Monotone Complete C*-algebras and Generic Dynamics

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- This talk is on joint work with Kazuyuki SAITÔ.
- I shall begin by talking about Monotone Complete C*-algebras.
- Then I will give a brief introduction to Generic Dynamics and its close connection to MCAs.

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commutative algebras

- When a monotone complete C^*-algebra is commutative, its lattice of projections is a complete Boolean algebra. Up to isomorphism, every complete Boolean algebra arises in this way.
- Let $C(X)$ be a commutative (unital) C^*-algebra Then $C(X)$ is monotone complete precisely when the compact Hausdorff space X is *extremally disconnected*, that is, the closure of each open subset of X is also open.
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introduction: generic dynamics

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Ergodic discrete group actions

- If there exists $x_0 \in X$ such that the orbit $G[x_0]$ is dense in X then every $G-$invariant open subset of X is either empty or dense.
- If every non-empty open $G-$invariant subset of X is dense then, for each x in X, the orbit $G[x]$ is either dense or nowhere dense.
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dense orbit

- We concentrate on the situation where, for some $x_0 \in X$, the orbit $\{\varepsilon_g(x_0) : g \in G\}$ is dense in X.
- This cannot happen unless X is separable.
- Let S be the Stone space of the (complete) Boolean algebra of regular open sets of X. Then, it can be shown that the action ε of G on X induces an action $\widehat{\varepsilon}$ of G as homeomorphisms of S; which will also have a dense orbit.
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orbit equivalence

- Let E be the orbit equivalence relation on S.
- That is, sEt, if, for some group element g, $\widehat{\varepsilon}_g(s) = t$.
- Then we can construct a monotone complete C^*-algebra M_E from the orbit equivalence relation.
- When there is a free dense orbit, the algebra will be a factor with a maximal abelian subalgebra, A, which is isomorphic to $C(S)$. There is always a faithful, normal, conditional expectation from M_E onto A.
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actions on C(S)

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recent work

- In recent work by Saitô and me, we consider 2^c algebras $C(S)$, each taking different values in the classification semi-group $\mathcal{W}$. (Here $c = 2^{\aleph_0}$, the cardinality of $\mathbb{R}$.)
- Each S is separable and a subspace of the (separable) compact space $2^{\mathbb{R}} = \{0, 1\}^{\mathbb{R}}$.
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Result

Let G be any countably infinite group. Let α be an action of G (as homeomorphisms) on S and suppose this action has at least one orbit which is dense and free. Then, modulo meagre sets, the orbit equivalence relation obtained can also be obtained by an action of $\bigoplus \mathbb{Z}_2$ as homeomorphisms of S.So $M(C(S), G)$ can be identified with $M(C(S), \bigoplus \mathbb{Z}_2)$.

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Construction

- We construct 2^c,essentially different, compact extremally disconnected spaces, S_η, where $ProjC(S_\eta)$ is countably generated.
- Simultaneously, we construct a natural action of $\bigoplus \mathbb{Z}_2$ with a free, dense orbit on each S_η This gives rise to a family of monotone complete C^*-algebras, $(B_\lambda, \lambda \in \Lambda)$ with the properties described below.
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classification semigroup

- Fix a large Hilbert space $H^{\#}$. only consider algebras which are isomorphic to subalgebras of $L(H^{\#})$.
- For (unital) small C*-algebras, their Pedersen Borel envelopes, or more generally any (unital) C*-algebra of cardinality $c = 2^{\aleph_0}$, it suffices if $H^{\#}$ has an orthonormal basis of cardinality $c = 2^{\aleph_0}$.

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- ϕ is *faithful* if $x \geq 0$ and $\phi(x) = 0$ implies $x = 0$.
- ϕ is *normal* if, whenever D is a downward directed set of positive elements of A then ϕ maps the infimum of D to the infimum of $\{\phi(d) : d \in D\}$.

