

# Calc3 – Exam One Notes

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## Chapter 11 – Parametric Equations, Polar Coord., Conic Sections

### 11.1

- Eliminate parameter to find Cartesian equation of curve
- Plot parametric curves:

**example** IMPORTANT EXAMPLE!

Identify what curve is represented by the following parametric equations.

$$x = \cos(t) \quad y = \sin(t) \quad 0 \leq t \leq 2\pi$$

Look at the  $x$  equation in terms of  $t$

Look at the  $y$  equation in terms of  $t$

Put them together:

**example** Describe the motion of a particle with position  $(x, y) = (3 + 2\cos(t), 1 + 2\sin(t))$  as  $t$  varies  $\pi/2 \leq t \leq 3\pi/2$ .

$y = -x/4 - 3/4$  ← whole line

↑ line segment

start →  $0 \leq t \leq 2\pi$

$C = (x, y) = (\cos(t), \sin(t))$  is a circle w/ center  $(0,0)$  and radius 1

Notice:  $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$   
 $\Rightarrow x^2 + y^2 = 1$   
 unit circle

this is a little easier

i will solve for  $\cos(t)$  and  $\sin(t)$

you should remember this example

- Use common parameterizations:

Line through  $P = (a, b) \rightarrow C = (a + rt, b + st)$  where  $m = \frac{s}{r}, r \neq 0$

Line through  $P, Q \rightarrow C = (a + t(c - a), b + t(d - b))$

Circle:  $C = (h + r \cos(t), k + r \sin(t)), \quad 0 \leq t < 2\pi \quad \text{or} \quad 0 \leq t \leq 2\pi$

Ellipse:  $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1 \rightarrow C = (a \cos(t), b \sin(t)), \quad -\pi \leq t \leq \pi \quad \text{or} \quad 0 \leq t < 2\pi$

- Match parametric equation to graphs
- Find equation of a tangent line:

$$y - y_0 = \frac{dy}{dx} \Big|_{(x_0, y_0)} (x - x_0)$$

For parametric functions,  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$  and  $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy/dt}{dx/dt})}{dx/dt}$

Horizontal tangent:  $\frac{dy}{dt} = 0, \frac{dx}{dt} \neq 0$ , Vertical tangent:  $\frac{dx}{dt} = 0, \frac{dy}{dt} \neq 0$

- Find area under a curve ( $x = f(t), y = g(t)$ ):

$$\left| \begin{array}{l} A = \int_a^b h(x) dx = \int_\alpha^\beta g(t) f'(t) dt \text{ (if } x = a \Rightarrow t = \alpha, \text{ else } \int_\beta^\alpha) \\ \text{else } \int_\beta^\alpha \end{array} \right| A = \int_c^d h(y) dy = \int_\alpha^\beta f(t) g'(t) dt \text{ (if " "}$$

## 11.2 Arc Length & Surface Area

- Arc Length  $L = \int_a^b \sqrt{1 + (dy/dx)^2} dx$

**Theorem** If a curve  $C = (x, y) = (f(t), g(t))$  for  $\alpha \leq t \leq \beta$  and  $f'$  and  $g'$  are continuous on  $[\alpha, \beta]$  and  $C$  is transversed exactly once as  $t$  increases from  $\alpha$  to  $\beta$ , then the length of  $C$  is

*is goes around exactly once*  $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$  *This integrand is the speed formula!*

- Surface Area ( $x = f(t), y = g(t)$ )

$$(x\text{-axis}) S = \int_\alpha^\beta 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$$

$$(y\text{-axis}) S = \int_\alpha^\beta 2\pi x \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$$

## 11.3-11.4 Polar Coordinates

- Convert from polar to Cartesian, Cartesian to polar
- Sketch and identify curves in polar coordinates. KNOW HOW TO FIND/CHOOSE LIMITS OF INTEGRATION.

