Exam 3 Notes

Alex Socarras

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Chapter 15 - Multiple Integration

15.1 - Two-Variable Integration Over Rectangles

$\left(\int_c^d \left(\int_a^b f(x,y)dx\right)dy\right)$

$f_{avg} = \dfrac{1}{A(R)} \iint_R f(x,y)\ dA$

$\iint_R f(x,y)\ dA \pm \iint_R g(x,y)\ dA = \iint_R f(x,y) \pm g(x,y)\ dA$

$\iint_R cf(x,y)\ dA = c\iint_R f(x,y)\ dA$

$\forall(x,y) \in \mathbb{R}^2,\quad f(x,y) \geq g(x,y) \implies \iint_R f(x,y)\ dA \geq \iint_R g(x,y)\ dA$

$V = \iint_R f(x,y) - g(x,y)\ dA$ where $f(x,y)$ is top surface, $g(x,y)$ bottom

Fubini's Theorem: If f is continuous on the rectangle $R = \{(x,y) \mid [a,b] \times [c,d]\}$, then:

$\iint_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$

Partial wrt y: $A(x) = \int_c^d f(x,y)dy \qquad (A(x)$ possibly in terms of $x)$

Partial wrt x: $\int_a^b A(x)dx\ = \int_a^b \int_c^d f(x,y)dydx$

15.2 - Double Integrals Over More General Regions

Vertically Simple (top, bottom terms of x)

$D = \{(x,y) \mid [a,b] \times [g_1(x), g_2(x)]\}$

If f is continuous on D, $\iint_R f(x,y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$

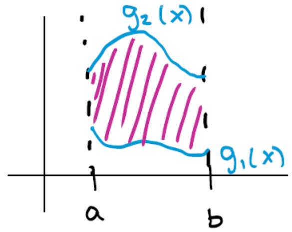
TYPE I REGION: VERTICALLY SIMPLE

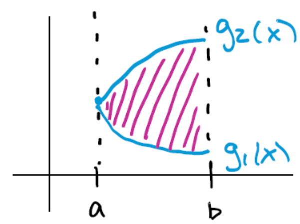
y is bounded by functions of x

Type I: (Top and Bottom in terms of x)

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

examples





Horizontally Simple (left, right terms of y)

$D = \{(x, y) \mid [h_1(y), h_2(y)] \times [c, d]\}$

If f is continuous D, $\iint_D f(x,y)\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$

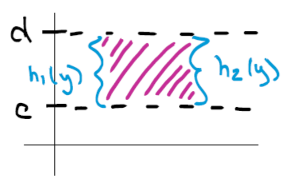
TYPE II REGION: HORIZONTALLY SIMPLE

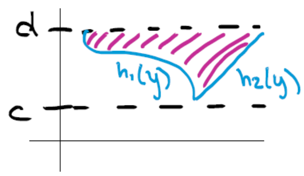
x is bounded by functions of y

Type II: (Left and Right in terms of y)

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

examples





With these regions, we are bounded on the left ; right by functions of y!

Properties of the Integral

Linearity, Homogeneity, Monotonicity

Additivity: $D = D_1 \cup D_2 \cdots \cup D_n$ (interior-disjoint)

$\iint_D f(x,y)\ dA = \iint_{D_1} f(x,y)\ dA + \cdots + \iint_{D_n} f(x,y)\ dA$ (*see pg. 81*)

Area of a Region: $\text{Area}(D) = \iint_D 1\ dA$

Estimation: If $m \le f(x,y) \le M$ for all $(x,y) \in D$,

$mA(D) \le \iint_D f(x,y)\ dA \le MA(D)$

15.3 - Triple Integrals ("Hyper-Volume")

$$\iiint_W f(x,y,z)\ dV \qquad W \subset \mathbb{R}^3$$

W = Box (*Fubini's Theorem*): If f continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$

$$\iiint_B f(x,y,z)\ dV = \int_r^s \int_a^b \int_c^d f(x,y,z)dydxdz$$

Once again, can be evaluated in any order. For

W = z-simple: $W = \{(x,y,z) \mid u_1(x,y) \le z \le u_2(x,y),\ (x,y) \in D\}$

W = x-simple (front/back): $W = \{(x,y,z) \mid u_1(y,z) \le x \le u_2(y,z),\ (y,z) \in D\}$

