Coarse geometry and quantum groups

Christian Voigt

University of Glasgow
christian.voigt@glasgow.ac.uk
http://www.maths.gla.ac.uk/~cvoigt/

Aberdeen
May 13, 2013

Noncommutative discrete spaces

Noncommutative discrete spaces

Definition

A noncommutative discrete space X is a triple $(\mathrm{Irr}(X), C_c(X), \phi)$ where

- $\mathrm{Irr}(X)$ is a set,
- $C_c(X)$ is a complex $*$-algebra of the form

$$C_c(X) \cong \text{alg-} \bigoplus_{x \in \mathrm{Irr}(X)} M_{n_x}(\mathbb{C})$$

  where $n_x \in \mathbb{N}$ for all $x \in \mathrm{Irr}(X)$,
- $\phi : C_c(X) \to \mathbb{C}$ is a faithful positive linear functional.

Noncommutative discrete spaces

Noncommutative discrete spaces

- We use the notation

$$C_0(X) = C^*\text{-}\bigoplus_{x \in \mathrm{Irr}(X)} M_{n_x}(\mathbb{C})$$

$$l^\infty(X) = l^\infty\text{-}\bigoplus_{x \in \mathrm{Irr}(X)} M_{n_x}(\mathbb{C})$$

$$C(X) = \prod_{x \in \mathrm{Irr}(X)} M_{n_x}(\mathbb{C})$$

  in the sequel.

- We denote by $l^2(X)$ the Hilbert space completion of $C_c(X)$ with respect to ϕ.

Noncommutative discrete spaces

We will be interested in certain bounded operators on $l^2(X)$.

Noncommutative discrete spaces

We will be interested in certain bounded operators on $l^2(X)$.

Notice first that any bounded operator $T \in \mathcal{L}(l^2(X))$ determines an (operator-valued) matrix

$$(T_{x,y})_{x,y \in \mathrm{Irr}(X)}$$

where

$$T_{x,y} = p_x T p_y \in \mathcal{L}(M_{n_y}(\mathbb{C}), M_{n_x}(\mathbb{C})).$$

Here

$$p_z \in \mathcal{L}(l^2(X))$$

for $z \in \mathrm{Irr}(X)$ is the orthogonal projection onto $M_{n_x}(\mathbb{C}) \subset l^2(X)$.

Noncommutative discrete spaces

Using the identification

$$\iota_\phi : M_{n_y}(\mathbb{C}) \cong M_{n_y}(\mathbb{C})^*$$

given by $\iota_\phi(f)(g) = \phi(fg)$ we identify

$$T = (T_{x,y})_{x,y \in \mathrm{Irr}(X)}$$

with its *kernel*, that is, with the corresponding element

$$K_T \in C(X \times X) = \prod_{x,y \in \mathrm{Irr}(X)} M_{n_x}(\mathbb{C}) \otimes M_{n_y}(\mathbb{C})$$

in the sequel.

Noncommutative discrete spaces

We say that $K = (K_{x,y}) \in C(X \times X)$ is a *finite kernel* if

- K defines a bounded operator on $l^2(X)$, that is, $K = K_T$ for some $T \in \mathcal{L}(l^2(X))$
- K is row-finite and column-finite, that is, for every $x \in \mathrm{Irr}(X)$

$$K_{x,y} \neq 0, \qquad K_{z,x} \neq 0$$

for only finitely many $y, z \in \mathrm{Irr}(X)$.

Let $K, L \in C(X \times X)$ be finite kernels. We write

$$(K \circ L)_{x,z} = \sum_{y \in \mathrm{Irr}(X)} K_{x,y} L_{y,z}, \qquad \sigma(K)_{x,y} = K^*_{y,x}$$

for their *composition* and for the *adjoint*, respectively.

Noncommutative discrete spaces

Let $K, L \in C(X \times X)$ be finite kernels. We write

$$(K \circ L)_{x,z} = \sum_{y \in \mathrm{Irr}(X)} K_{x,y} L_{y,z}, \qquad \sigma(K)_{x,y} = K^*_{y,x}$$

for their *composition* and for the *adjoint*, respectively.

