in the first two rows and in the top two places in the third row.




Then choose a frequency $f_b$ so that $m_b = f_b + max\{m_{\color{blue}\bullet}, m_{\color{red}\bullet}\} \leq k$,

$$m_{\color{blue}\bullet} = \max\{\sum_{a\in\mathcal{Z}} f_a \mid \mathcal{Z} \ \text{ ends in } {\color{blue}\bullet}\}, m_{\color{red}\bullet} = \max\{\sum_{a\in\mathcal{Z}} f_a \mid \mathcal{Z} \ \text{ ends in } {\color{red}\bullet}\}$$

# Thanks to Shashank and Andrej

Tomislav and I thank Shashank Kanade for a program in Maple for counting the number of colored partitions for $C_2^{(1)}$ and we thank Andrej Primc for programs[3] in Python for counting colored partitions for $C_\ell^{(1)}$ - see: https://github.com/aprimc/discretaly

It seems that there is no flaw in calculations and that conjectured identities for $C_2^{(1)}$-modules $L(k,0,0)$ agree for all numbers $\leq 30$ for levels $k = 2, \ldots, 10$.

We also checked that for $C_3^{(1)}$-modules $L(2,0,0,0)$ and $L(3,0,0,0)$ the number of partitions of $n \leq 30$ satisfying difference conditions equals the number of partitions of $n$ satisfying congruence conditions.

[3] The program part2.py counts for $n \leq N$ the number of all colored partitions of $n$ corresponding to $C_l^{(1)}$-module $L(k,0,\ldots,0)$. partition3.py counts for numbers in range (1, N) partitions in three colours satisfying congruence conditions.

## Can we extend the $C_1^{(1)}$ -identities to $C_\ell^{(1)}$ ?

Is the number of colored partitions[4] of $n = \sum_{a \in \mathcal{A}} f_a \cdot a$ such that

$$\sum_{a \in \mathcal{Z}} f_a \leq k \quad \text{for all downward paths } \mathcal{Z} \text{ in the array } \mathcal{A} \text{ and}$$

$$\sum_{a \in \mathcal{Z}} f_a \leq k_0 \text{ for all } \mathcal{Z} \subset \Delta, \quad \sum_{a \in \mathcal{Z}} f_a \leq k_\ell \text{ for all } \mathcal{Z} \subset \Delta,$$

$$\begin{array}{ccccccccccc}
1 & & 3 & & 5 & & 7 & & 9 & & \\
& 2 & & 4 & & 6 & & 8 & & 10 & \\
& & 3 & & 5 & & 7 & & 9 & & \dots \\
& 2 & & 4 & & 6 & & 8 & & 10 & \\
1 & & 3 & & 5 & & 7 & & 9 & &
\end{array}$$

equal to the number of colored partitions of $n$ given by the coefficient of $q^n$ in the principally specialized character of $L(k_0, 0, \dots, 0, k_\ell),\ k = k_0 + k_\ell$ ?

[4] part8.py counts the number of this kind of colored partitions of $n$.

For the following $C_\ell^{(1)}$-modules the number of partitions of $n \leq 40$ satisfying difference conditions equals the number of partitions of $n$ satisfying congruence conditions.

- $L(1,0,1) \sim \prod_{r\equiv 1,1,3,4,4,6,6,7,9,9 \mod 10}$
- $L(2,0,1) \sim \prod_{r \text{ odd};\ r\equiv 1,2,4,5,6,7,8,10,11 \mod 12}$
- $L(2,0,2) \sim \prod_{r\equiv 1,1,2,2,3,5,5,5,6,6,8,8,9,9,9,11,12,12,13,13 \mod 14}$
- $L(3,0,1) \sim \prod_{r \text{ odd};\ r\equiv 1,2,3,4,6,6,7,8,8,10,11,12,13 \mod 14}$
- $L(3,0,2) \sim \prod_{r \text{ odd};\ r\equiv 1,2,2,3,5,6,6,7,8,9,10,10,11,13,14,14,15 \mod 16}$
- $L(1,0,0,1) \sim \prod_{r \text{ odd};\ r\equiv 1,3,4,5,7,9,11 \mod 12}$
- $L(2,0,0,1) \sim \prod_{r \text{ odd};\ r\equiv 1,2,3,4,5,6,6,8,8,9,10,11,12,13 \mod 14}$
- $L(1,0,0,0,1) \sim \prod_{r\equiv 1,1,3,3,4,4,5,6,6,8,8,9,10,10,11,11,13,13 \mod 14}$

- Principally specialized $ch_q\, L(2,0,0,2)$ is the product

$$\prod_{r \text{ odd};\ r\equiv 1,2,2,3,4,5,6,6,6,7,9,10,10,10,11,12,13,14,14,15 \mod 16} (1-q^r)^{-1}$$

- The number of the corresponding partitions for $1 \leq n \leq 10$ is $2, 5, 10, 19, 34, 60, 100, 163, 260, 406$, whereas the difference conditions give $2, 5, 10, 19, 34, 61, 102, 168, 270, 426$.

$$\begin{array}{cccccccccccc}
1 & & 3 & & 5 & & 7 & & 9 & & & \\
& 2 & & 4 & & 6 & & 8 & & 10 & & \\
& & 3 & & 5 & & 7 & & 9 & & & \\
& & & 4 & & 6 & & 8 & & 10 & & \ldots \\
& & 3 & & 5 & & 7 & & 9 & & & \\
& 2 & & 4 & & 6 & & 8 & & 10 & & \\
1 & & 3 & & 5 & & 7 & & 9 & & &
\end{array}$$

**Thank you!**