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- Let Ω be the class of all monotone complete C*-algebras which are isomorphic to norm closed *-subalgebras of $L(H^{\#})$. Then $A \in \Omega$ precisely when A is isomorphic to an algebra in $\Omega^{\#}$

quasi-ordering

- We define a relation on Ω by $A \precsim B$ if there exists a positive linear map $\phi : A \to B$ which is faithful and normal.

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- It can be proved that $\precsim$ is a quasi-ordering of Ω.
- We define an equivalence relation $\sim$ on Ω by $A \sim B$ if $A \precsim B$ and $B \precsim A$.

isomorphism

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- If π is an isomorphism of A onto a subalgebra of B. Then π need not be normal.
- It will only be normal if its range is a monotone closed subalgebra of B. In particular, if A is a monotone closed subalgebra of B, then by taking the natural injection as π, we see that $A \precsim B$.

addition

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- We can try to define $[A] + [B]$ to be $[A \oplus B]$.
- It is not obvious this makes sense but it turns out to work OK. With this definition of addition, $\mathcal{W}$ is an abelian semi-group. It has a zero element $[\mathbb{C}]$

partial order

- We can partially order the semi-group $\mathcal{W}$ by:

$[A] \leq [B]$ if $A_1 \sim A$ and $B_1 \sim B$ with $A_1 \precsim B_1$.
We find, appropriately, that 0 is its smallest element.

Riesz

- The partially ordered semi-group has the Riesz Decomposition Property:

Riesz

- The partially ordered semi-group has the Riesz Decomposition Property:
- If $0 \leq x \leq a + b$ then $x = a_1 + b_1$ where $0 \leq a_1 \leq a$ and $0 \leq b_1 \leq b$.

normality weight

- For each $A \in \Omega$, A is isomorphic to $A^{\#} \in \Omega^{\#}$.
- We define $w(A) = [A^{\#}]$.

von N.

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- *Let A be an MCA in Ω.*
- Then $w(A) = 0$ if, and only if, A is a von Neumann algebra with a faithful normal state.

Proof

- The map $\lambda \to \lambda 1$ shows that $\mathbb{C} \precsim A$.So $A \sim \mathbb{C}$ if, and only if, $A \precsim \mathbb{C}$.

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- But this is equivalent to the existence of a faithful normal functional $\phi : A \to \mathbb{C}$.
- By Kadison, a faithful normal state on A implies A is von Neumann.
- When A is a von Neumann algebra with a faithful normal state, then $A \precsim \mathbb{C}$.

APPENDIX: Monotone complete C*-algebras and generic dynamics (JDM Wright)

The survey article [1] gives a bird's eye view of classifying monotone complete C*-algebras. In particular it describes the spectroid invariant for monotone complete algebras (there was insufficient time to do this in my talk). Full details of the classification are given in [2].

Just as the spectrum is a set which encodes information about an operator, the spectroid encodes information about a monotone complete C*-algebra. Since equivalent algebras have the same spectroid, it is an invariant for members of the classification semi-group. All this can be generalised to much more general partially ordered sets.

Generic dynamics for actions of countable groups on complete separable metric spaces were introduced in [3] and a strong uniqueness theorem was proved. The survey by Weiss [4] contains much more.

The recent paper [6] gives details of all generic dynamics results mentioned in this talk.

Imagine a city in which each building contains a small monotone complete C*-algebra, where algebras in different buildings are never isomorphic. Then by Hamana's marvellous breakthrough [5] we can suppose there are $2^{\mathbb{R}}$ buildings. In [2] we find the city can be organised into $2^{\mathbb{R}}$ avenues; each avenue is named by an element of the classification semigroup. The zero avenue is the one where the von Neumann algebras live. There is much to do before we can give a complete map of the city. But [2] brings some order out of chaos.

The work of [6] shows that generic dynamics can be applied to the construction of huge numbers of wild factors. In particular if we require that each algebra in our city is a small hyperfinite factor we still get $2^{\mathbb{R}}$ elements in the classification semi-group.

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4. B.Weiss, "A survey of generic dynamics", *Descriptive set theory and dynamical systems* (eds M.Formann et al.), London Math. Soc. Lecture Note Series **277**, 273-291 (C.U.P.), 2000.

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