These operations correspond to composition and taking adjoints of operators, that is,

$$K_{R \circ T} = K_R \circ K_T, \qquad K_{T^*} = \sigma(K_T),$$

for $R, T \in \mathcal{L}(l^2(X))$.

Coarse structures

Definition

Let $X = (\mathrm{Irr}(X), C_c(X), \phi)$ be a noncommutative discrete space. A coarse structure for X is a collection $\mathcal{E}$ of linear subspaces of $C(X \times X)$, called *controlled subspaces*, consisting of finite kernels such that

- If $E \in \mathcal{E}$ and $F \subset E$ then $F \in \mathcal{E}$.
- If $E_1, E_2 \in \mathcal{E}$ then $E_1 + E_2 \in \mathcal{E}$ and $E_1 \circ E_2 \in \mathcal{E}$.
- If $E \in \mathcal{E}$ then $\sigma(E) \in \mathcal{E}$.
- The space $C_c(X \times X)$ is contained in $\mathcal{E}$.
- All kernels corresponding to elements in the center $Z(l^\infty(X))$ are contained in $\mathcal{E}$.

A *noncommutative coarse space* is a noncommutative set X equipped with a coarse structure.

The uniform Roe algebra

By the definition of coarse structures, the collection of all operators associated to kernels in E for some $E \in \mathcal{E}$ forms a $*$-subalgebra $\mathbb{C}_u(X)$ of $\mathcal{L}(l^2(X))$.

Definition

Let $(X, \mathcal{E})$ be a noncommutative coarse space. The *uniform Roe algebra* $C_u^*(X) \subset \mathcal{L}(l^2(X))$ is the C^*-algebra obtained as the norm closure of $\mathbb{C}_u(X)$.

These definitions extend the standard definitions in the case that all matrix blocks in X have size one.

Discrete quantum groups

Discrete quantum groups

Definition

A discrete quantum group G is given by a unital C^*-algebra $S = C^*_{\text{red}}(G)$ together with a unital $*$-homomorphism $\Delta : S \to S \otimes S$ such that

$$\begin{tikzcd} S \arrow[r, "\Delta"] \arrow[d, "\Delta"] & S \otimes S \arrow[d, "\mathrm{id} \otimes \Delta"] \\ S \otimes S \arrow[r, "\Delta \otimes \mathrm{id}"] & S \otimes S \otimes S \end{tikzcd}$$

is commutative and $\Delta(S)(1 \otimes S)$ and $(S \otimes 1)\Delta(S)$ are dense subspaces of $S \otimes S$.

Example: Discrete groups

- If G is a discrete group then $S = C^*_{\mathrm{red}}(G)$ defines a discrete quantum group.

Example: Discrete groups

- If G is a discrete group then $S = C^*_{\mathrm{red}}(G)$ defines a discrete quantum group.

  The comultiplication $\Delta : C^*_{\mathrm{red}}(G) \to C^*_{\mathrm{red}}(G) \otimes C^*_{\mathrm{red}}(G)$ is given by

  $$\Delta(s) = s \otimes s$$

  for $s \in G \subset \mathbb{C}[G] \subset C^*_{\mathrm{red}}(G)$.

Example: Compact groups

- If G is a compact group then $S = C(G)$ determines a discrete quantum group.

Example: Compact groups

- If G is a compact group then $S = C(G)$ determines a discrete quantum group.

  The comultiplication $\Delta : C(G) \to C(G) \otimes C(G) = C(G \times G)$ is given by
  $$\Delta(f)(s,t) = f(st)$$

Example: Compact groups

- If G is a compact group then $S = C(G)$ determines a discrete quantum group.

  The comultiplication $\Delta : C(G) \to C(G) \otimes C(G) = C(G \times G)$ is given by
  $$\Delta(f)(s,t) = f(st)$$

- Every *commutative* discrete quantum group is of this form.

