Topological dimension of C^*-algebras

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The prototypical C^*-algebra:

$$\mathcal{B}(\mathcal{H}) := \{\text{continuous (=operator norm-bounded)} \\ \text{linear operators on } \mathcal{H}\},$$

where $\mathcal{H}$ is a complex Hilbert space.

Definition

A C^*-algebra is a norm-closed *-subalgebra of $\mathcal{B}(\mathcal{H})$.

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Examples of C^*-algebras

$M_n(\mathbb{C}) = \mathcal{B}(\mathcal{H})$ for $\mathcal{H} = \mathbb{C}^n$

$C_0(X, \mathbb{C})$ where X is a locally compact Hausdorff topological space

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C^*-algebras can be constructed from:

groups (via unitary representations)

coarse metric spaces

rings

graphs

topological dynamical systems (encoding orbit equivalence)

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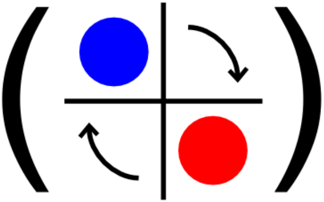

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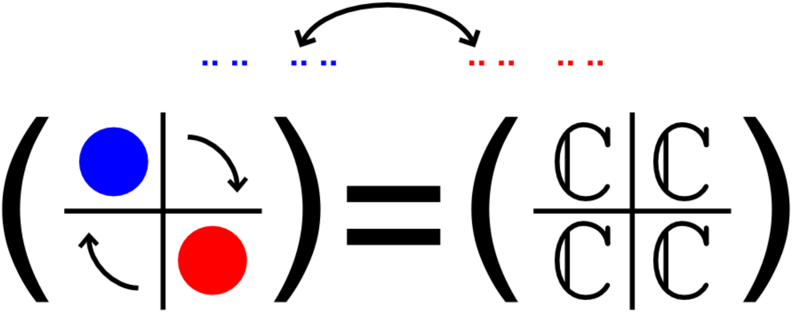

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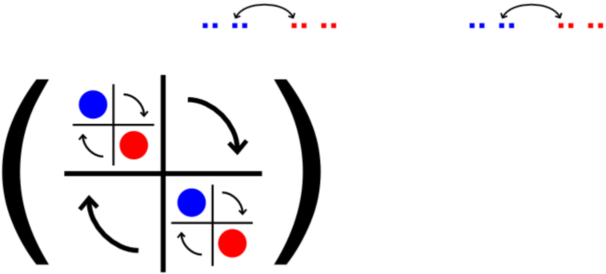

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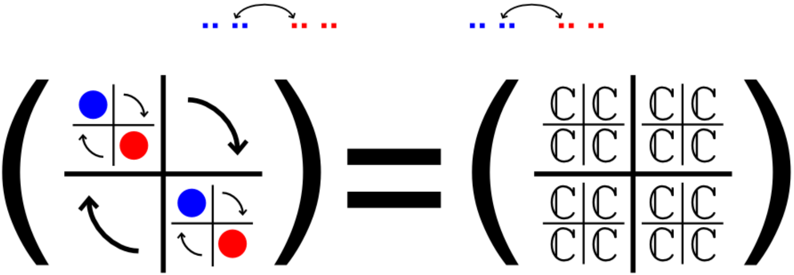

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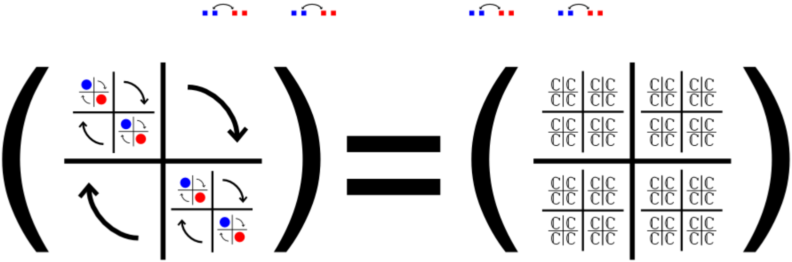

