As in the 1-d analysis, we investigate the stability of the extinct state. From Strogatz (2014), we can characterize the stability of this fixed point from the Jacobian matrix J, below, which describes our system of equations:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial s_1} & \frac{\partial f_1}{\partial s_2} \\ \frac{\partial f_2}{\partial s_1} & \frac{\partial f_2}{\partial s_2} \end{bmatrix}$$

We can characterize the stability of the extinct state by determining the eigenvalues of the Jacobian evaluated at that point. We evaluate the Jacobian at $(s_1^*, s_2^*) = (0, 0)$ below:

$$J_{0,0} = \begin{bmatrix} p_{11}e^r & p_{12}e^r \\ p_{21}e^r & p_{22}e^r \end{bmatrix}$$

We allow the carrying capacity K to be equal to one. As a result, the population terms are taken to represent a fraction of the total carrying capacity.

Setting up the characteristic equation for the Jacobian,

$$(p_{11}e^r - \lambda)(p_{22}e^r - \lambda) - p_{21}e^r p_{12}e^r = 0$$

we solve for the eigenvalues using the quadratic equation,

$$\lambda_+, \lambda_- = \frac{e^r \left(p_{11} + p_{22} \pm \sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}}\right)}{2} \tag{1}$$

where λ_+ denotes the root derived from adding the quantity under the square root, and λ_- denotes the result of subtracting it. We note that in our model, movement between nodes conserves population such that the column sums of the transition matrix $\mathbf{P}$ are equal to 1. Thus, $p_{11} = 1 - p_{21}$ and $p_{22} = 1 - p_{12}$. Substituting these into equation 1, we get:

$$\lambda_+, \lambda_- = \frac{e^r \left(p_{11} + p_{22} \pm (2 - p_{11} - p_{22})\right)}{2} \tag{2}$$

We can simplify this to get an expression for each eigenvalue:

$$\lambda_+ = e^r$$

$$\lambda_- = e^r(p_{11} + p_{22} - 1)$$

Recall that the elements of $\mathbf{P}$ represent the proportion of population transferred between nodes. As such, they are bounded between 0 and 1. Because $0 \leq p_{ll} \leq 1$ and $0 \leq p_{22} \leq 1$, it follows that $-1 \leq p_{11} + p_{22} - 1 \leq 1$. Thus the magnitude of λ_- will always be a fraction of λ_+. To determine stability, we need only look at the magnitude of ~~the larger eigenvalue,~~ λ_+. If the magnitude of λ_+ is less than 1, then the fixed point is stable, and conversely if the magnitude of λ_+ is greater than 1, the fixed point is unstable. Setting up the inequality $\lambda_+ > 1$, we find that for $r > 0$, the fixed point at $(0, 0)$ is unstable. We have then shown that in 2-D, the extinct state is unstable when the growth rate r is positive.