The Cuntz semigroup of $C(X, A)$

by

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Abstract

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The Cuntz semigroup is an isomorphism invariant for C^*-algebras consisting of a semigroup with a compatible (though not algebraic) ordering. Its construction is similar to that of the Murray-von Neumann semigroup (from which the ordered K_0-group arises by the Grothendieck construction), but using positive elements in place of projections. Both rich in structure and sensitive to subtleties of the C^*-algebra, the Cuntz semigroup promises to be a useful tool in the classification program for nuclear C^*-algebras. It has already delivered on this promise, particularly in the study of regularity properties and the classification of nonsimple C^*-algebras. The first part of this thesis introduces the Cuntz semigroup, highlights structural properties, and outlines some applications.

The main result of this thesis, however, contributes to the understanding of what the Cuntz semigroup looks like for particular examples of (nonsimple) C^*-algebras. We consider separable C^*-algebras given as the tensor product of a commutative C^*-algebra $C_0(X)$ with a simple, approximately subhomogeneous algebra A, under the regularity hypothesis that A is $\mathcal{Z}$-stable. (The $\mathcal{Z}$-stability hypothesis is needed even to describe the Cuntz semigroup of A.) For these algebras, the Cuntz semigroup is described in terms of the Cuntz semigroup of A and the Murray-von Neumann semigroups of $C(K, A)$ for compact subsets K of X. This result is a marginal improvement over one proven by the author in [49] (there, A is assumed to be unital), although improvements have been made to the techniques used.

The second part of this thesis provides the basic theory of approximately subhomogeneous algebras, including the important computational concept of recursive subhomogeneous algebras. Theory to handle nonunital approximately subhomogeneous algebras is novel here.

In the third part of this thesis lies the main result. The Cuntz semigroup computation is achieved by defining a Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ on the positive elements of the C^*-algebra, picking out certain data from a positive element which obviously contribute to determining its Cuntz class. The proof of the main result has two parts: showing that the invariant $\mathbb{I}(\cdot)$ is (order-)complete, and describing its range.

Dedication

This thesis is dedicated to my wife, Tara Tikuisis, for her endless support in my research endeavors.

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Contents

Chapter 1

Introduction

This thesis contributes to our understanding of the Cuntz semigroup as a C^*-algebra isomorphism invariant, and its relationships with other invariants. The Cuntz semigroup consists of equivalence classes of positive elements of the stabilized C^*-algebra, where the equivalence relation is determined by the Cuntz pre-order on positive elements. For projections, this pre-order agrees with Murray-von Neumann subequivalence, which is used to construct the Murray-von Neumann semigroup, and from this, the K_0-group. However, by using positive elements, the Cuntz semigroup is in many cases able to detect more from a C^*-algebra than the Murray-von Neumann semigroup (and therefore also more than the K_0-group).

This has enabled the Cuntz semigroup to be used to distinguish certain simple C^*-algebras where the classical invariant for classification – the Elliott invariant – fails. An attempt ensued to characterize which C^*-algebras are sufficiently well-behaved to be classified; to this end, certain regularity properties, including properties of the Cuntz semigroup, have been identified, and progress continues to be made in showing these properties to be both equivalent and sufficient for classification. This direction of applications of the Cuntz semigroup are discussed in Sections 2.2.1 and 2.2.2.

Some other important applications of the Cuntz semigroup come from recognizing that it is a useful invariant for dealing with ideal structure of a C^*-algebra. This leads one to view it as a promising candidate for the classification of nonsimple C^*-algebras. Some steps made in this direction are reviewed in Section 2.2.3.

In light of these applications of the Cuntz semigroup, and with an eye towards the likelihood of more applications in the future, one becomes interested in finding a workable picture of the Cuntz semigroup of particular C^*-algebras. For finite, simple, well-behaved C^*-algebras, a picture emerges of the Cuntz semigroup in terms of the Murray-von Neumann semigroup paired with the traces (a version of this result which makes "well-behaved" maximally inclusive is presented in Thm. 2.2.5). For stably finite C^*-algebras of real rank zero (simple or otherwise), the Cuntz semigroup can be constructed using the Murray-von Neumann semigroup alone [12]. Beyond these examples, there has also been a partial computation of the Cuntz semigroup of a commutative C^*-algebra with finite-dimensional spectrum in [43] (the computation is complete when the dimension of the spectrum is at most three). This computation relates the Cuntz semigroup of $C_0(X)$ to the Murray-von Neumann semigroups of $C(K)$ for compact subsets K of X.

The main result of this thesis provides a computation of the Cuntz semigroup for a certain class of nonsimple C^*-algebras, forming a certain natural next step in this investigation. The C^*-algebras in the class considered are separable algebras of the form $C_0(X, A)$ where X is a locally compact Hausdorff space and A is a simple, $\mathcal{Z}$-stable ASH algebra. In certain ways, this comes close to computing the Cuntz semigroup of any trivial field of finite simple C^*-algebras: the restriction that A be ASH is not known to exclude any nuclear simple finite C^*-algebra, while $\mathcal{Z}$-stability is necessary even to compute the Cuntz semigroup of A (at least, through known techniques).

This computation describes the Cuntz semigroup of such C^*-algebras $C_0(X, A)$ in terms of the Cuntz semigroup of A together with the Murray-von Neumann semigroups of $C(K, A)$ for compact subsets K of X. It is achieved by defining a Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ on the positive elements of $C_0(X, A \otimes \mathcal{K})$. First and foremost, $\mathbb{I}(a)$ tells us what the Cuntz class of $a(x)$ is (in the Cuntz semigroup of A) at each point $x \in X$ – that is, it contains the function $X \to \mathcal{C}u(A) : x \mapsto [a(x)]$. In addition, it tells us about the behaviour of a on any level set of this function, where the value on this level set is the Cuntz class of a projection. This is necessary since, for example, if such a level set (for a projection p) consists of a circle, then the Murray-von Neumann class of a restricted to this level set encodes the homotopy class of a unitary (with respect to p as a unit).

In Section 4.2, it is shown that $\mathbb{I}(\cdot)$ is a complete invariant (and in fact an order embedding) for the positive elements of $C_0(X, A \otimes \mathcal{K})$. In Section 4.3, the range of $\mathbb{I}(\cdot)$ is determined. Together with knowing that $\mathbb{I}(\cdot)$ is complete, this tells us exactly what the Cuntz semigroup of $C_0(X, A)$ is. Both of these results rely heavily on the ASH structure of the C^*-algebra A, as the chief computations occur in the finite stage algebras (which are subhomogeneous with a special form). Chapter 3 is devoted to introducing ASH algebras and their structure. Nonunital ASH algebras are considered here, though they have rarely been studied systematically in the literature; therefore, Section 3.2 contains novel results on the structure of such C^*-algebras.

Chapter 2

The Cuntz semigroup

2.1 Definitions and basic properties

Definition 2.1.1. *Let A be a C^*-algebra. The Cuntz relations on the positive elements of $A \otimes \mathcal{K}$ are given as follows. For $a, b \in (A \otimes \mathcal{K})_+$, we say that a is* ***Cuntz below*** *b if there exists a sequence $(s_n) \subset A \otimes \mathcal{K}$ such that*

$$\|a - s_n b s_n^*\| \to 0$$

as $n \to \infty$. If each of a and b is Cuntz below the other one, then a is ***Cuntz equivalent*** *to b.*

One may easily show that the Cuntz below relation just defined is transitive and reflexive, and therefore a pre-order. In particular, it induces an order on the Cuntz equivalence classes. The following easy-to-show facts are needed in order to establish an addition operation on the set of Cuntz equivalence classes: first, for any two elements $a, b \in (A \otimes \mathcal{K})_+$, there exist orthogonal elements $a', b' \in (A \otimes \mathcal{K})_+$ which are Cuntz equivalent to a, b respectively; secondly, if for each of $i = 1, 2$, $a_i, b_i \in (A \otimes \mathcal{K})_+$ is a pair of orthogonal elements, such that a_1, b_1 are respectively Cuntz equivalent to a_2, b_2, then $a_1 + b_1$ is Cuntz equivalent to $a_2 + b_2$.

Definition 2.1.2. *The* ***Cuntz semigroup*** *of a C^*-algebra A consists of the Cuntz equivalence classes of elements in $(A \otimes \mathcal{K})_+$. The Cuntz semigroup is denoted $\mathcal{C}u(A)$, and the equivalence class of $a \in (A \otimes \mathcal{K})_+$ is denoted $[a]$. For elements $[a], [b] \in \mathcal{C}u(A)$, their sum is given by*

$$[a] + [b] := [a' + b']$$

where $a', b' \in (A \otimes \mathcal{K})_+$ are orthogonal elements such that $[a] = [a']$ and $[b] = [b']$. The Cuntz semigroup is also endowed with the ordering induced by the Cuntz below relation.

The Cuntz semigroup is fairly rich in structure. To begin with, there is no better time than now to state the following vastly important lemma. The last statement of the lemma constitutes a weaker version of the lemma, yet is used much more frequently than the strong form. This weaker form appears earlier in the literature, in [44, Proposition 2.2].

Lemma 2.1.3. *[30, Lemma 2.2] Let A be any C^*-algebra and let $a, b \in (A \otimes \mathcal{K})_+$. For any real number ϵ satisfying $\|a - b\| < \epsilon$, we have $(a - \epsilon)_+ = dbd^*$ for some $d \in (A \otimes \mathcal{K})$. In particular, $[(a - \epsilon)_+] \leq [b]$ in $\mathcal{C}u(A)$*

We shall now recall some of the main useful facts regarding the order structure of the Cuntz semigroup. In the sequel, we will use these facts without explicit reference.

Proposition 2.1.4. *[12, Theorem 1 (i)] Let A be a C^*-algebra. Every increasing sequence in $\mathcal{C}u(A)$ has a supremum in $\mathcal{C}u(A)$.*

Definition 2.1.5. *Let S be an ordered set. For elements $x, y \in S$, we say that x is **far below** y if, for any increasing sequence*

$$z_1 \leq z_2 \leq \cdots$$

*with a supremum $z \geq y$, there exists an index i such that $x \leq z_i$. We use $x \ll y$ to denote that x is far below y. An element $x \in S$ is called **compact** if it is far below itself.*

It may be noted that the relation "far below" has also been called "way below" and "compactly contained" in the literature. For the ordered set of open subsets of a topological space, compact containment amounts to the same definition as far below, except allowing increasing nets instead of sequences (the two notions agree for second countable, locally compact Hausdorff spaces).

Proposition 2.1.6. *For any $\epsilon > 0$,*

$$[(a - \epsilon)_+] \ll [a].$$

In particular, every element of $\mathcal{C}u(A)$ is the supremum of a sequence of a $\ll$-increasing sequence.

Proof. This is essentially contained in the proof of [12, Theorem 1]. □

The last two propositions inspire us to define a category $\mathcal{O}rd\mathcal{C}u$ with the order-theoretic structure just revealed for Cuntz semigroups. (The symbol $\mathcal{O}rd\mathcal{C}u$ is used since this category is essentially given by taking only the order-theoretic aspects of the category $\mathcal{C}u$ defined in [12, Section 2].)

Definition 2.1.7. *The objects of the category $\mathcal{O}rd\mathcal{C}u$ are ordered sets S which satisfy the following two conditions.*

(i) Every increasing sequence in S has a supremum in S; and

(ii) Every element of S is the supremum of a $\ll$-increasing sequence.

The morphisms between S and T in $\mathcal{O}rd\mathcal{C}u$ are functions from S to T which preserve both the order and the far below relation.

We shall next define what we mean by a dense subset of an object in $\mathcal{O}rd\mathcal{C}u$. We warn that, although it has connections and resemblances to topological density, there is probably no way to define a topology on an object of $\mathcal{O}rd\mathcal{C}u$ so that the following definition amounts to topological density.

Definition 2.1.8. *For an object S of $\mathcal{O}rd\mathcal{C}u$ and a subset D of S, we shall say that D is* ***dense*** *in S if every element of S is the supremum of an increasing sequence of elements from D.*

Proposition 2.1.9. *Let $S \in \mathcal{O}rd\mathcal{C}u$ and suppose that $D \subseteq S$ is dense. Then in fact, every element of S is the supremum of a $\ll$-increasing sequence from D.*

Proof. Let $s \in S$. There exists a $\ll$-increasing sequence $(s_n) \subseteq S$ such that $s = \sup s_n$. For each k, since $s_{2k} \ll s_{2k+1}$ and s_{2k+1} is the supremum of an increasing sequence from D, there must exist $d_k \in D$ such that $s_{2k} \leq d_k \leq s_{2k+1}$. Since $s_{2k+1} \ll s_{2(k+1)}$, the elements d_k that we just found satisfy $d_k \ll d_{k+1}$. Moreover, since they intertwine the sequence (s_n), we see that $\sup d_k = \sup s_n = s$, as required. □

Proposition 2.1.10. *If A is a separable C^*-algebra then $\mathcal{C}u(A)$ contains a countable dense set.*

Proof. If S is a dense subset of $(A \otimes \mathcal{K})_+$ then

$$\{(a - 1/n)_+ : a \in S, n \geq 1\}$$

is dense in $\mathcal{C}u(A)$ (this can be proven using Lemma 2.1.3). □

The Cuntz semigroup is inspired by and related to the Murray-von Neumann semigroup, which we shall introduce next. Like the Cuntz semigroup, the Murray-von Neumann semigroup is constructed by imposing a pre-order relation on a subset of the elements of $A \otimes \mathcal{K}$.

Definition 2.1.11. *Let A be a C^*-algebra. The Murray-von Neumann relations on the projections of $A \otimes \mathcal{K}$ are given as follows. For two projections, $p, q \in A \otimes \mathcal{K}$, we say that p is **Murray-von Neumann equivalent** to q (and write $p \sim q$) if there exists an element $v \in A \otimes \mathcal{K}$ such that*

$$p = v^*v \text{ and } vv^* = q.$$

*We say that p is **Murray-von Neumann equivalent to a subprojection** of q (or, for conciseness, p is Murray-von Neumann subequivalent to q, and we write $p \preceq q$) if p is Murray-von Neumann equivalent to a subprojection of q; that is, if there exists an element $v \in A \otimes \mathcal{K}$ such that*

$$p = v^*v \text{ and } vv^* \leq q.$$

One may easily show that subequivalence is a pre-order while Murray-von Neumann equivalence is an equivalence relation. We once again need two simple results in order to be able to establish the addition operation on the Murray-von Neumann semigroup: first, for any two projections $p, q \in A \otimes \mathcal{K}$, there exist orthogonal projections $p', q' \in A \otimes \mathcal{K}$ which are Murray-von Neumann equivalent to p, q respectively; secondly, if $p_i, q_i \in A \otimes \mathcal{K}$

is a pair of orthogonal projections for $i = 1, 2$ such that p_1, q_1 are Murray-von Neumann equivalent to p_2, q_2 respectively then $p_1 + q_1$ is Murray-von Neumann equivalent to $p_2 + q_2$.

Definition 2.1.12. *The* ***Murray-von Neumann semigroup*** *of a C^*-algebra A consists of the Murray-von Neumann equivalence classes of projections in $A \otimes \mathcal{K}$. The Murray-von Neumann semigroup is denoted $V(A)$ and the equivalence class of a projection $p \in A \otimes \mathcal{K}$ is denoted $[p]$. For elements $[p], [q] \in V(A)$, their sum is given by*

$$[p] + [q] = [p' + q']$$

where $p', q' \in A \otimes \mathcal{K}$ are orthogonal projections such that $[p] = [p']$ and $[q] = [q']$.

In contrast to how the Cuntz below relation induces an order on $\mathcal{C}u(A)$, the Murray-von Neumann subequivalence relation may only give a pre-order on $V(A)$. The reason for this is that while two positive elements that are mutually Cuntz below each other are (by definition) Cuntz equivalent, two projections which are mutually Murray-von Neumann subequivalent to each other may not be Murray-von Neumann equivalent. If v, w are two elements implementing Murray-von Neumann subequivalences $p \preceq q$ and $q \preceq p$ respectively, then wv is an element implementing a Murray-von Neumann subequivalence $p \preceq p$; moreover, if we have

$$vv^* < q \text{ or } ww^* < p$$

then we must have $(wv)^*(wv) < p$, i.e. that p is Murray-von Neumann equivalent to a proper subprojection of itself. This brings up the notion of finiteness for a projection.

Definition 2.1.13. *Let A be a C^*-algebra. A projection $p \in A$ is* ***finite*** *if it is not Murray-von Neumann equivalent to any proper subprojection of itself. The C^*-algebra A is* ***stably finite*** *if every projection in $A \otimes \mathcal{K}$ is finite.*

From what was said above, if A is stably finite then whenever two projections are mutually Murray-von Neumann subequivalent to each other it follows that they are

Murray-von Neumann equivalent. Therefore, when A is stably finite, $V(A)$ is endowed with an order (the one induced by Murray-von Neumann subequivalence).

Here is a key well-known fact relating the Cuntz below relation and Murray-von Neumann subequivalence.

Proposition 2.1.14. *Let A be a C^*-algebra. Let $p, q \in A \otimes \mathcal{K}$ be projections. Then p is Cuntz below q if and only if p is Murray-von Neumann subequivalent to q.*

If A is stably finite then p is Cuntz equivalent to q if and only if p is Murray-von Neumann equivalent to q, and therefore, there is a natural embedding $V(A) \subseteq \mathcal{C}u(A)$.

Proof. If $[p] \leq [q]$ in $\mathcal{C}u(A)$ then we may find $x \in A \otimes \mathcal{K}$ such that xqx^* is close to p, say within ϵ for some $\epsilon < 1$.

By Lemma 2.1.3, there exists d such that $(p-\epsilon)_+ = dxqx^*d^*$. Since p is a projection, $(p-\epsilon)_+ = (1-\epsilon)p$, and hence if we set $s = (1-\epsilon)^{-1/2}qx^*d^*$, we have

$$s^*s = p \text{ and } ss^* \leq q,$$

as required.

The opposite implication is trivial, as are the remaining statements of this proposition.

□

As we will be dealing almost exclusively with stably finite C^*-algebras, the last proposition justifies using the same notation, $[p]$, for the Cuntz class and for the Murray-von Neumann class of a projection p.

A Cuntz element $[a] \in \mathcal{C}u(A)$ is in the image of the embedding $V(A) \to \mathcal{C}u(A)$ as long as the element a is Cuntz equivalent to a projection. For example, whenever $\chi_{(0,\|a\|]}$ is a continuous function on the spectrum of an element $a \in (A \otimes \mathcal{K})_+$, the element a is Cuntz equivalent to the projection $\chi_{(0,\|a\|]}(a)$. The next proposition shows, perhaps surprisingly, that the only way for a to be Cuntz equivalent to a projection is in this way, giving a useful characterization of when $[a] \in V(A)$.

Proposition 2.1.15. *[6, Theorem 3.5] Let A be a stably finite C^*-algebra and let $a \in (A \otimes \mathcal{K})_+$. Then the following are equivalent.*

(i) $[a] \in V(A)$;

(ii) $[a] \ll [a]$ *in* $\mathcal{C}u(A)$;

(iii) 0 *is an isolated point of the spectrum of* a.

In many cases (such as orderings on groups), ordered semigroups have the algebraic order, meaning that elements α, β satisfy

$$\alpha \leq \beta \text{ iff } \alpha + \gamma = \beta \text{ for some } \gamma.$$

The order on the Cuntz semigroup is not algebraic. However, Proposition 2.2 of [39] states that this condition holds at least for $\alpha = [p] \in V(A)$. A generalization of [39, Proposition 2.2] that does hold is the following weak-complementation type result.

Proposition 2.1.16. *[48, Lemma 7.1(i)] Let $[a], [b] \in \mathcal{C}u(A)$ be such that $[a] \leq [b]$. Then for any $[a'] \ll [a]$ there exists $[x] \in \mathcal{C}u(A)$ such that*

$$[a'] + [x] \leq [b] \leq [a] + [x]$$

Remark. To see that this proposition is a generalization of [39, Proposition 2.2], one need only recall that when $[a] \in V(A)$ we have $[a] \ll [a]$.