W = y-simple (left/right): $W = \{(x,y,z) \mid u_1(x,z) \le y \le u_2(x,z),\ (x,z) \in D\}$

15.4 - Integration in Polar, Cylindrical, Spherical Coordinates

Polar Rectangles

$$R = \{(r,\theta) | a \le r \le b, \alpha \le \theta \le \beta\}$$

examples

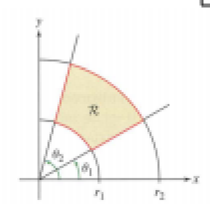


FIGURE 2 Polar rectangle.

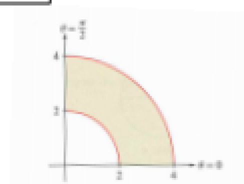


FIGURE 7 Quarter annulus $0 \le \theta \le \frac{\pi}{2}$, $2 \le r \le 4$.

General Polar Regions

$$R = \{(r,\theta)|, g_1(\theta) \le r \le g_2(\theta), \alpha \le \theta \le \beta\}$$

example

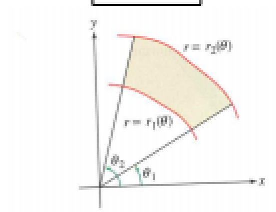


FIGURE 6 General polar region.

Rogawski, J.D. and Adams, C. Calculus: Third Edition. Macmillan Learning, 2015.

If f continuous on $D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ g_1(\theta) \le r \le g_2(\theta)\}$

$$\iint_D f(x,y)\ dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos(\theta), r\sin(\theta))r\ drd\theta$$

II. Triple Integration in Cylindrical Coordinates

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$z = z$

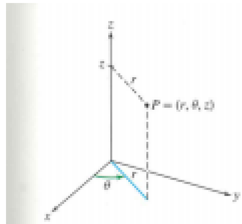


FIGURE 9 Cylindrical coordinates.

- symmetry about an axis
- circular regions in the plane
- $\underbrace{x^2+y^2}_{r^2}$ makes an appearance

Cylindrical Regions

$$W = \{(x,y,z) | u_1(x,y) \le z \le u_2(x,y), (x,y) \in D\}$$

where

$$D = \{(r,\theta) | \alpha \le \theta \le \beta, g_1(\theta) \le r \le g_2(\theta)\}$$

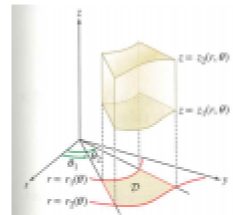


FIGURE 10 Region described in cylindrical coordinates.

If f continuous on D,

$$\iiint_W f(x,y,z)\, dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(r\cos(\theta), r\sin(\theta))}^{u_2(r\cos(\theta), r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z) r dz dr d\theta$$

III. Triple Integration in Spherical Coordinates

$x = \rho\sin(\phi)\cos(\theta)$

$y = \rho\sin(\phi)\sin(\theta)$

$z = \rho\cos(\phi)$

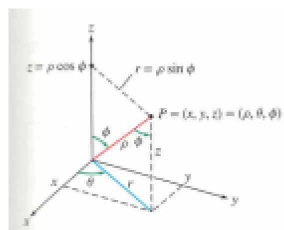


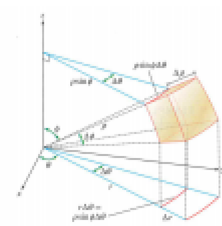
FIGURE 13 Spherical coordinates.