## Brief Trig Review

$$y = A \sin(Bx - c) + D \quad \text{OR} \quad y = A \cos(Bx - c) + D$$

Polar to Rectangular

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

Rectangular to Polar

$$r^2 = x^2 + y^2$$

$$r = \pm \sqrt{x^2 + y^2}$$

$$\tan(\theta) = y/x$$

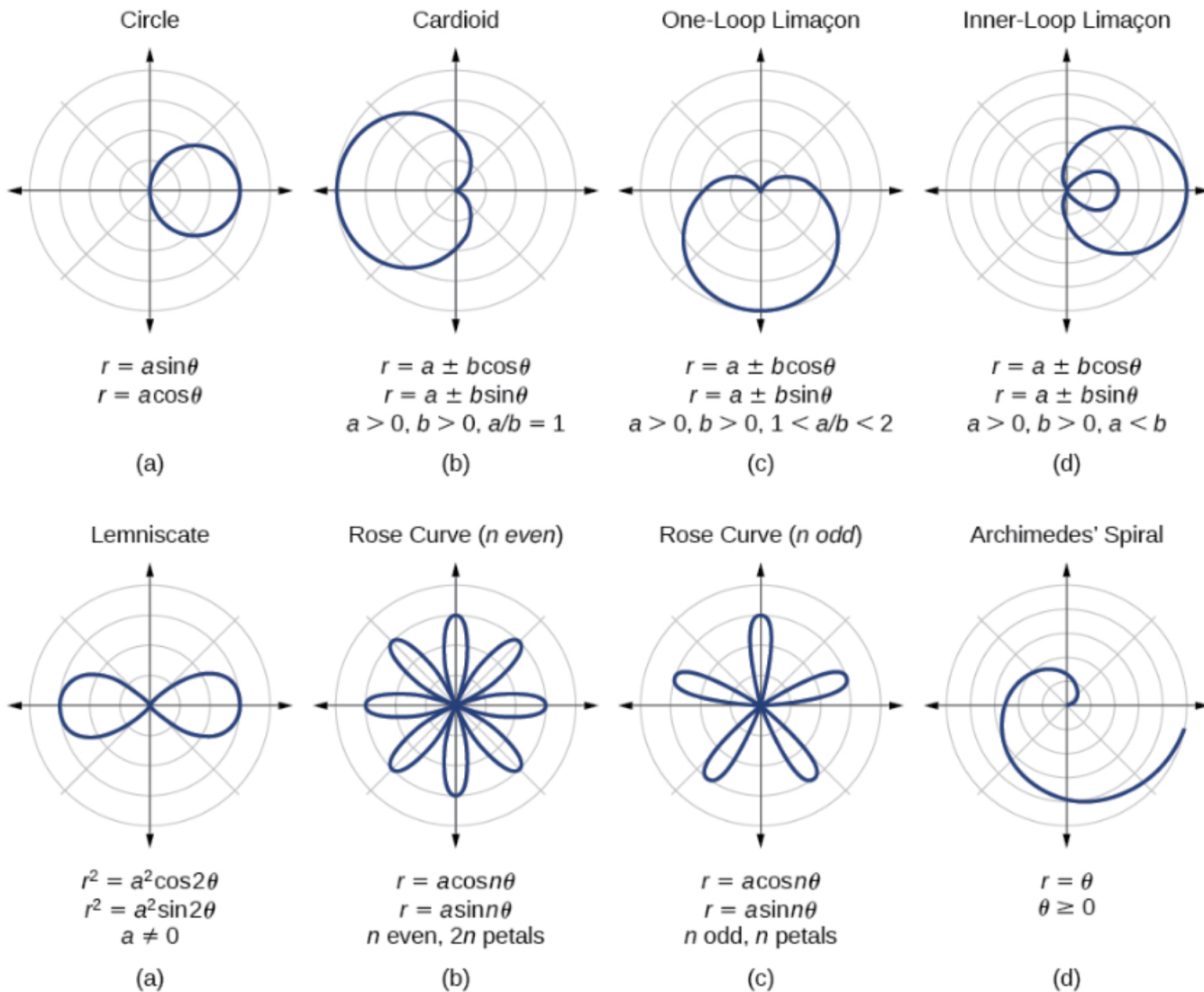
Amplitude =  $|A|$

Phase Shift =  $c/B$

Period =  $2\pi/B$

Vertical Shift =  $D$

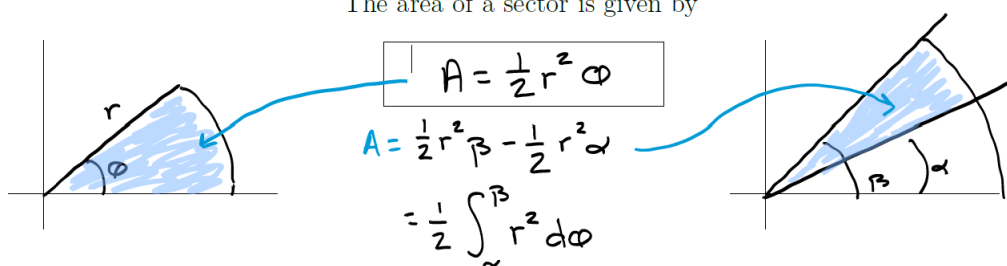
(period for tangent  $\pi/B$ )



- Area of sector ( $r = f(\theta)$ ), bounded by a curve and two rays:  $\theta = \alpha, \theta = \beta$ ):

$$\frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 d\theta$$

The area of a sector is given by

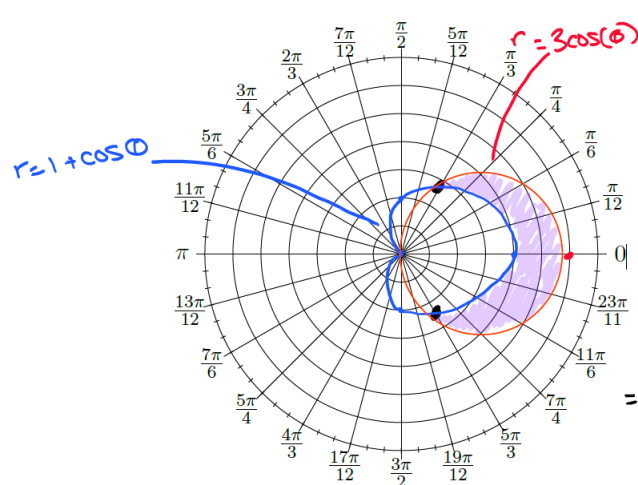


- Area between two polar curves  $r = f(\theta)$ ,  $r = g(\theta)$ ,  $g(\theta) \geq f(\theta)$ :

$$\frac{1}{2} \int_{\alpha}^{\beta} [g(\theta)]^2 - [f(\theta)]^2 d\theta$$

**example** Find the area of the region that lies inside  $r = 3 \cos(\theta)$  and outside of  $r = 1 + \cos(\theta)$ .

Graph the curves, find the intersection points, set up the integral, and compute:



$$\begin{aligned}
 3 \cos(\phi) &= 1 + \cos(\phi) \\
 \Rightarrow 2 \cos(\phi) &= 1 \quad \Rightarrow \\
 \phi &= \pm \pi/3 \quad (\text{just consider one trip around the curve})
 \end{aligned}$$

$$\begin{aligned}
 &\frac{1}{2} \int_{-\pi/3}^{\pi/3} [3 \cos(\phi)]^2 - [1 + \cos(\phi)]^2 d\phi \\
 &= 2 \cdot \frac{1}{2} \int_0^{\pi/3} 9 \cos^2(\phi) - [1 + 2 \cos(\phi) + \cos^2 \phi] d\phi
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^{\pi/3} 8 \cos^2(\phi) - 2 \cos(\phi) - 1 d\phi = \int_0^{\pi/3} 8 \left( \frac{1}{2} + \frac{\cos(2\phi)}{2} \right) - 2 \cos \phi - 1 d\phi \\
 &= 4\phi + \frac{4 \sin(2\phi)}{2} - 2 \sin(\phi) - \phi \Big|_0^{\pi/3} = \pi
 \end{aligned}$$

- Arc length:

$$L = \int_a^b \sqrt{f'(\theta)^2 + f(\theta)^2} d\theta = \int_a^b \sqrt{(dr/d\theta)^2 + r^2} d\theta$$

## 11.5 Conic Sections

- Hyperbolas:  $(\frac{x}{a})^2 - (\frac{y}{b})^2 = 1$ , asymptotes:  $y = \pm \frac{b}{a} x$        $(\frac{y}{a})^2 - (\frac{x}{b})^2 = 1$ , asymptotes:  $y = \pm \frac{a}{b} x$