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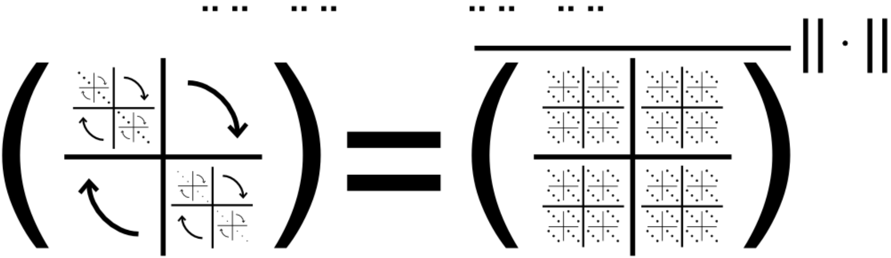

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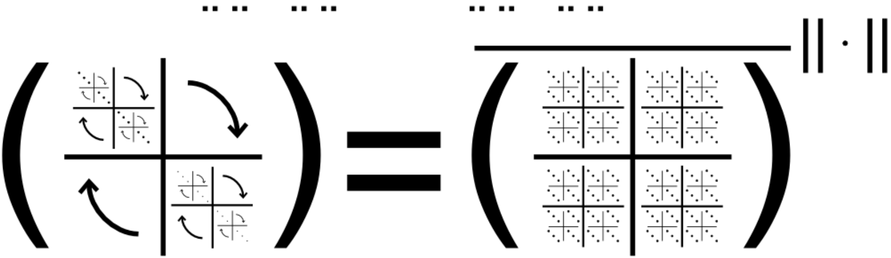

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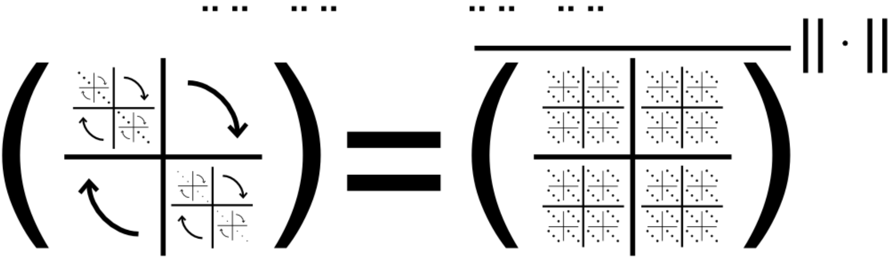

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Note: closure in a weaker topology gives a famous von Neumann algebra $\mathcal{R}$, the "hyperfinite II_1-factor."

UHF algebras

There is only one Cantor set, but:

Glimm (1959)

For p, q prime, $M_{p^\infty} \cong M_{q^\infty}$ if and only if $p = q$.

Note for later: $M_{p^\infty} \otimes M_{p^\infty} \cong M_{p^\infty}$.

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Amenability

"Amenability" for C^*-algebras is characterised by a finite approximation property:

Definition

A C^*-algebra A is amenable if there is a sequence (or net) of **approximately**$^{\triangle}$ commuting diagrams

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\psi_n"'] & & A \\ & F_n \arrow[ur, "\phi_n"'] & \end{tikzcd}$$

where F_n is **finite dimensional.**

Requiring $*$-homomorphisms would be too strong.
Instead, we require ϕ_n, ψ_n to be "completely positive contractions" (maps that order structure).
$^{\triangle}$: approximately means $\|\psi_n(\phi_n(a)) - a\| \xrightarrow{n\to\infty} 0 \ \forall a \in A$.

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Covering dimension

From Lebesgue, we have the following fruitful notion.

Definition

The covering dimension of a normal topological space X is the least number d such that, every finite open cover $\mathcal{U}$ of X has finite refinement $\mathcal{V}$ that can be coloured with $(d+1)$ colours.

