

Exam 3 Notes

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Chapter 15 - Multiple Integration

15.1 - Two-Variable Integration Over Rectangles

$$\left(\int_c^d \left(\int_a^b f(x, y) dx \right) dy \right)$$

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) dA$$

$$\iint_R f(x, y) dA \pm \iint_R g(x, y) dA = \iint_R f(x, y) \pm g(x, y) dA$$

$$\iint_R cf(x, y) dA = c \iint_R f(x, y) dA$$

$$\forall (x, y) \in \mathbb{R}^2, \quad f(x, y) \geq g(x, y) \implies \iint_R f(x, y) dA \geq \iint_R g(x, y) dA$$

$$V = \iint_R f(x, y) - g(x, y) dA \text{ where } f(x, y) \text{ is top surface, } g(x, y) \text{ bottom}$$

Fubini's Theorem: If f is continuous on the rectangle $R = \{(x, y) \mid [a, b] \times [c, d]\}$, then:

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

$$\text{Partial wrt } y: A(x) = \int_c^d f(x, y) dy \quad (A(x) \text{ possibly in terms of } x)$$

$$\text{Partial wrt } x: \int_a^b A(x) dx = \int_a^b \int_c^d f(x, y) dy dx$$

15.2 - Double Integrals Over More General Regions

Vertically Simple (top, bottom terms of x)

$$D = \{(x, y) \mid [a, b] \times [g_1(x), g_2(x)]\}$$

$$\text{If } f \text{ is continuous on } D, \quad \iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

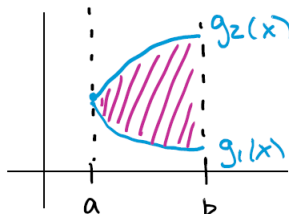
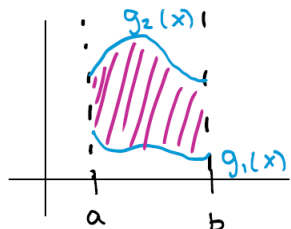
TYPE I REGION: VERTICALLY SIMPLE

y is bounded by functions of x
↓

Type I: (Top and Bottom in terms of x)

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

examples



Horizontally Simple (left, right terms of y)

$$D = \{(x, y) \mid [h_1(y), h_2(y)] \times [c, d]\}$$

If f is continuous D , $\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$

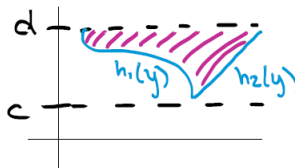
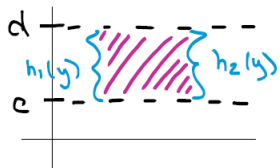
TYPE II REGION: HORIZONTALLY SIMPLE

x is bounded by functions of y
↓

Type II: (Left and Right in terms of y)

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

examples



with these regions, we are bounded on the left & right by functions of y !

Properties of the Integral

Linearity, Homogeneity, Monotonicity

Additivity: $D = D_1 \cup D_2 \cdots \cup D_n$ (interior-disjoint)

$$\iint_D f(x, y) dA = \iint_{D_1} f(x, y) dA + \cdots + \iint_{D_n} f(x, y) dA \text{ (see pg. 81)}$$

Area of a Region: $\text{Area}(D) = \iint_D 1 dA$

Estimation: If $m \leq f(x, y) \leq M$ for all $(x, y) \in D$,

$$mA(D) \leq \iint_D f(x, y) dA \leq MA(D)$$

15.3 - Triple Integrals (“Hyper-Volume”)

$$\iiint_W f(x, y, z) \, dV \quad W \subset \mathbb{R}^3$$

W = Box (*Fubini's Theorem*): If f continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$

$$\iiint_B f(x, y, z) \, dV = \int_r^s \int_a^b \int_c^d f(x, y, z) \, dy \, dx \, dz$$

Once again, can be evaluated in any order. For

W = z-simple: $W = \{(x, y, z) \mid u_1(x, y) \leq z \leq u_2(x, y), (x, y) \in D\}$

W = x-simple (front/back): $W = \{(x, y, z) \mid u_1(y, z) \leq x \leq u_2(y, z), (y, z) \in D\}$

W = y-simple (left/right): $W = \{(x, y, z) \mid u_1(x, z) \leq y \leq u_2(x, z), (x, z) \in D\}$

15.4 - Integration in Polar, Cylindrical, Spherical Coordinates

POLAR RECTANGLES

$$R = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

examples



FIGURE 2 Polar rectangle.



FIGURE 7 Quarter annulus $0 \leq \theta \leq \frac{\pi}{2}$, $2 \leq r \leq 4$.

Rogawski, J.D. and Adams, C. Calculus: Third Edition. Macmillan Learning, 2015.

GENERAL POLAR REGIONS

$$R = \{(r, \theta) \mid g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta\}$$

example



FIGURE 6 General polar region.

