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Canonical solutions and almost periodicity in a discrete logistic equation

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Abstract

Sufficient conditions are obtained for the existence of a globally attractive positive, almost periodic solution of a discrete system of the form

$$x(n+1) = \frac{\alpha(n)x(n)}{1+\beta(n)x(n)}$$

where $\{\alpha(n)\}$ and $\{\beta(n)\}$ denote almost periodic nonnegative sequences. The analysis generalises the results corresponding to the periodic case of the discrete system. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

A number of authors [1,3,8,12] have investigated the dynamical characteristics of the nonautonomous logistic differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = r(t)x(t)\left[1 - \frac{x(t)}{K(t)}\right] \qquad t > t_0 \in (-\infty, \infty) \tag{1.1}$$

in which $r(\cdot)$ denotes a nonnegative continuous real valued function defined for $t \in (-\infty, \infty)$ and $K(\cdot)$ denotes a continuous strictly positive function defined

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for $t \in (-\infty, \infty)$. In particular, Eq. (1.1) has been studied under the hypothesis that $r(\cdot)$ and $K(\cdot)$ denote periodic functions with a common period; extensions of these investigations to almost periodic cases of $r(\cdot)$ and $K(\cdot)$ are also found in the literature.

The Eq. (1.1) constitutes a prototype for the derivation of Lotka–Volterra systems of equations used in theoretical studies of multispecies dynamics. Due to the potential applications of Eq. (1.1) in population dynamics, discrete analogues of Eq. (1.1) have been proposed and studied by Clark and Gross [2]; the discrete analogue of Eq. (1.1) investigated by these authors is given by the nonautonomous difference equation

$$x(n+1) = \frac{a(n)x(n)}{1 + b(n)x(n)}, \tag{1.2}$$

where $a(n)$ and $b(n)$ are positive, bounded and periodic with integer period T. The relationship of Eq. (1.2) to Eq. (1.1) is derived from the fact that the dynamical characteristics of Eq. (1.2) are similar to those of Eq. (1.1). It is shown in Ref. [4] that if r and K are periodic with a common period, then all positive solutions of Eq. (1.1) converge to a strictly positive periodic solution. We note that in the analysis of Eq. (1.2) as in the case of Eq. (1.1), a change of variable is used to convert Eq. (1.2) to a linear nonhomogeneous equation which is then explicitly solved in the process of the derivation of the periodic solution.

We remark that in the process of discretization a periodic function such as $\sin(t)$, $t \in (-\infty, \infty)$ does not lead to a periodic sequence in the sense that the sequence $\{\sin(nh)\}$, $n \in Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $h \in (0, \infty)$ is not periodic with integer period. Such a sequence however is almost periodic (see Ref. [6]). It is thus necessary to investigate the dynamics of almost-periodic discrete analogues of Eq. (1.1). The purpose of this article is to propose a mechanistic formulation for the derivation of a discrete analogue of Eq. (1.1) and then derive sufficient conditions for the existence of a globally attractive positive almost periodic solution under the assumption that the resulting discrete sequences corresponding to $r(\cdot)$ and $K(\cdot)$ are almost periodic. As special cases, our results lead to certain discrete analogues of the results of Coleman [4].

2. A discrete nonautonomous logistic equation

There are several ways in which discrete-time analogues of continuous-time systems can be derived. Depending on the models used, the discrete analogues may mimic the dynamics of their continuous-time mother versions. In most cases, it is known that the discrete versions possess a wider spectrum of dynamical characteristics than their continuous-time counterparts.

One of the methods for the derivation of discrete versions of continuous time systems is by means of a suitable differential equation with piecewise constant arguments. In the case of autonomous systems, this method has been used by Gopalsamy et al. [10]. Application of this method for nonautonomous model systems does not seem to have been widely used in the literature. Let us consider the following differential equation with piecewise constant arguments:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)\left[r([t]) - \frac{r([t])}{K([t])}x(t)\right], \qquad t \neq 0, 1, 2, \ldots, \tag{2.1}$$

where $[t]$ denotes the integer part of $t \in (-\infty, \infty)$. On any interval of the form $[n, t),\ n \in Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ Eq. (2.1) becomes

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = r(n)x(t) - \frac{r(n)}{K(n)}x^2(t), \qquad t \in [n, t), \quad t < n+1. \tag{2.2}$$

An integration of Eq. (2.2) over $[n, t)$ leads to

$$\frac{1}{x(t)}\mathrm{e}^{r(n)t} - \frac{1}{x(n)}\mathrm{e}^{r(n)n} = \frac{\mathrm{e}^{r(n)t}}{K(n)} - \frac{\mathrm{e}^{r(n)n}}{K(n)}, \qquad n \leqslant t < n+1. \tag{2.3}$$

We let $t \to n+1$ in Eq. (2.3) and obtain after simplification,

$$x(n+1) = \frac{\mathrm{e}^{r(n)}x(n)}{1 + \left(\frac{\mathrm{e}^{r(n)}-1}{K(n)}\right)x(n)}, \qquad n = 0, 1, 2, \ldots. \tag{2.4}$$

The nonautonomous difference equation (2.4) is a discrete analogue of Eq. (1.1). If r and K denote positive constants in Eq. (2.4) one can show from

$$\frac{1}{x(n+1)} = \frac{\mathrm{e}^{-r}}{x(n)} + \frac{1-\mathrm{e}^{-r}}{K} \tag{2.5}$$

that

$$x(0) > 0 \quad \Rightarrow \quad x(n) > 0 \quad \text{and} \quad x(n) \to K \quad \text{as} \quad n \to \infty.$$

Also one notes that Eq. (2.5) leads to

$$\frac{1}{x(n+1)} - \frac{1}{K} = \left(\frac{1}{x(n)} - \frac{1}{K}\right)\mathrm{e}^{-r} \tag{2.6}$$

which implies that the equilibrium $x(n) \equiv K$ is asymptotically stable and is a global attractor in the sense that

$$x(0) > 0 \quad \Rightarrow \quad x(n) \to K \quad \text{as} \quad n \to \infty.$$

Discrete analogues which preserve the dynamics of continuous-time versions can be used as computational models for the numerical simulations.