Also, note the following important consequence of this proposition. If $[a] < [b]$ and $[a'] \ll [a]$ then there exists $[x] \neq 0$ such that $[a'] + [x] \leq [b]$.

Proof. This proof draws on the ideas in the proof of [39, Proposition 2.2]. It suffices to prove the result for the case that $a' = (a - \epsilon)_+$, since for arbitrary $[a'] \ll [a]$ we have $[a'] \leq [(a - \epsilon)_+] \ll [a]$ for some $\epsilon > 0$.

Let us first see why the result holds when $a \in \mathrm{Her}\,(b)$. In this case, let $f \in C_0([0, \epsilon))_+$ be such that $f(0) \neq 0$. Set

$$[x] = [f(a)bf(a)] = [f(a)^{1/2}bf(a)^{1/2}] = [b^{1/2}f(a)b^{1/2}] = [bf(a)b].$$

(Each step in the last line simply involves functional calculus for positive elements.) Since $f(a)bf(a)$ is orthogonal to $(a-\epsilon)_+$, we have

$$[x] + [(a-\epsilon)_+] = [f(a)bf(a) + (a-\epsilon)_+].$$

Now,

$$f(a)b^{1/2} = (f(0) - f(a))b^{1/2} + f(0)b^{1/2},$$

where $f(0) - f(a) \in C^*(a) \subseteq \mathrm{Her}\,(b)$; therefore, $f(a)b^{1/2} \in \mathrm{Her}\,(b)$ and so

$$f(a)bf(a) + (a-\epsilon)_+ \in \mathrm{Her}\,(b)$$

which implies that $[x] + [(a-\epsilon)_+] \leq [b]$.

On the other hand,

$$[x] + [a] = [bf(a)b] + [bab] \geq [b(f(a)+a)b] \geq [b^2] = [b],$$

since $f(a) + a$ is invertible.

Now, let us prove the result in full generality. Given that $[a] \leq [b]$, we may find $[c] = [(a-\epsilon/2)_+]$ such that $c \in \mathrm{Her}\,(b)$ by Lemma 2.1.3. Moreover, we may find $\delta > 0$ such that $[(a-\epsilon)_+] \leq [(c-\delta)_+]$. Finding $[x]$ that works for c and δ (in place of a and ϵ respectively) above, we have

$$[(a-\epsilon)_+] + [x] \leq [(c-\delta)_+] + [x] \leq [b] \leq [c] + [x] \leq [a] + [x],$$

as required. $\square$

Here is a simple result useful in working with the Cuntz semigroup of $\mathcal{Z}$-stable C^*-algebras.

Lemma 2.1.17. *Let A be a $\mathcal{Z}$-stable C^*-algebra and let $a, b \in (A \otimes \mathcal{K})_+$. Then $[a \otimes 1_{\mathcal{Z}}] \leq [b \otimes 1_{\mathcal{Z}}]$ in $\mathcal{C}u(A \otimes \mathcal{Z})$ if and only if $[a] \leq [b]$ in $\mathcal{C}u(A)$.*

Proof. The "if" direction is quite straightforward, and doesn't use the hypothesis that A is $\mathcal{Z}$-stable. Since $\mathcal{Z}$ is strongly self-absorbing, there exists an isomorphism $\sigma : A \to A \otimes \mathcal{Z}$ which is approximately unitarily equivalent to the first factor embedding; since this is an isomorphism, we must show that

$$[\sigma(a)] \leq [\sigma(b)]$$

in $\mathcal{C}u(A \otimes \mathcal{Z})$. However, since σ is approximately unitarily equivalent to the first factor embedding, it follows that

$$[\sigma(a)] = [a \otimes 1_{\mathcal{Z}}] \leq [b \otimes 1_{\mathcal{Z}}] = [\sigma(b)],$$

as required. □

2.2 A survey of some applications

Here, we shall look briefly at some ways in which the Cuntz semigroup has been used to further our understanding of the structure of simple C^*-algebras. One gets the sense that the era of using the Cuntz semigroup to solve problems is still in its infancy; although the applications mentioned here represent even the most recent developments, they tantalizingly suggest that there are vast reserves of results yet to be proven.

2.2.1 Measuring regularity

The Cuntz semigroup owes its recent prominence largely to Andrew Toms, as he found in [51] two (simple, nuclear, separable, unital) C^*-algebras which looked the same through every known invariant *except* the Cuntz semigroup. More precisely, he found an approximately homogeneous algebra A with a certain irregularity in its Cuntz semigroup such that $\mathcal{C}u(A) \not\cong \mathcal{C}u(A \otimes \mathcal{Z})$; however, A and $A \otimes \mathcal{Z}$ are shape equivalent (and therefore have the same value under any continuous homotopy invariant), have equal stable rank and real rank, and (automatically) have isomorphic tracial simplexes. The irregularity found by Toms in the Cuntz semigroup of A has been termed **perforation**; a positive ordered semigroup is called **almost unperforated** if whenever x, y satisfy $(n+1)x \leq ny$ for some n, it follows that $x \leq y$.

Proposition 2.2.1. *[47, Theorem 4.5] Let A be a $\mathcal{Z}$-stable C^*-algebra. Then $\mathcal{C}u(A)$ is almost unperforated.*

Toms' counterexample was born out of a construction developed by Jesper Villadsen in [57], and this construction was used to unearth other unexpected and unusual features that may occur in simple nuclear C^*-algebras (most notably, [45, 46]). This provided an impetus to formalize and study regularity properties for C^*-algebras, in the hopes that examples such as Toms' could be compartmentalized with a class of badly behaved C^*-algebras, and that the remaining well-behaved C^*-algebras might be amenable to strong positive results – classification in particular. The Cuntz semigroup continues to play a strong role in defining and understanding regularity properties.

For a simple C^*-algebra, the Cuntz semigroup being almost unperforated is equivalent to a condition that says roughly that the Cuntz ordering is largely determined by the functionals on the Cuntz semigroup. To make this precise, when we say "functionals on the Cuntz semigroup", we mean **dimension functions** – which are additive, order-preserving maps from $\mathcal{C}u(A)$ to $[0,\infty]$ which take finite values on elements coming from matrix algebras (i.e. for all $[a]$ for which $a \in M_n(A)_+$ for some n); these were introduced in [13]. We have:

Proposition 2.2.2. *[47, Proposition 3.2] Let A be a simple C^*-algebra. Then $\mathcal{C}u(A)$ is almost unperforated if and only if, for $[a],[b] \in \mathcal{C}u(A)$, if $f([a]) < f([b])$ for every dimension function f for which $f([b]) < \infty$ then $[a] \leq [b]$.*

Just as in functional analysis, once is interested in linear maps that are continuous, a continuity-type property enters here. We say that a dimension function $f : \mathcal{C}u(A) \to [0,\infty]$ is **lower semicontinuous** if the induced map

$$(A \otimes \mathcal{K})_+ \to \mathcal{C}u(A) \xrightarrow{f} [0,\infty]$$

is, with respect to the norm on $(A \otimes \mathcal{K})_+$. (We don't consider continuous maps because the only continuous dimension function is the constant zero map. This is because positive elements are Cuntz equivalent to positive scalar multiples of themselves, and in particular, to elements that are arbitrarily close to 0.) Dimension functions on $\mathcal{C}u(A)$ that are lower semicontinuous are in correspondence with the quasitraces on A (the traces when A is exact, by an unpublished result [26] of Uffe Haagerup, together with reductions to the unital case in [5, Remark 2.29(i)]). Lower semicontinuity of a dimension function f is equivalent to saying that f preserves suprema of increasing sequences, or that the corresponding quasitrace is lower semicontinuous (by [17, Proposition 4.2] and its proof).

Restricting to lower semicontinuous dimension functions still suffices to characterize being almost unperforated.

Proposition 2.2.3. *Let A be a simple C^*-algebra. Then $\mathcal{C}u(A)$ is almost unperforated if and only if, for $[a],[b] \in \mathcal{C}u(A)$, if $f([a]) < f([b])$ for every lower semicontinuous dimension function f for which $f([b]) < \infty$ then $[a] \leq [b]$.*

Proof. By Proposition 2.2.2, the "if" direction is automatic. Let us assume that $\mathcal{C}u(A)$ is almost unperforated, so that by Proposition 2.2.2, we have: for $[a],[b] \in \mathcal{C}u(A)$, if $f([a]) < f([b])$ for every dimension function f for which $f([b]) < \infty$ then $[a] \leq [b]$. If $[c],[d] \in \mathcal{C}u(A)$ are such that $f([c]) < f([d])$ holds for every lower semicontinuous dimension function (but not necessarily for the non-lower semicontinuous ones), then we must show that $[c] \leq [d]$.

Given any dimension function $f : \mathcal{C}u(A) \to [0,\infty]$, we may define $\bar{f} : \mathcal{C}u(A) \to [0,\infty]$ by

$$\bar{f}([x]) = \sup_{[x'] \ll [x]} f([x']).$$

Then by [17, Lemma 4.7], $\bar{f}$ is a lower semicontinuous dimension function on $\mathcal{C}u(A)$.

For $[a] \ll [c]$, we have

$$f([a]) \leq \bar{f}([c]) < \bar{f}([d]) \leq f([d]).$$

(The first and last inequalities are evident from the definition of $\bar{f}$ while the middle one is by hypothesis.) Therefore, $[a] \leq [d]$. But since $[c]$ is the supremum of $[a]$ satisfying $[a] \ll [c]$, we must have $[c] \leq [d]$, as required. □

In [55, Remark 3.5], it was suggested that certain regularity properties of very different natures for simple C^*-algebras could be equivalent (see also the expository article [22] and the introduction of [60] for more details). The different regularity properties include $\mathcal{Z}$-stability, almost unperforated Cuntz semigroup, and various notions of low dimension, such as finite nuclear dimension, low tracial rank, and slow dimension growth (the latter applying only to approximately subhomogeneous or AH algebras). The property of $\mathcal{Z}$-stability (that is, being isomorphic with the tensor product with $\mathcal{Z}$) is a strong and useful regularity property, its utility perhaps owing to the low dimensional nature of $\mathcal{Z}$ combined with the fact that $\mathcal{Z} \cong \mathcal{Z}^{\otimes\infty}$. For example, for simple unital approximately subhomogeneous algebras (see Chapter 3, being $\mathcal{Z}$-stable easily implies that the system has slow dimension growth. $\mathcal{Z}$-stability is used in a much deeper way in an approach by Winter to the classification of simple, finite C^*-algebras – an approach carried further by Lin to achieve very far-reaching results (this will be further discussed in Section 2.2.3).

As mentioned earlier (Proposition 2.2.1), $\mathcal{Z}$-stability implies that the Cuntz semigroup is almost unperforated, another regularity property. Philosophically, a regularity property at the level of the invariant, such as this one concerning the Cuntz semigroup, should be easier to verify than a property of the C^*-algebra itself. Indeed, as an example, it is easily observed that the Cuntz semigroups of approximately homogeneous (or even approximately subhomogeneous) algebras with slow dimension growth are almost unperforated ([52, Theorem 4.5] for the approximately homogeneous algebra case; [53, Theorem 5.3] for approximately subhomogeneous algebras).

Despite this, there is some hope that (for nuclear, separable, simple C^*-algebras) the Cuntz semigroup being almost unperforated could be equivalent to $\mathcal{Z}$-stability. Wilhelm Winter has proven a result that comes close to this in [61], although it requires the additional regularity property that the Cuntz semigroup is almost divisible (locally finite nuclear dimension of the C^*-algebra is also needed, but this is a very mild strengthening of nuclearity which no nuclear C^*-algebra is known to lack; see [61, Definition 3.1] for its definition). The Cuntz semigroup of A is **almost divisible** if for any $[a] \in \mathcal{C}u(A)$ and $k \geq 1$, there exists $[x] \in \mathcal{C}u(A)$ such that

$$k[x] \leq [a] \leq (k+1)[x].$$

Winter's result is formally stated as follows.

Theorem 2.2.4. *[61, Theorem 6.1] Let A be a simple, unital, separable C^*-algebra with locally finite nuclear dimension. If $\mathcal{C}u(A)$ is almost unperforated and almost divisible then $A \cong A \otimes \mathcal{Z}$.*

2.2.2 Regularity and Cuntz semigroup calculations

Being almost unperforated is a regularity condition concerning the question of when elements of the Cuntz semigroup are comparable (this is seen especially in the characterization in Proposition 2.2.3). Complementing this information, being almost divisible tells us about what elements there are in the Cuntz semigroup. In the case of a simple, exact, stably finite C^*-algebra, these two properties together allow a computation of the Cuntz semigroup.

To describe the result, we shall need the following notation. Let $T(A)$ denote the set of nonzero lower semicontinuous traces, meaning linear functionals $\tau : A_+ \to [0, \infty]$ (linear in the sense of being additive, commuting with positive scalar multiplication, and sending 0 to 0) which satisfy the tracial property, $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$. Every trace is determined by its values on the Pedersen ideal, where it is finite except for the singular "infinite" trace τ_∞ given by

$$\tau_\infty(a) = \begin{cases} \infty, & \text{if } a \neq 0, \\ 0, & \text{if } a = 0. \end{cases}$$

(This uses the fact that A is simple.) Let us endow $T(A)$ with the topology of pointwise convergence on the Pedersen ideal. Let us use $Lsc(T(A), (0, \infty])$ to denote the set of lower semicontinuous, linear functionals $T(A) \to (0, \infty]$

The following result, in the case of unital C^*-algebras with locally finite nuclear dimension, can be proven using Theorem 2.2.4, since it is a very familiar result in the case that A is $\mathcal{Z}$-stable, found in [7, Theorem 5.5] (see also [17, Corollary 6.8] for the nonunital case). (It should also be noted that, long before [7], a computation of $K_0^*(A)$, the

Grothendieck group of the non-stabilized Cuntz semigroup, was obtained in [3, Section III.3] for certain C^*-algebras A including unital simple AF algebras.) However, it has a proof using more elementary methods than those used to prove Theorem 2.2.4, and as such, it was known prior to Theorem 2.2.4 and it doesn't require that the algebra be unital or have locally finite nuclear dimension. The literature seems to lack a full proof, although [1, Theorem 5.27] gives a proof in the unital, stable rank one case.

Theorem 2.2.5. *Let A be a simple, exact, stably finite C^*-algebra for which $\mathcal{C}u(A)$ is both almost unperforated and almost divisible. Then*

$$\mathcal{C}u(A) \cong V(A) \amalg Lsc(T(A), (0, \infty]), \tag{2.2.1}$$

The identification is the identity on $V(A) \subseteq \mathcal{C}u(A)$ and sends $[a] \in \mathcal{C}u(A) \backslash V(A)$ to $\widehat{[a]}$ given by

$$\widehat{[a]}(\tau) = d_\tau(a) := \lim_{n\to\infty} \tau(a^{1/n}). \tag{2.2.2}$$

Addition within $V(A)$ comes from the semigroup structure on $V(A)$. Addition within $Lsc(T(A), (0, \infty])$ is done pointwise. For $[p] \in V(A)$, $f \in Lsc(T(A), (0, \infty])$, we set

$$[p] + f := \widehat{[p]} + f \in Lsc(T(A), (0, \infty]).$$

(Here, $\widehat{[p]}$ is defined exactly as in (2.2.2), though note that since p is a projection, we have $d_\tau(p) = \tau(p)$.)

Comparison within $V(A)$ is Murray-von Neumann comparison. Within $Lsc(T(A), (0, \infty])$, pointwise comparison is used. For $[p] \in V(A)$ and $f \in Lsc(T(A), (0, \infty])$, we have

$$\begin{aligned} &[p] \leq f \ \textit{if}\ \widehat{[p]}(\tau) < f(\tau)\ \forall \tau \in T(A), \ \textit{and} \\ &f \leq [p] \ \textit{if}\ f(\tau) \leq \widehat{[p]}(\tau)\ \forall \tau \in T(A). \end{aligned}$$

Proof. The statement of the proposition implicitly defines a map

$$\Phi : \mathcal{C}u(A) \to V(A) \amalg Lsc(T(A), (0, \infty]).$$

Let us verify that Φ is an order embedding, i.e. that $[a] \leq [b]$ if and only if $\Phi([a]) \leq \Phi([b])$. This will require only that $\mathcal{C}u(A)$ is almost unperforated. Four different cases need to be checked, depending on whether or not each of $[a], [b]$ is in $V(A)$.

It is trivial if both are in $V(A)$. By using Proposition 2.2.3, we obtain the "if" direction when $[a] \in V(A)$. However, if $[a] \in V(A)$ and $[a] < [b]$ then by Proposition 2.1.16, there exists a nonzero $[x]$ such that $[a] + [x] \leq [b]$. Since A is simple, $d_\tau(x) > 0$ and so $\widehat{[a]}(\tau) < \widehat{[b]}(\tau)$ for all τ.

The "only if" direction is automatic if $[a] \notin V(A)$. On the other hand, if $[a] \notin V(A)$ and $\widehat{[a]} \leq \widehat{[b]}$ pointwise then, using Proposition 2.1.16 and simplicity of A again, we have

$$\widehat{[a']}(\tau) < \widehat{[a]}(\tau) \leq \widehat{[b]}(\tau)$$

for all τ, and therefore by Proposition 2.2.3, $[a'] \leq [b]$. Since $[a]$ is the supremum of $[a'] \ll [a]$, we have $[a] \leq [b]$. This concludes the verification that Φ is an order embedding.

To show surjectivity, we shall use an adaptation of the proof of [7, Theorem 5.5] in which $\mathcal{C}u(A)$ being almost divisible takes the place of elements in $\mathcal{C}u(\mathcal{Z})$ of arbitrary trace.

By [37, Corollary 3.3] and [38, Theorem 3.1], $T(A)\backslash\{\tau_\infty\}$ forms a lattice cone. Moreover, [17, Theorem 3.7] shows that $T(A)$ is compact (we have defined the topology on $T(A)$ differently from how it is done in [17]; their definition is appropriate to nonsimple algebras, but for simple algebras as in the case here, [17, Proposition 3.10] can be used to show that the topologies are the same). Let a be any non-zero element of the Pedersen ideal of A and set

$$T_{a\mapsto 1}(A) := \{\tau \in T(A) : \tau(a) = 1\}.$$

Then $T_{a\mapsto 1}(A)$ is a base for the cone $T(A)\backslash\{\tau_\infty\}$ and it is clearly a closed, and therefore compact, subset of $T(A)$. Consequently, $T_{a\mapsto 1}(A)$ is a Choquet simplex.

Any extended linear functional $f : T(A) \to (0, \infty]$ is determined by its restriction to $T_{a\mapsto 1}(A)$, and f is lower semicontinuous if and only if $f|_{T_{a\mapsto 1}(A)}$ is. Thus, $Lsc(T(A), (0, \infty])$

can be identified with the set of lower semicontinuous affine functions $T_{a\mapsto 1}(A) \to (0,\infty]$. By [24, Theorem 11.8], every functional in $Lsc(T(A),(0,\infty])$ is the supremum of continuous linear functions $T(A) \to (0,\infty)$ (strictly speaking, [24, Theorem 11.8] deals with functions whose codomain is $\mathbb{R}$, but the same proof works for codomain $(0,\infty]$). By [15], we may in fact obtain each functional in $Lsc(T(A),(0,\infty])$ as an increasing net of continuous linear functions. Since A is separable, $T(A)$ is metrizable and we can in fact replace such a net by a sequence. The proof of this last statement doesn't quite go as one might expect, so we shall separate the argument as its own lemma.

Lemma 2.2.6. *Let X be a metrizable compact Hausdorff space. Suppose that $f : X \to [0,\infty]$ is a lower semicontinuous function which is the pointwise supremum of an increasing net (f_α) of lower semicontinuous functions. Then f is the pointwise supremum of an increasing sequence (f_{α_i}).*

Proof. Let $(q_k)_{k=1}^{\infty}$ be a dense sequence in $[0,\infty)$. For each k, $f^{-1}((q_k,\infty])$ is open, and since X is metrizable, it is σ-compact. Therefore, we can find an increasing sequence of open sets $(U_{k,i})_{i=1}^{\infty}$ whose union is $f^{-1}((q_k,\infty])$, yet for each i, $U_{k,i}$ is compactly contained in $f^{-1}((q_k,\infty])$.

