Analytic properties of groups and their actions.

Summary: amenable groups (discrete, topological groups), random walks on groups (spectral radius, Liouville property, etc.), sofic groups, hyperlinear, Property (T), hyperbolic groups.

Amenable groups

$\boxed{\text{Def 1}}$ A group G is called amenable if it admits a fin additive measure μ s.t. $\mu(G) = 1$ and $\mu(gE) = \mu(E)$ $\forall E \subset G$ and $g \in G$.

$\boxed{\text{Def 2.}}$ G is amenable if G admits G-invariant mean.

μ is a mean on X if μ is a lin. functional on $\ell^\infty(X)$, $\mu(\chi_G) = 1$, $\mu(f) \geq 0$ $f \geq 0$ $f \in \ell^\infty(G)$

$\boxed{G \curvearrowright X}$

G-inv. : $\mu(g.f) = \mu(f)$ $\forall f \in \ell^\infty(X)$, $g \in G$.

Def 1. due to von Neumann

$\Updownarrow$

Def 2.

G has paradoxical decomp. if $\exists H_i, V_i$

$G = \bigsqcup_{i=1}^n H_i \cup \bigsqcup_{i=1}^m V_i$, $\exists g_i$ $\exists h_i$

$G = \bigsqcup^n g_i H_i = \bigsqcup^m h_i V_i$

$i=1$ $\qquad$ $i=1$

If G has parad. deco $\overset{\Leftarrow}{\Rightarrow}$ G is not amenable

$$1 = m(G) = \mu\left(\bigsqcup_{i=1}^{n} H_i\right) + \mu\left(\bigsqcup_{i=1}^{m} V_i\right) =$$

$$= \mu\left(\bigsqcup_{i=1}^{n} g_i H_i\right) + \mu\left(\bigsqcup_{i=1}^{m} h_i V_i\right) = 2$$

Remark: $G \curvearrowright X$ — amenable

X admits G-invariant mean.

Ex.: define paradoxical decomp. on X
prove that if $G \curvearrowright X$ amenable
then it is not paradoxical.

If G is amenable $\Rightarrow$ $G \curvearrowright X$ is amen.
$\forall X$.

Def 3 (Følner set):

G is amenable, if $\exists$ Følner sets
For any finite set $E \subseteq G$, $\varepsilon > 0$
$\exists F \subseteq_{fin} G$ s.t

$$|gF \triangle F| < \varepsilon |F| \quad \forall g \in E.$$

Def 3 $\Rightarrow$ Def 2

$E_i \nearrow G$ sequence of fin sets

$\exists\, F_i \underset{fin}{\subseteq} G$ $\boxed{|gF_i \,\Delta\, F_i| < \frac{1}{i}|F_i| \quad \forall g \in E_i.}$

$$\mu_i = \frac{1}{|F_i|}\chi_{F_i}$$

cluster point in weak*-top.

gives a mean, which is G-invar.

Ozawa - Brown (on amenability).

Examples: ① Fin. groups.

② Subexponential growth

$$\limsup_n |B(n)|^{1/n} = 1.$$

$B(n)$ ↓ ball of radius n over a given fin. gen. set.

Ex: $\exists$ a subset of $B(n)$ which is a Følner set.

③ Abelian groups.

④ Free group is not amenable.

$\mathbb{F}_2 = \langle a, b \rangle$.

$\underline{w(x)}$ = all words in $\mathbb{F}_2$ which start with x.

$$\mathbb{F}_2 = \{e\} \cup w(a) \cup w(b) \cup w(a^{-1}) \cup w(b^{-1})$$

$$\underline{w(x) = x\left(\mathbb{F}_2 \setminus w(x^{-1})\right)}$$

$$1 = m(\mathbb{F}_2) = m(\{e\}) + m(w(a)) + m(w(b)) + m(w(a^{-1})) + m(w(b^{-1})) =$$

$$= m(\{e\}) + [1 - m(w(a^{-1}))] + [1 - m(w(a))] + [1 - m(w(b))] + [1 - m(w(b^{-1}))] = 2.$$

<u>OPEN ISSUE</u>: Find non-amenable groups that don't contain <u>$\mathbb{F}_2$</u>.

Examples of non-amenable groups without F_2

① Candidate: Thompson group F.

