Cohomological obstructions to lifting properties for full group C*-algebras *(joint work with A. Ioana and P. Spaas)*

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G will always be a discrete group.

Def^n: E - operator system
B - C^*-algebra
J - 2-sided, closed ideal of B.

A contractive completely positive (c.c.p.) map $\varphi: E \to B/J$ is liftable to B if $\exists$ c.c.p. map $\tilde{\varphi}: E \to B$ such that the diagram

$$\begin{tikzcd} & B \arrow[d, two heads] \\ E \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"'] \arrow[ur, phantom, shift right=3, "\circlearrowleft"] & B/J \end{tikzcd}$$

commutes.

Def^n: A C^*-alg A has the lifting property (LP) if every ccp map property (LP) if every ccp map from A into a quotient C^*-alg is liftable.

Theorem (Choi-Effros '76)

Every nuclear C^*-alg has LP.

Def^n: (Kirchberg '93)

A C^*-alg A has the local lifting property (LLP) if $\forall$ ccp map $\varphi : A \to B/J$ and every finite dimensional op. system $E \subset A$, $\varphi|_E : E \to B/J$ is liftable.

Theorem (Kirchberg '93)

A ~ C^*-alg

A has LLP $\iff$ $A \otimes_{min} B(H) = A \otimes_{max} B(H)$.

Open Problem:

Is LLP equivalent to LP for separable C^*-algs?

Main Problem (Ozawa '04, Pisier '16)

Find G where $C^*(G)$ does not have (L)LP.

Examples

Groups where $C^*(G)$ has LP:

→ amenable

→ free products of amenable group

↳ $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$

all known example of G where $C^*(G)$ is known to have LLP arise from this class.

Previous progress on main problem:

→ Ozawa '04: $\exists G$ s.t. $C^*(G)$ does not have LP.

→ Thom '10: 2 classes of groups where $C^*(G)$ fails to have LLP.

Def^n: $c: G \times G \to \mathbb{T}$ is a <u>2-cocycle</u> if

$$c(g,h)\,c(gh,k) = c(g,hk)\,c(h,k)$$

$\forall g,h,k \in G$.

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Example Let $b: G \to \mathbb{T}$ be a function. Then

$$c(g,h) = b(g)\,b(h)\,\overline{b(gh)}$$

is a 2-cocycle. Such examples are <u>2-coboundaries</u>.

Theorem (Ioana–Spaas–W. '20)

$H \leq G$, (G,H) has relative prop (T). Suppose $\exists\, c_n : G \times G \to \mathbb{T}$ 2-cocycles, s.t.

(1) $c_n|_{H\times H}$ is not a 2-coboundary of H.

(2) $\lim_{n\to\infty} c_n(g,h) = 1 \quad \forall g,h \in G$

(3) $\forall n$, $\exists$ projective rep^n $\pi_n : G \to \mathcal{U}(\mathcal{H}_n)$
where $\dim(\mathcal{H}_n) < \infty$,

$$\pi_n(g)\pi_n(h) = c_n(g,h)\,\pi_n(gh)$$

$\forall g,h \in G$.

Then $C^*(G)$ does not have LLP.

<u>Corollary</u>

$C^*(\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}))$ does not have LLP.

$\to$ $C^*(SL_n(\mathbb{Z}))$ $n \geq 3$ does not have LLP.

Also get analogous theorem for refuting LP for $C^*(G)$.

Theorem (Ioana - Spaas - W.)

If G has property (T) and is not finitely presentable, then $C^*(G)$ does not have LP.

Question:

Does $C^*(G)$ fail to have WLP for every property (T) group?