Sensor Network Localization

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CME 307

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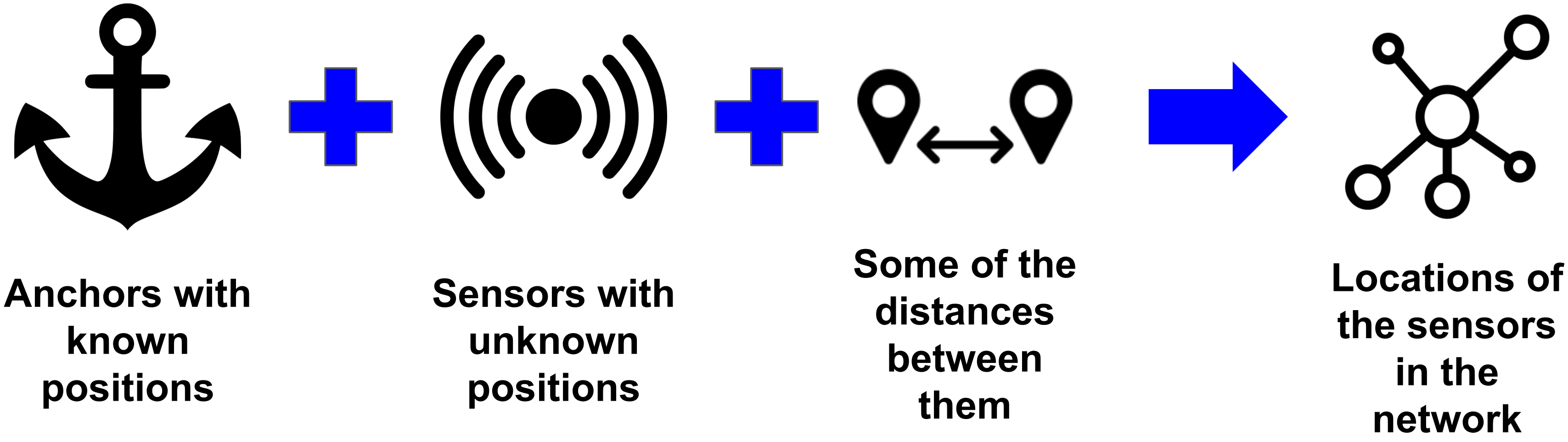
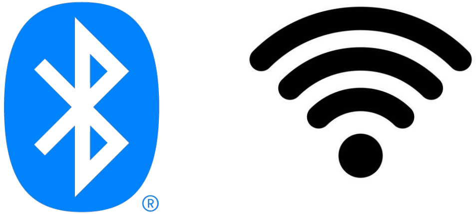
Problem Introduction



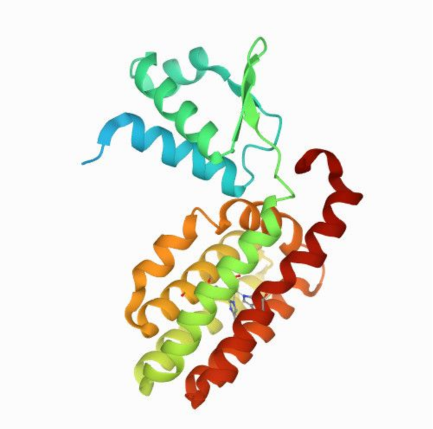
Performance Metrics

- 2-norm error: typical error of any coordinate in estimated locations
- ∞-norm error: inaccuracy of worst approximated sensor location

