Calc3 – Exam One Notes

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Chapter 11 – Parametric Equations, Polar Coord., Conic Sections

11.1

- Eliminate parameter to find Cartesian equation of curve
- Plot parametric curves:

$y = -x/4 - 3/4$ ← whole line

line segme

example IMPORTANT EXAMPLE!

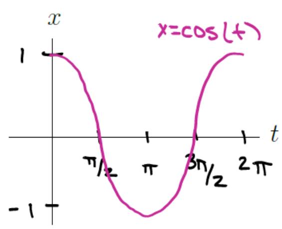
you should remember this example

Identify what curve is represented by the following parametric equations.

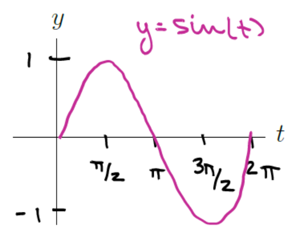
$$x = \cos(t) \qquad y = \sin(t) \qquad 0 \le t \le 2\pi$$

start

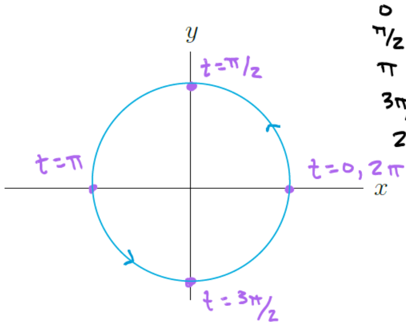
Look at the x equation in terms of t



Look at the y equation in terms of t



Put them together:



t	(x,y)
0	$(1,0)$
$\pi/2$	$(0,1)$
π	$(-1,0)$
$3\pi/2$	$(0,-1)$
2π	$(1,0)$

$C = (x,y) = (\cos(t), \sin(t))$ is $\boxed{\text{a circle w/ center } (0,0) \text{ and radius } 1}$

Notice: $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$

$\Rightarrow \underbrace{x^2 + y^2 = 1}_{\text{unit cirle}}$

example Describe the motion of a particle with position $(x, y) = (3 + 2\cos(t), 1 + 2\sin(t))$ as t varies $\boxed{\pi/2 \le t \le 3\pi/2}$.

this is a little easier

I will solve for cos(t) and sin(t)

- Use common parameterizations:

Line through $P=(a,b) \longrightarrow C=(a+rt, b+st)$ where $m=\dfrac{s}{r}, r \neq 0$
Line through $P,Q \longrightarrow C=(a+t(c-a), b+t(d-b))$

Circle: $C=(h+r\cos(t), k+r\sin(t)), \quad 0 \leq t < 2\pi \;$ or $\; 0 \leq t \leq 2\pi$

Ellipse: $(\frac{x}{a})^2+(\frac{y}{b})^2=1 \longrightarrow C=(a\cos(t), b\sin(t)), \quad -\pi \leq t \leq \pi \;$ or $\; 0 \leq t < \pi$

- Match parametric equation to graphs
- Find equation of a tangent line:

$y-y_0=\frac{d_y}{d_x}|_{(x_0,y_0)}(x-x_0)$

For parametric functions, $\dfrac{d_y}{d_x}=\dfrac{dy/dt}{dx/dt}$ and $\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}(\frac{dy/dt}{dx/dt})}{dx/dt}$
Horizontal tangent: $\frac{dy}{dt}=0, \frac{dx}{dt}\neq 0$, Vertical tangent: $\frac{dx}{dt}=0, \frac{dy}{dt}\neq 0$

- Find area under a curve $(x=f(t), y=g(t)$:
| $A=\displaystyle\int_a^b h(x)dx=\int_\alpha^\beta g(t)f'(t)dt$ (if $x=a \Rightarrow t=\alpha$, else $\int_\beta^\alpha$) | $A=\displaystyle\int_c^d h(y)dy=\int_\alpha^\beta f(t)g'(t)dt$ (if " " else $\int_\beta^\alpha$)

11.2 Arc Length & Surface Area

- Arc Length $L=\displaystyle\int_a^b \sqrt{1+(dy/dx)^2}dx$

Theorem If a curve $C=(x,y)=(f(t),g(t))$ for $\alpha \leq t \leq \beta$ and f' and g' are continuous on $[\alpha,\beta]$ and C is transversed exactly once as t increases from α to β, then the length of C is

goes around exactly once

$$L=\int_\alpha^\beta \sqrt{(dx/dt)^2+(dy/dt)^2}\,dt$$

This integrand is the speed formula!

