Math-205 Laplase-2set 8

$y = f(x) \rightarrow$ explicit

$f(x, y) \rightarrow$ implicit $\rightarrow z = e^{xy}$

$f(z) = f(x, y)$

$\frac{dy}{dx} = e^x \rightarrow$ ordinary differential equation

$\Rightarrow dy = e^x dx$

$\Rightarrow \int dy = \int e^x dx$

$\Rightarrow y = e^x + c$

~~Refe:~~

Reference:

1. Theory and Problems of Laplace Transform, Schaum's outline series, Murray R. Spiegel.

2. Mathematical methods by Prof. Abdur Rahman. (Vol. 2)

Laplace transform of $F(t)$ is denoted by $f(s)$ and is defined by,

$$\mathcal{L}\{F(t)\} = f(s) = \underbrace{\int_0^\infty F(t)\, e^{-st}\, dt}_{\text{improper integral}}$$

$\boxed{s = a + ib}$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma n+1 = n! = n \Gamma n$$

$$\Gamma \tfrac{1}{2} = \sqrt{\pi}$$

$$\Gamma \tfrac{3}{2} = \tfrac{1}{2} \Gamma \tfrac{1}{2} = \frac{\sqrt{\pi}}{2}$$

$$\mathcal{L}\{f(1)\} = \int_0^\infty e^{-st} \cdot 1 \, dt$$

$$= \left[\frac{-e^{-st}}{s} \right]_0^\infty$$

$$\frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

$$= -\frac{e^{-\infty}}{s} + \frac{e^0}{s}$$

$$= -0 + \frac{1}{s}$$

$$= \frac{1}{s}$$

$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$
1	$\frac{1}{s}$ $\quad s>0$
t	$\frac{1}{s^2}$ $\quad s>0$
t^n	$\frac{n!}{s^{n+1}}$ $\quad s>0$
e^{at}	$\frac{1}{s-a}$ $\quad s>a$
$\sin at$	$\frac{a}{s^2+a^2}$ $\quad s>0$
$\cos at$	$\frac{s}{s^2+a^2}$ $\quad s>0$

$$\mathcal{L}\{f(e^{at})\} = \int_0^\infty e^{-st}\cdot e^{at}\,dt$$

$$= \int_0^\infty e^{-st+at}\,dt$$

$$= \int_0^\infty e^{-t\left(s-a\right)}\,dt$$

$$= \left[\frac{-e^{-t(s-a)}}{\cancel{a-s}\ s-a}\right]_0^\infty$$

$$= \frac{-e^{-\infty}}{\cancel{a-s}\ s-a} + \frac{e^{0}}{\cancel{a-s}\ s-a}$$

$$= 0\ \cancel{\frac{e^{\infty}}{a-s}} + \frac{1}{\cancel{a-s}\ s-a}$$

$$= \frac{1}{s-a}$$

$$\int_0^\infty e^{(a-s)t}\,dt \qquad \boxed{s>a}$$

$$= \left[\frac{1}{a-s}\,e^{(a-s)t}\right]_0^\infty$$

$$= \frac{1}{a-s}\left[e^{-\infty} - e^{0}\right]$$

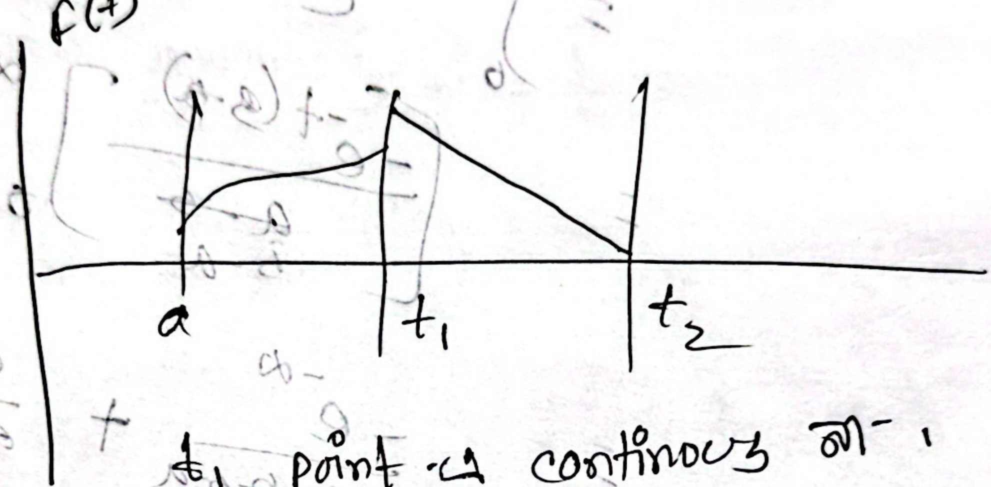
Math আসবে না।

defination আসবে।

sectional or piecewise continuity.

$[1, 2] \rightarrow$ interval

$$f(x) = \begin{cases} x, & x < 1 \\ 2 - x, & 1 \le x \le 2 \\ x^2, & x > 2 \end{cases}$$



t_1 point এ continous না।

Because L.H.L and R.H.L same না।

Functions of exponential order

Pg=2

Defination + Example

Piecewise continuous + exponential order হলেই function এর laplase transform exist করবে।

If real constants $M > 0$ and γ exist such that for all $t > N$

$$|e^{-\gamma t} F(t)| < M \quad \text{or} \quad |F(t)| < M e^{\gamma t}$$

Example 1: $F(t) = t^2$ is of exponential order 3, since $|t^2| = t^2 < e^{3t}$ for all $t > 0$.

Example 2: $F(t) = e^{t^2}$

$$\boxed{e^{t^2} < M e^{\gamma t}} \rightarrow \text{This can't be true.}$$

1. Linearity property:

$f(a+b) = f(a) + f(b) \rightarrow$ linear function.

$f(x) = x^2 \rightarrow$ Not linear function.

$f(\alpha a) = \alpha f(a) \rightarrow$ linear "

$$\mathcal{L}\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 \mathcal{L}\{F_1(t)\} + c_2 \{\mathcal{L}[F_2(t)]\} = c_1 f_1(s) + c_2 f_2(s)$$

Ex: $\mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\}$

3. First translation or shifting property.

$$\boxed{\mathcal{L}\{e^{at}F(t)\} = f(s-a)}$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4} = f(s)$$

$$\mathcal{L}\{e^{-t}\cos 2t\} = \frac{s-(-1)}{(s+1)^2+4}$$

$$= \frac{s+1}{(s+1)^2+4}$$

$\mathcal{L}\{e^{-t}\underbrace{\cos 2t}_{F(t)}\} = f(s-a)$

$\boxed{\mathcal{L}\{F(t)\} = f(s)}$

4. change of scale property:

If $\mathcal{L}\{F(t)\} = f(s)$ then,

$$\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$

$\mathcal{L}\{\cos 5 \cdot 2t\} = \frac{1}{5} \cdot \frac{s/5}{(s/5)^2+4}$

5. laplace transform of derivatives.

$$\mathcal{L}\{F(t)\} = f(s)$$

$$\mathcal{L}\{F'(t)\} = s f(s) - F(0)$$

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$$

$$\mathcal{L}\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)$$

$\boxed{+}$ $F(t) = 1$; $F'(t) = 0$; $F(0) = 1$

$\mathcal{L}(0) = 0$

$$\mathcal{L}\{F(t)\} = f(s)$$

$$\mathcal{L}\{F'(t)\} = s f(s) - F(0)$$

$$\Rightarrow 0 = s f(s) - 1$$

$$\Rightarrow f(s) = \frac{1}{s}$$

$\mathcal{L}\{1\} = \frac{1}{s}$

$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$

$\boxed{+}$ $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

$$\sin at = \mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}$$

$\mathcal{L}\{F(t)\} = f(s)$

$\mathcal{L}\{1\} = \frac{1}{s}$

Inverse laplase transform

If the laplase ~~fra~~ transform of a function $F(t)$ is $f(s)$ i.e. if $\mathcal{L}\{F(t)\} = f(s)$, then ~~ft~~ $F(t)$ is called an inverse laplase transform of $f(s)$ and we write symbolically $F(t) = \mathcal{L}^{-1}\{f(s)\}$ where $\mathcal{L}^{-1}$ is called the inverse transformation operator.

3. $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$

Ex: 1. $\mathcal{L}^{-1}\left\{\frac{1}{s^8}\right\} = \frac{t^7}{7!}$

2. $\mathcal{L}^{-1}\left\{\frac{1}{s^{7/2}}\right\} = \frac{t^{5/2}}{\Gamma 7/2}$

$$\boxed{\Gamma n+1 = n!} = n\Gamma n$$

$n+1 = \frac{7}{2}$

$n = \frac{7}{2} - 1$

$= \frac{7-2}{2} = \frac{5}{2}$

$\Gamma\frac{7}{2} = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2}$

1. Linearity property:

$$\mathcal{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\}$$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t$$

2. shifting property:

$\mathcal{L}, \mathcal{L}^{-1}, 3 \xrightarrow{\mathcal{L}^{-1}} \text{no need}, \boxed{1, 2, 4, 5}$

$$\mathcal{L}\{t^n\} = \frac{n!}{\cancel{1+}\, s^{n+1}}$$

$$\mathcal{L}\left\{\frac{t^n}{n!}\right\} = \frac{1}{s^{n+1}}$$

$$\frac{t^n}{n!} = \mathcal{L}^{-1}\left(\frac{1}{s^{n+1}}\right)$$

Pg-48 → 3 ~~All~~ All H.W

3. (a) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$

$= \frac{\sin 3t}{3}$ (Ans).

(b) $\mathcal{L}^{-1}\left\{\frac{4}{s-2}\right\}$

$= 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$

$= 4e^{2t}$

(c) $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^{3+1}}\right\}$

$= \frac{t^3}{3!} = \frac{t^3}{6}$

(d) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\right\} = \cos\sqrt{2}\,t$

(e) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-16}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{s}{s^2-4^2}\right\}$

$= \cos 4t$

(f) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-3}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{1}{s^2-(\sqrt{3})^2}\right\}$

$= \dfrac{1}{\sqrt{3}} \cdot \sin\sqrt{3}t$ (Ans).

(g) $\mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\right\}$

We know,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!} \text{ ; if } n \text{ is integer}$$

when n is a fraction number $\frac{t^n}{\Gamma(n+1)}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^{1/2+1}}\right\} = \frac{t^{1/2}}{\Gamma(3/2)}$$

$$= \frac{t^{1/2}}{\frac{1}{2}\Gamma(\frac{1}{2})}$$

$$= 2\sqrt{\frac{t}{\pi}} \text{ (Ans).}$$

Pg-49

5. (a) $\mathcal{L}^{-1}\left\{\frac{5s+4}{s^3} - \frac{2s-18}{s^2+9} + \frac{44-30\sqrt{s}}{s^4}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{5s+4}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{2s-18}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{44-30\sqrt{s}}{s^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{5s}{s^3} + \frac{4}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{2s}{s^2+9} - \frac{18}{s^2+9}\right\}$$

$$+ \mathcal{L}^{-1}\left\{\frac{44}{s^4} - \frac{30\sqrt{s}}{s^4}\right\}$$

$$\Gamma\left(\tfrac{5}{2}+1\right) = \tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right) = \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right), \quad s^{1/2}\ \tfrac{-4}{2}\ -\tfrac{7}{2}$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + 18\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} - 30\mathcal{L}^{-1}\left\{\frac{s^{1/2}}{s^4}\right\}$$

$$= 5t + 4\frac{t^2}{2!} - 2\cos 3t + \frac{18}{3}\cancel{\cos 3t}\sin 3t + 4\frac{t^3}{3!} - 30\mathcal{L}^{-1}\left\{\frac{1}{s^{7/2}}\right\} \qquad s^{n+1}$$

$$= 5t + 4\frac{t^2}{2!} - 2\cos 3t + 6\cancel{\cos 3t}\sin 3t + 4\frac{t^3}{3!} - 30\frac{t^{5/2}}{\Gamma(7/2)}$$

$$= 5t + 4\frac{t^2}{2!} - 8\cos 3t + 4\frac{t^3}{3!} - 30\frac{t^{5/2}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} + 6\sin 3t$$

$$= 5t + 4\frac{t^2}{2!} - 8\frac{t^{5/2}}{\sqrt{\pi}} + 6\sin 3t$$

$$= 5t + 4\frac{t^2}{2!} - 8\cos 3t + 4\frac{t^3}{3!} - 8$$

$$\frac{5*3+5\cdot 7}{(s+1)+5(s+3)} \qquad \frac{3}{s^2-2s-3}$$

(*) $\mathcal{L}^{-1}\left\{\frac{3s-7}{(s+1)(s-3)}\right\}$ — (1)

$= \mathcal{L}^{-1}\left\{\frac{3s-7}{s^2-3s+s-3}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{3s-7}{s^2-2s-3}\right\}$

$= \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$

$= \frac{5}{2}e^{-t} + \frac{1}{2}e^{3t}$ (Ans).

(*) $\mathcal{L}^{-1}\left\{\frac{A}{(s+1)^2}\right\} = e^{at}\frac{t^n}{n!}$

(*) $\mathcal{L}^{-1}\left\{\frac{As+B}{s^2+3}\right\} = \mathcal{L}^{-1}\left\{A\frac{s}{s^2+3} + B\frac{1}{s^2+3}\right\}$

$\frac{3s-7}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3} \Rightarrow \frac{5}{2(s+1)} + \frac{1}{2(s-3)}$

$\Rightarrow 3s-7 = A(s-3) + B(s+1)$ — (2)

a) $s-3=0 \Rightarrow s=3$

$\Rightarrow 2 = 4B \therefore B = \frac{1}{2}$

b) $s+1=0 \Rightarrow s=-1$

$\Rightarrow -10 = -4A$

$\therefore A = \frac{5}{2}$

Heaviside Expansion formula

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$$

$P(s) < Q(s)$

$Q'(s)$ exist , $Q'(s) \neq 0$

$\alpha_k \rightarrow$ root of $Q(s)$

For two roots,

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{2} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$$

$$= \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t}$$

Ex: Using Heaviside expansion formula

evalu[illegible]

Soln

For zeros of $Q(s)$

We may write, $Q(s) = 0$

$\Rightarrow (s+1)(s-3) = 0$

$\Rightarrow s = -1, 3$

$\therefore \alpha_1 = -1, \alpha_2 = 3$

$\therefore P(\alpha_1) = P(-1) = 3(-1)+7 = 4$

$P(\alpha_2) = P(3) = 3 \cdot 3 + 7 = 16$

$Q'(-1) = 2(-1) - 2 = -4$

$Q'(3) = 2 \cdot 3 - 2 = 4$

$$\mathcal{L}^{-1}\left\{\frac{3s+7}{(s+1)(s-3)}\right\} = \frac{P(-1)}{Q'(-1)} e^{(-1)t} + \frac{P(3)}{Q'(3)} e^{3t}$$

$$= \frac{4}{-4} e^{-t} + \frac{16}{4} e^{3t}$$

$$\frac{3s-7}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3}$$

$\Rightarrow 3s-7 = A(s-3) + B(s+1)$ — (2)

a) $s-3=0 \Rightarrow s=3$

$\Rightarrow 2 = 4B \therefore B = \frac{1}{2}$

b) $s+1=0 \Rightarrow s=-1$

$\Rightarrow -10 = -4A$

$\therefore A = \frac{5}{2}$

$\mathcal{L}\{F(t)\} = f(s)$	$\mathcal{L}^{-1}\{f(s)\} = F(t)$
1. $\cancel{\mathcal{L}\{\frac{1}{s}\} =}$ $\mathcal{L}\{1\} = \frac{1}{s}$	1. $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$
2. $\mathcal{L}\{t\} = \frac{1}{s^2}$	2. $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$
3. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$	3. $\mathcal{L}^{-1}\{\frac{1}{s^{n+1}}\} = \frac{t^n}{n!}$
4. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$	4. $\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$
5. $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$	5. $\mathcal{L}^{-1}\{\frac{1}{s^2+a^2}\} = \frac{\sin at}{a}$
6. $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$	6. $\mathcal{L}^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$
	7. $\mathcal{L}^{-1}\{\frac{1}{s^2-a^2}\} = \frac{\sinh(at)}{a}$
	8. $\mathcal{L}^{-1}\{\frac{s}{s^2-a^2}\} = \cosh(at)$

Online

Examples about laplace and Inverse laplace transform:

1. Find $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\}$

Sol^n: $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}$ $\left[\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)\right.$

$= e^{-t}\cos 2t$

Formula

a. $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \cancel{\frac{e^{at}\sin bt}{b}}\ \frac{1}{b}e^{at}\sin bt$

b. $\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

c. $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^2}\right\} = \frac{1}{2b}t\sin bt$

2. Find $\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$

<u>Soln.</u> $\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{6s-12+8}{s^2-2.2.s+2^2+16}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+4^2}\right\}$$

$$= 6\,\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+4^2}\right\}+8\,\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+4^2}\right\}$$

$$= 6e^{2t}\cos 4t + 8\cdot\frac{1}{4}e^{2t}\sin 4t$$

$$= 2e^{2t}(3\cos 4t+\sin 4t) \quad \text{(Ans)}.$$

3. Find the Laplace transform of $F(t)$, where $F(t) = e^{-2t}\sin 4t$

Let, $\mathcal{L}\{F(t)\} = f(s)$

We know that, If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$ ~~then~~

Here, $\mathcal{L}\{\sin 4t\} = \frac{4}{s^2+4^2}$

$\therefore \mathcal{L}\{e^{-2t}\sin 4t\} = \frac{4}{(s+2)^2+4^2}$

$= \frac{4}{s^2+4s+20}$ (Ans)

4. Laplace transform of integrals:

If $\mathcal{L}\{F(t)\} = f(s)$

then $\mathcal{L}\{\int_0^t F(u)\,du\} = \frac{f(s)}{s}$

For inverse Laplace transform:

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then

$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$

Example: Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$

We know that,

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u\,du$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \left[-\cos u\right]_0^t$$

$$= 1 - \cos t$$

Continuing ~~the~~ by the repeated process we can write,

since, $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos u)\, du$$

$$= \left[u - \sin u\right]_0^t$$

$$= t - \sin t \quad \text{(Ans)}.$$

Try yourself:

5. $\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{3s+\cancel{7}-3+10}{s^2-2.1.s+1^2-1^2-3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-4}\right\}$$

$s^2 - 3s + s - 3$

$s(s-3) + 1(s-3)$

$(s+1)(s-3)$

$\frac{1}{s+1} + \frac{1}{s-3}$

$$\cancel{= 3\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2}\right.}$$

5. $\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{3s+7}{(s+1)(s-3)}\right\}$

$$= \cancel{3\mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s-3)}\right\} + 7\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-3)}\right\}}$$

$$= \mathcal{L}^{-1}\left\{\frac{-1}{s+1} + \frac{4}{s-3}\right\}$$

$$= -e^{-t} + 4e^{3t}$$

Try yourself!

