# Equilibrium states on the Toeplitz-Cuntz-Krieger algebras of finite graphs

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This talk contains some results obtained in

A. an Huef, M. Laca, I. Raeburn and A. Sims, KMS states on $C^*$-algebras of finite graphs, *J. Math. Anal. Appl., 2013.*

Let $\alpha : \mathbb{R} \to \operatorname{Aut} A$ be an action of the real line $\mathbb{R}$ on a $C^*$-algebra $A$. (Today $A$ always has an identity.)

In physical models, observables of the system are represented by self-adjoint elements of $A$, and states of the system by positive functionals of norm 1 on $A$: $\phi(a)$ is the expected value of the observable $a$ in the state $\phi$ (which is real because $a = a^*$ and $\phi \geq 0$).

The action $\alpha$ represents the time evolution of the system: the observable $a$ at time 0 moves to $\alpha_t(a)$ at time $t$, or the state $\phi$ at time 0 moves to $\phi \circ \alpha_t$.

In statistical physics, an important role is played by *equilibrium states*, which are in particular invariant under the time evolution. In $C^*$-algebraic models equilibrium states are called *KMS states*, after Kubo, Martin and Schwinger.

Let $\alpha : \mathbb{R} \to \mathrm{Aut}\, A$ be an action. Then $a \in A$ is an *analytic element* if the function $t \mapsto \alpha_t(a)$ from $\mathbb{R}$ to $A$ has an extension to an entire function on $\mathbb{C}$. The set of analytic elements is always a dense subalgebra of $A$. A state $\phi$ on $A$ is a *KMS state at inverse temperature $\beta$* if

$$\phi(ab) = \phi(b\alpha_{i\beta}(a)) \text{ for all analytic } a, b.$$

- KMS states are $\alpha$-invariant.
- It suffices to check the KMS$_\beta$ condition on a set of analytic elements which span a dense subspace of $A$.
- The KMS$_\beta$ states always form a simplex.
- In a physical model we expect KMS states for most $\beta$.

Suppose that $E = (E^0, E^1, r, s)$ is a directed graph. Today it is always finite. A *Cuntz-Krieger $E$-family* consists of mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying $S_e^* S_e = P_{s(e)}$ and

$$P_v = \sum_{r(e)=v} S_e S_e^* \quad \text{whenever } v \text{ is not a source.}$$

The *graph algebra* is a $C^*$-algebra $C^*(E)$ which is generated by a universal Cuntz-Krieger family $\{p, s\}$.

Each graph algebra carries a *gauge action* $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$ characterised by $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$. This lifts to an action of $\mathbb{R}$ such that $\alpha_t(s_e) = e^{it} s_e$ and $\alpha_t(p_v) = p_v$.

We are going to compute the KMS states of $(C^*(E), \alpha)$.

The Cuntz-Krieger relation $p_v = \sum_{r(e)=v} s_e s_e^*$ and the orthogonality of the $p_v$ imply that the projections $\{s_e s_e^* : e \in E^1\}$ are mutually orthogonal: algebraically, $s_e^* s_f = 0$ whenever $e \neq f$.

Thus $C^*(E) = C^*(s, p)$ is spanned by the elements

$$s_\mu s_\nu^* := s_{\mu_1} s_{\mu_2} \cdots s_{\mu_{|\mu|}} s_{\nu_{|\nu|}}^* \cdots s_{\nu_1}^*$$

for paths $\mu$ and $\nu$ in $E$: we write $\mu, \nu \in E^*$. Crucial for us is that

$$t \mapsto \alpha_t(s_\mu s_\nu^*) = e^{it(|\mu|-|\nu|)} s_\mu s_\nu^*$$

extends to an analytic function (just replace $t$ by $z$).

Let $\phi$ be a KMS$_\beta$ state on $(C^*(E), \alpha)$. We have

$$\begin{aligned}\phi(s_\mu s_\nu^*) &= \phi(s_\nu^* \alpha_{i\beta}(s_\mu)) = e^{-\beta|\mu|}\phi(s_\nu^* s_\mu)\\ &= e^{-\beta(|\mu|-|\nu|)}\phi(s_\mu s_\nu^*).\end{aligned}$$

Thus $\phi(s_\mu s_\nu^*) = 0$ unless $|\mu| = |\nu|$, and then $s_\nu^* s_\mu = 0$ unless $\mu = \nu$.