- Surface Area ($x = f(t), y=g(t) $

(x-axis) $S=\displaystyle\int_\alpha^\beta 2\pi y\sqrt{(dx/dt)^2+(dy/dt)^2}dt$

(y-axis) $S=\displaystyle\int_\alpha^\beta 2\pi x\sqrt{(dx/dt)^2+(dy/dt)^2}dt$

11.3-11.4 Polar Coordinates

- Convert from polar to Cartesian, Cartesian to polar
- Sketch and identify curves in polar coordinates. KNOW HOW TO FIND/CHOOSE LIMITS OF INTEGRATION.

Brief Trig Review

$y = A\sin(Bx - c) + D$ OR $y = A\cos(Bx - c) + D$

Polar to Rectangular	Rectangular to Polar
$x = r\cos(\theta)$	$r^2 = x^2 + y^2$
$y = r\sin(\theta)$	$r = \pm\sqrt{x^2 + y^2}$
	$\tan(\theta) = y/x$

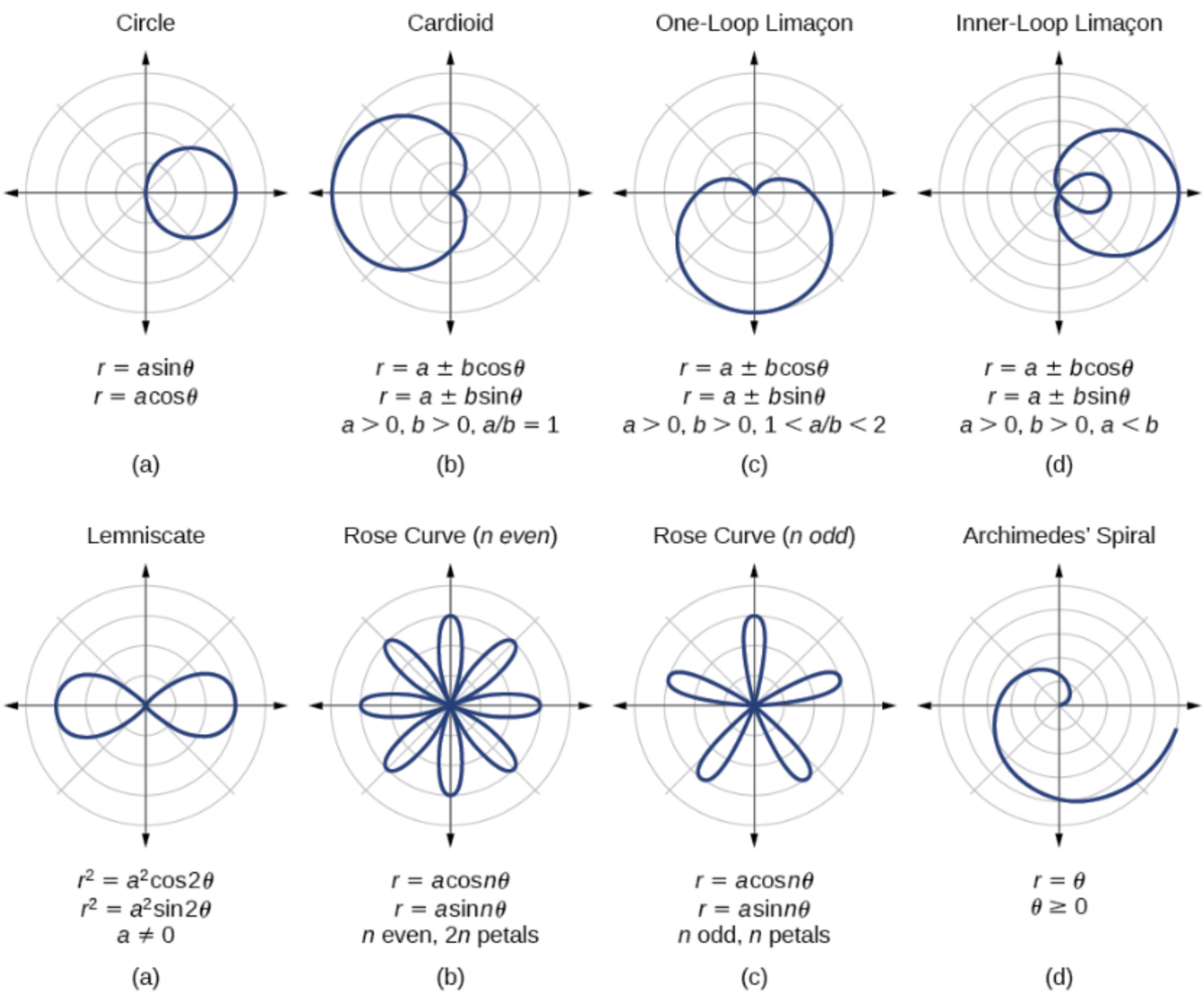
Amplitude $= |A|$

Phase Shift $= c/B$

Period $= 2\pi/B$

Vertical Shift $= D$

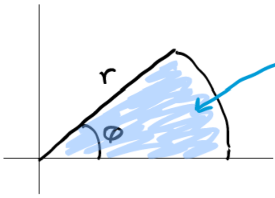
(Period for tangent π/B)



- Area of sector ($r = f(\theta)$, bounded by a curve and two rays: $\theta = \alpha, \theta = \beta$):

$$\frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2 d\theta$$

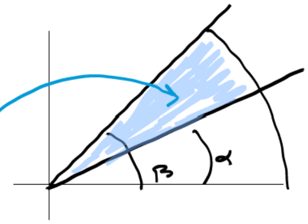
The area of a sector is given by



$$\boxed{A = \frac{1}{2} r^2 \theta}$$

$$A = \frac{1}{2} r^2 \beta - \frac{1}{2} r^2 \alpha$$

$$= \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta$$