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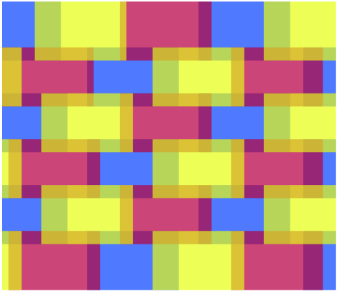

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Highly desirable to extend dimension to all C^*-algebras (from the class $\{C_0(X,\mathbb{C})\}$ of abelian C^*-algebras).

Nuclear dimension

Nuclear dimension marries the two previous concepts: amenability and covering dimension.

Definition (Kirchberg–Winter '04, Winter–Zacharias '10

The nuclear dimension of a C^*-algebra A is the least integer d such that there is a sequence (or net) of approximately commuting diagrams

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$d+1$-colourable: F_n decomposes as a direct sum $F_n^{(0)} \oplus \cdots \oplus F_n^{(d)}$ and $\phi_n|_{F_n^{(d)}}$ preserves orthogonality.

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Nuclear dimension: properties

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For a compact metrisable space X,

$$\dim_{nuc} C(X, \mathbb{C}) = \dim X = \dim_{nuc} C(X, M_p(\mathbb{C}))$$

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

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The Jiang-Su algebra

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

What is $\mathcal{Z}$?

It is a lot like M_{p^∞} except no nontrivial projections; in fact, is is more like an interpolation between M_{p^∞} and M_{q^∞} where p, q are coprime.

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Dimension reduction

What is $\dim_{nuc} C(X,\mathbb{C}) \otimes M_{p^\infty}$?

There is tension between the constant dimension of $C(X, M_{p^n}) = C(X,\mathbb{C}) \otimes M_{p^n}$ and the extra space afforded by the regularity of M_{p^∞}.

Theorem (T-Winter '14)

$\dim_{nuc}(C(X,\mathbb{C}) \otimes M_{p^\infty}) \leq 2$ for any compact space X.
(Also, $\dim_{nuc}(C(X,\mathbb{C}) \otimes \mathcal{Z}) \leq 2$.

This is a bit surprising, since

$$\dim_{nuc}(C(X,\mathbb{C}) \otimes C(Y,\mathbb{C})) = \dim(X \times Y) \geq \dim X = \dim_{nuc} C(X,\mathbb{C}),$$

so dimension reduction, where

$$\dim_{nuc}(C(X,\mathbb{C}) \otimes M_{2^\infty}) < \dim_{nuc} C(X,\mathbb{C})$$

is a purely noncommutative phenomenon.

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There is tension between the constant dimension of $C(X, M_{p^n}) = C(X, \mathbb{C}) \otimes M_{p^n}$ and the extra space afforded by the regularity of M_{p^∞}.

Theorem (T-Winter '14)

$\dim_{nuc}(C(X, \mathbb{C}) \otimes M_{p^\infty}) \leq 2$ for any compact space X.
(Also, $\dim_{nuc}(C(X, \mathbb{C}) \otimes \mathcal{Z}) \leq 2$.

This is a bit surprising, since

$$\dim_{nuc}(C(X, \mathbb{C}) \otimes C(Y, \mathbb{C})) = \dim(X \times Y) \geq \dim X = \dim_{nuc} C(X, \mathbb{C}),$$

so dimension reduction, where

$$\dim_{nuc}(C(X, \mathbb{C}) \otimes M_{2^\infty}) < \dim_{nuc} C(X, \mathbb{C})$$

is a purely noncommutative phenomenon.

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Theorem (T-Winter '12)

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More generally, inspired by what is known (and conjectured) for classifiable C^*-algebras:

Conjecture

If A is simple and nuclear then $A \otimes M_{p^\infty}$ has finite nuclear dimension (for any p).

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$\dim_{nuc}(A \otimes M_{p^\infty}) \leq 2$ if A is subhomogeneous, i.e., a subalgebra of $C(X, M_k)$.

Theorem (Matui-Sato '13, Sato-White-Winter '14, Bosa-Brown-Sato-T-White-Winter '14)

$\dim_{nuc}(A \otimes M_{p^\infty}) \leq 1$ if A is simple, unital, amenable, and not too many extreme traces. (Specifically, if $\partial_e T(A)$ is compact.)

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