By using compactness of $U_{k,i}$, lower semicontinuity of each f_α and the fact that the net (f_α) is increasing, we can find α_i such that $f_{\alpha_i}(x) > q_k$ for all $x \in \overline{U_{k,i}}$, $i = 1,\dots,k$. This condition, together with density of $\{q_k\}$, forces f to be the pointwise supremum of (f_{α_i}). As the net (f_α) is increasing, it is clear that we can arrange that (f_{α_i}) is increasing. $\square$

Since every functional in $Lsc(T(A),(0,\infty])$ is the pointwise supremum of an increasing sequence of continuous functions, [17, Theorem 5.7] shows that every such functional may in fact be realized as the supremum of an increasing sequence $(\hat{a}_i)$, where $a_i \in (A\otimes\mathcal{K})_+$ and $\hat{a}_i(\tau) := \tau(a_i)$. Therefore, to show surjectivity of Φ, we need only show that every $\hat{a} \in Lsc(T(A))$ is in the range of Φ; that is, given $a \in (A\otimes\mathcal{K})_+$, we need to show that there exists $[c] \in \mathcal{C}u(A)\backslash V(A)$ such that

$$\tau(a) = d_\tau(c) \text{ for all } \tau \in T(A). \tag{2.2.3}$$

By the argument in the first paragraph of the proof of [17, Lemma 6.5], it suffices to find $[c] \in \mathcal{C}u(A)$ satisfying (2.2.3) without necessarily having $[c] \notin V(A)$.

Here is where we need that $\mathcal{C}u(A)$ is almost divisible.

Lemma 2.2.7. *Let $\mathcal{C}u(A)$ be almost divisible. Then for any real numbers $r < s$, and any $[a] \in \mathcal{C}u(A)$, there exists $[b] \in \mathcal{C}u(A)$ such that*

$$rd_\tau(a) < d_\tau(b) \leq sd_\tau(a), \text{ for all } \tau \in T(A). \tag{2.2.4}$$

If $\mathcal{C}u(A)$ is almost unperforated then in fact we can arrange that

$$d_\tau(b) = sd_\tau(a), \text{ for all } \tau \in T(A). \tag{2.2.5}$$

In other words, $\{\widehat{[x]} : [x] \in \mathcal{C}u(A)\}$ is a cone.

Proof. We may find positive integers p and q so that $r < p/(q+1)$ and $p/q \leq s$. By using that $\mathcal{C}u(A)$ is almost divisible, we may find b such that

$$q[b] \leq p[a] \leq (q+1)[b].$$

From this, (2.2.4) follows.

For the second statement, let s be the supremum of the strictly increasing sequence (s_i). By the first part, we may find $[b_i]$ such that $d_\tau(b_i) \in (s_i d_\tau(a), s_{i+1} d_\tau(a)]$ for all i. Since $[b_i]$ generates the same ideal in $\mathcal{C}u(A)$ as $[a]$ (from the proof of the first part) and therefore the same ideal as $[b_{i+1}]$ and $d_\tau(b_i) < d_\tau(b_{i+1})$ for all i, it follows from [47, Proposition 3.2] along with the idea used in the proof of Proposition 2.2.3 that $[b_i] \leq [b_{i+1}]$. Hence $([b_i])$ is increasing, and its supremum $[b]$ will satisfy (2.2.5). □

Now, given $a \in (A \otimes \mathcal{K})_+$, we have by [17, Proposition 4.2] that for all $\tau \in T(A)$,

$$\tau(a) = \int_0^\infty d_\tau((a-t)_+)\, dt.$$

By using Riemann sums, the function $\tau \mapsto \tau(a)$ can be approximated by a strictly increasing (at each point) sequence in the real cone generated by the functions

$$\{[(a-t)_+]^\wedge : t \in [0,\infty)\}.$$

By Lemma 2.2.7, this sequence can be realized as $(\widehat{[c_i]})_{i=1}^\infty$. Since the sequence is strictly increasing and $\mathcal{C}u(A)$ is almost unperforated, $([c_i])$ is increasing, and so it has a supremum $[c] \in \mathcal{C}u(A)$. Evidently, $[c]$ satisfies (2.2.3). $\square$

Interestingly, this division of Cuntz semigroup regularity properties, one describing when Cuntz classes are comparable and another describing what Cuntz classes arise, reappears in the proof of the main result of this thesis.

Remark 2.2.8. So long as A is a simple, stably finite C^*-algebra for which $V(A)$ is almost unperforated, we can form an ordered semigroup $V(A) \amalg Lsc(T(A),(0,\infty])$ with the ordered semigroup structure as given in the statement of Theorem 2.2.5. We require that A be stably finite so that $V(A)$ is ordered and not merely pre-ordered. Simplicity of A and $V(A)$ being almost unperforated are needed so that the order is transitive – these hypotheses ensure that for $[p],[q] \in V(A)$, $[p] < [q]$ if and only if $d_\tau([p]) < d_\tau([q])$ for all $\tau \in T(A)$.

2.2.3 Classification using the Cuntz semigroup

An isomorphism invariant for (a class of) C^*-algebras is a functor I from the category of C^*-algebras and $*$-homomorphisms (or some full subcategory) to another category. Basic category theory provides that $I(A) \cong I(B)$ whenever A and B are isomorphic C^*-algebras (belonging to the domain of I). A class $\mathcal{C}$ of C^*-algebras is said to **classified** by an invariant I (defined on all of $\mathcal{C}$) if the converse holds: $A \cong B$ whenever $A, B \in \mathcal{C}$ satisfy $I(A) \cong I(B)$. (We may also say that I is a **complete invariant** for $\mathcal{C}$.)

In the classification program for nuclear C^*-algebras, the invariants used are K-theoretical in nature. In particular, classification of simple nuclear C^*-algebras usually means classification using the Elliott invariant, which consists of ordered K-theory paired with traces. It is readily apparent that the Elliott invariant is not suitable for classifying general nonsimple algebras (an exception can be made for certain classes of nonsimple algebras with real rank zero); an enlargement of the invariant involving the Cuntz semigroup is promising. Many would argue that the Cuntz semigroup is K-theoretical, since it is generated like the Murray-von Neumann semigroup (the precursor to K_0) except using positive elements instead of just projections (see Proposition 2.1.14).

We shall now review some important classification results for finite C^*-algebras, to provide some examples. Perhaps the best-known classification result is the classification of AF algebras using K_0 by George Elliott in [18, Theorem 4.3]. The invariant used there is (equivalent to)

$$I(A) = (K_0(A), K_0(A)_+, D(A)\},$$

where $D(A) = \{[p] \in K_0(A) : p \in A\})$ (called the dimension range of A). Effros, Handelman, and Shen gave in [16, Theorem 2.2] an axiomatic characterization of the ordered groups that arise as $(K_0(A), K_0(A)_+)$ for some AF algebra A, and together with [18, Section 5], we obtain a complete description of the range of the invariant.

This classification result was pushed further in [19], to include many real rank zero C^*-algebras with inductive limits involving nice finite stage algebras – namely, finite stages given by finite direct sums of algebras of continuous functions from a circle to a matrix algebra. Note that these finite stage algebras don't have real rank zero, so real rank zero in the limit certainly restricts the algebras covered by the classification. However, it was necessary to enlarge the invariant to include the K_1-group, which is unsurprising since $K_1(C(\mathbb{T})) \neq 0$, whereas $K_1(A) = 0$ for any finite dimensional C^*-algebra A. We may identify $K_0(A) \oplus K_1(A)$ canonically with $K_0(C(\mathbb{T}, A))$, and this identification gives us our ordering on $(K_0 \oplus K_1)$. In the simple case, each element of $[u] \in K_1(A)$ is an absolute infinitesimal, in the sense that, if $\alpha > 0$ in $K_0(A) \oplus K_1(A)$ then $\alpha \pm [u] \geq 0$. That is, for this classification, we use the invariant

$$I(A) = (K_0(C(\mathbb{T}, A)), K_0(C(\mathbb{T}, A))_+, D(C(\mathbb{T}, A))).$$

The result includes an axiomatic description of the range of the invariant.

The inclusion of nonsimple C^*-algebras in the classification just discussed is achieved only with the hypothesis of real rank zero. Consistent with earlier remarks about the Cuntz semigroup being an appropriate invariant in the nonsimple case, we note that it is somewhat unsurprising that we can make inroads into the classification of nonsimple C^*-algebras, since $\mathcal{C}u(A)$ can be determined from $V(A)$ when A has real rank zero (by [12, Corollary 5]).

However, if we restrict to simple C^*-algebras, then we can include significantly more general finite stages. A classification due to Elliott, Gong, and Li, in [20, 23], is given of unital C^*-algebras which arise as inductive limits of algebras of the form $pC(X, \mathcal{K})p$, where X is a finite dimensional compact metrizable space and $p \in C(X, \mathcal{K})$ is a projection, provided that there is a uniform bound on the dimension of the spaces X involved. Without the real rank zero hypothesis, it is possible for different traces on a C^*-algebra to induce the same map on $K_0(A)$; in order to detect this sort of behaviour, it becomes necessary to include the traces in the invariant. The invariant used is

$$I(A) = (K_0(A), K_0(A)_+, [1]_{K_0(A)}, K_1(A), T(A), \langle \cdot, \cdot \rangle_{K_0(A), T(A)}),$$

where $\langle \cdot, \cdot \rangle_{K_0(A), T(A)})$ is the pairing $K_0(A) \times T(A) \to \mathbb{R}$ given by

$$\langle [p], \tau \rangle = \tau(p).$$

The range of the invariant was determined by Villadsen in [56].

Huaxin Lin has proven the most far-reaching classification theorem to date for simple finite C^*-algebras in [31, Corollary 11.9], using results from [59] of Wilhelm Winter. With the same invariant $I(\cdot)$ as used in the result of Elliott, Gong, and Li, the class $\mathcal{C}$ of algebras classified by Lin includes all $\mathcal{Z}$-stable approximately homogeneous algebras and all $\mathcal{Z}$-stable approximately subhomogeneous algebras for which projections separate traces. Andrew Toms showed in [50, Theorem 1.2] that approximately subhomogeneous algebras with slow dimension growth satisfy the hypotheses of Theorem 2.2.4; therefore such approximately subhomogeneous algebras belong to $\mathcal{C}$ whenever either they are approximately homogeneous or their projections separate their traces. One consequence of this classification, noted in [61, Corollary 6.7], is that any approximately homogeneous algebra with slow dimension growth can actually be realized with no dimension growth. This makes use of [54, Theorem 3.6], where it is shown that for any unital, simple, $\mathcal{Z}$-stable approximately homogeneous algebra A, $I(A)$ is in the range described by Villadsen in [56]. In [34], Lin and Niu found the range of the invariant (for the entire class $\mathcal{C}$).

One property present in K-theoretical invariants described above (including the Cuntz semigroup) is that approximately inner homomorphisms (maps $A \to A$ which are point-norm limits of maps of the form $a \mapsto uau^*$, where $u \in A^\sim$ is unitary) are sent to identity maps. This property is quite natural; in fact, it holds whenever the invariant satisfies the following two very natural conditions:

(i) Every inner automorphism (that is, $a \mapsto uau^*$ for some unitary $u \in A^\sim$) is sent to the identity on $I(A)$; and

(ii) For any homomorphism $\alpha : I(A) \to I(B)$, its pre-image under I,

$$\{\phi : A \to B : I(\phi) = \alpha\}$$

is closed, in the point-norm topology on $Hom(A, B)$.

A classification theorem is often proven by showing one of the following stronger statements:

(i) Every isomorphism $\alpha : I(A) \to I(B)$ lifts to an isomorphism $\phi : A \to B$ (by "lifts," it is meant that ϕ satisfies $I(\phi) = \alpha$) (this is the case for all of the classification results cited above);

(ii) Every homomorphism $\alpha : I(A) \to I(B)$ lifts to a homomorphism $\phi : A \to B$ which is unique up to approximate unitary equivalence, i.e. I classifies homomorphisms (this holds for AF algebras, but not for the other classes mentioned above; the problem is uniqueness, and can be remedied by enlarging the invariant).

The latter statement implies the former. To see this, let $\alpha : I(A) \to I(B)$ be an isomorphism. Then there exists $\phi_0 : A \to B$ which lifts α and $\psi_0 : B \to A$ which lifts α^{-1}. Since $\alpha \circ \alpha^{-1}$ is the identity, $\phi_0 \circ \psi_0$ must be approximately unitarily equivalent to id_B (that is, $\phi_0 \circ \psi_0$ is approximately inner); likewise, $\psi_0 \circ \phi_0$ is approximately inner. Elliott's intertwining argument [19, Thms. 2.1 and 2.2] shows that when this holds, there are $\phi : A \to B, \psi : B \to A$ which are approximately unitarily equivalent to ϕ_0, ψ_0 respectively, and such that ϕ and ψ are inverses.

The Cuntz semigroup is a good candidate to take the role, in nonsimple classification, of the ordered K_0-group paired with traces. Indeed, not only does the Cuntz semigroup of a C^*-algebra A capture the Murray-von Neumann semigroup and tracial cone of A, but also it contains the lattice of ideals and the Cuntz semigroup of all ideals and quotients [11], and therefore captures their Murray-von Neumann semigroups and cones of traces. Predicting this role for the Cuntz semigroup, one would expect it to be sufficient in cases where there is no K_1-data (neither in A nor in any ideals or quotients). And, indeed, in [10, 8, 35, 9, 42], classification theorems have been proven for certain classes of nonsimple C^*-algebras where this is the case. The following result of Leonel Robert encompasses all of these classification theorems (and is proven using of the classification by Alin Ciuperca and George Elliott in [10]. In this result, a generalized mapping cone algebra means an algebra A arising from a pullback

$$\begin{array}{ccc} A & \to & C([0,1],E) \\ \downarrow & & \downarrow {\scriptstyle f\mapsto(f(0),f(1))} \\ F & \xrightarrow{\phi} & E \oplus E \end{array}$$

where E and F are finite dimensional C^*-algebras.

Theorem 2.2.9. *[42, Theorem 1] Let A, B be unital C^*-algebras which are inductive limits of generalized mapping cones with trivial K_1-groups. Then every homomorphism from $\mathcal{C}u(A)$ to $\mathcal{C}u(B)$ which preserves the class of the unit lifts to a $*$-homomorphism $A \to B$ which is unique up to approximate unitary equivalence.*

Robert in fact gives an invariant $\mathcal{C}u^{\sim}$ built using the Cuntz semigroup of the unitization of A, and shows that it classifies homomorphisms even in the non-unital case; the theorem as stated here is the result of specializing to the unital case (where $\mathcal{C}u^{\sim}$ and $\mathcal{C}u$ contain the same information). The generalized mapping cones with trivial K_1-groups include interval algebras, $C([0,1], M_n)$ and dimension-drop algebras, $Z_{p,q} := \{f \in C([0,1], M_p \otimes M_q) : f(0) \in M_p \otimes 1_q, f(1) \in 1_p \otimes M_q\}$, from which the Jiang-Su algebra $\mathcal{Z}$ is built.

Every simple algebra to which Theorem 2.2.9 applies is either finite-dimensional or $\mathcal{Z}$-stable. For the non-finite-dimensional ones, by Theorem 2.2.5 (and by noting that $V(A) = K_0(A)_+$ since these algebras have cancellation of projections), the Cuntz semigroup contains the same information as the K_0-group paired with the traces, which in turn is equivalent to the Elliott invariant (since the K_1-group is zero). Therefore, Theorem 2.2.9 has as a consequence a classification of certain simple C^*-algebras by the Elliott invariant. There is some overlap between this consequence and the classification by Huaxin Lin in [32]. However, the use of the Cuntz semigroup gives Robert's result a level of transparency greater than what is found in most past classification results for simple inductive limit C^*-algebras.

This is because, on using the Cuntz semigroup for inductive limits as in Theorem 2.2.9, an isomorphism of the invariant $\mathcal{C}u(\varinjlim A_i) \to \mathcal{C}u(\varinjlim B_i)$ lifts to an approximate intertwining of the invariants

$$\begin{array}{cccccc} \mathcal{C}u(A_{k_1}) & \to & \mathcal{C}u(A_{k_2}) & \to & \cdots \to & \mathcal{C}u(\lim A_i) \\ \downarrow & \nearrow & \downarrow & \nearrow & & \downarrow \\ \mathcal{C}u(B_{\ell_1}) & \to & \mathcal{C}u(A_{\ell_2}) & \to & \cdots \to & \mathcal{C}u(\lim B_i). \end{array}$$

In turn, this lifts to an approximate intertwining at the level of the C^*-algebras,

$$\begin{array}{ccccc} A_{k_1} & \to & A_{k_2} & \to & \cdots \\ \downarrow & \nearrow & \downarrow & \iddots & \\ B_{\ell_1} & \to & A_{\ell_2} & \to & \cdots \end{array}$$

and this induces the isomorphism $\varinjlim A_i \to \varinjlim B_i$ which lifts $\mathcal{C}u(\varinjlim A_i) \to \mathcal{C}u(\varinjlim B_i)$.

On the other hand, one certainly can't expect this feature in classification of simple C^*-algebras, where the (nonsimple) finite stages are not classified by the same invariant. On the other hand, we might wishfully expect this behaviour in future nonsimple classifications involving the Cuntz semigroup.

2.2.4 Riesz interpolation and minima in the Cuntz semigroup

Riesz interpolation is a property of ordered sets which is strictly weaker than being lattice- (or even semilattice-)ordered, yet which is often close enough in computational power to being lattice-ordered.

Definition 2.2.10. *Let S be an ordered set. S has* ***Riesz interpolation*** [1] *if, for any elements $x_1, x_2, y_1, y_2 \in S$ such that*

$$\begin{array}{c} x_1 \\ x_2 \end{array} \leq \begin{array}{c} y_1 \\ y_2, \end{array}$$

(meaning that $x_i \leq y_j$ for $i, j = 1, 2$), there exists an element $z \in S$ such that

$$\begin{array}{c} x_1 \\ x_2 \end{array} \leq z \leq \begin{array}{c} y_1 \\ y_2. \end{array}$$

Such an element z is called an interpolant.

[1] In the context of ordered sets, the name Riesz interpolation is somewhat historically misleading. The name Riesz interpolation was introduced, in the context of ordered groups, by Birkhoff in [2, Section 28] and justified there by showing that it is equivalent, for ordered groups, to a property studied by Riesz in [41] which we now call Riesz decomposition. Riesz decomposition does not make sense for general ordered sets. Unlike here, in the literature, Riesz interpolation is most often applied to ordered groups, or to ordered cancellative semigroups where the order is algebraic (these are exactly the positive cones of ordered groups, and as such, Riesz decomposition is still equivant to Riesz interpolation in this setting).

It is clear from this definition why being semilattice-ordered implies having Riesz interpolation (either $\max\{x_1, x_2\}$ or $\min\{y_1, y_2\}$ could serve as an interpolant). In the case of the Cuntz semigroup of a separable C^*-algebra, it is equivalent to being lattice-ordered, as pointed out to the author by Luis Santiago:

Proposition 2.2.11. *Let $S \in \mathcal{O}rd\mathcal{C}u$ such that S contains a countable dense subset. Then S has Riesz interpolation if and only if it is closed under taking finite infima.*

Proof. As just remarked the "if" direction is automatic. Conversely, suppose that S has Riesz interpolation. Let D be a countable dense subset of S.

