ALMOST FINITENESS, THE SMALL BOUNDARY PROPERTY, AND UNIFORM PROPERTY Γ

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Abstract. We prove that for a free action $\alpha : G \curvearrowright X$ of a countable discrete amenable group on a compact metrizable space, the small boundary property is equivalent to the uniform property Γ for the Cartan subalgebra $(C(X) \subseteq C(X) \rtimes_\alpha G)$, confirming a speculation of Kerr and Szabó. As an application we show that, under the additional minimality assumption, Kerr's almost finiteness is equivalent to the tracial $\mathcal{Z}$-stability of the subalgebra $(C(X) \subseteq C(X) \rtimes_\alpha G)$.

Introduction

Operator algebras and dynamical systems had long enjoyed a rich and sometimes surprising relationship. Murray and von Neumann's uniqueness theorem on hyperfinite II_1 factors ([36]) is paralleled by Dye's uniqueness of hyperfinite ergodic measure-preserving equivalence relations [11]. Connes' celebrated "injectivity implies hyperfiniteness" theorem ([7]) also has a direct analogue — the Connes–Feldman–Weiss theorem — which asserts that amenable nonsingular measured equivalence relations are hyperfinite ([8]). The connection between these two fields goes far beyond mere analogies. There is a natural construction of a von Neumann algebra from a measure-preserving equivalence relation, and amenable (respectively, hyperfinite) equivalence relations are exactly those whose associated von Neumann algebras are amenable (respectively, hyperfinite). In fact, Feldman and Moore showed in [13] that there is a full-fledged correspondence between Cartan subalgebras of von Neumann algebras and (twisted) measured equivalence relations.

The same types of analogies and connections are emerging between C^*-algebras and topological dynamics. Here the interest largely surrounds properties in the Toms–Winter conjecture — a C^*-algebraic conjecture designed to create a robust notion of regularity, primarily for the purpose of identifying classifiable C^*-algebras.[1] In the recent breakthrough ([5]), the important notion of *uniform property* Γ was identified, and we now know that for simple unital separable nuclear

[1]Amenability is also of interest, but developments on both the C^*- and dynamical sides (e.g., [48, 15]) show that amenability alone has weaknesses in terms of classifying C^*-algebras. Hence the question here is: under the base assumption of amenability, what additional conditions are needed to ensure regularity?

C^*-algebras, the following are equivalent (a statement which is close to the Toms–Winter conjecture; see [4]):

(C1) finite nuclear dimension;
(C2) $\mathcal{Z}$-stability;
(C3) strict comparison together with uniform property Γ.

On the dynamical side, Kerr defined *almost finiteness* for actions of countable discrete amenable groups on compact metrizable spaces, partially generalizing Matui's notion for topological groupoids with totally disconnected unit spaces (see [22]). Kerr proved that almost finiteness implies that the action has a comparison-type property, called *dynamical comparison*, that is analogous to strict comparison of positive elements for C^*-algebras. Moreover, he proved that for a free minimal action $G \curvearrowright X$, almost finiteness implies that the crossed product $C(X) \rtimes G$ is $\mathcal{Z}$-stable. In [24], Kerr and Szabó gave a detailed analysis of the *small boundary property* for group actions, originally introduced by Lindenstrauss and Weiss for purely dynamical reasons ([29, 30]; see also [28, 43]). They showed a relationship between almost finiteness, dynamical comparison, and the small boundary property which is reminiscent of the equivalence of (C2) and (C3) above. More precisely, for free actions the following are equivalent:

(D1) almost finiteness;
(D2) dynamical comparison together with the small boundary property.

They moreover proved that if an action has the small boundary property (and is free) then the crossed product has uniform property Γ, thus promoting the apparent analogy between these two properties to a formal (but one-way) relationship. It was noted in [24] that the proof in fact shows that the small boundary property implies a stronger property — uniform property Γ for the Cartan subalgebra $(C(X) \subseteq C(X) \rtimes G)$ — and speculated that the small boundary property may be equivalent to uniform property Γ for this Cartan subalgebra.

What makes the speculation particularly intriguing is the seemingly unrelated origins of these two notions. The small boundary property can be viewed as total disconnectedness for a dynamical system, and it plays a central role in some of the major problems in topological dynamics, such as the problem of identifying systems that are isomorphic to inverse limits of finite entropy systems (see [17, 29]). On the other hand, the study of central sequences – which is essentially what (uniform) property Γ is about – within Cartan subalgebras has a notable history in operator algebras. For example, Connes and Jones exhibited the first example of two non-conjugate Cartan subalgebras within a II_1-factor by showing that one has property Γ while the other does not ([9]).

Our first main result shows that, within the framework of free actions by amenable groups, the small boundary property is encoded in the Cartan subalgebra $(C(X) \subseteq C(X) \rtimes G)$ precisely as uniform property Γ, thus confirming the speculation in [24].

Theorem A. *Let G be a countable discrete amenable group, let X be a compact metrizable space, and let $\alpha : G \curvearrowright X$ be a free action. The following are equivalent:*

(i) α has the small boundary property;
(ii) $(C(X) \subseteq C(X) \rtimes_\alpha G)$ has uniform property Γ.

Next, inspired by the apparent analogy between almost finiteness and $\mathcal{Z}$-stability, we sought to establish a formal link here, provided the Cartan structure is taken into account on the C^*-algebraic side. We took a McDuff-type characterization of $\mathcal{Z}$-stability for nuclear C^*-algebras due to Hirshberg and Orovitz (called tracial $\mathcal{Z}$-stability [19], and their characterization makes key use of ideas of Matui and Sato [35]), and strengthened it in ways naturally related to the Cartan structure, to produce a property we call *tracial $\mathcal{Z}$-stability* for a sub-C^*-algebra. Our definition is, largely, inspired by a close analysis of Kerr's proof that minimal free actions of amenable groups which are almost finite give rise to $\mathcal{Z}$-stable crossed products. As an application of Theorem A and the equivalence between (D1) and (D2), we obtain the following.

Theorem B. *Let G be a countable discrete amenable group, let X be a compact metrizable space, and let $\alpha : G \curvearrowright X$ be a minimal free action. The following are equivalent:*

(i) α is almost finite;
(ii) $(C(X) \subseteq C(X) \rtimes_\alpha G)$ is tracially $\mathcal{Z}$-stable;
(iii) $(C(X) \subseteq C(X) \rtimes_\alpha G)$ has uniform property Γ and α has dynamical comparison.

Along the way, we establish a C^*-algebraic characterization of the dynamical subequivalence relation considered in [22] and [32], making use of "one-sided normalizers".

Proof of Theorem A. The proof of Theorem A requires several steps and manifests the deep connection between measurable and topological ideas that has appeared in the recent progress in C^*-algebra theory. To prove the small boundary property, we adapt a C^*-algebraic device which allows a local-to-global transfer (with respect to tracial states) of properties; the C^*-algebraic property is called complemented partitions of unity (CPoU), and was defined and used extensively in [5]. A characterization of the small boundary property in terms of disjoint open covers becomes a property that can be transferred by this mechanism, and the Ornstein–Weiss theorem allows it to be locally verified. In order to make this work, we needed a Cartan version of CPoU, and a proof that it arises for $(C(X) \subseteq C(X) \rtimes_\alpha G)$ (when G is an amenable group acting freely, and the Cartan subalgebra has uniform property Γ).

We look to the proof from [5] that a nuclear C^*-algebra A with uniform property Γ have CPoU. The groundwork lies in a strengthening of the completely positive approximation property (due to Brown, Carrión, and White [3], building on an earlier result of Hirshberg, Kirchberg, and White [18]), where they prove the existence of approximations of the form

$$\begin{tikzcd} A \arrow[rr, "="] \arrow[dr, "\psi"'] & & A \\ & F, \arrow[ur, "\phi"'] & \end{tikzcd}$$

where F is a finite-dimensional C^*-algebra, ψ is c.p.c. and approximately order zero, ϕ is a convex combination of c.p.c. order zero maps, and the diagram approximately commutes in point-norm. The proof of this approximation in [18] rests on the following facts:

(1) every GNS representation π_f of the C^*-algebra generates a hyperfinite von Neumann algebra (due to Connes [7]),
(2) any element in the strong* closure $\pi_f(A)''$ can be approximated strongly from a bounded set of A (the Kaplansky density theorem), and
(3) any c.p.c. order map from a finite-dimensional C^*-algebra into a quotient has a c.p.c. order zero lift (a combination of Loring's projectivity of the cone over a finite-dimensional algebra and Winter's characterization of order zero maps in terms of this cone).

Using these, the argument is roughly as follows. By taking strong*-limits, one gets something like a map from $\prod_{n=1}^\infty A$ to $\pi_f(A)''$, and this map is surjective by (2). Using (1), one can approximate the map (in point-weak*) by a conditional expectation onto a finite-dimensional subalgebra. Then (3) kicks in to allow the inclusion map from the finite-dimensional subalgebra to $\pi_f(A)''$ to be lifted to a c.p.c. order zero map to $\prod_{n=1}^\infty A$; this only gives a point-weak approximation, but then a Hahn–Banach argument upgrades it to a point-norm approximation, at the cost of taking a convex combination.

We adapt this argument, in the case $A = C(X) \rtimes_\alpha G$, to one that takes the structure of the Cartan subalgebra $(C(X) \subseteq C(X) \rtimes_\alpha G)$ into account. In $\pi_f(A)''$, we obtain that $\pi_f(C(X))''$ is a Cartan subalgebra (in the von Neumann algebraic sense), and the Connes–Feldmann–Weiss theorem becomes our replacement for (1). Viewing normalizers in $\pi_f(A)''$ as Borel functions (in an appropriate sense), we use Lusin's theorem to approximate them by continuous functions, giving a replacement for (2) (see Proposition 2.17). We also replace (3) by a lifting theorem for order zero, normalizer-preserving maps (see Proposition 2.7). We obtain the following (which is also of interest in connection to forthcoming work by the first author, K. Li, and W. Winter, on a concept called "diagonal dimension").

Theorem C. *Let G be a countable discrete amenable group, let X be a compact metrizable space, and let $\alpha : G \curvearrowright X$ be a free action. Then there exists a net of approximations*

$$\begin{tikzcd} C(X) \rtimes_\alpha G \arrow[rr, "="] \arrow[dr, "\psi"'] & & C(X) \rtimes_\alpha G \\ & F, \arrow[ur, "\phi"'] & \end{tikzcd}$$

where F is a finite-dimensional C^*-algebra, ψ is c.p.c. and approximately order zero, ϕ is a convex combination of c.p.c. order zero maps $\phi^1, \ldots, \phi^k$, and the diagram approximately commutes in point-norm. Moreover, F has a Cartan subalgebra D_F such that each ϕ^i sends normalizers of D_F to normalizers of $C(X)$; in particular, $\phi^i(D_F) \subseteq C(X)$.

After establishing this, we make use of the arguments in [5] (with small modifications) to obtain a Cartan version of CPoU for the sub-C^*-algebra $(C(X) \subseteq C(X) \rtimes_\alpha G)$, provided the sub-$C^*$-algebra has uniform property Γ.

Finally, inspired by the equivalence of uniform property Γ and CPoU for separable nuclear C^*-algebras ([4, Theorem 4.6]), we prove the following sub-C^*-algebra version, which, as in [4], adds a McDuff-type embedding condition to the equivalence.

Theorem D. *Let $(D \subseteq A)$ be a nondegenerate sub-C^*-algebra with A separable, $T(A)$ nonempty and compact, and D abelian. Suppose A has no nonzero finite-dimensional quotients and suppose $(D \subseteq A)$ satisfies the following two properties for every state f on A:*

(1) $(\pi_f(D)'' \subseteq \pi_f(A)'')$ is hyperfinite, and
(2) the unit ball of $\pi_f(\mathcal{N}_A(D))$ is strong*-dense in the unit ball of $\mathcal{N}_{\pi_f(A)''}(\pi_f(D)'')$.

Then the following are equivalent:

(i) $(D \subseteq A)$ has CPoU;
(ii) for every $n \in \mathbb{N}$ there is a unital embedding $\phi : M_n \to A^\omega \cap A'$ such that $\phi(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_{A^\omega \cap A'}((D, T(A)|_D)^\omega \cap A')$;
(iii) $(D \subseteq A)$ has uniform property Γ.

Organization of the paper. In Section 1, we establish our notations and record several facts about subalgebras and normalizers. In Section 2, we prove Theorem A (as Theorem 2.4, which adds CPoU as another equivalent condition), with Theorem C as a key step along the way. Section 3, is devoted to a C^*-characterization of the dynamical subequivalence defined in [22], [2] which is followed by Section 4 where we prove Theorem B (as Theorem 4.4). Finally in Section 5 we prove Theorem D (as Theorem 5.1).

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1. Preliminaries

For a C^*-algebra A we write A_+ for the set of positive elements in A, A^1 for the set of elements of norm at most 1, and A_+^1 for the intersection (the set of positive contractions). We write $T(A)$ for the set of tracial states on A.

Throughout the paper ω denotes a fixed free ultrafilter on $\mathbb{N}$. The ultrapower of A is defined by

$$A_\omega := \ell^\infty(\mathbb{N}, A)/\{(a_n)_n : \lim_{n\to\omega} \|a_n\| = 0\}.$$

Note that the algebra A embeds into A_ω as constant sequences. Given a nonempty subset $X \subseteq T(A)$, we define a seminorm $\|\cdot\|_{2,X}$ on A by

$$\|a\|_{2,X} := \sup_{\tau\in X} \tau(a^*a)^{\frac{1}{2}}.$$

The *tracial ultrapower of A with respect to X* is defined by

$$(A, X)^\omega := \ell^\infty(\mathbb{N}, A)/\{(a_n)_n : \lim_{n\to\omega} \|a_n\|_{2,X} = 0\},$$

which is always a quotient of A_ω. When $X = T(A)$ we recover the usual uniform tracial power A^ω.

A tracial state on A_ω or A^ω that is induced by a sequence $(\tau_n)_{n=1}^\infty$ of tracial states on A is called a *limit trace*. Following [5, Section 1.4] we write $T_\omega(A)$ for the set of limit traces on both A_ω and A^ω.

Definition 1.1. *A* sub-C^*-algebra $(D \subseteq A)$ *refers to a C^*-algebra A together with a C^*-subalgebra D. We say a sub-C^*-algebra $(D \subseteq A)$ is* nondegenerate *if D contains an approximate unit for A.*

For a sub-C^*-algebra $(D \subseteq A)$, every tracial state on A restricts to a positive tracial functional on D. We write $T(A)|_D$ for the set of such restrictions; if the sub-C^*-algebra $(D \subseteq A)$ is nondegenerate then $T(A)|_D \subseteq T(D)$.

When (D, A) is nondegenerate, we can form the relative commutant $(D, T(A)|_D)^\omega \cap S'$ for any subset $S \subseteq A^\omega$. When S is the image of A inside A^ω, we simply write $(D, T(A)|_D)^\omega \cap A'$.

[2]As mentioned in [22], this subequivalence relation and the definition of dynamical comparison first appeared in talks of Winter.

Definition 1.2. *Let $(D \subseteq A)$ be a sub-C^*-algebra. An element $a \in A$ is said to* normalize D *if $a^*Da + aDa^* \subseteq D$ (we also say a is a* normalizer *of D in A). The set of normalizers of D in A is denoted by $\mathcal{N}_A(D)$.*

It follows directly from the definition that the set of normalizers is closed under multiplication, involution, and norm-limits.

Although we won't explicitly need the definition, we recall the notion of a (C^*-algebra) Cartan subalgebra, as this is main context to keep in mind when we work with sub-C^*-algebras. A *Cartan subalgebra* is a nondegenerate sub-C^*-algebra $(D \subseteq A)$ where D is a maximal abelian subalgebra, such that there exists a faithful conditonal expectation $E : A \to D$, and such that $\mathcal{N}_A(D)$ generates A as a C^*-algebra.

When A is unital and D contains the unit of A, the normalizing condition guarantees that a^*a and aa^* belong to D for every $a \in \mathcal{N}_A(D)$. This is not true in general (for example, take $D = \{0\}$), but under the assumption of nondegeneracy everything works fine.

Lemma 1.3. *Let $(D \subseteq A)$ be a nondegenerate sub-C^*-subalgebra and $a \in \mathcal{N}_A(D)$. Then a^*a and aa^* belong to D.*

Proof. Let $(u_\lambda)_\lambda$ be an approximate unit in D for A. Then by definition $a^*u_\lambda a$ is in D for every λ. Since $\mathcal{N}_A(D)$ is closed under norm-limits, we see that $a^*a \in D$. The same argument shows that aa^* also belongs to D. □

Although a sum of normalizers is not necessarily a normalizer, it will be if the subalgebra is abelian and a certain orthogonality condition is satisfied.

Lemma 1.4. *Let $(D \subseteq A)$ be a sub-C^*-algebra with D abelian, and $x_1, ..., x_n$ be normalizers of D in A. Set $z = \sum_{i=1}^n x_i$.*

(i) If $x_i^*x_j = 0$ whenever $i \neq j$, then $z^*Dz \subseteq D$.

(ii) If $x_ix_j^* = 0$ whenever $i \neq j$, then $zDz^* \subseteq D$.

As a consequence, if $x_i^*x_j = 0 = x_ix_j^*$ whenever $i \neq j$, then z belongs $\mathcal{N}_A(D)$.

Proof. We only prove (i) as the other assertion is completely analogous. It suffices to show that if $i \neq j$ then $x_i^*dx_j = 0$ for every $d \in D$. For this observe that

$$\begin{aligned}\|x_i^*dx_j\|^4 &= \|x_j^*d^*x_ix_i^*dx_j\|^2 = \|x_j^*d^*x_ix_i^*dx_jx_j^*d^*x_ix_i^*dx_j\| \\ &\leq \|x_j\|^2\|x_j^*d^*x_ix_i^*dd^*x_ix_i^*dx_j\|.\end{aligned}$$

Note that

$$x_ix_i^*dd^*x_ix_i^* = x_i(x_i^*dd^*x_i)x_i^* \in x_iDx_i^* \subseteq D.$$

Since D is abelian,

$$x_j^*d^*(x_ix_i^*dd^*x_ix_i^*)dx_j = x_j^*d^*d(x_ix_i^*dd^*x_ix_i^*)x_j = 0.$$

It follows that $x_i^*dx_j = 0$ and the proof is complete. □

Throughout the paper we write M_n for the algebra of all complex-valued n-by-n matrices, and D_n for the subalgebra of M_n that consists of all diagonal matrices.

Example 1.5. [27, Example 2] An element $a \in M_n$ belongs to $\mathcal{N}_{M_n}(D_n)$ if and only if a has at most one nonzero entry in each row and each column.

In this paper, normalizer-preserving maps between sub-C^*-subalgebras will play an essential role.

Lemma 1.6. *Let $(D_A \subseteq A)$ and $(D_B \subseteq B)$ be two sub-C^*-algebras and $\phi : A \to B$ be a positive linear map. Suppose $(D_B \subseteq B)$ is nondegenerate.*

(i) If $\phi(\mathcal{N}_A(D_A)) \subseteq \phi_B(D_B)$, then $\phi(D_A) \subseteq D_B$.
(ii) Suppose in addition that $(D_A \subseteq A)$ is nondegenerate and ϕ is a *-homomorphism. If $\phi(\mathcal{N}_A(D_A)) = \mathcal{N}_B(D_B)$, then $\phi(D_A) = D_B$.

Proof. (i) Let d be a positive element in D_A. The $\phi(d)$ belongs to $\mathcal{N}_B(D_B)$ because D_A is contained in $\mathcal{N}_A(D_A)$. Since ϕ is positive, $\phi(d)^2 = \phi(d)^*\phi(d)$ and the later belongs to D_B by Lemma 1.3. It follows that $\phi(d) \in \mathcal{N}_B(D_B)$ and from linearity we conclude $\phi(D_A) \subseteq D_B$.

(ii) By (i) we only need to show that $D_B \subseteq \phi(D_A)$, so let e be a positive element in D_B. Since $D_B \subseteq \mathcal{N}_B(D_B)$, by assumption we can find an element $a \in \mathcal{N}_A(D_A)$ such that $\phi(a) = e$. Using the fact that ϕ is a *-homomorphism, we see that

$$\phi(|a|) = \phi((a^*a)^{\frac{1}{2}}) = (\phi(a)^*\phi(a))^{\frac{1}{2}} = (e^2)^{\frac{1}{2}} = e.$$

As $(D_A \subseteq A)$ is nondegenerate, $|a|$ belongs to D_A by Lemma 1.3 and the proof is complete.

□

Remark 1.7. In the previous lemma nondegeneracy is necessary in both assertions. For example, if we take $D_B = \{0\}$ then any map from A into B is normalizer-preserving (because $\mathcal{N}_B(D_B) = B$) but in most cases it does not map D_A into D_B. For (ii) we can take $D_A = \{0\}$ and $D_B = B$. Then any surjective *-homomorphism from A onto B maps $\mathcal{N}_A(D_A)$ (which is all of A) onto $\mathcal{N}_B(D_B)$ (which is all of B), but the image of D_A is $\{0\}$.

Recall that a c.p.c. map $\phi : A \to B$ be two C^*-algebras is *order zero* if $\phi(ab) = 0$ whenever a, b are positive elements in A satisfying $ab = 0$. Using the structure theorem ([51, Theorem 3.3]) we see that order zero maps in fact preserve arbitrary orthogonality: $\phi(ab) = 0$ if $ab = 0$. The next observation follows from Example 1.5 and Lemma 1.4.

Lemma 1.8. *Let $(D \subseteq A)$ be a nondegenerate sub-C^*-algebra, $\phi : M_n \to A$ be a c.p.c. order zero map. Then $\phi(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_A(D)$ if and only if $\phi(e_{ij}) \in \mathcal{N}_A(D)$ for every matrix unit e_{ij}.*

2. Uniform property Γ and the small boundary property

Definition 2.1. *Let G be a group, let X be a compact Hausdorff space, and let $\alpha : G \curvearrowright X$ be an action. We say that α has the* small boundary property *if for every point $x \in X$ and every open neighborhood U of x, there exists an open set V such that $x \in V \subseteq U$ and $\mu(\partial V) = 0$ for every α-invariant probability measure μ on X.*

Definition 2.2. *Let $(D \subseteq A)$ be a sub-C^*-algebra with A separable and $T(A)$ nonempty and compact. We say that $(D \subseteq A)$ has* uniform property Γ *if for any $k \in \mathbb{N}$, there exists a partition of unity consisting of projections $e_1, \ldots, e_k \in (D, T(A)|_D)^\omega \cap A'$ such that*

$$\tau(e_i a) = \frac{1}{k}\tau(a), \quad \tau \in T_\omega(A), a \in A. \tag{2.1}$$

We note that uniform property Γ for $(A \subseteq A)$ is exactly the same as uniform property Γ for the C^*-algebra A as defined in [5, Definition 2.1].

