Exam 2 Notes

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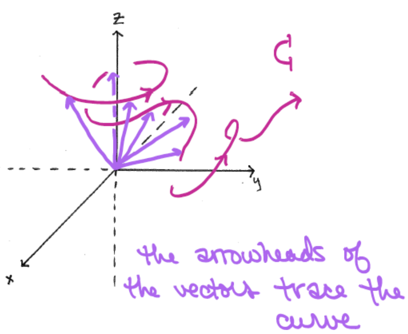
Chapter 13 - Vector-Valued Functions

13.1 - Space Curves

Space curves vs Vector functions

- A space curve is a curve in $\mathbb{R}^3$. If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then the set of points (x, y, z) where $x = f(t)$, $y = g(t)$, $z = h(t)$ is a space curve, C.



$\vec{r}(t)$ is the position vector of the point $P = (f(t), g(t), h(t))$ on C.

Any continuous vector function defines a space curve.

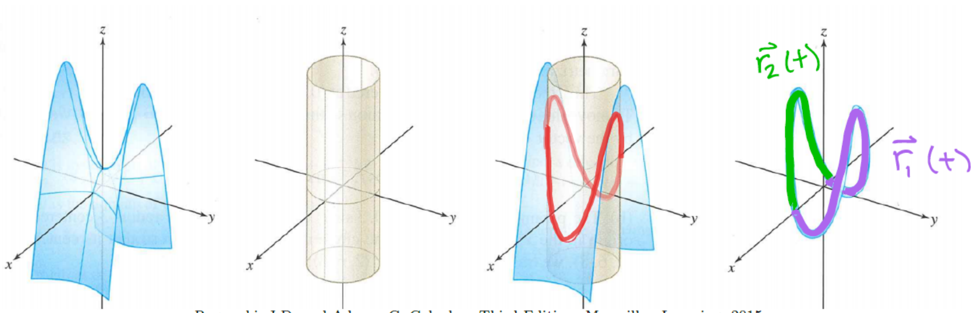
Note: $x = f(t)$, $y = g(t)$, $z = h(t)$ are again called parametric equations.

`Identify/sketch a curve from its vector equation`

`Show a curve lies on a given surface. Use this to sketch the curve.`

`Parameterize a curve given Cartesian equations`
`(solve/eliminate variable; use trig: x = rcos,y=rsin)`

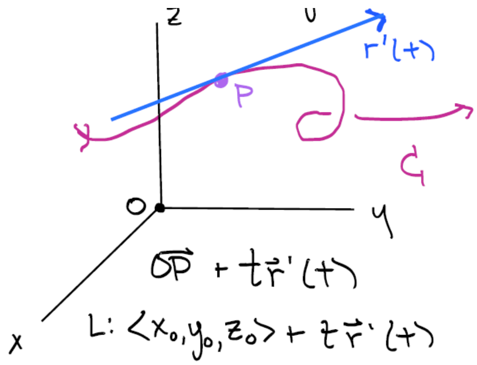
$\boxed{\text{example}}$ Parametrize the curve C obtained as the intersections of the surfaces $x^2-y^2 = z-1$ and $x^2 + y^2 = 4$.



Rogawski, J.D. and Adams, C. Calculus: Third Edition. Macmillan Learning, 2015.

13.2-13.3 - Arc Length & Speed

Derivatives, Tangent Vector, Tangent Line, Integrals



$\mathbf{r}'(t)$ is the tangent vector to the curve defined by $\mathbf{r}$ at the point P. $P = (x_0, y_0, z_0)$

($\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq 0$)

The tangent line to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$.

The unit tangent vector: $\vec{T}(t) = \dfrac{r'(t)}{\|r'(t)\|}$

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Find the tangent vector of $r(t)$ at a given point.
Find the line tangent to a curve at a given point.
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Derivative Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions, c a scalar, and f a real-valued function.