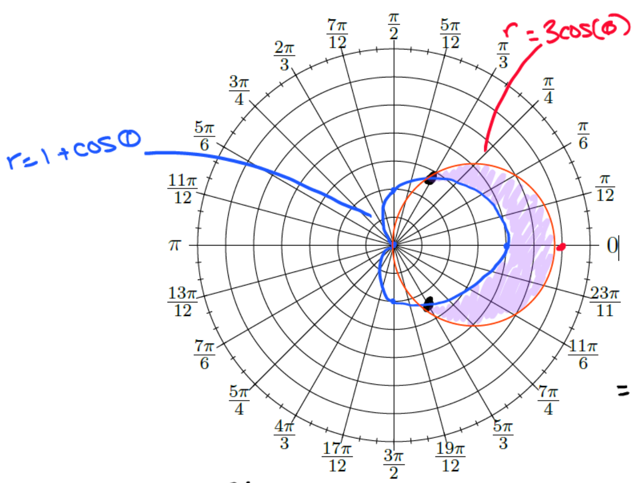


- Area between two polar curves $r = f(\theta), r = g(\theta), g(\theta) \geq f(\theta)$:

$$\frac{1}{2} \int_{\alpha}^{\beta} [g(\theta)]^2 - [f(\theta)]^2 d\theta$$

$\boxed{\text{example}}$ Find the area of the region that lies inside $r = 3\cos(\theta)$ and outside of $r = 1 + \cos(\theta)$.

Graph the curves, find the intersection points, set up the integral, and compute:



$$3\cos(\theta) = 1 + \cos(\theta)$$

$$\Rightarrow 2\cos(\theta) = 1 \Rightarrow$$

$$\theta = \pm \pi/3$$ (just consider one trip around the curve)

$$\frac{1}{2} \int_{-\pi/3}^{\pi/3} [3\cos(\theta)]^2 - [1 + \cos(\theta)]^2 d\theta$$

$$= 2 \cdot \frac{1}{2} \int_0^{\pi/3} 9\cos^2(\theta) - [1 + 2\cos(\theta) + \cos^2\theta] \, d\theta$$

$$= \int_0^{\pi/3} 8\cos^2(\theta) - 2\cos(\theta) - 1 \, d\theta = \int_0^{\pi/3} 8\left(\frac{1}{2} + \frac{\cos(2\theta)}{2}\right) - 2\cos\theta - 1 \quad d\theta$$

$$= 4\theta + \frac{4\sin(2\theta)}{2} - 2\sin(\theta) - \theta \Big|_0^{\pi/3} = \pi$$

- Arc length:

$$L = \int_a^b \sqrt{f'(\theta)^2 + f(\theta)^2} d\theta = \int_a^b \sqrt{(dr/d\theta)^2 + r^2} d\theta$$