Example: The quantum group $SU_q(2)$

Example: The quantum group $SU_q(2)$

Definition
Fix $q \in [-1, 1] \setminus \{0\}$.

Example: The quantum group $SU_q(2)$

Definition

Fix $q \in [-1, 1] \setminus \{0\}$.

The C^*-algebra $C(SU_q(2))$ is the universal C^*-algebra generated by elements α and γ satisfying the relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma,$$
$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

Example: The quantum group $SU_q(2)$

Definition

Fix $q \in [-1, 1] \setminus \{0\}$.
The C^*-algebra $C(SU_q(2))$ is the universal C^*-algebra generated by elements α and γ satisfying the relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma,$$
$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

These relations are equivalent to saying that the fundamental matrix

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is unitary.

Example: The quantum group $SU_q(2)$

If we write

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

then $\Delta : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$ is given by

$$\Delta(u_{ij}) = u_{i1} \otimes u_{1j} + u_{i2} \otimes u_{2j}.$$

Example: The quantum group $SU_q(2)$

If we write
$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$
then $\Delta : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$ is given by
$$\Delta(u_{ij}) = u_{i1} \otimes u_{1j} + u_{i2} \otimes u_{2j}.$$

For $q = 1$ one obtains in this way the C^*-algebra $C(SU(2))$ of functions on $SU(2)$ together with the group structure of $SU(2)$.

Peter-Weyl theory

Peter-Weyl theory

Let G be a discrete quantum group. A finite dimensional corepresentation of G is a unitary $u^\pi = (u^\pi_{ij}) \in C^*_{\mathrm{red}}(G) \otimes M_{\dim(\pi)}(\mathbb{C}) = M_n(C^*_{\mathrm{red}}(G))$ such that

$$\Delta(u^\pi_{ij}) = \sum_{k=1}^{n} u^\pi_{ik} \otimes u^\pi_{kj}.$$

Peter-Weyl theory

Let G be a discrete quantum group. A finite dimensional corepresentation of G is a unitary $u^\pi = (u^\pi_{ij}) \in C^*_{\mathrm{red}}(G) \otimes M_{\dim(\pi)}(\mathbb{C}) = M_n(C^*_{\mathrm{red}}(G))$ such that

$$\Delta(u^\pi_{ij}) = \sum_{k=1}^n u^\pi_{ik} \otimes u^\pi_{kj}.$$

A corepresentation is called irreducible if

$$(\mathrm{id} \otimes T)u^\pi = u^\pi(\mathrm{id} \otimes T)$$

for $T \in M_{\dim(\pi)}(\mathbb{C})$ implies $T \in \mathbb{C}\,\mathrm{id}$.

Peter-Weyl theory

Let G be a discrete quantum group. A finite dimensional corepresentation of G is a unitary $u^\pi = (u^\pi_{ij}) \in C^*_{\mathrm{red}}(G) \otimes M_{\dim(\pi)}(\mathbb{C}) = M_n(C^*_{\mathrm{red}}(G))$ such that

$$\Delta(u^\pi_{ij}) = \sum_{k=1}^{n} u^\pi_{ik} \otimes u^\pi_{kj}.$$

A corepresentation is called irreducible if

$$(\mathrm{id} \otimes T)u^\pi = u^\pi(\mathrm{id} \otimes T)$$

for $T \in M_{\dim(\pi)}(\mathbb{C})$ implies $T \in \mathbb{C}\,\mathrm{id}$.

We write $\mathrm{Irr}(G)$ for the set of all isomorphism classes of irreducible corepresentations of G.

Theorem (Peter-Weyl)

Let G be a discrete quantum group. There exists a canonical dense Hopf $*$-subalgebra $\mathbb{C}[G]$ inside $C^*_{\mathrm{red}}(G)$ such that

$$C_c(G) = \mathbb{C}[G]^* \cong \bigoplus_{\pi \in \mathrm{Irr}(G)} M_{\dim(\pi)}(\mathbb{C})$$

Moreover $C_c(G)$ is a multiplier Hopf $*$-algebra equipped with a faithful positive left invariant linear functional ϕ, uniquely determined up to a scalar.