5. $\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-4}\right\}$

$$= 3\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2-2^2}\right\} + 10\,\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2-4}\right\}$$

$$= 3e^t \underbrace{\cosh(2t)}_{\text{hyperbolic}} + 10 \cdot \frac{1}{2}\sinh(2t)$$

(Ans)

6. $\mathcal{L}^{-1}\left\{\dfrac{4s+12}{s^2+8s+16}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{4s+16-4}{s^2+2\cdot 4\cdot s+4^2-4^2+16}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2-16+16}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{s+4}{(s+4)^2}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}$$

$$= 4e^{-4t} - 4e^{-4t}t$$

$$= 4e^{-4t}(1-t) \quad \text{(Ans)}$$

(7) find $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}$

Sol^n:

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(2s+3)^{1/2}}\right\}$$

$$= \frac{1}{\sqrt{2}}\,\mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{3}{2})^{1/2}}\right\}$$

$$= \frac{1}{\sqrt{2}}\, e^{-\frac{3}{2}t}\, \frac{t^{-1/2}}{\sqrt{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{3t}{2}}\, t^{-1/2}$$

(8) Prove the followings.

(i) $\mathcal{L}\{1\} = \frac{1}{s}, \quad s>0$

(ii) $\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0$

(iii) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}; \quad s>a$

⑨ find $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}$

<u>Soln.</u> $\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}+\frac{1}{2}}{s^2+2\cdot\frac{1}{2}\cdot s+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}\right\}$

$=\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$

$=e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t+\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$

$=e^{-\frac{t}{2}}\left(\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$ (Ans).

11. find $\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^{5/2}}\right\}$.

<u>Soln.</u> $\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^{5/2}}\right\}$

$=e^{-4t}\frac{t^{3/2}}{\Gamma 5/2}$

Since, $\mathcal{L}\{e^{at}F(t)\}=f(s-a)$

$\Rightarrow \mathcal{L}\left\{e^{at}\frac{t^n}{n!}\right\}=\frac{1}{(s-a)^{n+1}}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^{n+1}}\right\} = e^{at}\frac{t^n}{n!} \quad (ii)$$

<u>Note:</u> $\mathcal{L}\left\{\frac{t^n}{n!}\right\} = \frac{1}{s^{n+1}}$

$$\mathcal{L}\left\{e^{at}\frac{t^n}{n!}\right\} = \frac{1}{(s-a)^{n+1}}$$

Properties 5 (Inverse laplace Transform)

1. $\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\}$

<u>Properties:</u> $\mathcal{L}^{-1}\{f(s)\} = F(t)$

$$\mathcal{L}^{-1}\{f^n(s)\} = (-1)^n t^n F(t)$$

let, $f(s) = \frac{1}{s^2+1}$

$$\therefore \mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$\frac{d}{ds}\left(f(s)\right) = \frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\frac{1}{(s^2+1)^2}\cdot 2s$$

By using the properties we can write,

$$\mathcal{L}^{-1}\{f'(s)\} = \mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\}$$

$$= (-1)^1 \cdot t^1 f(t)$$

$$= -t\sin t$$

$$\Rightarrow -\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} = -t\sin t$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} = t\sin t$$

Problem related to partial fraction/ Heavisides expansion formula:

1. Find $\mathcal{L}^{-1}\left\{\frac{s^2+1}{s^2(s+2)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{s^2}{s^2(s+2)} + \frac{1}{s^2(s+2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+2} + \frac{1}{s^2(s+2)}\right\}$$

$$= e^{-2t}$$

We know that,

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = \int_0^t e^{-2u}\,du$$

$$= \left[-2e^{-2u}\right]_0^t$$

$$= -2e^{-2t} + 2$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\} = \int_0^t \left(-2e^{-2u} + 2\right) du$$

$$= \left[-2\cdot(-2)e^{-2u} + 2u\right]_0^t$$

$$= 4e^{-2t} - 4 + 2t$$

(Ans).

2. Find $\mathcal{L}^{-1}\left\{\frac{s^2+1}{(s+1)^2(s+2)}\right\}$

Soln:

Let, $\frac{s^2+1}{(s+1)^2(s+2)} \equiv \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$ —— (1)

$\Rightarrow s^2+1 \equiv A(s+1)(s+2) + B\cancel{(s+1)}(s+2)$
$+ C(s+1)^2$ —— (2)

$\equiv A(s^2+3s+2) + B(s+2) + C(s^2+2s+1)$

$\equiv (A+C)s^2 + (3A+B+2C)s$
$+ 2A+2B+C$ —— (3)

Putting $s=-1$ in eqn (2)

$(-1)^2+1 = A\cdot 0 + B(-1+2) + C\cdot 0$

$\Rightarrow 2 = B$

Again, putting $s=-2$ in eqn (2) we can write, $(-2)^2+1 = A\cdot 0 + B\cdot 0 + C(-2+1)^2$

$\Rightarrow 5 = C$

Equating the co-efficient of s^2 from bothsides of eqn (3) we can write,

$A + C = 1$

$\Rightarrow A = 1 - 5 = -4$

Putting the value of A, B, C in (1) we can write,

$$\frac{s^2+1}{(s+1)^2(s+2)} = \frac{-4}{s+1} + \frac{2}{(s+1)^2} + \frac{5}{s+2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s^2+1}{(s+1)^2(s+2)}\right\} = -4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= -4e^{-t} + 2e^{-t}t + 5e^{-2t}$$

Note: $\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+4)(s+2)}\right\}$

Let, $\frac{s+1}{(s^2+4)(s+2)} = \frac{As+B}{s^2+4} + \frac{C}{s+2}$

Note: $\frac{x^3+2x+5}{x^2+2} = 1 + \frac{Ax+B}{x^2+2}$

$= 1 + \frac{P(s)}{Q(s)}$

OR

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+4)(s+2)}\right\}$$

$$P(s) = s+1$$
$$Q(s) = (s^2+4)(s+2)$$
$$Q(s) = s^3+2s^2+4s+8$$
$$Q'(s) = 3s^2+4s+4$$

for zeros of $Q(s)$,

$$Q(s)=0$$

$$\Rightarrow (s^2+4)(s+2)=0$$

$$\Rightarrow s^2+4=0 \quad \text{or} \quad s+2=0$$

$$\therefore s=\pm 2i \qquad \therefore s=-2$$

$$\alpha_1 = -2i, \quad \alpha_2 = 2i, \quad \alpha_3 = -2$$

$$P(\alpha_1) = -2i+1, \quad P(\alpha_2) = 2i+1, \quad P(\alpha_3) = -2+1 = -1$$

$$Q'(\alpha_1) = 3(-2i)^2 + 4(-2i)+4$$
$$= -12 - 8i + 4$$
$$= -8i - 8$$

W/H

$$Q'(\alpha_2) = 3(2i)^2 + 4(2i)+4 = -12+8i+4$$
$$= 8i-8$$

$$Q'(\alpha_3) = Q'(2) = 3\times(2)^2 + 4(-2) + 4$$
$$= 12 - 8 + 4$$
$$= 8$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+4)(s+2)}\right\} = \sum_{k=1}^{3} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$$

$$= \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t} + \frac{P(\alpha_3)}{Q(\alpha_3)} e^{\alpha_3 t}$$

$$= \frac{-2i+1}{-8i-8} e^{-2it} + \frac{2i+1}{8i-8} e^{2it} + \frac{-1}{8} e^{-2t}$$

$$= \frac{2i-1}{8i+8} e^{-2it} + \frac{2i+1}{8i-8} e^{2it} - \frac{1}{8} e^{-2t} \quad \text{(Ans).}$$

H/W

i) $\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$

ii) $\mathcal{L}^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$

iii) $\mathcal{L}^{-1}\left\{\frac{2s^2+5s-4}{s^3+s^2-2s}\right\}$

i) $\mathcal{L}^{-1}\left\{ \frac{3s+1}{(s-1)(s^2+1)} \right\}$

Let, $\frac{3s+1}{(s-1)(s^2+1)} \equiv \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$ ——①

$\Rightarrow 3s+1 \equiv A(s^2+1) + (Bs+C)(s-1)$ ——②

$\Rightarrow 3s+1 = As^2+A + Bs^2 - Bs + Cs - C$

$\Rightarrow 3s+1 = (A+B)s^2 + (C-B)s + A - C$ ——③

putting $s=1$ in eqn ② we can write,

$3.1+1 = A(1^2+1) + (Bs+C).0$

$\Rightarrow 4 = 2A$

$\therefore A = 2$

Equating the co-efficient of s^2 and constant from ~~eqn ③~~ bothsides of eqn ③ we can write,

$A - C = 1$

$\Rightarrow C = A - 1 = 2-1 = 1$

and, $A+B = 0$

$\Rightarrow B = -2$

Putting the value of A, B, C in eqn (1)

we can write,

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{-2s+1}{s^2+1}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{-2s+1}{s^2+1}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 2e^t - 2\cos t + \sin t \quad \text{(Ans).}$$

ii) $\mathcal{L}^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$

Let,

$$\frac{2s^2-6s+5}{(s-1)(s-3)(s-2)} \equiv \frac{A}{s-1} + \frac{B}{s-3} + \frac{C}{s-2} \quad \text{--- (1)}$$

$$\Rightarrow 2s^2-6s+5 \equiv A(s-3)(s-2) + B(s-1)(s-2) + C(s-1)(s-3) \quad \text{--- (2)}$$

Putting $s = 3$ in eqn ② we get,

$$2\cdot(3)^2 - 6\times 3 + 5 = A\cdot 0 + B(3-1)(3-2) + C\cdot 0$$

$$\Rightarrow 18 - 18 + 5 = B(2\times 1)$$

$$\Rightarrow 5 = 2B$$

$$\therefore B = \frac{5}{2}$$

Putting $s = 2$ in ~~eqn②~~ eqn ② we get,

$$2\cdot 2^2 - 6\cdot 2 + 5 = A\cdot 0 + B\cdot 0 + C(2-1)(2-3)$$

$$\Rightarrow 8 - 12 + 5 = C$$

$$\therefore C = 1$$

Putting $s = 1$ in eqn ② we get,

$$2\cdot 1^2 - 6\cdot 1 + 5 = A(1-3)(1-2) + B\cdot 0 + C\cdot 0$$

$$\Rightarrow 2 - 6 + 5 = A(-2)(-1)$$

$$\therefore A = \frac{1}{2}$$

Putting the value of A, B, C in eqn ①

we get,

① $$\frac{2s^2 - 6s + 5}{(s-1)\cdot(s-3)(s-2)} = \frac{1}{2(s-1)} + \frac{5}{2(s-3)} + \frac{1}{s-2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$+\frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$=\frac{1}{2}e^{t}+\frac{5}{2}e^{3t}+e^{2t} \quad \text{(Ans).}$$

(iii) $\mathcal{L}^{-1}\left\{\frac{2s^2+5s-4}{s^3+s^2-2s}\right\} = \mathcal{L}^{-1}\left\{\frac{2s^2+5s-4}{s^3-s^2+2s^2-2s}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{2s^2+5s-4}{s^2(s-1)+2s(s-1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s^2+5s-4}{(s-1)(s^2+2s)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s^2+5s-4}{s(s+2)(s-1)}\right\}$$

Let, $\frac{2s^2+5s-4}{s(s+2)(s-1)} = \frac{A}{s}+\frac{B}{s+2}+\frac{C}{s-1}$ ——— (1)

$\Rightarrow 2s^2+5s-4 \equiv A(s+2)(s-1)+B(s-1)s+Cs(s+2)$ ——— (2)

Putting $s=0$ in eqn (2) we get,

$$0+0-4 = A(0+2)(0-1)+B\cdot 0+C\cdot 0$$

$$\Rightarrow -4 = -2A$$

$$\therefore A = 2$$

Again, $s=1$, we get

$$2+5-4 = A\cdot 0 + B\cdot 0 + C\cdot(1+2)$$

$$\Rightarrow 3 = 3C$$

$$\therefore C = 1$$

$s=-2$, we get,

$$2(-2)^2+5(-2)-4 = A\cdot 0 + B(-2-1)\cdot(-2) + C\cdot 0$$

$$\Rightarrow 8-10-4 = 6B$$

$$\Rightarrow -6 = 6B$$

$$\therefore B = -1$$

Putting the value of A, B, C in eqn (1),

$$\frac{2s^2+5s-4}{s(s+2)(s-1)} = \frac{2}{s} + \frac{-1}{s+2} + \frac{1}{s-1}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{2s^2+5s-4}{s(s+2)(s-1)}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{2s^2+5s-4}{s(s+2)(s-1)}\right\} = 2 - e^{-2t} + e^{t}$$

(Ans).

Using Heaviside rules:

(i) $\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$

$P(s) = 3s+1$

$Q(s) = \cancel{s+1}\ (s-1)(s^2+1)$

$= s^3 + s - s^2 - 1$

$\therefore Q'(s) = 3s^2 - 2s + 1$

For zeros of $Q(s)$, $Q(s) = 0$

$\Rightarrow s^3 - s^2 + s - 1 = 0$

$\Rightarrow (s-1)(s^2+1) = 0$

$\therefore s = 1$ or, $s^2 = -1$

$\Rightarrow s = \pm i$

$\alpha_1 = 1$, $\alpha_2 = i$, $\alpha_3 = -i$

$P(\alpha_1) = 3+1 = 4$; $P(\alpha_2) = 3i+1$; $P(\alpha_3) = -3i+1$

$Q'(\alpha_1) = 3-2+1 = 2$; $Q'(\alpha_2) = 3i^2 - 2i + 1$

$= -3 - 2i + 1 = -2 - 2i$

$$Q'(\alpha_3) = 3i^2 + 2i + 1 = -3 + 2i + 1 = -2 + 2i$$

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = \sum_{k=1}^{3} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$$

$$= \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t} + \frac{P(\alpha_3)}{Q'(\alpha_3)} e^{\alpha_3 t}$$

$$= \frac{4}{2} e^{t} + \frac{3i+1}{-2-2i} e^{it} + \frac{-3i+1}{-2+2i} e^{-it}$$

(Ans)

ii) $$\mathcal{L}^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$$

$P(s) = 2s^2 - 6s + 5$

$Q(s) = s^3 - 6s^2 + 11s - 6$

$\therefore Q'(s) = 3s^2 - 12s + 11$

For zeros of $Q(s)$, $Q(s) = 0$

$\Rightarrow s^3 - 6s^2 + 11s - 6 = 0$

$\Rightarrow (s-1)(s-3)(s-2) = 0$

$\therefore s = 1, 2, 3$

$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 3$

$P(\alpha_1) = P(1) = 2 - 6 + 5 = 1$

$P(\alpha_2) = P(2) = 8 - 12 + 5 = 1$

$P(\alpha_3) = P(3) = 18 - 18 + 5 = 5$

$Q'(1) = 3 - 12 + 11 = 2$

$Q'(2) = 12 - 24 + 11 = -1$

$Q'(3) = 27 - 36 + 11 = 2$

$$\therefore L^{-1}\left\{\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}\right\} = \sum_{k=1}^{3} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$$

$$= \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t} + \frac{P(\alpha_3)}{Q'(\alpha_3)} e^{\alpha_3 t}$$

$$= \frac{1}{2} e^{t} + \frac{1}{-1} e^{2t} + \frac{5}{2} e^{3t}$$

$$= \frac{1}{2} e^{t} - e^{2t} + \frac{5}{2} e^{3t} \quad \text{(Ans).}$$

(iii) $\mathcal{L}^{-1}\left\{\dfrac{2s^2+5s-4}{s^3+s^2-2s}\right\}$.

$P(s) = 2s^2+5s-4$

$Q(s) = s^3+s^2-2s$

$\therefore Q'(s) = 3s^2+2s-2$

For zeros of $Q(s)$, $Q(s)=0$

$\Rightarrow s^3+s^2-2s=0$

$\Rightarrow s(s-1)(s+2)=0$

$\therefore s = 0, 1, -2$

$\therefore \alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = -2$

$P(\alpha_1) = P(0) = -4$

$P(\alpha_2) = P(1) = 2+5-4 = 3$

$P(\alpha_3) = P(-2) = 8-10-4 = -6$

$Q'(0) = -2$

$Q'(1) = 3+2-2 = 3$

$Q'(-2) = 12 - 4 - 2 = 6$

$\therefore \mathcal{L}^{-1}\left\{\dfrac{2s^2+5s-4}{s^3+s^2-2s}\right\} = \sum_{k=1}^{3} \dfrac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$

$$= \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t} + \frac{P(\alpha_3)}{Q'(\alpha_3)} e^{\alpha_3 t}$$

$$= \frac{-4}{-2} e^{0} + \frac{3}{3} e^{t} + \frac{-6}{6} e^{-2t}$$

$$= 2 + e^{t} - e^{-2t} \quad \text{(Ans).}$$

Convolution: Let, $F(t)$ and $G(t)$ be two functions that are ~~pro~~ piecewise continuous on every finite closed interval $0 \le t \le b$ and of exponential order.

Then, the convolution of the functions $F(t)$ and $G(t)$ is denoted by $F(t) * G(t)$ and is defined by.

$$F(t) \circledast G(t) = \int_0^t F(u)\, G(t-u)\, du$$

convolution operator

Properties: $F(t) * G(t) = G(t) * F(t)$

Theorem: $\mathcal{L}\{F(t) * G(t)\} = \mathcal{L}\{F(t)\} \cdot \mathcal{L}\{G(t)\}$

$$= f(s)\, g(s)$$

Convolution theorem: If $\mathcal{L}\{F(t)\} = f(s)$ and $\mathcal{L}\{G(t)\} = g(s)$ then,

$$\mathcal{L}^{-1}\{f(s).g(s)\} = F(t) * G(t)$$

$$= \int_0^t F(u)\, G(t-u)\, du$$

$$= \int_0^t G(u)\, F(t-u)\, du$$

where, $F(t) * G(t)$ is the convolution of the function $F(t)$ and $G(t)$

Ex-1: Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ using the convolution theorem.