Indiscriminate formulations of discrete versions of continuous time systems, motivated from the numerical solutions of differential equations have led to the discovery and prediction of irrelevant and erroneous dynamics involving spurious equilibria and chaotic behaviour in certain cases. Since population models can be formulated either in continuous-time or discrete-time, it is desirable that the dynamical characteristics are not model dependent.

In our analysis of Eq. (2.4), we obtain an explicit solution of Eq. (2.4) as follows: we first note that if $x(0) > 0$, then $x(n) > 0$ for $n = 1, 2, 3, \ldots$ We suppose that

$$0 \leqslant \inf_{n \in Z} r(n) \quad \text{and} \quad 0 < K_* \leqslant \inf_{n \in Z} K(n). \tag{2.7}$$

Since $x(n) > 0$, we let

$$y(n) = \frac{1}{x(n)}, \qquad n \geqslant 0, \tag{2.8}$$

in Eq. (2.4) and obtain

$$y(n+1) = \mathrm{e}^{-r(n)} y(n) + \frac{1 - \mathrm{e}^{-r(n)}}{K(n)}, \qquad n \geqslant 0. \tag{2.9}$$

It is now possible to show by an inductive argument that Eq. (2.9) leads to

$$\begin{aligned} y(n) = {} & y(0) \exp\left(-\sum_{i=0}^{n-1} r(i) \right) \\ & + \sum_{j=0}^{n-1} \frac{1}{K(j)} \left\{ \exp\left(-\sum_{l=j+1}^{n-1} r(l) \right) - \exp\left(-\sum_{l=j}^{n-1} r(l) \right) \right\}, \qquad n \geqslant 1. \end{aligned} \tag{2.10}$$

Changing the summation variables in Eq. (2.10),

$$\begin{aligned} y(n) = {} & y(0) \exp\left(-\sum_{i=0}^{n-1} r(i) \right) \\ & + \sum_{m=1}^{n} \left(\frac{1 - \mathrm{e}^{-r(n-m)}}{K(n-m)} \right) \exp\left(-\sum_{p=1}^{m-1} r(n-p) \right), \qquad n \geqslant 1. \end{aligned} \tag{2.11}$$

Since $y(n) > 0$ for all $n \geqslant 1$ a solution of Eq. (2.4) is now readily given by $x(n)$ where

$$x(n) = \frac{1}{y(n)}, \qquad n \geqslant 1. \tag{2.12}$$

The asymptotic behaviour of positive solutions of Eq. (2.4) is discussed below.

3. Canonical solution

We assume that the nonnegative sequence $\{r(n)\}$ satisfies the following:

$$\lim_{m \to \infty} \left\{ \sum_{j=1}^{m} r(n-j) - m\bar{r} \right\} = 0, \tag{3.1}$$

where $\bar{r}$ is a positive number and the convergence in Eq. (3.1) is uniform in $n \in Z$. One can think of $\bar{r}$ as the average value of $\{r(n)\}$ in the sense that (due to the uniformity with respect to n in Eq. (3.1))

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} r(j) = \bar{r}. \tag{3.2}$$

If the sequence $\{r(n)\}$ is periodic with an integer period σ then Eq. (3.2) implies

$$\frac{1}{\sigma} \sum_{j=1}^{\sigma} r(j) = \bar{r}. \tag{3.3}$$

If the sequence $\{r(n)\}$ is almost periodic then it is known that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} r(j) \tag{3.4}$$

exists and this can be called the mean of the almost periodic sequence. The existence of a canonical solution of Eq. (2.4) is established in the following.

Theorem 3.1. *Suppose* $\{K(n)\}$, $n \in Z$ *satisfies*

$$0 < K_* \leqslant K(n), \qquad n \in Z. \tag{3.5}$$

Let $\{r(n)\}$ *denote a nonnegative sequence and suppose that there exists* $\bar{r} > 0$ *such that*

$$\lim_{m\to\infty} \left\{ \sum_{j=1}^{m} r(n-j) - m\bar{r} \right\} = 0, \tag{3.6}$$

where the convergence in Eq. (3.6) *is required to be uniform in* $n \in Z$. *Then there exists a unique positive solution* $\{x^*(n)\}$ *of Eq.* (2.4) *such that any arbitrary positive solution* $\{x(n)\}$ *of Eq.* (2.4) *satisfies*

$$\lim_{n\to\infty} [x(n) - x^*(n)] = 0. \tag{3.7}$$

Proof. We note that

$$\exp\left(-\sum_{i=0}^{n-1} r(i) \right) = \exp\left(-n\bar{r} + \sum_{i=0}^{n-1} \delta(i) \right), \tag{3.8}$$

where

$$\sum_{i=0}^{n-1} \delta(i) = -\sum_{i=0}^{n-1} r(i) + n\bar{r} \to 0 \quad \text{as } n \to \infty, \tag{3.9}$$

it follows that $\sum_{i=0}^{n-1} \delta(i)$ is bounded and we let

$$\left|\sum_{i=0}^{n-1}\delta(i)\right| \leqslant M, \tag{3.10}$$

where M is independent of $n \geqslant 1$. We have from Eqs. (3.8), (3.10) and (2.11) that

$$\begin{aligned}\exp\left(-\sum_{i=0}^{n-1}r(i)\right) &= \exp\left(-n\bar{r}+\sum_{i=0}^{n-1}\delta(i)\right)\\ &\leqslant \mathrm{e}^{-n\bar{r}}\mathrm{e}^{M} \quad\to\quad 0 \quad \text{as } n\to\infty\end{aligned} \tag{3.11}$$

and

$$\exp\left(-\sum_{p=1}^{m-1}r(n-p)\right) = \exp\left(-(m-1)\bar{r}+\sum_{p=1}^{m-1}\delta(n-p)\right) \leqslant \mathrm{e}^{-(m-1)\bar{r}}\mathrm{e}^{M}. \tag{3.12}$$

Let us define $\{y^*(n)\}$ as follows:

$$y^*(n) = \sum_{m=1}^{\infty}\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\exp\left(-\sum_{p=1}^{m-1}r(n-p)\right), \qquad n\in Z. \tag{3.13}$$