For $s, t \in S$, let us describe how to find the infimum of S. Let us enumerate

$$\left\{ x \in D : x \leq \begin{array}{c} s \\ \\ t \end{array} \right\} = \{x_n : n = 1, 2, \dots\}.$$

Now, we shall produce an increasing sequence $(y_n)_{n=1}^\infty$ such that

$$x_n \leq y_n \leq \begin{array}{c} s \\ \\ t \end{array}$$

for each n. Quite simply, we set $y_1 = x_1$ and then inductively, given y_n, use Riesz interpolation to find y_{n+1} satisfying

$$\begin{array}{c} x_{n+1} \\ \\ y_n \end{array} \leq y_{n+1} \leq \begin{array}{c} s \\ \\ t. \end{array}$$

Since (y_n) is increasing, we may take $y = \sup y_n$. We then have

$$x_n \leq y \leq \begin{array}{c} s \\ \\ t \end{array}$$

for all n. If

$$r \leq \begin{array}{c} s \\ \\ t \end{array}$$

then, since r is a supremum of elements in D, there must be a subsequence $(x_{n_k})_{k=1}^\infty$ whose supremum is r. Since $x_{n_k} \leq y$ for each k, it follows that $r \leq y$. This shows that y is the infimum of s and t. $\square$

Example 2.2.12. Cuntz semigroups with Riesz interpolation may not be closed under maxima of finite sets, as the following example shows. Let A be a simple unital AF algebra with two extreme tracial states, τ_1 and τ_2. Then (τ_1, τ_2) gives an order-preservinge map from $K_0(A)$ to $\mathbb{R}^2$, the range of which is countable and dense. Since $V(A)$ has cancellation, it is equal to $K_0(A)_+$, which equals

$$\{0\} \cup \{\alpha \in K_0(A) : \tau_1(\alpha) > 0 \text{ and } \tau_2(\alpha) > 0\}.$$

We have

$$\mathcal{C}u(A) \cong V(A) \amalg Lsc(T(A), (0, \infty]) \cong V(A) \amalg (0, \infty]^2,$$

with the order given by Theorem 2.2.5. The proof that $\mathcal{C}u(A)$ is not a lattice uses the idea that $\mathbb{R}^2$ with the strict order (where $(x, y) < (x', y')$ if and only if $x < x'$ and $y < y'$) is not a lattice, and the fact that this strict order resembles the ordering in $\mathcal{C}u(A)$ when the smaller element is in $V(A)$.

Consider $[p_1], [p_2] \in V(A)$ such that $\tau_1(p_1) > \tau_1(p_2)$ and $\tau_2(p_1) < \tau_2(p_2)$. Let us show that $\max\{[p_1], [p_2]\}$ does not exist in $\mathcal{C}u(A)$.

By Theorem 2.2.5, for $[a] \in \mathcal{C}u(A)$, we have

$$\begin{matrix} [p_1] \\ \\ [p_2] \end{matrix} \leq [a] \tag{2.2.6}$$

if and only if $d_{\tau_i}(p_i) < d_{\tau_i}(a)$ for $i = 1, 2$. Given an element $[a]$ satisfying (2.2.6), we may find $(x_1, x_2) \in (0, \infty)$ such that

$$d_{\tau_i}(p_i) < x_i < d_{\tau_i}(a).$$

By Theorem 2.2.5, there exists $[b] \in \mathcal{C}u(A)$ such that $d_{\tau_i}(b) = x_i$ for $i = 1, 2$. Hence,

$$\begin{matrix} [p_1] \\ \\ [p_2] \end{matrix} \leq [b] < [a].$$

On the other hand, Theorem 2.2.13 (iv) will show that $\mathcal{C}u(A)$ has interpolation.

Theorem 2.2.13. *The following ordered sets (in fact, semigroups) have Riesz interpolation.*

(i) The space of lower semicontinuous affine functions from a Choquet simplex K to $(-\infty, \infty]$, with pointwise ordering;

(ii) $Lsc(T(A), (0, \infty])$ for any simple C^*-algebra A, with pointwise ordering;

(iii) $V(A) \amalg Lsc(T(A), (0, \infty])$ for any simple, stably finite C^*-algebra A for which $V(A)$ is almost unperforated, where the order structure is as described in Theorem 2.2.5.

(iv) $\mathcal{C}u(A)$ for any simple, exact, stably finite C^*-algebra A for which $\mathcal{C}u(A)$ is almost unperforated and almost divisible.

Proof. (i) Let $f_1, f_2, g_1, g_2 : K \to (-\infty, \infty]$ be lower semicontinuous affine functions satisfying

$$\begin{array}{ccc} f_1 & & g_1 \\ & \leq & \\ f_2 & & g_2. \end{array} \tag{2.2.7}$$

By [24, Theorem 11.8], we have that f_i is the pointwise supremum of all continuous affine functions $f : K \to \mathbb{R}$ which are below f_i, for $i = 1, 2$. Let I_i denote the net consisting of finite sets of continuous affine functions $f : K \to \mathbb{R}$ which are below f_i. For $(S_1, S_2) \in I_1 \times I_2$, define $f_{(S_1,S_2)} : K \to \mathbb{R}$ by

$$f_{(S_1,S_2)}(x) = \max\{f(x) : f \in S_1 \cup S_2\}.$$

This is a continuous convex function. Likewise, $g : K \to \mathbb{R}$ defined by $g(x) = \min\{g_1(x), g_2(x)\}$ is a lower semicontinuous concave function. Moreover, any lower semicontinous function $h : K \to \mathbb{R}$ satisfying

$$f_{(S_1,S_2)} \leq h \leq g \text{ for all } (S_1, S_2) \in I_1 \times I_2$$

will be an interpolant for (2.2.7).

Such an h will be found by taking the (pointwise) supremum of an increasing net of continuous affine functions $(h_{(S_1,S_2)})_{(S_1,S_2)\in I_1\times I_2}$ such that

$$f_{(S_1,S_2)} \leq h_{(S_1,S_2)} \leq g \text{ for all } (S_1,S_2) \in I_1 \times I_2; \tag{2.2.8}$$

(since the net is increasing, the pointwise supremum is indeed affine and lower semicontinuous.)

We obtain $h_{(S_1,S_2)}$ inductively. For $S_1 = S_2 = \varnothing$, the existence of $h_{(\varnothing,\varnothing)}$ is obtained from [15], which states that between any upper semicontinuous convex function and any lower semicontinuous concave function on a Choquet simplex, there exists a continuous affine function.

For the inductive step, having already found $h_{(S_1',S_2')}$ for any $(S_1',S_2') < (S_1,S_2)$ (meaning, of course, $S_i' \subseteq S_i$ for $i = 1,2$ yet $(S_1',S_2') \neq (S_1,S_2)$), define the continous convex function $f'_{(S_1,S_2)} : K \to \mathbb{R}$ by

$$f'_{(S_1,S_2)}(x) := \max\{f_{(S_1,S_2)}(x)\} \cup \{h_{(S_1',S_2')} : (S_1',S_2') < (S_1,S_2)\}.$$

Then, once again, the existence of $h_{(S_1,S_2)}$ satisfying

$$f'_{(S_1,S_2)} \leq h_{(S_1,S_2)} \leq g$$

follows from [15]. This produces an increasing net satisfying (2.2.8).

(ii) is a direct consequence of (i), since, as seen in the proof of Theorem 2.2.5, we may identify $Lsc(T(A),(0,\infty])$ with the lower semicontinuous affine functions $T_{a\mapsto 1}(A) \to (0,\infty]$ (where $a \in A_+$ is any nonzero element of the Pedersen ideal), and $T_{a\mapsto 1}(A)$ is a Choquet simplex. Moreover, interpolation passes from the set of lower semicontinuous affine functions $T_{a\mapsto 1}(A) \to (-\infty,\infty]$ to the subset $T_{a\mapsto 1}(A) \to (0,\infty]$ since the latter is order-convex in the former.

(iii) Suppose that $x_1, x_2, y_1, y_2 \in V(A) \amalg Lsc(T(A),(0,\infty])$ satisfies

$$\begin{matrix} x_1 & & y_1 \\ & \leq & \\ x_2 & & y_2. \end{matrix} \tag{2.2.9}$$

Let us assume that we have $x_i < y_j$ for all i, j, since otherwise we would automatically have an interpolant. For $j = 1, 2$, set

$$\hat{y}_j = \begin{cases} y_j, & \text{if } y_j \in Lsc(T(A), (0, \infty]) \\ \widehat{[p]}, & \text{if } y_j = [p] \in V(A). \end{cases}$$

We can see from the description of the order structure in Theorem 2.2.5 that $\hat{y}_j \leq y_j$ and, since $x_i < y_j$, we have $x_i \leq \hat{y}_j$, for $i, j = 1, 2$.

We shall now also define functions $\hat{x}_i \in Lsc(T(A), (0, \infty])$ such that we have

$$x_i \leq \hat{x}_i \leq \begin{array}{c} \hat{y}_1 \\ \\ \hat{y}_2 \end{array},$$

for $i = 1, 2$. Once again, if $x_i \in Lsc(T(A), (0, \infty])$ then we do the obvious thing, set $\hat{x}_i = x_i$. On the other hand, if $x_i = [p] \in V(A)$, then by the description of the order structure in Theorem 2.2.5, and using the fact that $\widehat{[p]}$ is continuous and $T(A)$ has a Choquet simplex as its base, we see that that there must exist $\gamma > 1$ such that

$$\gamma\widehat{[p]}(\tau) \leq \begin{array}{c} \hat{y}_1(\tau) \\ \\ \hat{y}_2(\tau). \end{array}$$

We then set $\hat{x}_i = \gamma\widehat{[p]}$.

We now have

$$\begin{array}{ccc} \hat{x}_1 & & \hat{y}_1 \\ & \leq & \\ \hat{x}_2 & & \hat{y}_2 \end{array}$$

in $Lsc(T(A), (0, \infty])$ so by (ii), there exists an interpolant $z \in Lsc(T(A), (0, \infty])$. Since $x_i \leq \hat{x}_i$ and $\hat{y}_i \leq y_i$, this interpolant works for (2.2.9).

(iv) follows from (iii) and Theorem 2.2.5. □

It was shown in [62, Corollary 1.3] that when A has real rank zero, $V(A)$ has the Riesz decomposition property (this is equivalent to Riesz interpolation for ordered groups, and therefore equivalent for semigroups with cancellation). The following observation shows that Theorem 2.2.13 provides different proof of this fact, under the additional assumption that A is simple, separable, and $\mathcal{Z}$-stable. In fact, it shows that Theorem 2.2.13 is a partial generalization of [62, Corollary 1.3]. To make the connection, note that when A has real rank zero, every element of $\mathcal{C}u(A)$ is the supremum of a sequence from $V(A)$, by [12, Corollary 5] (the statement of [12, Corollary 5] assumes that A has stable rank one, and concludes much more than the statement just cited, but does not use the stable rank one hypothesis to show that every element of $\mathcal{C}u(A)$ is the supremum of a sequence from $V(A)$). This connection between Riesz interpolation in $\mathcal{C}u(A)$ and in $V(A)$ when A has real rank zero was suggested to the author by Henning Petzka.

Proposition 2.2.14. *Let S be in $\mathcal{O}rd\mathcal{C}u$ and suppose that every element of S is the supremum of a sequence of compact elements (i.e. elements x satisfying $x \ll x$). Then S has Riesz interpolation if and only if the set of compact elements, $S_c := \{x \in S : x \ll x\}$, has Riesz interpolation.*

Proof. ($\Rightarrow$) Suppose that S has Riesz interpolation and $x_1, x_2, y_1, y_2 \in S_c$ satisfy

$$\begin{matrix} x_1 & & y_1 \\ & \leq & \\ x_2 & & y_2. \end{matrix}$$

Since S has Riesz interpolation, let $z \in S$ be an interpolant. Then since x_1, x_2 are compact, we have

$$\begin{matrix} x_1 & \\ & \ll z. \\ x_2 & \end{matrix}$$

As z is the supremum of a sequence from S_c, it follows that there exists $z' \in S_c$ such that

$$\begin{matrix} x_1 & \\ & \leq z' \leq z. \\ x_2 & \end{matrix}$$

Hence, z' is an interpolant.

($\Leftarrow$) Say S_c has Riesz interpolation and $x_1, x_2, y_1, y_2 \in S$ satisfy

$$\begin{matrix} x_1 & & y_1 \\ & \leq & \\ x_2 & & y_2. \end{matrix}$$

Let $(x_{i,n})_{n=1}^{\infty}$ be an increasing sequence from S_c whose supremum is x_i, for $i = 1, 2$. An interpolant z will be given as the supremum of an increasing sequence (z_n) from S_c such that

$$\begin{matrix} x_{1,n} \\ x_{2,n} \end{matrix} \leq z_n \leq \begin{matrix} y_1 \\ y_2 \end{matrix}$$

for each n.

But finding such z_n is easy; given $z_{n-1} \in S_c$, we have

$$\begin{matrix} x_{1,n} \\ x_{2,n} \\ z_{n-1} \end{matrix} \ll \begin{matrix} y_1 \\ y_2, \end{matrix}$$

and therefore, for $i = 1, 2$, there exists $y_i' \in S_c$ such that

$$\begin{matrix} x_{1,n} \\ x_{2,n} \\ z_{n-1} \end{matrix} \leq y_i' \leq y_i.$$

The interpolant z_n can now be found since S_c has Riesz interpolation. $\square$

2.3 Inductive limits

Here we shall see how the Cuntz semigroup behaves with respect to inductive limits. One might summarize the discussion by saying that it behaves well: if the Cuntz semigroup is put into an appropriate category, the Cuntz semigroup functor becomes continuous. This was proven by Coward, Elliott, and Ivanescu in [12, Theorem 2], using their pioneering approach of using Hilbert modules to describe the Cuntz semigroup (this is discussed briefly in Section 3.3); the results in this section may be found in the proof of that theorem. However, these results may also be proven using the classical description of the Cuntz semigroup, that is, the definition given in Def. 2.1.2 (see [35, Section 2.1]).

That the Cuntz semigroup functor is continuous means that the Cuntz semigroup of an inductive limit may be described purely in terms of the Cuntz semigroups of the finite stage algebras along with maps between these Cuntz semigroups induced by the maps of the inductive system. Let us here give this computation explicitly. First, we give a description of the elements of the Cuntz semigroup of an inductive limit.

Proposition 2.3.1. *Let*

$$A_1 \xrightarrow{\phi_1^2} A_2 \xrightarrow{\phi_2^3} \cdots \to A = \varinjlim A_i$$

be an inductive system of C^*-algebras and let $[a] \in \mathcal{C}u(A)$. Then there exist an element $[a_i] \in \mathcal{C}u(A_i)$ for each i such that $[\phi_i^{i+1}(a_i)] \ll [a_{i+1}]$ and

$$[a] = \sup[\phi_i^{\infty}(a_i)].$$

This last result means that every element of $\mathcal{C}u(A)$ can be represented by an increasing sequence $([a_i])$ from the Cuntz semigroups of the finite stage algebras. The next result describes the ordering on such representatives (and, in particular, says when two such sequences give rise to the same element of $\mathcal{C}u(A)$).

Proposition 2.3.2. *Let*

$$A_1 \xrightarrow{\phi_1^2} A_2 \xrightarrow{\phi_2^3} \cdots \to A = \varinjlim A_i$$

be an inductive limit of C^*-algebras. Let $[a_i], [b_i] \in \mathcal{C}u(A_i)$ be such that

$$[\phi_i^{i+1}(a_i)] \leq [a_{i+1}] \text{ and } [\phi_i^{i+1}(b_i)] \leq [b_{i+1}]$$

for each i, (i.e., be increasing), thus giving rise to elements

$$[a] = \sup[\phi_i^{\infty}(a_i)] \text{ and } [b] = \sup[\phi_i^{\infty}(b_i)].$$

Then $[a] \leq [b]$ if and only if, for every i and every $[a'] \ll [a_i]$ there exists $j \geq i$ such that

$$[\phi_i^j(a')] \ll [b_j].$$

Altogether, Propositions 2.3.1 and 2.3.2 say that we can identify $\mathcal{C}u(A)$ with the set of $\ll$-increasing sequences $([a_i] \in \mathcal{C}u(A_i))$, modulo the pre-order

$$([a_i]) \leq ([b_i]) \text{ if for all } i, \text{ there exists } j \geq i \text{ such that } [\phi_i^j(a_i)] \ll [b_j].$$

This description is mentioned to show explicitly that the Cuntz semigroup of A is expressed in terms of the Cuntz semigroup of the finite stages; however, we won't really make use of this description in the sequel (although we will appeal to Propositions 2.3.1 and 2.3.2 directly).

2.4 The Cuntz equivalence-invariant $\mathbb{I}(\cdot)$ for elements in algebras of the form $C_0(X, A)$

Let A be a simple C^*-algebra. Define a Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ on elements of $C_0(X, A\otimes\mathcal{K})_+$: $\mathbb{I}(a)$ consists of the map $x \mapsto [a(x)]$ along with the Murray-von Neumann class of $a|_K$ for any compact subset K of X for which $[a|_K] \in V(C(K, A))$. The order relation $\mathbb{I}(a) \leq \mathbb{I}(b)$ is determined by $[a(x)] \leq [b(x)]$ for all x and $[a|_K] \leq [b|_K]$ for any compact subset K of X for which $[a|_K], [b|_K] \in V(C(K, A))$; it is not hard to see that the relation $\mathbb{I}(a) \leq \mathbb{I}(b)$ can be determined by the information in $\mathbb{I}(a)$ and $\mathbb{I}(b)$ alone.

This section derives some basic facts about the data in $\mathbb{I}(a)$, and provides us with the codomain of $\mathbb{I}(\cdot)$.

Proposition 2.4.1. *Let A be any C^*-algebra and let $[a] \in \mathcal{C}u(C_0(X, A))$. Then the map $x \mapsto [a(x)]$ is $\ll$-lower semicontinuous; in other words, for any $[b] \in \mathcal{C}u(A)$, the set*

$$\{x \in X : [b] \ll [a(x)]\}$$

is open.

Proof. Let $x \in X$ be such that $[b] \ll [a(x)]$. Then, for some $\epsilon > 0$ we have $[b] \ll [(a - \epsilon)_+(x)]$. Let U be a neighbourhood of x such that for $y \in U$, $\|a(y) - a(x)\| < \epsilon$. Then by Lemma 2.1.3, we have

$$[(a(x) - \epsilon)_+] \leq [a(y)],$$

for all $y \in U$. Thus, U is an open neighbourhood of x contained in $\{x \in X : [b] \ll [a(x)]\}$. □

Note that since $\ll$ is not antireflexive, $\ll$-lower semicontinuity is a slightly unusual condition. However, we have the following.

Proposition 2.4.2. *Let A be any C^*-algebra and let $f : X \to \mathcal{C}u(A)$ be $\ll$-lower semicontinuous. Then for $[b] \in \mathcal{C}u(A)$, the set*

$$\{x \in X : [b] \ll f(x) \text{ and } [b] \neq f(x)\}$$

is open.

Proof. Suppose $[b] \ll [c] = f(x)$ yet $[b] \neq [c]$. Then for some $\epsilon > 0$ we must have

$$[b] < [(c - \epsilon)_+] \ll f(x).$$

Therefore, we have

$$\{x \in X : [b] \ll f(x) \text{ and } [b] \neq f(x)\} = \bigcup_{[d]>[b]} \{x \in X : [d] \ll f(x)\},$$

which is a union of open sets. □

Proposition 2.4.3. *Let K be a compact Hausdorff space, let A be a stably finite C^*-algebra, and let $[a] \in \mathcal{C}u(C(K, A))$. If $[p] \in V(A)$ and $[a(x)] = [p]$ for all $x \in K$ then $[a] \in V(C(K, A))$.*

Proof. For each $x \in K$, by Proposition 2.1.15, 0 is an isolated point in the spectrum of $a(x)$. Therefore, we may define $\lambda : K \to (0, \infty)$ by setting $\lambda(x)$ to be the least nonzero value in the spectrum of $a(x)$. We shall verify that λ is lower semicontinuous. From this, it follows that λ attains a minimum, say $\eta > 0$, and therefore $(0, \eta)$ does not intersect the spectrum of a. Consequently, $[a] \in V(C(K, A))$.