Definition 2.3. *Let $(D \subseteq A)$ be a sub-C^*-algebra, with A separable and $T(A)$ nonempty and compact. We say that $(D \subseteq A)$ has* CPoU *(complemented partitions of unit) if for any $\|\cdot\|_{2,T_\omega(A)}$-separable subset S of A^ω, any $a_1, \ldots, a_k \in (A^\omega)_+$, and any scalar*

$$\delta > \sup_{\tau \in T_\omega(A)} \min\{\tau(a_1), \ldots, \tau(a_k)\}, \tag{2.2}$$

there exists a partition of unity consisting of projections $e_1, \ldots, e_k \in (D, T(A)|_D)^\omega \cap S'$ such that

$$\tau(a_i e_i) \le \delta \tau(e_i), \quad i = 1, \ldots, k, \ \tau \in T_\omega(A). \tag{2.3}$$

The main result of this section is the following.

Theorem 2.4. *Let G be a countable discrete amenable group, let X be a compact metrizable space, and let $\alpha : G \curvearrowright X$ be a free action. The following are equivalent:*

(i) α has the small boundary property;
(ii) $(C(X) \subseteq C(X) \rtimes_\alpha G)$ has uniform property Γ;
(iii) $(C(X) \subseteq C(X) \rtimes_\alpha G)$ has CPoU.

The implication (i) ⇒ (ii) is contained in the proof of [24, Theorem 9.4]. The most involved part of the proof will be (ii) ⇒ (iii), which is contained in Section 2.4, making use of decomposable approximations developed in Section 2.2 and Section 2.3. In Section 2.5 we prove (iii) ⇒ (i).

We also record here the following versions of a key "tracial projectionization lemma" in [5].

Lemma 2.5 (cf. [5, Lemma 2.4])**.** *Let $(D \subseteq A)$ be a sub-C^*-algebra with A separable and $T(A)$ nonempty and compact, and let $S, T \subseteq A^\omega$ be $\|\cdot\|_{2,T_\omega(A)}$-separable subsets.*

If $(D \subseteq A)$ has uniform property Γ, then for any positive contraction $b \in (D, T(A)|_D)^\omega \cap S'$, there exists a projection $p \in (D, T(A)|_D)^\omega \cap S'$ such that

$$\tau(ap) = \tau(ab), \quad a \in T,\ \tau \in T_\omega(A) \tag{2.4}$$

Proof. Note that the definition of uniform property Γ can be upgraded, via Kirchberg's ϵ-test [25, Lemma A.1], to a version where centrality and tracial divisibility hold for any prescribed $\|\cdot\|_{2,T_\omega(A)}$-separable subset of A^ω (analogous to [5, Lemma 2.2]). The proof of [5, Lemma 2.4] then works verbatim. The element p defined right before [5, equation (2.9)] belongs to $(D, T(A)|_D)^\omega \cap S'$ because b and the projections $p_1, ..., p_n$ obtained from uniform property Γ are. □

2.1. **Lifting normalizer-preserving order zero maps.** Recall from [51, Corollary 4.1] there is a one-to-one correspondence between c.p.c. order zero maps $\phi : A \to B$ and *-homomorphisms $\rho : CA \to B$ given by

$$\rho(\mathrm{id}_{(0,1]} \otimes a) = \phi(a)$$

(here CA is the cone $C((0,1]) \otimes A$). This will be used in the next two lemmas.

Lemma 2.6. *Let $(D_A \subseteq A)$ and $(D_B \subseteq B)$ be nondegenerate sub-C^*-algebras and let $\pi : A \to B$ be a surjective *-homomorphism such that $\pi(\mathcal{N}_A(D_A)) = \mathcal{N}_B(D_B)$.*

(i) Suppose $a, b \in (D_B)_+$ and $x \in \mathcal{N}_B(D_B)$ satisfy

$$x^*x \leq a \quad \text{and} \quad xx^* \leq b. \tag{2.5}$$

Then for any lifts $\bar{a}, \bar{b} \in (D_A)_+$ of a, b respectively, there exists a lift $\bar{x} \in \mathcal{N}_A(D_A)$ of x such that

$$\bar{x}^*\bar{x} \leq \bar{a} \quad \text{and} \quad \bar{x}\bar{x}^* \leq \bar{b}. \tag{2.6}$$

(ii) Given $n \in \mathbb{N}$ and a c.p.c. order zero map $\phi : M_n \to B$ such that

$$\phi(e_{ij}) \in \mathcal{N}_B(D_B), \quad i, j = 1, \ldots, n, \tag{2.7}$$

there exists an order zero lift $\bar{\phi} : M_n \to A$ such that

$$\bar{\phi}(e_{ij}) \in \mathcal{N}_A(D_A), \quad i, j = 1, \ldots, n. \tag{2.8}$$

Proof. (i) The construction is identical to [31, Lemma 8.2.1]. For the reader's convenience we reproduce the argument here and verify explicitly that the lift $\bar{x}$ is indeed a normalizer. Given lifts $\bar{a}, \bar{b} \in (D_A)_+$ of a, b respectively, let $\tilde{x} \in \mathcal{N}_A(D_A)$ be any lift of x. Define

$$\tilde{a} := (\tilde{x}^*\tilde{x} - \bar{a})_+ + \bar{a}. \tag{2.9}$$

Then $\tilde{x}^*\tilde{x} \leq \tilde{a}$, $\bar{a} \leq \tilde{a}$, and $\pi(\tilde{a}) = a$. Moreover the element $\tilde{a}$ belongs to the subalgebra D_A by Lemma 1.3. By [31, Lemma 2.2.2] the norm-limit

$$\hat{x} := \lim_{n\to\infty} \tilde{x}\left(\tilde{a} + \frac{1}{n}\right)^{-\frac{1}{2}} \bar{a}^{\frac{1}{2}} \tag{2.10}$$

exists and satisfies $\hat{x}^*\hat{x} \leq \bar{a}$ and $\pi(\hat{x}) = x$. Since each term belongs to $\mathcal{N}_A(D_A)$, so does the limit $\hat{x}$. To arrange the analogous relation for $\bar{b}$, we set

$$\hat{b} := (\hat{x}\hat{x}^* - \bar{b})_+ + \bar{b} \in D_A, \tag{2.11}$$

As before $\hat{x}\hat{x}^* \leq \hat{b}$, $\bar{b} \leq \hat{b}$, and $\pi(\hat{b}) = b$. The norm-limit

$$\bar{x} = \lim_{n\to\infty} \bar{b}^{\frac{1}{2}}\left(\hat{b} + \frac{1}{n}\right)^{-\frac{1}{2}} \hat{x} \tag{2.12}$$

is a well-defined lift of x and belongs to $\mathcal{N}_A(D_A)$. Moreover, $\bar{x}^*\bar{x} \leq \hat{x}^*\hat{x} \leq \bar{a}$ and $\bar{x}\bar{x}^* \leq \bar{b}$.

(ii) First of all, we claim that for any elements $a_2, ..., a_n$ in $\mathcal{N}_B(D_B)$ satisfying

$$\|a_j\| \leq 1, \qquad a_j a_k = 0, \qquad a_j^* a_k = \delta_{jk} a_2^* a_2 \tag{2.13}$$

for all $j, k = 2, ..., n$, there exist lifts $\bar{a}_2, ..., \bar{a}_n$ in $\mathcal{N}_A(D_A)$ with the same properties. Indeed, following the proof of [31, Lemma 10.2.1] we define $h_1 := |a_2|$ and $h_j := |a_j^*|$ for $j = 2, ..., n$. These elements belong to D_B and satisfy the relations

$$\begin{aligned} &0 \leq h_j \leq 1 \quad (j = 1, ..., n),\\ &h_j h_k = 0 \quad (j \neq k),\\ &a_2^* a_2 \leq \cdots \leq a_n^* a_n \leq h_1^2,\\ &a_j a_j^* \leq h_j^2 \quad (j = 2, ..., n). \end{aligned} \tag{2.14}$$

Since $\pi(D_A) = D_B$, we may lift $h_1, \ldots, h_n$ to orthogonal positive contractions $\hat{h}_1, \ldots, \hat{h}_n \in D_A$. Then by applying (i) successively, we may lift $a_1, \ldots, a_n$ to $\hat{a}_1, \ldots, \hat{a}_n \in \mathcal{N}_A(D_A)$ such that

$$\hat{a}_2^*\hat{a}_2 \leq \cdots \leq \hat{a}_n^*\hat{a}_n \leq \hat{h}_1^2, \tag{2.15}$$

$$\hat{a}_j\hat{a}_j^* \leq \hat{h}_j^2 \quad (j = 2, ..., n). \tag{2.16}$$

Then the norm-limits

$$\bar{a}_j := \lim_{n\to\infty} \hat{a}_j\left(|\hat{a}_j| + \frac{1}{n}\right)^{-\frac{1}{2}} |\hat{a}_2| \tag{2.17}$$

(which belong to $\mathcal{N}_A(D_A)$) satisfy $\bar{a}_j^*\bar{a}_j = \hat{a}_2^*\hat{a}_2 \leq \hat{h}_1^2$ and $\bar{a}_j^*\bar{a}_j^* \leq \hat{a}_j\hat{a}_j^* \leq \hat{h}_j^2$. Since the elements $\hat{h}_j$ are orthogonal, the elements $\bar{a}_j$ provide the desired lifts of a_j.

Let $\rho : CM_n \to B$ be the *-homomorphism induced by the order zero map ϕ. Recall from [31, Proposition 3.3.1] that CM_n can be identified with the universal C^*-algebra on generators $a_2, ..., a_n$ subject to the relations

$$\|a_j\| \leq 1, \qquad a_j a_k = 0, \qquad a_j^* a_k = \delta_{jk} a_2^* a_2 \qquad (j, k = 2, ..., n) \tag{2.18}$$

via the map $\mathrm{id}_{(0,1]} \otimes e_{j1} \mapsto a_j$. The elements $\rho(\mathrm{id}_{(0,1]} \otimes e_{j1})$ satisfy the relations above, and by assumption they belong to $\mathcal{N}_B(D_B)$, so by the previous paragraph we can lift them to elements $\bar{a}_2, ..., \bar{a}_n$ in $\mathcal{N}_A(D_A)$ satisfying the same relations. The universal property then yields a well-defined *-homomorphism $\bar{\rho} : CM_n \to A$ mapping $\mathrm{id}_{(0,1]} \otimes e_{j1}$ to $\bar{a}_j$. The c.p.c. order zero map $\bar{\phi} : M_n \to A$ induced from $\bar{\rho}$ provides a required lift of ϕ. □

In the proof of the next proposition, for a C^*-algebra A and $a \in A$ let $\mathrm{her}_A(a)$ denote the hereditary subalgebra of A generated by a.

Proposition 2.7. *Let $(D_A \subseteq A)$ and $(D_B \subseteq B)$ be sub-C^*-algebras and let $\pi : A \to B$ be a surjective *-homomorphism such that $\pi(\mathcal{N}_A(D_A)) = \mathcal{N}_B(D_B)$. Let F be a finite-dimensional C^*-algebra with a maximal abelian subalgebra D_F.*

Then given a c.p.c. order zero map $\phi : F \to B$ such that $\phi(\mathcal{N}_F(D_F)) \subseteq \mathcal{N}_B(D_B)$ there exists an order zero lift $\bar{\phi} : F \to A$ such that $\bar{\phi}(\mathcal{N}_F(D_F)) \subseteq \mathcal{N}_A(D_A)$

Proof. The strategy is the same as [31, Theorem 10.1.11], although we need to exercise care to verify the condition on normalizers. We illustrate the idea by proving the case when $(D_F \subseteq F)$ is isomorphic to a direct sum $(D_{r_1} \oplus D_{r_2} \subseteq M_{r_1} \oplus M_{r_2})$ of two matrix summands. The general case is established by induction on the number of matrix summands in F.

Let $\rho : CF \cong CM_{r_i} \oplus CM_{r_2} \to B$ be the *-homomorphism induced by the c.p.c. order zero map ϕ. Define

$$h_i := \rho(\mathrm{id}_{(0,1]} \otimes \mathbf{1}_{r_i}) \in D_B \tag{2.19}$$

for $i = 1, 2$. Then $0 \leq h_1, h_2 \leq 1$ and $h_1 h_2 = 0$. Note that we also have $\rho(CM_{r_i}) \subseteq \mathrm{her}_B(h_i)$ because $\mathrm{id}_{(0,1]} \otimes \mathbf{1}_{r_i}$ is strictly positive in CM_{r_i}. Since $\pi(D_A) = D_B$, [31, Proposition 10.1.10] asserts that there are lifts $k_1, k_2 \in D_A$ of h_1, h_2 respectively so that

$$0 \leq k_1, k_2 \leq 1 \qquad \text{and} \qquad k_1 k_2 = 0. \tag{2.20}$$

By [31, Corollary 8.2.4] the map π surjects $\mathrm{her}_A(k_i)$, $\mathrm{her}_{D_A}(k_i)$ onto $\mathrm{her}_B(h_i)$, $\mathrm{her}_{D_B}(h_i)$ respectively, whence π maps $\mathcal{N}_{\mathrm{her}_A(k_i)}(\mathrm{her}_{D_A}(k_i))$ into $\mathcal{N}_{\mathrm{her}_B(h_i)}(\mathrm{her}_{D_B}(h_i))$.

We claim that in fact $\pi(\mathcal{N}_{\mathrm{her}_A(k_i)}(\mathrm{her}_{D_A}(k_i))) = \mathcal{N}_{\mathrm{her}_B(h_i)}(\mathrm{her}_{D_B}(h_i))$. To see this, let $x \in \mathcal{N}_{\mathrm{her}_B(h_i)}(\mathrm{her}_{D_B}(h_i)) \subseteq \mathcal{N}_B(D_B)$ and $\bar{a}, \bar{b} \in \mathrm{her}_{D_A}(k_i)$ be positive lifts of $x^*x, xx^* \in \mathrm{her}_{D_B}(h_i)$ respectively. By Lemma 2.6(i) there exists a lift $\bar{x} \in \mathcal{N}_A(D_A)$ such that $\bar{x}^*\bar{x} \leq \bar{a}$ and $\bar{x}\bar{x}^* \leq \bar{b}$. Since

$\mathrm{her}_{D_A}(k_i)$ is hereditary in D_A, $\bar{x}^*\bar{x}$ and $\bar{x}\bar{x}^*$ actually belong to $\mathrm{her}_{D_A}(k_i)$. It follows from the identity

$$\bar{x} = \lim_n (\bar{x}\bar{x}^*)^{\frac{1}{n}} \bar{x} (\bar{x}^*\bar{x})^{\frac{1}{n}}, \tag{2.21}$$

which is a consequence of polar decomposition, that $\bar{x}$ belongs to the hereditary subalgebra $\mathrm{her}_A(k_i)$. For any element $d \in \mathrm{her}_{D_A}(k_i)$ we have $\bar{x}d\bar{x}^*, \bar{x}^*d\bar{x} \in D_A \cap \mathrm{her}_A(k_i) = \mathrm{her}_{D_A}(k_i)$. Therefore $\bar{x}$ normalizes $\mathrm{her}_{D_A}(k_i)$ and the claim is proved.

Now by the proof of Lemma 2.6(ii) each restriction $\rho_i := \rho|_{CM_{r_i}}$ lifts to a *-homomorphism $\bar{\rho}_i : CM_{r_i} \to \mathrm{her}_A(k_i)$ in a way that $\bar{\rho}_i(\mathrm{id}_{(0,1]} \otimes e_{j1})$ normalizes $\mathrm{her}_{D_A}(k_i)$ for $j = 1, 2, ..., r_i$. Since k_1 and k_2 are orthogonal, the map

$$\bar{\rho} := \bar{\rho}_1 + \bar{\rho}_2 : CM_{r_1} \oplus CM_{r_2} \cong CF \to A \tag{2.22}$$

remains a *-homomorphism, and the c.p.c. order zero map $\bar{\phi} : F \to A$ induced from $\bar{\rho}$ is a lift of ϕ satisfying $\bar{\phi}(\mathcal{N}_F(D_F)) \subseteq \mathcal{N}_A(D_A)$. □

2.2. Decomposable approximations for Cartan subalgebras.

Here we prove a Cartan subalgebra version of [3, Theorem 3.1], where we ask that our order zero maps additionally preserve normalizers. The broad framework of the argument is similar to the approach used in [3]: first establish an approximation in the weak topology, then use the Hahn–Banach theorem to upgrade it to a norm approximation. Whereas the finite-dimensional approximations in [3] come essentially from Connes' Theorem, we instead use Connes–Feldman–Weiss, to allow finite-dimensional approximations which relate to the Cartan structure.

The following improvement of [18, Lemma 1.1] was observed by Chris Schafhauser, and we thank him for allowing us to include it here with a proof. We refer the reader to [25, 26] for the definition and discussion of σ-ideals.

Lemma 2.8. *Let $\mathcal{M}$ be a von Neumann algebra and let $(B_n)_{n=1}^\infty$ be a sequence of C^*-algebras, together with representations $\pi_n : B_n \to \mathcal{M}$ such that $(\pi_n(B_n))_{n=1}^\infty$ is an increasing sequence and $\mathcal{M} = (\bigcup_n \pi_n(B_n))''$. Define*

$$\begin{aligned} B := \{ &(a_n)_{n=1}^\infty \in \prod_{n=1}^\infty B_n : \\ &(\pi_n(a_n))_{n=1}^\infty \text{ converges strong}^* \text{ in } \mathcal{M}\}/c_0 \end{aligned} \tag{2.23}$$

and define $q : B \to \mathcal{M}$ by

$$q((a_n)_{n=1}^\infty) = \text{strong}^*\text{-}\lim_{n\to\infty} \pi_n(a_n). \tag{2.24}$$

Then B is a C^*-algebra, q is a surjective *-homomorphism, and $J := \ker(q)$ is a σ-ideal of B (as defined in [25, Definition 1.5]*).*

Proof. Define

$$B_0 := \{(a_n)_{n=1}^\infty \in \ell^\infty(B) : (\pi_n(a_n))_{n=1}^\infty \text{ converges strong}^* \text{ in } \mathcal{M}\}. \tag{2.25}$$

Since multiplication is jointly strong* continuous on bounded subsets of $\mathcal{M}$, it follows that B_0 is a *-subalgebra of $\ell^\infty(B)$; let us also show that it is norm closed and thus a C^*-subalgebra (following roughly an argument in the proof of [18, Lemma 1.1]).

Let $b = (b_n)_{n=1}^\infty \in \ell^\infty(A)$ be the $\|\cdot\|$-limit of a sequence $(\alpha_k)_{k=1}^\infty$ in B_0. For each k, write $\alpha_k = (a_n^k)_{n=1}^\infty$ and set $a_\infty^k :=$ strong*-$\lim_{n\to\infty} \pi_n(a_n^k) \in \mathcal{M}$. Then we see that $\|a_\infty^k - a_\infty^l\| \leq \|\alpha_k - \alpha_l\|$, and thus $(a_\infty^k)_{k=1}^\infty$ is a $\|\cdot\|$-Cauchy sequence, so it converges to some $b_\infty \in \mathcal{M}$. Moreover, by writing

$$\pi_n(b_n) - b_\infty = (\pi_n(b_n) - \pi_n(a_n^k)) + (\pi_n(a_n^k) - a_\infty^k) + (a_\infty^k - b_\infty), \tag{2.26}$$

and by judiciously choosing k (so that $\|\alpha_n - b\|$ is small) and then n (so that $\pi_n(a_n^k)$ is strong* close to a_∞^k), we see that $\pi_n(b_n) \to b_\infty$ in the strong* topology, and thus $b \in B_0$.

Since B_0 is a C^*-algebra it follows that $B = B_0/c_0(A)$ is as well. The map q is evidently a *-homomorphism, and its surjectivity is a direct consequence of the Kaplansky Density Theorem.

Finally, to prove that J is a σ-ideal, we will use the ϵ-test. Given a finite set $\mathcal{H}$ of B, an element $a \in J_+$, and $\epsilon > 0$, using a quasicentral approximate unit for J, there exists $e = (e_n)_{n=1}^\infty \in J$ such that $\|[e,b]\| < \epsilon$ and $\|ea - a\| < \epsilon$. Let d_{s^*} be a metric on the unit ball of $\mathcal{M}$ which induces the strong* topology, so that we can encode the condition $e \in J$ by $\lim_{n\to\infty} d_{s^*}(\pi_n(e_n), 0) = 0$. Then it is clear that for any separable subalgebra C of B, we may use the ϵ-test to produce some $e \in J \cap C'$ such that $ea = a$. Thus, J is a σ-ideal as required. □

We emphasize that in the next two definitions, counter to notation commonly used with von Neumann algebras, we continue to use $\mathcal{N}_\mathcal{M}(\mathcal{D})$ to refer to not-necessarily-unitary normalizers (as in Definition 1.2).

Definition 2.9. *Let $\mathcal{M}$ be a von Neumann algebra with separable predual and let $\mathcal{D}$ be an abelian von Neumann subalgebra. We say that the sub-von Neumann algebra $(\mathcal{D} \subseteq \mathcal{M})$ is* hyperfinite *if there is an increasing sequence $(F_i)_{i=1}^\infty$ of finite-dimensional C^*-subalgebras of $\mathcal{M}$, together with a maximal abelian subalgebra D_{F_i} of F_i for each i such that*

$$\mathcal{M} = \Big(\bigcup_{i=1}^\infty F_i\Big)'' \quad \text{and} \quad \mathcal{D} = \Big(\bigcup_{i=1}^\infty D_{F_i}\Big)'', \tag{2.27}$$

and such that $\mathcal{N}_{F_i}(D_{F_i}) \subseteq \mathcal{N}_\mathcal{M}(\mathcal{D})$ for all i.

For a C^*-algebra A and a state f on A, we write $(\pi_f, L^2(A,f))$ for the associated GNS representation.

Lemma 2.10. *Let $(D \subseteq A)$ be a sub-C^*-algebra with A separable nuclear and D abelian. Suppose $(D \subseteq A)$ satisfies the following two properties for every state f on A:*

(1) $(\pi_f(D)'' \subseteq \pi_f(A)'')$ is hyperfinite, and
(2) the unit ball of $\pi_f(\mathcal{N}_A(D))$ is strong-dense in the unit ball of $\mathcal{N}_{\pi_f(A)''}(\pi_f(D)'')$.*

Then there exists a net of finite-dimensional C^*-algebras F_i together with maximal abelian subalgebras D_{F_i} and c.p.c. maps

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} A \tag{2.28}$$

such that:

(i) $\phi_i(\psi_i(a)) \to a$ in the relative σ-strong topology (from A^{**}), for all $a \in A$,*
(ii) for each i, ϕ_i has order zero and sends $\mathcal{N}_{F_i}(D_{F_i})$ to $\mathcal{N}_A(D)$, and
(iii) (ψ_i) is asymptotically order zero, that is, $\lim_i \|\psi_i(a)\psi_i(b)\| = 0$ for all orthogonal $a, b \in A_+$.