THE DISCRIMINATE:  $D = b^2 - 4ac$

We have the following cases:

- $D = 0$  Parabola
- $D > 0$  Hyperbola
- $D < 0$  Ellipse or Circle

## Chapter 12: Vector Geometry

### 12.1-12.2

- Identify & sketch the 3D curve from its equation (often, complete the square)

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 \text{ (circle)}$$

$$(x - h)^2 + (y - k)^2 = r^2 \text{ (vertical right cylinder through } (h, k, 0))$$

- Normalize a vector

### 12.3 - Dot Product, Projections, Work

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = ||v|| ||w|| \cos \theta$$

Geometrically, the dot product calculates the “net movement” in each axis given by the two vectors. This is why perpendicular vectors give a dot product of zero. Note that  $\vec{v} \cdot \vec{v} = ||\vec{v}||^2$  and  $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ .

Angle b/t vectors:  $\vec{v} \cdot \vec{w} < 0 \Rightarrow \theta$  obtuse,  $\vec{v} \cdot \vec{w} > 0 \Rightarrow \theta$  acute

**Scalar Projection:**  $\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||} = ||b|| \cos \theta$  – the signed magnitude of the vector projection

**Vector Projection:**  $\text{proj}_{\vec{a}} \vec{b} = \left( \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||} \right) \left( \frac{\vec{a}}{||\vec{a}||} \right)$

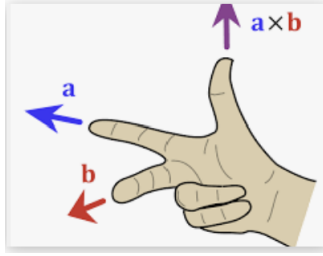
**Work:**  $W = \vec{F} \cdot \vec{D} = ||\vec{D}|| ||\vec{F}|| \cos \theta$

- Examples/types of Work Problems

### 12.4 Cross Product

Basic Properties of Cross Product

- Orthogonal:  $(\vec{v} \times \vec{w}) \cdot \vec{v} = 0 = (\vec{v} \times \vec{w}) \cdot \vec{w}$
- Length:  $||\vec{v} \times \vec{w}|| = ||\vec{v}|| ||\vec{w}|| \sin \theta$



- Direction: *right-hand rule*
- Parallel: non-zero  $\vec{v}, \vec{w}$  parallel iff  $\vec{v} \times \vec{w} = 0$

$$\begin{array}{ll}
 1. \vec{v} \times \vec{w} = -\vec{w} \times \vec{v} & 2. (c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w}) = \vec{v} \times (c\vec{w}) \\
 3. \vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w} & 4. (\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w} \\
 5. \vec{v} \cdot (\vec{u} \times \vec{w}) = (\vec{v} \times \vec{u}) \cdot \vec{w} & 6. \vec{v} \times (\vec{u} \times \vec{w}) = (\vec{v} \cdot \vec{w})\vec{u} - (\vec{v} \cdot \vec{u})\vec{w}
 \end{array}$$

$$1. \vec{v} \times \vec{w} = -\vec{w} \times \vec{v} \quad 5. \vec{v} \cdot (\vec{u} \times \vec{w}) = (\vec{v} \times \vec{u}) \cdot \vec{w} \quad 6. \vec{v} \times (\vec{u} \times \vec{w}) = (\vec{v} \cdot \vec{w})\vec{u} - (\vec{v} \cdot \vec{u})\vec{w}$$

### Scalar triple product:

The quantity  $\vec{v} \cdot (\vec{u} \times \vec{w})$  is called the scalar triple product of  $\vec{v}, \vec{u}, \vec{w}$ .

$$\vec{v} \cdot (\vec{u} \times \vec{w}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

The volume of a parallelepiped determined by the vectors  $\vec{u}, \vec{v}, \vec{w}$  is the magnitude of their scalar triple product.