$f(s) = \frac{1}{s}$

$g(s) = \frac{1}{s^2+1}$

$\mathcal{L}^{-1}\{f(s)\} = F(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathcal{L}^{-1}\{g(s)\} = G(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

By using convolution theorem,

$$\mathcal{L}^{-1}\{f(s)\, g(s)\} = F(t) * G(t)$$

$$= \int_0^t F(u)\, G(t-u)\, du$$

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t 1 \cdot \sin(t-u)\,du$$

$$= \left[\frac{-\cos(t-u)}{-1}\right]_0^t$$

$$= \cos(t-t) - \cos(t-0)$$

$$= \cos 0 - \cos t$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1-\cos t \text{ (Ans).}$$

$\boxed{E}$ Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\}$ by using convolution theorem

$f(s) = \frac{1}{s^2}$

$g(s) = \frac{1}{s^2+4}$

$\mathcal{L}^{-1}\{f(s)\} = F(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

$\mathcal{L}^{-1}\{g(s)\} = G(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{\sin 2t}{2}$

By using convolution theorem, we can write,

$u\int_0 v\,dv - \int \frac{vd}{d}$

$$\mathcal{L}^{-1}\{f(s)g(s)\} = F(t) * G(t)$$

$$= \int_0^t F(u)\, G(t-u)\,du$$

$$= \int_0^t u \, \frac{\sin 2(t-u)}{2}\, du$$

$$= \frac{1}{2}\int_0^t u \cdot \sin 2(t-u)\, du$$

$$= \frac{1}{2} \cancel{t \int_0^t \sin}$$

$$= \frac{1}{2}\left[\sin 2(t-u)\int_0^t u\,du - \int_0^t \left[\frac{d}{du}\sin 2(t-u) \int_0^t u\,du\right] du\right]$$

$$= \frac{1}{2}\left[\sin 2(t-u)\cdot\left[\frac{u^2}{2}\right]_0^t - \int_0^t 2\cos 2(t-u) \cdot \frac{u^2}{2}\,du\right]$$

$$= \frac{1}{2} \sin(2t-2u)\cdot\left(\frac{t^2}{2} - 0\right)$$

$$- \int \frac{\sin 2(t-u)}{2}$$

$$u \sin 2(t-u)$$

$$- \quad u^2 \int \cos 2(t-u)$$

$$= \int_0^t (t-u)\, \frac{\sin 2u}{2}\, du$$

$$= \frac{1}{2}\left[(t-u) \int_0^t \sin 2u \, du - \int_0^t \left[\frac{d}{du}(t-u) \int \sin 2u \, du\right] du\right]$$

$$= \frac{1}{2}\left[(t-u) \left[-\frac{\cos 2u}{2}\right]_0^t - \int_0^t -1 \cdot \left[\frac{-\cos 2u}{2}\right] du\right]$$

$$= \frac{1}{2}\left[(t-u)\left[-\frac{\cos 2t}{2} + \frac{\cos 0}{2}\right] - \frac{1}{2}\int_0^t \cos 2u \, du\right]$$

$$= \frac{1}{2}\left[(t-u)\left(\frac{1}{2} - \frac{\cos 2t}{2}\right) - \frac{1}{2}\left(\frac{\sin 2u}{2}\right)_0^t\right]$$

$$= \frac{1}{2}\left[(t-u)\,\frac{1}{2}\left(1 - \cos 2t\right) - \frac{1}{2}\left(\frac{\sin 2t}{2} - \frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{4}\left[(t-u)(1-\cos 2t) - \left(\frac{\sin 2t}{2} - 0\right)\right]$$

$$= \frac{1}{4}\left[(t-u)_0^t (1-\cos 2t) - \frac{\sin 2t}{2}\right]$$

$$= \frac{1}{4}t - \frac{1}{8}\sin 2t \quad \text{(Ans).}$$

$$(t-t) - (t-0)$$

$$- t(1-\cos 2t)$$

$$\int (t-u)\frac{\sin 2u}{2}\,du$$

$$= (t-u)\int \frac{\sin 2u}{2}\,du - \int\left\{\frac{d}{du}(t-u)\int \frac{\sin 2u}{2}\,du\right\}du$$

$$= (t-u)\,\frac{-\cos 2u}{4} - \int (-1)\left(-\frac{\cos 2u}{4}\right)du$$

$$= (t-u)\left(\frac{-\cos 2u}{4}\right) - \frac{\sin 2u}{8}$$

Now,

$$\int_0^t (t-u)\,\frac{\sin 2u}{2}\,du$$

$$= \left[(t-u)\left(\frac{-\cos 2u}{4}\right) - \frac{\sin 2u}{8}\right]_0^t$$

$$= \left[(t-t)\,\frac{-\cos 2t}{4} - \frac{\sin 2t}{8}\right]$$

$$- \left[(t-0)\left(\frac{-\cos 0}{4}\right) - \frac{\sin 0}{8}\right].$$

$$= -\frac{\sin 2t}{8} + \frac{1}{4}t$$

$$= \cancel{\frac{1}{4}t - \sin 2} \quad \frac{1}{4}t - \frac{1}{8}\sin 2t$$

Ex-3: Evaluate $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ using convolution theorem

$$f(s) = \frac{s}{s^2+a^2}$$

$$g(s) = \frac{1}{s^2+a^2}$$

$$\mathcal{L}^{-1}\{f(s)\} = F(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}$$

$$= \cos at$$

$$\mathcal{L}^{-1}\{g(s)\} = G(t) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^2+a^2}\right)\right\}$$

$$= \frac{\sin at}{a}$$

$$\mathcal{L}^{-1}\{f(s)\,g(s)\} = \int_0^t \cos au \cdot \frac{\sin(at - au)}{a}\,du$$

$$\therefore \int_0 \cos au \cdot \sin a(t-u)\,du$$

$$= \cos au \int_t \sin a(t-u)\,du - \int\left\{\frac{d}{du}(\cos au)\right.$$

$$\left.\int \sin a(t-u)\,du\right\}du$$

$$= \cos au \left\{ \frac{-\cos a(t-u)}{-a} \right\} - \int (-a\sin au) \left\{ \frac{-\cos a(t-u)}{-a} \right\} du$$

$$= \cos au \cdot \frac{\cos a(t-u)}{a} + \int \sin au \cos a(t-u)\, du$$

$$= \cos au \cdot \frac{\cos a(t-u)}{a} + \cos a(t-u) \int \sin au\, du$$

$$- \int \left\{ \frac{d}{du} \{\cos a(t-u)\} \int \sin au\, du \right\} du$$

$$= \cos au \cdot \frac{\cos a(t-u)}{a} + \cos a(t-u) \cancel{\sin}$$

$$\left(-\frac{\cos au}{a} \right) - \int a \sin a(t-u) \left(\frac{-\cos au}{-a} \right) du$$

$$= \frac{1}{2a} \int_0^t 2 \cos au \sin(at - au)\, du$$

$$= \frac{1}{2a} \int_0^t \{\sin(au + at - au) - \sin(au - at + au)\}\, du$$

$$= \frac{1}{2a} \int_0^t \{\sin at - \sin(2au - at)\}\, du$$

$$= \frac{1}{2a}\left\{\sin at \cdot [u]_0^t - \left[\frac{-\cos(2au-at)}{2a}\right]_0^t\right\}$$

$$= \frac{1}{2a}\left[\sin at\,(t-0) - \left[\frac{\cos(2at-at)}{2a} - \frac{\cos(-at)}{2a}\right]\right]$$

$$= \frac{1}{2a}\left[t\sin at - \frac{1}{2a}\left(\cos at - \cos at\right)\right]$$

$$= \frac{1}{2a}\, t\sin at \qquad \text{(Ans).}$$

Ex: 4: $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

$$f(s) = \frac{s}{s^2+a^2}$$

$$g(s) = \frac{s}{s^2+b^2}$$

$$\mathcal{L}^{-1}\{f(s)\} = F(t) = \cos at$$

$$\mathcal{L}^{-1}\{g(s)\} = G(t) = \cos bt$$

$$\mathcal{L}^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\, G(t-u)\,du$$

$$= \int_0^t \cos au \cdot \cos(bt-bu)\,du$$

$$= \frac{1}{2}\int_0^t 2\cos au \cdot \cos(bt - bu)\, du$$

$$= \frac{1}{2}\int_0^t \left[\cos(au + bt - bu) + \cos(au - bt + bu)\right] du$$

$$= \frac{1}{2}\left[\frac{\sin(au+bt-bu)}{a-b} + \frac{\sin(au-bt+bu)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\left\{\frac{\sin(at+bt-bt)}{a-b} + \frac{\sin(at-bt+bt)}{a+b}\right\} - \left\{\frac{\sin bt}{a-b} + \frac{\sin(-bt)}{a+b}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin(at \cancel{-2bt})}{a+b}\right] - \left[\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b} + \frac{1}{a+b}\right) - \sin bt\left[\frac{2a}{a^2-b^2}\right]\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{a+b+a-b}{a^2-b^2}\right) - \sin bt\left(\frac{2a}{a^2-b^2}\right)\right]$$

$$= \frac{1}{2}\cancel{\sin at}\left(\frac{2a}{a^2-b^2}\right)(\sin at - \sin bt)$$

$$= \frac{a}{a^2-b^2}\cancel{\sin at}\ \cancel{(Ans)}\,.(\sin at - \sin bt)$$

(*) Evaluate the followings using convolution theorem.

5. $\mathcal{L}^{-1}\left\{\frac{3}{s^2(s+2)}\right\}$ Ans: $\frac{3}{2}t+\frac{3}{4}e^{-2t}-\frac{3}{4}$

6. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$ Ans: $t e^{-t}+2e^{-t}+t-2$

5: $3\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\}$

$f(s)=\frac{1}{s^2}$

$g(s)=\frac{1}{s+2}$

$\mathcal{L}^{-1}\{f(s)\}=t$

$\mathcal{L}^{-1}\{g(s)\}=e^{-2t}$

$3\mathcal{L}^{-1}\{f(s)\,g(s)\}=3\int_0^t(t-u)\cdot e^{-2u}\,du$

$\int(t-u)\,e^{-2u}\,du=(t-u)\int e^{-2u}\,du-\int\left\{\frac{d}{du}(t-u)\int e^{-2u}\,du\right\}du$

$=(t-u)\left(\frac{e^{-2u}}{-2}\right)-\int(-1)\left(\frac{e^{-2u}}{-2}\right)du$

$$= -\cancel{8}(t-u)\frac{e^{-2u}}{-2} - \cancel{8}\frac{1}{2}\left(-\cancel{8}\frac{e^{-2u}}{-2}\right)$$

$$= -\cancel{8}(t-u)\frac{e^{-2u}}{2} + \cancel{8}\frac{e^{-2u}}{4}$$

$$3\int_0^t (t-u)e^{-2u}\,du = 3\left[-\cancel{8}(t-u)\frac{e^{-2u}}{2} + \cancel{8}\frac{e^{-2u}}{4}\right]_0^t$$

$$= 3\left[\left\{-\cancel{8}\frac{(t-t)e^{-2t}}{2} + \cancel{8}\frac{e^{-2t}}{4}\right\} - \left\{-\cancel{8}\frac{(t-0)}{2}e^{-2\cdot 0} + \cancel{8}\frac{e^{0}}{4}\right\}\right]$$

$$= 3\left\{\cancel{8}\frac{e^{-2t}}{4} - \left(-\cancel{8}\frac{t\,\cancel{e^{-2t}}}{2} + \frac{\cancel{4}}{4}\right)\right\}$$

$$= \cancel{3\left\{4e^{-2t} + 4te^{-2t} - 4\right)}$$

$$= \cancel{12}\;3\left(\frac{e^{-2t}}{4} + \frac{t}{2}\cancel{e^{-2t}} - \frac{1}{4}\right)$$

$$= \frac{3}{2}t + \frac{3}{4}e^{-2t} - \frac{3}{4} \quad \text{(Ans)}.$$

$$= (t-u)\frac{e^{-2u}}{-2} - \frac{1}{2}\left(\frac{e^{-2u}}{-2}\right)$$

$$= -(t-u)\frac{e^{-2u}}{2} + \frac{e^{-2u}}{4}$$

$$3\int_0^t (t-u)e^{-2u}\,du = 3\left[-(t-u)\frac{e^{-2u}}{2} + \frac{e^{-2u}}{4}\right]_0^t$$

$$= 3\left[\frac{e^{-2t}}{4} + (t-0)\frac{e^{0}}{2} - \frac{e^{0}}{4}\right]$$

$$= \frac{3}{4}e^{-2t} + \frac{3}{2}t - \frac{3}{4} \quad \text{(Ans)}.$$

6. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

$$f(s) = \frac{1}{s^2}$$

$$g(s) = \frac{1}{(s+1)^2}$$

$$\mathcal{L}^{-1}\{f(s)\} = t$$

$$\mathcal{L}^{-1}\{g(s)\} = e^{-t}\cdot\frac{t}{1!} = e^{-t}\cdot t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t ue^{-u}\cdot(t-u)\,du$$

$$\int u e^{-u} (t-u)\, du$$

$$= \int \left[u e^{-u} t - u^2 e^{-u}\right] du$$

$$= \int (ut - u^2) e^{-u}\, du$$

$$= (ut - u^2) \int e^{-u}\, du - \int \left\{ \frac{d}{du} (ut - u^2) \int e^{-u}\, du \right\} du$$

$$= (ut - u^2)(-e^{-u}) - \int (t-2u)(-e^{-u})\, du$$

$$= (ut - u^2)(-e^{-u}) + (t-2u) \int e^{-u}\, du$$

$$- \int \left\{ \frac{d}{du} (t-2u) \int e^{-u}\, du \right\} du$$

$$= (ut - u^2)(-e^{-u}) + (t-2u)(-e^{-u})$$

$$- \int (-2)(-e^{-u})\, du$$

$$= (ut - u^2)(-e^{-u}) - (t-2u) e^{-u} - 2(-e^{-u})$$

$$= (ut - u^2)(-e^{-u}) - (t-2u) e^{-u} + 2e^{-u}$$

$$= -u e^{-u} - \int e \quad du$$

$$- \left[-u e^{-u} + t e^{u}\right.$$

$$\therefore \int_0^t ue^{-u}(t-u)\,du$$

$$= \left[(ut-u^2)(-e^{-u}) - (t-2u)e^{-u} + 2e^{-u}\right]$$

$$= \left\{0 - (t-2t)e^{-t} + 2e^{-t}\right\}$$

$$- \left\{0 - (t-0)e^{0} + 2e^{0}\right\}$$

$$= te^{-t} + 2e^{-t} + t - 2 \quad \text{(Ans)}$$

$\int u e^{-u} (t-u)\, du$

$= (t-u)\int u e^{-u}\, du + \int\left\{\frac{d}{du}(t-u)\int u e^{-u} du\right\} du$

$= (t-u)\left[u\int e^{-u} du - \int e^{-u} du\right] - \int\left\{(-1)\int u e^{-u} du\right\} du$

$= (t-u)\left[-u e^{-u} + e^{-u}\right] + \int\left[-u e^{-u} + e^{-u}\right] du$

$= (t-u)\left[-u e^{-u} + e^{-u}\right] - \int u e^{-u} du + \int e^{-u} du$

$= (t-u)\left[-u e^{-u} + e^{-u}\right] - \left[-u e^{-u} + e^{-u}\right]$

$- e^{-u}$

$= (t-u)\left[-u e^{-u} + e^{-u}\right] + u e^{-u} - e^{-u} - e^{-u}$

$= (t-u)\left[-u e^{-u} + e^{-u}\right] + u e^{-u} - 2e^{-u}$

$\therefore \int_0^t u e^{-u} (t-u)\, du$

$= \Big[(t-u)\left[-u e^{-u} + e^{-u}\right] + u e^{-u} - 2e^{-u}\Big]_0^t$

$= -u e^{-u} - \int e^{-u} du$

$= -u e^{-u} + e^{-u}$

$$= \left[(t-t)\left[ue^{-u}+e^{-u}\right] + te^{-t} - 2e^{-t}\right]$$

$$-\left[(t-0)\left[0+e^{0}\right] + 0 - 2e^{0}\right]$$

$$= te^{-t} - 2e^{-t} - t + 2$$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} =$

$\int u e^{-u}(t-u)\,du = (t-u)\int u e^{-u}\,du - \int\left\{\frac{d}{du}(t-u)\int u e^{-u}\,du\right\}du$

$= (t-u)\left[u\int e^{-u}\,du - \int e^{-u}\,du\right] - \int\left\{-\int u e^{-u}\,du\right\}du$

$= (t-u)\left[-ue^{-u} - e^{-u}\right] + \int\left(-ue^{-u} - e^{-u}\right)du$

$= (t-u)\left[-ue^{-u} - e^{-u}\right] + ue^{-u} + e^{-u} + e^{-u}$

$\therefore \int_0^t u e^{-u}(t-u)\,du = \Big[(t-u)\left(-ue^{-u} - e^{-u}\right) + 2e^{-u} + ue^{-u}\Big]_0^t$

$= 2e^{-t} + te^{-t} - \left[(t-0)\left(0 - e^{0}\right) + 2e^{0}\right]$

$= te^{-t} + 2e^{-t} + t - 2$ (Ans).

$\int u e^{-u}\,du = u\int e^{-u}\,du - \int\frac{d}{du}(u)\int e^{-u}\,du$

$= -ue^{-u} - \int e^{-u}\,du$

$= -ue^{-u} + e^{-u}$ ...