Proof. Recall that the normal states on A^{**} are precisely those coming from the predual A^*, and from [1, III.2.2.19] that on bounded subsets of A^{**} the σ-strong* topology coincides with the topology generated by the seminorms $\{\|\cdot\|_\rho^\#\}_{\rho\in A^*}$, where $\|x\|_\rho^\# := \rho\left(\frac{1}{2}(xx^* + x^*x)\right)^{\frac{1}{2}}$.

We prove the local version of the lemma. That is, given a finite subset $\mathcal{F} \subseteq A$, a finite subset of normal states $S \subseteq A^*$, and $\epsilon > 0$, we will find a finite-dimensional C^*-algebra F with a maximal abelian subalgebra D_F and c.p.c. maps

$$A \xrightarrow{\psi} F \xrightarrow{\varphi} A \tag{2.29}$$

such that

(i) $\|\phi\psi(a) - a\|_\rho^\# < \epsilon$ for all $a \in \mathcal{F}$ and $\rho \in S$,
(ii) ϕ is order zero,
(iii) $\phi(\mathcal{N}_F(D_F)) \subseteq \mathcal{N}_A(D)$, and
(iv) $\|\psi(a)\psi(b)\| < \epsilon$ if a, b are orthogonal positive elements in $\mathcal{F}$.

By working with the average of the states in S, we may assume that S contains a single element ρ. If we let $\pi_\rho : A \to B(L^2(A, \rho))$ be the GNS representation associated to ρ, then by assumption the sub-von Neumann algebra $(\mathcal{D} \subseteq \mathcal{M}) := (\overline{\pi_\rho(D)}^{\text{strong}^*} \subseteq \overline{\pi_\rho(A)}^{\text{strong}^*})$ is hyperfinite. Let $(D_{F_n} \subseteq F_n)$ be an increasing sequence of finite-dimensional sub-C^*-algebras witnessing the hyperfiniteness of $(\mathcal{D} \subseteq \mathcal{M})$.

Define

$$B := \{(x_n)_n \in \prod_n F_n : (x_n)_n \text{ converges in the strong}^* \text{ topology}\}/c_0 \tag{2.30}$$

and

(2.31)
$$J := \{(x_n)_n \in B : (x_n)_n \text{ converges to 0 in the strong}^* \text{ topology}\}$$

as in the statement of Lemma 2.8 (note that we are abusing the notation here; we do not distinguish a sequence with its equivalence class). Then J is a σ-ideal of B, so by [25, Proposition 1.6] the map $\pi_\rho : A \to \mathcal{M} \cong B/J$ lifts to a c.p.c. order zero map from A into B, which further lifts to a c.p.c. map

$$\psi = (\psi_n)_n : A \to \prod_n F_n \tag{2.32}$$

by the Choi–Effros lifting theorem [6]. Notice that the sequence (ψ_n) is asymptotically order zero along the ultrafilter ω, and that by construction, $\psi_n(a) \to \pi_\rho(a)$ in the strong* topology for every $a \in A$.

For each $n \in \mathbb{N}$, let $\iota_n : (D_{F_n} \subseteq F_n) \to (\mathcal{D} \subseteq \mathcal{M})$ be the inclusion map. To obtain the incoming map ϕ, define

(2.33)
$$B' := \{(y_m)_m \in \prod_m A : (\pi_\rho(y_m))_m \text{ converges in the strong}^* \text{ topology}\}$$

and

(2.34)
$$J' := \{(y_m)_m \in \prod_m A : (\pi_\rho(y_m))_m \text{ converges to 0 in the strong}^* \text{ topology}\}$$

Then B' is a C^*-subalgebra of $\prod_m A$ and the (strong*-limit) evaluation map $q' : B' \to \mathcal{M}$ is surjective with $\ker(q') = J'$. Note that q' is surjective because the GNS space $L^2(A,\rho)$ is separable (A is assumed to be separable) and so the strong* topology is metrizable on bounded sets. We also define

(2.35)
$$D_{B'} := \{(z_m)_m \in \prod_m D : (\pi_\rho(z_m))_m \text{ converges in the strong}^* \text{ topology}\}.$$

Then q' maps $D_{B'}$ onto the abelian subalgebra $\mathcal{D}$ of $\mathcal{M}$. By assumption given any $x \in \mathcal{N}_{\mathcal{M}}(\mathcal{D})$ we can find a bounded sequence $(x_m)_m$ in $\mathcal{N}_A(D)$ such that $(\pi_\rho(x_m))_m$ converges strong* to x. Since the sequence $(x_m)_m$ belongs to $\mathcal{N}_{B'}(D_{B'})$, the quotient map q' maps $\mathcal{N}_{B'}(D_{B'})$ onto $\mathcal{N}_{\mathcal{M}}(\mathcal{D})$. Using Proposition 2.7 we can lift the inclusion map $\iota_n : F_n \to \mathcal{M} \cong q'(B')$ to a c.p.c. order zero map

$$\phi_n = (\phi_{n,m})_m : F_n \to B' \subseteq \prod_m A \tag{2.36}$$

such that ϕ_n maps $\mathcal{N}_{F_n}(D_{F_n})$ into $\mathcal{N}_{B'}(D_{B'})$. As a consequence, for each m the image $\phi_{n,m}(\mathcal{N}_{F_n}(D_{F_n}))$ is contained in $\mathcal{N}_A(D)$. By construction we have

$$\pi_\rho(\phi_{n,m}\psi_n(a)) \to \pi_\rho(a) \qquad \text{strong}^* \tag{2.37}$$

for every $a \in A$ as $n, m \to \infty$. Since strong* convergence in $B(L^2(A,\rho))$ implies convergence in the seminorm $\|\cdot\|_\rho^{\#}$, the proof is finished by choosing suitably large indices n and m. $\square$

Remark 2.11. In Lemma 2.10, even though A is assumed to be separable, in general we still need a net of approximations because A^* may not be separable.

Theorem 2.12 (cf. [3, Theorem 3.1]). *Let $(D \subseteq A)$ be a sub-C^*-algebra with A separable nuclear and D abelian. Suppose $(D \subseteq A)$ satisfies the following two properties for every state f on A:*

(1) $(\pi_f(D)'' \subseteq \pi_f(A)'')$ is hyperfinite, and

(2) the unit ball of $\pi_f(\mathcal{N}_A(D))$ is strong-dense in the unit ball of $\mathcal{N}_{\pi_f(A)''}(\pi_f(D)'')$.*

Then there exists a sequence of finite-dimensional C^*-algebras F_n together with maximal abelian subalgebras D_{F_n} and c.p.c. maps

$$A \xrightarrow{\psi_n} F_n \xrightarrow{\phi_n} A \tag{2.38}$$

such that:

(i) $\phi_n(\psi_n(a)) \to a$ in $\|\cdot\|$, for all $a \in A$,

(ii) for each n, ϕ_n is a convex combination of finitely many c.p.c. order zero maps, each sending $\mathcal{N}_{F_n}(D_{F_n})$ to $\mathcal{N}_A(D)$, and

(iii) (ψ_n) is asymptotically order zero, that is, $\lim_{n\to\infty} \|\psi_n(a)\psi_n(b)\| = 0$ for all orthogonal $a, b \in A_+$.

In particular, if $(D \subseteq A)$ is nondegenerate then $\phi_n(D_{F_n}) \subseteq D$ for all n (by (ii) and Lemma 1.3).

Proof. This follows from Lemma 2.10 and a standard Hahn–Banach convexity argument as in the proof of [18, Theorem 1.4] (see also the paragraph after [3, Proposition 2.3]). To be more precise, consider the set $\mathcal{B}(A)$ of all bounded linear maps from A to itself. By the Hahn–Banach theorem the point-norm and point-weak closures coincide for any convex subset of $\mathcal{B}(A)$. Given a finite subset $\mathcal{F}$ of A and $\epsilon > 0$, let $K_0 \subseteq \mathcal{B}(A)$ be the set of all c.p.c. maps for which there is a finite-dimensional C^*-algebra F with a maximal abelian subalgebra D_F and a factorization $A \xrightarrow{\psi} F \xrightarrow{\phi} A$, where

- ψ is c.p.c. with $\|\psi(ab) - \psi(a)\psi(b)\| < \epsilon$ for all orthogonal positive elements $a, b \in \mathcal{F}$, and
- ϕ is c.p.c. order zero and sends $\mathcal{N}_F(D_F)$ into $\mathcal{N}_A(D)$.

By Lemma 2.10 the identity map id_A lies in the point-weak closure of K_0, and hence the point-norm closure of the convex hull $\mathrm{conv}(K_0)$. This establishes the local version of the theorem. Since A is separable, it is enough to take a sequence (instead of a net) of approximations. $\square$

A standard reindexing argument yields the following version of Theorem 2.12.

Lemma 2.13 (cf. [5, Lemma 3.5]). *Let $(D \subseteq A)$ be a sub-C^*-algebra with A separable nuclear and D abelian. Suppose $(D \subseteq A)$ satisfies the following two properties for every state f on A:*

(1) $(\pi_f(D)'' \subseteq \pi_f(A)'')$ is hyperfinite, and

(2) the unit ball of $\pi_f(\mathcal{N}_A(D))$ is strong*-dense in the unit ball of $\mathcal{N}_{\pi_f(A)''}(\pi_f(D)'')$.

Let $\hat{S} \subseteq A_\omega$ be a $\|\cdot\|$-separable subalgebra, and let $S \subseteq A^\omega$ be the $\|\cdot\|_{2,T_\omega(A)}$-closure of the image of $\hat{S}$ in A^ω. Then there exists a sequence of finite-dimensional C^*-algebras F_n together with maximal abelian subalgebras D_{F_n} and c.p.c. maps

$$A \xrightarrow{\psi_n} F_n \xrightarrow{\phi_n} A \tag{2.39}$$

which induce c.p.c. maps $\psi : A_\omega \to \prod_\omega F_n$ and $\phi : \prod_\omega F_n \to A^\omega$ such that:

(i) the restriction of $\phi \circ \psi$ to $\hat{S}$ agrees with the quotient map $\hat{S} \to S \subseteq A^\omega$,

(ii) for each n, ϕ_n maps D_{F_n} into D and restricts to an order zero map on each full matrix direct summand of F_n, and

(iii) the map ψ restricts to an order zero map on $\hat{S}$.

2.3. **Decomposable aproximations for crossed products.** In this subsection we apply the results obtained in Section 2.2 to sub-C^*-algebras arising from actions of amenable groups on compact metrizable spaces. The main goal is to verify the two assumptions in Theorem 2.12 for these sub-C^*-algebras.

Let G be a countable discrete group, let X be a compact metrizable space, let $\alpha : G \curvearrowright X$ be an action, and let μ be a Borel probability measure on X. Identifying $\ell^2(G)\otimes L^2(X,\mu)$ with $\bigoplus_{g\in G} L^2(X,\mu)$, define a *-representation $\rho_\alpha : C(X) \to B(\ell^2(G) \otimes L^2(X,\mu))$ by

$$\rho_\alpha(f) := \bigoplus_{g\in G} \alpha_{g^{-1}}(f). \tag{2.40}$$

Next, fix a function $w : G \to (0,1)$ such that $\sum_{g\in G} w(g) = 1$, and use this to define a Borel probability measure $\bar{\mu}$ on X by

$$\bar{\mu}(E) := \sum_{g\in G} w(g)\mu(g^{-1}.E), \tag{2.41}$$

i.e., $\bar{\mu}$ is a weighted average of the pushforward measure $g.\mu$ over all $g \in G$ (that is, $(g.\mu)(E) = \mu(g^{-1}.E)$). Since $\bar{\mu}$ is G-quasi-invariant (meaning that for all $g \in G$ and all Borel sets E of X, $\bar{\mu}(E) = 0$ if and only if $(g.\bar{\mu})(E) = 0$), so that α induces an action $\bar{\alpha}$ on $L^\infty(X, \bar{\mu})$.

Lemma 2.14. *Let X be a compact metrizable space and let μ, ν be Borel probability measures on X such that $\mu \le k\nu$ for some $k \in (0,\infty)$. Then the canonical map $\Phi : L^\infty(X,\nu) \to L^\infty(X,\mu)$ is a well-defined surjective normal *-homomorphism.*

Proof. We only prove normality since the rest is straightforward. Let both algebras be represented as multiplication operators on their L^2-spaces. Since Φ is a *-homomorphism, it maps the unit ball $L^\infty(X,\nu)^1$ into the unit ball $L^\infty(X,\mu)^1$. A standard result in von Neumann algebras (for example, see [1, Proposition III.2.2.2]) says that normality of Φ is equivalent to the restriction of Φ to the unit ball $L^\infty(X,\nu)^1$ being continuous with respect to the strong operator topologies. Recall that the strong operator topology is metrizable on bounded sets, so let $(f_n)_n$ be a sequence in $L^\infty(X,\nu)^1$ that converges in the strong operator topology to an element f that also lies in $L^\infty(X,\nu)^1$. Then for any continuous function $h \in C(X)$, we have

$$\begin{aligned}\|f_n h - fh\|_{L^2(\mu)}^2 &= \int_X |f_n h - fh|^2 \; d\mu \leq k \cdot \int_X |f_n h - fh|^2 \; d\nu \\ &= k \cdot \|f_n h - fh\|_{L^2(\nu)}^2 \to 0.\end{aligned}$$

As $C(X)$ is dense in $L^2(X,\mu)$ and the sequence $(f_n)_n$ is uniformly bounded in the L^∞-norm, a standard 3ϵ-argument shows that $\|f_n\xi - f\xi\|_{L^2(\mu)}$ converges to zero for every $\xi \in L^2(X,\mu)$. □

By Lemma 2.14 the canonical map $L^\infty(X,\bar{\mu}) \to L^\infty(X,\mu)$ is a well-defined surjective normal *-homomorphism, so we can define a normal representation

$$L^\infty(X,\bar{\mu}) \to B(\ell^2(G) \otimes L^2(X,\mu)), \quad b \mapsto \bigoplus_{t\in G} \bar{\alpha}_{t^{-1}}(b). \tag{2.42}$$

If $\bigoplus_{t\in G} \bar{\alpha}_{t^{-1}}(b) = 0$ then $b = 0$ $(t.\mu)$-a.e. for every $t \in G$. So by construction $b = 0$ $\bar{\mu}$-a.e. and the representation is faithful. We use it to define the crossed product $L^\infty(X,\bar{\mu}) \rtimes_{\bar{\alpha}} G$, which is a von Neumann algebra on the Hilbert space $\ell^2(G) \otimes (\ell^2(G) \otimes L^2(X,\mu))$.

Lemma 2.15. *Let G be a countable discrete amenable group, let X be a compact metrizable space, let $\alpha : G \curvearrowright X$ be a free action, and let φ be a state on $C(X) \rtimes_\alpha G$. Let μ be the Borel probability measure on X associated to $\varphi|_{C(X)}$. Then the following von Neumann algebras are *-isomorphic.*

- *(i)* $\pi_\varphi(C(X) \rtimes_\alpha G)''$;
- *(ii)* $L^\infty(X,\bar{\mu}) \rtimes_{\bar{\alpha}} G$, *where $w : G \to (0,1)$ is any function such that $\sum_{g\in G} w(g) = 1$, $\bar{\mu}$ is then defined by (2.41), and $\bar{\alpha}$ is the action on $L^\infty(X,\bar{\mu})$ induced by α; and*
- *(iii)* $(\rho_\alpha(C(X)) \cup \lambda(G) \otimes \mathrm{id}_{L^2(X,\mu)})'' \subseteq B(\ell^2(G) \otimes L^2(X,\mu))$, *where ρ_α is defined by (2.40) and $\lambda : G \to B(\ell^2(G))$ is the left regular representation.*

Moreover, there is a *-isomorphism $\pi_\varphi(C(X) \rtimes_\alpha G)'' \cong L^\infty(X,\bar{\mu}) \rtimes_{\bar{\alpha}} G$ sending $\pi_\varphi(C(X))''$ onto $L^\infty(X,\bar{\mu})$.

Proof. We first identify the von Neumann algebras in (i) and (iii). Let $L^2(C(X)\rtimes_\alpha G,\varphi)$ be the GNS space associated to the state φ and view $C_c(G,C(X))$ as a dense subspace. Define a linear map

$$U : C_c(G,C(X)) \to \ell^2(G)\otimes L^2(X,\mu) \tag{2.43}$$

by setting $U(f_t t) := \delta_t \otimes \alpha_{t^{-1}}(f_t)$. Then U is an isometry onto a dense subspace of $\ell^2(G)\otimes L^2(X,\mu)$, and hence extends to a unitary between the Hilbert spaces. A straightforward computation shows that the conjugation by U maps $\pi_\varphi(C_c(G,C(X)))$ onto the *-algebra generated by $\rho_\alpha(C(X))$ and $\lambda(G)\otimes \mathrm{id}_{L^2(X,\mu)}$, and in fact maps $\pi_\varphi(C(X))$ onto $\rho_\alpha(C(X))$. Therefore the corresponding weak-closures are (spatially) *-isomorphic.

To see that the crossed product $L^\infty(X,\bar\mu)\rtimes_{\bar\alpha} G$ is isomorphic to the von Neumann algebra in (iii), it is enough to show that $L^\infty(X,\bar\mu)\rtimes_{\bar\alpha}G$ is isomorphic to the von Neumann algebra generated by $\mathrm{id}_{\ell^2(G)}\otimes L^\infty(X,\bar\mu)$ and $\mathrm{id}_{\ell^2(G)}\otimes(\lambda(G)\otimes \mathrm{id}_{L^2(X,\mu)})$ within $B(\ell^2(G)\otimes\ell^2(G)\otimes L^2(X,\mu))$. This is essentially a consequence of Fell's absorption principle. To be more explicit, consider the unitary

$$V : \ell^2(G)\otimes\ell^2(G)\otimes L^2(X,\bar\mu) \to \ell^2(G)\otimes\ell^2(G)\otimes L^2(X,\bar\mu) \tag{2.44}$$

defined by $V(\delta_t\otimes\delta_s\otimes\xi) := \delta_s\otimes\delta_{ts}\otimes\xi$. For any $f\in L^\infty(X,\bar\mu)\subseteq L^\infty(X,\bar\mu)\rtimes G$, we have

$$\begin{aligned} Vf(\delta_t\otimes\delta_s\otimes\xi) &= V(\delta_t\otimes\alpha_{t^{-1}}(f)(\delta_s\otimes\xi)) \\ &= V(\delta_t\otimes\delta_s\otimes\alpha_{s^{-1}t^{-1}}(f)\xi) \\ &= \delta_s\otimes\delta_{ts}\otimes\alpha_{s^{-1}t^{-1}}(f)\xi. \end{aligned} \tag{2.45}$$

On the other hand,

$$\begin{aligned} (\mathrm{id}_{\ell^2(G)}\otimes f)V(\delta_t\otimes\delta_s\otimes\xi) &= (\mathrm{id}_{\ell^2(G)}\otimes f)(\delta_s\otimes\delta_{ts}\otimes\xi) \\ &= \delta_s\otimes f(\delta_{ts}\otimes\xi) \\ &= \delta_s\otimes\delta_{ts}\otimes\alpha_{s^{-1}t^{-1}}(f)\xi. \end{aligned} \tag{2.46}$$

Similarly for the operators λ_g we have

$$\begin{aligned} V(\lambda_g\otimes\mathrm{id}_{\ell^2(G)\otimes L^2(X,\mu)})(\delta_t\otimes\delta_s\otimes\xi) &= V(\delta_{gt}\otimes\delta_s\otimes\xi) \\ &= \delta_s\otimes\delta_{gts}\otimes\xi \end{aligned} \tag{2.47}$$

and

$$\begin{aligned} &(\mathrm{id}_{\ell^2(G)}\otimes\lambda_g\otimes\mathrm{id}_{L^2(X,\mu)})V(\delta_t\otimes\delta_s\otimes\xi) \\ &\quad= (\mathrm{id}_{\ell^2(G)}\otimes\lambda_g\otimes\mathrm{id}_{L^2(X,\mu)})(\delta_s\otimes\delta_{ts}\otimes\xi) \\ &\quad= \delta_s\otimes\delta_{gts}\otimes\xi. \end{aligned}$$

It follows that V is an intertwining unitary.

It follows from the explicit identifications from (i) to (iii) and from (iii) to (ii) that $\pi_\varphi(C(X))''$ is mapped onto $\rho_\alpha(C(X))''$, and then onto $L^\infty(X,\bar\mu)$. ∎

Lemma 2.16. *Let (X,μ) be a Borel probability space, and let $\xi \in L^2(X,\mu)$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever E is a Borel subset of X with $\mu(E) < \delta$ we have $\|\xi \mathbf{1}_E\|_2 < \epsilon$*

Proof. As a consequence of the Dominated Convergence Theorem, there exists an $M > 0$ such that the set $A := \{x \in X : |\xi(x)|^2 \geq M\}$ satisfies $\|\mathbf{1}_A \xi\|_2 < \frac{\epsilon}{2}$. Then for every subset $E \subseteq X$ with $\mu(E) < \frac{\epsilon}{2M}$ we have

$$\|\mathbf{1}_E \xi\|_2 = \|\mathbf{1}_{E\cap A}\xi\|_2 + \|\mathbf{1}_{E\cap A^c}\xi\|_2 < \frac{\epsilon}{2} + M \cdot \frac{\epsilon}{2M} = \epsilon, \tag{2.48}$$

so we can take $\delta := \frac{\epsilon}{2M}$. □

To verify the assumptions in Theorem 2.12 for the sub-C^*-algebra $(C(X) \subseteq C(X) \rtimes_\alpha G)$, we use Connes–Feldman–Weiss theorem to establish hyperfiniteness. For the other assumption we give an explicit description of the normalizers in the von Neumann algebra setting and approximate them by C^*-normalizers. Since these rely on the language of Borel equivalence relations, in what follows we give a quick review of the theory of countable Borel equivalence relations. For details we refer the reader to [8] and [12].

A Borel equivalence relation R on a standard Borel space X is *countable* (respectively, *finite*) if each R-equivalence class is countable (respectively, finite). The set

$$R^{(0)} := \{(x,x) : x \in X\} \tag{2.49}$$

is identified with the space X. A measure μ on X is *R-quasi-equivalent* if for every μ-null Borel subset $A \subseteq X$ the set (called the *saturation*)

$$A_R := \{y \in X : xRy \text{ for some } x \in A\} \tag{2.50}$$

is μ-null. A quasi-invariant measure μ induces a σ-finite measure

$$\nu := \int \nu^x \, d\mu(x) \tag{2.51}$$

on R, where ν^x is the (right) counting measure on the fiber $R^x := \{(x,y) : xRy\}$. We will mostly be interested in the orbit equivalence relation that comes from an action $G \curvearrowright X$ of a countable group on a compact metrizable space (which is standard Borel). More precisely, the orbit equivalence relation on X is defined by

$$R_{G\curvearrowright X} := \{(x, g.x) : x \in X, g \in G\}, \tag{2.52}$$

which is countable because G is countable. In this case a measure μ on X is $(R_{G\curvearrowright X})$-quasi-invariant if and only if it is G-quasi-invariant.