- $\dfrac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$
- $\dfrac{d}{dt}(c\mathbf{u}) = c\mathbf{u}'(t)$
- $\dfrac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u} + f(t)\mathbf{u}'(t)$
- $\dfrac{d}{dt}(\mathbf{u}(f(t)) = f'(t)\mathbf{u}'(f(t))$ chain rule
- similar to the product rule $\dfrac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
- $\dfrac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

Integrals

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

Fundamental Theorem of Calculus...vector function edition

If $\mathbf{r}(t)$ is continuous on $[a, b]$, and $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$, then

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(b) - \mathbf{R}(a)$$

Arc Length: $L = \displaystyle\int_a^b \|\vec{r}'(t)\|dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

Parametric: $s(t) = \displaystyle\int_a^t \|\vec{r}'(u)\|du$
(u is "placeholder")

Speed: $\dfrac{ds}{dt} = \|\vec{r}'(t)\|$
(at time t)

Chapter 14 - Differentiation in Several Variables

14.1 - Functions of 2+ Variables

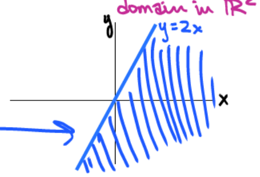
`Find and sketch the domain of the function`

1. $f(x,y) = \sqrt{2x-y}$ $\quad f: \underbrace{\mathbb{R}^2}_{\text{domain in } \mathbb{R}^2} \to \mathbb{R}$

$2x - y \geq 0$

$2x \geq y$

$D: \{(x,y) \mid y \leq 2x\}$



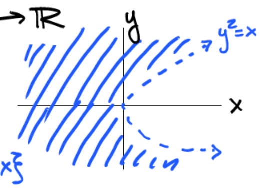
2. $f(x,y) = \ln(y^2 - x)$

$f: \mathbb{R}^2 \to \mathbb{R}$

$y^2 - x > 0$

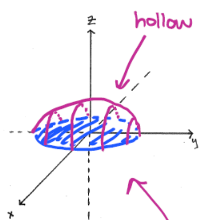
$y^2 > x$

$D: \{(x,y) \mid y^2 > x\}$



`Sketch graph of function`

$\boxed{\text{example}}$ Sketch the graph of the function $\underbrace{f(x,y)}_{z} = \sqrt{4-4x^2-y^2}$.



$z = \sqrt{4-4x^2-y^2}$ so $z \geq 0$

Notice

$z^2 = 4 - 4x^2 - y^2$

$\Rightarrow \underbrace{x^2 + y^2/4 + z^2/4 = 1}$ $\Rightarrow$ ellipsoid

Domain

$4 - 4x^2 - y^2 \geq 0$

$4 \geq 4x^2 + y^2$

$\underbrace{1 \geq x^2 + y^2/4}$

filled in ellipse in the xy-plane

`Sketch level curves and contour maps of a function`

$f(x,y) = k \Rightarrow z = k$

$\boxed{\text{example}}$ Sketch the level curves of the function $\underbrace{f(x,y)}_{z = x^2+y^2} = x^2 + y^2$ for constant values $k = -1, 0, 1, 4$

$\underline{k=-1}$

$-1 = x^2 + y^2$

no such point exists!

$\underline{k=0}$

$0 = x^2 + y^2$

$(0,0)$

$\underline{k=1}$

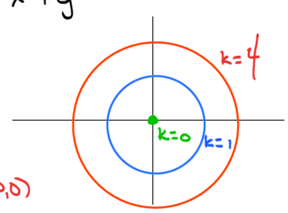
$1 = x^2 + y^2$

circle w/ $r=1$ & center $(0,0)$

$\underline{k=4}$

$4 = x^2 + y^2$

circle w/ $r=2$ & center $(0,0)$



`Sketch *sections* (vertical traces) of a graph`

14.2 - Limits and Continuity in Several Variables

$$\lim_{x \to b} f(x) = L \iff \left(\forall \epsilon > 0, \exists \delta > 0 \ s.t. \sqrt{(x_1-b_1)^2 + (x_2-b_2)^2 + \cdots + (x_n-b_n)^2} < \delta \implies |f(x) - L| < \epsilon\right)$$

$x = (x_1, x_2, \ldots, x_n) \in D(f) \subset \mathbb{R}^n$

Continuity: $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Limit Existence:

> If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and
>
> if $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2 and
>
> $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ DNE

Disclaimer:
C_1 & C_2 might be difficult to pick.

