C*-algebras: quasidiagonality and amenability

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Outline

- C*-algebras

- Examples

- Isomorphism of C*-algebras

- Quasidiagonality

- Main result

- Applications

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C*-algebras

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$.

$\mathcal{B}(\mathcal{H})$ denotes the set of linear operators $T : \mathcal{H} \to \mathcal{H}$ that are continuous, or equivalently bounded in the operator norm,
$$\|T\| := \sup\{\|T\xi\| : \xi \in \mathcal{H}, \|\xi\| \leq 1\}.$$

$\mathcal{B}(\mathcal{H})$ has a lot of structure:
- scalar multiplication and addition
- composition (as multiplication) $\Big\}$ an algebra over $\mathbb{C}$
- adjoint operation, $T \mapsto T^*$
- it is complete under $\|\cdot\|$.

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Definition

A *C*-algebra* is a subset $A \subseteq \mathcal{B}(\mathcal{H})$ closed under addition, scalar multiplication, composition, adjoints, and closed in the topology from the norm $\| \cdot \|$.

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Example. $M_n(\mathbb{C})$ is a C*-algebra; multiplication is matrix multiplication, adjoint is conjugate-transpose.

Example. Let X be a compact topological space. Then $C(X, \mathbb{C})$ can be identified with a C*-algebra either in $\mathcal{B}(l^2(X))$, or more generally in $\mathcal{B}(L^2(X, \mu))$ for a measure μ with full support. Here, composition corresponds to pointwise multiplication, $f^*(x) = \overline{f(x)}$, and

$$\|f\| = \max\{|f(x)| : x \in X\}.$$

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Gelfand's Theorem

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Let G be a group. For $g \in G$, define $\lambda_g : l^2(G) \to l^2(G)$ by
$$\lambda_g(\xi)(h) := \xi(g^{-1}h)$$
(the *left regular representation* of G).

Note that $\mathbb{C}[G] \cong \operatorname{span}\{\lambda_g : g \in G\}$.

Define
$$C_r^*(G) := \overline{\operatorname{span}\{\lambda_g : g \in G\}},$$
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Isomorphism of C*-algebras

Isomorphism of C*-algebras means algebraic $*$-isomorphism preserving $\|\cdot\|$, potentially ignoring how the C*-algebras sit inside $\mathcal{B}(\mathcal{H})$'s.

(In fact, algebraic $*$-isomorphisms automatically preserve $\|\cdot\|$.)

Underpinning the reasonableness of this notion of $\cong$ is the Gelfand–Naimark Theorem:

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C*-algebras can be described axiomatically as those Banach $*$-algebras A satisfying
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C*-algebra constructions

In addition to $C(X, \mathbb{C})$, $C_r^*(G)$, there are many constructions of C*-algebras from other mathematical objects: locally compact groups (harmonic analysis), group actions on topological spaces (dynamical systems), metric spaces (coarse geometry), directed graphs (symbolic dynamics), foliations, integral domains, ...

The general pattern

A map (often functor)

$$\Omega \mapsto C^*(\Omega)$$

from some class of mathematical objects to the land of C*-algebras.

Open-ended question

Given an interesting property P of input objects Ω, is there a C*-algebraic property P_{C^*} such that

$$\Omega \text{ satisfies } P \Leftrightarrow C^*(\Omega) \text{ satisfies } P_{C^*}?$$

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A number of properties of G are reflected in $C_r^*(G)$. One is amenability. Recall:

Definition

A group G is *amenable* if there exists a left-translation invariant finitely additive probability measure on G.

Amenability has *many* characterizations.

Theorem (classical)

A group G is amenable if and only if $C_r^*(G)$ is amenable.

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Quasidiagonality

Quasidiagonality is an external approximation property for C*-algebras, saying roughly that it can be modelled approximately inside finite dimensional algebras.

Completely positive: $\phi_i^{(n)}(a^*a)$ is a non-negative definite matrix, for any $a \in M_n(A), n \in \mathbb{N}$.