Applications



Wireless Communication



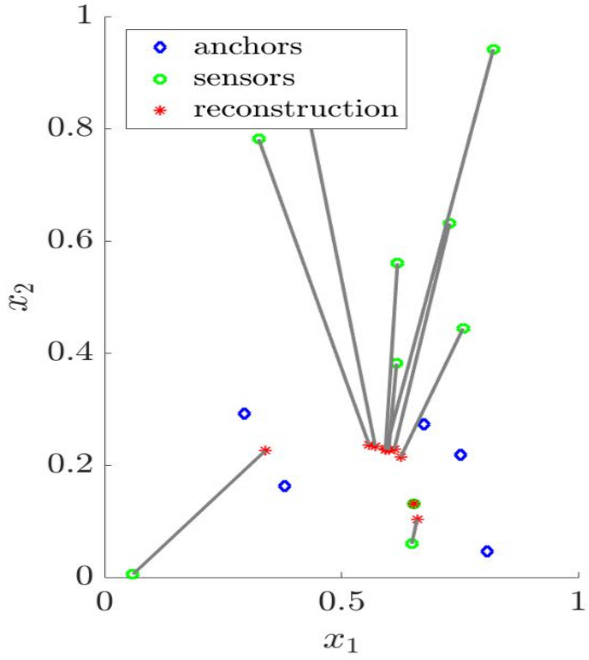
Molecular Conformations of Proteins

Wildfire Tracking

Traffic Monitoring

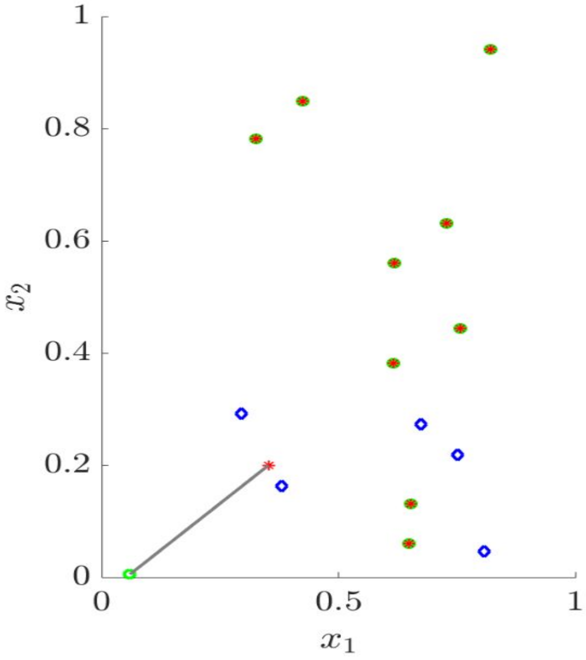
Relaxations

SOCP



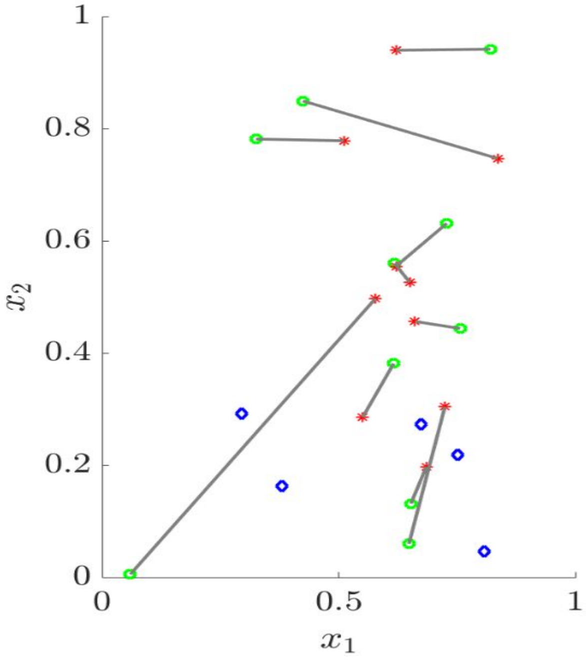
$$\begin{aligned}\underset{\{\boldsymbol{x}_i\}_{i=1}^{n_s}}{\textit{minimize}} \quad & \sum_{i=1}^{n_s} \mathbf{0}^\top \boldsymbol{x}_i,\\ \textit{subject to} \quad & \|\boldsymbol{x}_i - \boldsymbol{x}_j\|_2^2 \leqslant d_{ij}^2, \forall (i,j) \in N_x, i<j,\\ & \|\boldsymbol{x}_i - \boldsymbol{a}_j\|_2^2 \leqslant \hat{d}_{ij}^2, \forall (i,j) \in N_a.\end{aligned}$$

SDP



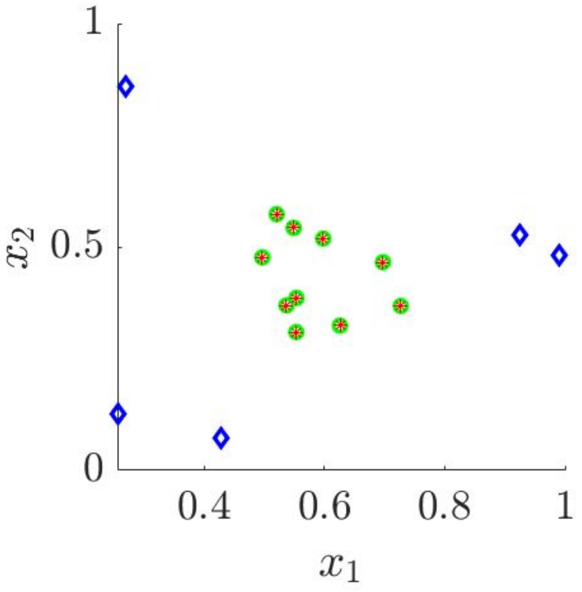
$$\begin{aligned}\underset{\boldsymbol{Z}}{\textit{minimize}} \quad & \mathbf{0} \bullet \boldsymbol{Z},\\ \textit{subject to} \quad & \boldsymbol{Z}[1:d, 1:d] = \boldsymbol{I},\\ & (\mathbf{0}; \boldsymbol{e}_i - \boldsymbol{e}_j)(\mathbf{0}; \boldsymbol{e}_i - \boldsymbol{e}_j)^\top \bullet \boldsymbol{Z} = d_{ij}^2, \forall (i,j) \in N_x, i<j,\\ & (\boldsymbol{a}_j; -\boldsymbol{e}_i)(\boldsymbol{a}_j; -\boldsymbol{e}_i)^\top \bullet \boldsymbol{Z} = \hat{d}_{ij}^2, \forall (i,j) \in N_a,\\ & \boldsymbol{Z} \succeq \mathbf{0}.\end{aligned}$$