The sequence $\{y^*(n)\}$ is well-defined for all $n \in Z$ in the sense that

$$y^*(n) \leqslant \sum_{m=1}^{\infty}\left(\frac{1}{K_*}\right)\mathrm{e}^{M}\ \mathrm{e}^{-(m-1)\bar{r}} = \frac{\mathrm{e}^{M}}{K_*}\frac{1}{1-\mathrm{e}^{-\bar{r}}}. \tag{3.14}$$

We derive from Eq. (2.11),

$$\begin{aligned}&\left|y(n)-\sum_{m=1}^{\infty}\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\exp\left(-\sum_{p=1}^{m-1}r(n-p)\right)\right|\\ &\quad=\left|y(0)\exp\left(-\sum_{i=0}^{n-1}r(i)\right)-\sum_{m=n+1}^{\infty}\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\exp\left(-\sum_{p=1}^{m-1}r(n-p)\right)\right|\\ &\quad\leqslant y(0)\mathrm{e}^{-n\bar{r}}\mathrm{e}^{M}+\frac{\mathrm{e}^{M}}{K_*}\sum_{m=n+1}^{\infty}\mathrm{e}^{-(m-1)\bar{r}}\\ &\quad= y(0)\mathrm{e}^{-n\bar{r}}\mathrm{e}^{M}+\frac{\mathrm{e}^{M}}{K_*}\frac{\mathrm{e}^{-n\bar{r}}}{1-\mathrm{e}^{-\bar{r}}} \quad\to\quad 0 \quad\text{as}\quad n\to\infty.\end{aligned} \tag{3.15}$$

Thus we have from Eq. (3.15) that

$$\lim_{n\to\infty}\left\{y(n)-\sum_{m=1}^{\infty}\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\exp\left(-\sum_{p=1}^{m-1}r(n-p)\right)\right\}=0. \tag{3.16}$$

It is found from Eq. (3.13), after a change of the summation index that

$$
\begin{aligned}
y^*(n+1) &= \frac{1-\mathrm{e}^{-r(n)}}{K(n)} + \sum_{l=1}^{\infty}\left(\frac{1-\mathrm{e}^{-r(n-l)}}{K(n-l)}\right)\exp\left(-\sum_{p=1}^{l} r(n+1-p)\right) \\
&= \frac{1-\mathrm{e}^{-r(n)}}{K(n)} + \mathrm{e}^{-r(n)}\sum_{l=1}^{\infty}\left(\frac{1-\mathrm{e}^{-r(n-l)}}{K(n-l)}\right)\exp\left(-\sum_{q=1}^{l-1} r(n-q)\right) \\
&= \frac{1-\mathrm{e}^{-r(n)}}{K(n)} + \mathrm{e}^{-r(n)}y^*(n), \qquad n \in Z.
\end{aligned}
\tag{3.17}
$$

Thus $\{y^*(n)\}$ is a solution of Eq. (2.9). The uniqueness of $\{y^*(n)\}$ is a consequence of Eq. (3.16) since an arbitrary solution $\{y(n)\}$ satisfies

$$|y(n) - y^*(n)| \to 0 \quad \text{as } n \to \infty.$$

From the nonnegativity of $\{r(n)\}$ and the positivity of $\bar{r}$, it will follow that $\{r(n)\}$ is not identically zero; as a consequence we have that $y^*(n) > 0$ for $n \in Z$. We define

$$x^*(n) = \frac{1}{y^*(n)}, \qquad n \in Z \tag{3.18}$$

and note that an arbitrary solution of Eq. (2.4) satisfies

$$\lim_{n\to\infty}[x(n) - x^*(n)] = \lim_{n\to\infty}\left[\frac{1}{y(n)} - \frac{1}{y^*(n)}\right] = 0.$$

With $\{x^*(n)\}$ identified as the canonical solution of Eq. (2.4), the proof is complete. $\square$

4. Almost periodic discrete logistic equation

The temporal variation of the environment and resources of a biological population is usually incorporated through time dependent parameters of equations modelling the dynamics of the species. If the variations in $r(\cdot)$ and $K(\cdot)$ are periodic with a common period or if their periods are rationally dependent, then the continuous time model (1.1) becomes a periodic differential equation. A natural question then is related to the existence of a periodic solution with the same period as that of $r(\cdot)$ and $K(\cdot)$. If $r(\cdot)$ and $K(\cdot)$ have rationally independent periods, then the Eq. (1.1) is not periodic and it belongs to the class of almost periodic differential equations [7,18]. Dynamics of almost periodic equations modelling population systems have been investigated extensively by Gopalsamy and He [9], Seifert [17], Hamaya [13]. The existence and stability of almost periodic solutions will be of interest in such models. If an almost periodic solution exists and if it is asymptotically stable, then the species is said to track the temporal variations of the environment.

For computer simulation purposes and to model the dynamics of species with nonoverlapping generations with temporal variability in the environment, one needs discrete analogues of models such as those of Eq. (1.1). If $r(t)$ and $K(t)$ are periodic in $t \in (-\infty, \infty)$, then sequences of the form $\{r(n)\}$, $\{K(n)\}$, $n \in Z$ are not periodic with integer period; however $\{r(n)\}$, and $\{K(n)\}$ are almost periodic in $n \in Z$ (see Refs. [5,6,15]). Thus it is necessary to study almost periodic discrete analogues of Eq. (1.1) or equivalently study Eq. (2.4) under the assumption that $\{r(n)\}$ and $\{K(n)\}$ are almost periodic sequences. For the benefit of the reader, we narrate a few important facts about almost periodic real sequences.

Let $\{x(n)\}$ be a sequence of real numbers defined for $n \in Z$ where Z denotes the set of integers. An integer p is called an ϵ-almost period of the sequence $\{x(n)\}$ if for any $n \in Z$,

$$|x(n+p) - x(n)| < \epsilon.$$

If the sequence $\{x(n)\}$ is periodic with period p, i.e. $x(n+p) = x(n)$, $n \in Z$, then for any $\epsilon > 0$, the numbers jp, $j \in Z$ are ϵ-almost periods of this sequence. It is not difficult to see that if p is an ϵ-almost period of a sequence $\{x(n)\}$, then $-p$ is also an ϵ-almost period of this sequence.