As for groups there is a notion of amenability for measured Borel equivalence relations (see [8, Definition 6]). Instead of giving the precise definition, we only mention that the orbit equivalence relation $R_{G\curvearrowright X}$ is amenable whenever G is amenable (see, for example, [46, Chapter XIII, Proposition 4.9]). The celebrated Connes–Feldman–Weiss theorem ([8, Theorem 10]) says that every amenable countable Borel equivalence relation R is *hyperfinite*, i.e., there is an increasing sequence $(T_n)_n$ of finite subequivalence relations such that

$$R = \bigcup T_n \qquad \nu\text{-a.e.}. \tag{2.53}$$

Note that in the definition of hyperfiniteness we may assume $T_n^{(0)} = R^{(0)}$ for each n by taking union with $R^{(0)}$. Moreover, by [49, p. 420] (see also [10, Theorem 5.1]) we may assume that each T_n is uniformly bounded, i.e., there is a positive integer m_n such that every equivalence class in T_n has cardinality at most m_n.

We briefly recall the foundations of von Neumann algebras associated to measured equivalence relations as in [13], and we restrict ourselves to the case where the 2-cocycle σ is trivial. Let R be a measured countable Borel equivalence relation on a standard probability space (X, μ), where μ is R-quasi-invariant. A bounded Borel function a on R is *band-limited* if there is a constant $C > 0$ such that for each x and y,

$$|\{z : a(z, y) \neq 0\}| + |\{z : a(x, z) \neq 0\}| \leq C. \tag{2.54}$$

The set of band-limited Borel functions is made into a *-algebra by declaring

$$ab(x, y) = \sum_z a(x, z)b(z, y), \qquad a^*(x, y) = \overline{a(y, x)}\delta(y, x) \tag{2.55}$$

(here δ is the Radon–Nikodym derivative of μ with respect to R; see [12, Definition 2.1] for details). This *-algebra can be represented on $L^2(R, \nu)$ by the same convolution formula. i.e.,

$$a\xi(x, y) = \sum_z a(x, z)\xi(z, y). \tag{2.56}$$

Its weak closure, the von Neumann algebra associated to R, is denoted by $\mathbf{M}(R)$. By [13, Proposition 2.6] the elements in $\mathbf{M}(R)$ can be identified with bounded, square-integrable functions on R and the convolution and involution formulas continue to hold.

Following [13, p. 332] we identify $L^\infty(X, \mu)$ with the maximal abelian subalgebra of $\mathbf{M}(R)$ consisting of the operators supported on the diagonal $\Delta_X \subseteq X \times X$. When $R = R_{G \curvearrowright X}$ for a *free* action of a countable group, there is a canonical *-isomorphism $L^\infty(X, \mu) \rtimes_{\bar{\alpha}} G \cong \mathbf{M}(R)$ that restricts to identity on $L^\infty(X, \mu)$.

Proposition 2.17. *Let G be a countable discrete amenable group, let X be a compact metrizable space, and let $\alpha : G \curvearrowright X$ be a free action. Then for every state f on $C(X) \rtimes_\alpha G$ the sub-von Neumann algebra $(\pi_f(C(X))'' \subseteq \pi_f(C(X) \rtimes_\alpha G)'')$ is hyperfinite.*

Proof. To lighten the notation, we set

$$A := C(X) \rtimes_\alpha G \quad \text{and} \quad D := C(X). \tag{2.57}$$

Fix a state f on A. By Lemma 2.15 we can identify the sub-von Neumann algebra $(\pi_f(D)'' \subseteq \pi_f(A)'')$ with $(L^\infty(X,\bar{\mu}) \subseteq L^\infty(X,\bar{\mu}) \rtimes_{\bar{\alpha}} G)$, where $\bar{\mu}$ is defined by (2.41) and $\bar{\alpha}$ is the action on $L^\infty(X,\bar{\mu})$ induced by α. This subalgebra can further be identified with $(L^\infty(X,\bar{\mu}) \subseteq \mathbf{M}(R))$, where $R = R_{G\curvearrowright X}$ is the orbit equivalence relation obtained from the action. We will switch between these two pictures whenever convenient.

By the Connes–Feldman–Weiss theorem there is an increasing sequence $(T_n)_n$ of finite subequivalence relations of R such that $R = \bigcup T_n$ ν-a.e., where ν is the measure on R induced by $\bar{\mu}$. By the discussion preceding the proposition, we may assume that $T_n^{(0)} = R^{(0)}$ $(= X)$ for all n and that each T_n has uniformly bounded equivalence classes. Let us fix an $n \in \mathbb{N}$ and suppose the equivalence classes of T_n have cardinality at most m_n. By collecting equivalence classes of the same cardinality, we can decompose X into a disjoint union

$$X = \bigsqcup_{k=1}^{m_n} X_{n,k} \tag{2.58}$$

of T_n-invariant subsets such that each restriction $T_n|_{X_{n,k}}$ is isomorphic to the equivalence relation $S_{n,k}$ on $\{1,...,k\} \times (X_{n,k}/T_n)$ given by

$$(s,x') \sim_{S_{n,k}} (t,y') \iff x' = y' \tag{2.59}$$

(this is saying that $T_n|_{X_{n,k}}$ is *type* I_k; see [12, Definition 3.4]). Note that the quotient space $X_{n,k}/T_n$ is standard Borel by [21, Example 6.1, Proposition 6.3]. Since the trivial equivalence relation on $\{1,...,k\}$ gives rise to the full matrix algebra $M_k(\mathbb{C})$, the von Neumann algebra associated to T_n has the form

$$\mathbf{M}(T_n) \cong \bigoplus_k M_k(\mathbb{C}) \otimes L^\infty(X_{n,k}/T_n). \tag{2.60}$$

The isomorphism comes from the level of equivalence relations, so it preserves the canonical abelian subalgebras and hence the normalizers. A standard approximation argument shows that $\mathbf{M}(T_n)$ is approximately finite-dimensional and the finite-dimensional subalgebras can be chosen in a way that every matrix unit is a normalizer. Then an induction proves that the sub-von Neumann algebra $(L^\infty(X,\bar{\mu}) \subseteq \mathbf{M}(R))$ is hyperfinite in the sense of Definition 2.9. $\square$

Lemma 2.18. *Let $\mathcal{M}$ be a separably acting von Neumann algebra and $\mathcal{D}$ be an abelian von Neumann subalgebra of $\mathcal{M}$ such that $\mathbf{1}_{\mathcal{M}} \in \mathcal{D}$. Let $r \in \mathcal{N}_{\mathcal{M}}(\mathcal{D})$ and $r = v|r|$ be the unique polar decomposition in $\mathcal{M}$ such that* $\ker v = \ker r$. *Then $v \in \mathcal{N}_{\mathcal{M}}(\mathcal{D})$.*

Proof. Since v^*v is the left support projection of $|r|$ and vv^* is the right support projection of $|r^*|$, we see that v^*v and vv^* belong to $\mathcal{D}$ $(= \mathcal{D}'')$. Identifying $\mathcal{D}$ with some $L^\infty(X,\mu)$, we can find a bounded positive measurable function h on X that represents $|r|$. Then v^*v is represented by the characteristic function $p_\ell := \mathbf{1}_{\{x\in X: h(x)>0\}}$. Notice that for every $d \in \mathcal{D}$ we have

$$v|r|d|r|v^* = rdr^* \in \mathcal{D}. \tag{2.61}$$

If we define $X_n := \{x \in X : h(x) \geq \frac{1}{n}\}$ then each characteristic function $\mathbf{1}_{X_n}$ belongs to $|r|\mathcal{D}|r|$ and the sequence $(\mathbf{1}_{X_n})_n$ converges to p_ℓ strongly. Therefore $|r|\mathcal{D}|r|$ is strongly dense in $(v^*v)\mathcal{D}(v^*v)$, and we have

$$vdv^* = v(v^*v)d(v^*v)v^* \in \overline{v|r|\mathcal{D}|r|v^*}^{\text{strong}} \subseteq \overline{\mathcal{D}}^{\text{strong}} = \mathcal{D} \tag{2.62}$$

for any $d \in \mathcal{D}$. An entirely analogous argument shows that $v^*\mathcal{D}v \subseteq \mathcal{D}$, so v normalizes $\mathcal{D}$ in $\mathcal{M}$. □

The following lemma generalizes a well-known characterization of unitary normalizers (see [13, Proposition 2.10]) in the von Neumann algebra setting to the non-unitary case.

Lemma 2.19. *Let G be a countable discrete amenable group, let (X, μ) be a standard Borel space, and let $\alpha : G \curvearrowright X$ be a free action under which μ is quasi-invariant. If $r \in \mathcal{N}_{L^\infty(X,\mu)\rtimes_\alpha G}(L^\infty(X, \mu))$ then there exists a $\|\cdot\|_\infty$-bounded, orthogonal family of functions $f_g \in L^\infty(X, \mu)$ for $g \in G$ such that $\{\alpha_g^{-1}(f_g)\}_{g\in G}$ is orthogonal and*

$$r = \sum_{g\in G} f_g u_g,$$

converging in the strong*-topology.

Proof. Again to lighten the notation we set

$$\mathcal{M} := L^\infty(X, \mu) \rtimes_\alpha G \quad \text{and} \quad \mathcal{D} := L^\infty(X, \mu). \tag{2.63}$$

Suppose $r \in \mathcal{M}$ normalizes $\mathcal{D}$ and let $r = v|r|$ be the unique polar decomposition in $\mathcal{M}$ such that $\ker v = \ker r$. Then $v \in \mathcal{N}_{\mathcal{M}}(\mathcal{D})$ by Lemma 2.18. Let h and k be bounded positive measurable functions on X that represent v^*v and vv^*, respectively. If we set $X_1 := \{x \in X : h(x) > 0\}$ and $X_2 := \{x \in X : k(x) > 0\}$, then the abelian von Neumann algebras $v^*v\mathcal{D}v^*v$ and $vv^*\mathcal{D}vv^*$ are identified with $L^\infty(X_1, \mu|_{X_1})$ and $L^\infty(X_2, \mu|_{X_2})$, respectively. The conjugation map $\mathrm{Ad}(v)$ is a *-isomorphism from $L^\infty(X_1, \mu|_{X_1})$ onto $L^\infty(X_2, \mu|_{X_2})$. Since (X, μ) is standard Borel, so is each of its subspaces (see, e.g., [23, Corollary A.14]). Therefore by [45, Chapter IV, Lemma 8.22] the isomorphism $\mathrm{Ad}(v)$ induces a Borel isomorphism $\Phi : Y_2 \to Y_1$ between two Borel co-null subsets of X_2 and X_1, respectively, so that the pushforward measure $\Phi(\mu|_{X_2})$ is equivalent to $\mu|_{X_1}$. Moreover, by the same lemma, for $a \in L^\infty(Y_1, \mu|_{X_1})$,

$$vav^*(x) = (a \circ \Phi)(x) \qquad \mu\text{-a.e. on } Y_2. \tag{2.64}$$

Following [13, p. 334] (and using the equivalence relation picture) we compute

$$vav^*(x) = \sum_y |v(x,y)|^2 a(y) = (a \circ \Phi)(x). \tag{2.65}$$

Since a is arbitrary, we conclude that $v(x,y) = 0$ unless $y = \Phi(x)$ and that the graph of Φ is contained in the equivalence relation $R_{G \curvearrowright X}$, i.e., $\Phi(x) = g.x$ for some $g \in G$. For each $g \in G$, define

$$Y_g := \{x \in Y_2 : \Phi(x) = g^{-1}.x\}. \tag{2.66}$$

Then $\{Y_g\}_{g \in G}$ partitions Y_2 into disjoint Borel sets and similarly the sets $\{g^{-1}.Y_g\}_{g \in G}$ form a Borel partition of Y_1. If we define

$$v_\Phi := \sum_{g \in G} \mathbf{1}_{Y_g} u_g, \tag{2.67}$$

where the convergence takes place in the strong* topology because of the pairwise orthogonality and Lemma 2.16, then a straightforward computation shows that $\mathrm{Ad}(v)$ and $\mathrm{Ad}(v_\Phi)$ agree on $\mathcal{D}$. In particular, $v_\Phi(x,y) = 0$ unless $y = \Phi(x)$ by the same argument as for v. Since

$$v_\Phi^* v(x,y) = \sum_z \overline{v_\Phi(z,x)} v(z,y) \delta(z,x), \tag{2.68}$$

the product $v_\Phi^* v$ assumes nonzero values only when $x = y$. It follows that $v_\Phi^* v$ belongs to $\mathcal{D}$ and hence there is a norm-one measurable function d such that $v_\Phi^* v = d$. It follows that

$$v = vv^*v = v_\Phi v_\Phi^* v = v_\Phi d. \tag{2.69}$$

Putting everything together we arrive at

$$r = |r^*|v = |r^*|v_\Phi d = \sum_{g \in G} f_g u_g, \tag{2.70}$$

for some $f_g \in L^\infty(X,\mu)$ supported in the set Y_g, such that the collection $\{f_g\}_{g \in G}$ is $\|\cdot\|_\infty$-bounded. Since the Y_g are disjoint, the f_g are orthogonal; likewise, since the $g^{-1}.Y_g$ are disjoint, the $\alpha_g^{-1}(f_g)$ are orthogonal. □

Proposition 2.20. *Let G be a countable discrete amenable group, let X be a compact metrizable space, and let $\alpha : G \curvearrowright X$ be a free action. Then for every state f on $C(X) \rtimes_\alpha G$ the unit ball of $\pi_f(\mathcal{N}_{C(X) \rtimes_\alpha G}(C(X)))$ is strong*-dense in the unit ball of $\mathcal{N}_{\pi_f(C(X) \rtimes_\alpha G)''}(\pi_f(C(X))'')$.*

Proof. Fix a state f on $C(X) \rtimes_\alpha G$. As in the proof of Proposition 2.17 we identify the sub-von Neumann algebra $(\pi_f(C(X))'' \subseteq \pi_f(C(X) \rtimes_\alpha G)'')$ with either $(L^\infty(X, \bar\mu) \subseteq L^\infty(X, \bar\mu) \rtimes_{\bar\alpha} G)$ or $(L^\infty(X, \bar\mu) \subseteq \mathbf{M}(R))$.

Let $r \in \mathcal{N}_{L^\infty(X,\mu)\rtimes_\alpha G}(L^\infty(X,\mu))$. By Lemma 2.19, we obtain a family $f_g \in L^\infty(X,\bar{\mu})$ such that

$$r = \sum_{g\in G} f_g u_g.$$

The fact that such r can be approximated by normalizers of $C(X)$ in $C(X) \rtimes_\alpha G$ is essentially a consequence of Lusin's theorem; although the argument is entirely elementary, for the reader's convenience we include the details. Since the sum $\sum_{g\in G} f_g u_g$ is strong* convergent it is enough to approximate a finite sum $\sum_{g\in F} f_g u_g$, where F is a finite subset of G. Let $\xi \otimes \delta_s$ be a vector in $L^2(X,\bar{\mu}) \otimes \ell^2(G)$, let $\epsilon > 0$, and let $\delta > 0$ be a small constant yet to be determined. For each $g \in F$ we apply Lusin's theorem (see, for example, [14, Theorem 7.10]) to the function f_g and the pushforward measure $(gs).\bar{\mu}$ to obtain a compactly supported continuous function h_g whose support is contained in an open set U_g such that $\|h_g\|_\infty \leq \|f_g\|_\infty$ and

$$(gh).\bar{\mu}\left(\{x \in X : h_g(x) \neq f_g(x)\}\right) < \delta. \tag{2.71}$$

Since X is normal, the proof of [14, Theorem 7.10] can be modified in a way that the open sets $\{U_g\}_{g\in F}$ are pairwise disjoint. Upon shrinking the sets further, we may also assume pairwise disjointness for the collection $\{g^{-1}.U_g\}_{g\in F}$. In particular the element $\sum_{g\in F} h_g u_g$ normalizes $C(X)$ in $C(X) \rtimes_\alpha G$.

To see that $\sum_{g\in F} h_g u_g$ approximates $\sum_{g\in F} f_g u_g$ on the vector $\xi \otimes \delta_h$, we compute

$$\begin{aligned}
&\left\| \left(\sum_{g\in F} h_g u_g - \sum_{g\in F} f_g u_g \right) (\xi \otimes \delta_s) \right\| \\
&\qquad = \left\| \sum_{g\in F} (h_g - f_g)(\xi \otimes \delta_{gs}) \right\| \\
&\qquad = \left\| \sum_{g\in F} \alpha_{s^{-1}g^{-1}}(h_g - f_g)\xi \otimes \delta_{gs} \right\| \\
&\qquad = \sum_{g\in F} \|\alpha_{s^{-1}g^{-1}}(h_g - f_g)\xi\|.
\end{aligned} \tag{2.72}$$

Note that $\alpha_{s^{-1}g^{-1}}(h_g - f_g) \neq 0$ precisely on the set $s^{-1}g^{-1}.\{x \in X : h_g(x) \neq f_g(x)\}$. By construction this set has $\bar{\mu}$-measure at most δ, so by Lemma 2.16 the expression (2.72) is at most ϵ provided δ is sufficiently small. Although we have only dealt with one vector and only the strong (instead of strong*) operator topology, the same argument works for finitely many vectors and the adjoint because Lusin's theorem can be applied to finitely many equivalent measures at the same time. Finally, since all the operators considered are uniformly bounded in norm, the approximation in fact works for the entire Hilbert space $L^2(X, \bar{\mu}) \otimes \ell^2(G)$. □

Proof of Theorem C. Proposition 2.17 and Proposition 2.20 show that the sub-C^*-algebra $(C(X) \subseteq C(X) \rtimes_\alpha G)$ satisfies the assumptions of Theorem 2.12. The conclusion of that theorem matches the conclusion of Theorem C. □

2.4. **Uniform property Γ implies CPoU.** In order to prove that uniform property Γ implies CPoU, we adapt the proof of [5, Theorem 3.8], using Theorem 2.12 in place of [3, Theorem 3.1].

Lemma 2.21 (cf. [5, Lemma 3.6])**.** *Let $(D \subseteq A)$ be a sub-C^*-algebra with A separable, $T(A)$ nonempty and compact, and D abelian. Suppose $(D \subseteq A)$ satisfies the following two properties for every state f on A:*

(1) $(\pi_f(D)'' \subseteq \pi_f(A)'')$ is hyperfinite, and
(2) the unit ball of $\pi_f(\mathcal{N}_A(D))$ is strong-dense in the unit ball of $\mathcal{N}_{\pi_f(A)''}(\pi_f(D)'')$.*

Let $S \subseteq A^\omega$ be a $\|\cdot\|_{2,T_\omega(A)}$-separable subset and let $q \in A^\omega \cap A'$ be a projection such that there exists $\mu \in (0,1]$ with $\tau(q) = \mu$ for all $\tau \in T_\omega(A)$. Let $a_1, \ldots, a_k \in A_+$ and $\delta > 0$ be such that

$$\delta > \sup_{\tau \in T(A)} \min_{i=1,\ldots,k} \tau(a_i). \tag{2.73}$$

Then there exist positive contractions $b_1, \ldots, b_k \in ((D, T_\omega(A)|_D)^\omega \cap S'$ such that

$$\tau\Big(\sum_{i=1}^{k} b_i q\Big) = \lambda, \qquad \tau \in T_\omega(A), \text{ and} \tag{2.74}$$

$$\tau(a_i b_i q) \leq \delta \tau(b_i q), \qquad \tau \in T_\omega(A),\ i = 1, \ldots, k. \tag{2.75}$$

Proof. The proof follows that of [5, Lemma 3.6], using Lemma 2.13 in place of [5, Lemma 3.5]. The elements $f_m^{(i)}$ defined in [5, Equation (3.29)] belong to D_{F_m}. Since each ϕ_m maps D_{F_m} into D, each $b_m^{(i)} := \phi_m(f_m^{(i)})$, as given by [5, Equation (3.29)], is in D. It follows that the elements $b_i := (b_m^{(i)})_{m=1}^\infty \in A^\omega$ defined in the line after [5, Equation (3.29)] indeed belong to $(D, T(A)|_\omega)^\omega$. □

Theorem 2.22. *Let $(D \subseteq A)$ be a sub-C^*-algebra with A separable, $T(A)$ nonempty and compact, and D abelian. Suppose $(D \subseteq A)$ is nondegenerate and satisfies the following two properties for every state f on A:*

(1) $(\pi_f(D)'' \subseteq \pi_f(A)'')$ is hyperfinite, and
(2) the unit ball of $\pi_f(\mathcal{N}_A(D))$ is strong-dense in the unit ball of $\mathcal{N}_{\pi_f(A)''}(\pi_f(D)'')$.*

If $(D \subseteq A)$ has uniform property Γ then it has CPoU.

Proof. The proof is essentially the same as [5, Lemma 3.7], so we only outline the key steps and indicate necessary changes. Let $S \subseteq A^\omega$ be a $\|\cdot\|_{2,T_\omega(A)}$-separable subset that contains A, and let $a_1,...,a_k \in A_+$ and $\delta > 0$ be such that

$$\delta > \sup_{\tau\in T(A)} \min\{\tau(a_1),...,\tau(a_k)\}. \tag{2.76}$$

The main idea is to consider the set I of all $\alpha \in [0,1]$ such that there are pairwise orthogonal projections $p_1,...,p_k \in (D,T(A)|_D)^\omega \cap S'$ with

$$\tau(\sum_{i=1}^{k} p_i) = \alpha \qquad \text{and} \qquad \tau(a_ip_i) \leq \delta\tau(p_i) \qquad \tau \in T_\omega(A). \tag{2.77}$$

In other words, we are asking how much trace the projections with the desired property (from the definition of CPoU, Definition 2.3) can occupy. The goal is then to show that $\beta := \sup I = 1$, which is enough because I is closed by Kirchberg's ϵ-test.

The claim of $\beta = 1$ follows from a maximality argument; that is, we assume $\beta < 1$ and try to construct projections satisfying (2.77) in the orthogonal corner. Then summing the two families would yield projections that safisfy (2.77) and take up more traces than β and hence contradicts the assumption that β is the supremum.