= group of all piece-wise linear homeo of $[0,1]$ with breaking points of the first derivative in $\mathbb{Z}[\frac{1}{2}]$ and slopes are powers of 2.

$F_2 \not< $ Thom. F.

(1965).

1980 Ol'shanskii: not amenable, all proper subgroups are torsion.

1983 Adyan: free Burnside group $B(n,m)$ is not amenable, n odd

$n \geq 665$

$m \geq 2$.

$B(n,m) = \langle x_1, \dots, x_n : x^m = e \quad \forall$ words x in $x_1, \dots, x_n \rangle$.

$\not\supset \mathbb{F}_2$

<u>Monod - Ozawa</u> (with restrict on n).
$n \geq 665$.

2013 Ol'shanski, Sapir: finitely presented not am.
no $\mathbb{F}_2$.

Golod - Shafarevych, Ershov it has Property (T).

<u>Osin</u>: G without $\mathbb{F}_2$, with positive ℓ^2- Betti number.

<u>Monod</u>: piece-wise $PSL_2(A)$-homeo of projective line that fix ∞
A is a dense subring of $\mathbb{R}$.

<u>OPEN</u>: Dixmier similarity problem
G is amenable $\Leftrightarrow$ G is unitarizable.

$\forall$ bounded representation
$\pi : G \to B(H)$ $\exists$ $S \in B(H)$ - invert.

$S\pi(g)S^{-1}$ is unitary $\forall\ g \in G$.

$\|\pi(g)\| \leq \underline{C} \quad \forall g \in G$.

Osin's is not unitarizable.

Pisier : Similarity problems & completely bdd maps.

Maria Gerasimova, Andreas Thom, Nicolas Monod.

<u>Combination of algebra & amenability</u>

① • groups of intermidiate growth.

② • fin. gen., ∞, simple, amenable

topology + analysis.

OPEN • fin presented, ∞, simple, amenable (Nekrashevych)

③ • fin gen. intermidiate growth, simple

topol. + agebra

OPEN • fin presented intermidiate growth simple.

<u>Intermidiate growth</u> :

Milnor 1968 : a group
not polynom not exponential.

Grigorchuk 1985.
Kate's book : all known example.

Algebra and amenability

- Topological full group.

X - Cantor set.

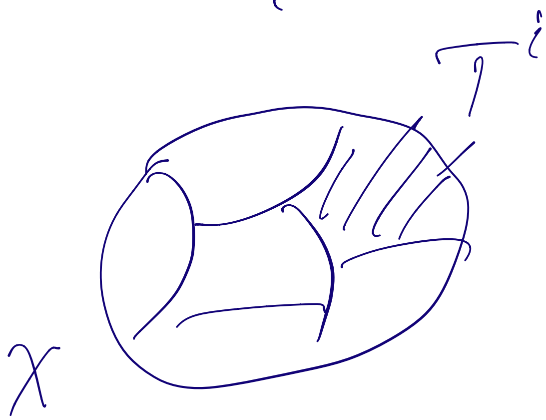
$Homeo(X) \ni T$

$[[T]]$ - topological full group of T, piece-wise powers of T.

$\varphi \in [[T]]$ if $\varphi \in Homeo(X)$
$\exists$ clopen decomposition of X