NLS

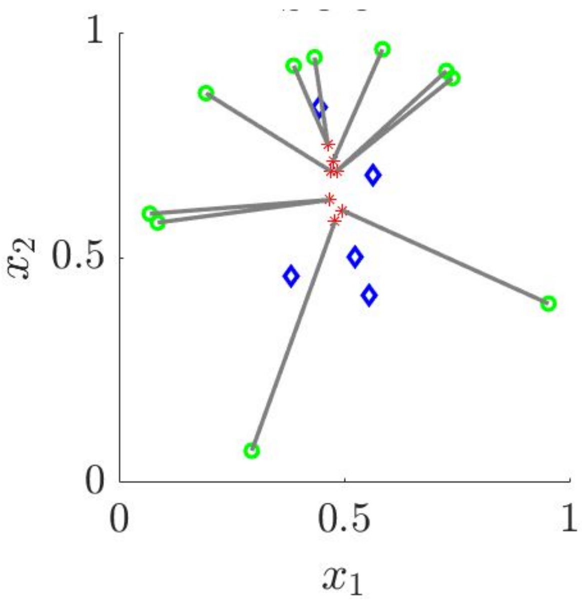


$$\underset{\{\boldsymbol{x}_i\}_{i=1}^{n_s}}{\textit{minimize}} \quad \sum_{(i,j)\in N_x} \left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|_2^2 - d_{ij}^2\right)^2 + \sum_{(i,j)\in N_a} \left(\|\boldsymbol{x}_i - \boldsymbol{a}_j\|_2^2 - \hat{d}_{ij}^2\right)^2$$

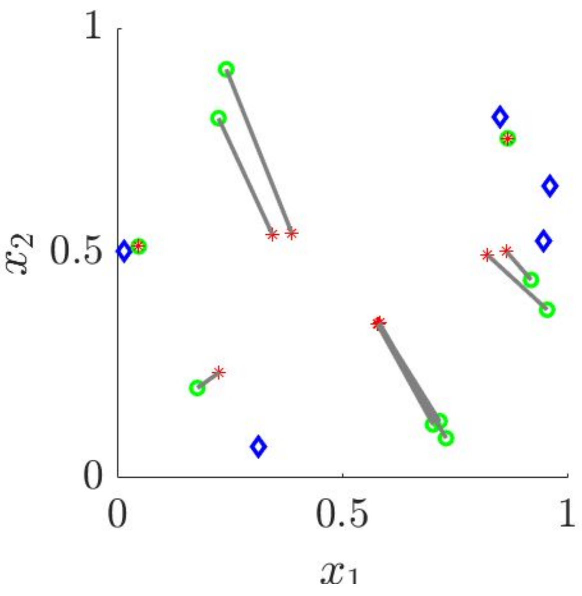
Impact of Position of Anchors and Sensors on SOCP Performance