**Lemma.** A state $\phi$ on $C^*(E)$ is KMS$_\beta$ for $\alpha$ if and only if

$$\phi(s_\mu s_\nu^*) = \begin{cases} 0 & \text{when } \nu \neq \mu \\ e^{-\beta|\mu|}\phi(p_{s(\mu)}) & \text{when } \nu = \mu. \end{cases}$$

We haven't used the Cuntz-Krieger relation yet:

Let $\phi$ be a KMS$_\beta$ state on $(C^*(E), \alpha)$. Suppose $v \in E^0$ is not a source. Then

$$\phi(p_v) = \sum_{r(e)=v} \phi(s_e s_e^*) = \sum_{r(e)=v} e^{-\beta}\phi(p_{s(e)}).$$

The *vertex matrix* of $E$ is the $E^0 \times E^0$ integer matrix $A$ with entries $A(v, w) = |r^{-1}(v) \cap s^{-1}(w)|$. We can rearrange the above sum as

$$e^\beta \phi(p_v) = \sum_{w \in E^0} \sum_{r(e)=v,\ s(e)=w} \phi(p_w) = \sum_{w \in E^0} A(v, w)\phi(p_w). \quad (1)$$

So if $E$ has no sources, then the vector $m = (m_v) := (\phi(p_v)) \in [0, \infty)^{E^0}$ satisfies $Am = e^\beta m$.

If $E$ is strongly connected, then $A$ is irreducible, and $e^\beta$ has to be the Perron-Frobenius eigenvalue of $A$. Since $1 = \phi(1) = \sum_v \phi(p_v) = \sum_v m_v$, the vector $m$ is the unique PF eigenvector with $\|m\|_1 = 1$.

**Theorem (Enomoto-Fujii-Watatani 1984).** Let $E$ be a strongly connected finite graph with vertex matrix $A$. Then $(C^*(E), \alpha)$ has a unique KMS state. This state has inverse temperature $\beta = \ln \rho(A)$, where $\rho(A)$ is the spectral radius of $A$.

We have shown there is at most one, so we need to show existence. The easiest way to construct the state is to use the idea from Exel-Laca (2003), Laca-Neshveyev (2004): the Toeplitz algebra $\mathcal{T}C^*(E)$ of $E$ has a much richer supply of KMS states.

$\mathcal{T}C^*(E)$ is generated by a universal Toeplitz-Cuntz-Krieger family of mutually orthogonal projections $\{q_v\}$ and partial isometries $\{t_e\}$ such that $t_e^* t_e = q_{s(e)}$ and, if $v$ is not a source,

$$q_v \geq \sum_{e \in F} t_e t_e^* \quad \text{for } F \subset r^{-1}(v).$$

Again $\mathcal{T}C^*(E) = \overline{\operatorname{span}}\{t_\mu t_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu)\}$ and there is a gauge action $\gamma : \mathbb{T} \to \operatorname{Aut}(\mathcal{T}C^*(E))$ satisfying $\gamma_z(t_e) = z t_e$ and $\gamma_z(q_v) = q_v$, which we can lift to $\mathbb{R}$.

Let $I$ be the ideal of $\mathcal{T}C^*(E)$ generated by

$$\{q_v - \sum_{r(e)=v} t_e t_e^* : v \text{ is not a source}\}.$$

Then $C^*(E) \cong \mathcal{T}C^*(E)/I$.

It is again easy to recognise a $\mathrm{KMS}_\beta$ state $\phi$ on $(\mathcal{T}C^*(E), \alpha)$, and the formulas look the same.

But now $m = (m_v) := (\phi(q_v))$ is a unit vector in $\ell^1(E^0)$ satisfying $Am \leq e^\beta m$.

PF theory says for $A$ irreducible:

- $Am = e^\beta m \Longrightarrow e^\beta = \rho(A) \Longrightarrow \beta = \ln \rho(A)$;
- $Am \leq e^\beta m$ and $\beta = \ln \rho(A) \Longrightarrow m$ is the PF eigenvector;
- $Am \leq e^\beta m$ and $Am \neq e^\beta m \Longrightarrow \beta > \ln \rho(A)$.

Note $Am \leq e^\beta m \Longleftrightarrow (I - e^{-\beta}A)m \geq 0$, and consider $\beta > \ln \rho(A)$. Then $\sum_{n=0}^\infty e^{-\beta n}A^n$ converges to $(1 - e^{-\beta}A)^{-1}$.

Take $\epsilon := (1 - e^{-\beta}A)m$. Which $\epsilon \in [0, \infty]^{E^0}$ arise?
For $v \in E^0$, set

$$y_v := \sum_{n=0}^\infty \sum_{w \in E^0} e^{-\beta n} A^n(w, v)$$

(which again converges because $e^\beta > \rho(A)$), and take $y = (y_v)$. Then:

**Lemma.** Let $\beta > \ln \rho(A)$. Then $m := (1 - e^{-\beta}A)^{-1}\epsilon$ is a unit vector in $\ell^1(E^0)$ satisfying $Am \leq e^\beta m$ if and only if $\epsilon \cdot y = 1$.