To make this work, let $p_1^{(1)},...,p_k^{(1)}$ be pairwise orthogonal projections in $(D,T(A)|_D)^\omega \cap S'$ witnessing (2.77) for β. Using Lemma 2.21 we can produce positive elements $b_1^{(2)},...,b_k^{(2)}$ in $(D,T(A)|_D)^\omega \cap S' \cap \{q\}'$, where $q := \mathbf{1}_{A^\omega} - \sum p_i^{(1)}$, that satisfy the property of a CPoU, but within the cutdown by q. Note that q belongs to $(D,T(A)|_D)^\omega$ since $(D \subseteq A)$ is assumed to be nondegenerate. The projectionization lemma (Lemma 2.5), which is a consequence of uniform property Γ, then yields projections $p_1^{(2)},...,p_k^{(2)} \in (D,T(A)|_D)^\omega \cap S' \cap \{q\}'$ that replace the positive elements $b_1^{(2)},...,b_k^{(2)}$. Now the projections $p_i^{(1)} + qp_i^{(2)}$ almost work, except that $p_1^{(2)},...,p_k^{(2)}$ are not necessarily pairwise orthogonal. To remedy the issue we apply uniform property Γ one more time to obtain pairwise orthogonal projections $r_1,...,r_k \in (D,T(A)|_D)^\omega$ that commute with and tracially divide everything appearing thus far. Then as the calculations in [5, Lemma 3.7] showed, the projections

$$p_i := p_i^{(1)} + qp_i^{(2)}r_i \tag{2.78}$$

satisfy the desired condition in (2.77) and occupy more trace than β, so we have reached the intended contradiction. Note that p_i indeed belongs to $(D,T(A)|_D)^\omega \cap S'$ because all the components do. $\square$

Proof of Theorem 2.4 (ii) $\Rightarrow$ (iii)*.* the sub-C^*-algebra $(C(X) \subseteq C(X) \rtimes_\alpha G)$ is nondegenerate. By Proposition 2.17 and Proposition 2.20 it satisfies the assumptions of Theorem 2.22. $\square$

2.5. **CPoU implies the small boundary property.** We note the following useful criterion for the small boundary property. To the authors' knowledge, the result was conveyed by Gábor Szabó to the C^*-algebras community, although according to Szabó, it is likely well-known to experts in dynamics.

Proposition 2.23 ([33, Proposition 3.8]; see also [24, Theorem 5.4])**.** *Let G be a group, let X be a compact metric space, and let $\alpha : G \curvearrowright X$ be an action. Suppose that for every $\delta > 0, \epsilon > 0$ there is a collection $\mathcal{U}$ of open sets of X such that each $U \in \mathcal{U}$ has diameter at most δ, and for all α-invariant probability measures μ on X,*

$$\mu\Big(\bigcup_{U\in\mathcal{U}} U\Big) > 1 - \epsilon. \tag{2.79}$$

Then α has the small boundary property.

For a continuous function $f \in C_0(X)$ on a locally compact Hausdorff space X, we denote

$$\mathrm{supp}^\circ(f) := f^{-1}(\mathbb{C}\backslash\{0\}) \tag{2.80}$$

(the *open support* of f).

Proof of Theorem 2.4 (iii) $\Rightarrow$ (i). We shall verify the condition in Proposition 2.23, so let $\delta > 0$ and $\epsilon > 0$ be given. Let λ be a small constant yet to be determined. [33, Lemma 3.3] implies that for any α-invariant probability measure μ there is a finite collection $\mathcal{U}$ of disjoint open sets such that

(i) $\max_{U\in\mathcal{U}}\{\mathrm{diam}(U)\} < \delta$, and
(ii) $\mu(X\backslash \bigcup_{U\in\mathcal{U}} U) < \lambda$.

Since the action is free, there is a one-to-one correspondence between α-invariant probability measures and tracial states on $C(X) \rtimes_\alpha G$ (see, for example [16, Theorem 11.1.22]). By inner regularity of μ and Urysohn's lemma, the previous sentence can be translated to the following statement about continuous functions: for every $\tau \in T(C(X) \rtimes_\alpha G)$, there is a finite collection $\Omega_\tau \subseteq C(X)^1_+$ consisting of pairwise orthogonal members such that

(i) $\max_{f\in\Omega_\tau}\{\mathrm{diam}(\mathrm{supp}^\circ f)\} < \delta$, and
(ii) $\tau\left(\mathbf{1} - \sum_{f\in\Omega_\tau} f\right) < \lambda$.

Define $g_\tau := \mathbf{1} - \sum_{f\in\Omega_\tau} f$. Since $T(A)$ is compact, there exist $\tau_1, ..., \tau_k \in T(A)$ such that

$$\sup_{\tau\in T(A)} \min\{\tau(g_{\tau_1}), ..., \tau(g_{\tau_k})\} < \lambda. \tag{2.81}$$

Applying CPoU, we obtain pairwise orthogonal positive contractions $e_1, ..., e_k$ in $C(X)$ such that for every $\tau \in T(C(X) \rtimes_\alpha G)$ and $i \in \{1, ..., k\}$ one has

(i) $\tau\left(\sum_{i=1}^k e_i\right) > 1 - \frac{\epsilon}{4}$ and
(ii) $\tau(e_i g_{\tau_i}) < \lambda\tau(e_i) + \frac{\epsilon}{4(k+1)}$.

Consider the collection

$$\Omega := \{e_i f : i = 1, ..., k;\ f \in \Omega_{\tau_i}\} \subseteq C(X)^1_+. \tag{2.82}$$

Then Ω consists of pairwise orthogonal members. Moreover, for every $\tau \in T(C(X) \rtimes_\alpha G)$ one has

$$\begin{aligned}
\tau\left(\mathbf{1} - \sum_{i=1}^k \sum_{f\in\Omega_{\tau_i}} e_i f\right) &< \tau\left(\sum_{i=1}^k e_i\left(\mathbf{1} - \sum_{f\in\Omega_{\tau_i}} f\right)\right) + \frac{\epsilon}{4} \\
&= \tau\left(\sum_{i=1}^k e_i g_{\tau_i}\right) + \frac{\epsilon}{4} \\
&< \sum_{i=1}^k \lambda\tau(e_i) + k\cdot\frac{\epsilon}{4(k+1)} + \frac{\epsilon}{4} \\
&< \lambda + \lambda\cdot\frac{\epsilon}{4} + \frac{\epsilon}{2} < \epsilon, \qquad (2.83)
\end{aligned}$$

provided that λ is sufficiently small. Now take the collection

$$\mathcal{U} := \{\mathrm{supp}^\circ(e_i f) : i = 1, ..., k;\ f \in \Omega_{\tau_i}\}. \tag{2.84}$$

Then $\mathcal{U}$ consists of disjoint open sets whose diameters are uniformly bounded by δ. Finally, for every α-invariant probability measure μ one has

$$\mu\left(\bigcup_{U\in\mathcal{U}} U\right) \geq \tau_\mu\left(\sum_{i=1}^k \sum_{f\in\Omega_{\tau_i}} e_i f\right) > 1 - \epsilon. \tag{2.85}$$

Therefore $\mu\left(X \setminus \bigcup_{U\in\mathcal{U}} U\right) < \epsilon$, and we have verified the assumption of Proposition 2.23. □

3. Dynamical comparison

Given a group acting on a compact Hausdorff space, one obtains a preorder on the open sets of the space which encodes certain information about the dynamics. This preorder was first defined in talks of W. Winter, and features prominently in [22, 24]. An important application of it is to define a notion of "dynamical comparison" for such group actions.

In [32], a variant of the type semigroup was defined, building on this preorder; it was then shown that almost-unperforation of the type semigroup implies dynamical comparison.

We explore these concepts further in this section, and produce a C^*-algebraic characterization of the preorder, making use of "r-normalizers" in the crossed product.

For a continuous function $f \in C_0(X)$ on a locally compact Hausdorff space X, we recall that $\operatorname{supp}^\circ(f)$ is the open support of f, defined in (2.80).

Definition 3.1 ([32, Definitions 1.4 and 2.1])**.** *Let G be a countable discrete group, let X be a compact Hausdorff space, and let $\alpha : G \curvearrowright X$ be a free action by homeomorphisms. For a tuple of compact sets $F_1, \ldots, F_n \subseteq X$ and a tuple of open sets $V_1, \ldots, V_m \subseteq X$, write $(F_1, \ldots, F_n) \prec (V_1, \ldots, V_m)$ if there are open sets $U_{i,j} \subseteq X$, group elements $s_{i,j} \in G$, and indices $k_{i,j} \in \{1, \ldots, m\}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, J_i$ such that:*

(i) For each i,

$$F_i \subseteq U_{i,1} \cup \cdots \cup U_{i,J_i}, \tag{3.1}$$

(ii)

$$\coprod_{i=1}^{n} \coprod_{j=1}^{J_i} s_{i,j} U_{i,j} \times \{k_{i,j}\} \subseteq \coprod_{l=1}^{m} V_l \times \{l\}. \tag{3.2}$$

For $a = \operatorname{diag}(a_1, \ldots, a_n) \in (D_n \otimes C(X))_+$ and $b = \operatorname{diag}(b_1, \ldots, b_m) \in (D_m \otimes C(X))_+$, we write $a \preceq b$ if for every tuple of compact sets $F_1, \ldots, F_n$ with $F_i \subseteq \operatorname{supp}^\circ(a_i)$ for all i, we have $(F_1, \ldots, F_n) \prec (\operatorname{supp}^\circ(b_1), \ldots, \operatorname{supp}^\circ(b_m))$. Define $a \approx b$ to mean that both $a \preceq b$ and $b \preceq a$, and set

$$W(X, G) := \bigcup_{n=1}^{\infty} (D_n \otimes C(X))_+ / \approx . \tag{3.3}$$

For $a \in (D_n \otimes C(X))_+$, we use $[a]$ to denote its equivalence class in $W(X, G)$. The preorder $\preceq$ induces an order $\leq$ on $W(X, G)$, and there is a well-defined addition operation on $W(X, G)$ given by

$$[a] + [b] = [a \oplus b]. \tag{3.4}$$

Given $\tau \in T(C(X) \rtimes_\alpha G)$, and $a \in C(X)_+$, define

$$d_\tau(a) := \lim_{n \to \infty} \tau(a^{1/n}), \tag{3.5}$$

i.e., the value of the measure associated to τ evaluated on $\operatorname{supp}^\circ(a)$*. We say that the action α has* (dynamical) comparison *if for any $a, b \in C(X)_+$, if $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(C(X) \rtimes_\alpha G)$ then $a \preceq b$.*

In order to give this an algebraic formulation, we recall the following definition.

Definition 3.2. *Let $(D \subseteq A)$ be a sub-C^*-algebra. An element $a \in A$ is an r-*normalizer *of D if $a^* D a \subseteq D$. It is an s-*normalizer *of D if $a D a^* \subseteq D$.*

The set of r-normalizers of D in A is denoted $\mathcal{RN}_A(D)$, and the set of s-normalizers of D in A is denoted $\mathcal{SN}_A(D)$.

The names "r-normalizer" and "s-normalizer" is motivated by a connection to r- and s-sections for groupoids, established in Proposition 3.5 below. The following facts are evident:

- The product of two r-normalizers is an r-normalizer, and likewise for s-normalizers;
- $\mathcal{RN}_A(D) = \mathcal{SN}_A(D)^*$;
- an element is a normalizer if and only if it is both an r- and an s-normalizer.

Here is a useful descriptions of r-normalizers in matrix amplifications.

Lemma 3.3. *Let $(D \subseteq A)$ be a sub-C^*-algebra and let*

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in M_n \otimes A. \tag{3.6}$$

Then $x \in \mathcal{RN}_{M_n \otimes A}(D_n \otimes D)$ if and only if

(i) $x_{ij} \in \mathcal{RN}_A(D)$ for all i, j, and
(ii) for all i, j, k with $i \neq j$ and all $a \in D$, $x_{ki}^* a x_{kj} = 0$.

Proof. For $b = \operatorname{diag}(b_1, \ldots, b_n) \in D_n \otimes D$, we compute that the (i, j)-entry of x^*bx is

$$\sum_{k=1}^{n} x_{ki}^* b_k v_{kj}. \tag{3.7}$$

Suppose that $x \in \mathcal{RN}_{M_n \otimes A}(D_n \otimes D)$, so this must always be in D, and it must moreover be 0 whenever $i \neq j$. By setting $b_k := a$ and $b_l := 0$ for $l \neq k$, we thus get that $x_{ki}^* a x_{kj} \in D$, and is moreover 0 if $i \neq j$. This shows both (i) and (ii).

Conversely, suppose that (i) and (ii) hold. By (ii), it follows that the (i, j)-entry of x^*bx is 0 whenever $i \neq j$, i.e., x^*bx is a diagonal matrix. Moreover by (i), it follows that $x_{ki}^* b_k x_{ki} \in D$ for all i, k, and thus $x^*bx \in D_n \otimes D$. This shows that $x \in \mathcal{RN}_{M_n \otimes A}(D_n \otimes D)$. ☐

These "one-sided" normalizers are best understood in the context of groupoids. For a locally compact Hausdorff étale groupoid $\mathcal{G}$ we write $\mathcal{G}^{(0)}$ for its unit space, r and s for the range and source map, respectively. If x is a point in $\mathcal{G}^{(0)}$ then we write $\mathcal{G}_x := \{\gamma \in \mathcal{G} : s(\gamma) = x\}$ and $\mathcal{G}^x := \{\gamma \in \mathcal{G} : r(\gamma) = x\}$. We refer the readers to [44] for more on étale groupoids and their C^*-algebras.

Definition 3.4 ([39, Section 3])**.** *Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid. A subset A of $\mathcal{G}$ is an* r-section *if $r|_A : A \to \mathcal{G}^{(0)}$ is injective; it is an* s-section *if $s|_A : A \to \mathcal{G}^{(0)}$ is injective.*

We note for context that the more familiar notion of a *bisection* is a subset $A \subseteq \mathcal{G}$ which is both an r-section and an s-section.

It is known that normalizers can be characterized in terms of bisections (in the case of principal étale groupoids, see [27, Proposition 1.6]), a fact classically proven using the polar decomposition of a normalizer. We generalize this fact here to r-normalizers and r-sections; however, we give a completely different argument, since the partial isometry in the polar decomposition of an r-normalizer does not lead to an r-section.

Proposition 3.5. *Let $\mathcal{G}$ be a locally compact Hausdorff principal étale groupoid, and let $a \in C_r^*(\mathcal{G})$. Then $a \in \mathcal{RN}_{C_r^*(\mathcal{G})}(C_0(\mathcal{G}^{(0)}))$ if and only if $\operatorname{supp}^\circ(a)$ is an r-section. Likewise, $a \in \mathcal{SN}_{C_r^*(\mathcal{G})}(C_0(\mathcal{G}^{(0)}))$ if and only if $\operatorname{supp}^\circ(a)$ is an s-section.*

Proof. Using the adjoint, the second statement is equivalent to the first, which is the one we'll prove. Set $A := \operatorname{supp}^\circ(a)$.

For $f \in C_0(\mathcal{G}^{(0)})$ and $\gamma \in \mathcal{G}$, we compute

$$\begin{aligned}(a^* f a)(\gamma) &= \sum_{s(\alpha)=r(\gamma)} a^*(\alpha^{-1}) f(r(\alpha)) a(\alpha\gamma) \\ &= \sum_{s(\alpha)=r(\gamma)} \overline{a(\alpha)} f(r(\alpha)) a(\alpha\gamma), \qquad (3.8)\end{aligned}$$

and note that the summand can only be nonzero when both α and $\alpha\gamma$ are in A.

Thus, if A is an r-section, then nonzero summands can only arise if γ is a unit, so that $a^* f a \in C_0(\mathcal{G}^{(0)})$.

For the other direction, suppose for a contradiction that there exist distinct elements $\gamma_1, \gamma_2 \in A$ such that $r(\gamma_1) = r(\gamma_2)$; then we set

$$\gamma := \gamma_1^{-1}\gamma_2 \in \mathcal{G}\setminus\mathcal{G}^{(0)}. \qquad (3.9)$$

By [38, Proposition II.4.1 (i)], the sums $\sum_{s(\alpha)=r(\gamma)} |a(\alpha)|^2$ and $\sum_{s(\alpha)=r(\gamma)} |a(\alpha\gamma)|^2$ converge; thus by the Cauchy–Schwarz inequality, so does

$$\sum_{s(\alpha)=r(\gamma)} |a(\alpha)a(\alpha\gamma)|. \qquad (3.10)$$

Therefore we may find a finite set F of $\mathcal{G}_{r(\gamma)}$ such that

$$\sum_{\alpha \in G_{r(\gamma)}\setminus F} |a(\alpha)a(\alpha\gamma)| < |a(\gamma_1)a(\gamma_2)|. \qquad (3.11)$$

Choose a function $f \in C_0(\mathcal{G}^{(0)})$ of norm 1 such that $f(r(\gamma_1)) = 1$ and $f(r(\alpha)) = 0$ for $\alpha \in F\setminus\{\gamma_1\}$. (Since $\mathcal{G}$ is principal, $r(\gamma_1) \neq r(\alpha)$ for any $\alpha \in F\setminus\{\gamma_1\}$, so this is possible.) Then

$$\begin{aligned}|(a^* f a)(\gamma)| &\overset{(3.8)}{=} \Big|\sum_{s(\alpha)=r(\gamma)} \overline{a(\alpha)} f(r(\alpha)) a(\alpha\gamma)\Big| \\ &\overset{(3.11)}{>} \Big|\sum_{\alpha\in F} \overline{a(\alpha)} f(r(\alpha)) a(\alpha\gamma)\Big| - |a(\gamma_1)a(\gamma_2)|\end{aligned}$$

$$
\begin{aligned}
&= |\overline{a(\gamma_1)}f(r(\gamma_1))a(\gamma_1\gamma)| - |a(\gamma_1)a(\gamma_2)| \\
(3.12) \qquad &= 0.
\end{aligned}
$$

Since $\gamma \notin \mathcal{G}^{(0)}$, this implies that $a^* f a \notin C_0(\mathcal{G}^{(0)})$, which contradicts the hypothesis that $a \in \mathcal{RN}_{C_r^*(\mathcal{G})}(C_0(\mathcal{G}^{(0)}))$. □

Specializing the above to the case of interest in this paper – that $\mathcal{G}$ is a transformation groupoid $G \times X$ – yields the next corollary. In the following, for an element a of the crossed product $C(X) \rtimes_\alpha G$, we write

$$a = \sum_{g \in G} a_g u_g \tag{3.13}$$

to mean that $E(a u_g^*) = a_g$ (an element of $C(X)$) for all $g \in G$, where $E : C(X) \rtimes_\alpha G \to C(X)$ is the canonical conditional expectation. We do *not* mean that the sum converges in any sense.

Corollary 3.6. *Let G be a countable discrete group, let X be a compact Hausdorff space, let $\alpha : G \curvearrowright X$ be a free action, and let*

$$a = \sum_{g \in G} a_g u_g \in C(X) \rtimes_\alpha G. \tag{3.14}$$

Then $a \in \mathcal{RN}_{C(X)\rtimes_\alpha G}(C(X))$ if and only if the collection

$$\{\operatorname{supp}^\circ(a_g) : g \in G\} \tag{3.15}$$

is pairwise disjoint.

Proof. The crossed product $C(X) \rtimes_\alpha G$ is the groupoid C^*-algebra of the transformation groupoid $\mathcal{G} = G \times X$ (see [44, Example 2.1.15] for example), and upon making this identification, one can easily compute

$$\operatorname{supp}^\circ(a) = \bigcup_{g \in G} \{g\} \times g^{-1}.\operatorname{supp}^\circ(a_g).$$

Moreover, since $r(g, g^{-1}x) = x$ for $(g, x) \in \mathcal{G}$, this is an r-section if and only if $\{\operatorname{supp}^\circ(a_g) : g \in G\}$ is pairwise disjoint. □

We also use the above to give an interpretation of the conditions in Lemma 3.3 in the case of a free group action.

Corollary 3.7. *Let G be a countable discrete group, let X be a compact Hausdorff space, let $\alpha : G \curvearrowright X$ be a free action, and let*

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in M_n \otimes (C(X) \rtimes_\alpha G), \tag{3.16}$$

where for each $i, j = 1, \ldots, n$,

$$x_{ij} = \sum_{g \in G} x_{i,j,g} u_g. \tag{3.17}$$

Then $x \in \mathcal{RN}_{M_n \otimes (C(X) \rtimes_\alpha G)}(D_n \otimes C(X))$ *if and only if, for every* $i = 1, \ldots, n$*, the collection*

$$\{\operatorname{supp}^\circ(x_{i,j,g}) : j = 1, \ldots, n,\ g \in G\} \tag{3.18}$$

is pairwise disjoint.

Proof. Consider the action $(\pi \times \alpha) : \mathbb{Z}/n \times G \curvearrowright \{1, \ldots, n\} \times X$ where π is the canonical action of the cyclic group $\mathbb{Z}/n$ on $\{1, \ldots, n\}$; this product action is free since both π and α are. The sub-C^*-algebra $(D_n \otimes C(X) \subseteq M_n \otimes (C(X) \rtimes_\alpha G))$ identifies canonically with $(C(\{1, \ldots, n\} \times X) \subseteq C(\{1, \ldots, n\} \times X) \rtimes_{\pi\times\alpha} (\mathbb{Z}/n \times G))$, and this identification maps x to

$$y := \sum_{g \in G} \sum_{i,j=1}^{n} (\chi_{\{i\}} \otimes x_{i,j,g}) u_{(i-j,g)} \in C(\{1, \ldots, n\} \times X) \rtimes_{\pi\times\alpha} (\mathbb{Z}/n \times G). \tag{3.19}$$

Thus, $x \in \mathcal{RN}_{M_n \otimes (C(X) \rtimes_\alpha G)}(D_n \otimes C(X))$ if and only if

$$y \in \mathcal{RN}_{C(\{1,\ldots,n\}\times X) \rtimes_{\pi\times\alpha} (\mathbb{Z}/n \times G)}(C(\{1, \ldots, n\} \times X)). \tag{3.20}$$

By Corollary 3.6, this is equivalent to the collection

$$\{\operatorname{supp}^\circ(\chi_{\{i\}} \otimes x_{i,j,g}) : i, j = 1, \ldots, n,\ g \in G\} \tag{3.21}$$

of subsets of $\{1, \ldots, n\} \times G$ being pairwise disjoint. Since $\operatorname{supp}^\circ(\chi_{\{i\}} \otimes x_{i,j,g}) = \{i\} \times \operatorname{supp}^\circ(x_{i,j,g})$, this is the same as requiring that

$$\{\operatorname{supp}^\circ(x_{i,j,g}) : j = 1, \ldots, n,\ g \in G\} \tag{3.22}$$

be pairwise disjoint, for each i. □

Here is our algebraic characterization of the comparison $\preceq$ from Definition 3.1. Note that although $\preceq$ is defined for diagonal matrices in $C(X)$ of different sizes, we may always pad one of them with zeroes to arrange that they have the same size.

Proposition 3.8. *Let G be a countable discrete group, let X be a compact Hausdorff space, and let $\alpha : G \curvearrowright X$ be a free action. Let $a, b \in (D_n \otimes C(X))_+$. The following are equivalent:*

(i) $a \preceq b$;

(ii) there exists a sequence $(t_k)_{k=1}^\infty$ *in* $\mathcal{RN}_{M_n \otimes (C(X) \rtimes_\alpha G)}(D_n \otimes C(X))$ *such that*

$$\lim_{k \to \infty} \|t_k^* b t_k - a\| = 0; \tag{3.23}$$

(iii) For every $\epsilon > 0$ *there exists* $\delta > 0$ *and* $t \in \mathcal{RN}_{M_n \otimes (C(X) \rtimes_\alpha G)}(D_n \otimes C(X))$ *such that*

$$t^*(b - \delta)_+ t = (a - \epsilon)_+. \tag{3.24}$$

Proof. Let us write $a = \operatorname{diag}(a_1, \dots, a_n)$ and $b = \operatorname{diag}(b_1, \dots, b_n)$.