Definition. A real valued sequence $\{x(n)\}$, $n \in Z$ is said to be almost periodic if for any $\epsilon > 0$ there exists a relatively dense set of ϵ-almost periods, i.e. there exists a natural number $N(\epsilon)$ such that for an arbitrary $m \in Z$, one can find an integer $p \in [m, m+N]$ for which the following inequality holds:

$$|x(n+p) - x(n)| < \epsilon, \quad n \in Z.$$

The following results will be used in our analysis below and proofs of the following can be found in Ref. [16].

Theorem A. *An almost periodic sequence is bounded.*

Theorem B. *For each $m \in N = \{0, 1, 2, \ldots\}$ let the sequence $\{x_m(i)\}$, $i \in Z$ be almost periodic. If the sequence $\{x_m(i)\}$ converges to $\{y(i)\}$ as $m \to \infty$, then $\{y(i)\}$, $i \in Z$ is almost periodic.*

Theorem C. *A sequence $\{x(n)\}$ is almost periodic if and only if for any sequence of integers $\{m_i\}$ there exists a subsequence $\{m_{i_k}\}$ such that the sequence $x(n + m_{i_k})$ converges for $k \to \infty$ uniformly with respect to $n \in Z$.*

Theorem D. *If $\{x(n)\}$ and $\{y(n)\}$ are almost periodic real sequences then $\{x(n) + y(n)\}$, $\{x(n)y(n)\}$ are also almost periodic; also $\{1/x(n)\}$ is almost periodic if $x(n) \neq 0$, $n \in Z$. If $\epsilon > 0$ is an arbitrary real number, then there exists a relatively dense set of ϵ-almost periods common to both $\{x(n)\}$ and $\{y(n)\}$.*

Definition. A sequence $\{x(n)\}$ of real numbers is said to be asymptotically almost periodic if and only if there exists two sequences $\{u(n)\}$ and $\{v(n)\}$ such that

$$x(n) = u(n) + v(n), \quad n \in Z$$

where $\{u(n)\}$ is almost periodic and $v(n) \to 0$ *as* $n \to \infty$.

We note that the concept of an asymptotically almost periodic sequence has been used by Halanay [11]. Apart from this, asymptotic almost periodic sequences have not been used in the literature on difference equations. However, asymptotically almost periodic functions have been used extensively in differential and functional differential equations by many authors (see, for instance, Refs. [7,9,14,18]).

We can now proceed to establish the existence of a globally asymptotically stable almost periodic solution of Eq. (2.4).

Theorem 4.1. *Suppose that* $\{r(n)\}$ *is an almost periodic real sequence such that*

$$0 \leqslant \inf_{n\in Z} r(n) \quad \textit{and} \quad \left[\sum_{i=0}^{m-1} r(i) - m\bar{r}\right] \to 0 \quad \textit{as } m \to \infty, \tag{4.1}$$

where $\bar{r}$ *is a positive number. Let* $\{K(n)\}$ *denote an almost periodic sequence satisfying*

$$0 < K_* \leqslant \inf_{n\in Z} K(n). \tag{4.2}$$

Then Eq. (2.4) *has a unique globally attracting almost periodic solution which also is locally asymptotically stable.*

Proof. Let $\{x(n)\}$ be a solution of Eq. (2.4) with $x(0) > 0$. Such a sequence $\{x(n)\}$ remains positive for all $n \geqslant 1$ and so we can let

$$y(n) \equiv \frac{1}{x(n)}, \qquad n \geqslant 0 \tag{4.3}$$

and obtain that $y(n)$ is governed by

$$y(n+1) = e^{-r(n)} y(n) + \frac{1 - e^{-r(n)}}{K(n)}, \qquad n \geqslant 0. \tag{4.4}$$

It is sufficient to prove the existence of a locally asymptotically stable and globally attractive almost periodic solution of Eq. (4.4). Let $\{\tau_p\}$, $\{\tau_q\}$ be integer valued sequences such that $\tau_p \to \infty$, $\tau_q \to \infty$ as $p, q \to \infty$. We have from Eq. (4.4),

$$\left.\begin{aligned} y(n+1+\tau_p) &= y(n+\tau_p)e^{-r(n+\tau_p)} + \frac{1-e^{-r(n+\tau_p)}}{K(n+\tau_p)} \\ y(n+1+\tau_q) &= y(n+\tau_q)e^{-r(n+\tau_q)} + \frac{1-e^{-r(n+\tau_q)}}{K(n+\tau_q)} \end{aligned}\right\} \quad \text{for } \begin{array}{l} n+\tau_p \geqslant 0, \\ n+\tau_q \geqslant 0, \end{array}$$

that

$$y(n+1+\tau_p) - y(n+1+\tau_q) = [y(n+\tau_p) - y(n+\tau_q)]\mathrm{e}^{-r(n+\tau_p)}$$
$$+ y(n+\tau_q)[\mathrm{e}^{-r(n+\tau_p)} - \mathrm{e}^{-r(n+\tau_q)}] + \left[\frac{1-\mathrm{e}^{-r(n+\tau_p)}}{K(n+\tau_p)} - \frac{1-\mathrm{e}^{-r(n+\tau_q)}}{K(n+\tau_q)}\right]. \quad (4.5)$$

Let ϵ be an arbitrary positive number. By the almost periodicity of $\{r(n)\}$ and $\{K(n)\}$ and the boundedness of $\{y(n)\}$, it will follow (by Theorem D above) that there exists a positive integer $N = N(\epsilon)$ such that

$$\left.\begin{aligned} & y(n+\tau_q)|\mathrm{e}^{-r(n+\tau_p)} - \mathrm{e}^{-r(n+\tau_q)}| < \frac{\epsilon}{2} \\ & \left|\frac{1-\mathrm{e}^{-r(n+\tau_p)}}{K(n+\tau_p)} - \frac{1-\mathrm{e}^{-r(n+\tau_q)}}{K(n+\tau_q)}\right| < \frac{\epsilon}{2} \end{aligned}\right\} \quad \text{for} \quad \begin{aligned} & n+\tau_p \geqslant N, \\ & n+\tau_q \geqslant N. \end{aligned} \quad (4.6)$$