Sensors inside convex hull of anchors

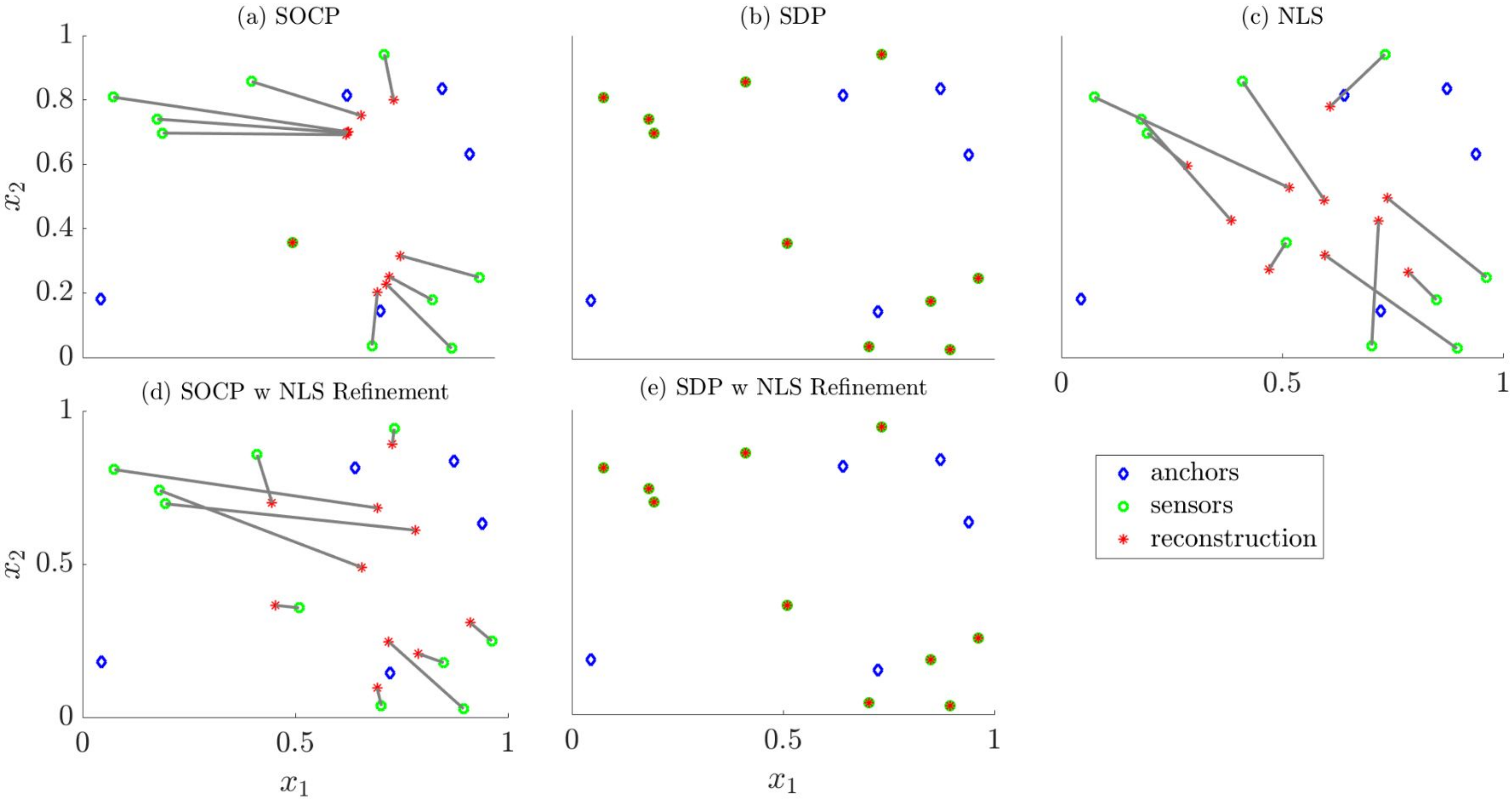


Sensors outside convex hull of anchors

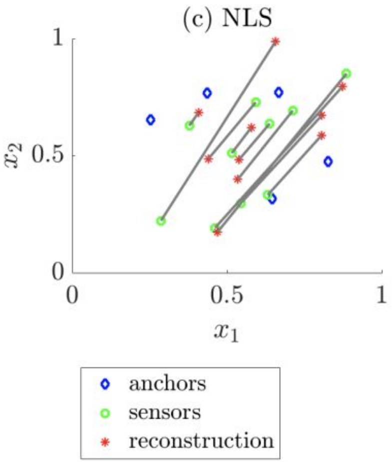


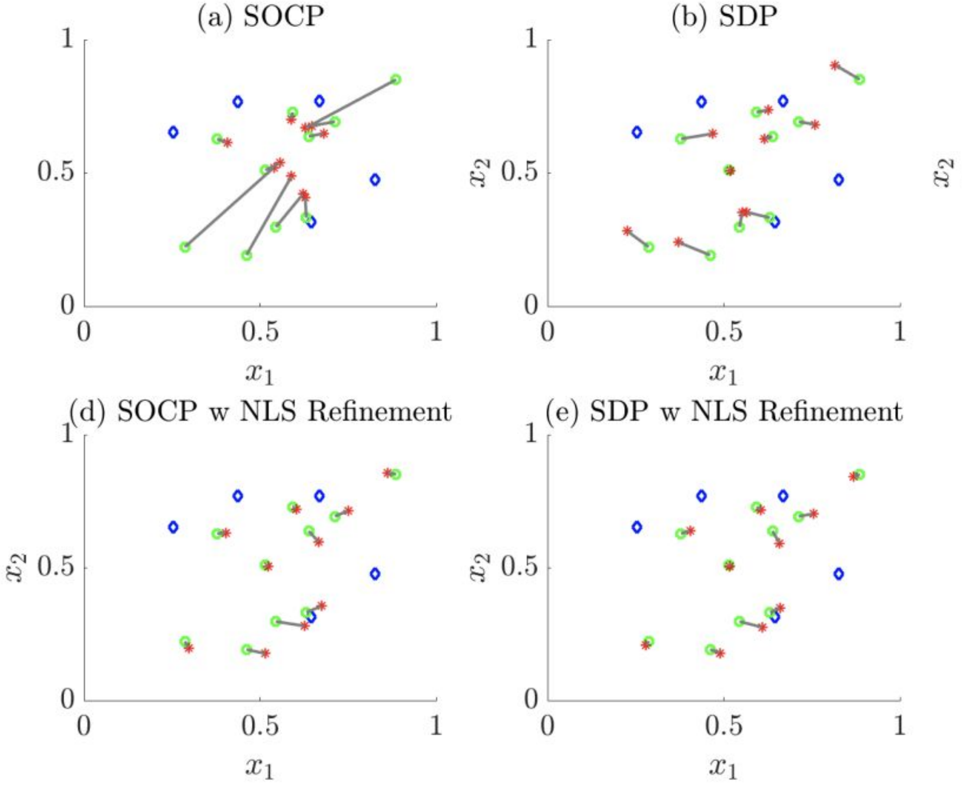
Sensors and anchors along perimeter of circle

