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Parameter Estimation Assignment

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Ques 1. Let (X_1, X_2 ....) be a random sample of size n taken from a Normal population with parameters: mean = θ_1 and variance = θ_2. Find the Maximum Likelihood Estimates of these two parameters.

Ans: A sample of size 'n' is taken: (X_1, X_2, ... X_n)

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f(x) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x-\theta_1)^2}{2\theta_2}}$$

$$P(x_i, \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x-\theta_1)^2}{2\theta_2}}$$

~~Now~~ ~~Lik~~ LL. $$P(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

Likelihood ~~func~~ $$\Rightarrow P(\theta_1, \theta_2) = \prod_{i=1}^{n} (\theta_2)^{-\frac{1}{2}} \prod_{i=1}^{n} (2\pi)^{-1/2} \prod_{i=1}^{n} e^{\left(-\frac{(x_i-\theta_1)^2}{2\theta_2}\right)}$$

$$P(\theta_1, \theta_2) = (\theta_2)^{-\frac{n}{2}} (2\pi)^{-\frac{n}{2}} e^{-\left(\sum_{i=1}^{n} \frac{(x_i-\theta_1)^2}{2\theta_2}\right)}$$

$$P(\theta_1, \theta_2) = (\theta_2)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i-\theta_1)^2}$$

Taking log.

$$\ln P(\theta_1, \theta_2) = \ln\left[(\theta_2)^{-\frac{n}{2}} (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i-\theta_1)^2}\right]$$

$$= \ln (\theta_2)^{\frac{-n}{2}} + \ln (2\pi)^{\frac{-n}{2}} + \ln \left(e^{\frac{-1}{2\theta_2} \sum_{i=1}^{n} (x_i-\theta_1)^2}\right)$$

$$= \frac{-n}{2} \ln(\theta_2) - \frac{n}{2} \ln 2\pi - \frac{1}{2\theta_2} \left\{ \sum_{i=1}^{n} (x_i-\theta_1)^2 \right. - ①$$

Diff. both sides w.r.t. θ_1

$$\frac{\partial P(\theta_1,\theta_2)}{\partial \theta_1} = \frac{1}{\theta_2} \sum_{i=1}^{n} (x_i-\theta_1)$$

$$\frac{1}{\theta_2} \sum_{i=1}^{n} (x_i-\theta_1) = 0$$

$$\frac{1}{\theta_2} \sum_{i=1}^{n} x_i - n\theta_1 = 0$$

$$\theta_1 = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\theta_1 = \bar{x}_n$$

$$\theta_{1\,MLE} = \bar{x}_n \quad \rightarrow ②$$

Diff. w.r.t. θ_2

$$\frac{\partial P(\theta_1,\theta_2)}{\partial \theta_2} = \frac{-n}{2\theta_2} - \frac{-1}{2(\theta_2)^2} \sum_{i=1}^{n} (x_i-\theta_1)^2$$

Putting equals to 0.

$$\frac{-n}{2\theta_2} - \frac{-1}{2(\theta_2)^2} \sum_{i=1}^{n} (x_i-\theta_1)^2 = 0$$

$$\Rightarrow \frac{1}{2(\theta_2)^2} \sum_{i=1}^{n} (x_i - \theta_1)^2 = \frac{-n}{2\theta_2}$$

$$\theta_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \theta_1)^2 \quad - (3)$$

from (2) & (3)

$$\theta_{2\,MLE} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2$$

Ques2 Let $X_1, X_2, \ldots X_n$ be a random sample from $B(m, \theta)$ distribution where $\theta \in (0,1)$ is unknown and m is a known positive integer. compute values of θ using the M.L.E.

Ans For binomial distribution.

$$PMF = {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$$

Given: $n = m \qquad p = \theta.$

$$= {}^{m}C_{x}\, \theta^{x} (1-\theta)^{m-x}$$

Likelihood func. $L(p) = \prod_{i=1}^{n} (P(x_i \mid m, \theta))$

$$= \prod_{i=1}^{n} \left({}^{m}C_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i}\right)$$

$$= \prod_{i=1}^{n} {}^{m}C_{x_i} \prod_{i=1}^{n} \theta^{x_i} \prod_{i=1}^{n} (1-\theta)^{m-x_i}$$

$$= \prod_{i=1}^{n} {}^{m}C_{x_i}\, \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{nm - \sum_{i=1}^{n} x_i}$$

Taking log

$$\ln(L(p)) = \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i} \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{nm-\sum_{i=1}^{n} x_i}\right)$$

$$= \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\right) + \ln\left(\theta^{\sum_{i=1}^{n} x_i}\right) + \ln\left((1-\theta)^{nm-\sum_{i=1}^{n} x_i}\right)$$

$$\sum_{i=1}^{n} x_i \ln(\theta)$$

Diff. w.r.t. θ

$$\frac{\partial \ln(Lp)}{\partial \theta} = \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}\left(nm - \sum_{i=1}^{n} x_i\right) = 0$$

$$= \frac{1}{\theta}\sum_{i=1}^{n} x_i = \frac{1}{1-\theta}\left(nm - \sum_{i=1}^{n} x_i\right)$$

$$\frac{1-\theta}{\theta} = \frac{nm - \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i}$$

$$\frac{nm}{\sum_{i=1}^{n} x_i} - 1 = \frac{1}{\theta} - 1$$

$$\theta = \sum_{i=1}^{n} \frac{x_i}{nm}$$

$$\theta = \frac{\bar{x}_n}{m}$$

$$\theta_{MLE} \in (0,1) = \frac{\bar{x}_n}{m}$$