For convenience, we define $v(n, \tau_p, \tau_q)$ by the following:

$$v(n, \tau_p, \tau_q) = y(n+\tau_p) - y(n+\tau_q), \qquad n+\tau_p \geqslant 0, \; n+\tau_q \geqslant 0. \quad (4.7)$$

We have from Eqs. (4.5)–(4.7) that

$$v(n, \tau_p, \tau_q) < v(0, \tau_p, \tau_q) \exp\left(-\sum_{i=0}^{n-1} r(i+\tau_p)\right)$$
$$+ \epsilon\left[1 + \mathrm{e}^{-r(n-1+\tau_p)} + \mathrm{e}^{-r(n-1+\tau_p)-r(n-2+\tau_p)}\right.$$
$$\left. + \cdots + \exp\left(-\sum_{j=1}^{n-1} r(j+\tau_p)\right)\right], \quad n \geqslant 1, \; \tau_p, \tau_q \geqslant N. \quad (4.8)$$

Since $\bar{r} > 0$ and $\sum_{i=0}^{n-1} \delta(i+\tau_p) \to 0$ as $n \to \infty$,

$$\exp\left(-\sum_{i=0}^{n-1} r(i+\tau_p)\right) = \exp\left(-n\bar{r} + \sum_{i=0}^{n-1} \delta(i+\tau_p)\right) \to 0 \quad \text{as } n \to \infty. \quad (4.9)$$

Consider the infinite series

$$\sum_{i=1}^{\infty} u(i) = 1 + \mathrm{e}^{-r(n-1+\tau_p)}$$
$$+ \mathrm{e}^{-r(n-2+\tau_p)-r(n-1+\tau_p)} + \cdots + \exp\left(-\sum_{j=1}^{n-1} r(j+\tau_p)\right) + \cdots \quad (4.10)$$

This series converges by the ratio test; for instance we have the ratio,

$$\begin{aligned}
\lim_{n\to\infty}\frac{u(n+1)}{u(n)} &= \lim_{n\to\infty}\frac{\exp\Big(-\sum_{j=1}^{n} r(j+\tau_p)\Big)}{\exp\Big(-\sum_{j=1}^{n-1} r(j+\tau_p)\Big)}\\
&= \lim_{n\to\infty}\frac{\exp\Big(-n\bar r+\sum_{j=1}^{n}\delta(j+\tau_p)\Big)}{\exp\Big(-(n-1)\bar r+\sum_{j=1}^{n-1}\delta(j+\tau_p)\Big)}\\
&= \lim_{n\to\infty} \mathrm{e}^{-\bar r+\delta(n+\tau_p)}\\
&< 1 \quad \text{since } \bar r > 0 \text{ and } \delta(n+\tau_p)\to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Thus it will follow that for all $n+\tau_p$ and $n+\tau_q$ large enough and arbitrary $\epsilon_1 > 0$,

$$|v(n,\tau_p,\tau_q)| = |y(n+\tau_p)-y(n+\tau_q)| < \epsilon_1. \tag{4.11}$$

It follows that the sequence $\{y(n)\}$ is asymptotically almost periodic. Hence we have

$$y(n) = w(n) + z(n),$$

where $w(n)$ is almost periodic in $n\in Z$ and $z(n)\to 0$ as $n\to\infty$.

We show now that $\{w(n)\}$ is a solution of Eq. (4.4); we let $y(n) = w(n) + z(n)$ in

$$y(n+1) = \mathrm{e}^{-r(n)}y(n) + \frac{1-\mathrm{e}^{-r(n)}}{K(n)}$$

and obtain

$$w(n+1) + z(n+1) = \mathrm{e}^{-r(n)}[w(n)+z(n)] + \frac{1-\mathrm{e}^{-r(n)}}{K(n)}$$

implying

$$w(n+1) - \left(\mathrm{e}^{-r(n)}w(n) + \frac{1-\mathrm{e}^{-r(n)}}{K(n)}\right) = -z(n+1) + \mathrm{e}^{-r(n)}z(n),$$

by using the boundedness of the almost periodic sequence $\{r(n)\}$ and the fact that $z(n)\to 0$ as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\left[w(n+1) - \left(\mathrm{e}^{-r(n)}w(n) + \frac{1-\mathrm{e}^{-r(n)}}{K(n)}\right)\right] = 0.$$

We show next that

$$w(n+1) = \mathrm{e}^{-r(n)}w(n) + \frac{1-\mathrm{e}^{-r(n)}}{K(n)}, \qquad n\in Z.$$

Let

$$v(n) = w(n+1) - \left(\mathrm{e}^{-r(n)}w(n) + \frac{1-\mathrm{e}^{-r(n)}}{K(n)}\right), \qquad n\in Z.$$

Suppose $v(n^*) \neq 0$ for some integer n^*; let $\bar{\epsilon} = (|v(n^*)|)/2$ and note that there exists $L = L(\bar{\epsilon}) > 0$ such that any set of L consecutive integers contains an integer say m for which

$$|v(n^* + m) - v(n^*)| \leqslant \frac{|v(n^*)|}{2}$$

implying that

$$|v(n^* + m)| \geqslant \frac{|v(n^*)|}{2}.$$

Let $I_j = [jm, (j+1)m]$ where $j = 0, 1, 2, \ldots$ denote intervals of length m. In each interval I_j, there is a corresponding m_j such that

$$|v(n^* + m_j)| \geqslant \frac{|v(n^*)|}{2}, \qquad j = 0, 1, 2, \ldots$$

Let $m_j \to \infty$ as $j \to \infty$ and note that

$$\lim_{j\to\infty} |v(n^* + m_j)| \geqslant \frac{|v(n^*)|}{2}$$

which contradicts the fact $v(n) \to 0$ as $n \to \infty$. Thus we conclude that

$$v(n) \equiv 0, \qquad n \in Z$$

which implies

$$w(n+1) = \mathrm{e}^{-r(n)} w(n) + \frac{1 - \mathrm{e}^{-r(n)}}{K(n)}, \qquad n \in Z. \tag{4.12}$$