- cones and spheres
- combinations of the above
- symmetry about a point

Spherical Wedges and More General Regions

$$W = \{(\rho,\theta,\phi) | a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

$$W = \{(\rho,\theta,\phi) | g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi), \alpha \le \theta \le \beta, c \le \phi \le d\}$$



If f continuous on D,

$$\iiint_W f(x,y,z)\, dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\rho^2\sin(\phi) d\rho d\theta d\phi$$

15.5 - Applications of Multiple Integrals

Center of mass: Given a lamina occupying D in $\mathbb{R}^2$ with variable density $\rho(x,y)$,

$$(X_{CM}, Y_{CM}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) \text{ where } m = \iint_D \rho(x,y)\, dA \quad M_x = \iint_D y\rho(x,y)\, dA \quad M_y = \iint_D x\rho(x,y)\, dA$$

3-D: $(X_{CM}, Y_{CM}, Z_{CM}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)$ where $m = \iiint \rho(x,y,z)\ dV$,

$$M_{xy} = \iiint z\rho(x,y,z)\ dV \quad M_{xz} = \iiint y\rho(x,y,z)\ dV \quad M_{yz} = \iiint x\rho(x,y,z)\ dV$$

Probability and Wait Times

$P(a \le x \le b) = \int_a^b f(x)dx$ where $\int_{-\infty}^{\infty} f(x)dx = 1$ and $f(x) \ge 1$ for all $x \in X$

Joint Probability: $P((X,Y) \text{ in } D) = \iint_D f(x,y)\ dA$ where $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ dA = 1$ and $f(x,y) \ge 0$

Independent variables: if joint density $f(x,y) = g(x)h(y)$, where $g(x)$ df(X), $h(y)$ df(Y)

Wait-times example:

We can model waiting times by using exponential probability density functions:

$$f(t) = \begin{cases} 0 & t < 0 \\ \mu^{-1}e^{-t/\mu} & t \ge 0 \end{cases}$$

where μ is the mean or average waiting time. Let's check out an example:

$\boxed{\text{example}}$ The manager at local coffee shop determines that the average patron's wait in line to order coffee and a snack is 5 minutes and the average time they wait to receive their refreshments and caffeine is 10 minutes. Assuming that the waiting times are independent, find the probability that a patron waits a total of 20 minutes or less before being able to start consuming their items.

Start with the density functions:

X = waiting time in line

$$g(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{5}e^{-x/5} & x \ge 0 \end{cases}$$

Y waiting for food & coffee

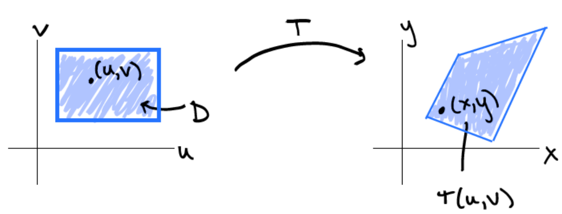
$$h(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{10}e^{-y/10} & y \ge 0 \end{cases}$$

$$f(x,y) = g(x)h(y) = \begin{cases} \frac{1}{50}e^{-y/10}e^{-x/5} & x \ge 0, y \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

Then compute $P(X+Y<20) = \iint f(x,y)\ dA = \int_0^{20}\int_0^{20-x} \frac{1}{50}e^{-y/10}e^{-x/5}dydx$

15.6 - Change of Variables

$T(u,v) = (x,y)$ where $x = x(u,v)$ and $y = y(u,v)$



It is important to know that any transformation T will map the segment joining any two points P and Q to the segment joining $T(P)$ and $T(Q)$.

$$\text{Jac}(T) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Other Notation:

$$\text{Jac}(T) = \frac{\partial(x,y)}{\partial(u,v)}$$ slightly more common!

The **Jacobian Determinant** (Jacobian) of $T(u,v) = (x(u,v), y(u,v))$ helps calculate the area under a transformation (given numerous assumptions...):

$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

It can also *estimate* the area: $\text{Area}(T(D)) \approx \left|\text{Jac}(T)\right| \text{Area}(D)$, $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$

There is a generalization of the change of variables formula for triple integrals. Suppose $T(u,v,w) = (x(u,v,\mathbf{w}), y(u,v,w), z(u,v,w))$.

Then Jac(T)= $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$ determinant of a 3x3 matrix (see 12.4 on the cross product)

Then we have the Change of Variable formula (triple integrals):

$$\iiint_R f(x,y,z)\, dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$