SDP and SOCP initialized least squares



Noisy SNL





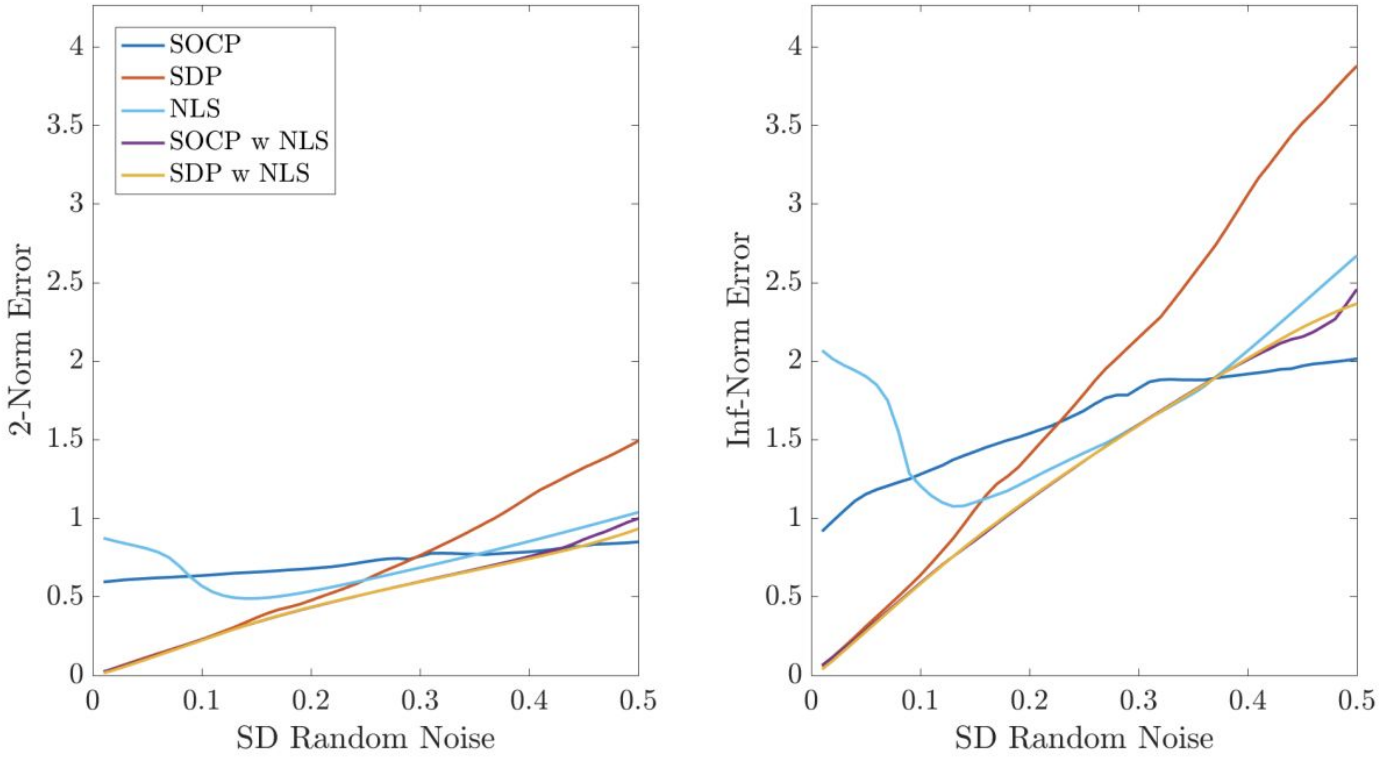
Noisy SOCP

$$
\begin{aligned}
\min_{\mathbf{x}_i,\delta'',\hat{\delta}''} \quad & \sum_{(i,j)\in N_x} (\delta''_{ij}) + \sum_{(k,j)\in N_a} (\hat{\delta}''_{kj}) \\
s.t. \quad & \|\mathbf{x}_i - \mathbf{x}_j\|^2 - \delta''_{ij} \leqslant d_{ij}^2, \ \forall\ (i,j)\in N_x,\ i<j, \\
& \|\mathbf{a}_k - \mathbf{x}_j\|^2 - \hat{\delta}''_{kj} \leqslant \hat{d}_{kj}^2, \ \forall\ (k,j)\in N_a.
\end{aligned}
$$

Noisy SDP

$$
\begin{aligned}
\min_{Z,\delta',\delta'',\hat{\delta}',\hat{\delta}''} \quad & \sum_{(i,j)\in N_x} (\delta'_{ij} + \delta''_{ij}) + \sum_{(k,j)\in N_a} (\hat{\delta}'_{kj} + \hat{\delta}''_{kj}) \\
s.t. \quad & Z_{1:d,1:d} = I, \\
& (\mathbf{0};\mathbf{e}_i - \mathbf{e}_j)(\mathbf{0};\mathbf{e}_i - \mathbf{e}_j)^T \bullet Z + \delta'_{ij} - \delta''_{ij} = d_{ij}^2, \ \forall\ i,j\in N_x,\ i<j, \\
& (\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \bullet Z + \hat{\delta}'_{kj} - \hat{\delta}''_{kj} = \hat{d}_{kj}^2, \ \forall\ k,j\in N_a, \\
& Z \succeq \mathbf{0} \\
& \delta', \delta'', \hat{\delta}', \hat{\delta}'' \geqslant 0.
\end{aligned}
$$

Noisy SNL cont.



