Simple amenable operator algebras

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Unital classification theorem (many hands)

Simple, separable, unital, nuclear, $\mathcal{Z}$-stable C^*-algebras satisfying the UCT are classified by K-theory and traces.

A tiny bit of history

- Unital classification theorem first proved in 2015,[a] by combining work of Elliott-Gong-Lin-Niu, Gong-Lin-Niu, Winter, Tikusis-W-Winter with the Kirchberg-Phillips classification theorem.
- I'm discussing a new proof combining Chris Schafhauser's breakthrough approach to classification through the trace kernel extension with methods for handling many traces simultaneously (Carrión, Gabe, Schafhasuer, Tikusis, W.).
- All of this builds, and relies, on decades of research by large numbers of people.

[a] 2015 version of classification theorem had hypothesis of finite nuclear dimension in place of $\mathcal{Z}$-stability.

Dichotomy

Theorem (Kirchberg)

Let A be a simple separable unital and exact $\mathcal{Z}$-stable C^*-algebra. Then A is purely infinite or stably finite.

- Kirchberg actually proved his dichotomy for simple exact algebras with non-trivial decomposition $A \otimes B$.
- Relatively easy proof for UHF-stable C^*-algebras (due to Rørdam) using results from 80's.

Unital Kirchberg Phillips Theorem (94-00)

[illegible] classification of unital simple separable nuclear purely infinite C^*-algebras with UCT by K-theory.

- Classification theorem proceeds using this dichotomy.
- Purely infinite case of classification done.

AF algebras / hyperfinite finite vNAs

1. Classify maps from finite dimensional algebras

$F \to A$ has cancellation by ordered K_0. $[p]_0 = [q]_0 \Rightarrow p \sim q$.

s.t. $F \xrightarrow{\varphi,\psi} \mathcal{M}$ II_1 vNA. by traces. $\tau \circ \varphi = \tau \circ \psi$ $\forall \tau \in T(\mathcal{M})$

$\boxed{\exists \varphi \text{ inducing a map on traces}}$ $\Updownarrow$ $\varphi \sim_u \psi$

2. Classify maps from approximately finite dimensional algebras

$\overline{\bigcup F_n} \to A$

AF C^*-algebras $\to$ $\{C^*$-alg with cancellation$\}$

by ordered K_0.

$(\bigcup F_n)'' \to \mathcal{M}$ II_1

by traces.

3. Symmetrise assumptions: classify approximately finite dimensional algebras

AF $\Rightarrow$ cancellation.

Classification of AF C^*-algs.

Classification of hyperfinite II_1 vNAs

Splitting hypotheses

Hypotheses split

	domain side A	codomain side B
	separable	unital
exactness	(nuclear)	(simple)
	UCT class	$\mathcal{Z}$-stable
		quasi traces are traces

nuclearity, full

- Can go further too, and move some hypotheses to the maps

Ideally, classify maps $A \hookrightarrow B$ by K-theory and traces

But it's not true!

$\mathcal{O}_3 \otimes \mathcal{O}_3$

$x \otimes y \mapsto y \otimes x$

this is trivial on K-theory
but the flip is not approx inner

!ness fails.

Sol^n: Enlarge the invariant.

- Existence becomes even harder.
- $A \hookrightarrow B_\omega$

Additional Invariants

- Total K-theory

$$\underline{K}(A) = K_*(A) \oplus K_*(A;\mathbb{Z}/n\mathbb{Z})$$

$$K_*(A;\mathbb{Z}/n\mathbb{Z}) = K_*(A \otimes \mathcal{O}_{n+1})$$

Thomsen: $\overline{K}_1^{alg}(A) \cong K_1(A) \oplus \dfrac{\mathrm{Aff}\, T(A)}{\overline{\mathrm{Im}\, K_0(A)}}$

non canonical.

- (Hausdorffised, Unitary) Algebraic K_1

$K_1(A)$ abelian gp.

$$U_\infty(A) = \bigcup_{n=1}^{\infty} U_n(A)$$

$DU_\infty(A)$ derived gp

$= \langle uvu^*v^* : u, v \in U_\infty(A) \rangle$.

$$\overline{K}_1^{alg}(A) = U_\infty(A) / \overline{DU_\infty(A)}$$

in the inductive limit top on $U_\infty(A)$.

THE EXTENDED INVARIANT

- $\underline{K}T_u$: K-theory, traces, total K-theory, algebraic K_1.
- and all the natural maps connecting these objects.

THEOREM (CLASSIFICATION OF EMBEDDINGS)

Let A be unital separable nuclear in the UCT class, and let unital B be simple, separable $\mathcal{Z}$-stable (quasitraces are traces). Then injective maps $A \to B$ are classified by $\underline{K}T_u$.

eg) $\underbrace{\left(\bigotimes_{-\infty}^{\infty} \mathcal{Z}\right) \rtimes \mathbb{Z}}_{A}$ Bernoulli shift action

- A has ! trace
- $K_*(A) = (\mathbb{Z}, \mathbb{Z})$

U canonical unitary implementing shift $K_1(A) = \langle [u]_1 \rangle$.

$K_1^{alg}(A) \cong K_1(A) \oplus \underbrace{\mathrm{Aff}\, T(A)}_{\overline{\mathrm{Im}\, \kappa}} = \mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} = \mathbb{Z} \oplus \mathbb{T}$

$(\lambda, 1)$
$\downarrow$
$(1, \lambda)$

$(\lambda, 1)$ gives a copy of $\mathbb{T}$ in A realizing $\mathbb{T}$ in $K_1^{alg}(A)$.