Ex-01://

□ Application of laplace transform solving the IVP (Initial value problem)

Ex-01: $y'' - y' = t;\quad y(0) = 2,\ y'(0) = -3$

Taking laplace transform on bothsides we can write,

$$\mathcal{L}\{y''(t)\} - \mathcal{L}\{y'(t)\} = \mathcal{L}\{t\}$$

$$\Rightarrow s^2 y(s) - s\,y(0) - y'(0) - s\,y(s) + y(0) = \frac{1}{s^2} \quad [\because \mathcal{L}\{y(t)\} = y(s)]$$

$$\Rightarrow s^2 y(s) - 2s + 3 - s\,y(s) + 2 = \frac{1}{s^2}$$

$$\Rightarrow (s^2 - s)\,y(s) = \frac{1}{s^2} + 2s - 3 - 2$$

$$\Rightarrow y(s) = \frac{1 + 2s^3 - 5s}{s^2(s^2 - s)}$$

$$= \frac{1}{(s^2 - s)s^2} + \frac{2s}{s^2 - s} - \frac{5}{s^2 - s}$$

$$= \frac{1}{s^3(s-1)} + \frac{2s - 5}{s^2 - s}$$

Taking inverse Laplace of bothsides ~~of (i)~~

$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3(s-1)}\right\} + \mathcal{L}^{-1}\left\{\frac{2s-5}{s(s-1)}\right\}$

$\cancel{\Rightarrow y(t) =}$

let's consider, $\cancel{F(t) =}$

$$f(s) = \frac{1}{s-1}$$

$$\mathcal{L}^{-1}\{f(s)\} = e^t$$

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = \int_0^t e^u \, du$$

$$= \left[e^u\right]_0^t$$

$$= e^t - 1$$

Again, $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = \int_0^t (e^u - 1)\, du$

$$= \left[e^u - u\right]_0^t$$

$$= e^t - t - 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s-1)}\right\} = \int_0^t (e^u - u - 1)\, du$$

$$= \left[e^u - \frac{u^2}{2} - u\right]_0^t$$

$$= e^t - \frac{t^2}{2} - t - 1$$

Using partial fraction,

$$\frac{2s-5}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$$

$\Rightarrow 2s-5 = A(s-1)+Bs$

$\Rightarrow$ NOW, $s=1$,

$2-5 = B$

$\therefore B = -3$

Again, $s=0$,

$-5 = A(0-1)$

$A = 5$

$$\therefore \frac{2s-5}{s(s-1)} = \frac{5}{s} + \frac{-3}{s-1}$$

$$\therefore Y(t) = e^t - \frac{t^2}{2} - t - 1 + \cancel{\frac{5}{s} - \frac{3}{s-1}}$$

$$+ \mathcal{L}^{-1}\left\{\frac{5}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= e^t - \frac{t^2}{2} - t - 1 + 5 - 3e^t$$

$$= \cancel{4t - \frac{t^2}{2} - 2e^t - 1} \text{ (Ans)}.$$

$$= 4 - 2e^t - t - \frac{t^2}{2}$$

which is the required solution.

H.W

Solve the following IVP using laplace transform, $y'' - y' - 2y = 0$,

$$y(0) = 1,\ y'(0) = 0$$

Taking laplace transform on bothsides we can write,

$$\mathcal{L}\{y''(t)\} - \mathcal{L}\{y'(t)\} - 2\mathcal{L}\{y(t)\} = 0$$

$$\Rightarrow s^2 y(s) - s\,y(0) - y'(0) - s\,y(s) + y(0) - 2y(s) = 0$$

$$\Rightarrow (s^2 - s - 2)\,y(s) - s - 0 + 1 = 0$$

$$\Rightarrow y(s) = \frac{s - 1/2 - 1/2}{s^2 - s - 2}$$

$$= \frac{s - 1/2 - 1/2}{s^2 - 2s\cdot\frac{1}{2} + \frac{1}{2}^2 - \frac{1}{2}^2 - 2}$$

$$= \frac{s - 1/2 - 1/2}{(s - \frac{1}{2})^2 - 9/4}$$

$$= \frac{s-1}{s^2 - s - 2}$$

$$= \frac{s-1}{(s-2)(s+1)}$$

$-\frac{1}{4} - 2$

$-\left(\frac{1}{4} + 2\right)$

$-\left(\frac{1+8}{4}\right)$

$= \frac{9}{4}$

$s - 2 +$

$\frac{1}{s-2} + \frac{1}{s+1}$

$s+1 + s+2$

Taking inverse laplace of bothsides

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{s - 1/2 - 1/2}{(s - \frac{1}{2})^2 - (\frac{3}{2})^2}\right\}$$

$$\therefore y(t) = \cancel{e^t \cos\sqrt{3}\,t} \text{ ~~(Ans)~~ } e^{\frac{1}{2}t}\cosh\frac{3}{2}t - \frac{1}{3}\sinh\frac{3}{2}t \text{ (Ans)}$$

Property:

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{F(t)\}$$

$$= (-1)^n \frac{d^n}{ds^n} f(s)$$

$F(x, y)$

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

Steps:

Step-1: $y'(s)$ এর আগে কিছু থাকবে না।

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$IF: e^{\int P(x)\,dx}$$

Taking inverse laplace;

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s-2} + \frac{B}{s+1}\right\}$$

$$\Rightarrow y(t) = A e^{2t} + B e^{-t}$$

$$= \frac{1}{2} e^{2t} + \frac{2}{3} e^{-t}$$

$s-1 = A(s+1) + B(s-2)$

$-2 = -3B$

$\therefore B = \frac{2}{3}$

$1 = 2A$

$A = \frac{1}{2}$

Solving ODE with variable co-efficient using laplace transform.

Ex: Solve the following BVP (Boundary value problem)

$$tY'' + (t-1)Y' - Y = 0, \quad Y(0) = 5, \; Y(\infty) = 0$$

Taking laplace transform

$$\mathcal{L}\{tY''\} + \mathcal{L}\{tY'\} - \mathcal{L}\{Y'\} - \mathcal{L}\{Y\} = \mathcal{L}\{0\}$$

$$\Rightarrow (-1)^1 \frac{d}{ds}\mathcal{L}\{Y''(t)\} + (-1)\frac{d}{ds}\mathcal{L}\{Y'(t)\}$$
$$- \mathcal{L}\{Y'(t)\} - y(s) = 0$$

$$\Rightarrow -\frac{d}{ds}\mathcal{L}\{Y''(t)\} - \frac{d}{ds}\mathcal{L}\{Y'(t)\} - s\,y(s) + Y(0) - y(s) = 0$$

$$\Rightarrow -\frac{d}{ds}\{s^2 y(s) - sY(0) - \underbrace{Y'(0)}_{\text{constant}}\} - \frac{d}{ds}\{s\,y(s) - Y(0)\} - s\,y(s) + Y(0) - y(s) = 0$$

$$\Rightarrow -\frac{d}{ds}\{s^2 y(s) - 5s - c\} - \frac{d}{ds}\{s\,y(s) - 5\} - s\,y(s) - 5 - y(s) = 0$$

[Let, $Y'(0) = c$]

$$\Rightarrow -s^2 y'(s) - 2s\,y(s) + 5 - s\,y'(s) - y(s)$$
$$- s\,y(s) + 5 - y(s) = 0$$

$$\Rightarrow (-s^2 - s)\,y'(s) - 3s\,y(s) - 2y(s) + 10 = 0$$

$$\Rightarrow (-s^2 - s)\,y'(s) - (3s+2)\,y(s) + 10 = 0$$

$$\Rightarrow (s^2+s)\,y'(s) + (3s+2)\,y(s) - 10 = 0$$

$$\Rightarrow y'(s) + \frac{3s+2}{s^2+s}\,y(s) = \frac{10}{s^2+s}$$

$\longrightarrow$ 1st order ODE

$$\Rightarrow y'(s) + \frac{(2s+1)+(s+1)}{s(s+1)}\,y(s) = \frac{10}{s^2+s}$$

$$\Rightarrow y'(s) + \left\{\frac{2s+1}{s^2+s} + \frac{1}{s}\right\} y(s) = \frac{10}{s^2+s} \quad \text{--- (2)}$$

$\therefore$ Integrating factor, $IF = e^{\int\left(\frac{2s+1}{s^2+s} + \frac{1}{s}\right) ds}$

$$\Rightarrow IF = e^{\ln(s^2+s) + \ln s}$$

$$= e^{\ln\{s(s^2+s)\}}$$

$$= s(s^2+s)$$

$$IF = s^2(s+1)$$

Multiplying bothsides of eqn (2) by

$s^2(s+1)$, we get,

$$y'(s) \cdot s^2(s+1) + y(s)\left[\frac{(2s+1)+s+1}{s(s+1)}\right] s^2(s+1)$$

$$= \frac{10}{s^2+s} \cdot s^2(s+1)$$

$$\Rightarrow y'(s)(s^3+s^2) + y(s)(3s^2+2s) = 10s$$

$$\Rightarrow d\left[y(s)(s^3+s^2)\right] = 10s$$

Integrating bothsides we can write,

$$y(s)(s^3+s^2) = \int 10s \, ds$$

$$\Rightarrow y(s)(s^3+s^2) = 5s^2 + c_1$$

$$\Rightarrow y(s) = \frac{5s^2}{s^3+s^2} + \frac{c_1}{s^3+s^2}$$

$$\Rightarrow y(s) = \frac{5}{s+1} + \frac{c_1}{s^2(s+1)}$$

Now taking inverse laplace on bothsides,

$$\mathcal{L}^{-1}\{y(s)\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + c_1 \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$$

$$\Rightarrow y(t) = 5e^{-t} + c_1 \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}\right\}$$

$$= 5e^{-t} + c_1\left[A + Bt + Ce^{-t}\right]$$

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$\Rightarrow \quad 1 = As(s+1) + B(s+1) + Cs^2$

$s = 0, \quad 1 = B$

$s = -1, \quad 1 = \cancel{A}\, C$

$0 = (A + C)$

$\Rightarrow A = -1$

$\therefore y(t) = 5e^{-t} + c_1\left[-1 + t + e^{-t}\right]$

Now, applying the condition $y(\infty) = 0$

$y(\infty) = 5e^{-\infty} + c_1\left[-1 + \infty + e^{-\infty}\right]$

$\Rightarrow 0 = c_1(\infty)$

$\Rightarrow c_1 = \frac{0}{\infty} = 0$

$y(t) = 5e^{-t}$ (Ans).

H.W]

$$y'' + y' + y = 1; \quad y(0) = 1, \quad y'(0) = 2$$

by using laplace transform.

Taking laplace transform,

$$\Rightarrow \mathcal{L}\{y''(t)\} + \mathcal{L}\{y'(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{1\}$$

$$\Rightarrow s^2 y(s) - s y(0) - y'(0) + s y(s) - y(0) + y(s) = \frac{1}{s}$$

$$\Rightarrow (s^2 + s + 1)\, y(s) - s - 2 - 1 = \frac{1}{s}$$

$$\Rightarrow (s^2+s+1)\, y(s) = \frac{1}{s} + s + 2 + 1$$

$$\Rightarrow (s^2+s+1)\, y(s) = \frac{1 + s^2 + 2s + s}{s}$$

$$\Rightarrow y(s) = \frac{1 + s^2 + 3s}{s(s^2+s+1)}$$

Taking inverse laplace transform,

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{1+s^2+3s}{s(s^2+s+1)}\right\}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s^2+s+1+2s}{s(s^2+s+1)}\right\}$$

$1-\frac{1}{4}=\frac{4-1}{4}=\frac{3}{4}$

W.H

$$\Rightarrow y(t)=\mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s(s^2+s+1)}+\frac{2s}{s(s^2+s+1)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{2}{s^2+s+1}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{2}{s^2+2\cdot s\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+2\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2-\frac{1}{4}+1}\right\}$$

$$=1+2\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$=1+2e^{-1/2t}\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}$$

$$=1+\frac{4}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$

(Ans).

Fourier Transform

Periodic function: The function $f(x)$ of real variable x is said to be periodic if there exists a number $P > 0$ independent of x such that, $f(P+x) = f(x)$ holds for all values of x. The least value of $P > 0$ is called the period of $f(x)$.

As for example $f(x) = \sin x$ and $f(x) = \cos x$ are periodic function having period 2π.

$f(x) = \tan x$ is a periodic function having period π.

Note: $f(\underbrace{\boxed{2\pi}}_{\text{period}} + x) = \underbrace{\sin x}_{\text{function}}$

Real form of Fourier series: The trigonometric series, $f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$ is a fourier series if its coefficient a_0, a_n and b_n are given by the following formulas.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx \quad (n = 1, 2, 3, \ldots)$$

where $f(x)$ is any real valued function defined on the interval $(-\pi, \pi)$

<u>Note:</u> $\frac{n\pi x}{L}$; $\pi = L$; $\cos nx$

$f(-x) = f(x) \rightarrow b_n$ থাকবে না।
↓
even

$f(-x) = -f(x) \rightarrow \boxed{a_0, a_n} = 0$
↓
odd

Alternative form: The trigonometric series,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

is a Fourier series if its co-efficients a_0, a_n and b_n are given by the following formulas,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

where $f(x)$ is any real valued ~~fourier~~ function defined on the interval $(-L, L)$

The Fourier cosine series: A function $f(x)$ is called even function if $f(-x) = f(x)$

We know that,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

Here, $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$= \frac{1}{2L}\cdot 2\int_{0}^{L} f(x)\,dx$ [$\because f(x)$ is even]

$= \frac{1}{L}\int_{0}^{L} f(x)\,dx$

$$\boxed{\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx; & \text{when } f(x) \text{ is even} \\ 0, & \text{when } f(x) \text{ is odd} \end{cases}}$$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

$= \frac{1}{L}\int_{L}^{-L} f(-x)\cos\left(-\frac{n\pi x}{L}\right)d(-x)$

[Replacing x by $-x$]

$= \frac{1}{L}\int_{x=L}^{-L} f(x)\cos\frac{n\pi x}{L}(-)\,dx$

$= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_{L}^{-L} f(-x) \sin\left(-\frac{n\pi x}{L}\right) d(-x)$$

$$= \frac{1}{L}\int_{L}^{-L} f(x) \left(-\sin\frac{n\pi x}{L}\right)(-)\,dx$$

$$= \frac{1}{L}\int_{L}^{-L} f(x) \sin\frac{n\pi x}{L}\,dx$$

$$= -\frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\,dx$$

$$\cancel{b_n = -\frac{2}{L}\int_{0}^{L} f(x) \sin\frac{n\pi x}{L}\,dx}$$

$$b_n = -b_n$$

$$\Rightarrow 2b_n = 0$$

$$\Rightarrow b_n = 0$$

Fourier sine series: We know the Fourier series of a function $f(x)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

where, $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

if $f(x)$ is an odd function, then,

$f(-x) = -f(x)$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$= 0$ [since $f(x)$ is an odd function]

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

$= \frac{1}{L}\int_{L}^{-L} f(-x)\left(\cos\frac{-n\pi x}{L}\right)d(-x)$

[replacing x by $-x$]

$$= \frac{1}{L}\int_{L}^{-L} (-)\, f(x) \cos\frac{n\pi x}{L} \cancel{dx}\, (-)\, dx$$

$$= -\frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\, dx$$

$$= -a_n$$

$$\Rightarrow 2a_n = 0$$

$$\Rightarrow a_n = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{L}\int_{L}^{-L} f(-x) \sin\left(\frac{-n\pi x}{L}\right) d(-x)$$

$$= \frac{1}{L}\int_{L}^{-L} -f(x) \left(-\sin\frac{n\pi x}{L}\right)(-)\, dx$$

$$= -\frac{1}{L}\int_{L}^{-L} f(x) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\, dx$$

Ex-01: The function x^2 is periodic with period $2l$ on the interval $[-l, l]$. Find its fourier series.

Soln: Given the function is,

$$f(x) = x^2$$

Since the function x^2 is an even function

$\therefore$ its fourier series is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l} \int_0^l f(x)\,dx = \frac{1}{l} \int_0^l x^2\,dx$$

$$= \frac{1}{l} \left[\frac{x^3}{3}\right]_0^l = \frac{1}{l}\left[\frac{l^3}{3} - 0\right]$$

$$a_0 = \frac{l^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\frac{l}{n\pi}x^2 \sin\frac{n\pi x}{l}\right]_0^l - \frac{2}{l}\int_0^l 2x\,\frac{l}{n\pi}\sin\frac{n\pi x}{l}\,dx$$

$$= 0 - \frac{4}{n\pi}\left[\left[-x\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right]_0^l + \int_0^l \cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\,dx\right]$$

$$= \frac{4l}{n^2\pi^2}\left[l\cos n\pi\right] - \frac{4l^2}{n^3\pi^3}\left[\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n - 0$$

$n = \text{odd}, -1$

$n = \text{even}, 1$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n$$

$$f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{l}$$

$\boxed{H/W}$ Find the fourier series expansion of the function $f(x) = x^2$ in the interval $-\pi \le x \le \pi$, and hence evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^\pi x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^\pi$$

$$= \frac{1}{\pi}\cdot\frac{\pi^3}{3}$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

$$\therefore \int f(x)\cos nx\,dx = \int x^2 \cos nx\,dx$$

$$= x^2 \int \cos nx \, dx - \int \left\{ \frac{d}{dx}(x^2) \int \cos nx \, dx \right\} dx$$

$$= x^2 \frac{\sin nx}{n} - \int 2x \frac{\sin nx}{n} dx$$

$$= x^2 \frac{\sin nx}{n} - \frac{2}{n} \left[x \int \sin nx \, dx - \int \left\{ \frac{d}{dx}(x) \int \sin nx \, dx \right\} dx \right]$$

$$= x^2 \frac{\sin nx}{n} - \frac{2}{n} \left[x \left(\frac{-\cos nx}{n} \right) + \int \cos \frac{nx}{n} dx \right]$$

$$= \left[\frac{x^2}{n} \sin nx + \frac{2x}{n^2} \cos nx - \frac{\sin nx}{n^2} \right.$$

$$a_n = \frac{2}{\pi} \left[\frac{x^2}{n} \sin nx + \frac{2x}{n^2} \cos nx - \frac{\sin nx}{n^2} \right]_0^\pi$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{n} \sin n\pi + \frac{2\pi}{n^2} \cos n\pi - \frac{\sin n\pi}{n^2} \right]$$

$$= \frac{2}{\pi} \cdot \frac{2\pi}{n^2} \cos n\pi$$

$$= \frac{4}{n^2} (-1)^n$$

$f(x) = x^2$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$f(x) = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \frac{1}{4^2}\cos 4x + \cdots\right]$$

$$\Rightarrow f(0) = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\Rightarrow 0 = \frac{\pi^2}{3} - 4\left[\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) - 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \cdots\right)\right]$$

$$\Rightarrow 0 = \frac{\pi^2}{3} - 4\left[\sum_{n=1}^{\infty}\frac{1}{n^2} - 2\cdot\frac{1}{2^2}\left(\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)\right]$$

$$\Rightarrow 0 = \frac{\pi^2}{3} - 4\left[\sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\right]$$

$$= \frac{\pi^2}{3} - 4\cdot\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{(Ans).}$$

Complex form of Fourier series:

We know that,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \text{ and } \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

The real form of fourier series is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right],$$

$$[-L<x<L]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}\right) + \frac{b_n}{2i}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}\right) - \frac{ib_n}{2}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[e^{\frac{in\pi x}{L}}\left(\frac{a_n}{2} - \frac{ib_n}{2}\right) + e^{-\frac{in\pi x}{L}}\left(\frac{a_n}{2} + \frac{ib_n}{2}\right)\right]$$

$$\Rightarrow f(x) = c_0 + \sum_{n=1}^{\infty} \left[c_n e^{\frac{in\pi x}{L}} + c_{-n} e^{\frac{in\pi x}{L}} \right]$$

where, $c_0 = a_0$; $c_n = \frac{1}{2}(a_n - ib_n)$

and $c_{-n} = \frac{1}{2}(a_n + ib_n)$

This eqn is known as the complex form of the Fourier series.