11.5 Conic Sections

- Hyperbolas: $(\frac{x}{a})^2 - (\frac{y}{b})^2 = 1$, asymptotes: $y = \pm \frac{b}{a} x$ $\quad$ $(\frac{y}{a})^2 - (\frac{x}{b})^2 = 1$, asymptotes: $y = \pm \frac{a}{b} x$

THE DISCRIMINATE: $D = b^2 - 4ac$

We have the following cases:

- $D = 0$ Parabola
- $D > 0$ Hypertola
- $D < 0$ Ellipse OR Circle

Chapter 12: Vector Geometry

12.1-12.2

- Identify & sketch the 3D curve from its equation (often, complete the square)

$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$ (circle)

$(x-h)^2 + (y-k)^2 = r^2$ (vertical right cylinder through $(h, k, 0)$)

- Normalize a vector

12.3 - Dot Product, Projections, Work

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = ||v||\ ||w|| \cos\theta$

Geometrically, the dot product calculates the "net movement" in each axis given by the two vectors. This is why perpendicular vectors give a dot product of zero. Note that $\vec{v} \cdot \vec{v} = ||\vec{v}||^2$ and $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.

Angle b/t vectors: $\vec{v} \cdot \vec{w} < 0 \Rightarrow \theta$ obtuse, $\vec{v} \cdot \vec{w} > 0 \Rightarrow \theta$ acute

Scalar Projection: $comp_{\vec{a}}\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{||\vec{a}||} = ||b|| \cos\theta$ – the signed magnitude of the vector projection

Vector Projection: $proj_{\vec{a}}\vec{b} = \left(\dfrac{\vec{a} \cdot \vec{b}}{||\vec{a}||}\right)\left(\dfrac{\vec{a}}{||\vec{a}||}\right)$

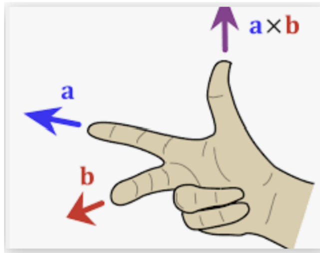
Work: $W = \vec{F} \cdot \vec{D} = ||\vec{D}||\ ||\vec{F}|| \cos\theta$

- Examples/types of Work Problems

12.4 Cross Product

Basic Properties of Cross Product

- Orthogonal: $(\vec{v} \times \vec{w}) \cdot \vec{v} = 0 = (\vec{v} \times \vec{w}) \cdot \vec{w}$
- Length: $||\vec{v} \times \vec{w}|| = ||\vec{v}||\ ||\vec{w}|| \sin\theta$



- Direction: *right-hand rule*
- Parallel: non-zero $\vec{v}, \vec{w}$ parallel iff $\vec{v} \times \vec{w} = 0$

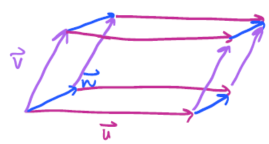
1. $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
2. $(c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w}) = \vec{v} \times (c\vec{w})$
3. $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$
4. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
5. $\vec{v} \cdot (\vec{u} \times \vec{w}) = (\vec{v} \times \vec{u}) \cdot \vec{w}$
6. $\vec{v} \times (\vec{u} \times \vec{w}) = (\vec{v} \cdot \vec{w})\vec{u} - (\vec{v} \cdot \vec{u})\vec{w}$

1. $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$ **5.** $\vec{v} \cdot (\vec{u} \times \vec{w}) = (\vec{v} \times \vec{u}) \cdot \vec{w}$ **6.** $\vec{v} \times (\vec{w} \times \vec{w}) = (\vec{v} \cdot \vec{w})\vec{u} - (\vec{v} \cdot \vec{u})\vec{w}$

Scalar triple product:

The quantity $\vec{v} \cdot (\vec{u} \times \vec{w})$ is called the scalar triple product of $\vec{v}, \vec{u}, \vec{w}$.

$$\vec{V} \cdot (\vec{u} \times \vec{w}) = \begin{vmatrix} V_1 & V_2 & V_3 \\ u_1 & u_2 & u_3 \\ W_1 & W_2 & W_3 \end{vmatrix}$$



The volume of a parallelepiped determine by the vectors $\vec{u}, \vec{v}, \vec{w}$ is the magnitude their scalar triple product.