We now know that the KMS condition on a state $\phi$ places restraints on $m := (\phi(p_v))$. We still need to construct KMS states. We need a concrete representation of $\mathcal{T}C^*(E)$:

**Example.** Consider the usual orthonormal basis $\{h_\mu : \mu \in E^*\}$ for $\ell^2(E^*)$ (by convention $E^0 \subset E^*$). There are projections $Q_v$ and partial isometries $T_e$ on $\ell^2(E^*)$ such that

$$Q_v h_\mu = \begin{cases} 0 & \text{unless } r(\mu) = v \\ h_\mu & \text{if } r(\mu) = v \text{, and} \end{cases}$$

$$T_e h_\mu = \begin{cases} 0 & \text{unless } r(\mu) = s(e) \\ h_{e\mu} & \text{if } r(\mu) = s(e). \end{cases}$$

Then $(Q, T)$ is a Toeplitz-CK family, and we have a representation $\pi_{Q,T}$ of $\mathcal{T}C^*(E)$ on $\ell^2(E^*)$ (in fact injective).

**Theorem (an Huef-Laca-Raeburn-Sims, 2013).** Suppose $E$ is a finite graph with vertex matrix $A$, and $\beta > \ln \rho(A)$. Take $y = (y_v) \in [1, \infty)^{E^0}$ as above, and suppose $\epsilon \cdot y = 1$. Then there is a KMS$_\beta$ state $\phi_\epsilon$ of $\mathcal{T}C^*(E)$ such that

$$\phi_\epsilon(a) = \sum_{\mu \in E^*} e^{-\beta|\mu|} \epsilon_{s(\mu)} (\pi_{Q,T}(a) h_\mu \mid h_\mu).$$

The map $\epsilon \mapsto \phi_\epsilon$ is an affine isomorphism of $\Delta_\beta = \{\epsilon \in [0,1]^{E^0} : \epsilon \cdot y = 1\}$ onto the simplex of KMS$_\beta$ states.

Notice there is no hypothesis on $E$, hence no irreducibility assumption on $A$. So what happens at $\beta = \ln \rho(A)$? When $A$ is irreducible, the series defining $y$ diverges, so the simplex $\Delta_\beta$ contracts to $\{0\}$ as $\beta \to \ln \rho(A)+$.

**Corollary (Enomoto-Fujii-Watatani).** If $E$ is strongly connected, then $(C^*(E), \alpha)$ has a $\mathrm{KMS}_{\ln \rho(A)}$ state.

**Proof.** Choose $\beta_n$ decreasing to $\ln \rho(A)$, and $\mathrm{KMS}_{\beta_n}$ states $\phi_n$ of $\mathcal{T}C^*(E)$. By passing to a subsequence, $\phi_n \to \phi$, and $\phi$ is a $\mathrm{KMS}_{\ln \rho(A)}$ state of $\mathcal{T}C^*(E)$. Then $m := (\phi(q_v))$ satisfies $Am \leq \rho(A)m$. PF implies $Am = \rho(A)m$. Thus

$$\begin{aligned}
\rho(A)\phi(q_v) = \rho(A)m_v = (Am)_v &= \sum_{w \in E^0} A(v,w)\phi(q_w) \\
&= \sum_{r(e)=v} \phi(q_{s(e)}) = \sum_{r(e)=v} \rho(A)\phi(t_e t_e^*) \\
&= \rho(A)\phi\Big(\sum_{r(e)=v} t_e t_e^*\Big).
\end{aligned}$$

So for all $v \in E^0$ which are not sources,

$$\phi\Big(q_v - \sum_{r(e)=v} t_e t_e^*\Big) = 0.$$

Now a technical lemma implies that $\phi$ factors through $C^*(E) = \mathcal{T}C^*(E)/I$.

This completes the proof of:

**Theorem (Enomoto-Fujii-Watatani 1984).** Let $E$ be a strongly connected finite graph with vertex matrix $A$. Then $(C^*(E), \alpha)$ has a unique KMS state. This state has inverse temperature $\beta = \ln \rho(A)$, where $\rho(A)$ is the spectral radius of $A$.

- T. Carlsen and N. Larsen, work in progress.
- M. Enomoto, M. Fujii and Y. Watatani, KMS states for gauge action on $\mathcal{O}_A$, *Math. Japon.* **29** (1984).
- R. Exel and M. Laca, Partial dynamical systems and the KMS condition, *Comm. Math. Phys.* **232** (2003).
- N.J. Fowler and I. Raeburn, The Toeplitz algebra of a Hilbert bimodule, *Indiana Univ. Math. J.* **48** (1999).
- A. an Huef, M. Laca, I. Raeburn and A. Sims, KMS states on $C^*$-algebras of finite graphs, *J. Math. Anal. Appl.*
- A. an Huef, M. Laca, I. Raeburn and A. Sims, KMS states on the $C^*$-algebras associated to higher-rank graphs, arXiv.
- T. Kajiwara and Y. Watatani, KMS states on finite-graph $C^*$-algebras, arXiv.
- M. Laca and S. Neshveyev, KMS states of quasi-free dynamics on Pimsner algebras, *J. Funct. Anal.* **211** (2004), 457–482.