Thus there exists a positive almost periodic solution $\{y^*(n)\}$ of Eq. (4.4) where

$$y^*(n) = w(n) \quad \text{for all } n \in Z.$$

To prove the local asymptotic stability and global attractivity of $\{y^*(n)\}$, we proceed as follows: let $\{y(n)\}$, $n \geqslant 0$ denote an arbitrary solution of Eq. (4.4). Then

$$\begin{aligned} y^*(n+1) - y(n+1) &= \mathrm{e}^{-r(n)} [y^*(n) - y(n)] \\ &= \mathrm{e}^{-r(n)}\ \mathrm{e}^{-r(n-1)} [y^*(n-1) - y(n-1)] \\ &= \exp\left(-\sum_{j=0}^{n} r(j) \right) [y^*(0) - y(0)] \\ &= [y^*(0) - y(0)] \exp\left(-(n+1)\bar{r} - \sum_{j=0}^{n} \delta(j) \right) \\ &\to 0 \quad \text{as } n \to \infty \end{aligned} \tag{4.13}$$

since $\bar{r} > 0$ and $\sum_{j=0}^{n} \delta(j) \to 0$ as $n \to \infty$. It follows that $\{y^*(n)\}$ is globally attractive. The local asymptotic stability of $\{y^*(n)\}$ is a consequence of the following:

$$
\begin{aligned}
|y^*(n+1) - y(n+1)| &\leqslant |y^*(0) - y(0)| \exp\left(- \sum_{j=0}^{n} r(j) \right), \qquad n \geqslant 0 \\
&< |y^*(0) - y(0)| \mathrm{e}^{-(n+1)\bar{r}/2} \\
&< \epsilon \quad \text{if } |y^*(0) - y(0)| < \epsilon \, \mathrm{e}^{(n+1)\bar{r}/2}.
\end{aligned} \tag{4.14}
$$

The global asymptotic stability of $\{y^*(n)\}$ is established. Solutions $\{x(n)\}$ and $\{y(n)\}$ are related by Eq. (4.3) and hence the proof of Theorem 4.1 is complete. □

We shall obtain an explicit representation of the almost periodic solution of Eq. (2.4).

Theorem 4.2. *Suppose the assumptions of Theorem* 4.1 *are satisfied. Then the sequence* $\{x^*(n)\}$, $n \in Z$ *defined by*

$$
x^*(n) = \left[\sum_{m=1}^{\infty} \left(\frac{1 - \mathrm{e}^{-r(n-m)}}{K(n-m)} \right) \exp\left(- \sum_{p=1}^{m-1} r(n-p) \right) \right]^{-1}, \qquad n \in Z \tag{4.15}
$$

is the unique positive almost periodic solution of Eq. (2.4).

Proof. Since the solutions $\{x^*(n)\}$ and $\{y^*(n)\}$ of Eqs. (2.4) and (4.4) are related by

$$
x^*(n) = \frac{1}{y^*(n)}, \qquad n \in Z \tag{4.16}
$$

it suffices to prove (see Theorem D above) the almost periodicity of the solution $\{y^*(n)\}$ where

$$
y^*(n) = \sum_{m=1}^{\infty} \left(\frac{1 - \mathrm{e}^{-r(n-m)}}{K(n-m)} \right) \exp\left(- \sum_{p=1}^{m-1} r(n-p) \right), \qquad n \in Z. \tag{4.17}
$$

Let $\epsilon > 0$ be given. Consider the difference $|y^*(n+\tau) - y^*(n)|$ for some integer τ; it is sufficient to show that this difference can be made sufficiently small depending on ϵ. Let τ be an integer such that

$$
\left.
\begin{aligned}
&\left| \frac{1 - \mathrm{e}^{-r(n+\tau-m)}}{K(n+\tau-m)} - \frac{1 - \mathrm{e}^{-r(n-m)}}{K(n-m)} \right| < \left(\frac{\epsilon}{2} \right) \left(\frac{1 - \mathrm{e}^{-\bar{r}}}{\mathrm{e}^{M}} \right) \\
&\left| \exp\left(- \sum_{p=1}^{m-1} r(n+\tau-p) + \sum_{p=1}^{m-1} r(n-p) \right) - 1 \right| < \left(\frac{\epsilon}{2} \right) \left(\frac{K_*(1 - \mathrm{e}^{-\bar{r}})}{\mathrm{e}^{M}} \right)
\end{aligned}
\right\}. \tag{4.18}
$$

The existence of an integer τ satisfying these inequalities uniformly in $n \in Z$ is a consequence of the almost periodicity of $\{r(n)\}$ and $\{K(n)\}$ (see Theorem D above). It follows from Eq. (4.17),

$$\begin{aligned}
y^*(n+\tau)-y^*(n) &= \sum_{m=1}^{\infty}\Bigg\{\left(\frac{1-\mathrm{e}^{-r(n+\tau-m)}}{K(n+\tau-m)}\right)\exp\left(-\sum_{p=1}^{m-1}r(n+\tau-p)\right) \\
&\quad -\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\exp\left(-\sum_{p=1}^{m-1}r(n-p)\right)\Bigg\} \\
&= \sum_{m=1}^{\infty}\left\{\left(\frac{1-\mathrm{e}^{-r(n+\tau-m)}}{K(n+\tau-m)}\right)-\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\right\} \\
&\quad \times\exp\left(-\sum_{p=1}^{m-1}r(n+\tau-p)\right)+\sum_{m=1}^{\infty}\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right) \\
&\quad \times\left\{\exp\left(-\sum_{p=1}^{m-1}r(n+\tau-p)\right)-\exp\left(-\sum_{p=1}^{m-1}r(n-p)\right)\right\}
\end{aligned}$$

and hence

$$\begin{aligned}
&|y^*(n+\tau)-y^*(n)| \\
&\leqslant \sum_{m=1}^{\infty}\left|\left(\frac{1-\mathrm{e}^{-r(n+\tau-m)}}{K(n+\tau-m)}\right)-\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\right|\exp\left(-\sum_{p=1}^{m-1}r(n+\tau-p)\right) \\
&\quad +\sum_{m=1}^{\infty}\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\exp\left(-\sum_{p=1}^{m-1}r(n-p)\right) \\
&\quad \times\left|\exp\left(-\sum_{p=1}^{m-1}r(n+\tau-p)+\sum_{p=1}^{m-1}r(n-p)\right)-1\right|. \qquad (4.19)
\end{aligned}$$