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| \quad \text{absolute value}$$

**Torque:**  $\vec{\tau} = \vec{r} \times \vec{F} \implies \|\vec{\tau}\| = \|\vec{r} \times \vec{F}\| = \|\vec{r}\| \|\vec{F}\| \sin(\theta)$

### 12.5 - Planes & Equations of Lines (from 12.2)

Find line through a point  $P_0(x_0, y_0, z_0)$  in the direction of  $\vec{v} = \langle a, b, c \rangle$

$$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \quad -\infty < t < \infty$$

Find line through two points  $P(x_0, y_0, z_0), Q(x_1, y_1, z_1)$ :

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle = (1-t)\langle x_0, y_0, z_0 \rangle + t\langle x_1, y_1, z_1 \rangle$$

Determine if lines are parallel, intersecting, or skew (neither)

**Planes** – The plane through  $P_0(x_0, y_0, z_0)$  with normal vector  $\vec{n} = \langle a, b, c \rangle$ :

- vector form:  $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$
- scalar form:  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$  or  $ax + by + cz = d$

Find the equation of the plane given...

- ... a point on the plane and a normal vector
- ... a point and parallel plane (parallel normal vectors)
- ... a point and two planes perpendicular to it (orthogonal normal vectors)
- ... three points on the plane

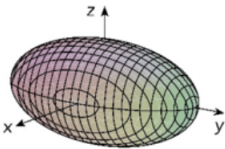
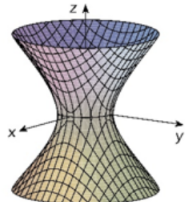
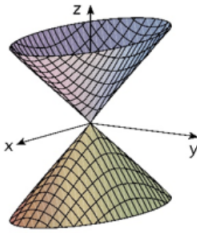
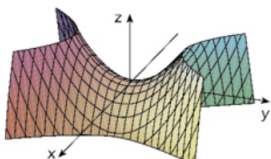
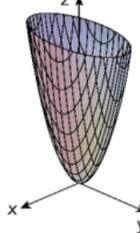
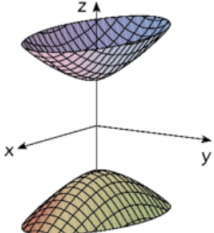
Find point where line intersects the plane

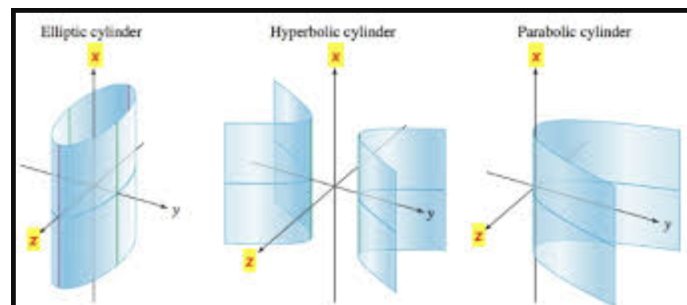
Find angle between planes: if not parallel or perpendicular,  $\theta = \arccos\left(\frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \|\vec{n}_2\|}\right)$

## 12.6 Quadratic Surfaces

General quadratic surface:  $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$

Identify quadratic surfaces by formula and sketch using traces:

<p><b>Ellipsoid</b></p> 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p>"A bunch of ellipses stacked together"</p> <p>Special case: If <math>a = b = c</math>, we have a sphere</p>	<p><b>Hyperboloid of One Sheet</b></p> 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <p>In the xy plane, the traces are ellipses.</p> <p>In the xz or yz planes, the traces are hyperbolas.</p> <p>*Whichever variable is negative corresponds to the axis of symmetry</p>
<p><b>Cone</b></p> 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p>In the xy plane, the traces are ellipses.</p> <p>In the xz or yz planes, the traces are hyperbolas, except when <math>x = 0</math> or <math>y = 0</math>, then the traces are pairs of lines</p>	<p><b>Hyperbolic Paraboloid</b></p> 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ <p>In the xy plane, the traces are hyperbolas.</p> <p>In the xz or yz plane, the traces are parabolas.</p>
<p><b>Elliptic Paraboloid</b></p> 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p>In the xy plane, the traces are ellipses.</p> <p>In the xz or yz planes, the traces are parabolas.</p>	<p><b>Hyperboloid of Two Sheets</b></p> 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p>In the xy plane, the traces are ellipses if <math>z &gt; c</math> or <math>z &lt; -c</math></p> <p>In the xz or yz planes, the traces are hyperbolas.</p>