This eqn also be written as,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}};$$

where, $c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} dx$;

$n = 0, \pm 1, \pm 2, \ldots$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$2\cos\theta = e^{i\theta} + e^{-i\theta}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Example: (a) Obtain the Fourier series of the function

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 \le x \le \pi \end{cases}$$

(b) Verify the result found in (a) by assuming the complex form of the Fourier series.

Soln: We know that, real form of Fourier Series is:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right], \quad -\pi \le x \le \pi$$

Here, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= 0 + \frac{1}{2\pi}\int_{0}^{\pi} dx = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_{0}^{\pi}\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = \frac{1}{\pi}\cdot\frac{\sin n\pi}{n} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_{0}^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{1}{n\pi}\left[-\cos n\pi + \cos 0\right]$$

$$= \frac{1}{n\pi}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 & \text{when } n \text{ is even} \\ \frac{2}{n\pi} & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\left[1-(-1)^n\right]\sin nx$$

$$= \frac{1}{2} + \left[\frac{2}{\pi}\sin x + 0 + \frac{2}{3\pi}\sin 3x + 0 + \cdots\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \cdots\right]$$

2nd part

Complex form:

We know that, $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$; $-\pi < x < \pi$

where, $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{0} f(x)\, dx + \frac{1}{2\pi} \int_{0}^{\pi} f(x)\, dx$$

$$= 0 + \frac{1}{2\pi} \int_0^{\pi} dx = \frac{1}{2}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \quad (n \neq 0)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{0} f(x) e^{-inx} dx + \frac{1}{2\pi} \int_{0}^{\pi} f(x) e^{-inx} dx$$

$$= 0 + \frac{1}{2\pi} \int_0^{\pi} e^{-inx} dx$$

$$= \frac{1}{i2\pi n} \left[1 - e^{-in\pi} \right]_0^{\pi}$$

$$= \frac{1}{i2\pi n}\left[-e^{-in\pi} + e^{0}\right]$$

$$= \frac{1}{i2\pi n}\left[-\cos n\pi + 1\right]$$

$$= \frac{1}{i2\pi n}\left[1-(-1)^{n}\right]$$

$$= \begin{cases} \frac{1}{n\pi i}, & \text{where } n = \pm 1, \pm 3, \dots \\ 0, & \text{when } n = \pm 2, \pm 4, \dots \end{cases}$$

Then,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$= c_0 + \sum_{n=-\infty}^{\infty} \frac{1}{i2\pi n}\left[1-(-1)^{n}\right] e^{-inx}$$

$$= \frac{1}{2} + \frac{1}{\pi i}\left(\frac{e^{ix}}{1} + 0 + \frac{e^{i3x}}{3} + 0 + \frac{e^{i5x}}{5} + \dots\right)$$

$$+ \frac{1}{\pi i}\left(\frac{e^{ix}}{-1} + 0 + \frac{e^{-i3x}}{-3} + 0 + \frac{e^{-i5x}}{-5} + \dots\right)$$

$$= \frac{1}{2} + \frac{1}{\pi i}\left(e^{ix} - e^{-ix}\right) + \frac{1}{3\pi i}\left(e^{i3x} - e^{-i3x}\right)$$

$$+ \frac{1}{5\pi i}\left(e^{i5x} - e^{-i5x}\right) + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \cdots$$

$$\therefore f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

which coincides with the result obtain from (a) [verified]

$$\sin\theta = \frac{e^{i\theta} - \bar{e}^{i\theta}}{2i}$$

$$\Rightarrow \frac{e^{i\theta} - \bar{e}^{i\theta}}{i} = 2\sin\theta$$

(*) Dirichlet condition for the existance of a fourier series of a function:

Any function $f(x)$ can be developed as a fourier series if:

1. $f(x)$ is finite single valued with a finite number of discontinuity.

2. $f(x)$ has a finite number of extrema in $[-L, L]$

   min or max value থাকবে

3. $\int_0^L |f(x)|\, dx$ exists.

4. $f(x)$ is $2L$ periodic.

Theorem: let, $f(x)$ be a function such that,

1. $f(x)$ is periodic of period $2L$ and.
2. $f(x)$ is piecewise smooth on the interval $-L \leq x \leq L$

then the trigonometric Fourier series of $f(x)$, converges at every point x to the value $f(x+) + f(x-)$ where $f(x+)$ is the right hand limit of $f(x)$ at x and $f(x-)$ is the left hand limit of $f(x)$ at x, (if x is the point of discontinuity). In particular, if f is also continuous at x, the value $\dfrac{f(x+)+f(x-)}{2}$ reduces to $f(x)$ and the trigonometric Fourier series of $f(x)$ at x converges to $f(x)$.

Ex: Consider the function;

$$f(x) = \begin{cases} \pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Find the trigonometric Fourier series of the function $f(x)$ and hence evaluate

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Soln: The Fourier series is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where, $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx \right]$$

$$= \frac{1}{2\pi} \left[\pi [x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi} \right]$$

$$= \frac{1}{2\pi} \left[\pi (0+\pi) + \frac{\pi^2}{2} \right]$$

$$= \frac{1}{2\pi} \left[\pi^2 + \frac{\pi^2}{2} \right] = \frac{1}{2\pi} \cdot \frac{3\pi^2}{2}$$

$$a_0 = \frac{3\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \pi \cos nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} x \cos nx \, dx$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} \frac{-2}{n^2\pi}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= -\frac{1}{n}$$

Thus, the Fourier series is,

$$f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nx - \frac{1}{n}\sin nx\right]$$

Now putting $x = 0$ on bothsides we can write,

$$f(0) = \frac{3\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2}$$

where, $f(0) = \frac{f(0^+) + f(0^-)}{2}$

$$= \frac{0+\pi}{2} = \frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2} = \frac{3\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2}$$

$$\Rightarrow -\frac{\pi}{4} = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$ when n is odd

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$ (Ans).

and cosine

(*) Finite Fourier Sine transform:

The finite Fourier sine transform of $F(x)$ in the interval $0 < x < l$, is defined as,

$$f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{l} dx \text{ ; where } n \text{ is an integer.}$$

(*) The finite Fourier cosine transform:

$$f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{l} dx; \quad n \text{ is an integer.}$$

(*) Inverse finite Fourier sine and cosine transform:

Inverse finite Fourier sine transform:

$$F(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{l}$$

Inverse finite Fourier cosine transform:

$$F(x) = \frac{1}{l} f_c(0) \cancel{F(x)} + \frac{2}{l} \sum_{n=1}^{\infty} f_c(n) \cos \frac{n\pi x}{l}$$

Final -এ লাগবে ~~না~~।

Infinite Fourier sine and cosine transform:

Infinite sine transform:

$$f_s(n) = \int_0^\infty F(x) \sin nx \, dx$$

Infinite cosine transform:

$$f_c(n) = \int_0^\infty F(x) \cos nx \, dx$$

Inverse Infinite sine transform:

$$F(x) = \frac{2}{\pi} \int_0^\infty f_s(n) \sin nx \, dn$$

" " cosine " :

$$F(x) = \frac{2}{\pi} \int_0^\infty f_c(n) \cos nx \, dn$$

Alternative forms of Fourier transform:

If $F(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} dx$ then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} du$$

where, $F(u) = \mathcal{F}\{f(x)\}$

$\mathcal{F}$ → Fourier transform symbol.

(18-20) marks

Ex: Use Finite Fourier sine transform to solve:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \rightarrow \text{(Heat equation)}$$

$\boxed{u(x,t)=?}$

subject to the conditions:

$u(0,t)=0, \quad u(4,t)=0$

$u(x,0)=2x$ where $0<x<4, \; t>0$

and interpret physically.

Soln: The given differential eqn is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

Taking the finite fourier sine transform with $(l=4)$ of bothsides of (1) we get,

$$\int_0^4 \frac{\partial u}{\partial t} \sin\frac{n\pi x}{4}\,dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{4}\,dx \quad \text{--- (2)}$$

Let, $u = u(n,t) = \int_0^4 u(x,t) \sin\frac{n\pi x}{4}\,dx \quad \text{--- (3)}$

$$\Rightarrow \frac{du}{dt} = \int_0^4 \frac{\partial u}{\partial t} \sin\frac{n\pi x}{4}\,dx$$

$$\Rightarrow \frac{du}{dt} = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi x}{4} \, dx$$

$$\Rightarrow \frac{du}{dt} = \left[\sin \frac{n\pi x}{4} \cdot \frac{\partial u}{\partial x} \right]_0^4$$

$$- \frac{n\pi}{4} \int \cos \frac{n\pi x}{4} \cdot \frac{\partial u}{\partial x} \, dx$$

$$= 0 - \frac{n\pi}{4} \int \cos \frac{n\pi x}{4} \cdot \frac{\partial u}{\partial x} \, dx$$

$$= - \frac{n\pi}{4} \left[\cos \frac{n\pi x}{4} \cdot u(x,t) \right]_0^4$$

$$- \frac{n^2\pi^2}{16} \int_0^4 u(x,t) \sin \frac{n\pi x}{4} \, dx$$

$$= 0 - \frac{n^2\pi^2}{16} u$$

$$\Rightarrow \frac{du}{dt} = - \frac{n^2\pi^2}{16} u$$

$$\Rightarrow \frac{du}{u} = - \frac{n^2\pi^2}{16} dt$$

\# Integrating bothsides we get,

$$\ln u = - \frac{n^2\pi^2}{16} t + \ln A$$

$\Rightarrow \ln u - \ln A = \frac{-n^2\pi^2}{16} t$

$\Rightarrow \ln\left(\frac{u}{A}\right) = -\frac{n^2\pi^2}{16} t$

$\Rightarrow u = A e^{-\frac{n^2\pi^2}{16} t}$

$\Rightarrow u(n,t) = A e^{-\frac{n^2\pi^2}{16} t}$

$\Rightarrow u(n,0) = A e^{0} = A$

Again, $u(n,0) = \int_0^4 u(x,0) \sin \frac{n\pi x}{4}\, dx$

$\therefore A = \int_0^4 2x \sin \frac{n\pi x}{4}\, dx$

$= \left[2x \cdot \frac{4}{n\pi}\left(-\cos \frac{n\pi x}{4}\right) - \frac{8}{n\pi} \cdot \frac{4}{n\pi} \left(-\sin \frac{n\pi x}{4}\right)\right]_0^4$

$= -2 \cdot 4 \cdot \frac{4}{n\pi} \cos n\pi - 0$

$= -\frac{32}{n\pi} \cos n\pi$

$\therefore\ u(n,t) = -\frac{32}{n\pi}\cos n\pi\, e^{-\frac{n^2\pi^2}{16}t}$

Now applying inversion formula for finite Fourier sine transform. we get,

$$U(x,t) = \frac{2}{4}\sum_{n=1}^{\infty} -\frac{32}{n\pi}\cos n\pi\, e^{-\frac{n^2\pi^2}{16}t}\sin\frac{n\pi x}{4}$$

$$= \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\, e^{-\frac{n^2\pi^2}{16}t}\sin\frac{n\pi x}{4}$$

☐ Solve the integral equation,

$$\int_0^\infty F(x)\cos ux\, dx = \begin{cases} 1-u, & 0\leq u\leq 1 \\ 0, & u>1 \end{cases}$$

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FAILURE
IS NOT THE
OPPOSITE
OF SUCCESS.
IT'S PART OF I

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Class ____________ Section ________ Roll No. ________
Subject ____________________ Year ________

Sec-A

In the equation, the derivative or differential is involved is called the differential equation.

An equation involving derivatives or differential of one or more dependent variable with respect to one or more independent variable is called differential equation.

derivative or differential co-oficient $= \frac{dy}{dx}$

dx or dy is called differential.

Δx = small change $= \delta x$

Example: $\frac{dy}{dx} - xy = 0$

$(x+y)\,dx + (y+x)\,dy = 0$

(i) $\frac{dy}{dx} = ky$ (Exponential growth)

(ii) $\frac{dy}{dx} = k(A - y)$ (Newton's law of cooling)

(iii) $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = \underset{\text{weight}}{f(t)}$

(Mechanical vibrations)

DE

$\downarrow$

ODE $\qquad\qquad$ PDE

<u>ODE:</u> An equation involving derivatives or differential of one dependent variable with respect to one independent variable is called ordinary differential equation.

$y = x^2$

$\boxed{\frac{dy}{dx} = 2x} \rightarrow$ ODE

PDE: An equation involving derivatives or differential of one dependent variable with respect to one or more independent variable is called partial D.E.

$$z = f(x, y)$$

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}$$

Example: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

ODE

Linear (straight line)

Non-linear (curve)

Non-linear: (1) In DE dependent variable product $y \cdot y \sim y^2$

(2) Derivatives of dependent variable product $\frac{dy}{dx} \cdot \frac{dy}{dx}$ or $\left(\frac{dy}{dx}\right)^2$

③ Product of dependent variable and derivative $y\frac{dy}{dx}$

④ Transcendental equation e^y, $\sin y$, $\cos y$ (dependent variable)

$$\frac{dy}{dx} - y^2 = 0$$

$$\left(\frac{dy}{dx}\right)^2 - y = 0$$

$$y\frac{dy}{dx} - y = 0$$

$$\frac{dy}{dx} - \cos y = 0$$

$$\frac{dy}{dx} - e^y = 0$$

→ Non-linear ODE

<u>Order</u>: The order of the differential equation is the order of ~~the high~~ highest derivative involved.

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0$$

Degree: The degree of the differential equation is the power (degree) of the highest derivative after the equation has ~~beeen~~ been made rational.

$$\sin\left(\frac{dy}{dx}\right) + x = 0$$

$$\Rightarrow \sin\left(\frac{dy}{dx}\right) = -x$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1}(-x)$$

$$\frac{dy}{dx} + \sin^{-1} x = 0$$

$$\frac{d^2y}{dx^2} = \frac{m}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{d^2y}{dx^2}\right)^2 = \frac{m^2}{H^2}\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^{2} - \frac{m^2}{H^2}\left(\frac{dy}{dx}\right)^2 = \frac{m^2}{H^2}$$

degree

order

constant = order = differentiate number

Formation of differential equation:

$y = e^{mx}(A\cos nx + B\sin nx)$ —— ①

constant = 2   order = 2

by eliminating arbitrary constant A and B

Now differentiating eqn ① w.r.to x

$\frac{dy}{dx} = e^{mx} \cdot \frac{d}{dx}(A\cos nx + B\sin nx)$

$+ (A\cos nx + B\sin nx) \frac{d}{dx}(e^{mx})$

$= e^{mx}(-nA\sin nx + nB\cos nx)$

$+ (A\cos nx + B\sin nx)\, me^{mx}$

$= e^{mx}(-nA\sin nx + nB\cos nx)$

$+ my$ [Using eqn ①] —— ②

$\Rightarrow \frac{d^2y}{dx^2} = e^{mx}[-n^2A\cos nx - n^2B\sin nx]$

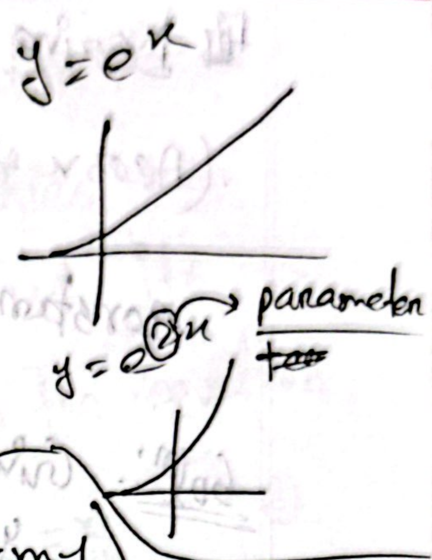
$+ m \cdot \frac{dy}{dx} + (-nA\sin nx$

$+ nB\cos nx) \cdot me^{mx}$

$$\Rightarrow \frac{d^2y}{dx^2} = -n^2y + m\frac{dy}{dx}$$

[using eqn (1)]

$$\Rightarrow \quad + me^{mx}(-nA\sin nx + nB\cos nx)$$



$$\Rightarrow \frac{d^2y}{dx^2} = -n^2y + m\frac{dy}{dx} + m\left(\frac{dy}{dx} - my\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -y(n^2+m^2) + 2m\frac{dy}{dx}$$

$$\therefore \frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + y(n^2+m^2) = 0$$

Working Value

(1) Put the given eqn in number
(2) differentiate upto n times if the number of arbitrary constant n.
(3) Eliminating arbitrary constant from $(n+1)$ equations.

Derive a differential equation from $y = e^x (A\cos x + B\sin x)$ by eliminating arbitrary constant A & B.