Steepest Descent Projection Method (SDPM)

SDP Relaxation

$$\begin{aligned}
&\underset{\boldsymbol{Z}}{\textit{minimize}} && \boldsymbol{0} \bullet \boldsymbol{Z}, \\
&\textit{subject to} && \boldsymbol{Z}[1:d, 1:d] = \boldsymbol{I}, \\
& && (\boldsymbol{0}; \boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{0}; \boldsymbol{e}_i - \boldsymbol{e}_j)^\top \bullet \boldsymbol{Z} = d_{ij}^2, \forall (i,j) \in N_x, i < j, \\
& && (\boldsymbol{a}_j; -\boldsymbol{e}_i)(\boldsymbol{a}_j; -\boldsymbol{e}_i)^\top \bullet \boldsymbol{Z} = \widehat{d}_{ij}^2, \forall (i,j) \in N_a, \\
& && \boldsymbol{Z} \succeq \boldsymbol{0}.
\end{aligned}$$

Recast as a Least Squares Problem

$$\begin{aligned}
&\underset{\boldsymbol{Z}}{\textit{minimize}} && f(\boldsymbol{Z}) := \frac{1}{2}\|\mathcal{A}\boldsymbol{Z} - \boldsymbol{b}\|_2^2, \\
&\textit{subject to} && \boldsymbol{Z} \succeq \boldsymbol{0}.
\end{aligned}$$

Steepest Descent Projection Method (SDPM)

Steepest Descent Projection Method

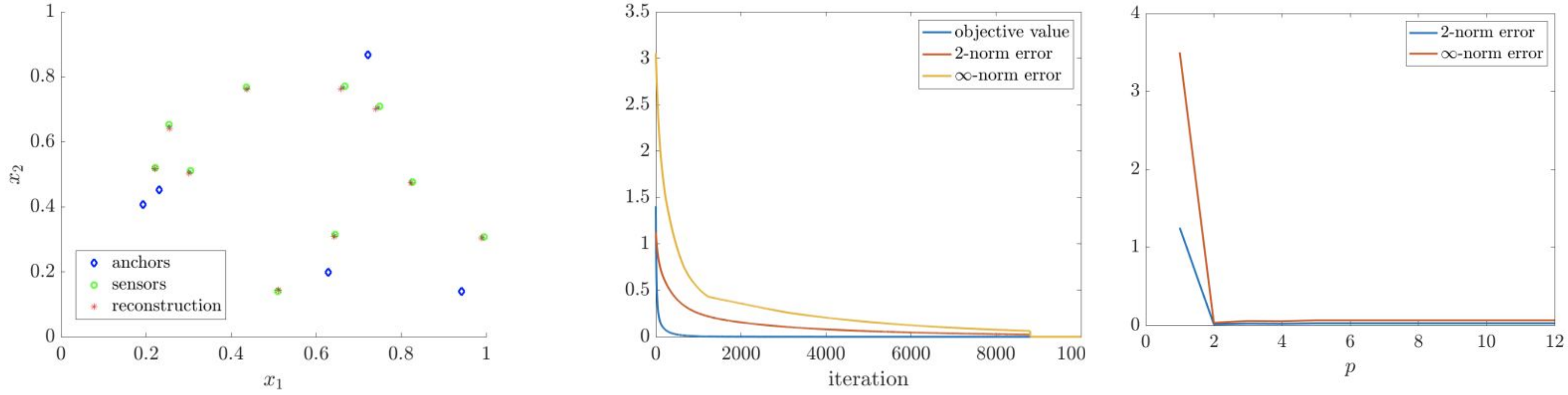
$$\begin{array}{ll} \underset{\boldsymbol{Z}}{\textit{minimize}} & f(\boldsymbol{Z}) := \frac{1}{2}\|\mathcal{A}\boldsymbol{Z} - \boldsymbol{b}\|_2^2, \\ \textit{subject to} & \boldsymbol{Z} \succeq \mathbf{0}. \end{array}$$

(1) Descent step: $\widehat{\boldsymbol{Z}}^{k+1} = \boldsymbol{Z}^k - \epsilon \nabla f(\boldsymbol{Z}^{\boldsymbol{k}})$, where ϵ is a step-size;
(2) Projection step: $Z^{k+1} = \boldsymbol{V} \max(\mathbf{0}, \boldsymbol{\Lambda}) V^\top$, where $\widehat{\boldsymbol{Z}}^{k+1} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^\top$ is the orthogonal eigen-decomposition of $\widehat{\boldsymbol{Z}}^{k+1}$ (In practice, use only the first p eigenpairs to compute this projection!)