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A C*-algebra A is *quasidiagonal* if there exists a net of linear, *-preserving, completely positive maps $\phi_i : A \to M_{k_i}(\mathbb{C})$ (some $k_i \in \mathbb{N}$) such that:

- $\|\phi_i(a)\phi_i(b) - \phi_i(ab)\| \to 0$, for all $a, b \in A$ (ϕ_i is *approximately multiplicative*), and
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• $\|\phi_i(a)\phi_i(b) - \phi_i(ab)\| \to 0$, for all $a, b \in A$ (ϕ_i is *approximately multiplicative*), and

• $\|\phi_i(a)\| \to \|a\|$, for all $a \in A$ (ϕ_i is *approximately isometric*).

Completely positive: $\phi_i^{(n)}(a^*a)$ is a non-negative definite matrix, for any $a \in M_n(A)$, $n \in \mathbb{N}$.

Example. $M_n(\mathbb{C})$ is quasidiagonal.

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$C(X, \mathbb{C})$ is quasidiagonal: Assume X is separable. Take a dense
sequence $(x_n)_{n=1}^{\infty}$, and define a sequence
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In fact, these maps are exactly multiplicative.

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Example. $\mathcal{B}(\mathcal{H})$ is not quasidiagonal when $\mathcal{H}$ is infinite dimensional.

To see this:

Fact

If A is unital and quasidiagonal, then it has a nonzero *trace*, i.e., a positive functional $\tau : A \to \mathbb{C}$ such that

$$\tau(ab) = \tau(ba), \quad a, b \in A.$$

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Quasidiagonality: non-trivial examples

When is $C_r^*(G)$ quasidiagonal?

It has a trace, so can't fail for the same reason as $\mathcal{B}(\mathcal{H})$.

Theorem (Rosenberg, '87)

If $C_r^*(G)$ is quasidiagonal then G is amenable.

Conjecture (Rosenberg, '87)

The converse holds: $C_r^*(G)$ is quasidiagonal if and only if G is amenable.

Open question (Blackadar–Kirchberg, '97)

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Main result

Theorem (T–White–Winter, Ann. of Math. '17)

Let A be a C*-algebra that is separable, amenable, has a faithful trace, and satisfies the Universal Coefficient Theorem for KK-theory. Then A is quasidiagonal.

Faithful trace: a trace τ is *faithful* if $\tau(a^*a) \neq 0$ whenever $a \neq 0$.

Universal Coefficient Theorem for KK-theory (UCT): a technical cohomological condition due to Rosenberg and Schochet ('87), which is ubiquitous in the classification of C*-algebras. Known to hold for many amenable C*-algebras ("if you can build it, the UCT will come"). In particular, it holds for $C_r^*(G)$ when G is amenable and countable (Tu '99).

It is open whether it holds for all amenable C*-algebras.

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Rosenberg's conjecture
Our theorem resolves Rosenberg's conjecture.

Corollary

Rosenberg's conjecture holds: G is amenable if and only if $C_r^*(G)$ is quasidiagonal.

Classification of C*-algebras

Our theorem (in a slightly stronger form) shows that a formerly mysterious hypothesis automatically holds in a far-reaching classification theorem:

Corollary (with Elliott–Gong–Lin–Niu)

K-thoery paired with traces is a complete invariant for the class of simple, separable, unital C*-algebras with finite nuclear dimension that satisfy the Universal Coefficient Theorem for KK-theory.

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Approximation properties

Our theorem fills a hypothesis in previous work on the Toms–Winter conjecture for C*-algebras (concerning covering dimension and approximation properties):

Corollary (with Bosa–Brown–Sato–T–White–Winter, Mem. Amer. Math. Soc., to appear)

Let A be a simple separable amenable $\mathcal{Z}$-stable C*-algebra such that $T(A)$ is a Bauer simplex. Then:
(i) A has nuclear dimension at most 1;
(ii) If A satisfies the Universal Coefficient Theorem for KK-theory, then A has decomposition rank at most 1.

Applications

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