Applying the assumptions (4.1), (4.2) and (4.18) to Eq. (4.19) we obtain that

$$\begin{aligned}
|y^*(n+\tau)-y^*(n)| &\leqslant \sum_{m=1}^{\infty}\left(\frac{\epsilon}{2}\right)\left(\frac{1-\mathrm{e}^{-\bar{r}}}{\mathrm{e}^{M}}\right)\mathrm{e}^{-(m-1)\bar{r}+M} \\
&\quad +\sum_{m=1}^{\infty}\left(\frac{\mathrm{e}^{-(m-1)\bar{r}+M}}{K_*}\right)\left(\frac{\epsilon}{2}\right)\left(\frac{K_*(1-\mathrm{e}^{-\bar{r}})}{\mathrm{e}^{M}}\right)=\epsilon
\end{aligned} \qquad (4.20)$$

uniformly for all $n\in Z$. Hence the sequence $\{y^*(n)\}$ is a unique positive almost periodic solution of Eq. (4.4). By the relation (4.16) and Theorem D the sequence $\{x^*(n)\}$ is a unique positive almost periodic solution of Eq. (2.4) and hence the proof of Theorem 4.2 is complete. □

Corollary 4.3. *Suppose that* $\{r(n)\}$ *and* $\{K(n)\}$ *are periodic real sequences with a common period* $\sigma > 0$, *i.e.* $r(n+\sigma) = r(n)$ *and* $K(n+\sigma) = K(n)$, $n \in Z$. *Suppose further that*

$$0 \leqslant \inf_{n\in Z} r(n) \quad \text{and} \quad \sum_{m=1}^{\sigma} r(j) = \sigma\bar{r}, \tag{4.21}$$

where $\bar{r}$ *is a positive number*; *and*

$$0 < K_* \leqslant \inf_{n\in Z} K(n). \tag{4.22}$$

Then the sequence $\{x_\sigma^*(n)\}$, $n \in Z$ *defined by*

$$x_\sigma^*(n) = [1 - \mathrm{e}^{-\sigma\bar{r}}]\left[\sum_{m=1}^{\sigma}\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\exp\left(-\sum_{p=1}^{m-1} r(n-p)\right)\right]^{-1}, \quad n \in Z \tag{4.23}$$

is the unique positive σ*-periodic solution of Eq.* (2.4) *which is globally attractive and is also locally asymptotically stable.*

Proof. We have from Theorem 3.1 that Eq. (2.4) has a canonical solution given by

$$x^*(n) = \left[\sum_{m=1}^{\infty}\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\exp\left(-\sum_{p=1}^{m-1} r(n-p)\right)\right]^{-1}, \qquad n \in Z. \tag{4.24}$$

We then rewrite Eq. (4.24) as follows:

$$\begin{aligned} x^*(n) = \Bigg[&\sum_{m=1}^{\sigma}\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\exp\left(-\sum_{p=1}^{m-1} r(n-p)\right) \\ &+ \sum_{m=\sigma+1}^{2\sigma}\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\exp\left(-\sum_{p=1}^{m-1} r(n-p)\right) \\ &+ \sum_{m=2\sigma+1}^{3\sigma}\left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)}\right)\exp\left(-\sum_{p=1}^{m-1} r(n-p)\right) + \cdots\Bigg]^{-1}, \\ &n \in Z. \end{aligned} \tag{4.25}$$

Applying the periodicity of $\{r(n)\}$ and $\{K(n)\}$ and the assumption (4.21) one can derive from Eq. (4.25) that

$$
\begin{aligned}
x^*(n) = \Bigg[& \sum_{m=1}^{\sigma} \left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)} \right) \exp\left(-\sum_{p=1}^{m-1} r(n-p) \right) \\
& + \mathrm{e}^{-\sigma\bar{r}} \sum_{m=1}^{\sigma} \left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)} \right) \exp\left(-\sum_{p=1}^{m-1} r(n-p) \right) \\
& + \mathrm{e}^{-2\sigma\bar{r}} \sum_{m=1}^{\sigma} \left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)} \right) \exp\left(-\sum_{p=1}^{m-1} r(n-p) \right) + \cdots \Bigg]^{-1} \\
= & \left[\sum_{j=0}^{\infty} \mathrm{e}^{-j\sigma\bar{r}} \right]^{-1} \left[\sum_{m=1}^{\sigma} \left(\frac{1-\mathrm{e}^{-r(n-m)}}{K(n-m)} \right) \exp\left(-\sum_{p=1}^{m-1} r(n-p) \right) \right]^{-1}, \\
& n \in Z. \qquad (4.26)
\end{aligned}
$$

Since $\sigma > 0$ and $\bar{r} > 0$ one can then obtain Eq. (4.23) from Eq. (4.26). The stability criteria for the solution $\{x^*_\sigma(n)\}$ is a consequence of Theorems 3.1 and 4.1; hence the proof is complete. □

We conclude with the presentation below of a few computer simulations of illustrative examples of the discrete system investigated. We have included simulations corresponding both periodic and almost periodic cases. Sets of solutions corresponding to two distinct initial values are graphically illustrated.