Lemma *Let $\boldsymbol{A} \in S^{n\times n}$ be a symmetric matrix, and let $\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^\top$ be its orthogonal eigen-decomposition. Let $\boldsymbol{A}_{proj} := \boldsymbol{V}\max(\mathbf{0}, \boldsymbol{\Lambda})\boldsymbol{V}^\top$. Then $\boldsymbol{A}_{proj}$ solves*

$$\begin{array}{l} \underset{\boldsymbol{B}}{\textit{minimize}}\ \|\boldsymbol{A} - \boldsymbol{B}\|_F, \\ \textit{subject to}\ \boldsymbol{B} \succeq \mathbf{0}. \end{array}$$

Projection Method Results



Projected steepest descent often performs comparably to the SDP relaxation but is much faster!

Alternating Direction Method of Multipliers (ADMM)

Variable Splitting to Establish Constrained Bi-Convex Minimization Problem

$$
\begin{aligned}
&\textit{minimize} && \sum_{(i,j)\in N_x} \left[(\boldsymbol{x}_i - \boldsymbol{x}_j)^T(\boldsymbol{z}_i - \boldsymbol{z}_j) - d_{ij}^2\right]^2 + \sum_{(k,j)\in N_a} \left[(\boldsymbol{a}_k - \boldsymbol{x}_j)^T(\boldsymbol{a}_k - \boldsymbol{z}_j) - \hat{d}_{kj}^2\right]^2, \\
&\textit{subject to} && \boldsymbol{x}_j - \boldsymbol{z}_j = \boldsymbol{0}, \forall j.
\end{aligned}
$$

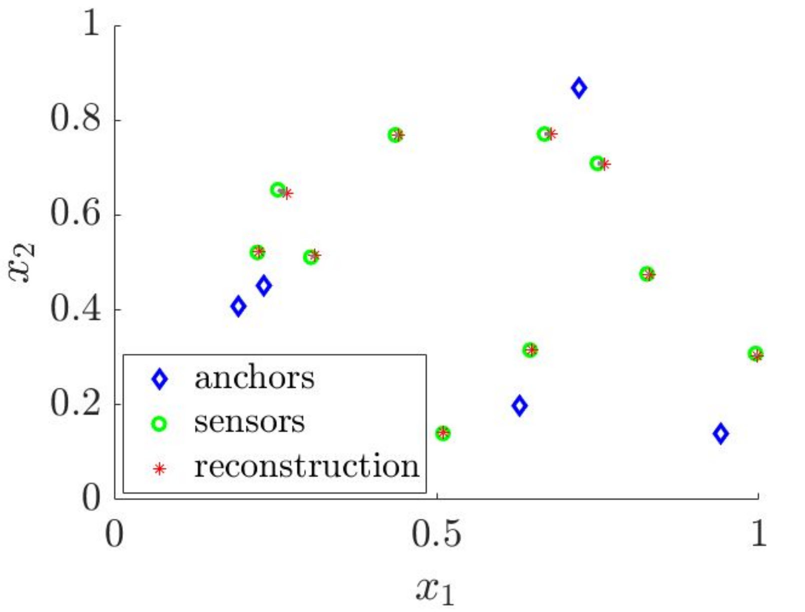
Augmented Lagrangian

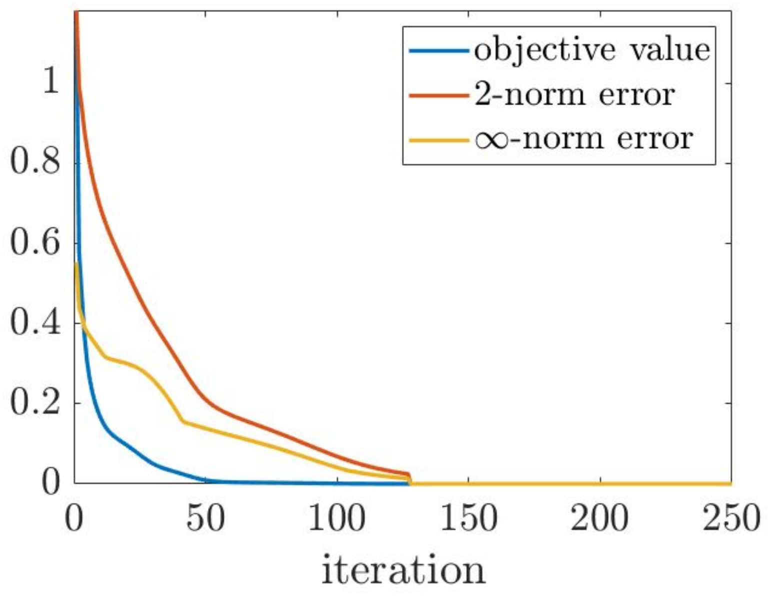
$$
\begin{aligned}
L_a(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}) = &\sum_{(i,j)\in N_x} \left[(\boldsymbol{x}_i - \boldsymbol{x}_j)^T(\boldsymbol{z}_i - \boldsymbol{z}_j) - d_{ij}^2\right]^2 + \sum_{(k,j)\in N_a} \left[(\boldsymbol{a}_k - \boldsymbol{x}_j)^T(\boldsymbol{a}_k - \boldsymbol{z}_j) - \hat{d}_{kj}^2\right]^2 \\
&- \sum_i \boldsymbol{y}_i^T(\boldsymbol{x}_i - \boldsymbol{z}_i) + \frac{\beta}{2}\sum_i \|\boldsymbol{x}_i - \boldsymbol{z}_i\|_2^2,
\end{aligned}
$$