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \quad y = ax^2$$

## 12.7 - Cylindrical and Spherical Coordinates

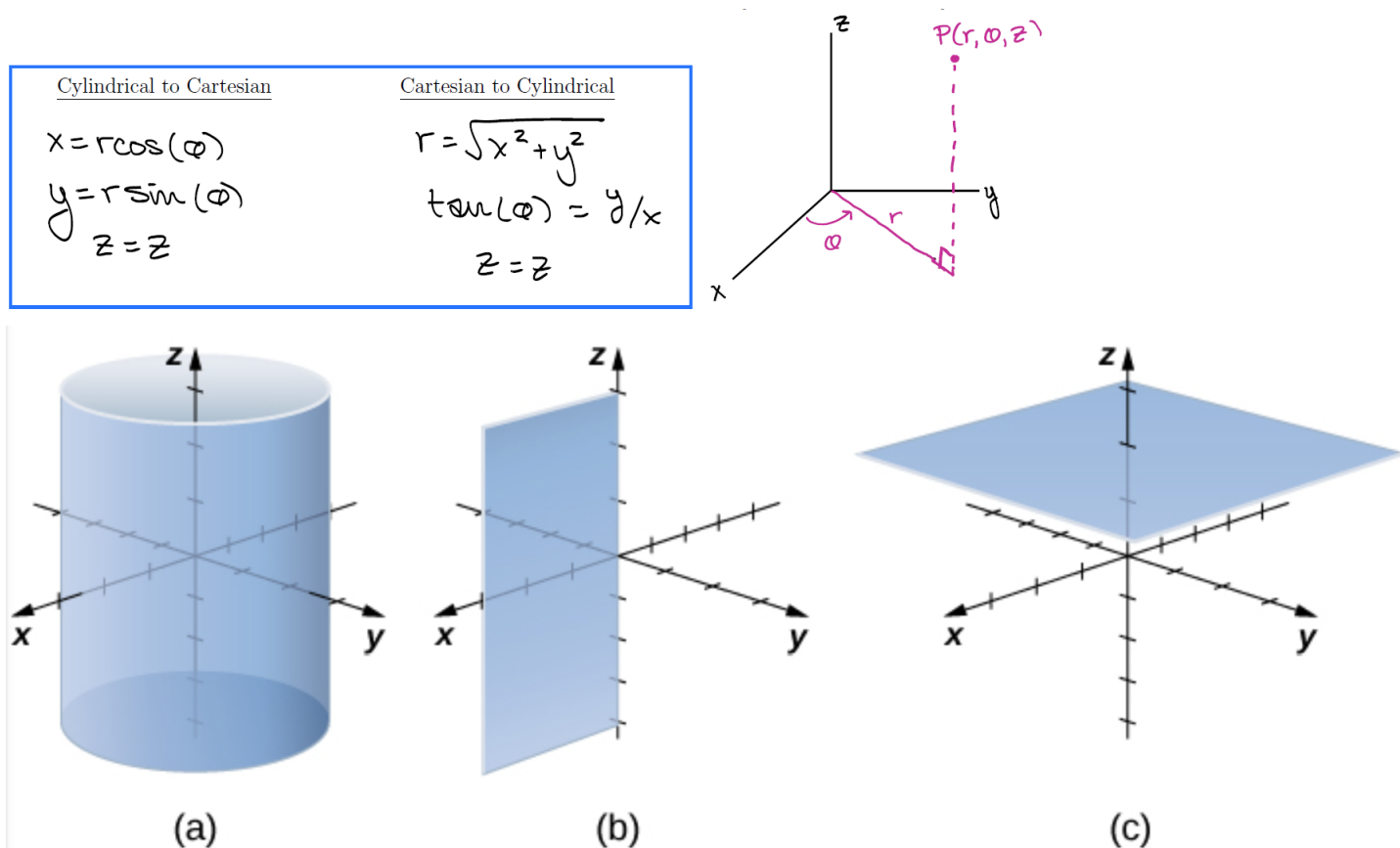


Figure 11.7.4: In cylindrical coordinates, (a) surfaces of the form  $r = c$  are vertical cylinders of radius  $r$ , (b) surfaces of the form  $\theta = c$  are **half-planes** at angle  $\theta$  from the  $x$ -axis, and (c) surfaces of the form  $z = c$  are planes parallel to the  $xy$ -plane.