(i) $\Rightarrow$ (iii): This is a variant on the proofs of [22, Lemma 12.3] and [32, Proposition 2.3]. In both of those proofs, it is shown (roughly) that $a \preceq b$ implies that a is Cuntz below b in $C(X) \rtimes_\alpha G$;[3] the main novelty here is to verify that the Cuntz subequivalence can be witnessed using r-normalizers.

Let $\epsilon > 0$ be given. Set $F_i := \overline{\operatorname{supp}^\circ((a_i - \epsilon)_+)}$ for $i = 1, \dots, n$, so that F_i is a compact set contained in $\operatorname{supp}^\circ(a_i)$. Then apply Definition 3.1 to obtain open sets $U_{i,j} \subseteq X$, group elements $s_{i,j} \in G$, and indices $k_{i,j} \in \{1, \dots, m\}$ for $i = 1, \dots, n$ and $j = 1, \dots, J_i$ such that

$$\begin{aligned} &F_i \subseteq U_{i,1} \cup \cdots \cup U_{i,J_i}, \qquad i = 1, \dots, n, \text{ and} \\ &\coprod_{i=1}^{n} \coprod_{j=1}^{J_i} s_{i,j} U_{i,j} \times \{k_{i,j}\} \subseteq \coprod_{l=1}^{m} \operatorname{supp}^\circ(b_l) \times \{l\}. \end{aligned} \tag{3.25}$$

Next find open sets $V_{i,j}$ such that $\overline{V_{i,j}} \subseteq U_{i,j}$ and

$$F_i \subseteq V_{i,1} \cup \cdots \cup V_{i,J_i}, \quad i = 1, \dots, n. \tag{3.26}$$

It follows that

$$\coprod_{i=1}^{n} \coprod_{j=1}^{J_i} \overline{s_{i,j} V_{i,j}} \times \{k_{i,j}\} \subseteq \coprod_{l=1}^{m} \operatorname{supp}^\circ(b_l) \times \{l\}, \tag{3.27}$$

so by compactness of the left-hand side, there exists $\delta > 0$ such that

$$\coprod_{i=1}^{n} \coprod_{j=1}^{J_i} \overline{s_{i,j} V_{i,j}} \times \{k_{i,j}\} \subseteq \coprod_{l=1}^{m} \operatorname{supp}^\circ((b_l - 2\delta)_+) \times \{l\}. \tag{3.28}$$

By (3.26) and the definition of F_i, we may choose a continuous function $h_{i,j} \in C_0(V_{i,j})_+$ for each $i = 1, \dots, n$ and $j = 1, \dots, J_i$ such that

$$\sum_{j=1}^{J_i} h_{i,j}^2 = (a_i - \epsilon)_+, \quad i = 1, \dots, n. \tag{3.29}$$

Using functional calculus, let $\hat{b}_l \in C^*(b_l)$ be a function such that

$$\hat{b}_l(x)^2 (b_l - \delta)_+(x) = 1, \quad x \in \operatorname{supp}^\circ((b_l - 2\delta)_+). \tag{3.30}$$

Now define

$$t = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nn} \end{pmatrix} \in M_n \otimes (C(X) \rtimes_\alpha G) \tag{3.31}$$

[3] In [22, Lemma 12.3], the hypothesis is formally stronger than just $a \preceq b$.

by

$$t_{li} := \sum_{j:k_{i,j}=l} \hat{b}_l u_{s_{i,j}} h_{i,j} = \sum_{j:k_{i,j}=l} \hat{b}_l (h_{i,j} \circ \alpha_{s_{i,j}}^{-1}) u_{s_{i,j}}, \quad i,l = 1,\dots,n. \tag{3.32}$$

By (3.25) and since $h_{i,j} \in C_0(V_{i,j})$, for each l the collection

$$\{\mathrm{supp}^\circ(h_{i,j} \circ \alpha_{s_{i,j}}^{-1}) : k_{i,j} = l\} = \{\alpha_{s_{i,j}}(\mathrm{supp}^\circ(h_{i,j})) : k(i,j) = l\} \tag{3.33}$$

is pairwise disjoint. Therefore by Corollary 3.7, $t \in \mathcal{RN}_{M_n \otimes (C(X) \rtimes_\alpha G)}(D_n \otimes C(X))$.

In particular, $t^*(b-\delta)_+ t$ is a diagonal matrix. Moreover, for $i = 1,\dots,n$, we compute the (i,i)-entry of $t^*(b-\delta)_+ t$ to be

$$\begin{aligned} \sum_{k=1}^n t_{ki}^*(b_k - \delta)_+ t_{ki} &= \sum_{j,j':k_{i,j}=k_{i,j'}} h_{i,j} u_{s_{i,j}}^* \hat{b}_{k_{i,j}}^2 (b_{k_{i,j}} - \delta)_+ u_{s_{i,j'}} h_{i,j'} \\ &\overset{(3.30)}{=} \sum_{j,j':k_{i,j}=k_{i,j'}} u_{s_{i,j}}^* (h_{i,j} \circ \alpha_{s_{i,j}}^{-1})(h_{i',j'} \circ \alpha_{s_{i,j'}}^{-1}) u_{s_{i,j'}}. \end{aligned} \tag{3.34}$$

By pairwise disjointness of the collection (3.33), we have that $(h_{i,j} \circ \alpha_{s_{i,j}}^{-1})(h_{i,j'} \circ \alpha_{s_{i,j'}}) = 0$ whenever $k_{i,j} = k_{i,j'}$ and $j \neq j'$; thus the above simplifies to

$$\sum_{j=1}^{J_i} u_{s_{i,j}}^* (h_{i,j} \circ \alpha_{s_{i,j}}^{-1})^2 u_{s_{i,j}} \overset{(3.29)}{=} \sum_{j=1}^{J_i} h_{i,j}^2 = (a_i - \epsilon)_+, \tag{3.35}$$

as required.

(iii) $\Rightarrow$ (i): As in Definition 3.1, let F_i be a compact subset of $\mathrm{supp}^\circ(a_i)$ for $i = 1,\dots,n$. By compactness, there exists $\epsilon > 0$ such that $F_i \subseteq \mathrm{supp}^\circ((a_i - \epsilon)_+)$. By (iii), let $t \in \mathcal{RN}_{M_n \otimes (C(X) \rtimes_\alpha G)}(D_n \otimes C(X))$ and $\delta > 0$ be such that $t^*(b-\delta)_+ t = (a-\epsilon)_+$. Write

$$t = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nn} \end{pmatrix} \tag{3.36}$$

and for each i, j, write

$$t_{ij} = \sum_{g \in G} t_{i,j,g} u_g \tag{3.37}$$

where $t_{i,j,g} \in C(X)$ for each i, j, g. By Corollary 3.7, for each i

$$\{\mathrm{supp}^\circ(t_{i,j,g}) : j = 1,\dots,n,\ g \in G\} \text{ is pairwise disjoint.} \tag{3.38}$$

We now compute that $(a_i - \epsilon)_+$, which is the (i,i)-entry of $t^*(b-\delta)_+ t$, is equal to

$$\sum_{k=1}^n t_{ki}^*(b_k - \delta)_+ t_{ki} = \sum_{k=1}^n \sum_{g,h \in G} u_g^* t_{k,i,g}^* (b_k - \delta)_+ t_{k,i,h} u_h$$

$$\overset{(3.38)}{=} \sum_{k=1}^{n} \sum_{s \in G} u_s^* |t_{k,i,s}|^2 b_k u_s$$

$$= \sum_{k=1}^{n} \sum_{s \in G} (|t_{k,i,s}|^2 (b_k - \delta)_+) \circ \alpha_s. \tag{3.39}$$

Since $F_i \subseteq \mathrm{supp}^\circ((a_i - \epsilon)_+)$, it follows that

$$F_i \subseteq \bigcup_{k=1}^{n} \bigcup_{s \in G} \mathrm{supp}^\circ((|t_{k,i,s}|^2 b_k) \circ \alpha_s). \tag{3.40}$$

By compactness, we may choose $k_{i,1}, \dots, k_{i,J_i} \in \{1, \dots, n\}$ and $s_{i,1}, \dots, s_{i,J_i} \in G$ such that, upon setting

$$U_{i,j} := \mathrm{supp}^\circ((|t_{k_{i,j},i,s_{i,j}}|^2 b_{k_{i,j}}) \circ \alpha_{s_{i,j}}) = \alpha_{s_{i,j}}^{-1}(\mathrm{supp}^\circ(t_{k_{i,j},i,s_{i,j}}) \cap \mathrm{supp}^\circ(b_{k_{i,j}})), \tag{3.41}$$

we have

$$F_i \subseteq \bigcup_{j=1}^{J_i} U_{i,j}. \tag{3.42}$$

Also, the collection of sets of the form

$$\{k_{i,j}\} \times \alpha_{s_{i,j}}(U_{i,j}) = \{k_{i,j}\} \times (\mathrm{supp}^\circ(t_{k_{i,j},i,s_{i,j}}) \cap \mathrm{supp}^\circ(b_{k_{i,j}})) \tag{3.43}$$

(where i ranges over $\{1, \dots, n\}$ and j ranges over $\{1, \dots, J_i\}$) is contained in the collection of sets of the form

$$\{k\} \times (\mathrm{supp}^\circ(t_{k,i,s}) \cap \mathrm{supp}^\circ(b_k)). \tag{3.44}$$

Each of these is evidently contained in $\coprod_k \{k\} \times \mathrm{supp}^\circ(b_k)$, and by (3.38), they are pairwise disjoint. Therefore, we have

$$\coprod_{i,j} \{k_{i,j}\} \times \alpha_{s_{i,j}}(U_{i,j}) \subseteq \coprod_k \{k\} \times \mathrm{supp}^\circ(b_k), \tag{3.45}$$

as required.

(iii) $\Rightarrow$ (ii) is immediate.

(ii) $\Rightarrow$ (iii): Assume (ii) holds and let $\epsilon > 0$ be given. Then for some k we have $\|t_k^* b t_k - a\| < \epsilon/2$, and thus there exists $\delta > 0$ such that $\|t_k^*(b - \delta)_+ t_k - a\| < \epsilon$. Since $t_k \in \mathcal{RN}_{M_n \otimes (C(X) \rtimes_\alpha G)}(D_n \otimes C(X))$, it follows that $t_k^*(b - \delta)_+ t_k \in D_n \otimes C(X)$, so applying [40, Proposition 2.2] to this algebra (and using that it is commutative), we see that there exists $s \in D_N \otimes C(X)$ such that

$$s^* t_k^* (b - \delta)_+ t_k s = (a - \epsilon)_+. \tag{3.46}$$

Thus (iii) holds with $t := t_k s$. $\square$

Remark 3.9. As noted in the above proof, the argument for (iii) $\Rightarrow$ (i) is a variant on the proof of [22, Lemma 12.3]. We note for use later that, in fact, the element v constructed in the proof of [22, Lemma 12.3] is an r-normalizer, for example by writing it as

$$v = \sum_{i=1}^{n} ((fh_i)^{1/2} \circ \alpha_{s_i}^{-1}) u_{s_i} \tag{3.47}$$

(where we use $\alpha : G \curvearrowright X$ to denote the action) and then using Corollary 3.6.

4. Almost finiteness and tracial $\mathcal{Z}$-stability

In this section we prove an equivalence between almost finiteness – a dynamical property for a countable discrete amenable group acting by homeomorphisms defined by Kerr – and a version of $\mathcal{Z}$-stability for the crossed product that takes the Cartan subalgebra into account. We begin with the definition of our $\mathcal{Z}$-stability property.

In C^*-algebra theory, $\mathcal{Z}$-stability is the property of tensorially absorbing a certain canonical C^*-algebra, called the Jiang–Su algebra $\mathcal{Z}$. This C^*-algebra was defined in [20], but in practice it is often a McDuff-type characterization of $\mathcal{Z}$-stability that is used (see [50, Proposition 2.14], a combination of results by Rørdam–Winter [42], Kirchberg [25], and Toms–Winter [47]). Building on ideas of Matui and Sato ([35]), Hirshberg and Orovitz defined "tracial $\mathcal{Z}$-stability", an a priori weakening of this McDuff-type condition, and proved that it is equivalent to $\mathcal{Z}$-stability for simple separable unital nuclear C^*-algebras (see [19, Definition 2.1, Proposition 2.2, and Theorem 4.1]) (nuclearity is the key hypothesis that enables this equivalence). The following is a version of tracial $\mathcal{Z}$-stability for a sub-C^*-algebra, where a key role is played by (one-sided) normalizers of the smaller algebra; tracial $\mathcal{Z}$-stability for the algebra A is precisely the case $D = A$.

Definition 4.1. *Let $(D \subseteq A)$ be a sub-C^*-algebra with A unital and simple, and such that $\mathbf{1}_A \in D$. For $a, b \in D_+$, write $a \preceq_{(D\subseteq A)} b$ if there is a sequence $(t_k)_{k=1}^\infty$ in $\mathcal{RN}_A(D)$ such that $\lim_{k\to\infty} \|t_k^* b t_k - a\| = 0$.*

We say that $(D \subseteq A)$ is tracially $\mathcal{Z}$-stable *if for every $n \in \mathbb{N}$, every tolerance $\epsilon > 0$, every finite set $\mathcal{F} \subset A$, and every $h \in D_+ \backslash \{0\}$, there exists a c.p.c. order zero map $\phi : M_n \to A$ such that:*

(i) $\phi(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_A(D)$,
(ii) $\mathbf{1}_A - \phi(\mathbf{1}_n) \preceq_{(D\subseteq A)} h$,[4] and
(iii) $\|[a, \phi(x)]\| < \epsilon$ for all $a \in \mathcal{F}$ and every contraction $x \in M_n$.

The following definition of a "castle" is borrowed from David Kerr, except that (for later use) we allow the castle to possibly have infinitely many towers.

Definition 4.2 (cf. [22, Definitions 4.1 and 5.7])**.** *Let G be a countable discrete group, let X be a compact Hausdorff space, and let $\alpha : G \curvearrowright X$*

---

[4] By (i), it follows that $\phi(\mathbf{1}_n) \in D$, so this makes sense.

be a free action. A castle *is a collection* $\{(V_i, S_i)\}_{i\in I}$ *where each* V_i *is a subset of* X *and each* S_i *is a finite subset of* G*, such that the collection*

$$\{sV_i : s \in S_i,\, i \in I\} \tag{4.1}$$

is pairwise disjoint. Each (V_i, S_i) *is called a* tower*, the sets* S_i *are called* shapes*, and the sets* sV_i *(where* $s \in S_i$*) are called* levels *of the castle.*

We now recall the definition of almost finiteness for a group acting by homeomorphisms. Kerr's definition partially generalizes an earlier concept for locally compact étale groupoids with compact totally disconnected unit spaces, which Matui defined and used to prove strong results about the associated topological full group (see [34], particularly Definition 6.2).

Definition 4.3 ([22, Definition 8.2])**.** *Let* G *be a countable discrete group, let* X *be a compact metric space, and let* $\alpha : G \curvearrowright X$ *be a free action. The action is* almost finite *if for every finite subset* $K \subset G$*, and every* $\delta > 0$*, there exists:*

(i) a castle $\{(V_i, S_i)\}_{i\in I}$ *such that* I *is finite, each level is open with diameter at most* δ*, and each shape is* (K, δ)*-invariant (i.e.,* $|gS_i \triangle S_i|/|S_i| < \delta$ *for all* $g \in K$ *and all* $i \in I$*), and*

(ii) a set $S_i' \subseteq S_i$ *for each* $i \in I$ *such that* $|S_i'| < \delta|S_i|$ *and*

$$X \setminus \coprod_{i\in I} S_iV_i \prec \coprod_{i\in I} S_i'V_i, \tag{4.2}$$

using $\prec$ *from Definition 3.1.*

As an application of Theorem 2.4, we establish the following in this section.

Theorem 4.4. *Let* G *be a countable discrete amenable group, let* X *be a compact metrizable space, and let* $\alpha : G \curvearrowright X$ *be a minimal free action. The following are equivalent:*

(i) α *is almost finite;*

(ii) $(C(X) \subseteq C(X) \rtimes_\alpha G)$ *is tracially* $\mathcal{Z}$*-stable;*

(iii) $(C(X) \subseteq C(X) \rtimes_\alpha G)$ *has uniform property* Γ *and* α *has dynamical comparison.*

The above theorem has strong connections to the state-of-the-art in the Toms–Winter regularity conjecture in C^*-algebra theory. This conjecture predicts that three very different-looking regularity conditions coincide among the class of separable simple unital nuclear C^*-algebras. Two of these conditions, $\mathcal{Z}$-stability and strict comparison of positive elements, are directly analogous to the conditions of tracial $\mathcal{Z}$-stability for the sub-C^*-algebra and dynamical comparison for the action. The full equivalence between these two conditions, however, has not been fully established; the strongest general result is that uniform property Γ

(for the C^*-algebra) and strict comparison is equivalent to $\mathcal{Z}$-stability. This is a direct analogue of the equivalence of (ii) and (iii) in the above.

Interestingly, in the C^*-algebraic context, it is known that (for separable simple unital nuclear C^*-algebras), strict comparison can fail to hold ([48]), whereas it is open whether every nuclear C^*-algebra has uniform property Γ ([4, Question C]). On the other hand, (among amenable groups acting freely and minimally on compact metrizable spaces) uniform property Γ for the sub-C^*-algebra $(C(X) \subseteq C(X) \rtimes_\alpha G)$ can fail to hold,[5] while, to the authors' knowledge, no examples are known not to have dynamical comparison.

Our definition of tracial $\mathcal{Z}$-stability for a sub-C^*-algebra is largely inspired by Kerr's proof that almost finiteness implies $\mathcal{Z}$-stability of $C(X) \rtimes_\alpha G$ ([22, Theorem 12.4]), and indeed his proof shows the implication (i) ⇒ (ii) (we flesh out the details below). For the implication (iii) ⇒ (i), one need only combine Theorem 2.4 (iii) ⇒ (i) with Kerr and Szabó's proof that dynamical comparison combined with the small boundary property implies almost finiteness ([24, Theorem 6.1]).

The main content of this section is therefore the implication (ii) ⇒ (iii). We begin with proving uniform property Γ from tracial $\mathcal{Z}$-stability.

Lemma 4.5. *Let G be a countable discrete amenable group, let X be a compact metrizable space, and let $\alpha : G \curvearrowright X$ be a minimal free action. If $(C(X) \subseteq C(X) \rtimes_\alpha G)$ is tracially $\mathcal{Z}$-stable then it has uniform property Γ.*

Proof. Let $k \in \mathbb{N}$. Using tracial $\mathcal{Z}$-stability with any $h \in C(X)_+$ which is not Cuntz equivalent to $\mathbf{1}_{C(X)}$ and then setting $h_n := \phi(e_{11})$, we have

$$d_\tau(h_n) \leq \frac{1}{n}, \quad \tau \in T(C(X) \rtimes_\alpha G). \tag{4.3}$$

For notational convenience, set

$$A := C(X) \rtimes_\alpha G. \tag{4.4}$$

Let $(a_n)_{n=1}^\infty$ be a dense sequence in A.

Using the definition of tracial $\mathcal{Z}$-stability again – now with k in place of n, $\frac{1}{n}$ in place of ϵ, $\{a_1, \ldots, a_n\}$ in place of $\mathcal{F}$, and h_n in place of h – we obtain an order zero map $\phi_n : M_k \to A$ such that

$$\begin{aligned} \phi_n(D_k) &\subseteq C(X), \\ \mathbf{1}_A - \phi_n(\mathbf{1}_k) &\precsim_{(C(X)\subseteq A)} h_n, \quad \text{and} \end{aligned}$$

[5]By Theorem 2.4, uniform property Γ for the sub-C^*-algebra is equivalent to the small boundary property of the action. The small boundary property implies mean dimension zero ([30, Theorem 5.4]) (in fact, these conditions are equivalent, at least for minimal $\mathbb{Z}$-systems, by [29, Theorem 6.2]). There are minimal $\mathbb{Z}$-systems which don't have mean dimension zero ([30, Proposition 3.5]).

$$\|[a_i, \phi_n(x)]\| < \frac{1}{n}, \quad \text{for } i = 1, \ldots, n,\ x \in (M_k)^1. \tag{4.5}$$

The second of these conditions implies that $\tau(\phi_n(\mathbf{1}_k)) > 1 - \frac{1}{n}$ for all $\tau \in T(A)$.

Define $\Phi := (\phi_n)_{n=1}^{\infty} : M_k \to A^\omega$, an order zero map, and set

$$e_i := \Phi(e_{ii}), \quad i = 1, \ldots, k. \tag{4.6}$$

Since $\phi_n(e_{ii}) \in C(X)$ for all n, we have that $e_i \in (C(X), T(A)|_{C(X)})^\omega$. The tracial condition of ϕ_n ensures that $\tau(\Phi(\mathbf{1}_k)) = 1$ for all $\tau \in T_\omega(A)$, which implies that $\Phi(\mathbf{1}_k) = \mathbf{1}_{A^\omega}$,[6] so Φ is a unital *-homomorphism. By the approximate centrality of ϕ_n, the image of Φ commutes with A, and in particular, $e_i \in (C(X), T(A)|_{C(X)})^\omega \cap A'$.

Finally, for $\tau \in T_\omega(A)$ and $a \in A$, the map

$$x \mapsto \tau(\Phi(x)a) \tag{4.7}$$

defines a tracial functional on M_k, which is therefore a scalar multiple of the unique trace. Thus we have

$$\tau(\Phi(x)a) = \tau(\Phi(\mathbf{1}_k)a)\tau_{M_k}(x) = \tau(a)\tau_{M_k}(x), \quad x \in M_k. \tag{4.8}$$

Using $x := e_{ii}$, we obtain $\tau(e_i a) = \frac{1}{k}\tau(a)$, as required. □

We now turn to dynamical comparison. In [41, Corollary 4.6], Rørdam proved that simple exact unital $\mathcal{Z}$-stable C^*-algebra has strict comparison of positive elements. The following can be viewed as a dynamical analogue of this.

Lemma 4.6. *Let G be a countable discrete infinite amenable group, let X be a compact metrizable space, and let $\alpha : G \curvearrowright X$ be a minimal free action. If $(C(X) \subseteq C(X) \rtimes_\alpha G)$ is tracially $\mathcal{Z}$-stable then α has dynamical comparison.*

Proof. Let $f, g \in C(X)_+$, and assume that $d_\tau(f) < d_\tau(g)$ for all $\tau \in T(C(X) \rtimes_\alpha G)$. We need to show that $f \precsim g$ (as in Definition 3.1).