To see that λ is lower semicontinuous, pick a point $x \in X$ and a number $\epsilon \in (0, \lambda(x))$. Using Lemma 2.1.3 and continuity of $(a - \epsilon)_+$, we see that there exists an open neighbourhood U of x such that

$$[(a - \lambda(x))_+(x)] \leq [(a - \epsilon)_+(y)]$$

for all $y \in U$. But, since $[(a - \lambda(x))_+(x)] = [a(x)] = [a(y)] \in V(A)$, this means that

$$\chi_{(0,\infty)}(a(y)) \sim \chi_{(\epsilon,\infty)}(a(y)) \leq \chi_{(0,\infty)}(a(y)),$$

and so by stable finiteness, we have equality. This implies that $(0, \epsilon)$ does not intersect the spectrum of $a(y)$, and therefore, $\lambda(y) \geq \epsilon$ for all $y \in U$. Hence, λ is indeed lower semicontinuous. □

We shall now introduce a set $V_c^{[p]}(Y, A)$ which is important for understanding the possibilities for the range of the invariant $\mathbb{I}(\cdot)$ – and in fact plays a crucial role in describing this range. If we fix $[p] \in V(A)$ then, in general, the set S of all $x \in X$ for which $[a(x)] = [p]$ is not compact. It is, however, the difference of open sets, and can also be seen to be σ-compact. The invariant $\mathbb{I}(a)$ includes $[a|_K]$ for compact subsets K of S, and we see that this information (over all compact K) constitutes an element of

$$\varprojlim_{K \text{compact}, K \nearrow S} V(C(K, A)),$$

where the connecting maps are given by restriction, $V(C(K, A)) \to V(C(L, A)) : p \mapsto p|_L$ when $L \subseteq K$.

Set

$$V^{[p]}(K, A) := \{[q] \in V(C(K, A)) : [q(x)] = [p] \ \forall x \in K\};$$

then evidently, $[a|_K] \in V^{[p]}(K, A)$ when $K \subseteq S$. Therefore, the element described above is in fact in

$$V_c^{[p]}(S, A) := \varprojlim_{K \text{compact}, K \nearrow S} V^{[p]}(K, A). \tag{2.4.1}$$

The following gives equivalent descriptions of this set.

Proposition 2.4.4. *Let A be a separable stably finite C^*-algebra and X a σ-compact, locally compact Hausdorff space. Fix $[p] \in V(A)$. The following sets may be canonically identified:*

(i) The inverse limit

$$\varprojlim_{K \text{compact}, K \nearrow X} V^{[p]}(K, A).$$

where the connecting maps are given by restriction, $V^{[p]}(K, A)) \to V^{[p]}(L, A) : p \mapsto p|_L$ when $L \subseteq K$.

(ii) *The subsemigroup of $\mathcal{C}u(C_0(X, A))$ consisting of all $[a]$ for which $[a(x)] = [p]$ for all $x \in X$.*

(iii) *Equivalence classes of projection-valued functions $q \in C_b(X, A \otimes \mathcal{K})$ for which $[q(x)] = [p]$ for all $x \in X$, under the equivalence relation $q_1 \sim_c q_2$ if $q_1|_K \sim q_2|_K$ for every compact subset $K \subseteq X$.*

(iv) *Equivalence classes of functions $q \in C(X, A \otimes \mathcal{K})_+$ for which $[q(x)] = [p]$ for all $x \in X$, under the equivalence relation $q_1 \sim_c q_2$ if $\chi_{(0,\infty)}(q_1|_K) \sim \chi_{(0,\infty)}(q_2|_K)$ for every compact subset $K \subseteq X$.*

Remark 2.4.5. In the proof, as well as the sequel, we shall use $\langle q \rangle$ to denote the equivalence class of a function $q \in C(X, A \otimes \mathcal{K})$ as in (iv). Following the proof, we shall consider $\langle q \rangle$ as an element of $V_c^{[p]}(X, A)$.

Proof. Let us label the sets $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ respectively. Obviously, $\Omega_3 \subseteq \Omega_4$. For $\langle q \rangle \in \Omega_4$, by Proposition 2.4.3, $x \mapsto \chi_{(0,\infty)}(q(x))$ define a projection in $C_b(X, A \otimes \mathcal{K})$. From the definition of the equivalence relation used to define Ω_4, we have $\langle q \rangle = \langle \chi_{(0,\infty)}(q) \rangle \in \Omega_3$; thus, $\Omega_3 = \Omega_4$.

We have already seen that $q \mapsto (q|_K)$ defines an injection $\psi_3^1 : \Omega_3 \to \Omega_1$.

Let $[a] \in \mathcal{C}u(C_0(X, A))$ satisfy $[a(x)] = [p]$ for all $x \in X$. Then $a \in C_0(X, A \otimes \mathcal{K})_+ \subseteq C(X, A \otimes \mathcal{K})_+$ and so $\langle a \rangle \in \Omega_4$. If $[a] = [b]$ in $\mathcal{C}u(C_0(X, A))$ then since Cuntz equivalence passes to quotients, $[a|_K] = [b|_K]$ for all compact sets K, and so by Proposition 2.1.14, $\langle a \rangle = \langle b \rangle$. This shows that we have a well-defined map $\psi_2^3 : \Omega_2 \to \Omega_4 = \Omega_3$.

To see that this map is injective, suppose $[a], [b] \in \mathcal{C}u(C_0(X, A))$ are such that $[a(x)] = [p] = [b(x)]$ and $\langle a \rangle = \langle b \rangle$. For any $\epsilon > 0$, $(a - \epsilon)_+$ is supported on a compact set K and we have

$$[(a - \epsilon)_+|_K] \leq [\chi_{(0,\infty)}(a_K)] = [\chi_{(0,\infty)}(b|_K)] = [b|_K].$$

Therefore, $[(a - \epsilon)_+] \leq [b]$ (the middle equality is by Proposition 2.1.14). Since ϵ is arbitrary, $[a] \leq [b]$ and by symmetry, $[a] = [b]$.

Now let us define a map from Ω_1 to Ω_2. Given

$$([q_K]) \in \varprojlim_{K\text{compact}, K \nearrow X} V^{[p]}(K, A),$$

let us express X as a countable increasing union of the interiors of compact sets,

$$X = \bigcup_n K_n^\circ.$$

Let $f_n \in C_0(K_n^\circ)$ be strictly positive for each n, allowing us to define $f_n q_{K_n} \in C_0(X, A)$. Since $K_n \subseteq K_{n+1}$ and $[q_{K_n}] = [q_{K_{n+1}}|_{K_n}]$, we see that

$$[f_n q_{K_n}] \leq [f_{n+1} q_{K_{n+1}}]$$

in $\mathcal{C}u(C_0(X, A))$. Since $\mathcal{C}u(C_0(X, A))$ is closed under countable increasing suprema, let

$$[a] = \sup_n [f_n q_{K_n}] \in \mathcal{C}u(C_0(X, A)).$$

For any compact set $K \subseteq X$, we have $K \subseteq K_n^\circ$ for some n (by compactness) and therefore

$$[q_K] = [q_{K_m}|_K] = [f_m q_{K_m}|_K]$$

for all $m \geq n$. Since restriction to K is a homomorphism of C^*-algebras, and C^*-algebra homomorphisms produce Cuntz semigroup maps that preserve countable increasing suprema, we must have

$$[q_K] = \sup_m [f_m q_{K_m}|_K] = [a|_K].$$

This shows that $\psi_3^1 \circ \psi_2^3([a]) = ([q_K])$; since ψ_3^1 and ψ_2^3 are injective, it follows that $[a]$ does not depend on the choice of sets K_n (nor on the functions f_n), so $([q_n]) \mapsto [a]$ gives a well-defined map $\psi_1^2 : \Omega_1 \to \Omega_2$.

One can now verify with the three maps $\psi_1^2, \psi_2^3, \psi_3^1$ that composing all three of them (in any of the three valid ways) gives the identity. $\square$

To summarize and conclude this section, we find that the codomain of the invariant $\mathbb{I}(\cdot)$ consists of pairs $(f, (\langle a_{[p]}\rangle)_{[p]\in V(A)})$ where $f : X \to \mathcal{C}u(A)$ is a $\ll$-lower semicontinuous function, and for each $[p] \in V(A)$, $a_{[p]} \in V_c^{[p]}(f^{-1}([p]), A)$ (this part of $\mathbb{I}(a)$ captures the [nonstable] K-theory arising through a moving continuously in the Murray-von Neumann class of p).

Chapter 3

Approximately subhomogeneous C^*-algebras

3.1 Definition

Definition 3.1.1. *A C^*-algebra is **subhomogeneous** if there is a finite upper bound on the dimension of its irreducible representations. An **approximately subhomogeneous** C^*-algebra is one that can be written as an inductive limit of subhomogeneous algebras.*

The name "approximately subhomogeneous" may suggest that the algebra is locally approximable by subhomogeneous algebras, in the sense that for any finite subset F of the algebra and any $\epsilon > 0$, there is a subalgebra which is subhomogeneous and has distance at most ϵ from each element of F. However, this property is a priori weaker than the inductive limit property which "approximately subhomogeneous" is being used to mean; it is not known if in fact the two properties are equivalent for separable algebras.

It should be mentioned, however, that there are examples of separable C^*-algebras which are locally approximable by homogeneous algebras but cannot be written as an inductive limit of homogeneous algebras [14]. On the other hand, turning to the simple, well-behaved case, Huaxin Lin very recently showed, in [33], that every separable, unital C^*-algebra which is locally approximable by homogeneous algebras can be expressed as an inductive limit of homogeneous algebras, so long as the homogeneous local approximants have slow dimension growth (meaning that they can be chosen to have an arbitrarily small ratio of the topological dimension to the matricial dimension).

The following result is fundamental to the techniques used for the fine analysis of ASH algebras.

Proposition 3.1.2. *[36, Corollary 2.1] Every separable ASH algebra is an inductive limit of subhomogeneous algebras (A_i) for which $Prim_n(A_i)$ has finite dimension for all n.*

Proof. We can write any separable, unital ASH algebra as an inductive limit of finitely generated subhomogeneous algebras. Generalizing the fact that if $C(X)$ is finitely generated then X is finite-dimensional, it is proven in [36, Theorem 1.5] that if A is a finitely generated subhomogeneous algebra then $Prim_n(A)$ has finite dimension for all n. $\square$

Definition 3.1.3. *The class of recursive subhomogeneous (RSH) algebras is the smallest class $\mathcal{RSH}$ containing M_n for all n and closed under a certain pullback construction as follows. If $R \in \mathcal{RSH}$, Ω is a locally compact Hausdorff space, $\Omega^{(0)}$ is a closed subset of Ω, $n \in \{1, 2, \dots\}$, $\rho : R \to C_0(\Omega^{(0)}, M_n)$ is a $*$-homomorphism, and*

$$\begin{array}{ccc} R' & \to & C_0(\Omega, M_n) \\ \downarrow & & \downarrow_{f \mapsto f|_{\Omega^{(0)}}} \\ R & \xrightarrow{\rho} & C_0(\Omega^{(0)}, M_n); \end{array}$$

is a pull-back then $R' \in \mathcal{RSH}$. Explicitly, we may identify the pullback R' with the amalgamated direct sum,

$$\{(f, a) \in C_0(\Omega, M_n) \oplus R : f|_{\Omega^{(0)}} = \rho(a)\}.$$

Notation 3.1.4. *An RSH algebra R has a (non-unique) decomposition involving a sequence of algebras $R_0, \dots, R_\ell$, with $R = R_\ell$, $R_0 = M_{n_0}$ and, recursively, R_i is given by the pull-back*

$$\begin{array}{ccc} R_i & \xrightarrow{\sigma_i} & C_0(\Omega_i, M_{n_i}) \\ \lambda_i^{i-1} \downarrow & & \downarrow_{f \mapsto f|_{\Omega_i^{(0)}}} \\ R_{i-1} & \xrightarrow{\rho_i} & C_0(\Omega_i^{(0)}, M_{n_i}); \end{array} \tag{3.1.1}$$

here, Ω_i is a locally compact Hausdorff space, $\Omega_i^{(0)}$ is a closed subset and ρ_i is some specified $*$-homomorphism. The diagram (3.1.1) defines the $*$-homomorphisms σ_i and λ_i^{i-1}. For $i \geq j$, define

$$\lambda_i^j := \lambda_{j+1}^j \circ \cdots \circ \lambda_i^{i-1} : R_i \to R_j;$$

this is consistent with the definition of λ_i^{i-1}.

*The **total space** of (this RSH decomposition for) R is*

$$\Omega := \Omega_1 \amalg \cdots \amalg \Omega_\ell,$$

and the canonical representation of R is

$$\sigma : R \to C_0(\Omega, \mathcal{K})$$

given by

$$\sigma(a)|_{\Omega_i} = \sigma_i \circ \lambda_\ell^i(a).$$

We define $d_{top} : \Omega \to \{0, 1, \ldots, \infty\}$ by $d_{top}(\omega) = \dim \Omega_i$ when $\omega \in \Omega_i$.

Proposition 3.1.5. *[40, Theorem 2.15] A separable unital C^*-algebra A has an RSH decomposition with finite-dimensional total space if and only if A is subhomogeneous and $Prim_n(A)$ has finite dimension for all n.*

Combining the last two propositions gives the following corollary, which is extremely useful for computations involving approximately subhomogeneous algebras.

Corollary 3.1.6. *Every unital separable ASH algebra can be written as an inductive limit of RSH algebras.*

3.2 Nonunital ASH algebras and ideals

Nonunital ASH algebras have not been studied as thoroughly as their unital (simple) counterparts. It seems that this is partly because they aren't considered important, as there are few true examples (only stably projectionless examples count, since if there is a projection p in the stabilization of the simple algebra A then the hereditary subalgebra generated by p is unital and stably isomorphic to A). In many contexts, having a unit also allows simplifications, making stronger results possible or at least easier.

Nonetheless, there is a void left in our understanding of the structure of ASH algebras. One technique that can allow us to make progress quickly is to view a nonunital ASH algebra as an ideal in its unitization. To get anywhere with this technique, it helps to establish that an algebra is RSH if and only if its unitization is. The forward direction is easily seen, and the reverse follows from the following more general result. See also [40, Corollary 3.3] where a related, but strictly weaker, result is proven using Proposition 3.1.5.

Proposition 3.2.1. *An ideal of an RSH algebra is itself an RSH algebra.*

Proof. Let R be an RSH algebra with a decomposition as in Notation 3.1.4. Suppose J is an ideal in R. For $j = 0, \dots, \ell$, let

$$J_j := \lambda_\ell^j(J) \subseteq R_j.$$

Note that since restriction is surjective, so are the maps λ_i^{i-1} and therefore λ_i^j for all $i \geq j$. In particular, since J is an ideal, it follows that J_j is an ideal for each j.

Now for each i, let Z_i be the open subset of Ω_i such that $C_0(Z_i, M_{n_i})$ is the ideal generated by $\sigma_i(J_i)$. By commutativity of (3.1.1), $\rho_i(J_{i-1})$ is contained in $C_0(Z_i \cap \Omega_i^{(0)}, M_{n_i})$, and therefore

$$\begin{array}{ccc} J_i & \xrightarrow{\sigma_i|_{J_i}} & C_0(Z_i, M_{n_i}) \\ \lambda_i^{i-1}|_{J_i} \downarrow & & \downarrow f \mapsto f|_{\Omega_i^{(0)} \cap Z_i} \\ J_{i-1} & \xrightarrow{\rho_i|_{J_{i-1}}} & C_0(Z_i \cap \Omega_i^{(0)}, M_{n_i}) \end{array} \tag{3.2.1}$$

commutes. All that remains is to show that (3.2.1) is a pullback, and for this, we need only show that for any $a \in R_i$, if $\sigma_i(a) \in C_0(Z_i, M_{n_i})$ and $\lambda_i^{i-1}(a) \in J_{i-1}$ then $a \in J_i$.

Let $\epsilon > 0$. Since

$$\sigma_i(a)|_{\Omega_i^{(0)}} \in \rho_i(\lambda_i^{i-1}(J_i)) = \sigma_i(J_i)|_{\Omega_i^{(0)}},$$

there must be some open set U containing $\Omega_i^{(0)}$ such that $\sigma_i(a)|_{Z_i \cap \Omega_i^{(0)}}$ is approximately contained (to within ϵ) in $\sigma_i(J_i)|_{\overline{U}}$. On the other hand, since U contains $\Omega_i^{(0)}$, the map $a \mapsto \sigma_i(a)|_{\Omega_i \setminus U}$ is surjective, so that $\sigma_i(J_i)|_{\Omega_i \setminus U}$ is an ideal and therefore

$$\sigma_i(J_i)|_{\Omega_i \setminus U} = C_0(Z_i \setminus U).$$

In particular, we see that $\sigma_i(a)|_{\Omega_i \setminus U} \in \sigma_i(J_i)|_{\Omega_i \setminus U}$.

Making use of an approximate identity for J_i, we can see that a is close (a distance of at most ϵ) to an element of J_i. Since ϵ is arbitrary, it follows that $a \in J_i$. □

We may apply Proposition 3.2.1 to generalize Proposition 3.1.5 and Corollary 3.1.6 to the nonunital case; in both cases, the basic method is to apply these results from the unital case to the unitization, then use Proposition 3.2.1 to the ideal given by an algebra within its unitization.

Corollary 3.2.2. *A C^*-algebra A has an RSH decomposition with finite-dimensional total space iff A is subhomogeneous and $Prim_n(A)$ has finite dimension for all n.*

Corollary 3.2.3. *Every separable ASH algebra can be written as an inductive limit of RSH algebras with finite dimensional total space.*

We shall now include a consequence for simple ASH algebras of a deep result of Blackadar and Cuntz. The consequence is that simplicity of the inductive limit is particularly transparent in the finite stages: for any nonzero element a in one finite stage, there is a later finite stage which is generated as an ideal by a. This can be proven easily in the unital case, since as one moves out in the system, the unit becomes approximately contained in the ideal generated by a; but if the unit is sufficiently close then this ideal contains an invertible element and is therefore the whole algebra. In the nonunital case, Blackadar and Cuntz's result is necessary; it says that if A is simple then $A \otimes \mathcal{O}_2 \otimes \mathcal{K}$ contains a nonzero projection (indeed, this holds with any simple purely infinite algebra in place of $\mathcal{O}_2$). One can roughly view such a projection as standing in for the unit.

Proposition 3.2.4. *Let A be a separable simple ASH algebra. Then there exists an inductive limit decomposition*

$$A_1 \xrightarrow{\phi_1^2} A_2 \xrightarrow{\phi_2^3} \cdots \to A = \varinjlim A_i$$

such that each A_i is an RSH algebra, each connecting map is full (i.e. $\phi_i^j(A_i)$ generates A_j as an ideal) and for any i and any nonzero $a \in A_i$, there exists $j \geq i$ such that A_j is the ideal generated by $\phi_i^j(a)$.

Proof. By Corollary 3.2.3, let

$$B_1 \xrightarrow{\phi_1^2} B_2 \xrightarrow{\phi_2^3} \cdots$$

be any inductive system of RSH algebras with A as its limit.

By [4, Corollary 5.2], $A \otimes \mathcal{O}_2 \otimes \mathcal{K}$ contains a nonzero projection. Hence, $B_i \otimes \mathcal{O}_2 \otimes \mathcal{K}$ contains a nonzero projection p for some i, which without loss of generality we take to be 1.

Let us approximate p by a finite sum of elementary tensors, so that there exist $b_1, \ldots, b_n \in B_1$ and $c_1, \ldots, c_n \in \mathcal{O}_2 \otimes \mathcal{K}$ such that

$$\|(\sum_{i=1}^n b_i \otimes c_i) - p\| < 1/2$$

and $\|c_i\| = 1$ for all i. Set $b = \sum_{i=1}^n |b_i|$ and take A_i to be the ideal of B_i generated by $\phi_1^i(b)$ for each i. Since $\overline{\bigcup \phi_i^\infty(A_i)}$ is a nonzero ideal in the simple algebra A, we see that A is the inductive limit of the A_i's.