Soln: Given that,

$$y = e^x (A\cos x + B\sin x) \quad \text{——— (1)}$$

Differentiate eqn (1) w.r.to x,

$$\frac{dy}{dx} = e^x(-A\sin x + B\cos x) + (A\cos x + B\sin x)e^x$$

$$\Rightarrow \frac{dy}{dx} = e^x(-A\sin x + B\cos x) + y \quad [\text{using (1)}] \quad \text{——— (11)}$$

$\Rightarrow$ Again differentiate eqn (1) w.r.to x,

$$\frac{d^2y}{dx^2} = e^x(-A\cos x - B\sin x) + (-A\sin x + B\cos x)e^x + \frac{dy}{dx}$$

$$= -y + \frac{dy}{dx} - y + \frac{dy}{dx} \quad [\text{using (11) \& (1)}]$$

$$\therefore \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

↳ order: 2, degree: 1

Solution of D.E

⇒ A solution of differential equation is a function of $y = f(x)$ that satisfies the differential equation when $f(x)$ & its derivative are substituted into the equation.

$$y = x^2$$

$$\Rightarrow \frac{dy}{dx} = 2x$$

$$\Rightarrow dy = 2x\,dx$$

$$\Rightarrow y = 2\cdot\frac{x^2}{2} + c$$

$$y = x^2 + c$$

$$\frac{dy}{dx} = x \quad \text{---} (1)$$

$$dy = x\,dx$$

$$y = \frac{x^2}{2} + c \rightarrow \text{sol}^n \text{ of } (1)$$

$$\frac{dy}{dx} = x$$

$$(1) \Rightarrow \boxed{x = x}$$

Solution of first order & first degree:

D.E can be written in form

$$M\,dx + N\,dy = 0 \quad \text{or,} \quad M + N\frac{dy}{dx} = 0$$

or, $M\frac{dy}{dx} + N = 0$, where M & N are function of x & y

$$M(x,y) \text{ or } N(x,y)$$

$$\int \frac{1}{2\sqrt{t}}\,dt = \sqrt{t} + c$$

• Several Method:

(I) Seperable variable

(II) Homogeneous equation

(III) Exact differential equation

(IV) Linear D.E

(V) Bernoulli D.E

(I) Seperable var: $Mdx + Ndy = 0$ → first order D.E

$$f(x)\,dx + \phi(y)\,dy = 0$$

Solve: $y\sqrt{1+x^2}\,dy - x\sqrt{1+y^2}\,\cancel{dy}\,dx = 0$

⇒ Given that,

$$y\sqrt{1+x^2}\,dy - x\sqrt{1+y^2}\,dx = 0 \quad \text{--- (1)}$$

$$y\sqrt{1+x^2}\,dy = x\sqrt{1+y^2}\,\cancel{dy}\,dx$$

$$\Rightarrow \frac{y}{\sqrt{1+y^2}}\,dy = \frac{x}{\sqrt{1+x^2}}\,dx$$

$$\Rightarrow \frac{1}{2}\int \frac{1}{2\sqrt{t}}\,\cancel{dy}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{z}}\,dz$$

$$\Rightarrow \sqrt{t} = \sqrt{z} + c$$

let, $1+y^2 = t$

$\Rightarrow 2y\,dy = dt$

$\Rightarrow y\,dy = \frac{dt}{2}$

&

$1+x^2 = z$

$\Rightarrow 2x\,dx = dz$

$\Rightarrow x\,dx = \frac{dz}{2}$

$\Rightarrow \int \frac{y\,dy}{\sqrt{1+y^2}} = \int \frac{x\,dx}{\sqrt{1+x^2}}$

$\Rightarrow \int \frac{\frac{dt}{2}}{\sqrt{t}} = \int \frac{\frac{dz}{2}}{\sqrt{z}}$

$\Rightarrow \int \frac{dt}{2\sqrt{t}} = \int \frac{dz}{2\sqrt{z}}$

$\Rightarrow \sqrt{t} = \sqrt{z} + c$

$\boxed{\text{Q}}$ $\left(\frac{dy}{dx}\right)^{3/2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$\Rightarrow \left(\frac{dy}{dx}\right)^3 = 1+\left(\frac{dy}{dx}\right)^2$

$\Rightarrow \left(\frac{dy}{dx}\right)^3 - \left(\frac{dy}{dx}\right)^2 - 1 = 0$

order = 1

degree = 3

$\boxed{\text{Q}}$ $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) + x + 2y = 0$

we can't ~~or~~ obtain the order and degree.

Solve: $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

$\Rightarrow y\,dx - x\,dy = ay^2\,dx + a\,dy$

$\Rightarrow (y - ay^2)\,dx = (a+x)\,dy$

$\Rightarrow \frac{1}{a+x}\,dx = \frac{1}{y(1-ay)}\,dy$

$\Rightarrow \frac{dx}{a+x} = \left(\frac{1}{y} + \frac{a}{1-ay}\right)dy$

$\Rightarrow \ln(x+a) = \ln y - \ln(1-ay) + \ln c$

$\Rightarrow \ln\{(x+a)(1-ay)\} = \ln\{yc\}$

$\Rightarrow (x+a)(1-ay) = yc$ (Ans).

---

or//

$\Rightarrow \ln\{(x+a)(1-ay)\} = \ln y + c$

$\Rightarrow \ln\left\{\frac{(x+a)(1-ay)}{y}\right\} = c$

$\Rightarrow \frac{(x+a)(1-ay)}{y} = e^c = c_1$

$\Rightarrow (x+a)(1-ay) = yc_1$

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☐ check whether it is homogeneous or not?

Homogeneous equation → power is same

A homogeneous equation can be written in the form $f(tx, ty) = t\, f(x, y)$

$ax^2 + bx + c = 0$ ✗

$\frac{ax^2}{2} + \frac{2hxy}{2} + \frac{by^2}{2} = 0$ ✓

☐ Homogeneous differential equation can be written as ~~ft~~ follows $\frac{dy}{dx} = \frac{\phi(x,y)}{\psi(x,y)}$

where, $\phi(x,y)$ and $\psi(x,y)$ are both homogeneous equation of same degree.

let $y = vx$

☐ Solve: $(x^2 - y^2)\, dy = 2xy\, dx$ —— (1)

~~Given that,~~ $\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$ —— (2)

let, $y = vx$ —— (3)

Differentiate eqn (3) w.r.to x,

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ —— (4)

from ②, ③, ④, we get,

$$v + x\frac{dv}{dx} = \frac{2x\,vx}{x^2 - (vx)^2}$$

$$\Rightarrow v + \frac{x\,dv}{dx} = \frac{2x^2 v}{x^2 - v^2x^2}$$

$$= \frac{2x^2 v}{x^2(1-v^2)}$$

$$= \frac{2v}{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v}{1-v^2} - v = \frac{2v - v + v^3}{1-v^2}$$

$$= \frac{v+v^3}{1-v^2}$$

$$\Rightarrow \frac{1-v^2}{v+v^3}\,dv = \frac{dx}{x}$$

$$\cancel{\Rightarrow \frac{1}{v} - \frac{2v}{1+v^2}}$$

$$\Rightarrow \frac{1-v^2}{v(1+v^2)}\,dv = \frac{dx}{x}$$

$$\Rightarrow \left\{\frac{1}{v} - \frac{2v}{(1+v^2)}\right\} dv = \frac{dx}{x}$$

$$\Rightarrow \ln v - \ln(1+v^2) = \ln x \; \cancel{+\ln c} \; + \ln c$$

$$\Rightarrow \ln x = \ln\left(\frac{v}{1+v^2}\right) + \ln C$$

$\Rightarrow \ln x = \ln \frac{y}{x} - \ln\left(1+ y^{?}(x^2)\right) + \ln c$ (Ans).

<u>Solve:</u> $(2\sqrt{xy} - x)\,dy + y\,dx = 0$ ——— (1)

$\Rightarrow (2\sqrt{xy} - x)\frac{dy}{dx} = -y$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{2\sqrt{xy} - x}$ ——— (2)

Let, $y = vx$ ——— (3)

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{-vx}{2\sqrt{xvx} - x}$

$\Rightarrow v + x\frac{dv}{dx} = -\frac{vx}{2\sqrt{vx^2} - x}$

$= -\frac{vx}{2x\sqrt{v} - x}$

$= -\frac{v}{2\sqrt{v} - 1}$

$\Rightarrow x\frac{dv}{dx} = -v - \frac{v}{2\sqrt{v} - 1}$

$= \frac{-2v\sqrt{v} + v - v}{2\sqrt{v} - 1}$

$$\Rightarrow x\frac{dv}{dx} = -\frac{2v\sqrt{v}}{2\sqrt{v}-1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\sqrt{v}}{1-2\sqrt{v}}$$

$$\Rightarrow \frac{dx}{x} = \frac{1-2\sqrt{v}}{2v\sqrt{v}}\,dv$$

$$\Rightarrow \ln x = \left(\frac{1}{2v\sqrt{v}} - \frac{1}{v}\right)dv$$

$$= \left(\frac{1}{2}v^{-3/2} - \frac{1}{v}\right)dv$$

$$= \frac{1}{2}\cdot\frac{v^{-3/2+1}}{-\frac{3}{2}+1} - \ln v + \ln c$$

$$= \frac{1}{2}\frac{v^{-1/2}}{-\frac{1}{2}} - \ln v + \ln c$$

$$= -\frac{1}{\sqrt{v}} - \ln v + \ln c$$

$$\Rightarrow \ln x = -\frac{1}{\sqrt{\frac{y}{x}}} - \ln\frac{y}{x} + \ln c$$

$$\therefore \ln x = -\frac{\sqrt{x}}{\sqrt{y}} - \ln\frac{y}{x} + \ln c$$

(Ans).

$f(x)\,dx + f(y)\,dy = 0 \rightarrow$ sep. val

$M(x,y)\,dx + N(x,y)\,dy = 0 \rightarrow$ Exact

Exact Differential Equation

If the DE $M dx + N dy = 0$ ① where M and N are function of x and y then the eqn ① is called exact if

$$d(x,y) = M dx + N dy$$

<u>Note:</u> $z = f(x,y)$

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$f(x,y) = 0$$

$$d\{f(x,y)\} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = M dx + N dy$$

$$M = \frac{\partial f}{\partial x} \text{ and } N = \frac{\partial f}{\partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \rightarrow \text{if the following condition hold.}$$

Step-1: Integrate M w.r.to x treating y as constant.

Step-2: Integrate N w.r.to y only those term of N which do not contain x.

Step-3: Equate the sum of those two integrals to an arbitrary constant

$$\int\limits_{y \text{ as constant}} M\,dx + \int (\text{term in N not contain } x)\,dy = c$$

Solve: $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

can be done in homogenea but tedious

Compare the given eqn with $M\,dx + N\,dy = 0$

$M = x^2 - 4xy - 2y^2$ and

$N = y^2 - 4xy - 2x^2$

$$\frac{\partial M}{\partial y} = -4x - 4y$$

$$\frac{\partial N}{\partial x} = -4y - 4x$$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -4y - 4x$

The given eqn 1 is exact.

$$\int M\,dx = \int_{y \text{ as constant}} (x^2 - 4xy - 2y^2)\,dx$$

$$= \frac{x^3}{3} - \frac{4x^2y}{2} - 2y^2x$$

$$= \frac{x^3}{3} - 2x^2y - 2xy^2$$

Integrating N which is free from x

$$\int N\,dy = \int y^2\,dy = \frac{y^3}{3}$$

$$\therefore \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c$$ (Ans).

Find the value of the constant δ such that the $(2xe^y + 3y^2)\frac{dy}{dx} + (3x^2 + \delta e^y) = 0$ is ~~eac~~ exact and hence solve.

$$(3x^2 + \delta e^y)\,dx + (2xe^y + 3y^2)\,dy = 0$$

$M = 3x^2 + \delta e^y$ and $N = 2xe^y + 3y^2$

$\frac{\delta M}{\delta y} = \cancel{3x^2}\ \cancel{+}\ \delta e^y$ and $\frac{\delta N}{\delta x} = 2e^y$

Therefore, $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$

$\Rightarrow \delta e^y = 2e^y$

$\Rightarrow \delta = 2$

$\therefore$ $\cancel{2xe}$ $(3x^2 + 2e^y)\,dx + (2xe^y + 3y^2)\,dy = 0$

$$\int\limits_{y \text{ as constant}} M\,d\cancel{y}x = \int (3x^2 + 2e^y)\,dx$$

$$= 3\frac{x^3}{3} + 2e^y x$$

$$= x^3 + 2xe^y$$

$\int$ (term in N not contain x) $dy = \int 3y^2\,dy = y^3$

$$x\,dy + y\,dx = 0$$
$$d(xy) = 0$$

$\therefore\ x^3 + 2xe^y + y^3 = c$ (Ans).

Non-exact differential equation

$$M dx + N dy = 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

A non-exact D.E can always be made exact by multiplying it by some function of x and y, such a function is called integrating factor.

Rule-1: If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only say, $f(x)$, then $e^{\int f(x)\,dx}$ is an Integrating factor of $Mdx + Ndy = 0$

Rule-2: If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y only say $f(y)$, then $e^{\int f(y)\,dy}$ is an IF of $Mdx + Ndy = 0$

OR $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$ is a function of y

$$IF = e^{\int -f(x)\,dx}$$

Solve: $(x^2+y^2+1)\,dx + x(x-2y)\,dy = 0$ —— (1)

$M\,dx + N\,dy = 0$

$\therefore M = x^2+y^2+1$

$N = x^2 - 2xy$

$\dfrac{\partial M}{\partial y} = 2y$ and $\dfrac{\partial N}{\partial x} = 2x - 2y$

$\therefore \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

Consider,

$\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 2y - 2x + 2y = 4y - 2x$

$= 2(2y - x) = -2(x-2y)$

$\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{-2(x-2y)}{x(x-2y)} = -\dfrac{2}{x}$

(which is x function)

$$\therefore \ I \cdot F = e^{\int -\frac{2}{x} dx}$$

$$= e^{-2\ln x}$$

$$= e^{-\ln x^2} = \frac{1}{e^{\ln x^2}}$$

$$= \frac{1}{x^2}$$

Now, multiplying by I·F in eqn ①

$$\therefore \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx + \left(1 - \frac{2y}{x}\right) dy = 0$$

$$\int_{y \text{ as constant}} M \, dx = \int \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx$$

$$= x + y^2 \frac{x^{-2+1}}{-2+1} + \frac{x^{-2+1}}{-2+1}$$

$$\int N \, dy = \int 1 \, dy = y$$

$$\therefore x - \frac{y^2}{x} - \frac{1}{x} + y = C$$

$\therefore$ I.F $= e^{\int -\frac{2}{x} dx}$

$= e^{-2\ln x}$

$= e^{-\ln x^2} = \frac{1}{e^{\ln x^2}}$

$= \frac{1}{x^2}$

Now, multiplying by I.F in eqn ①

$\therefore \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx + \left(1 - \frac{2y}{x}\right) dy = 0$

$\int_{y \text{ as constant}} M\,dx = \int\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx$

$= x + y^2 \frac{x^{-2+1}}{-2+1} + \frac{x^{-2+1}}{-2+1}$

$\int N\,dy = \int 1\,dy = y$

$\therefore x - \frac{y^2}{x} - \frac{1}{x} + y = c$

Solve: $(2xy^4e^y + 2xy^3 + y)\,dx$

$+ (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$ — ①

$M\,dx + N\,dy = 0$

$M = 2xy^4e^y + 2xy^3 + y$

$\frac{\partial M}{\partial y} = \cancel{8x}\; 2x(y^4e^y + e^y \cdot 4y^3)$

$+ 2x \cdot 3y^2 + 1$

$= 2xy^4e^y + 8xy^3e^y + 6xy^2 + 1$

$N = x^2y^4e^y - x^2y^2 - 3x$

$\cancel{\frac{\partial N}{\partial x} = x^2(y^4e^y + e^y \cdot 4y^3) - x^2 2y - 0}$

$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Consider, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 8xy^3e^y$

$+ 8xy^2 + 4$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{4(2xy^3e^y + 2xy^2 + 1)}{y(2xy^3e^y + 2xy^2 + 1)}$$

$$= \frac{4}{y}$$

~~Multiplying~~

$$I.F = e^{\int -\frac{4}{y}dy}$$

$$= e^{-4\ln y} = e^{-\ln y^4}$$

$$= \cancel{y^4} \quad \frac{1}{y^4}$$

Multiplying by I.F in eq^n 1

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0$$

$$\int_{y \text{ as constant}} M\,dx = 2\frac{x^2}{2}\cdot e^y + \frac{2x^2}{2y} + \frac{1}{y^3}\cdot x$$

$$= \cancel{x^2y}\; x^2e^y + \frac{x^2}{y} + \frac{x}{y^3}$$

$\int$ ~~N~~ (term in x not contain x) dy

$= \int 0 \, dy = 0$

$\therefore \; x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$

The linear differential equation of first order ~~of~~ is of the form $\boxed{\frac{dy}{dx} + Py = Q}$ where P and Q are function of x alone or constant.

$\boxed{\frac{dx}{dy} + Px = Q}$

$I.F = e^{\int P\,dx}$

$IF = e^{\int P\,dy}$

$I.F \frac{dy}{dx} + IF\,Py = IFQ$ [UV method]

$IFy = \int IF\,Q\,dx$

<u>Solve:</u> $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$

Compare $\frac{dy}{dx} + Py = Q$

Here, $P = \frac{2x+1}{x}$, $Q = e^{-2x}$

$\therefore I.F = e^{\int \frac{2x+1}{x}dx}$

$= e^{\int (2+\frac{1}{x})dx}$

$= e^{2x+\ln x} = e^{2x} \cdot e^{\ln x} = x \cdot e^{2x}$

Multiplying with eqn (1) by I.F

$$xe^{2x}\frac{dy}{dx} + xe^{2x}\left(\frac{2x+1}{x}\right)y = xe^{2x}\cdot e^{-2x}$$

$$\Rightarrow xe^{2x}\frac{dy}{dx} + e^{2x}(2x+1)y = xe^{2x}e^{-2x}$$

$$\Rightarrow \frac{d}{dx}(\underbrace{xe^{2x}}_{u}\cdot \underbrace{y}_{v}) = xe^{2x}\cdot e^{-2x}$$

$$\Rightarrow d(xe^{2x}\cdot y) = x\,dx$$

$$\Rightarrow \int d(xe^{2x}y) = \int x\,dx$$

$$\Rightarrow xye^{2x} = \frac{x^2}{2} + c$$

Solve: $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$$\Rightarrow (\tan^{-1}y - x)\frac{dy}{dx} = (1+y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x}$$

$$\Rightarrow \frac{dy}{dx}\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2}$$

compare $\frac{dx}{dy} + Px = Q$

$$P = \frac{1}{1+y^2}, \quad Q = \frac{\tan^{-1}y}{1+y^2}$$

$$I.F = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy}$$

$$= e^{\tan^{-1}y}$$

Multiplying with eqn (1) by I.F

$$e^{\tan^{-1}y}\frac{dx}{dy} + \frac{e^{\tan^{-1}y}}{1+y^2}x = e^{\tan^{-1}y} \cdot \frac{\tan^{-1}y}{1+y^2}$$

$$\Rightarrow \frac{d}{dy}\left(e^{\tan^{-1}y} x\right) = e^{\tan^{-1}y} \cdot \frac{\tan^{-1}y}{1+y^2}$$

$$\Rightarrow e^{\tan^{-1}y} \cdot x = \int e^{\tan^{-1}y} \cdot \frac{\tan^{-1}y}{1+y^2}\,dy$$

$$\Rightarrow e^{\tan^{-1}y} x = \int e^{z} \cdot z\,dz$$

$x(1)=0 \begin{Bmatrix} y=1 \\ x=0 \end{Bmatrix}$ $y(1)=0 \begin{Bmatrix} x=1 \\ y=0 \end{Bmatrix}$

$\Rightarrow e^z x = ze^z - \int e^z\,dz$

$= ze^z - e^z + c$

let, $\tan^{-1}y = z$

$\Rightarrow \frac{1}{1+y^2}dy = dz$

$\Rightarrow xe^{\tan^{-1}y} = \tan^{-1}y\, e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$

$\therefore xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$ (Ans)