Since the action is free minimal and G is infinite, X has no isolated points. Choose any point $x_0 \in X$ such that $g(x_0) \neq 0$, let $h \in C(X)_+$ be a positive contraction which vanishes at x_0 and is nonzero everywhere else, and consider $g' := hg$. Since the action is minimal, $\mu(\{x_0\}) = 0$ for every G-invariant measure on X. Since the G-invariant probability measures on X correspond to traces on $C(X) \rtimes_\alpha G$ (see [16, Theorem 11.1.22], for example), it follows that $d_\tau(g) = d_\tau(g')$ for all $\tau \in T(C(X) \rtimes_\alpha G)$. Also, 0 is not an isolated point of the spectrum of g'. Thus by replacing g with g', we may assume that 0 is not an isolated point in the spectrum of g.

Now by [32, Theorem 3.8], and since the G-invariant probability measures on X correspond to traces on $C(X) \rtimes_\alpha G$, the condition

[6]This follows since $\|\mathbf{1}_{A^\omega} - \Phi(\mathbf{1}_k)\|_{2,T_\omega(A)}^2 \leq \|\mathbf{1}_{A^\omega} - \Phi(\mathbf{1}_k)\|\, \tau(\mathbf{1}_{A_\omega} - \Phi(\mathbf{1}_k)) = 0$, and $\|\cdot\|_{2,T_\omega(A)}$ is a norm on A^ω.

$d_\tau(f) < d_\tau(g)$ for all $\tau \in T(C(X) \rtimes_\alpha G)$ is equivalent to $(n+1)[f] \leq n[g]$ in $W(X,G)$, for some $n \in \mathbb{N}$.

Since $n[f] \leq (n+1)[f] \leq n[g]$, this means that $\mathbf{1}_{M_n} \otimes f \preceq \mathbf{1}_{M_n} \otimes g$. Let $\epsilon > 0$; we will show that there exists $t \in \mathcal{RN}_{C(X)\rtimes_\alpha G}(C(X))$ such that $\|t^*gt - f\| < \epsilon$, which suffices to show $f \preceq g$ by Proposition 3.8. By that same proposition, we have that there exists $\delta > 0$ and

$$v = \begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \cdots & v_{nn} \end{pmatrix} \in \mathcal{RN}_{M_n \otimes C(X)\rtimes_\alpha G}(D_n \otimes C(X)) \tag{4.9}$$

such that $v^*(\mathbf{1}_{M_n} \otimes (g-\delta)_+)v = \mathbf{1}_{M_n} \otimes (f - \frac{\epsilon}{2})_+$. By Lemma 3.3, we have

$$v_{ij} \in \mathcal{RN}_{C(X)\rtimes_{\bar\alpha} G}(C(X)), \qquad i,j = 1,\ldots,n, \text{ and} \tag{4.10}$$

$$v_{ki}^* a v_{kj} = 0, \qquad i \neq j,\ a \in C(X). \tag{4.11}$$

By looking at the entries of $v^*(\mathbf{1}_{M_n} \otimes (g-\delta)_+)v = \mathbf{1}_{M_n} \otimes (f - \frac{\epsilon}{2})_+$, we obtain for all j

$$\sum_{i=1}^n v_{ij}^*(g-\delta)_+ v_{ij} = (f - \frac{\epsilon}{2})_+. \tag{4.12}$$

Since 0 is not an isolated point of the spectrum of g, we may use functional calculus to find a nonzero element $d \in C^*(g)_+ \subseteq C(X)_+$, along with orthogonal elements $\hat{d}, \hat{g} \in C^*(g)_+ \subseteq C(X)_+$ such that

$$g\hat{d}^2 = d \quad \text{and} \quad g\hat{g}^2 = (g-\delta)_+. \tag{4.13}$$

Set

$$\eta := \frac{\epsilon}{8n^2+3} \tag{4.14}$$

and using tracial $\mathcal{Z}$-stability of $(C(X) \subseteq C(X) \rtimes_\alpha G)$, let $\phi : M_n \to C(X) \rtimes_\alpha G$ be an order zero map such that:

(i) $\phi(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_{C(X)\rtimes_\alpha G}(C(X))$,
(ii) $\mathbf{1}_A - \phi(\mathbf{1}_n) \preceq d$,[7] and
(iii) $\|[a, \phi(x)]\| < \eta$ for all $a \in \{v_{ij} : i,j = 1,\ldots,n\} \cup \{(g-\delta)_+\}$ and every contraction $x \in M_n$.

Define

$$r := \sum_{i,j=1}^n \phi(e_{ii}) v_{ij} \phi(e_{ij}). \tag{4.15}$$

Note that by (i), $\phi(e_{ii}) \in C(X)$. Using this, for $a \in C(X)$, we have

$$r^*ar = \sum_{i,j,k,l=1}^n \phi(e_{ji}) v_{ij}^* \phi(e_{ii}) a \phi(e_{kk}) v_{kl} \phi(e_{kl})$$

[7] Note that $\preceq_{(C(X)\subseteq C(X)\rtimes_\alpha G)}$ in Definition 4.1 is the same as $\preceq$ from Definition 3.1, by Proposition 3.8.

$$
\begin{aligned}
&= \sum_{i,j,k,l=1}^{n} \phi(e_{ji})v_{ij}^*\phi(e_{ii})\phi(e_{kk})av_{kl}\phi(e_{kl}) \\
&= \sum_{i,j,l=1}^{n} \phi(e_{ji})v_{ij}^*\phi(e_{ii})^2 av_{il}\phi(e_{il}) \\
&\overset{(4.11)}{=} \sum_{i,j=1}^{n} \phi(e_{ji})v_{ij}^*\phi(e_{ii})^2 av_{ij}\phi(e_{ij}), \qquad (4.16)
\end{aligned}
$$

and by (4.10), and (i), this is in $C(X)$. This shows that $r \in \mathcal{RN}_{C(X)\rtimes_\alpha G}(C(X))$.

Next, in the case that $a = (g-\delta)_+$, we get

$$
\begin{aligned}
r^*(g-\delta)_+ r &= \sum_{i,j=1}^{n} \phi(e_{ji})v_{ij}^*\phi(e_{ii})(g-\delta)_+ v_{ij}\phi(e_{ij}) \\
&\approx_{4n^2\eta} \sum_{i,j=1}^{n} \phi(e_{ji})\phi(e_{ii})^2\phi(e_{ij})v_{ij}^*(g-\delta)_+ v_{ij} \\
&= \sum_{i,j=1}^{n} \phi(e_{jj})^4 v_{ij}^*(g-\delta)_+ v_{ij} \\
&\overset{(4.12)}{=} \sum_{j=1}^{n} \phi(e_{jj})^4 (f-\frac{\epsilon}{2})_+ \\
&\approx_{\epsilon/2} \phi(\mathbf{1}_{M_n})^4 f. \qquad (4.17)
\end{aligned}
$$

Next, we note that $\mathbf{1}_{C(X)\rtimes_\alpha G} - \phi(\mathbf{1}_{M_n})$ and $\mathbf{1}_{C(X)\rtimes_\alpha G} - \phi(\mathbf{1}_{M_n})^4$ are Cuntz equivalent (in $C^*(\mathbf{1}_{C(X)\rtimes_\alpha G}, \phi(\mathbf{1}_{M_n}))$), and so combining this with (ii), we obtain some $s \in \mathcal{RN}_{C(X)\rtimes_\alpha G}(C(X))$ such that

$$
s^*ds \approx_\eta (\mathbf{1}_{C(X)\rtimes_\alpha G} - \phi(\mathbf{1}_{M_n})^4)f. \qquad (4.18)
$$

Define $t := \hat{g}r + \hat{d}s$ (using $\hat{g}, \hat{d}$ defined just above (4.13)). Since $\hat{g}, \hat{d}$ are orthogonal and in $C(X)$, it follows that $(\hat{g}r)^*a(\hat{d}s) = 0$ for all $a \in C(X)$, and thus $t \in \mathcal{RN}_{C(X)\rtimes_\alpha G}(C(X))$. Moreover,

$$
\begin{aligned}
t^*gt &= r^*\hat{g}g\hat{g}r + s^*\hat{d}g\hat{d}s \\
&\overset{(4.13)}{=} r^*(g-\delta)_+ r + s^*ds \\
&\approx_{4n^2\eta+\frac{\epsilon}{2}+\eta} \phi(\mathbf{1}_{M_n})^4 f + (1-\phi(\mathbf{1}_{M_n})^4)f = f. \qquad (4.19)
\end{aligned}
$$

Since $4n^2\eta + \eta < \frac{\epsilon}{2}$, we are done. □

We next take a closer look at the normalizer-preserving condition in the definition of $\mathcal{Z}$-stability for a sub-C^*-algebra. We shall consider a general construction of a c.p.c. map into a crossed product, given the following data. Let G be a countable discrete group, let X be a compact metrizable space, and let $\alpha : G \curvearrowright X$ be an action. Let

T be a countable set, and for each $t \in T$, let $f_t \in C(X)_+$ and let $S_t = \{s_{t,1}, \dots, s_{t,n}\}$ be a subset of G of size n. Suppose that:

(i) $\lim_{t\to\infty} \|f_t\| = 0$, and
(ii) $((\mathrm{supp}^\circ(f_t), S_t)_{t=1}^\infty$ is a castle.

Also for each $t \in T$ and $i = 1, \dots, n$, let $\theta_{t,i} : \mathrm{supp}^\circ(f_t) \to \mathbb{T}$ be a continuous function. Define $\phi : M_n \to C(X) \rtimes_\alpha G$ by

$$\phi(e_{ij}) := \sum_{t\in T} u_{s_{t,i}} \theta_{t,i} \bar{\theta}_{t,j} f_t u^*_{s_{t,j}} \tag{4.20}$$

and extending linearly. We call such a map a *castle order zero map.*

It is not hard to check the following facts.

(i) The sum defining $\phi(e_{ij})$ converges in norm.
(ii) ϕ is c.p.c. order zero and normalizer-preserving, i.e., $\phi(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_{C(X)\rtimes_\alpha G}(C(X))$.

Remark 4.7. We note that c.p.c. order zero, normalizer-preserving maps $M_n \to C(X) \rtimes_\alpha G$ arising earlier in this paper are always explicitly castle order zero maps (with a finite sum rather than an infinite sum). The next proposition will show that this is no coincidence.

We also note that the map φ defined in Equation (18) in the proof of [22, Theorem 12.4] is a castle order zero map (again with a finite sum). To see this, using the notation of that proof, set

$$\begin{aligned} I := \{(k, l, m, q, c, t) : k = 1, \dots, K,\ l = 1, \dots, L, \\ m = 1, \dots, M,\ c \in C^{(1)}_{k,l,m},\ q = 1, \dots, Q,\ t \in B_{k,l,c,q}\}, \end{aligned} \tag{4.21}$$

and for $(k, l, m, q, c, t) \in I$, set

$$\begin{aligned} S_{(k,l,m,q,c,t)} &:= \{t\Lambda_{k,1}(c), \dots, t\Lambda_{k,n}(c)\}, \\ f_{(k,l,m,q,c,t)} &:= \frac{q}{Q} h_k, \quad \text{and} \quad \theta_{t,i} \equiv 1. \end{aligned} \tag{4.22}$$

We note that for each k, the sets $S_{(k,l,m,q,c,t)}$ are pairwise disjoint[8] and contained in S_k (where S_k is defined in [22])[9] Since $h_k \in C_0(U_k)$ and the sets sU_k for $k = 1, \dots, K$ are pairwise disjoint, it follows that

$$(\mathrm{supp}^\circ(f_{(k,l,m,q,c,t)}), S_{(k,l,m,q,c,t)})_{(k,l,m,q,c,t)\in I} \tag{4.23}$$

[8]Suppose that $x \in S_{(k,l,m,q,c,t)} \cap S_{(k,l',m',q',c',t')}$. Then $x = t\Lambda_{k,i}(c) = t'\Lambda_{k,i'}(c')$ for some $i, i' \in \{1, \dots, n\}$. We have $t \in B_{k,l,c,q} \subseteq T'_{k,l,c} \subseteq T_{k,l,c}$ and $t' \in B_{k',l',c',q'} \subseteq T'_{k',l',c'} \subseteq T_{k',l',c'}$, while $\Lambda_{k,i}(c) \in C^{(i)}_{k,l,m} \subseteq C_{k,l}$ and $\Lambda_{k',i'}(c') \in C^{(i')}_{k',l',m'} \subseteq C_{k',l'}$. Since the collection of sets $T_{k,l,c}\gamma$ for $l = 1, \dots, L$ and $\gamma \in C_{k,l}$ are disjoint, it follows that $l = l'$, $t = t'$, and $\Lambda_{k,i}(c) = \Lambda_{k,i'}(c')$. Since $C^{(i)}_{k,l,m}$ for $i = 1, \dots, n$ and $m = 1, \dots, M$ are pairwise disjoint, it then follows that $i = i'$ and $m = m'$. Injectivity of $\Lambda_{k,i}$ then implies that $c = c'$. Finally, the sets $B_{k,l,c,q}$ for $q = 1, \dots, Q$ are pairwise disjoint, so $q = q'$.

[9]Continuing from the previous footnote, if $x = t\Lambda_{k,i}(c)$ for some $i \in \{1, \dots, n\}$ then $x \in T_{k,l,c}C_{k,l,m} \subseteq S_k$ (by the use of [22, Theorem 12.2] right after [22, Eq. (14)]).

is a castle, and so defines a castle order zero map.

To see that the map it defines is φ, using that $\Lambda_{k,i,j} = \Lambda_{k,i} \circ \Lambda_{k,j}^{-1}$ and that $\Lambda_{k,j} : C_{k,l,m}^{(1)} \to C_{k,l,m}^{(j)}$ is a bijection, we see that

$$\begin{aligned} h_{k,l,c,i,j} &= \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q}} \frac{q}{Q} u_{t\Lambda_{k,i,j}(c)c^{-1}t^{-1}}(h_k \circ \alpha_{tc}) \\ &= \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q}} u_{t\Lambda_{k,i}(c)} f_{(k,l,m,q,c,t)} u_{t\Lambda_{k,j}(c)}^{*}. \end{aligned} \tag{4.24}$$

Thus

$$\varphi(e_{ij}) = \sum_{(k,l,m,q,c,t) \in I} u_{t\Lambda_{k,i}(c)} f_{(k,l,m,q,c,t)} u_{t\Lambda_{k,j}(c)}^{*}. \tag{4.25}$$

Proposition 4.8. *Let G be a countable discrete amenable group, let X be a compact metrizable space, and let $\alpha : G \curvearrowright X$ be a free action. If $\phi : M_n \to C(X) \rtimes_\alpha G$ is a c.p.c. order zero map such that $\phi(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_{C(X)\rtimes_\alpha G}(C(X))$, then ϕ is a castle order zero map.*

Proof. Let

$$\phi(e_{1i}) = \sum_{g \in G} h_{i,g} u_g^{*} \tag{4.26}$$

for $h_{i,g} \in C(X)$. Since $\phi(e_{1i})$ is a normalizer, by Corollary 3.6, the collections $\{\mathrm{supp}^\circ(h_{i,g}) : g \in G\}$ and $\{\mathrm{supp}^\circ(h_{i,g} \circ \alpha_g^{-1}) : g \in G\}$ are both pairwise disjoint. Since ϕ is order zero, we have

$$\phi(e_{11}) = (\phi(e_{1i})\phi(e_{1i})^*)^{\frac{1}{2}} = \sum_{g \in G} |h_{i,g}| \tag{4.27}$$

(where the sum is orthogonal and norm-convergent). Since this is true for all i, it follows that we may find a pairwise disjoint family $(f_t)_{t \in T}$ in $C(X)_+$ (indexed by some countable set T) along with a function $s : T \times \{1, \dots, n\} \to G$ such that

$$|h_{i,g}| = \sum_{t \in T : s(t,i) = g} f_t. \tag{4.28}$$

Since the orthogonal sum $\sum_{g \in G} |h_{i,g}| = \sum_{t \in T} f_t$ converges, we must have $\|f_t\| \to 0$ as $t \to \infty$. For each $t \in T$ and $i = 1, \dots, n$, we have that $f_t = |h_{i,s(t,i)}|$ on $\mathrm{supp}^\circ(f_t)$ (by the orthogonality of the f_t), so we may define $\theta_{t,i} : \mathrm{supp}^\circ(f_t) \to \mathbb{T}$ by

$$\theta_{t,i}(x) := \frac{f_t(x)}{h_{i,s(t,i)}(x)}, \tag{4.29}$$

and this is a continuous function. We obtain

$$h_{i,g} = \sum_{t \in T : s(t,i) = g} \bar{\theta}_{t,i} f_t. \tag{4.30}$$

Since $\phi(e_{11}) \in C(X)_+$, we have $s(t,1) = e$ and $\theta_{t,1} \equiv 1$ for all $t \in T$. We may therefore rewrite

$$\phi(e_{1j}) = \sum_{t\in T} \bar{\theta}_{t,j} f_t u^*_{s(t,j)} = \sum_{t\in T} u_{s(t,1)}\theta_{t,i}\bar{\theta}_{t,j} f_t u^*_{s(t,j)}. \tag{4.31}$$

Since ϕ is order zero, we also obtain

$$\phi(e_{ij}) := \sum_{t\in T} u_{s(t,i)}\theta_{t,i}\bar{\theta}_{t,j} f_t u^*_{s(t,j)}. \tag{4.32}$$

It remains only to show that when we set

$$S_t := \{s(t,1), \ldots, s(t,n)\}, \tag{4.33}$$

we have that $((\mathrm{supp}^\circ(f_t), S_t))_{t\in T}$ is a castle.

For this, first since

$$\phi(e_{i1}) = \sum_{t\in T} u_{s(t,i)}\theta_{t,i} f_t = \sum_{t\in T}((\theta_{t,i} f_t) \circ \alpha^{-1}_{s(t,i)}) u_{s(t,i)} \tag{4.34}$$

is a normalizer, it follows from Corollary 3.6 (and the fact that the f_t are orthogonal) that

$$\{\mathrm{supp}^\circ(f_t \circ \alpha^{-1}_{s(t,i)}) : t \in T\} = \{\alpha_{s(t,i)}(\mathrm{supp}^\circ(f_t)) : t \in T\} \tag{4.35}$$

is pairwise orthogonal. Moreover, we compute

$$\phi(e_{ii}) = \sum_{t\in T} u_{s(t,i)} f_t u^*_{s(t,i)} = \sum_{t\in T} f_t \circ \alpha^{-1}_{s(t,i)}, \tag{4.36}$$

so using the orthogonality of the above family,

$$\mathrm{supp}^\circ(\phi(e_{ii})) = \coprod_{t\in T} \alpha_{s(t,i)}(\mathrm{supp}^\circ(f_t)). \tag{4.37}$$

Since ϕ is order zero, we know that $\phi(e_{ii})$ and $\phi(e_{jj})$ are orthogonal for all $i \neq j$. Consequently, we find that the entire family

$$\{\alpha_{s(t,i)}(\mathrm{supp}^\circ(f_t)) : t \in T, i = 1, \ldots, n\} \tag{4.38}$$

is pairwise disjoint, which means that $((\mathrm{supp}^\circ(f_t), S_t)_{t\in T}$ is a castle. □

Proof of Theorem 4.4. As explained earlier, (iii) ⇒ (i) is simply a combination of Theorem 2.4 (iii) ⇒ (i) with [24, Theorem 6.1]. (ii) ⇒ (iii) is a combination of Lemmas 4.5 and 4.6.

(i) ⇒ (ii): The proof of [22, Theorem 12.4] essentially shows this implication, although since tracial $\mathcal{Z}$-stability for a sub-C^*-algebra is not defined there, it is not explicitly stated in this way. Let us explain carefully how to obtain (ii) from the proof of [22, Theorem 12.4].

The proof begins with $a \in C(X)_+$ nonzero, a finite set Γ of the unit ball of $C(X)$, a finite symmetric set F of G, and a tolerance $\epsilon > 0$. It produces a c.p.c. order zero map $\phi : M_n \to C(X) \rtimes_\alpha G$ such that:

- $\mathbf{1}_{C(X)\rtimes_\alpha G} - \phi(\mathbf{1}_n)$ is Cuntz subequivalent to a, and
- $\|[x, \phi(b)]\| < \epsilon$ for all $x \in \Gamma \cup \{u_g : g \in F\}$.

As argued in the proof of [22, Theorem 12.4], since the crossed product is generated by $C(X)$ and the canonical unitaries, given any finite subset F' of $C(X) \rtimes_\alpha G$, by choosing Γ, F, and ϵ appropriately, the second condition will imply that $\|[x, \phi(b)]\| < \epsilon$ for all $x \in F'$. By Remark 3.9, the order zero map ϕ is in fact a "castle order zero map" and therefore $\phi(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_{C(X)\rtimes_\alpha G}(C(X))$. Finally, we note that the first of the above conditions is obtained in the proof of [22, Theorem 12.4] by invoking [22, Lemma 12.3], and so by Remark 3.9, we get the stronger conclusion that $\mathbf{1}_{C(X)} - \phi(\mathbf{1}_n) \precsim_{(C(X)\subseteq C(X)\rtimes_\alpha G)} a$. Hence, we get precisely our definition of $(C(X) \subseteq C(X) \rtimes_\alpha G)$ being tracially $\mathcal{Z}$-stable, as required. $\square$

5. Uniform property Γ and CPoU for sub-C^*-algebras

For a separable nuclear C^*-algebra A with $T(A)$ compact and nonempty, it was proved in [4] that uniform property Γ is equivalent to CPoU. Moreover, either of these properties is equivalent to a McDuff-type condition which says that any matrix algebra can be unitally embedded into $A^\omega \cap A'$ (see [4, Theorem 4.6]). In this section we establish a completely analogous result for a sub-C^*-algebra, which in particular generalizes the equivalence between (ii) and (iii) in Theorem 2.4.

Theorem 5.1. *Let $(D \subseteq A)$ be a nondegenerate sub-C^*-algebra with A separable, $T(A)$ nonempty and compact, and D abelian. Suppose A has no nonzero finite-dimensional quotients and suppose $(D \subseteq A)$ satisfies the following two properties for every state f on A:*

1. *$(\pi_f(D)'' \subseteq \pi_f(A)'')$ is hyperfinite, and*
2. *the unit ball of $\pi_f(\mathcal{N}_A(D))$ is strong*-dense in the unit ball of $\mathcal{N}_{\pi_f(A)''}(\pi_f(D)'')$.*

Then the following are equivalent:

- *(i) $(D \subseteq A)$ has CPoU;*
- *(ii) for every $n \in \mathbb{N}$ there is a unital embedding $\phi : M_n \to A^\omega \cap A'$ such that $\phi(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_{A^\omega\cap A'}((D, T(A)|_D)^\omega \cap A')$;*
- *(iii) $(D \subseteq A)$ has uniform property Γ.*

The implication (iii) $\Rightarrow$ (i) is Theorem 2.22, and (ii) $\Rightarrow$ (iii) is straightforward. Indeed given $k \in \mathbb{N}$ there is a unital embedding $\phi : M_k \to A^\omega \cap A'$ such that $\phi(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_{A^\omega\cap A'}((D, T(A)|_D)^\omega \cap A')$. Then the projections $e_i := \phi(e_{ii})$ for $i = 1, ..., k$ belong to the subalgebra $(D, T(A)|_D)^\omega \cap A'$ and form a partition of unity for $A^\omega \cap A'$. Moreover, since M_n has a unique tracial state, for every $\tau \in T_\omega(A)$ and $a \in A$ we have

$$\tau(e_i a) = \tau(\phi(e_{ii})a) = \tau(a)\tau_{M_n}(e_{ii}) = \frac{1}{k}\tau(a),$$

which shows that $(D \subseteq A)$ has uniform property Γ.