If f continuous on $D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, g_1(\theta) \leq r \leq g_2(\theta)\}$

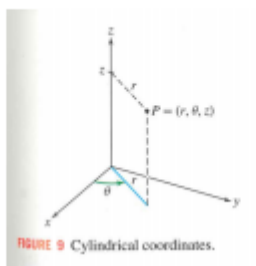
$$\iint_D f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos(\theta), r \sin(\theta)) r \, dr \, d\theta$$

II. TRIPLE INTEGRATION IN CYLINDRICAL COORDINATES

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

$$z = z$$



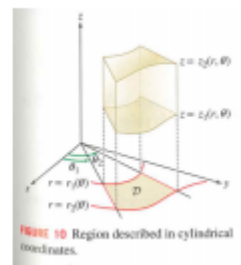
- symmetry about an axis
- circular regions in the plane
- $\underbrace{x^2 + y^2}_{r^2}$ makes an appearance

CYLINDRICAL REGIONS

$$W = \{(x, y, z) | u_1(x, y) \leq z \leq u_2(x, y), (x, y) \in D\}$$

where

$$D = \{(r, \theta) | \alpha \leq \theta \leq \beta, g_1(\theta) \leq r \leq g_2(\theta)\}$$



If f continuous on D ,

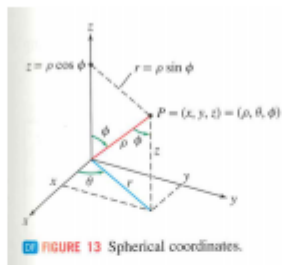
$$\iiint_W f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(r \cos(\theta), r \sin(\theta))}^{u_2(r \cos(\theta), r \sin(\theta))} f(r \cos(\theta), r \sin(\theta), z) r dz dr d\theta$$

III. TRIPLE INTEGRATION IN SPHERICAL COORDINATES

$$x = \rho \sin(\phi) \cos(\theta)$$

$$y = \rho \sin(\phi) \sin(\theta)$$

$$z = \rho \cos(\phi)$$

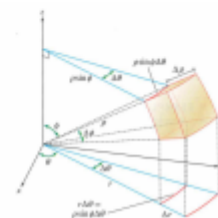


- cones and spheres
- combinations of the above
- symmetry about a point

SPHERICAL WEDGES AND MORE GENERAL REGIONS

$$W = \{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

$$W = \{(\rho, \theta, \phi) | g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi), \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$



If f continuous on D ,

$$\iiint_W f(x, y, z) dV = \int_c^d \int_{\alpha}^{\beta} \int_{g_1(\theta, \phi)}^{g_2(\theta, \phi)} f(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi)) \rho^2 \sin(\phi) d\rho d\theta d\phi$$

15.5 - Applications of Multiple Integrals

Center of mass: Given a lamina occupying D in \mathbb{R}^2 with variable density $\rho(x, y)$,

$$(X_{CM}, Y_{CM}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right) \text{ where } m = \iint_D \rho(x, y) dA \quad M_x = \iint_D y \rho(x, y) dA \quad M_y = \iint_D x \rho(x, y) dA$$

3-D: $(X_{CM}, Y_{CM}, Z_{CM}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$ where $m = \iiint \rho(x, y, z) dV$,

$$M_{xy} = \iiint z\rho(x, y, z) dV \quad M_{xz} = \iiint y\rho(x, y, z) dV \quad M_{yz} = \iiint x\rho(x, y, z) dV$$

Probability and Wait Times

$P(a \leq x \leq b) = \int_a^b f(x)dx$ where $\int_{-\infty}^{\infty} f(x)dx = 1$ and $f(x) \geq 0$ for all $x \in X$

Joint Probability: $P((X, Y) \text{ in } D) = \iint_D f(x, y) dA$ where $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dA = 1$ and $f(x, y) \geq 0$

Independent variables: if joint density $f(x, y) = g(x)h(y)$, where $g(x) df(X)$, $h(y) df(Y)$

Wait-times example:

We can model waiting times by using exponential probability density functions:

$$f(t) = \begin{cases} 0 & t < 0 \\ \mu^{-1}e^{-t/\mu} & t \geq 0 \end{cases}$$

where μ is the mean or average waiting time. Let's check out an example:

example The manager at local coffee shop determines that the average patron's wait in line to order coffee and a snack is 5 minutes and the average time they wait to receive their refreshments and caffeine is 10 minutes. Assuming that the waiting times are independent, find the probability that a patron waits a total of 20 minutes or less before being able to start consuming their items.

Start with the density functions:

X = waiting time in line

$$g(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{5}e^{-x/5} & x \geq 0 \end{cases}$$

Y waiting for food & coffee

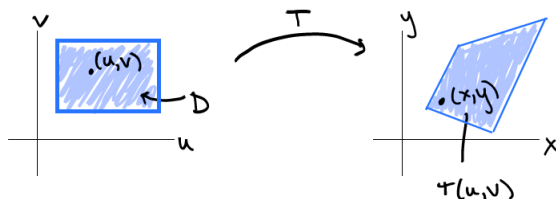
$$h(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{10}e^{-y/10} & y \geq 0 \end{cases}$$

$$f(x, y) = g(x)h(y) = \begin{cases} \frac{1}{50}e^{-y/10}e^{-x/5} & x \geq 0, y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Then compute $P(X + Y < 20) = \iint f(x, y) dA = \int_0^{20} \int_0^{20-x} \frac{1}{50}e^{-y/10}e^{-x/5} dy dx$

15.6 - Change of Variables

$T(u, v) = (x, y)$ where $x = x(u, v)$ and $y = y(u, v)$



It is important to know that any transformation T will map the segment joining any two points P and Q to the segment joining $T(P)$ and $T(Q)$.

$$\text{Jac}(T) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

Other Notation:

$$\text{Jac}(T) = \frac{\partial(x,y)}{\partial(u,v)} \quad \text{slightly more common!}$$

The **Jacobian Determinant** (Jacobian) of $T(u,v) = (x(u,v), y(u,v))$ helps calculate the area under a transformation (given numerous assumptions...):

$$\iint_R f(x,y) dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

It can also *estimate* the area: $\text{Area}(T(D)) \approx |\text{Jac}(T)| \text{Area}(D)$, $dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$

There is a generalization of the change of variables formula for triple integrals. Suppose $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$.

Then $\text{Jac}(T) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$ determinant of a 3x3 matrix
(see 12.4 on the cross product)

Then we have the Change of Variable formula (triple integrals):

$$\iiint_R f(x,y,z) dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du dv dw$$