Potential Options

$x=0$, $y=0$, $y=kx^2$
$x=ky^2$, $y=mx$, etc

14.3 - Partial Derivatives

$*$ Partial Derivatives : the rates of change with respect to each variable separately.

The partial derivative of f with respect to x at (a,b) is

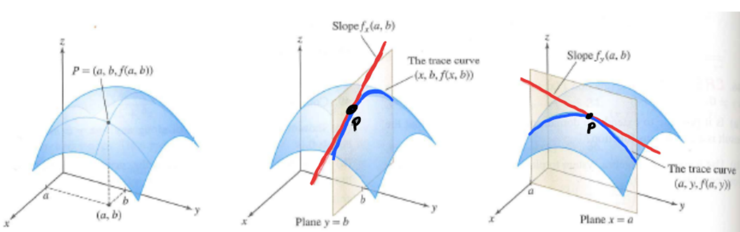
$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

similarly for y

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Graphically, we have the following:

Let $P = (a,b)$. Then the partial derivatives at P are <u>Slopes of tangent lines</u> to the <u>vertical traces</u> curves through the point $(a, b, f(a,b))$.



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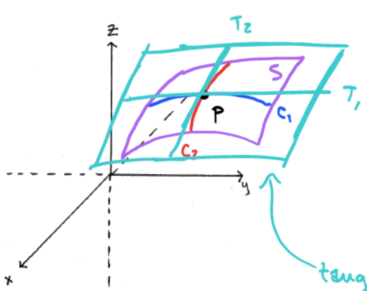
14.4 - Differentiability and Tangent Planes



tangent plane at P.

<u>Tangent Planes</u>

Let C_1 and C_2 be curves on the surface obtained by obtaining the intersection with the surface and vertical planes.

T_1 and T_2 be tangents to C_1 and C_2 at the point P.

$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

- Compare: $y - y_0 = f'(x_0)(x - x_0)$

Linear Approximation: The *linearization* of f at (a, b) — $L(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$

Differentials and Increments

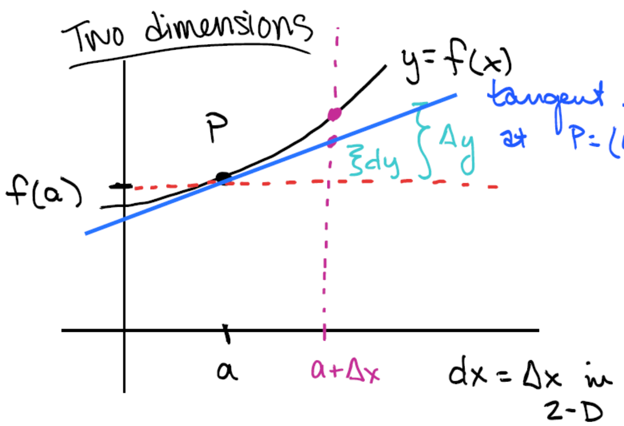
Increments: $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$

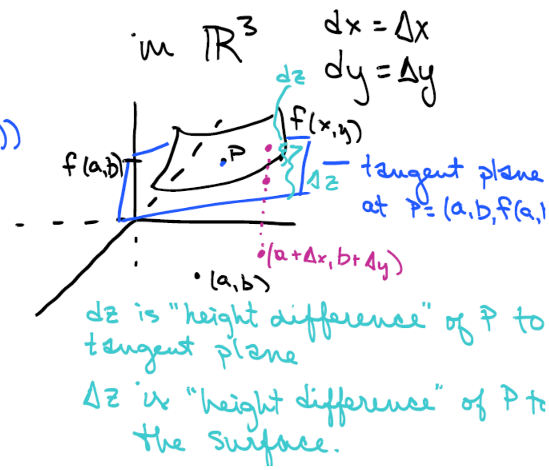
Differentiability: If $z = f(x, y)$, f is differentiable at (a, b) if