Now, $x(1)=0$

$0 = e^{\pi/4}\left(\frac{\pi}{4} - 1\right) + c$

$\Rightarrow e^{\pi/4}\left(\frac{\pi-4}{4}\right) + c = 0$

$\Rightarrow c = -e^{\pi/4}\left(\frac{\pi-4}{4}\right)$

Bernoulli's equation

$$\boxed{\frac{dy}{dx} + Py = ay^n} \rightarrow \frac{dy}{dx} + Py = a$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + \frac{Py}{y^n} = a$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P y^{1-n} = a$$

let, $y^{1-n} = v$

$$(1-n) y^{-n} \frac{dy}{dx} = \frac{dv}{dx}$$

or,

$$\frac{1}{y^n}\frac{dy}{dx} + P\cdot\frac{1}{y^{n-1}} = a$$

let, $y^{n-1} = v$

$$\Rightarrow (n-1) y^{n-2} \frac{dy}{dx} = \frac{dv}{dx}$$

Solve: $\frac{dy}{dx} + y = xy^3$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + y^{-2} = x$$

let, $y^{-2} = v$

$$\Rightarrow -2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

$$\Rightarrow -\frac{1}{2}\frac{dv}{dx} + v = x$$

$$\Rightarrow \frac{dv}{dx} - 2v = -2x$$

$$\cancel{\Rightarrow \frac{dv}{dx} + 2x = 2v}$$

$$\therefore IF = e^{\int -2\,dx}$$

$$= e^{-2x}$$

Multiplying by IF,

$$e^{-2x}\frac{dv}{dx} - 2v \cdot e^{-2x} = -e^{-2x}\,2x$$

$$\Rightarrow \frac{d}{dx}\left(e^{-2x} \cdot v\right) = -e^{-2x}\,2x$$

$$\Rightarrow ve^{-2x} = \int -2x \cdot e^{-2x}\,dx$$

$$\Rightarrow ve^{-2x} = -2\left[\frac{e^{-2x}}{-2} \cdot x - \int \frac{e^{-2x}}{-2}\,dx\right]$$

$$= -2\left[-\frac{x}{2}e^{-2x} + \frac{e^{-2x}}{-4}\right]$$

$$= xe^{-2x} + \frac{e^{-2x}}{2} + c$$

$\Rightarrow ve^{-2x} = e^{-2x}\left(x+\frac{1}{2}\right)+c$

~~$\Rightarrow v = x+1$~~ $\Rightarrow 2y^{-2}e^{-2x} = e^{-2x}(2x+1)+c$

~~$\Rightarrow y^{-2} = x+1$~~

Solve: $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

$= e^{x-y}\cdot e^x - e^{x-y}\cdot e^y$

$= e^{2x-y} - e^x$

$= e^{2x}\cdot e^{-y} - e^x$

$\Rightarrow \frac{dy}{dx} + e^x = e^{2x}e^{-y}$

$\Rightarrow \frac{1}{e^{-y}}\frac{dy}{dx} + \frac{e^x}{e^{-y}} = e^{2x}$

$\Rightarrow e^y\frac{dy}{dx} + e^x e^y = e^{2x}$

let, $e^y = v$

$\Rightarrow e^y\cdot\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} + ve^x = e^{2x}$

$$IF = e^{\int e^x dx}$$
$$= e^{e^x}$$

Multiplying by IF,

$$e^{e^x}\frac{dv}{dx} + e^{e^x} v e^x = e^{e^x} e^{2x}$$

$$\Rightarrow \frac{d}{dx}\left(v.e^{e^x}\right) = e^{e^x} e^{2x}$$

$$\Rightarrow v e^{e^x} = \int e^{e^x} e^{2x} dx$$

let, $\cancel{e^{e^x}} = z.$

$\cancel{e^x . e^{e^x} dx = dz}$

$e^x = z$

$e^x dx = dz$

$$= \int \left(e^{e^x + 2x}\right) dx$$

$$= \int e^{e^x} . e^x . e^x dx$$

$$= \int e^z . z\, dz$$

$$= z e^z - \int e^z dz$$

$$= ze^z - e^z + c$$

$$\Rightarrow e^y . e^{e^x} = e^x e^{e^x} - e^{e^x} + c$$

$$\Rightarrow e^{(y+e^x)} = e^{e^x}(e^x - 1) + c$$

Linear equation

D

Solve: $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$

The ~~ax~~ auxeilary equation is

$m^3 + 6m^2 + 11m + 6 = 0$

$\Rightarrow m^3 + m^2 + 5m^2 + 5m + 6m + 6 = 0$

$\Rightarrow m^2(m+1) + 5m(m+1) + 6(m+1) = 0$

$\Rightarrow (m+1)(m^2 + 5m + 6) = 0$

$\Rightarrow (m+1)(m+2)(m+3) = 0$

$\therefore m = -1, -2, -3$

Hence the general sol^n is

$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

Solve: $(D^4 - 5D^2 + 4)y = 0$

A.E is,

$m^4 - 5m^2 + 4 = 0$

$\Rightarrow m^3(m-1) + m^2(m-1) - 4m(m-1) - 4(m-1) = 0$

$\Rightarrow (m-1)(m^3 + m^2 - 4m - 4) = 0$

$\Rightarrow (m-1)\{m^2(m+1) - 4(m+1)\} = 0$

$\Rightarrow (m-1)(m+1)(m^2-4) = 0$

$m = 1, -1, -2, 2$

$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{-2x} + c_4 e^{2x}$

Higher order linear differential equation or Differential eqn with constant co-efficient.

The general form of the eqn is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = X \quad \text{--- (1)}$$

where X is function of x alone and $a_0 \ldots a_n$ are constant.

Operator, $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$ $\therefore$ $\frac{d^n}{dx^n} = D^n$

$$\underbrace{(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + a_n)}_{f(D)} y = X \quad \text{--- (2)}$$

$f(D) y = X$

$f(D)y = X$; $y = C.F + P.I$

$f(D)y = 0$

$y = C.F$

$(a_0 D^n + a_1 D^{n-1} + \dots a_n) y = 0$ —— (3)

$(a_0 m^n + a_1 m^{n-1} + \dots a_n) e^{mx} = 0$

$\Rightarrow a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$ —— (4)

let, $y = e^{mx}$ be trail soln of (3)

$\frac{dy}{dx} = me^{mx}$

$\frac{d^2y}{dx^2} = m^2 e^{mx}$

$\frac{d^ny}{dx^n} = m^n e^{mx}$

Since, $e^{mx} \neq 0$

eqn (4) is represented by as Auxiliary eqn,

(I) All roots real and distinct

(II) All roots real and some may be repeated

(III) All roots are imaginary.

$CF = y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_n e^{m_n x}$

$\{(D - m_1)(D - m_2)(D - m_3) \dots\} y = 0$

$(D - m_1) y = 0$

$$\frac{dy}{dx} - m_1 y = 0$$

$$\Rightarrow \frac{dy}{dx} = m_1 y$$

$$\Rightarrow \frac{dy}{y} = m_1 dx \Rightarrow \ln y = m_1 x + c$$

$$\Rightarrow y = e^{m_1 x + c} = e^{m_1 x} \cdot e^{c}$$

$$\Rightarrow y = A e^{m_1 x}$$

Q. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

$(D^2 + 5D + 6) y = 0$ — (1) since $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$

Let $y = e^{mx}$ be trail sol^n of (1)

$(m^2 + 5m + 6) e^{mx} = 0$

$\Rightarrow m^2 + 3m + 2m + 6 = 0$

$\Rightarrow (m+3)(m+2) = 0$

Case - II: Auxiliary eq^n having repeated real roots.

The roots of the eq^n having m_1 and m_2 repeated then the general sol^n is

$$y = (c_1 + xc_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$ $\boxed{c_1 + xc_2 + x^2 c_3}$

Case - III: Having imaginary root, $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$; $y = e^{\alpha}(A\cos\beta x + B\sin\beta x)$

$y = C.F + P.I$

$\boxed{+}$ $X = F(x)$

Type-1: If $F(x) = e^{ax}$ then

$P.I$ = Particular Integrate = $\dfrac{1}{f(D)} e^{ax}$

$= \dfrac{1}{f(a)} e^{ax}$; $f(a) \neq 0$

If $f(a) = 0$ then

$P.I = \dfrac{x}{f'(a)} e^{ax}$ and $f'(a) \neq 0$

Type-2: If ~~f(x~~ $F(x)$ = $\sin ax / \cos ax$ then

$P.I = \dfrac{1}{f(D^2)} \sin ax = \dfrac{1}{f(-a^2)} \sin ax$ if $f(-a^2) \neq 0$

If $f(-a^2) = 0$ then, $P.I = \dfrac{1}{f'(D^2)}$

Type-3: If $F(x) = x$, polynomial function then,

$P.I = \dfrac{1}{f(D)} x = [f(D)]^{-1} x$

$\boxed{1}$ $(4D^2+12D+9)y = 144e^{3x}$

The auxilary equation is

$4m^2+12m+9 = 0$

$\Rightarrow (2m+3)^2 = 0$

$\therefore m = -\frac{3}{2}, -\frac{3}{2}$

Hence, the complementary function

$C.F = (c_1 + x c_2)e^{-3/2 x}$

$P.I = \frac{1}{4D^2+12D+9} 144 e^{3x}$

$= \frac{144}{4\times3^2+12\times3+9} e^{3x}$

$= \frac{144}{81} e^{3x}$

Hence the general solution is,

$y = C.F + P.I$

$= (c_1 + xc_2) e^{-3/2 x} + \frac{144}{81} e^{3x}$

$\boxed{}$ $(D^2+4D+4)\,y = e^{2x} - e^{-2x}$

A.E is,

$m^2+4m+4=0$

$\Rightarrow (m+2)^2 = 0$

$\therefore m = -2, -2$

$C.F = (c_1 + x c_2)\, e^{-2x}$

$$P.I = \frac{1}{D^2+4D+4}\left(e^{2x} - e^{-2x}\right)$$

$$= \frac{1}{D^2+4D+4} e^{2x} - \frac{1}{D^2+4D+4} e^{-2x}$$

$$= \frac{1}{4+8+4} e^{2x} - \frac{1}{4-8+4} e^{-2x}$$

$$= \frac{1}{16} e^{2x} - \frac{1}{0} e^{-2x} \text{ (2nd term case failure)}$$

$$= \frac{1}{16} e^{2x} - \frac{x}{\frac{\partial}{\partial D}(D^2+4D+4)} e^{-2x}$$

$$= \frac{e^{2x}}{16} - \frac{x}{2D+4} e^{-2x}$$

$$P.I = \frac{e^{2x}}{16} - \frac{x}{-4+4} e^{-2x}$$

$$= \frac{e^{2x}}{16} - \frac{x}{0} e^{-2x} \quad \text{[3rd term case failure]}$$

$$= \frac{e^{2x}}{16} - \frac{x^2}{2} e^{-2x}$$

$\therefore y = C.F. + P.I$

$$= (c_1 + xc_2) e^{-2x} + \frac{e^{2x}}{16} - \frac{x^2}{2} e^{-2x}$$

(Ans).

<u>Note:</u> $(D^2+4D+4)y = (e^{2x} - e^{-2x})^2$

$$= e^{4x} - 2 + e^{-4x}$$

$$P.I = \frac{1}{D^2+4D+4}(e^{4x} - 2 + e^{-4x})$$

$$= \frac{1}{D^2+4D+4} e^{4x} + \frac{1}{D^2+4D+4} e^{-4x} - \frac{1}{D^2+4D+4} 2e^{0x}$$

$$= \frac{1}{36} e^{4x} + \frac{1}{4} e^{-4x} - \frac{2}{4}$$

$(D^2 - 4D + 4)y = \sin 2x$

A.E is,

$m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

$\therefore m = 2, 2$

$C.F = (c_1 + xc_2)e^{2x}$

$P.I = \frac{1}{D^2 - 4D + 4} \sin 2x$

$= \frac{1}{(-2^2)^2 - 4(-2^2) + 4} \sin 2x$

$= \frac{1}{16 + 16 + 4} \sin 2x$

$= \frac{1}{36} \sin 2x$

$D = \frac{d}{dx}$

$\frac{1}{D} = \int$

$= \frac{\sin 2x}{-2^2 - 4D + 4}$

$= \frac{\sin 2x}{-4 - 4D + 4}$

$= \frac{\sin 2x}{-4D}$

$= -\frac{1}{4} \frac{1}{D} \sin 2x$

$= -\frac{1}{4}\left(-\frac{\cos 2x}{2}\right)$

$= \frac{1}{8}\cos 2x$

OR,

$P.I = -\frac{1}{4}\frac{D}{D^2}\sin 2x$

$= -\frac{1}{4}\frac{D}{-4}\sin 2x$

$= \frac{1}{16} 2\cos 2x$

$= \frac{1}{8}\cos 2x$

$y = C.F + P.I$

~~$= (c_1 + xc_2)e^{2x} + \frac{1}{36}\sin 2x$~~

$= (c_1 + xc_2)e^{2x} + \frac{1}{8}\cos 2x$ (Ans).

03 July 2024

$(D^3+D^2-D-1)Z = \cos 2x$

A.E is

$m^3+m^2-m-1=0$

$\Rightarrow m^2(m+1)-1(m+1)=0$

$\Rightarrow (m^2-1)(m-1)=0$

$\Rightarrow (m+1)(m-1)(m+1)=0$

$\therefore m = -1, -1, 1$

$C.F = C_1 e^{x} + (c_2 + c_3 x e^{-x})$

$\therefore P.I = \frac{1}{D^3+D^2-D-1} \cos 2x$

$= \frac{1}{-2^2 D - 2^2 - D - 1} \cos 2x$

$= \frac{1}{-4D - D - 4 - 5} \cos 2x$

$= \frac{1}{-5D - 5} \cos 2x$

$= -\frac{1}{5} \frac{1}{D+1} \cos 2x$

$= -\frac{1}{5} \frac{D-1}{D^2-1} \cos 2x$

$\therefore P.I = \frac{-1}{5} \frac{D-1}{-2^2-1} \cos 2x = \frac{1}{25}(D-1)\cos 2x$

$$= \frac{1}{25}(-\sin 2x \cdot 2 - \cos 2x)$$

$$= -\frac{1}{25}(2\sin 2x + \cos 2x)$$

$$y = C.F + P.I$$

$$= c_1 e^{x} + (c_2 + x/3)e^{-x}$$

$$- \frac{1}{25}(2\sin 2x + \cos 2x)$$

Type 3:

$$(D^2+2D+1)y = 2x+x^2$$

$$m = -1, 1$$

$$C.F = (c_1 + xc_2)e^{-x}$$

$$P.I = \frac{1}{(D^2+2D+1)}(2x+x^2)$$

$$= \frac{1}{(D+1)^2}(2x+x^2)$$

$$\therefore P.I = [1+D]^{-2}(2x+x^2)$$

$$\boxed{(1+x)^2 = 1 - 2x - 3x^2 - 4x^2}$$

$$= (1 - 2D + 3D^2 - 4D^3 \cdots)(2x+x^2)$$

$$(1+x)^{m} = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

$$= (1 - 2D + 3D^2 - 4D^3 + \cdots)(2x+x^2)$$

$= 2x + x^2 - 2D(2x + x^2) + 3D(2x + x^2) - 4D^3(2x + x^2) + \ldots$

$= 2x + x^2 - 2(2 + 2x) + 3 \cdot 2 - 4D + \ldots$

$= 2x + x^2 - 4 - 4x + 6 \qquad = 2x + x^2 - 4 - 4x + 6$

$= x^2 - 2x + 2 \qquad = x^2 - 2x + 6$

$$y = C.F + P.I = (C_1 + xC_2)e^{-x} + x^2 - 2x + 2$$

2

$P.I. = \frac{1}{D^2 + 2D + 1}(2x + x^2)$

$= \frac{1}{[1 + (2D + D^2)]}(2x + x^2)$

$= [1 + (2D + D^2)]^{-1}[2x + x^2]$

$= \{1 - (2D + D^2) + (2D + D^2)^2 - \ldots\}(2x + x^2)$

$= [2x + x^2 - (2D + D^2)(2x + x^2) + (4D^2 + 4D^3 + D^4)(2x + x^2) \ldots]$

$= 2x + x^2 - (4 + 4x + 2) + 8$

$= 2x + x^2 - 6 - 4x + 8$

$= x^2 - 2x + 2$

$2D(2x + x^2) = 2(2 + 2x) = 4 + 4x$

$D^2(2x + x^2) = D(2 + 2x) = 2$

$(2D + D^2)(2x + x^2)$

$= 4 + 4x + 2$

$\boxed{5}$ $(D^2+2D+2)y = x^2$

A.E

m^2+2m+2

$$\therefore m = \frac{-2\pm\sqrt{4-8}}{2}$$

$$= -1\pm i$$

$$= \alpha \pm i\beta$$

$\alpha = -1$

$\beta = 1$

$C.F = e^{-\alpha x}(A\cos\beta x + B\sin\beta x)$

(complex form)

$\therefore C.F = e^{-x}(A\cos x + B\sin x)$

$$P.I = \frac{1}{2(1+D+D^2/2)}x^2$$

$$= \frac{1}{2}\frac{1}{\{1+(D+\frac{D^2}{2})\}}x^2$$

$$= \frac{1}{2}\left[1+(D+\frac{D^2}{2})\right]^{-1}x^2$$

$$= \frac{1}{2}\left[1 - (D+D^2/2) + (D+D^2/2)^2 - (D+D^2/2)^3 \ldots\right]x^2$$

$$= \frac{1}{2}\left[x^2 - (D+D^2/2)x^2\right.$$

Type-4:

if $f(x) = e^{\alpha x} x$ where x is a function of z and $\sin ax$ or $\cos ax$.

if $f(x) = e^{\alpha x} x$

$\therefore P.I = \frac{1}{F(D)} e^{\alpha x} \cdot x$

$= \frac{e^{\alpha x}}{F(D+\alpha)} x$

$\boxed{} \ (D^2+3D+2) y = e^{2x} \sin x$

A.E is,

$m^2+3m+2 = 0$

$\Rightarrow (m+2)(m+1) = 0$

$\therefore m = -2, -1$

$C.F = c_1 e^{-x} + c_2 e^{-2x}$

$P.I = \frac{1}{D^2+3D+2} e^{2x} \sin x$

$= \frac{e^{2x}}{(D+2)^2 + 3(D+2) + 2} \sin x$

$= \frac{e^{2x}}{D^2+4D+4+3D+6+2} \sin x$

$$P.I = \frac{e^{2x}}{D^2+7D+12}\sin x$$

$$= \frac{e^{2x}}{-1^2+7D+12}\sin x$$

$$= \frac{e^{2x}}{7D+11}\sin x$$

$$= e^{2x}\,\frac{7D-11}{49D^2-121}\sin x$$

$$= e^{2x}\,\frac{7D-11}{49(-1^2)-121}\sin x$$

$$= e^{2x}\,\frac{7D-11}{-49-121}\sin x$$

$$= e^{2x}\,\frac{7D-11}{-170}\sin x$$

$$= -\frac{e^{2x}}{170}(7\cos x - 11\sin x)$$

$$y = C.F + P.I$$

$$= c_1 e^{-x} + c_2 e^{-2x} - \frac{e^{2x}}{170}(7\cos x - 11\sin x)$$

$$(D^3 - 3D - 2)\, y = 540 x^3 e^{-x}$$

A.E is,

$$m^3 - 3m - 2 = 0$$

$$\Rightarrow m^2(m+1) - m(m+1) - 2(m+1) = 0$$

$$\Rightarrow (m+1)(m^2 - m - 2) = 0$$

$$\Rightarrow (m+1)(m^2 - 2m + m - 2) = 0$$

$$\Rightarrow (m+1)\{m(m-2) + 1(m-2)\} = 0$$

$$\Rightarrow (m+1)(m+1)(m-2) = 0$$

$$\therefore m = -1, -1, 2$$

C.F =

$$P.I = \frac{1}{D^3 - 3D - 2} 540 e^{-x} x^3$$

$$= e^{-x} \frac{540}{(D-1)^3 - 3(D-1) - 2} x^3$$

Partial differential Equation

$z = f(x, y)$

$\frac{\partial z}{\partial x} = p$ where y as constant

$\frac{\partial z}{\partial y} = q$

$\frac{\partial^2 z}{\partial x^2} = r$

$\frac{\partial^2 z}{\partial y^2} = t$

$\frac{\partial^2 z}{\partial x \partial y} = s$

$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$
$= p\,dx + q\,dy$

Formation of PDE

(I) Eliminating arbitrary constant
(II) " " function

Form a partial differential equation by eliminating arbitrary function ϕ from the eqn $\phi(u, v) = 0$ where u and v is a function of x, y and z.