$X = \bigsqcup_{i=1}^{n} C_i$, C_i - clopen

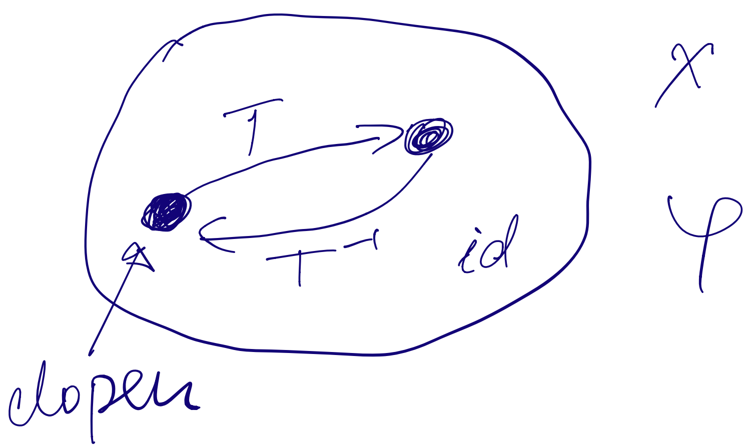
$$\varphi|_{C_i} = T^{k_i}$$



$\varphi \in [[T]]$

$\cap$

$\mathrm{Homeo}(X).$



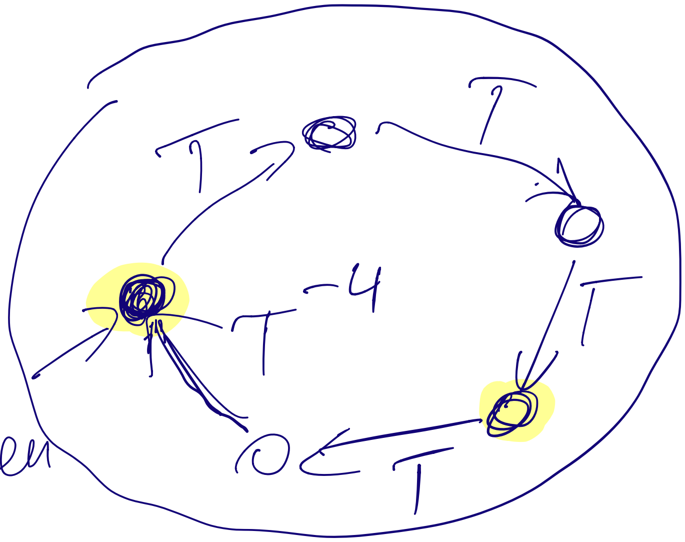
$T \curvearrowright X$ minimally if

$\overline{\{T^i x : i \in \mathbb{Z}\}} = X \quad \forall x \in X.$

$S(n).$

$\forall n$

$\exists$ clopen



$S(n) \subset [[T]]$.

<u>Matui</u>: $[[T]]'$ – simple. (commutator)

fin. generated.

<u>Monod, J</u>: $[[T]]'$ amenable

<u>Nekrashevych ('19, Annals)</u>

$\exists$ subgroup $[[T]]$ s.t

intermediate growth.

(every element is of fin

order), fin. generated,
<u>Simple</u>.

Analog $A(G)$

Construction of the group

$a \in \mathrm{Homeo}(X)$, $a^2 = 1$.

A - finite group of $\mathrm{Homeo}(X)$

$\forall h \in A$, $z \in X$

$h(z) = z$ or $h(z) = a(z)$

and $\forall z \in X \ \exists h \in A$

$h(z) = a(z)$.

A - fragmentation of a.

$D_\infty = \langle a, b : a^2 = b^2 = e \rangle$

A - fr. of a,

B - fragm. of b.

$\langle A, B \rangle < \mathrm{Homeo}(X)$,

Rem.: Grigorchuk is such.

Nekrashevych: Assume ζ is a fixed point of a and for every $h \in A$ s.t $h(\zeta) = \zeta$ the interior of the set of fixed points of h is accumulated around ζ. Then $\langle A, B \rangle$ - period group.

+ on B <u>then</u> $\langle A, B\rangle$

is simple + intermediate growth.

<u>Ex:</u> Show $\langle A, B\rangle < [[T]]$ for some $T \in \mathrm{Homeo}(X)$.

<u>Finitely presented</u> ?

<u>Random walks, Kesten's criteria, inverted orbits.</u>

<u>Thm (Kesten)</u>: A group Γ is gen. by a symmetric set S $S^{-1} = S$

$\boxed{\begin{array}{l}\mathrm{supp}\,\mu_1 \\ = \mathrm{supp}\,\mu_2\end{array}}$

$\iff \left\|\sum_{t\in S} \mu(t)\lambda(t)\right\| = 1$

Γ is amenable $\iff \frac{1}{|S|}\left\|\sum_{t\in S}\lambda(t)\right\|_{B(\ell^2(\Gamma))} = 1$

$\lambda : \Gamma \to B(\ell^2(\Gamma))$, $f \in \ell^2(\Gamma)$

$[\lambda(g) f](t) = f(g^{-1}t)$.