$\varphi : A \to A$
φ gives $\bigotimes_{-\infty}^{\infty} \mathcal{Z}$
& sends $u \mapsto \lambda u$

Any endo of $A = \left(\bigotimes_{-\infty}^{\infty} Z \right) \rtimes \mathbb{Z}$

is approx unitary equiv to

φ_λ : fixes $\otimes Z$

$u \mapsto \lambda u$

or ψ_λ

$u \mapsto \lambda u^*$

& reverses the order of the infinite tensor product.

A little look at uniqueness

Let $\phi, \psi : A \to B_\omega$ agreeing on invariants. $\underline{K}T_u$

$$\begin{tikzcd} & & & A \arrow[dl, "\phi,\psi"'] \arrow[d, "q\circ\phi,\, q\circ\psi"] & \\ 0 \arrow[r] & J_{B_\omega} \arrow[r] & B_\omega \arrow[r, "q"'] & (\overline{B}^{T(B)})^\omega \arrow[r] & 0 \end{tikzcd}$$

McDuff.

$q\circ\phi$, $q\circ\psi$ agree on traces then $q\circ\phi \approx_{au} q\circ\psi$

$\Rightarrow$ $q\circ\phi$ unitarily equivalent to $q\circ\psi$.

$\exists\ \bar{u} \in \mathcal{U}((\overline{B}^{T(B)})^\omega)$

$\bar{u}\, q\circ\phi\, \bar{u}^* = q\circ\psi$.

- $\bar{u} = e^{i\pi h}$ for some self-adjoint $h \in (\overline{B}^{T(B)})^\omega$

$\therefore$ $\bar{u} = q(u)$ for some $u \in \mathcal{U}(B_\omega)$.

A little look at uniqueness

Let $\phi, \psi : A \to B_\omega$.

1. Agreement on traces: adjust by a unitary in B_ω so that $q \circ \phi = q \circ \psi$

$$\begin{tikzcd}
 & & & A \arrow[dl, "{\phi,\psi}"'] \arrow[d, "q\circ\phi = q\circ\phi"] & \\
0 \arrow[r] & J_{B_\omega} \arrow[r] & B_\omega \arrow[r] & (\overline{B}^{T(B)})^\omega \arrow[r] & 0
\end{tikzcd}$$

$$J_{B_\omega} = \{ (x_n) \in \ell^\infty(B) \mid \lim_{n\to\omega} \|x_n\|_{2,T(B)} = 0 \}$$

- $\phi(a) - \psi(a) \in J_{B_\omega}$ for all $a \in A$. $[\phi, \psi]$ is a Cuntz pair rep an element of $KK(A, J_{B_\omega})$.

For a $KK(A, J)$ ← want J to be stable, σ-unital

↯ J_{B_ω} is not stable ↯ Prop J_{B_ω} is separably stable. $\forall$ sep $J_0 \subseteq J_{B_\omega}$ $\exists$ stable sep $J_1 \subseteq J_{B_\omega}$ s.t. $J_0 \subseteq J_1$

$\mathcal{Z}$-STABLE KK-UNIQUENESS THEOREM

THEOREM

Let A be separable, J be separable and stable. Suppose $\phi, \psi : A \to \mathcal{M}(J)$ are absorbing and define a Cuntz pair, i.e. $\phi(a) - \psi(a) \in J$ for all $a \in J$.

$$[\phi, \psi] = 0 \text{ in } KK(A, J) \Longrightarrow \phi(a) \otimes 1_{\mathcal{Z}} \approx \phi(a) \otimes 1_{\mathcal{Z}}$$

$\exists (u_t)_{t \geq 0} \in (I \otimes \mathcal{Z})^{\sim}$

$u_t(\varphi(a) \otimes 1_{\mathcal{Z}}) u_t^* \to \psi(a) \otimes 1_{\mathcal{Z}}$ as $t \to \infty$

- Key idea: Paschke duality

$$KK(A, J) \cong K_1\left(\mathcal{M}(J)/J \cap q(\varphi(A))' \right)$$

$q : \mathcal{M}(J) \to \mathcal{M}(J)/J$. $\quad \underset{\overset{\shortparallel}{0}}{[\varphi, \psi]} \mapsto [u]_1 = 0. \Rightarrow u \otimes 1_{\mathcal{Z}} \sim_h 1 \otimes 1_{\mathcal{Z}}$

in $\left(\mathcal{M}(J)/J \cap q(\varphi(A))'\right) \otimes 1_{\mathcal{Z}}$

A little look at uniqueness

Let $\phi, \psi : A \to B_\omega$ agreeing on invariants.

$$
\begin{tikzcd}
 & & & A \arrow[dl, "\phi{,}\psi"'] & \\
0 \arrow[r] & J_{B_\omega} \arrow[r] & B_\omega \arrow[r] & (\overline{B}^{T(B)})^\omega \arrow[r] & 0
\end{tikzcd}
$$

$$\begin{tikzcd}
K_0(A, \mathbb{Z}/n) \arrow[rr] \arrow[dr, "\gamma_A^{(n)}"'] & & K_1(A) \\
& K_1^{alg}(A) \arrow[ur, "\alpha_A"'] &
\end{tikzcd}$$