Note that for any nonzero representation π of B_1, $(\pi \otimes id_{\mathcal{O}_2 \otimes \mathcal{K}})(p) = 0$, in which case $\pi(A_1) = 0$, or else $\|(\pi \otimes id_{\mathcal{O}_2 \otimes \mathcal{K}})(p)\| = 1$ and so

$$\|\sum_{i=1}^n \pi(b_i) \otimes c_i\| \geq \|(\pi \otimes id_{\mathcal{O}_2 \otimes \mathcal{K}})(p)\| - \|(\pi \otimes id_{\mathcal{O}_2 \otimes \mathcal{K}})((\sum_{i=1}^n b_i \otimes c_i) - p)\| > 1/2.$$

Now, for any $a \in A_i$, since A is simple, we can find some $j \geq i$ such that there is an element c in the ideal generated by $\phi_i^j(a)$ with a distance at most $1/2$ to $\phi_1^j(b)$. But then, for every nonzero representation π of A_j (since it induces a representation of B_1), we have $\|\pi(\phi_1^j(b))\| > 1/2$ and therefore $\|\pi(c)\| \neq 0$. It follows that c (and therefore $\phi_i^j(b)$) generates A_j as an ideal. $\square$

3.3 Embedding theorems for Hilbert C^*-modules over RSH and RSH $\otimes\mathcal{Z}$ algebras

There is a relationship between positive elements of a stable C^*-algebra and countably generated Hilbert C^*-modules over the algebra. To each positive element $a \in A_+$, the right ideal generated by a, $\overline{aA}$, is a countably (in fact, singly) generated Hilbert A-module. Moreover, every countably generated Hilbert A-module is isomorphic to one of the form $\overline{aA}$.

Whenever one Hilbert module $\overline{aA}$ embeds (as a Hilbert A-module) into another Hilbert module $\overline{bA}$, it follows that a is Cuntz below b. This is a key ingredient to the Hilbert module picture of the Cuntz semigroup taken in [12] (see [12, Appendix]).

This section contains results which can be interpreted as saying that there is an embedding of one Hilbert A-module into another under certain conditions. The C^*-algebra A here is RSH (in the first result) or the $\mathcal{Z}$-stabilization of an RSH algebra (in the second result) (actually, the Hilbert modules are over their stabilizations). Although this Hilbert module interpretation is possible, the results are stated in terms of positive elements, since it is in this form that they will be later used. To see how to interpret them as Hilbert module results, we note the following well-known result.

Proposition 3.3.1. *(cf. [3, Proposition III.2.1]) Let A be a stable C^*-algebra and let $a, b \in A_+$. Then there is an embedding of $\overline{aA}$ into $\overline{bA}$ (as Hilbert A-modules) if and only if there exists $s \in A$ such that $s^*s = a$ and $ss^* \in \mathrm{Her}\,(b)$.*

Here is the first embedding result. This result generalizes, in a number of ways, a well-known embedding result [28, Theorem 9.1.2] for vector bundles over a finite dimensional space X; the vector bundle result is recovered if we assume that $R = C(X, \mathcal{K})$, a and b are constant rank projections, and $I = 0$. In [40, Proposition 4.2 (1)], Chris Phillips gave a relativized version of [28, Theorem 9.1.2], which is to say that his result has the same restrictions as just stated except that I may be any ideal (which is necessarily of the form $C_0(U,\mathcal{K})$ for some open set U). Building on this, the author and Leonel Robert generalized the result further in [43, Theorem 3.2], to the case that R is still $C(X,\mathcal{K})$ yet a and b needn't be projections. The result here is proven using [43, Theorem 3.2] and induction.

Proposition 3.3.2. *Let R be the stabilization of an RSH algebra with total space Ω, such that $\dim(\Omega) < \infty$, and let I be an ideal of R. Suppose that $a, b \in R_+$ and $s \in R/I$ are such that*

(i) $s^*s = \pi_I(a)$ and $ss^* \in \mathrm{Her}\,(\pi_I(b))$; and

(ii) For all $\omega \in \Omega$ for which $\sigma(I)(\omega) \neq 0$, we have

$$\mathrm{Rank}\,\sigma(a)(\omega) + \frac{d_{top}(\omega) - 1}{2} \leq \mathrm{Rank}\,\sigma(b)(\omega).$$

Then there exists $\tilde{s} \in R$ such that $\tilde{s}^*\tilde{s} = a$, $\tilde{s}\tilde{s}^* \in \mathrm{Her}\,(b)$ and $\pi_I(\tilde{s}) = s$.

Proof. This will be proven by induction on the length of the RSH decomposition for R. The case where $R = \mathcal{K}$ is trivial. For the inductive step, we may express R as a pullback

$$\begin{array}{ccc} R & \xrightarrow{\sigma_\ell} & C_0(\Omega_\ell, M_{n_\ell}) \\ \lambda_\ell^{\ell-1} \downarrow & & \downarrow f \mapsto f|_{\Omega_\ell^{(0)}} \\ R_{\ell-1} & \xrightarrow{\rho_\ell} & C_0(\Omega_\ell^{(0)}, M_{n_\ell}). \end{array}$$

By (the proof of) Proposition 3.2.1, $I_{\ell-1} = \lambda_\ell^{\ell-1}(I)$ is an ideal, and there exists a closed set $Y_\ell \subseteq \Omega_\ell$ such that we have a pullback

$$\begin{array}{ccc} R/I & \rightarrow & C_0(Y_\ell, M_{n_\ell}) \\ \downarrow & & \downarrow f \mapsto f|_{Y_\ell \cap \Omega_\ell^{(0)}} \\ R_{\ell-1}/I_{\ell-1} & \xrightarrow{\rho'_\ell} & C_0(Y_\ell \cap \Omega_\ell^{(0)}, M_{n_\ell}), \end{array}$$

where each map is induced by the ones in the pullback for R. Hence, we may associate s with a pair $(f,t) \in C_0(Y_\ell, M_{n_\ell}) \oplus R_{\ell-1}/I_{\ell-1}$ such that $f|_{Y_\ell \cap \Omega_\ell^{(0)}} = \rho'_\ell(t)$.

By induction, there exists $\tilde{t} \in R_{\ell-1}$ such that $\tilde{t}^*\tilde{t} = \lambda_\ell^{\ell-1}(a)$, $\tilde{t}\tilde{t}^* \in \operatorname{Her}\left(\lambda_\ell^{\ell-1}(b)\right)$ and $\pi_{I_{\ell-1}}(\tilde{t}) = t$. The element $\tilde{s}$ that we want will be of the form $(\tilde{g}, \tilde{t}) \in C(\Omega_\ell, M_{n_\ell}) \oplus R_{\ell-1}$. We simply need to define $\tilde{g}$. In order that $(\tilde{g}, \tilde{t}) \in R$, we require that

$$\tilde{g}|_{\Omega_\ell^{(0)}} = \rho_\ell(\tilde{t});$$

on the other hand, in order that $\pi_I(\tilde{s}) = s$, we need to have

$$\tilde{g}|_{Y_\ell} = f.$$

Let us therefore define $g \in C_0(\Omega_\ell^{(0)} \cup Y_\ell, M_{n_\ell})$ by

$$\begin{aligned} g|_{\Omega_\ell^{(0)}} &= \rho_\ell(\tilde{t}), \text{ and} \\ g|_{Y_\ell} &= f. \end{aligned}$$

To see that this is well defined, notice that

$$\begin{aligned} \rho_\ell(\tilde{t})|_{\Omega_\ell^{(0)} \cap Y_\ell} &= \rho_\ell'(\pi_{I_\ell}(\tilde{t})) \\ &= \rho_\ell'(t') \\ &= f|_{Y_\ell \cap \Omega_\ell^{(0)}}. \end{aligned}$$

For every $\omega \in \Omega_\ell \backslash (\Omega_\ell^{(0)} \cup Y_\ell)$, we know that $\sigma(I)(\omega) \neq 0$ and therefore,

$$\operatorname{Rank} \sigma_\ell(a)(\omega) + \frac{\dim \Omega_\ell - 1}{2} \leq \operatorname{Rank} \sigma_\ell(b)(\omega).$$

Hence, by [43, Corollary 3.3], we may extend g to a function $\tilde{g} \in C_0(\Omega_\ell, M_{n_\ell})$ such that

$$\tilde{g}^*\tilde{g} = \sigma_\ell(a)$$

and

$$\tilde{g}\tilde{g}^* \in \operatorname{Her}\left(\sigma_\ell(b)\right).$$

Now, $\tilde{s} = (\tilde{t}, \tilde{g})$ is exactly as required. $\square$

Corollary 3.3.3. *Let R be the stabilization of an RSH algebra with total space Ω, such that $\dim(\Omega) < \infty$, and let I be an ideal of R. Suppose that $a, b \in R_+$ and $s \in R/I$ are such that*

(i) $s^*s = \pi_I(a)$ and $ss^* \in \mathrm{Her}\,(\pi_I(b))$; and

(ii) For all $\omega \in \Omega$ for which $\sigma(I)(\omega) \neq 0$, we have

$$\mathrm{Rank}\,\sigma(a)(\omega) < \mathrm{Rank}\,\sigma(b)(\omega).$$

Then there exists $\tilde{s} \in R \otimes \mathcal{Z}$ such that $\tilde{s}^*\tilde{s} = a \otimes 1_{\mathcal{Z}}$, $\tilde{s}\tilde{s}^* \in \mathrm{Her}\,(b \otimes 1_{\mathcal{Z}})$ and $\pi_{I\otimes\mathcal{Z}}(\tilde{s}) = s \otimes 1_{\mathcal{Z}}$.

Proof. Let $p, q \geq \dim(\Omega)/2$ be coprime integers. Then we will in fact find $\tilde{s} \in R \otimes Z_{p,q}$, where $Z_{p,q}$ is the dimension drop algebra

$$Z_{p,q} := \{f \in C([0,1], M_p \otimes M_q : f(0) \in M_p \otimes 1_q, f(1) \in 1_p \otimes M_q\}.$$

This will suffice since $Z_{p,q}$ embeds into $\mathcal{Z}$ by [47, Theorem 2.1 (i)] (due essentially to Jiang and Su, [29]).

By [40, Proposition 3.4], $R \otimes Z_{p,q}$ is an RSH algebra whose total space has dimension at most $\dim(\Omega) + 1$. We have

$$R \otimes Z_{p,q} \cong \{f \in C([0,1], R \otimes M_p \otimes M_q) : f(0) \in R \otimes M_p \otimes 1_q, f(1) \in R \otimes 1_p \otimes M_q\}.$$

and from this one quickly realizes that every irreducible representation of $R \otimes Z_{p,q}$ is unitarily equivalent to one of the form

$$f \mapsto \sigma(f(t))(\omega)$$

for some $t \in (0,1)$ and some $\omega \in \Omega$, or else, it is unitarily equivalent to the unique representation which when taken with multiplicity q or p gives

$$f \mapsto \sigma(f(0))(\omega) \text{ or } f \mapsto \sigma(f(1))(\omega)$$

respectively, for some $\omega \in \Omega$.

For each such irreducible representation π, we see then that either $\pi(I \otimes Z_{p,q}) = 0$ (in the case that $\sigma(I)(\omega) = 0$) or

$$\operatorname{Rank} \pi(b \otimes 1_{Z_{p,q}}) - \operatorname{Rank} \pi(a \otimes 1_{Z_{p,q}}) = k(\operatorname{Rank} \sigma(b)(\omega) - \operatorname{Rank} \sigma(b)(\omega)) \geq k$$

where k is either pq or q or p (depending on the possible cases above).

Hence, in particular, for any ω in the total space of $R \otimes Z_{p,q}$, since evaluation by ω is a direct sum of irreducible representations, we have either

$$\sigma(I \otimes Z_{p,q}) = 0$$

or

$$\operatorname{Rank} \sigma(a \otimes 1_{Z_{p,q}})(\omega) + \frac{\dim \Omega}{2} \leq \operatorname{Rank} \sigma(b \otimes 1_{Z_{p,q}})(\omega).$$

The existence of $\tilde{s}$ then follows by Proposition 3.3.2. $\square$

Corollary 3.3.4. *Let R be the stabilization of an RSH algebra with total space Ω, such that $\dim(\Omega) < \infty$, and let I be an ideal of R. Suppose that $a, b \in R_+$ are such that*

(i) $[\pi_I(a)] \leq [\pi_I(b)]$ in $\mathcal{C}u(R/I)$, and

(ii) For all $\omega \in \Omega$ for which $\sigma(I)(\omega) \neq 0$, we have

$$\operatorname{Rank} \sigma(a)(\omega) < \operatorname{Rank} \sigma(b)(\omega).$$

Then $[a] \leq [b]$ in $\mathcal{C}u(R)$.

Proof. For $\epsilon > 0$, we have by Lemma 2.1.3 some $s \in R/I$ such that

$$\pi_I((a-\epsilon)_+) = s^*s \text{ and } ss^* \in \operatorname{Her}(\pi_I(b))\,.$$

(To get this from Lemma 2.1.3, let $d \in R/I$ be such that $\pi_I((a-\epsilon)_+) = d\pi_I(b)d^*$, and set $s = \pi_I(b)^{1/2}d^*$.) Therefore, Corollary 3.3.2 applies with $(a-\epsilon)_+$ in place of a, and thus there exists $\tilde{s} \in R$ such that

$$(a-\epsilon)_+ = \tilde{s}^*\tilde{s} \text{ and } \tilde{s}\tilde{s}^* \in \operatorname{Her}(b)\,.$$

In particular, $[(a-\epsilon)_+] \leq [b]$ in $\mathcal{C}u(R)$.

Since ϵ is arbitrary, $[a] \leq [b]$. $\square$

Chapter 4

The Cuntz semigroup of $C_0(X, A)$ where A is a $\mathcal{Z}$-stable, simple, ASH algebra

4.1 The main result

Here, we present the main result:

Theorem 4.1.1. *Let A be a simple, $\mathcal{Z}$-stable ASH algebra and let X be a second countable, locally compact Hausdorff space. Then $\mathcal{C}u(C_0(X, A))$ may be identified with pairs $\left(f, \left(\langle q_{[p]}\rangle\right)_{[p]\in V(A)}\right)$, where*

(i) $f : X \to \mathcal{C}u(A)$ is a function which is lower semicontinuous with respect to $\ll$, and

(ii) for each $[p] \in V(A)$, $\langle q_{[p]}\rangle$ is an element of $V_c^{[p]}(f^{-1}([p]), A)$ (as defined in (2.4.1)).

The ordering is given by $\left(f, \left(\langle q_{[p]}\rangle\right)_{[p]\in V(A)}\right) \le \left(g, \left(\langle r_{[p]}\rangle\right)_{[p]\in V(A)}\right)$ if $f(x) \le g(x)$ for each x, and for each $[p] \in V(A)$,

$$\langle q_{[p]}|_{f^{-1}([p])\cap g^{-1}([p])}\rangle = \langle r_{[p]}|_{f^{-1}([p])\cap g^{-1}([p])}\rangle.$$

The addition is given by

$$\left(f, \left(\langle q_{[p]}\rangle\right)_{[p]\in V(A)}\right) + \left(g, \left(\langle r_{[p]}\rangle\right)_{[p]\in V(A)}\right) = \left(f+g, \left(\langle s_{[p]}\rangle\right)_{[p]\in V(A)}\right),$$

where for every pair of projections $0 \le p' \le p \in A\otimes\mathcal{K}$, we have

$$s_{[p]}|_{f^{-1}([p'])\cap g^{-1}([p-p'])} = q_{[p']} + r_{[p-p']}.$$

(We have that $(f+g)^{-1}([p])$ breaks into disjoint components of the form $f^{-1}([p'])\cap g^{-1}([p-p'])$, one component for each $[p'] \le [p]$, and so this definition of $s_{[p]}$ is continuous.)

The proof of this result takes two steps. First, we shall show that the invariant $\mathbb{I}(\cdot)$ is complete, in the sense that $[a] \le [b]$ if and only if $\mathbb{I}(a) \le \mathbb{I}(b)$. Second, we shall describe the range of the invariant $\mathbb{I}(\cdot)$, establishing that $\mathbb{I}(\cdot)$ attains every $\left(f, \left(\langle q_{[p]}\rangle\right)_{[p]\in V(A)}\right)$ as described in Theorem 4.1.1.

4.2 Completeness of the invariant $\mathbb{I}(\cdot)$

Here, we show that the invariant $\mathbb{I}(\cdot)$ is a complete Cuntz equivalence invariant. In fact, we prove something stronger, that $\mathbb{I}(\cdot)$ is an order embedding: not only does $\mathbb{I}(a) = \mathbb{I}(b)$ only happen when $[a] = [b]$, but also,

$$\mathbb{I}(a) \leq \mathbb{I}(b) \text{ implies that } [a] \leq [b]. \tag{4.2.1}$$

This stronger statement is a necessary ingredient in the proof of Theorem 4.1.1, the computation of the Cuntz semigroup of $C_0(X, A)$.

We draw on the inductive limit structure of $C_0(X, A)$ in order to prove (4.2.1). The proof goes along the following lines. Suppose that $a, b \in C_0(X, A \otimes \mathcal{K})_+$ satisfy $\mathbb{I}(a) \leq \mathbb{I}(b)$. We can express $C_0(X, A \otimes \mathcal{K})$ as the inductive limit of a sequence

$$C_0(X, A_1 \otimes \mathcal{K}) \xrightarrow{\phi_1^2} C_0(X, A_2 \otimes \mathcal{K}) \xrightarrow{\phi_2^3} \cdots .$$

Applying Proposition 2.3.1 to this, we can find elements $a_i \in C_0(X, A_i \otimes \mathcal{K})_+$ for each i such that

$$[\phi_i^j(a_i)] \leq [a_j]$$

for all $i \leq j$ and

$$[a] = \sup[\phi_i^\infty(a_i)].$$

For each i, we have $\mathbb{I}(\phi_i^\infty(a_i)) \leq \mathbb{I}(b)$, and if we show that $[\phi_i^\infty(a_i)] \leq [b]$ then it follows that $[a] \leq [b]$. Hence, we see that we can reduce the proof of (4.2.1) to the case that $a = \phi_i^\infty(a_i)$ for some i and some $a_i \in C_0(X, A_i \otimes \mathcal{K})_+$.

Now let us consider the invariant $\mathbb{I}(\cdot)$. The inequality $\mathbb{I}(a) \leq \mathbb{I}(b)$ means, first, that $[a(x)] \leq [b(x)]$ for all x in X. It tells us more, related to the points where $[a(x)] = [b(x)]$, but to avoid obscuring this sketch of the argument, let us consider the case where $[a(x)] < [b(x)]$ for all $x \in X$. Let us also make the simplifying assumption that $b = \phi_i^\infty(b_i)$ (for some i).

We shall show, in Lemma 4.2.1, that given $\epsilon > 0$, there is some $j \geq i$ for which the rank of $\phi_i^j((a_i - \epsilon)_+)$ is strictly smaller than that of $\phi_i^j(b_i)$ at each point in the total space of A_j. Assuming that X itself has finite dimension, it follows from Proposition 3.3.3 (with $I = 0$) that $[\phi_i^j((a_i - \epsilon)_+) \otimes 1_{\mathcal{Z}}] \leq [\phi_i^j(b_i) \otimes 1_{\mathcal{Z}}]$ in $\mathcal{C}u(C_0(X, A_i) \otimes \mathcal{Z})$. Since ϵ is arbitrary, we have $[\phi_i^\infty(a_i) \otimes 1_{\mathcal{Z}}] \leq [\phi_i^\infty(b_i) \otimes 1_{\mathcal{Z}}]$ in $\mathcal{C}u(C_0(X, A) \otimes \mathcal{Z})$. Moreover, since A is $\mathcal{Z}$-stable, it follows that $[\phi_i^\infty(a_i)] \leq [\phi_i^\infty(b_i)]$ as required.

In the above sketch, three simplifying assumptions were made, and the technicalities of the full proof are required in order to remove these assumptions. Before going into these technicalities, let us highlight the assumptions and briefly explain how they will be removed.

- Assumption: $[b] = [\phi_i^\infty(b_i)]$.