The rest of the section will therefore focus on proving (i) $\Rightarrow$ (ii). The basic idea is the same as in [4]: establish McDuffness for each tracial GNS representation, and then glue the fiberwise result using CPoU.

The next few lemmas are subalgebra versions of well-known results in the theory of tracial von Neumann algebras.

Lemma 5.2. *Let $(D \subseteq A)$ be a sub-C^*-algebra and let $F \subseteq A$ be a finite-dimensional subalgebra with a maximal abelian subalgebra D_F such that $\mathcal{N}_F(D_F) \subseteq \mathcal{N}_A(D)$. If we identify $F = \bigoplus_{k=1}^K M_{n_k}$ and write $e_{ij}^{(k)}$ for the matrix units with respect to D_F, then there is a *-isomorphism*

$$\bigoplus_{k=1}^{K} e_{11}^{(k)} A e_{11}^{(k)} \cong A \cap F'$$

which maps $\bigoplus_{k=1}^K e_{11}^{(k)} D e_{11}^{(k)}$ onto $D \cap F'$.

Proof. The isomorphism is given by [2, Equation (3.47)]. More precisely, the map $\varphi : \bigoplus_{k=1}^K e_{11}^{(k)} A e_{11}^{(k)} \to A \cap F'$ defined by

$$\varphi(a_1, a_2, ..., a_K) := \sum_{k=1}^{K} \sum_{i=1}^{n_k} e_{i1}^{(k)} a_k e_{1i}^{(k)}.$$

is an isomorphism that preserves the diagonals because each $e_{i1}^{(k)}$ normalizes D in A. □

Lemma 5.3. *Let $(\mathcal{M}, \tau)$ be a separable nonabelian von Neumann algebra with a faithful normal tracial state τ, and let $\mathcal{D}$ be a maximal abelian subalgebra of $\mathcal{M}$. Then there exists a projection $p \in \mathcal{D}$ such that $p\mathcal{M}(1-p) \neq \{0\}$.*

Proof. Assume that $p\mathcal{M}(1-p) = \{0\}$ for all projections $p \in \mathcal{D}$. Then $px = pxp$ for every $x \in \mathcal{M}$ and projection $p \in \mathcal{D}$. Since $\mathcal{D}$ is hyperfinite, by [37, Lemma 1.2] (and the paragraph after that lemma), for every $\epsilon > 0$ we can find a partition of unity $(e_k)_{k=1}^m$ in $\mathcal{D}$ such that $\| \sum_{k=1}^m e_k x e_k - E_{\mathcal{D}}(x)\|_{2,\tau} < \epsilon$, where $E_{\mathcal{D}}$ is the canonical conditional expectation from $\mathcal{M}$ onto $\mathcal{D}$ ([45, Proposition 2.36]). It follows that

$$\|x - E_{\mathcal{D}}(x)\|_{2,\tau} = \left\| \sum_{k=1}^{m} e_k x - E_{\mathcal{D}}(x) \right\|_{2,\tau} = \left\| \sum_{k=1}^{m} e_k x e_k - E_{\mathcal{D}}(x) \right\|_{2,\tau} < \epsilon.$$

Since ϵ is arbitrary, we conclude that $x = E_{\mathcal{D}}(x)$ and hence $\mathcal{M} = \mathcal{D}$, a contradiction. □

Note that if $\mathcal{D}$ is maximal abelian in $\mathcal{M}$ and e is a projection in $\mathcal{D}$ then $e\mathcal{D}e$ is maximal abelian in $e\mathcal{M}e$. Indeed, let x be in the relative commutant $(e\mathcal{D}e)' \cap e\mathcal{M}e$. Then $(ede)x = x(ede)$ for all $d \in \mathcal{D}$. Since $ex = xe = x$ and e commutes with d, we must have $dx = xd$. Therefore $x \in \mathcal{D} \cap e\mathcal{M}e = e\mathcal{D}e$.

Lemma 5.4. *Let $\mathcal{D} \subseteq \mathcal{M}$ be an abelian subalgebra of a von Neumann algebra $\mathcal{M}$ such that $\mathcal{N}_{\mathcal{M}}(\mathcal{D})$ generates $\mathcal{M}$. Suppose $e, f \in \mathcal{D}$ are two projections such that $e\mathcal{M}f \neq \{0\}$. Then there is a nonzero partial isometry v in $\mathcal{N}_{\mathcal{M}}(\mathcal{D})$ such that $v^*v \leq e$ and $vv^* \leq f$.*

Proof. Let x be an element in $\mathcal{N}_{\mathcal{M}}(\mathcal{D})$ such that $r := exf \neq 0$. Let $r = v|r|$ be the unique polar decomposition in $\mathcal{M}$ such that $\ker v = \ker r$. Then by Lemma 2.18, v is a partial isometry in $\mathcal{N}_{\mathcal{M}}(\mathcal{D})$ that satisfies the desired properties. □

Recall from [13] that an abelian subalgebra $\mathcal{D}$ of a von Neumann algebra $\mathcal{M}$ is a *Cartan subalgebra* if

(1) $\mathcal{D}$ is maximal abelian;
(2) $\mathcal{N}_{\mathcal{M}}(\mathcal{D})$ generates $\mathcal{M}$ as a von Neumann algebra;
(3) there exists a faithful normal conditional expectation of $\mathcal{M}$ onto $\mathcal{D}$.

We note this is different than a Cartan subalgebra of a C^*-algebra, but it will be clear from context which type of Cartan subalgebra we mean.

Proposition 5.5. *(cf.* [45, Proposition V.1.35]*) Let $\mathcal{D}$ be a Cartan subalgebra of a type II_1 von Neumann algebra $\mathcal{M}$. Then for every $n \in \mathbb{N}$ there is a unital embedding $M_n \to \mathcal{M}$ such that $\mathcal{N}_{M_n}(D_n) \subseteq \mathcal{N}_{\mathcal{M}}(\mathcal{D})$.*

Proof. We only prove the proposition for $n = 2$ since the proof can be easily adapted to the general case (although it becomes notationally unpleasant). Let e be a nonzero projection in $\mathcal{D}$. Since $e\mathcal{M}e$ is non-abelian, applying Lemma 5.3 to the subalgebra $e\mathcal{D}e \subseteq e\mathcal{M}e$ we obtain a projection p in $e\mathcal{D}e$ such that $p\mathcal{M}(e-p) \neq \{0\}$. Hence by Lemma 5.4 there exists a partial isometry v in $\mathcal{N}_{e\mathcal{M}e}(e\mathcal{D}e) = e\mathcal{N}_{\mathcal{M}}(\mathcal{D})e \subseteq \mathcal{N}_{\mathcal{M}}(\mathcal{D})$ such that

$$v^*v \leq p \qquad \text{and} \qquad vv^* \leq e - p.$$

In other words, every nonzero projection in $\mathcal{D}$ has two nonzero orthogonal equivalent subprojections where the equivalence is witnessed by a normalizer.

Consider the collection of families $\{p_i, q_i : i \in I\}$ of mutually orthogonal projections where $p_i \sim q_i$ and the equivalence is witnessed by normalizers. Order the collection by set-theoretic inclusion. Then by Zorn's lemma there is a maximal family $\{p_j, q_j : j \in J\}$. If we set $p := \sum_j p_j$ and $q := \sum_j q$, then p and q are equivalent projections in $\mathcal{D}$. Moreover the equivalence is witnessed by a normalizer since the set of normalizers is closed under orthogonal sums. Finally, a standard maximality argument shows that $p + q = \mathbf{1}_{\mathcal{M}}$. □

Proposition 5.6. *Let $\mathcal{D}$ be a Cartan subalgebra of a type II_1 separable von Neumann algebra $\mathcal{M}$. Suppose $(\mathcal{D} \subseteq \mathcal{M})$ is hyperfinite (in the sense of Definition 2.9). Then for each $n \in \mathbb{N}$ there is a sequence of unital embeddings $\phi_i : M_n \to \mathcal{M}$ such that $\phi_i(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_{\mathcal{M}}(\mathcal{D})$ for* *each $i \in \mathbb{N}$ and the induced map $\Phi : M_n \to \mathcal{M}^\omega$ satisfies $\Phi(M_n) \subseteq \mathcal{M}^\omega \cap \mathcal{M}'$.*

Proof. Fix $n \in \mathbb{N}$. By hyperfiniteness there is an increasing sequence $(F_i)_{i=1}^\infty$ of finite-dimensional subalgebras of $\mathcal{M}$ with diagonals $D_{F_n} \subseteq F_n$ such that

$$\mathcal{M} = \left(\bigcup_i F_i\right)'', \qquad \mathcal{D} = \left(\bigcup_i D_{F_i}\right)'',$$

and $\mathcal{N}_{F_i}(D_{F_i}) \subseteq \mathcal{N}_{\mathcal{M}}(\mathcal{D})$ for all $i \in \mathbb{N}$. By Lemma 5.2 and Proposition 5.5, for each $i \in \mathbb{N}$ there is a unital embedding

$$\phi_i : M_n \to \mathcal{M} \cap F_i'$$

such that $\phi_i(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_{\mathcal{M}\cap F_i'}(\mathcal{D} \cap F_i') \subseteq \mathcal{N}_{\mathcal{M}}(\mathcal{D})$. Since the union $\bigcup_i F_i$ is dense in $\mathcal{M}$, the induced map $\Phi : M_n \to \mathcal{M}^\omega$takes values in the relative commutant. □

Proof of Theorem 5.1 (i) ⇒ (ii). The proof is essentially the same as [4, Lemma 4.2], with the assumption of $\pi_\tau(A)''$ being McDuff replaced by our Proposition 5.6. We include full details for the reader's convenience. Fix $n \in \mathbb{N}$. Let $\mathcal{F} \subseteq (M_n)^1$ be a finite generating set of M_n, $\mathcal{G} \subseteq A$ a finite subset, and $\epsilon \in (0, 1)$, and set $\delta := \frac{\epsilon}{5+|\mathcal{F}||\mathcal{G}|}$. Then by the proof of [5, Proposition 1.11] there is a positive contraction $e \in A$ such that $\|e - \mathbf{1}_{A^\omega}\|_{2,T_\omega(A)} < \delta$. We will produce a c.p.c. map $\phi : M_n \to A$ such that

(1) $\phi(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_A(D)$,
(2) $\|\phi(x)\phi(y)\| < \epsilon$ for all orthogonal positive elements x, y in $\mathcal{F}$,
(3) $\|[\phi(x), a]\|_{2,T(A)} < 3\epsilon^{\frac{1}{2}}$ for all $x \in \mathcal{F}$ and $a \in \mathcal{G}$, and
(4) $\|e - \phi(\mathbf{1}_{M_n})\|_{2,T(A)} < 3\epsilon^{\frac{1}{2}}$.

The existence of such a map then implies (ii) since the induced map from M_n into $A^\omega \cap A'$ would be a unital c.p.c. order zero map, which must be a *-homomorphism.

Let us fix a tracial state $\tau \in T(A)$ for now, and write $(\mathcal{D} \subseteq \mathcal{M})$ for the sub-von Neumann algebra $(\pi_\tau(D)'' \subseteq \pi_\tau(A)'')$, where π_τ is the GNS representation associated to τ. Since $\mathcal{M}$ is type II_1, by Proposition 5.6 there is a unital embedding $\theta : M_n \to \mathcal{M}$ such that $\theta(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_{\mathcal{M}}(\mathcal{D})$ and $\|[\theta(x), \pi_\tau(a)]\|_{2,\tau} < \delta$ for all $x \in \mathcal{F}$ and $a \in \mathcal{G}$. Define

$$B' := \{(a_m)_m \in \prod_m A : (\pi_\tau(a_m))_m \text{ converges strong}^*\}$$

and

$$D_{B'} := \{(d_m)_m \in \prod_m D : (\pi_\tau(d_m))_m \text{ converges strong}^*\}.$$

Then the strong* limit evaluation map $q' : B' \to \mathcal{M}$ is surjective by Kaplansky's theorem. Moreover, by hypothesis (2) in Theorem 5.1, q'

maps $\mathcal{N}_{B'}(D_{B'})$ onto $\mathcal{N}_{\mathcal{M}}(\mathcal{D})$. Therefore by Proposition 2.7, θ lifts to a c.p.c. order zero map

$$\tilde{\theta} = (\tilde{\theta}_m)_m : M_n \to B' \subseteq \prod_m A$$

such that $\tilde{\theta}_m(\mathcal{N}_{M_n}(D_n) \subseteq \mathcal{N}_A(D))$ for all $m \in \mathbb{N}$. By construction

$$\|[\tilde{\theta}_m(x), a]\|_{2,\tau} \xrightarrow{m\to\infty} \|[\theta(x), \pi_\tau(a)]\|_{2,\tau} < \delta$$

for all $x \in \mathcal{F}$ and $a \in \mathcal{G}$. Thus by setting $\phi_\tau := \tilde{\theta}_m$ for m sufficiently large we obtain a c.p.c. order zero map $\phi_\tau : M_n \to A$ such that

- $\phi_\tau(\mathcal{N}_{M_n}(D_n)) \subseteq \mathcal{N}_A(D)$,
- $\|[\phi_\tau(x), a]\|_{2,\tau} < \delta$ for all $x \in \mathcal{F}$ and $a \in \mathcal{G}$, and
- $1 - \tau(\phi_\tau(\mathbf{1}_{M_n})) < \delta$.

By compactness of $T(A)$ there are normalizer-preserving c.p.c. order zero maps $\phi_1, ..., \phi_k : M_n \to A$ such that for every $\tau \in T(A)$ there exists an $i \in \{1, ..., k\}$ with the properties

$$\|[\phi_i(x), a]\|_{2,\tau} < \delta, \qquad \text{for } x \in \mathcal{F},\ a \in \mathcal{G}$$

and

$$1 - \tau(\phi_i(\mathbf{1}_{M_n})) < \delta.$$

For this particular index i we have

$$\begin{aligned}
\tau(|e - \phi_i(\mathbf{1}_{M_n})|^2) &= \|e - \phi_i(\mathbf{1}_{M_n})\|_{2,\tau}^2 \\
&\leq \left(\|e - \mathbf{1}_{A^\omega}\|_{2,\tau} + \|\mathbf{1}_{A^\omega} - \phi_i(\mathbf{1}_{M_n})\|_{2,\tau}\right)^2 \\
&< [\delta + \tau((\mathbf{1}_{A^\omega} - \phi_i(\mathbf{1}_{M_n}))^2)^{\frac{1}{2}}]^2 \\
&\leq [\delta + \tau(\mathbf{1}_{A^\omega} - \phi_i(\mathbf{1}_{M_n}))^{\frac{1}{2}}]^2 \\
&= [\delta + (1 - \tau(\phi_i(\mathbf{1}_{M_n})))^{\frac{1}{2}}]^2 \\
&< (\delta + \delta^{\frac{1}{2}})^2 < 4\delta.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\tau\left(\sum_{x\in\mathcal{F}}\sum_{a\in\mathcal{G}} |[\phi_i(x), a]|^2\right) &= \sum_{x\in\mathcal{F}}\sum_{a\in\mathcal{G}} \tau\left(|[\phi_i(x), a]|^2\right) \\
&= \sum_{x\in\mathcal{F}}\sum_{a\in\mathcal{G}} \|[\phi_i(x), a]\|_{2,\tau}^2 \\
&< |\mathcal{F}| \cdot |\mathcal{G}| \cdot \delta^2.
\end{aligned}$$

From these computations we see that if we set

$$a_i := |e - \phi_i(\mathbf{1}_{M_n})|^2 + \sum_{x\in\mathcal{F}}\sum_{g\in\mathcal{G}} |[\phi_i(x), a]|^2 \in A_+$$

then

$$\sup_{\tau\in T(A)} \min\{\tau(a_1), ..., \tau(a_k)\} < 4\delta + |\mathcal{F}| \cdot |\mathcal{G}| \cdot \delta^2$$

$$< (4 + |\mathcal{F}| \cdot |\mathcal{G}|)\delta.$$

Since $(D \subseteq A)$ has CPoU, we can find pairwise orthogonal positive contractions $e_1, ..., e_k \in D_+$ such that

- $\max\{\|[e_i, b]\|, \|[e_i^{\frac{1}{2}}, b\|\} < \frac{\epsilon}{2k+1}$ for all $i \in \{1, ..., k\}$ and all $b \in \left(\bigcup_{i=1}^k \phi_i(\mathcal{F})\right) \cup \mathcal{G} \cup \{e\}$,
- $\tau(e_1 + \cdots + e_k) > 1 - \epsilon$ for all $\tau \in T(A)$, and
- $\tau(a_i e_i) < (4 + |\mathcal{F}| \cdot |\mathcal{G}|)\delta\tau(e_i) + \epsilon$ for all $i \in \{1, ..., k\}$ and $\tau \in T(A)$.

Define a c.p.c. map $\phi : M_n \to A$ by

$$\phi(x) := \sum_{i=1}^k e_i^{\frac{1}{2}} \phi_i(x) e_i^{\frac{1}{2}} \qquad (x \in M_n).$$

Let us show that ϕ satisfies properties (1) to (4). First of all, if v belongs to $\mathcal{N}_{M_n}(D_n)$ then the image $\phi(v) = \sum_{i=1}^k e_i^{\frac{1}{2}} \phi_i(v) e_i^{\frac{1}{2}}$ normalizes D because $\phi_i(v) \in \mathcal{N}_A(D)$ and the elements $e_i \in D$ are pairwise orthogonal.

For (2), let $x, y \in \mathcal{F}$ be orthogonal positive elements and compute

$$\begin{aligned}
\|\phi(x)\phi(y)\| &= \left\| \left(\sum_{i=1}^k e_i^{\frac{1}{2}} \phi_i(x) e_i^{\frac{1}{2}} \right) \left(\sum_{j=1}^k e_j^{\frac{1}{2}} \phi_j(y) e_j^{\frac{1}{2}} \right) \right\| \\
&= \left\| \sum_{i=1}^k e_i^{\frac{1}{2}} \phi_i(x) e_i \phi_i(y) e_i^{\frac{1}{2}} \right\| \\
&< \left\| \sum_{i=1}^k e_i^{\frac{1}{2}} [\phi_i(x), e_i] \phi(y) e_i^{\frac{1}{2}} \right\| + \left\| \sum_{i=1}^k e_i^{\frac{3}{2}} \phi_i(x) \phi_i(y) e_i^{\frac{1}{2}} \right\| \\
&< k \cdot \frac{\epsilon}{2k+1} + 0 < \epsilon.
\end{aligned}$$

To verify (3), let $x \in \mathcal{F}$ and $a \in \mathcal{G}$. We first make the following estimate for $\tau \in T(A)$:

$$\begin{aligned}
&\tau\left(\Big| \sum_{i=1}^k e_i^{\frac{1}{2}} [\phi_i(x), a] e_i^{\frac{1}{2}} \Big|^2 \right) \\
&\qquad = \tau\left(\sum_{i=1}^k |e_i^{\frac{1}{2}} [\phi_i(x), a] e_i^{\frac{1}{2}}|^2 \right) \\
&\qquad \leq \tau\left(\sum_{i=1}^k e_i^{\frac{1}{2}} |[\phi_i(x), a]|^2 e_i^{\frac{1}{2}} \right) \\
&\qquad \leq \sum_{i=1}^k \tau(a_i e_i) < \sum_{i=1}^k (4 + |\mathcal{F}| \cdot |\mathcal{G}|)\delta \cdot \tau(e_i) + \epsilon
\end{aligned}$$

$$< 2\epsilon.$$

Then

$$\begin{aligned}
\|[\phi(x),a]\|_{2,T(A)} &= \left\|\sum_{i=1}^{k}[e_i^{\frac{1}{2}}\phi_i(x)e_i^{\frac{1}{2}},a]\right\|_{2,T(A)} \\
&= \left\|\sum_{i=1}^{k}e_i^{\frac{1}{2}}[\phi_i(x),a]e_i^{\frac{1}{2}}\right\|_{2,T(A)} + (2k)\cdot\frac{\epsilon}{2k+1} \\
&< \sqrt{2}\epsilon^{\frac{1}{2}} + \epsilon < 3\epsilon^{\frac{1}{2}},
\end{aligned}$$

which is (3). For the last condition (4) we write

$$\begin{aligned}
&\|e-\phi(\mathbf{1}_{M_n})\|_{2,T(A)} = \left\|e-\sum_{i=1}^{k}e_i^{\frac{1}{2}}\phi_i(\mathbf{1}_{M_n})e_i^{\frac{1}{2}}\right\|_{2,T(A)} \\
&\leq \left\|e-\sum_{i=1}^{k}e_i^{\frac{1}{2}}ee_i^{\frac{1}{2}}\right\|_{2,T(A)} + \left\|\sum_{i=1}^{k}e_i^{\frac{1}{2}}(e-\phi_1(\mathbf{1}_{M_n}))e_i^{\frac{1}{2}}\right\|_{2,T(A)} \\
&< \left\|e-\left(\sum_{i=1}^{k}e_i\right)e\right\|_{2,T(A)} + \frac{k\epsilon}{2k+1} + \left\|\sum_{i=1}^{k}e_i^{\frac{1}{2}}(e-\phi_1(\mathbf{1}_{M_n}))e_i^{\frac{1}{2}}\right\|_{2,T(A)}.
\end{aligned}$$

Straightforward computations similar to the one for (3) show that

$$\left\|e-\left(\sum_{i=1}^{k}e_i\right)e\right\|_{2,T(A)} < \epsilon^{\frac{1}{2}}$$

and

$$\left\|\sum_{i=1}^{k}e_i^{\frac{1}{2}}(e-\phi_1(\mathbf{1}_{M_n}))e_i^{\frac{1}{2}}\right\|_{2,T(A)} < \sqrt{2}\epsilon^{\frac{1}{2}}.$$

Therefore

$$\|e-\phi(\mathbf{1}_{M_n})\|_{2,T(A)} < \epsilon^{\frac{1}{2}} + \frac{\epsilon}{2} + \sqrt{2}\epsilon^{\frac{1}{2}} < 3\epsilon^{\frac{1}{2}},$$

and the proof is complete. □

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