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$$ absolute Value

Torque: $\vec{\tau} = \vec{r} \times \vec{F} \implies \|\vec{\tau}\| = \|\vec{r} \times \vec{F}\| = \|\vec{r}\| \|\vec{F}\| \sin(\theta)$

12.5 - Planes & Equations of Lines (from 12.2)

Find line through a point $P_0(x_0, y_0, z_0)$ in the direction of $\vec{v} = \langle a, b, c \rangle$

- $\vec{r}(t) = \vec{r_0} + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \quad -\infty < t < \infty$

Find line through two points $P(x_0, y_0, z_0), Q(x_1, y_1, z_1)$:

- $\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle = (1 - t)\langle x_0, y_0, z_0 \rangle + t\langle x_1, y_1, z_1 \rangle$

Determine if lines are parallel, intersecting, or skew (neither)

Planes – The plane through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$:

- vector form: $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$
- scalar form: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or $ ax + by+cz = d$

Find the equation of the plane given…

…a point on the plane and a normal vector
…a point and parallel plane (parallel normal vectors)
…a point and two planes perpendicular to it (orthogonal normal vectors)
…three points on the plane

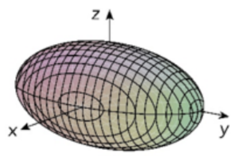
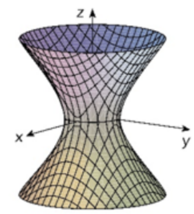
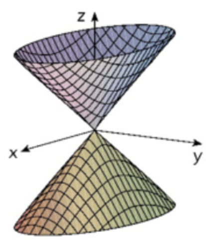
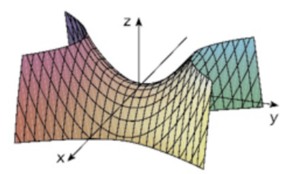
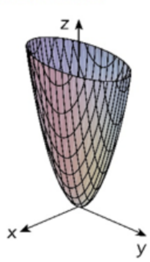
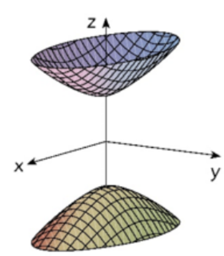
Find point where line intersects the plane

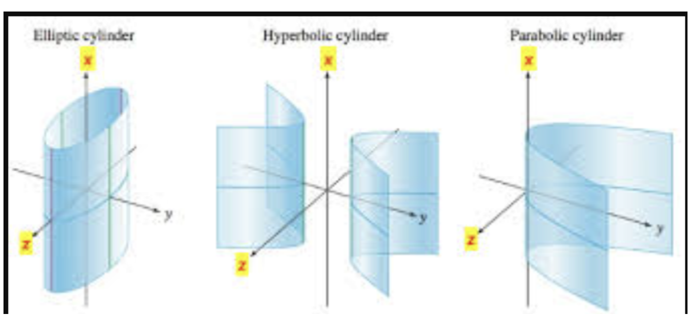
Find angle between planes: if not parallel or perpendicular, $\theta = \arccos\left(\dfrac{\vec{n_1} \cdot \vec{n_2}}{\|\vec{n_1}\|\|\vec{n_2}\|}\right)$

12.6 Quadratic Surfaces

General quadratic surface: $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$

Identify quadratic surfaces by formula and sketch using traces:

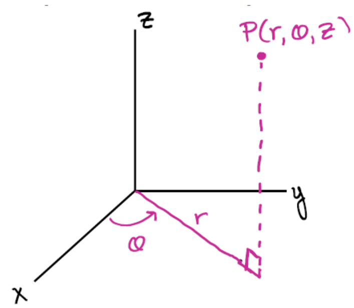
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ "A bunch of ellipses stacked together" Special case: If $a=b=c$, we have a sphere	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ In the xy plane, the traces are ellipses. In the xz or yz planes, the traces are hyperbolas. *Whichever variable is negative corresponds to the axis of symmetry
Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ In the xy plane, the traces are ellipses. In the xz or yz planes, the traces are hyperbolas, except when $x=0$ or $y=0$, then the traces are pairs of lines	Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ In the xy plane, the traces are hyperbolas. In the xz or yz plane, the traces are parabolas.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ In the xy plane, the traces are ellipses. In the xz or yz planes, the traces are parabolas.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ In the xy plane, the traces are ellipses if $z>c$ or $z<-c$ In the xz or yz planes, the traces are hyperbolas.