  Asymmetrically, we cannot reduce to this case as we can for a, since $\mathbb{I}(a) \leq \mathbb{I}(b)$ doesn't imply that $\mathbb{I}(a) \leq \mathbb{I}(\phi_i^\infty(b_i))$ for some i. However, this assumption doesn't simplify the argument, only the presentation. Lemma 4.2.1 is where we go to the building blocks and turn a pointwise strict inequality $[a(x)] < [b(x)]$ into a large difference in rank, and it works equally well without assuming $[b] = [\phi_i^\infty(b_i)]$.

- Assumption: $[a(x)] < [b(x)]$.

  We need to make use of the other part of the invariant $\mathbb{I}(\cdot)$ in order to remove this assumption. This other part tells us that $[a|_Y] = [b|_Y]$ for certain closed sets where $[a(x)] = [b(x)] \in V(A)$, and using the theory of inductive limits of Cuntz semigroups, that therefore

  $$[\phi_i^j(a_i)|_Y] = [\phi_i^j(b_i)|_Y].$$

  In particular, we apply this to the case that Y is the closure of

  $$\{x : [(a - \epsilon)_+(x)] = [b(x)]\},$$

so that, for the set of the points x where we can't get a difference in the ranks of $[\phi_i^j(a_i)]$ and $[\phi_i^j(b_i)]$, we instead have a Murray-von Neumann equivalence. We then use the full force of Proposition 3.3.2, where $I = C_0(X\backslash Y, A_i)$ instead of 0.

- Assumption: $\dim X < \infty$.

  A general space X can be written as an inverse limit of finite dimensional spaces, giving an inductive limit of the C^*-algebras of continuous functions. Inductive limit theory of the Cuntz semigroup allows a reduction to the case that X has finite dimension.

Lemma 4.2.1. *Let A be a simple separable unital ASH algebra. Suppose an inductive system*

$$A_1 \xrightarrow{\phi_1^2} A_2 \xrightarrow{\phi_2^3} \cdots \to A = \varinjlim A_i$$

satisfies the conclusion of Proposition 3.2.4. Let X be a compact Hausdorff space which is given as a projective limit of the system

$$X_1 \xleftarrow{\alpha_2^1} X_2 \xleftarrow{\alpha_3^2} \cdots$$

where each space X_i is compact and Hausdorff and each map α_{i+1}^i is proper. For $i \leq j$, let $\Phi_i^j : C(X_i, A_i) \to C(X_j, A_j)$ be given by

$$\Phi_i^j(f) = \phi_i^j \circ f \circ \alpha_j^i.$$

Let $a \in C(X_1, A_1 \otimes \mathcal{K})_+$, $b \in C(X, A \otimes \mathcal{K})_+$, and $b_i \in C(X_i, A_i \otimes \mathcal{K})_+$ for each i, such that

$$[(\Phi_i^{i+1})(b_i)] \leq [b_{i+1}]$$

in $\mathcal{C}u(C(X_{i+1}, A_{i+1}))$ and

$$[b] = \sup[\Phi_i^\infty(b_i)].$$

Suppose that for every $x \in X$,

$$[\phi_1^\infty(a)(x)] < [b(x)].$$

Then for $\epsilon > 0$, there exists $i \geq 1$ such that, for every $x \in X_i$ and every ω in the total space of A_i,

$$[\sigma(\Phi_1^i((a-\epsilon)_+)(x))(\omega)] < [\sigma(b_i(x))(\omega)].$$

Proof. We shall prove this result locally (in X), then use compactness to turn local results into a global one. Therefore, let us begin by fixing $x \in X$, and set $x_i = \alpha_\infty^i(x)$ for each i. Set $\eta = \epsilon/3$. Let us make the simplifying assumption that A and A_i are stable for each i (we may always replace A by $A \otimes \mathcal{K}$ with no detriment to the statement of the result).

By Lemma 2.1.16, we can find a nonzero element $c' \in A_+$ such that

$$[\phi_1^\infty((a-\eta)_+(x_1))] + [c'] \leq [b(x)].$$

By Proposition 2.3.1, we may find a nonzero element $c \in (A_{i_1})_+$ for some i_1 such that $[\phi_1^\infty(c)] \leq [c']$. Let $\delta > 0$ be sufficiently small that

$$\phi_{i_1}^\infty((c-\delta)_+) > 0.$$

Then since $[\phi_1^\infty((a-\eta)_+(x_1))] + [\phi_{i_1}^\infty(c)] \leq [b(x)]$, it follows by Proposition 2.3.2 that there exist $i_x \geq i_1$ and $\zeta > 0$ such that

$$[\phi_1^{i_x}((a-2\eta)_+(x_1))] + [\phi_{i_1}^{i_x}((c-\delta)_+)] \leq [(b_{i_x}-\zeta)_+(x)]. \tag{4.2.2}$$

Using Proposition 3.2.4, we may increase i_x to find that $\phi_{i_1}^{i_x}((c-\delta)_+)$ generates A_{i_x} as an ideal.

By Lemma 2.1.3, let U_x be a neighbourhood of x in X_{i_x} such that for $y \in U_x$,

$$[(a-\epsilon)_+(\alpha_{i_x}^1(y))] \leq [(a-\eta)_+(x_1)]$$

and

$$[(b_{i_x}-\zeta)_+(x)] \leq [b_{i_x}(y)];$$

combining these with (4.2.2) and setting

$$c_x := \phi_{i_1}^{i_x}((c-\delta)_+) \in A_{i_x},$$

we have

$$[\Phi_1^{i_x}((a-\epsilon)_+)(y)] + [c_x] \leq [b_{i_x}(y)]$$

for all $y \in U_x$.

In particular, for any $i \geq i_x$ and any ω in the total space of A_i, since $\phi_{i_x}^i(c_x)$ generates A_i as an ideal,

$$[\sigma(\Phi_1^i((a-\epsilon)_+)(y))(\omega)] < [\sigma(b_i(y))(\omega)].$$

We have that $(\alpha_\infty^{i_x})^{-1}(U_x)$ is an open cover of X, so by compactness, there exists a finite subcover indexed by $x_1, \ldots, x_n$. The conclusion of the lemma follows by taking $i = \max\{i_{x_1}, \ldots, i_{x_n}\}$. □

Theorem 4.2.2. *Let A be a simple $\mathcal{Z}$-stable ASH algebra and let X be a second countable locally compact Hausdorff space. For $a, b \in C_0(X, A \otimes \mathcal{K})_+$, we have $[a] \leq [b]$ in $\mathcal{C}u(C_0(X, A))$ if and only if $\mathbb{I}(a) \leq \mathbb{I}(b)$.*

Proof. By Lemma 2.1.17, it suffices to show that $[a \otimes 1_{\mathcal{Z}}] \leq [b \otimes 1_{\mathcal{Z}}]$ in $\mathcal{C}u(C_0(X, A) \otimes \mathcal{Z})$.

Let

$$A_1 \xrightarrow{\phi_1^2} A_2 \xrightarrow{\phi_2^3} \cdots \to A = \varinjlim A_i$$

be an inductive system as in Proposition 3.2.4. Let us further express X as the projective limit of the system

$$X_1 \xleftarrow{\alpha_2^1} X_2 \xleftarrow{\alpha_3^2} \cdots .$$

where each X_i is a second countable locally compact Hausdorff space and each map α_{i+1}^i is proper and surjective. Let $\Phi_i^j : C_0(X_i, A_i) \to C_0(X_j, A_j)$ be given by $\Phi_i^j(f) = \phi_i^j \circ f \circ \alpha_j^i$.

By Proposition 2.3.1, let $[a_i], [b_i] \in \mathcal{C}u(C_0(X_i, A_i))$ be such that

$$[\Phi_i^{i+1}(a_i)] \leq [a_{i+1}] \text{ and } [\Phi_i^{i+1}(b_i)] \leq [b_{i+1}] \text{ in } \mathcal{C}u(C_0(X_{i+1}, A_{i+1}))$$

and

$$[a] = \sup[\Phi_i^\infty(a_i)] \text{ and } [b] = \sup[\Phi_i^\infty(b_i)] \text{ in } \mathcal{C}u(C_0(X, A)).$$

Let $i \in \mathbb{N}$ and $\epsilon > 0$ be given. We shall show that $[\Phi_i^\infty((a_i - \epsilon)_+) \otimes 1_{\mathcal{Z}}] \leq [b \otimes 1_{\mathcal{Z}}]$. Since i and ϵ are arbitrary, it will follow that $[a \otimes 1_{\mathcal{Z}}] \leq [b \otimes 1_{\mathcal{Z}}]$. For the sake of simpler presentation, let us assume that $i = 1$.

Set

$$Y = Y_{\epsilon/2} := \{x \in X : [\Phi_1^\infty((a_1 - \epsilon/2)_+)(x)] = [b(x)]\}.$$

Note that $\overline{Y_{I,\epsilon/2}} \subseteq Y_{I,\eta}$ for any $\eta < \epsilon/2$. Also, for any $x \in Y_{I,\eta} \supseteq \overline{Y_I}$, we may find a neighbourhood U of x such that, for $y \in U \cap Y_{I,\eta}$,

$$[b(y)] = [\Phi_1^\infty((a_1 - \eta)_+)(y)] \ll [\Phi_1^\infty(a_1)(x)] \leq [a(x))] \leq [b(x)]$$

and likewise, $[b(x)] \leq [a(y)] \leq [b(y)]$. Hence, using Proposition 2.1.15, we must have

$$[a(x)] = [a(y)] = [b(x)] = [b(y)] \in V(A). \tag{4.2.3}$$

It follows from the definition of $\mathbb{I}(\cdot)$ that

$$[a|_Y] = [b|_Y] \in V(C_0(Y, A)).$$

If we set $Y' = Y \cup \{x \in X : [\Phi_1^\infty((a_1 - \epsilon/2)_+)(x) = 0\}$ then clearly

$$[\Phi_1^\infty((a_1 - \epsilon/2)_+)|_{Y'}] \leq [b|_{Y'}]$$

in $\mathcal{C}u(C_0(Y', A))$.

As Y' is closed, we may choose a decreasing sequence of open sets (U_n) such that $Y' \subseteq U_n$ for each n and

$$Y' = \bigcap \overline{U_n}.$$

Then by using Proposition 2.3.2, we have

$$[\Phi_1^\infty((a_1 - \epsilon)_+)|_{\overline{U_n}}] \leq [b|_{\overline{U_n}}] \tag{4.2.4}$$

for some n.

On the other hand, $Y_\infty := X \backslash U_n$ is compact (since $\|a(y)\| \geq \epsilon/2$ only on a compact set), and for every $x \in Y_\infty$, we have

$$[\Phi_1^\infty((a_1 - \epsilon/2)_+)(x)] < [b(x)].$$

Also, observe that Y_∞ is the projective limit of the system

$$\alpha_\infty^1(Y_\infty) \xleftarrow{\alpha_2^1|_{\alpha_\infty^2(Y_\infty)}} \alpha_\infty^2(Y_\infty) \xleftarrow{\alpha_3^2|_{\alpha_\infty^3(Y_\infty)}} \cdots .$$

Therefore, by Lemma 4.2.1, there exists some i such that for every $x \in \alpha_\infty^i(Y_\infty)$ and every ω in the total space of A_i,

$$[\sigma(\Phi_1^i((a_1 - \epsilon)_+)(x))(\omega)] < [\sigma(b_i(x))].$$

Further, using (4.2.4) and Proposition 2.3.2, by possibly increasing i we have

$$[\Phi_1^i((a_1 - \epsilon)_+)|_{\alpha_\infty^i(\overline{U_n})}] \leq [b_i|_{\alpha_\infty^i(\overline{U_n})}].$$

(Here we are using the fact that the maps α_j^i are proper in order to get that $\overline{U_n} = \varprojlim \alpha_\infty^i(\overline{U_n})$.)

Since α_∞^i is surjective, $X_i \backslash \alpha_\infty^i(\overline{U_n}) \subseteq \alpha_\infty^i(Y_\infty)$, and so Corollary 3.3.4 applies with $R = C_0(X_i, A_i)$ and $I = C_0(X_i \backslash \alpha_\infty^i(\overline{U_n}), A_i)$. The result is that $[\Phi_1^i((a_1 - \epsilon)_+)] \leq [b_i]$ and in particular, $[\Phi_1^\infty((a_1 - \epsilon)_+)] \leq [b]$. Since ϵ is arbitrary, $[\Phi_1^\infty(a_1)] \leq [b]$ as required. $\square$

4.3 The range of the invariant $\mathbb{I}(\cdot)$

Here, we show that the range of $\mathbb{I}(\cdot)$ is everything described as its codomain in Section 2.4. The first component of this proof is Proposition 4.3.2, where we show that a $\ll$-lower semicontinuous function $f : X \to \mathcal{C}u(A)$ can be approximated nicely by $\ll$-lower semicontinuous functions with finite ranges. This ties into our goal since we show, in Corollary 4.3.5, that a lower semicontinuous function from a finite dimensional space X to a finite subset of the Cuntz semigroup of a finite stage algebra A_i (i.e. an RSH algebra with finite dimensional total space) can be obtained as a function $x \mapsto [a(x)]$ where $a \in C_0(X, A_i \otimes \mathcal{Z} \otimes \mathcal{K})$ (note that this is at a cost of having to $\mathcal{Z}$-stabilize). The main proof puts those pieces together, with some care to handle of the other part of the invariant, namely the $V_c^{[p]}$-classes on the level sets of the $\ll$-lower semicontinuous function.

The following derives a formally stronger interpolation property from Riesz interpolation.

Lemma 4.3.1. *Let X be a compact Hausdorff space, let $S \in \mathcal{O}rd\mathcal{C}u$, and let $f : X \to S$ be a $\ll$-lower semicontinuous function. Suppose that S has Riesz interpolation and that $s_1, \ldots, s_n \in S$ are such that*

$$\begin{matrix} s_1 \\ \vdots \\ s_n \end{matrix} \ll f(x)$$

for all $x \in X$. Then there exists $t \in S$ such that

$$\begin{matrix} s_1 \\ \vdots \\ s_n \end{matrix} \ll t \leq f(x)$$

for all $x \in X$.

Proof. For $x \in X$, since we have

$$\begin{matrix} s_1 \\ \vdots \\ s_n \end{matrix} \ll f(x),$$

there must be some $a_x \in S$ such that

$$\begin{matrix} s_1 \\ \vdots \\ s_n \end{matrix} \leq a_x \ll f(x).$$

By $\ll$-lower semicontinuity, there exists an open neighbourhood U_x of x such that $a_x \ll f(y)$ for all $y \in U_x$.

By using compactness of X, let $U_{x_1}, \dots, U_{x_k}$ be a finite cover of X. Then by Riesz interpolation, let $t \in S$ satisfy

$$\begin{matrix} s_1 \\ \vdots \\ s_n \end{matrix} \leq t \leq \begin{matrix} a_{x_1} \\ \vdots \\ a_{x_k}. \end{matrix}$$

Then, for each $x \in X$, since there exists i such that $a_{x_i} \ll f(x)$, we must have $t \ll f(x)$. $\square$

Proposition 4.3.2. *Let X be a second countable locally compact Hausdorff space, let $S \in \mathcal{O}rd\mathcal{C}u$, and let $f : X \to S$ be a $\ll$-lower semicontinuous function. If S has Riesz interpolation and S contains a countable dense set then there exists a sequence of $\ll$-lower semicontinuous functions $f_n : X \to S$ with finite range, such that for each $x \in X$, $(f_n(x))_{n=1}^{\infty}$ is a $\ll$-increasing sequence with supremum $f(x)$.*

If A is a simple, stably finite C^*-algebra for which $V(A)$ is almost unperforated and $S = V(A) \amalg Lsc(T(A), [0, \infty])$ with order structure as described in Theorem 2.2.5 then we can arrange that, if for each $[p] \in V(A)$, $W_{[p]}$ is an open set containing $f^{-1}([p])$ then for each n and $[p] \in V(A)$, $f_n^{-1}([p])$ is compactly contained in $W_{[p]}$.

Proof. We shall first attempt to construct a sequence that is pointwise increasing (and not necessarily $\ll$-increasing).

Suppose that (A_n) is an increasing sequence of finite subsets of S whose union is dense. The idea of the proof here is that we would like to set $f_n(x)$ to be

$$\sup\{s : s \in A_n \text{ and } s \ll f(x)\}.$$

However, as seen in Example 2.2.12, such suprema may not exist in S.

Riesz interpolation is used to make up for the absence of such suprema. For a subset $B = \{s_1, \ldots, s_\ell\}$ of A_n, we shall use Lemma 4.3.1 to find an element t_B satisfying

$$\begin{matrix} s_1 \\ \vdots \\ s_\ell \end{matrix} \leq t_B,$$

and also $t_B \ll f(x)$ on a suitable subset $U_{n,B}$ of

$$\left\{ x \in X : \begin{matrix} s_1 \\ \vdots \\ s_\ell \end{matrix} \ll f(x) \right\}.$$

In order to apply Lemma 4.3.1, we shall need $U_{n,B}$ to be compactly contained in the aforementioned set. Additional care is needed so that f_n is $\ll$-lower semicontinuous and the sequence (f_n) is pointwise increasing

For each $s \in S$, let us choose an increasing sequence $(V_{s,i})_{i=1}^\infty$ of open sets which are compactly contained in

$$\{x \in X : s \ll f(x)\},$$

and whose union is this entire set. Let us also choose a sequence $(s_i)_{i=1}^\infty$ with union dense in S.

Let us now construct the functions f_n inductively, with the following properties:

(i) For $i \leq n$ and $x \in V_{s_i,n}$, we have $f_n(x) \geq s_i$ (this will ensure that $f \geq \sup f_n$);

(ii) For each element s of the range of f_n, the set $\{x \in X : s \leq f_n(x)\}$ is compactly contained in $\{x \in X : s \ll f(x)\}$ (this ensures that $f_n \leq f$, but is in fact stronger, as needed to allow the construction of f_{n+1}).

To construct f_{n+1} given f_n, we first set

$$A_{n+1} = A_n \cup f_n(X) \cup \{s_{n+1}\}.$$

For $s \in A_{n+1}$, set

$$U_{s,n+1} = \{x \in X : s \leq f_n(x)\} \cup V_{s,n+1}.$$

By the inductive hypothesis and choice of $V_{s,n+1}$, we have that $U_{s,n+1}$ is compactly contained in $\{x : s \ll f(x)\}$. Now, for any subset B of A_{n+1}, set

$$U_{B,n+1} = \bigcap_{s \in B} U_{s,n+1}.$$

Let us now associate a value $t_B \in S$ to each subset B of A_{n+1}. The value t_B will satisfy

$$s \leq t_B \ll f(x)$$

for all $s \in B$ and all $x \in \overline{U_{B,n+1}}$. Also, if $B' \subseteq B$ then $t_B \leq t_{B'}$; this condition will be arranged by finding t_B inductively on the size of the set B.

Having defined $t_{B'}$ for all $B' \subsetneq B$, we have

$$\begin{matrix} t_{B'} \\ s \end{matrix} \ll f(x)$$

for all $B' \subsetneq B$, all $s \in B$, and all x in the compact set $\overline{U_{B,n+1}}$. So, by Lemma 4.3.1, we may find t_B such that

$$\begin{matrix} t_{B'} \\ s \end{matrix} \leq t_B \ll f(x)$$

for all $B' \subsetneq B$, all $s \in B$, and all $x \in \overline{U_{B,n+1}}$, as required.

Now that we have specified all the values t_B, for each $x \in X$, if

$$B_x = \{s \in A_{n+1} : x \in U_{s,n+1}\}$$

then we see that

$$f_{n+1}(x) := t_{B_x} = \max\{t_{B'} : x \in U_{B',n+1}\}.$$

It is evident from this that f_{n+1} is lower semicontinuous with respect to $\leq$ and therefore also with respect to $\ll$.

This construction gave us a sequence (f_n) which may not be pointwise $\ll$-increasing, yet is pointwise increasing and also has the property that for each n and each value $[a]$ in the range of f_n, $[a] \ll f(x)$ for all $x \in \overline{f_n^{-1}([a])}$, which is compact (as long as $[a] \neq 0$). Let us see why there exists an $\ll$-increasing subsequence of such a sequence (f_n).