$\boxtimes$ Γ - amenable $\Rightarrow$

$\exists$ an almost invariant vector for λ. $\|\xi\|=1$ $\|\lambda(t)\xi - \xi\|_2 \to 0$

S - gen. $\exists F$ $\dfrac{|sF \triangle F|}{|F|} \to 0$.

$$\xi = \frac{\chi_F}{|F|^{\frac{1}{2}}} \longrightarrow$$

$\xi^{1/2}$

$\Downarrow$

$$\frac{1}{|S|} \Big\| \sum_{t \in S} \lambda(t) \Big\| = 1$$

Assume $\dfrac{1}{|S|} \Big\| \sum_{t \in S} \lambda(t) \Big\| = 1$

self adjoint

$\forall \varepsilon > 0$ $\exists \xi \in \ell^2(\Gamma)$

$$\frac{1}{|S|}\left|\left\langle \sum_{t\in S} \lambda(t)\xi,\xi\right\rangle\right| > 1-\varepsilon$$

$\xi \rightsquigarrow |\xi|$ $\leftarrow$ pointwise

$$\frac{1}{|S|}\left\langle \sum_{t\in S} \lambda(t)|\xi|,|\xi|\right\rangle \geq$$

$$\geq \frac{1}{S}\left\langle \sum_{t\in S} \lambda(t)\xi,\xi\right\rangle > 1-\varepsilon$$

$\Rightarrow$ $\| \lambda(t)|\xi_i| - |\xi_i| \| \to 0.$

$\Rightarrow$ λ has an almost inv. vector.

$$\mu_i = \xi_i^2 \in \ell^1(\Gamma)$$

cluster point in weak*-top

$\mu_i \Rightarrow \mu$ — mean.
μ is Γ-invariant. $\boxtimes$

<u>Kesten's criteria in terms of random walks</u>

(Woess & random walks)

S – symm. fin., $\langle S \rangle = \Gamma$,

$\mu(g^{-1}) = \mu(g)$, $g \in \Gamma$,

$\mu \in \ell^1(\Gamma)$ $\|\mu\|_1 = 1$

$\mu \geq 0$.

$\mu(m) f(x) =$

$$= \sum_{t \in \Gamma} f(t^{-1}x)\, \mu(t)$$

$\partial(-)$

$f \in \ell^2(\Gamma)$.

$$M(\mu)\cdot M(\nu) = M(\mu * \nu)$$

$$\mu * \nu(x) = \sum_{t\in\Gamma} \mu(xt^{-1})\nu(t)$$

$$\|M(\mu)\| \leq 1$$

$$\Downarrow$$

$M(\mu)$ is self-adjoint.

(μ before – uniform measure.)

$\sigma(\mu)$ – spectrum of $M(\mu)$

$$\rho(\mu) = \max\{|t| : t\in\sigma(\mu)\}$$

$$\|M(\mu)\| = \rho(\mu).$$

Thm: let Γ be fin. gen.

μ – symmetric measure

with fin support.

$$\|M(\mu)\| = \lim_n \mu^{*2n}(e)^{1/2n}.$$

<u>Corollary</u>: Γ is amenable

$$\Longleftrightarrow \lim_n \mu^{*2n}(e)^{1/2n} = 1.$$

$\boxtimes$ $$M(\mu^{*n}) = M(\mu)^n$$

$$\mu^{*n}(e) = M(\mu)^n \delta_e$$

$$= \langle M(\mu)^n \delta_e, \delta_e \rangle$$

$$\mu^{*n}(e) \leq \|M(\mu)^n\|.$$

$$\langle M(\mu)^n \delta_e, \delta_e \rangle = \int_{[-1,1]} t^n \nu(dt)$$

$$\mu^{*2n}(e)^{1/2n} = \langle M(\mu)^{2n}\delta_e, \delta_e\rangle^{1/2n}$$

$$= \left(\int_{[-1,1]} t^{2n}\, \nu(dt)\right)^{1/2n}$$

$$\lim_n \mu^{*2n}(e)^{1/2n} = \lim_n$$

$$= \|t\|_\infty = \max(|t| : t \in \sigma(M(\mu)))$$

$$= \rho(M(\mu)) = \|M(\mu)\| \quad \boxtimes$$

$\boxed{\text{Inverted orbit}}$

Schreier graphs

if $\Gamma \curvearrowright X$ is amenable

$\Rightarrow \Gamma$ is amenable

Liouville property.