Unit 4

2022-04-27

16.1 - Vector Fields

$\mathbf{F} : D \longrightarrow \mathbb{R}^n$, $\mathbf{F}(x_1, x_2, ..., x_n) = \langle v_1, v_2, \ldots, v_n \rangle$

Scalar fields: The real-valued scalar functions which are coefficients of the gradient field.

3-D: $F(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ $F = \langle P, Q, R \rangle$, $F = \langle F_1, F_2, F_3 \rangle$

Gradient Vector Field: $\nabla f = \langle f_x, f_y, f_z \rangle = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k}$

If $f(x,y,z)$ differentiable, its gradient vector field is a **conservative vector field** (*aka* a *potential function*).

$\mathbf{F} = \nabla f = \langle f_x, f_y, f_z \rangle$

Operations on Vector Fields

Let $\mathbf{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field on $\mathbb{R}^3$.

Vector differential operator: $\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle$

Curl: $\text{curl}(\mathbf{F}) = \nabla \times F = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$ is a *vector field* on $\mathbb{R}^3$

$\boxed{\text{Theorem}}$ If f is a function of three variables that has continuous second-order partial derivatives, then $\text{curl}(\nabla f) = \mathbf{0}$. In other words:

$F = \nabla f$ $\Leftarrow$ If $\mathbf{F}$ is conservative, then $\text{curl}(\mathbf{F}) = \mathbf{0}$

$\boxed{\text{Theorem}}$ If $\mathbf{F}$ is a vector fields defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\text{curl}(\mathbf{F}) = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Definition: Let $\mathbf{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, $\partial R/\partial z$ exist, then the divergence of $\mathbf{F}$ is the function of three variables defined by

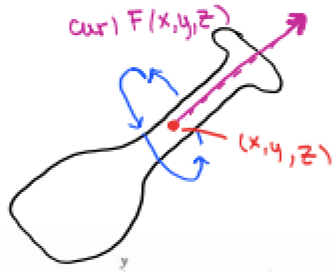
$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle P, Q, R \rangle$$
$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = P_x + Q_y + R_z$$

scalar field

$\boxed{\text{Theorem}}$ Let $\mathbf{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field on $\mathbb{R}^3$ and P, Q, R have continuous second-order partial derivatives, then

$$\text{div}(\text{curl}(F)) = 0$$

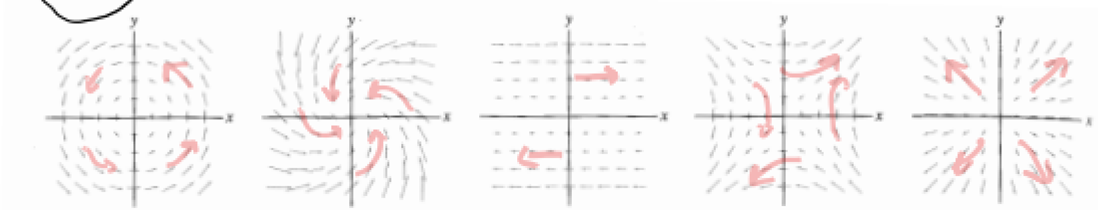
scalar field



- If $\text{curl}(\mathbf{F}) = \mathbf{0}$ at a point P, then the fluid is free from rotations. (no spin)

  F is called irrotational at P

- If $\text{curl}(\mathbf{F}) \neq \mathbf{0}$ at a point P, then the paddle rotates about its axis.



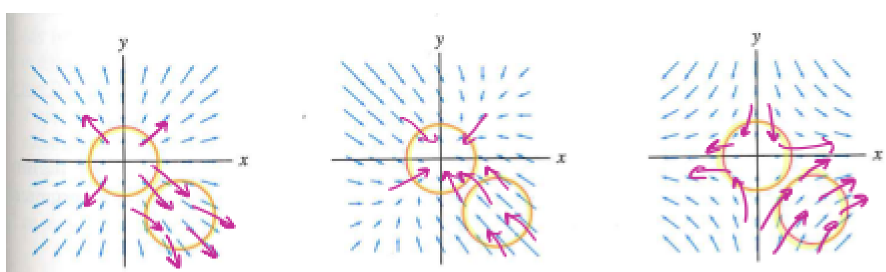
(A) $\mathbf{F} = \langle -y, x\rangle$ $\text{curl}_z(\mathbf{F}) = 2$ (B) $\mathbf{F} = \langle -x-y, x-y\rangle$ $\text{curl}_z(\mathbf{F}) = 2$ (C) $\mathbf{F} = \langle y, 0\rangle$ $\text{curl}_z(\mathbf{F}) = -1$ (D) $\mathbf{F} = \langle y, x\rangle$ $\text{curl}_z(\mathbf{F}) = 0$ (E) $\mathbf{F} = \langle x, y\rangle$ $\text{curl}_z(\mathbf{F}) = 0$

Physical Interpretation of Divergence

The div $\mathbf{F}$ measures how much a vector field will "spread out" or diverge from a given point.

If $\mathbf{F}(x, y, z) = \langle P, Q, R\rangle$ is the vector field which gives the velocity of a fluid flowing at point (x, y, z) then $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$ gives the amount of fluid being repelled by the point (x, y, z) minus the amount begin attracted to that point.

If $\nabla \cdot \mathbf{F} = 0$ then $\mathbf{F}$ is **incompressible**. The amount coming into a point is the same as the amount coming out. It is not being compressed.



(A) The field $\mathbf{F} = \langle x, y\rangle$ with $\text{div}(\mathbf{F}) = 2$. There is a net outflow through every circle. (B) The field $\mathbf{F} = \langle y-2x, x-2y\rangle$ with $\text{div}(\mathbf{F}) = -4$. There is a net inflow into every circle. (C) The field $\mathbf{F} = \langle x, -y\rangle$ with $\text{div}(\mathbf{F}) = 0$. The flux through every circle is zero.