$(\frac{x}{a})^2+(\frac{y}{b})^2=1$ $\quad$ $(\frac{x}{a})^2-(\frac{y}{b})^2=1$ $\quad$ $y=ax^2$

12.7 - Cylindrical and Spherical Coordinates

Cylindrical to Cartesian	Cartesian to Cylindrical
$x = r\cos(\theta)$	$r = \sqrt{x^2+y^2}$
$y = r\sin(\theta)$	$\tan(\theta) = y/x$
$z = z$	$z = z$



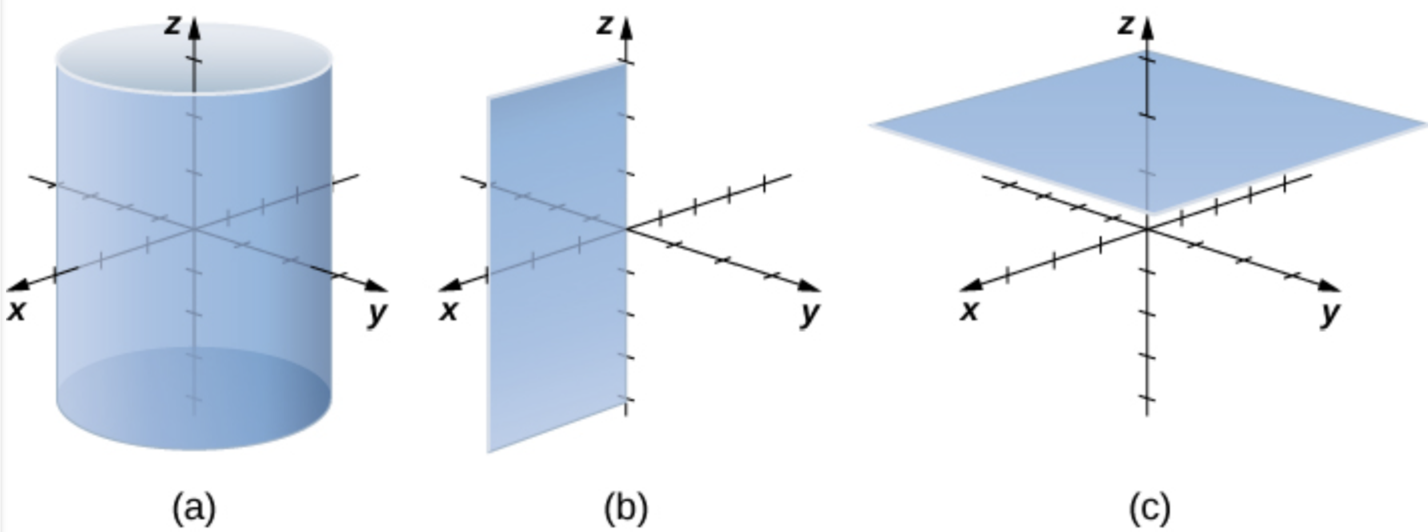
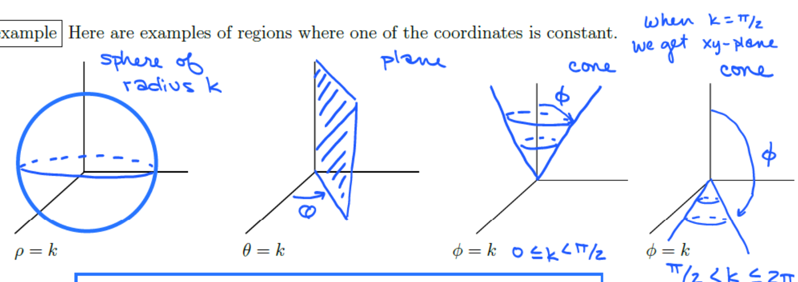


Figure 11.7.4: In cylindrical coordinates, (a) surfaces of the form $r = c$ are vertical cylinders of radius r, (b) surfaces of the form $\theta = c$ are half-planes at angle θ from the x-axis, and (c) surfaces of the form $z = c$ are planes parallel to the xy-plane.

$\boxed{\text{example}}$ Here are examples of regions where one of the coordinates is constant.



Spherical to Cartesian	Cartesian to Spherical
$x = \rho\sin(\phi)\cos(\theta)$	$\rho = \sqrt{x^2+y^2+z^2}$
$y = \rho\sin(\phi)\sin(\theta)$	$\tan(\theta) = y/x$
$z = \rho\cos(\phi)$	$\cos(\phi) = z/\rho$