Computing Iterates:

$$
\begin{aligned}
\boldsymbol{x}_{k+1} &:= \arg\min_{\boldsymbol{x}} L_a(\boldsymbol{x}, \boldsymbol{z}_k, \boldsymbol{y}_k), \\
\boldsymbol{z}_{k+1} &:= \arg\min_{\boldsymbol{z}} L_a(\boldsymbol{x}_{k+1}, \boldsymbol{z}, \boldsymbol{y}_k), \\
\boldsymbol{y}_{k+1} &:= \boldsymbol{y}_k - \beta(\boldsymbol{x}_{k+1} - \boldsymbol{z}_{k+1}).
\end{aligned}
$$

ADMM Results





ADMM often performs comparably to the SDP relaxation but is much faster, and converges in fewer iterations than the steepest descent projection method.

Further Directions

- Using more sophisticated step-size functions to further refine SDP, SDPM, and ADMM
- Using different distributions for the sensor and anchor locations
- Using different distributions for the noise or different objectives to model noisy SNL
- Fixed vs. variable radii for noisy SNL

Thank you!

Questions?

Happy Pi Day!

References

[1] David G. Luenberger and Yinyu Ye. *Linear and Nonlinear Programming*. Springer Publishing Company, Incorporated. 2021.

[2] Michael Grant and Stephen Boyd. CVX: Matlab software for disciplined convex programming, September 2013.

[3] Michael Grant and Stephen Boyd. "Graph implementations for nonsmooth convex programs," Recent Advances in Learning and Control (a tribute to M. Vidyasagar), pages 95-110, Lecture Notes in Control and Information Sciences, Springer, 2008.

[4] P. Biswas, K-C. Toh, and Y. Ye. "A Distributed SDP Approach for Large-Scale Noisy Anchor-Free Graph Realization with Applications to Molecular Conformation." *SIAM J. Scientific Computing*. 30. 1251-1277. 10.1137/05062754X, 2008.

[5] P. Biswas and Y. Ye, "Semidefinite programming for ad hoc wireless sensor network localization," *Proceedings of the Third International Symposium on Information Processing in Sensor Networks*, ACM Press, 2004, pp. 46–54.

[6] Pratik Biswas, T-C Liang, K-C Toh, Y. Ye, T-C Wang, "Semidefinite programming approaches for sensor network localization with noisy distance measurements," *IEEE Transactions on Automation Science and Engineering*, 3(4), 2006, pp. 360-371.

[7] L. Doherty, L. E. Ghaoui, and K. S. J. Pister, "Convex position estimation in wireless sensor networks." in *Proceedings of IEEE Infocom*, Anchorage, Alaska, April 2001, pp. 1655 –1663.

[8] A. So. A Semidefinite Programming Approach to the graph realization problem: Theory, Applications and Extensions *PhD Thesis, Stanford University*, 2007.

Relaxations

SOCP

$$\begin{aligned} \underset{\{\boldsymbol{x}_i\}_{i=1}^{n_s}}{\textit{minimize}} \quad & \sum_{i=1}^{n_s} \mathbf{0}^\top \boldsymbol{x}_i, \\ \textit{subject to} \quad & \|\boldsymbol{x}_i - \boldsymbol{x}_j\|_2^2 \leqslant d_{ij}^2, \forall (i,j) \in N_x, i < j, \\ & \|\boldsymbol{x}_i - \boldsymbol{a}_j\|_2^2 \leqslant \widehat{d}_{ij}^2, \forall (i,j) \in N_a. \end{aligned}$$

SDP

$$\begin{aligned} \underset{\boldsymbol{Z}}{\textit{minimize}} \quad & \mathbf{0} \bullet \boldsymbol{Z}, \\ \textit{subject to} \quad & \boldsymbol{Z}[1:d, 1:d] = \boldsymbol{I}, \\ & (\mathbf{0}; \boldsymbol{e}_i - \boldsymbol{e}_j)(\mathbf{0}; \boldsymbol{e}_i - \boldsymbol{e}_j)^\top \bullet \boldsymbol{Z} = d_{ij}^2, \forall (i,j) \in N_x, i < j, \\ & (\boldsymbol{a}_j; -\boldsymbol{e}_i)(\boldsymbol{a}_j; -\boldsymbol{e}_i)^\top \bullet \boldsymbol{Z} = \widehat{d}_{ij}^2, \forall (i,j) \in N_a, \\ & \boldsymbol{Z} \succeq \mathbf{0}. \end{aligned}$$

NLS

$$\underset{\{\boldsymbol{x}_i\}_{i=1}^{n_s}}{\textit{minimize}} \quad \sum_{(i,j)\in N_x} \left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|_2^2 - d_{ij}^2\right)^2 + \sum_{(i,j)\in N_a} \left(\|\boldsymbol{x}_i - \boldsymbol{a}_j\|_2^2 - \widehat{d}_{ij}^2\right)^2$$