We may therefore view $(\mathrm{Irr}(G), C_c(G), \phi)$ as a noncommutative discrete space.

The standard coarse structure

The standard coarse structure

Let G be a discrete quantum group.

Let G be a discrete quantum group.

If $F \subset \mathrm{Irr}(G)$ is a subset we write $\mathbb{C}[F] \subset \mathbb{C}[G]$ for the linear span of all matrix coefficients of corepresentations in F.

The standard coarse structure

Let G be a discrete quantum group.

If $F \subset \mathrm{Irr}(G)$ is a subset we write $\mathbb{C}[F] \subset \mathbb{C}[G]$ for the linear span of all matrix coefficients of corepresentations in F.

Let $\pi : C_0(G) \to \mathcal{L}(l^2(G))$ and $\hat{\pi} : C^*_{\mathrm{red}}(G) \to \mathcal{L}(l^2(G))$ be the left regular representations.

The standard coarse structure

Let G be a discrete quantum group.

If $F \subset \mathrm{Irr}(G)$ is a subset we write $\mathbb{C}[F] \subset \mathbb{C}[G]$ for the linear span of all matrix coefficients of corepresentations in F.

Let $\pi : C_0(G) \to \mathcal{L}(l^2(G))$ and $\hat{\pi} : C^*_{\mathrm{red}}(G) \to \mathcal{L}(l^2(G))$ be the left regular representations.

We consider as *basic controlled subspaces* in $C(G \times G)$ the spaces of kernels of operators on $l^2(G)$ of the form

$$\hat{\pi}(x)\pi(f) \in \mathcal{L}(l^2(G))$$

where $x \in \mathbb{C}[F]$ for some *finite set* $F \subset \mathrm{Irr}(G)$ and $f \in Z(l^\infty(G))$, the *center* of $l^\infty(G)$.

The standard coarse structure

Definition

The standard coarse structure on G is the coarse structure generated by all basic controlled subspaces.

We write $C^*_u(G)$ for the uniform Roe algebra associated to the standard coarse structure on G.

The standard coarse structure

Definition

The standard coarse structure on G is the coarse structure generated by all basic controlled subspaces.

We write $C^*_u(G)$ for the uniform Roe algebra associated to the standard coarse structure on G.

We always have

$$G \ltimes_{\mathrm{red}} C_0(G) \subset C^*_u(G) \subset G \ltimes_{\mathrm{red}} l^\infty(G),$$

and these inclusions are typically *strict*.

Coarse equivalence

The notion of a $*$-homomorphism compatible with coarse structures is too restrictive to encompass all interesting morphisms between noncommutative coarse spaces.

Coarse equivalence

The notion of a $*$-homomorphism compatible with coarse structures is too restrictive to encompass all interesting morphisms between noncommutative coarse spaces.

Definition

Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be noncommutative coarse spaces. A coarse equivalence between X and Y is a coarse structure $\mathcal{E}_f$ on $C_c(X) \oplus C_c(Y)$ which is generated by $\mathcal{E}_f \cap C(X \times (X \cup Y))$ and restricts to $\mathcal{E}_X$ and $\mathcal{E}_Y$ on $C_c(X)$ and $C_c(Y)$, respectively.

Coarse equivalence

Proposition

The discrete quantum groups dual to $SU_q(2)$ are all mutually coarsely equivalent for $q \in (-1, 1) \setminus \{0\}$.

We note that these quantum groups are all pairwise monoidally inequivalent.

The Roe algebra and exactness

Theorem

Let G be a discrete quantum group. Then the following conditions are equivalent.

- *G is exact.*
- *The uniform Roe algebra $C_u^*(G)$ is nuclear.*

We point out again that we have

$$C_0(G) \rtimes_{\mathrm{red}} G \subset C_u^*(G) \subset l^\infty(G) \rtimes_{\mathrm{red}} G,$$

and these inclusions are typically *strict*.