1. eliminate ϕ
2. then construct PDE

$z = f(x, y)$ → two ~~id.~~ independent function

প্রত্যেককে differentiate হয়না।

$\phi(u, v) = 0$ — ①

Differentiate eqn ① partially w.r.to x

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)$$

$$+ \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot 0 + \frac{\partial u}{\partial z}\cdot p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot p\right) =$$

$$\Rightarrow \frac{\partial \phi}{\partial u} \Big/ \frac{\partial \phi}{\partial v} = -\frac{\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}} \quad \text{— ②}$$

Form a partial DE by eliminating arbitrary function ϕ from $\phi(x+y+z, x^2+y^2+z^2) = 0$. Find order and degree $\phi(u, v) = 0$.

Given that,

$$\phi\,(x+y+z,\; x^2+y^2+z^2) = 0 \quad \text{—— (1)}$$

let, $\phi(u,v) = 0$ where

$$u = x+y+z\;;\quad v = x^2+y^2+z^2$$

$$\frac{\partial u}{\partial x} = 1\;;\quad \frac{\partial u}{\partial y} = 1\;;\quad \frac{\partial u}{\partial z} = 1\;;\quad \frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial v}{\partial y} = 2y\,,\quad \frac{\partial v}{\partial z} = 2z$$

$\phi(u,v) = 0$

now dif it partially w.r.to x,

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)$$

$$+ \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\cdot\frac{\partial v}{\partial z}\right) = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial u}(1+p) + \frac{\partial \phi}{\partial v}(2x+2zp) = 0$$

$$\frac{\partial \phi}{\partial u} \Big/ \frac{\partial \phi}{\partial v} = -\frac{2(x+pz)}{1+p} \quad \text{--- (2)}$$

Now diff it partially w.r.to y

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)$$

$\frac{\partial x}{\partial y} \to 0$

$$+ \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\cdot\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial u}(1+q) + \frac{\partial \phi}{\partial v}(2y+2qz) = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial u} \Big/ \frac{\partial \phi}{\partial v} = -\frac{(2y+2qz)}{1+q} \quad \text{--- (3)}$$

From (2) and (3),

$$-\frac{2(x+pz)}{1+p} = -\frac{2(y+qz)}{1+q}$$

$$\Rightarrow x + xq + pz + \cancel{pqz} = y + py + qz + \cancel{pqz}$$

$\Rightarrow pz - py + xq - qz = y - x$

$\Rightarrow (z-y)p + (x-z)q = y-x$

$\Rightarrow Pp + Qq = R$ $\boxed{\therefore P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R}$

where P, Q and R are function of x, y, z which ~~leg~~ ranges standard eqn.

$x+y+z = f(x^2+y^2+z^2)$

Now. diff it w.r.to x partially,

$$1 + \frac{\partial z}{\partial x} = f'(x^2+y^2+z^2).\left(2x + 2z\cdot\frac{\partial z}{\partial x}\right)$$

$$1+p = f'(x^2+y^2+z^2)\,(2x+2zp)$$

$$\frac{1+p}{2x+2zp} = f'(x^2+y^2+z^2) \quad \text{--- (1)}$$

Now, diff it w.r. to y partially,

$$0+1+\frac{\partial z}{\partial y} = f'(x^2+y^2+z^2)\left(0+2y+2z\frac{\partial z}{\partial y}\right)$$

$$\Rightarrow \frac{1+q}{2y+2zq} = f'(x^2+y^2+z^2) \quad \text{--- (2)}$$

$1.12 \rightarrow 4, 6, 12$

$$\therefore \frac{1+p}{2(x+zp)} = \frac{1+q}{2(y+zq)}$$

$\Rightarrow y+zq+py+pqz = x+zp+xq+pzq$

$\Rightarrow y-x = (z-y)p + q(x-z)q$

The general solution of Lagrange equation,

$Pp+Qq=R$ is $\qquad p=\frac{dz}{dx}$

$\qquad q=\frac{dz}{dy}$

$\phi(u,v)=0$

where, $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$

are two ~~inp~~ independent sol^n of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Type-1:

Ex-2, 3, 4, 6

Solve: $\frac{y^2 z}{x} p + xzq = y^2$ ——(i) [order 1; degree = 1]

Comparing eq^n (i) with Lagrange standard P.D.E

$Pp + Qq = R$

$\therefore P = \frac{y^2 z}{x}$; $Q = xz$; $R = y^2$

We know, Lagrange auxilary equation,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \quad \text{--- (ii)}$$

Taking first and second term from (ii)

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz}$$

$$x^2\,dx = y^2\,dy$$

$$\Rightarrow \frac{x^3}{3} - \frac{y^3}{3} = \frac{c_1}{3}$$

$$\therefore x^3 - y^3 = c_1$$

Taking first and third term from (ii)

$$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2} \Rightarrow x\,dx = z\,dz$$

$$\Rightarrow \frac{x^2}{2} - \frac{z^2}{2} = \frac{c_2}{2}$$

$$\therefore x^2 - z^2 = c_2$$

Hence the sol^n is

$\phi(c_1, c_2) = 0$ or, $\phi(x^3 - y^3,\ x^2 - z^2) = 0$

(Ans).

Solve: $y^2 p - xyq = x(z - 2y)$ —— (i)

Comparing eq^n (i) with lagrange standard PDE, $Pp + Qq = R$

$P = y^2$; $Q = -xy$; $R = x(z - 2y)$

We know,

lagrange auxilary equation,

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \quad \text{—— (ii)}$$

Taking 1st and 2nd term from (ii)

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

Do 4 maths from book.

$$\Rightarrow \frac{dx}{y} = \frac{dy}{-x}$$

$$\Rightarrow -x\,dx = y\,dy$$

$$\Rightarrow y\,dy + x\,dx = 0$$

$$\Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = \frac{c_1}{2}$$

$$\Rightarrow x^2 + y^2 = c_1$$

Taking 2nd and 3rd term from eqn (II)

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\Rightarrow -\frac{dy}{y} = \frac{dz}{z-2y}$$

$$\Rightarrow z\,dy - 2y\,dy = -y\,dz$$

$$z\,dy + y\,dz = 2y\,dy$$

$$\Rightarrow \int d(yz) = \int 2y\,dy$$

$$\Rightarrow yz = 2\cdot\frac{y^2}{2} + c_2$$

$$\Rightarrow yz - y^2 = c_2$$

Hence the solution is, $\phi(x^2+y^2,\ zy-y^2) = 0$ (Ans).

3, 4, 5, 6, 7, 18

Q-1 → book Ex-1

Imp $\boxed{\text{Type-2}}$

$$p + 3q = 5z + \tan(y-3x) \quad \text{--- (1)}$$

The lagrange auxiliary eqn's are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)}$$

Taking first two terms from (1)

$$dx = \frac{dy}{3}$$

$$\Rightarrow 3dx = dy$$

$$\Rightarrow dy - 3x = 0$$

$$\therefore y - 3x = c_1$$

Taking 1^{st} & 3^{rd} term from (2)

$$dx = \frac{dz}{5z + \tan(y-3x)}$$

$$= \frac{dz}{5z + \tan c_1}$$

$$\int \frac{1}{x+a}\,dx = \ln|x+a| + c$$

$$\Rightarrow x = \frac{1}{5} \ln(5z + \tan c_1) + c_2$$

$$\phi \quad \tan(y-3x)) = c_2$$

$$\Rightarrow 5x - \ln(5z + \tan(y-3x))$$

Hence, the soln is

$$\cancel{\phi(y-3x)}\ \phi\left(y-3x, \{5x - \ln 5z + \tan(y-3x)\}\right) = 0$$

$\boxed{Q-2}$ $z(z^2+xy)(px - qy) = x^4$

$$\Rightarrow zx(z^2+xy)p - zy(z^2+xy)q = x^4$$

$\therefore$ Lagrang's auxiliary eqn:

$$\frac{dx}{zx(z^2+xy)} = \frac{dy}{-zy(z^2+xy)} = \frac{dz}{x^4} \quad \text{--- (1)}$$

Taking first two terms,

$$\frac{dx}{x} = \frac{dy}{-y} \Rightarrow \frac{dy}{y} + \frac{dx}{x} = 0$$

$$\Rightarrow \ln y + \ln x = \ln c_1$$

$$\Rightarrow yx = c_1$$

Taking 1st & 3rd terms:

$$\therefore \frac{dx}{zx(z^2+xy)} = \frac{dz}{x^4}$$

$$\Rightarrow \frac{dx}{z(z^2+c_1)} = \frac{dz}{x^3}$$

$$\Rightarrow x^3\, dx = (z^3 + zc_1)\, dz$$

$$\Rightarrow \frac{x^4}{4} = \frac{z^4}{4} + \frac{z^2}{2} c_1 + c_2$$

$$\Rightarrow \frac{x^4}{4} = \frac{z^4}{4} + \frac{z^2}{2} xy + c_2$$

$$\Rightarrow \frac{x^4}{4} - \frac{z^4}{4} - \frac{z^2 xy}{2} = c_2$$

$$\phi(c_1, c_2) = 0$$

$$\phi\left(yx, \frac{x^4}{4} - \frac{z^4}{4} - \frac{xyz^2}{2}\right) = 0 \text{ (Ans).}$$

DO 6 maths from book.

Ex - 4

Type-3

Solve: $x(y^2-z^2)p - y(z^2+x^2)q = z(x^2+y^2)$ —— (1)

Given that,

Lagrange auxilary equation are

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)} \quad \text{—— (2)}$$

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{2+6+9}{4+12+18} = \frac{17}{34} = \frac{1}{2}$$

Taking x, y, z as multipliers of each fraction,

$$\Rightarrow \frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)}$$

$$= \frac{dz}{z(x^2+y^2)} = \frac{x\,dx + y\,dy + z\,dz}{\cancel{x^2y^2} - \cancel{x^2z^2} - \cancel{y^2z^2} - \cancel{y^2x^2} + \cancel{z^2x^2} + \cancel{z^2y^2}}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$= 0$

if $x\,dx + y\,dy + z\,dz = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$$

$$\therefore x^2 + y^2 + z^2 = c_1$$

Taking $\frac{1}{x}, \frac{-1}{y}, \frac{1}{z}$ as multipliers of each fraction,

$$\Rightarrow \frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$$

$$= \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{y^2-z^2+z^2+x^2-x^2-y^2}$$

$$\therefore \frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz = 0$$

$$\Rightarrow \ln x - \ln y - \ln z = \ln c_2$$

$$\therefore \frac{x}{yz} = c_2$$

$$\phi\left(x^2+y^2+z^2, \frac{x}{yz}\right) = 0$$

$\boxed{\text{Book - (1-9)}}$ 2.8

2 imp

5 "

Sec-B

Example-01: $y'' - ty' + y = 1$, $y(0)=1$, $y'(0)=2$

Soln: Given, $y''(t) - ty'(t) + y = 1$ —(1)

Taking laplace transform on bothsides of eqn (1) we can write,

$$\mathcal{L}\{y''(t)\} - \mathcal{L}\{t\,y'(t)\} + \mathcal{L}\{y\} = \mathcal{L}\{1\}$$

$$\Rightarrow s^2 y(s) - s\,y(0) - y'(0) - \left\{-\frac{d}{ds}\left(sy(s) - y(0)\right)\right\} + y(s) = \frac{1}{s}$$

$$\Rightarrow s^2 y(s) - s - 2 + sy'(s) + y(s) + y(s) = \frac{1}{s}$$

$$\Rightarrow (s^2+2)\,y(s) + sy'(s) = s + 2 + \frac{1}{s}$$

$$\Rightarrow sy'(s) + \left(s + \frac{2}{s}\right) y(s) = \frac{(s+1)^2}{s}.$$

$$\Rightarrow y'(s) + \left(s + \frac{2}{s}\right) y(s) = \frac{(s+1)^2}{s^2} \quad \text{—(2)}$$

$$I.F = e^{\int \left(s + \frac{2}{s}\right) ds}$$

$$= e^{\frac{s^2}{2} + 2\ln s}$$

$$= e^{s^2/2} \cdot e^{\ln s^2} = s^2 e^{s^2/2}$$

Multiplying bothsides of eqn (2) by $s^2 e^{s^2/2}$ we can write,

$$s^2 e^{s^2/2}\, y'(s) + s^2 e^{s^2/2}\left(s+\frac{2}{s}\right)y(s) = s^2 e^{s^2/2}\left(\frac{s+1}{s}\right)^2$$

$$\Rightarrow d\left[y(s)\, s^2 e^{s^2/2}\right] = e^{s^2/2}\left(s^2+2s+1\right)$$

$$\Rightarrow d\left[y(s)\, s^2 e^{s^2/2}\right] = \left(s(s+2)+1\right)e^{s^2/2}$$

$$\Rightarrow d\left[y(s)\, s^2 e^{s^2/2}\right] = (s+2)\,s e^{s^2/2} + 1\cdot e^{s^2/2}$$

$$\Rightarrow d\left[y(s)\, s^2 e^{s^2/2}\right] = d\left[(s+2)\,e^{s^2/2}\right]$$

Integrating bothsides we can write,

$$y(s)\, s^2 e^{s^2/2} = (s+2)\, e^{\frac{s^2}{2}} + c$$

$$\Rightarrow y(s) = \frac{s+2}{s^2} + \frac{c}{s^2}\, e^{-\frac{s^2}{2}}$$

$$= \frac{s}{s^2} + \frac{2}{s^2} + \frac{c}{s^2}\left[1 - \frac{\left(\frac{s^2}{2}\right)}{1!} + \frac{\frac{s^4}{4}}{2!} - \cdots\right]$$

$$= \frac{1}{s} + \frac{2}{s^2} + \frac{c}{s^2} - \frac{c}{2} + \frac{c}{8}s^2 - \cdots$$

Taking inverse Laplace on bothsides we can write,

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + c\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$- \frac{c}{2}\mathcal{L}^{-1}\{1\} + \frac{c}{8}\mathcal{L}^{-1}\{s^2\} - \cdots$$

$$Y(t) = 1 + 2t + ct - 0 \quad \left[\text{Set } \mathcal{L}^{-1}\{s^n\} = 0\right]$$

$$\Rightarrow Y(t) = 1 + (2+c)t$$

Now, applying the condition $Y'(0) = 2$

$$Y'(t) = 2+c$$

$$\Rightarrow Y'(0) = 2+c$$

$$\Rightarrow 2 = 2+c$$

$$\therefore c = 0$$

$$\therefore Y(t) = 1 + 2t \quad \text{(Ans).}$$

Note: If $F(0) = 0$ and $\mathcal{L}^{-1}\{f(s)\} = F(t)$

then $\mathcal{L}^{-1}\{s f(s)\} = F'(t)$

$$\therefore \mathcal{L}^{-1}\{s^n\} = \mathcal{L}^{-1}\{s^{n-2}\, s^2\}$$

$$= F^{n-2}(t) = 0$$

$$IF = e^{\int \frac{s^2+2}{s} ds}$$

$$= e^{\int \left(s + \frac{2}{s}\right) ds}$$

$$= e^{\frac{s^2}{2} + 2\ln s}$$

$$= e^{\frac{s^2}{2}} \cdot e^{\ln s^2}$$

$$= s^2 e^{\frac{s^2}{2}}$$

$$y'(s) \cdot s^2 e^{\frac{s^2}{2}} + \frac{s^2+2}{s} \cdot s^2 e^{\frac{s^2}{2}} y(s) = (1+s^2+2s) s^2 e^{\frac{s^2}{2}}$$

$$\Rightarrow s^2 e^{s^2/2} y'(s) + s e^{s^2/2} (s^2+2) y(s) = (1+s^2+2s) s^2 e^{s^2/2}$$

$$\mathcal{L}\{F'(t)\} = s f(s) - F(0)$$

If $F(0) = 0$

$$F'(t) = \mathcal{L}^{-1}\{s f(s)\}$$

$$F^n(t) = \mathcal{L}^{-1}\{s^n f(s)\}$$

$$= \mathcal{L}^{-1}\left\{s^{n+2} \cdot \frac{1}{s^2}\right\}$$

$$= \mathcal{L}^{-1}\{s^n\}$$

$$F^n(t) = \mathcal{L}^{-1}\{s^n\}$$

$$F^{n+2}\ t = \mathcal{L}^{-1}\{s^n\}$$

t ↳ Double Derivative w.r.to t.