Example 1 (see Fig. 1).

$$r(n) = 1.0 + 1.0\sin(\pi n), \quad \text{period } \sigma = 2, \quad n \geqslant 0;$$
$$K(n) = 6.0 + 4.0\cos(\sqrt{2}\pi n), \quad \text{no integer period}, \quad n \geqslant 0;$$
$$x_1(0) = 1.0, \quad x_2(0) = 15.0.$$

Example 2 (see Fig. 2).

$$r(n) = 1.0 + 1.0\sin(\sqrt{2}\pi n), \quad \text{no integer period}, \quad n \geqslant 0;$$
$$K(n) = 6.0 + 4.0\cos(\pi n), \quad \text{period } \sigma = 2, \quad n \geqslant 0;$$
$$x_1(0) = 1.0, \quad x_2(0) = 15.0.$$

Example 3 (see Fig. 3).

$$r(n) = 1.0 + 1.0\sin((\pi/3)n), \quad \text{period } \sigma_r = 6, \quad n \geqslant 0;$$
$$K(n) = 6.0 + 4.0\cos((\pi/2)n), \quad \text{period } \sigma_K = 4, \quad n \geqslant 0;$$
$$x_1(0) = 1.0, \quad x_2(0) = 15.0, \quad x^*(0) = x^*(12) = 7.5342.$$

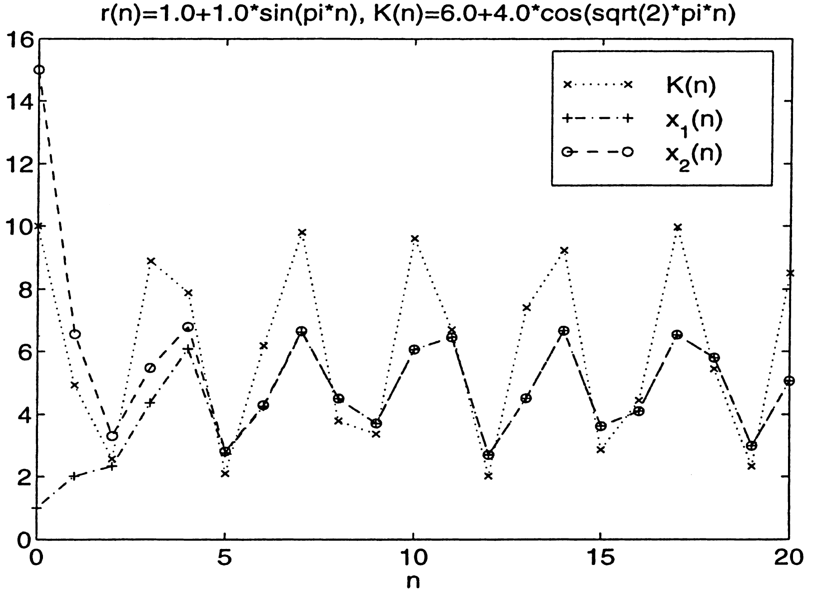

Fig. 1. Solutions of Eq. (2.4) approaching the unique almost periodic solution $\{x^*(n)\}$. The parameters are given in Example 1.

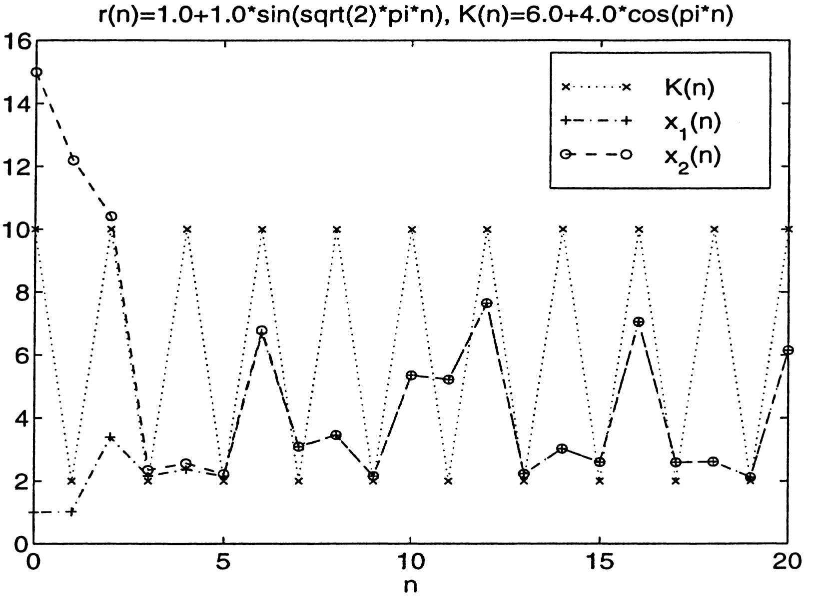

Fig. 2. Solutions of Eq. (2.4) approaching the unique almost periodic solution $\{x^*(n)\}$. The parameters are given in Example 2.

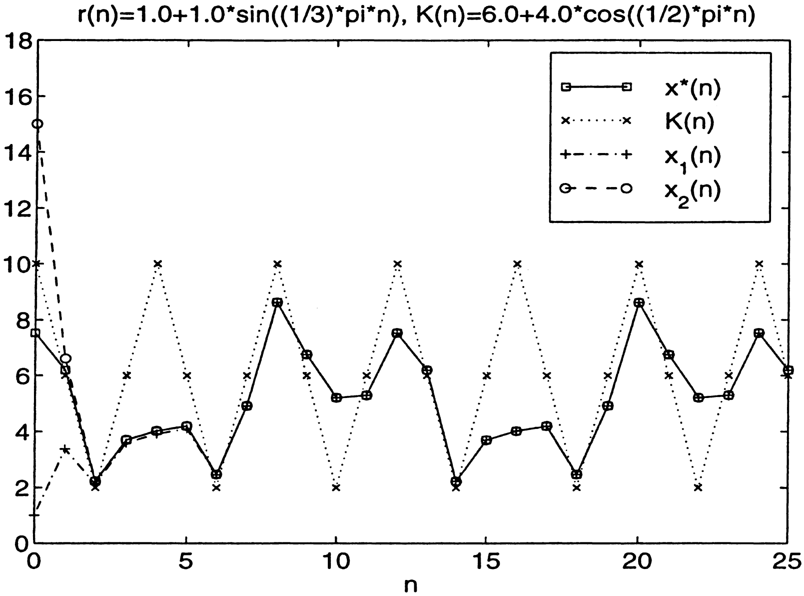

Fig. 3. Solutions of Eq. (2.4) approaching the unique periodic solution $\{x^*(n)\}$ of period $\sigma = 12$. The parameters are given in Example 3. In this case the canonical solution $\{x^*(n)\}$ is explicitly illustrated.

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