Given any n and any $[a]$ in the range of f_n, at each point $x \in \overline{f_n^{-1}([a])}$, since $[a] \ll f(x)$, we may find some $m \geq n$ such that $[a] \ll f_m(x)$; moreover, by $\ll$-lower semicontinuity of f_m, we have $[a] \ll f(y)$ for all y in some neighbourhood of x. By using compactness of $\overline{f_n^{-1}([a])}$, it follows that there exists m such that, in fact, $[a] \leq f_m(x)$ for all $x \in \overline{f_n^{-1}([a])}$. Doing this for all (finitely many) values in the range of f_n, we see that there exists m such that $f_n(x) \ll f_m(x)$ for all x. It is evident from this that the desired subsequence exists.

To prove the last statement of the proposition, first note that by Theorem 2.2.13 (iii), we can apply the first part to get an $\ll$-increasing sequence (g_n) such that $f(x) = \sup g_n(x)$ for all x. We shall define f_n to agree with g_n except at points $x \notin W_{[p]}$ where $f_n(x) = [p]$ (for any $[p]$); at such points, we shall instead define $f_n(x) = \widehat{[p]} \in Lsc(T(A), (0, \infty])$. The resulting f_n is still lower semicontinuous with respect to $\leq$.

Also, note that whenever an element $[a] \in \mathcal{C}u(A)$ is the supremum of an increasing sequence $([p_n]) \subseteq V(A)$, as long as the sequence isn't eventually constant, it is also the supremum of $(\widehat{[p_n]})$. This shows that $\sup g_n(x) = \sup f_n(x) = f(x)$. $\square$

We now begin to show how $\ll$-lower semicontinuous functions with finite range can be realized, for Cuntz semigroups of the finite stage algebras.

Lemma 4.3.3. *Let X be a locally compact Hausdorff space such that* $\dim(X) < \infty$. *Let*

R be an RSH algebra with total space Ω such that $\dim(\Omega) < \infty$. Suppose that we are given

(i) an open cover $U_1, \ldots, U_n$ of X, such that each set U_i is σ-compact.

(ii) for each $i = 1, \ldots, n$, an element $[a_i] \in \mathcal{C}u(C_0(\overline{U_i}, R))$.

Suppose that, if $i \leq j$ then for $x \in \overline{U_i} \cap \overline{U_j}$ and $\omega \in \Omega$,

$$\operatorname{Rank} \sigma(a_i(x))(\omega) + \frac{\dim X + d_{top}(\omega) - 1}{2} \leq \operatorname{Rank} \sigma(a_j(x))(\omega). \tag{4.3.1}$$

Then there exists $[a] \in \mathcal{C}u(C_0(X, R))$ such that, for each i, if $f_i \in C_0(U_i \backslash \bigcup_{j>i} U_j)_+$ is strictly positive, then

$$[f_i a|_{U_i \backslash \bigcup_{j>i} U_j}] = [f_i a_i|_{U_i \backslash \bigcup_{j>i} U_j}] \tag{4.3.2}$$

in $\mathcal{C}u(C_0(U_i \backslash \bigcup_{j>i} U_j, R))$.

Remark 4.3.4. As seen in (the proof of) Proposition 2.4.4, if $[a_i(x)] = [p] \in V(R)$ for all $x \in U_i$ and U_i is σ-compact then (4.3.2) amounts to

$$\langle a|_{U_i \backslash \bigcup_{j>i} U_j} \rangle = \langle a_i|_{U_i \backslash \bigcup_{j>i} U_j} \rangle$$

in $V_c^{[p]}(U_i \backslash \bigcup_{j>i} U_j, R)$.

Proof. We shall find elements $s_i \in C_0(\overline{U_i}, R)$ such that $s_i^* s_i = a_i$ and, for $i \leq j$,

$$s_i s_i^*|_{\overline{U_i} \cap \overline{U_j}} \in \operatorname{Her}\left(s_j s_j^*|_{\overline{U_i} \cap \overline{U_j}}\right).$$

Then, using a strictly positive element λ_i of $C_0(U_i)_+$ for each i, we shall set

$$a = \sum_{i=1}^{n} \lambda_i s_i s_i^*.$$

It follows that, if $f_i \in C_0(U_i \backslash \bigcup_{j>i} U_j)_+$ then

$$f_i a|_{U_i \backslash \bigcup_{j>i} U_j} = f_i \lambda_i s_i s_i^* + c$$

where $c \in \mathrm{Her}\,(f_i s_i s_i^*)$, and so

$$[f_i a|_{U_i \setminus \bigcup_{j>i} U_j}] = [f_i s_i s_i^*|_{U_i \setminus \bigcup_{j>i} U_j}] = [f_i a_i|_{U_i \setminus \bigcup_{j>i} U_j}].$$

To find the elements s_i, we use induction on i, beginning at n and decreasing. For $i = n$, we simply set $s_n = a_n^{1/2}$. Having defined $s_n, \ldots, s_{i+1}$, let now define s_i. This will be done first on $\overline{U_i} \cap \overline{U_{i+1}}$, and then extended to add the set $\overline{U_i} \cap \overline{U_{i+2}}$, and so on until we add the set $\overline{U_i} \cap \overline{U_n}$, and then finally the rest of $\overline{U_i}$.

For the step where we extend the definition to include the set $\overline{U}_i \cap \overline{U}_j$ (for $j > i$), we can assume that s_i is already defined on some closed (possibly empty) subset K of $\overline{U_i} \cap \overline{U_j}$, such that the definition already satisfies $s_i \in \mathrm{Her}\,(s_j)$. By (4.3.1), we can apply Proposition 3.3.2 (with $A = C_0(\overline{U_i} \cap \overline{U_j}, R)$ and $I = C_0(\overline{U_i} \cap \bigcap_{j'=j}^{n} \overline{U_{j'}}, R)$), providing the extension of s_i to $\overline{U_i} \cap \overline{U_j}$, as required. □

Corollary 4.3.5. *Let X be a locally compact Hausdorff space such that $\dim(X) < \infty$. Let R be an RSH algebra with total space Ω such that $\dim(\Omega) < \infty$. Suppose that we are given*

(i) an open cover $U_1, \ldots, U_n$ of X, such that each set U_i is σ-compact.

(ii) for each $i = 1, \ldots, n$, an element $[a_i] \in \mathcal{C}u(C_0(\overline{U_i}, R))$.

Suppose that, if $i \leq j$ then for $x \in \overline{U_i} \cap \overline{U_j}$ and $\omega \in \Omega$,

$$\mathrm{Rank}\,\sigma(a_i(x))(\omega) < \mathrm{Rank}\,\sigma(a_j(x))(\omega).$$

Then there exists $[a] \in \mathcal{C}u(C_0(X, R \otimes \mathcal{Z}))$ such that, for each i, if $f_i \in C_0(U_i \setminus \bigcup_{j>i} U_j)_+$ is strictly positive, then

$$[f_i a|_{U_i \setminus \bigcup_{j>i} U_j}] = [f_i a_i \otimes 1_{\mathcal{Z}}|_{U_i \setminus \bigcup_{j>i} U_j}]$$

in $\mathcal{C}u(C_0(U_i \setminus \bigcup_{j>i} U_j, R))$.

Proof. If p, q are sufficiently large coprime integers then the family $(a_i \otimes 1_{Z_{p,q}})$ satisfies the hypotheses of Lemma 4.3.3 (the argument for this is exactly as in the proof of Corollary 3.3.3), and therefore we can in fact find $[a] \in \mathcal{C}u(C_0(X, R \otimes Z_{p,q}))$. $\square$

The next lemma will help enable the application of Corollary 4.3.5, in terms of getting the elements a_i to be defined on open sets.

Lemma 4.3.6. *Let X be a locally compact Hausdorff space and let Y be a closed subset. Let A be a C^*-algebra. Let $p \in C_b(Y, A)$ such that $p(x)$ is a projection for all $x \in Y$, and $p(x) \sim p(y)$ for all $x, y \in Y$. Then there exists an open set U with $Y \subseteq U \subseteq X$ and some $\tilde{p} \in C_b(U, A)$ such that $\tilde{p}|_Y = p$, $\tilde{p}(x)$ is a projection for all $x \in U$ and $\tilde{p}(x) \sim \tilde{p}(y)$ for all $x, y \in U$.*

Proof. We may find a continuous extension $a \in C_b(X, A)$ of p. Moreover, we may find an open set U containing Y such that, for every $x \in \overline{U}$ there exists $y \in Y$ such that $\|a(x) - a(y)\| < 1/2$. It follows that the spectrum of $a|_U$ (in the algebra $C_b(U, A)$) does not contain the point $1/2$, and so by using functional calculus, $\tilde{p} := \chi_{(1/2,\infty)}(a|_U) \in C_b(U, A)$. Clearly $\tilde{p}|_Y = p$ and, for $x \in U$ there exists $y \in Y$ such that $\tilde{p}(x) \sim p(y)$. Consequently, the Murray-von Neumann class of $\tilde{p}(x)$ is constant over all of U. $\square$

Theorem 4.3.7. *Let A be a simple $\mathcal{Z}$-stable ASH algebra and let X be a second countable locally compact Hausdorff space. Let there be given a map $f : X \to \mathcal{C}u(A)$ which is lower semicontinuous with respect to $\ll$ and, for each $[p] \in V(A)$, some*

$$\langle a_{[p]} \rangle \in V_c^{[p]}(f^{-1}(\{[p]\}), A).$$

Then there exists $[a] \in \mathcal{C}u(C_0(X, A))$ such that

$$\mathbb{I}([a]) = \big(f, (\langle a_{[p]} \rangle)_{[p] \in V(A)}\big);$$

which is to say, $[a(x)] = f(x)$ for all $x \in X$ and $\langle a|_{f^{-1}([p])} \rangle = \langle a_{[p]} \rangle$ for all $p \in V(A)$.

Proof. By Lemma 2.1.17 and Proposition 2.4.4 (ii), it suffices to find $[a] \in \mathcal{C}u(C_0(X, A \otimes \mathcal{Z}))$ such that $[a(x)] = e(f(x))$ where $e : \mathcal{C}u(A) \to \mathcal{C}u(A \otimes \mathcal{Z})$ is the map induced by the first factor embedding $A \to A \otimes \mathcal{Z}$, and $\langle a|_{f^{-1}([p])}\rangle = \langle a_{[p]} \otimes 1_{\mathcal{Z}}\rangle$

For each $[p] \in V(A)$, let $W_{[p]}$ be an open set containing $f^{-1}([p])$ and $\tilde{a}_{[p]}$ an extension of $a_{[p]}$ as given by Lemma 4.3.6. By making use of Lemma 4.3.2, choose an increasing sequence $(f_n)_{n=1}^{\infty}$ of $\ll$-lower semicontinuous functions with finite range which $\ll$-increase pointwise to f, and such that $f_n^{-1}([p]) \subseteq W_{[p]}$ for each $[p] \in V(A)$.

Let us construct, for each n, $[b_n] \in \mathcal{C}u(C_0(X, A \otimes \mathcal{Z}))$ such that, for each $x \in X$,

$$f_n(x) \leq [b_n(x)] \leq f_{n+1}(x),$$

and also, for each $[p] \in V(A)$, $\{x : [b_n(x)] = [p]\} = f_{n+1}^{-1}([p])$ and

$$\langle b_n|_{\{x:[b_n(x)]=[p]\}}\rangle = \langle \tilde{a}_{[p]}|\rangle.$$

Then by Theorem 4.2.2, the sequence $([b_n])$ is increasing and its supremum $[a]$ satisfies the desired conclusions.

Fix n. We may write X as an inverse limit of the system

$$X_1 \xleftarrow{\alpha_2^1} X_2 \xleftarrow{\alpha_3^2} \cdots,$$

where each X_i is a finite-dimensional locally compact Hausdorff space and the maps α_j^i are proper and surjective. By Proposition 3.2.4, we may write A as an inductive limit of the system

$$A_1 \xrightarrow{\phi_1^2} A_2 \xrightarrow{\phi_2^3} \cdots$$

where each A_i as an RSH algebra with finite dimensional total space.

Let us list the elements of the range of f_{n+1} in non-decreasing order as $[c_1], \dots, [c_m]$. For each $i = 1, \dots, m$, we can, by Proposition 2.3.1 and induction, find some d_i in a finite stage algebra, $A_{\ell_i} \otimes \mathcal{K}$, and some $\epsilon_i > 0$ such that

(i) $f_n(x) \leq [\phi_{\ell_i}^{\infty}((d_i - \epsilon_i)_+)] \leq [\phi_{\ell_i}^{\infty}(d_i)] \ll [c_i]$ (for $[c_i] \in V(A)$, we can get $[\phi_{\ell_i}^{\infty}(d_i)] = [c_i]$); and

(ii) whenever $[c_j] < [c_i]$, we have $[\phi_{\ell_j}^{\infty}(d_j)] < [\phi_{\ell_i}^{\infty}((d_i - \epsilon_i)_+)]$.

By using Proposition 2.1.16 and Proposition 2.3.2, we may take $\ell \geq \max\{\ell_1, \dots, \ell_m\}$ sufficiently large so that, by replacing each d_i with its image in $A_\ell \otimes \mathcal{K}$, we have for each i, j for which $[c_j] < [c_i]$, there exists a nonzero $e_{i,j} \in (A_\ell \otimes \mathcal{K})_+$ such that $[(d_j - \epsilon_j)_+] + [e_{i,j}] < [(d_i - \epsilon_i)_+]$ whenever $[c_j] < [c_i]$. But then, by Proposition 3.2.4, we may increase ℓ so that, whenever $[c_j] < [c_i]$, we have for all ω in the total space of A_ℓ,

$$\operatorname{Rank} \sigma((d_j - \epsilon_j)_+)(\omega) < \operatorname{Rank} \sigma((d_i - \epsilon_i)_+)(\omega).$$

If ℓ is sufficiently large, we may also have that, on the closure of some open set $W'_{[p]}$ containing $\overline{f_{n+1}^{-1}([p])}$ and compactly contained in $W_{[p]}$, $a_{[p]}|_{\overline{W'_{[p]}}}$ is the image of a projection in $C(\alpha_\infty^\ell(\overline{W'_{[p]}}), A_\ell \otimes \mathcal{K})$. Therefore, we may apply Corollary 4.3.5 to obtain some $[b'_n] \in \mathcal{C}u(C_0(X_\ell, A_\ell))$ such that, for each $x \in X$,

$$[b'_n(\alpha_\infty^\ell(x))] = [(d_i - \epsilon_i)_+]$$

where $f_{n+1}(x) = [c_i]$, and

$$\langle \phi_\ell^\infty \circ b'_n \circ \alpha_\infty^\ell|_{f_{n+1}^{-1}([p])} \rangle = \langle a_{[p]}|_{f_{n+1}^{-1}([p])} \rangle.$$

Hence, $[b_n] = [\phi_\ell^\infty \circ b'_n \circ \alpha_\infty^\ell]$ is the desired element in $\mathcal{C}u(C_0(X, A))$. $\square$

4.4 Counterexamples with non-$\mathcal{Z}$-stable algebras

Here, we give some examples showing that without the hypothesis of $\mathcal{Z}$-stability, the Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ may not be complete.

Lemma 4.4.1. *Suppose that A is a stably finite C^*-algebra, Y is a compact Hausdorff space, and $C(Y, A)$ contains projections p_1, p_2, q such that $p_1 \sim p_2$ and $p_1, p_2 \leq q$ yet p_1 is not homotopic to p_2 within the space of projections in $qC(Y, A)q$. Then the Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ is not complete on $C([0, 1] \times Y, A)$.*

Proof. Replacing Y by a component, if necessary, we may assume that $[p_1(x)] = [p_2] = [p'] \in V(A)$ and $[q(x)] = [q'] \in V(A)$ for all $x \in Y$.

Let $f \in C_0([0, 1/4))_+, g \in C_0((3/4, 1])_+, h \in C_0((0, 1))_+$ be strictly positive, and in addition, let us impose the condition that $\|h\| = \epsilon < 1$. Define $a, b \in C([0, 1] \times Y, A) \cong C([0, 1], C(Y, A))$ by

$$\begin{aligned} a(t) &= f(t)p_1 + g(t)p_2 + h(t)q, \text{ and} \\ b(t) &= p_1 + h(t)q, \end{aligned}$$

for $t \in [0, 1]$.

Let us show that $\mathbb{I}(a) = \mathbb{I}(b)$. First, note that, for $(t, x) \in [0, 1] \times Y$, we have

$$[a(x)] = [b(x)] = \begin{cases} [p'], & \text{if } t = 0, 1 \\ [q], & \text{if } t \in (0, 1). \end{cases}$$

Moreover, since $[p_1] = [p_2]$, we see that

$$\langle a|_{\{0,1\}\times Y} \rangle = \langle b|_{\{0,1\}\times Y} \rangle$$

in $V_c^{[p']}(\{0, 1\} \times Y, A) = V^{[p']}(\{0, 1\} \times Y, A)$. On the other hand, clearly

$$\langle a|_{(0,1)\times Y} \rangle = \langle 1_{C_b((0,1))} \otimes q \rangle = \langle b|_{(0,1)\times Y} \rangle$$

in $V_c^{[q']}((0, 1) \times Y, A)$. Thus, we see that $\mathbb{I}(a) = \mathbb{I}(b)$.

Now let us show that $[a] \neq [b] \in \mathcal{C}u(C([0,1] \times Y, A))$. In fact, we shall show that $[b] \not\leq [a]$. For this part, we shall once again view the ambient algebra as $C([0,1], C(Y, A))$. If it were true that $[b] \leq [a]$, then we could find $t \in C([0,1], C(Y, A \otimes \mathcal{K}))$ such that

$$\|b - tat^*\| \leq \epsilon.$$

But then, by Lemma 2.1.3, there exists $d \in C([0,1], C(Y, A \otimes \mathcal{K}))$ such that

$$(b - \epsilon)_+ = dtat^*d^*.$$

However, by the definition of b, we have $(1-\epsilon)p_1 = (b-\epsilon)_+$, and therefore, if we set $s = (1-\epsilon)^{-1/2}dta^{1/2}$, we have $s^*s = p_1$ and so $h := ss^*$ is a projection in $\mathrm{Her}\,(a)$.

We have $p_1 \sim h(0) \leq p_1$ (since $h(0) \in \mathrm{Her}\,(a(0)) = \mathrm{Her}\,(p_1)$), and so $h(0) = p_1$. Likewise, $h(1) = p_2$. For $t \in (0,1)$, we have $h(t) \in qC(Y, A)q$. Therefore, h is a homotopy from p_1 to p_2 in the space of projections in $qC(Y, A)q$, which contradicts the hypothesis. $\square$

Failure of an ASH algebra A to be $\mathcal{Z}$-stable can occur in two different ways. It could be that A is of type I (equal to either M_n for some n or $\mathcal{K}$). The other possibility is that the only way of expressing A as an inductive limit of subhomogeneous C^*-algebras involves large topological dimension compared to matricial dimension in the finite stages.

In the first case, the Cuntz semigroup of $C_0(X, A)$ is equal to the Cuntz semigroup of $C_0(X)$. This situation was studied in [43], and the results there show that $\mathbb{I}(\cdot)$ is in fact complete, so long as X has dimension at most three. We may find projections p_1, p_2, q in $C(S^3, \mathcal{K})$ satisfying the hypotheses of Lemma 4.4.1, and therefore, $\mathbb{I}(\cdot)$ is not complete for $C([0,1] \times S^3)$ (the spectrum here has dimension four). ([43, Section 6.1] also shows that $\mathbb{I}(\cdot)$ is incomplete for a four dimensional space, S^4.)

For an example from the second case, [58, Corollary 18] gives an example of a simple unital AH algebra and projections which are unitarily equivalent but not homotopic. Thus, by Lemma 4.4.1, $\mathbb{I}(\cdot)$ is not complete for $C([0,1], A)$.

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