$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta z$ and $\varepsilon_1, \varepsilon_2 \to 0$ as $(a, b) \to (0, 0)$

Theorem — if f_x, f_y exist "near" (a, b) and are continuous at (a, b), then f is differentiable at (a, b)

Differentials: $dz = f_x dx + f_y dy$, $dz \ \approx \Delta z$





14.5 - Directional Derivatives and the Gradient of a Vector-Valued Function

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} = \langle a, b, c\rangle$:

$$D_{\vec{u}} f(x, y, z) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0)}{h}$$

Theorem — $f(x, y, z)$ differentiable $\implies \forall \vec{u} = \langle a, b, c\rangle$, f has a directional derivative:

$$D_{\vec{u}} f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

Gradient: $\nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j} + \dfrac{\partial f}{\partial z}\vec{k}$
(similar for function of two variables, $\nabla f(x, y) = \langle f_x, f_y\rangle$)

Re: Directional Derivative: $D_{\vec{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u}$

Assume $\nabla f(p) \neq 0$:

- $\pm\nabla f(p)$ points in the direction of max rate of increase/decrease of f at p
- $\nabla f(p)$ is normal to the level curve (or surface) of f at p
  — *tangent plane* of $f(x, y, z) = k$ at $p = (x_0, y_0, z_0)$: $\quad \nabla f(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$
- $\|\nabla f(p)\|$ gives *max slope of a tangent line* (max rate of change) to the surface $z = f(x, y)$ at $(p, f(p))$

14.6 - The Chain Rule

1.) z is a differentiable function of two single-parameter differentiable functions

$$z = f(x(t), y(t)) \longrightarrow \frac{dz}{dt} = (\frac{\partial f}{\partial x})(\frac{dx}{dt}) + (\frac{\partial f}{\partial y})(\frac{dy}{dt})$$

2.) z is a differentiable function of two “double”-parameter differentiable functions

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

3.) Implicit Differentiation

For instance, consider the implicit function $x^2y - xy^3 = 3$. We learned to use the following steps to find $\frac{dy}{dx}$:

$$\begin{aligned} \frac{d}{dx}\left(x^2y - xy^3\right) &= \frac{d}{dx}\left(3\right) \\ 2xy + x^2\frac{dy}{dx} - y^3 - 3xy^2\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{2xy - y^3}{x^2 - 3xy^2}. \end{aligned} \tag{13.5.1}$$

Instead of using this method, consider $z = x^2y - xy^3$. The implicit function above describes the level curve $z = 3$. Considering x and y as functions of x, the Multivariable Chain Rule states that

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}. \tag{13.5.2}$$

Note that $\frac{dx}{dx} = 1$ and, if z is constant, $\frac{dz}{dx} = 0$. Hence $\frac{\partial z}{\partial y} \neq 0 \implies \frac{dy}{dx} = -\frac{\partial z}{\partial x}/\frac{\partial z}{\partial y} = \frac{f_x}{f_y}$

Types of Questions

13.1

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Identify/sketch a curve from its vector equation

Show a curve lies on a given surface. Use this to sketch the curve.

Parameterize a curve given Cartesian equations
(solve/eliminate variable; use trig: x = rcos,y=rsin)
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13.2-13.3

Find the tangent vector of $r(t)$ at a given point.

Find the line tangent to a curve at a given point.

14.1

Find and sketch the domain of the multi-valued function

14.2

Where is the function continuous?

Determine if the limit exists and compute the limit if possible