

Topological dimension of C^* -algebras

Aaron Tikuisis

a.tikuisis@abdn.ac.uk

University of Aberdeen

The prototypical C^* -algebra:

$\mathcal{B}(\mathcal{H}) := \{\text{continuous (=operator norm-bounded)} \text{ linear operators on } \mathcal{H}\},$

where \mathcal{H} is a complex Hilbert space.

Definition

A C^* -algebra is a norm-closed *-subalgebra of $\mathcal{B}(\mathcal{H})$.

(C^* -algebras can also be defined abstractly, as Banach *-algebras satisfying $\|a^*a\| = \|a\|^2$ for all a .)

The prototypical C^* -algebra:

$\mathcal{B}(\mathcal{H}) := \{\text{continuous (=operator norm-bounded)} \\ \text{linear operators on } \mathcal{H}\},$

where \mathcal{H} is a complex Hilbert space.

Definition

A C^* -algebra is a norm-closed *-subalgebra of $\mathcal{B}(\mathcal{H})$.

(C^* -algebras can also be defined abstractly, as Banach
*-algebras satisfying $\|a^*a\| = \|a\|^2$ for all a .)

The prototypical C^* -algebra:

$\mathcal{B}(\mathcal{H}) := \{\text{continuous (=operator norm-bounded)} \\ \text{linear operators on } \mathcal{H}\},$

where \mathcal{H} is a complex Hilbert space.

Definition

A **C^* -algebra** is a norm-closed $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$.

(C^* -algebras can also be defined abstractly, as Banach
 $*$ -algebras satisfying $\|a^*a\| = \|a\|^2$ for all a .)

The prototypical C^* -algebra:

$\mathcal{B}(\mathcal{H}) := \{\text{continuous (=operator norm-bounded)} \\ \text{linear operators on } \mathcal{H}\},$

where \mathcal{H} is a complex Hilbert space.

Definition

A **C^* -algebra** is a norm-closed $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$.

(C^* -algebras can also be defined abstractly, as Banach $*$ -algebras satisfying $\|a^*a\| = \|a\|^2$ for all a .)

Examples of C^* -algebras

$M_n(\mathbb{C}) = \mathcal{B}(\mathcal{H})$ for $\mathcal{H} = \mathbb{C}^n$

$C_0(X, \mathbb{C})$ where X is a locally compact Hausdorff topological space

$C_0(X, M_n(\mathbb{C}))$

Examples of C^* -algebras

$M_n(\mathbb{C}) = \mathcal{B}(\mathcal{H})$ for $\mathcal{H} = \mathbb{C}^n$

$C_0(X, \mathbb{C})$ where X is a locally compact Hausdorff topological space

$C_0(X, M_n(\mathbb{C}))$

Examples of C^* -algebras

$M_n(\mathbb{C}) = \mathcal{B}(\mathcal{H})$ for $\mathcal{H} = \mathbb{C}^n$

$C_0(X, \mathbb{C})$ where X is a locally compact Hausdorff topological space

$C_0(X, M_n(\mathbb{C}))$

Examples of C^* -algebras

$M_n(\mathbb{C}) = \mathcal{B}(\mathcal{H})$ for $\mathcal{H} = \mathbb{C}^n$

$C_0(X, \mathbb{C})$ where X is a locally compact Hausdorff topological space

$C_0(X, M_n(\mathbb{C}))$

Constructions of C^* -algebras

C^* -algebras can be constructed from:

groups (via unitary representations)

coarse metric spaces

rings

graphs

topological dynamical systems (encoding orbit equivalence)

Constructions of C^* -algebras

C^* -algebras can be constructed from:

groups (via unitary representations)

coarse metric spaces

rings

graphs

topological dynamical systems (encoding orbit equivalence)

Constructions of C^* -algebras

C^* -algebras can be constructed from:

groups (via unitary representations)

coarse metric spaces

rings

graphs

topological dynamical systems (encoding orbit equivalence)

Constructions of C^* -algebras

C^* -algebras can be constructed from:

groups (via unitary representations)

coarse metric spaces

rings

graphs

topological dynamical systems (encoding orbit equivalence)

Constructions of C^* -algebras

C^* -algebras can be constructed from:

groups (via unitary representations)

coarse metric spaces

rings

graphs

topological dynamical systems (encoding orbit equivalence)

C^* -algebras can be constructed from:

groups (via unitary representations)

coarse metric spaces

rings

graphs

topological dynamical systems (encoding orbit equivalence)

Constructions of C^* -algebras

Idea: encode input data as continuous operators. Example:

Constructions of C^* -algebras

Idea: encode input data as continuous operators. Example:

...

Orientation-preserving local isometries of the Cantor set

Constructions of C^* -algebras

Idea: encode input data as continuous operators. Example:



Orientation-preserving local isometries of the Cantor set

Constructions of C^* -algebras

Idea: encode input data as continuous operators. Example:



Orientation-preserving local isometries of the Cantor set

Constructions of C^* -algebras

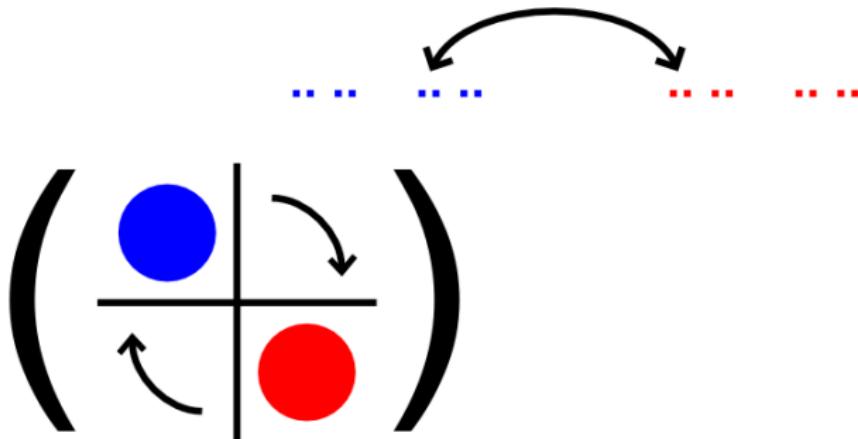
Idea: encode input data as continuous operators. Example:



Orientation-preserving local isometries of the Cantor set

Constructions of C^* -algebras

Idea: encode input data as continuous operators. Example:



Constructions of C^* -algebras

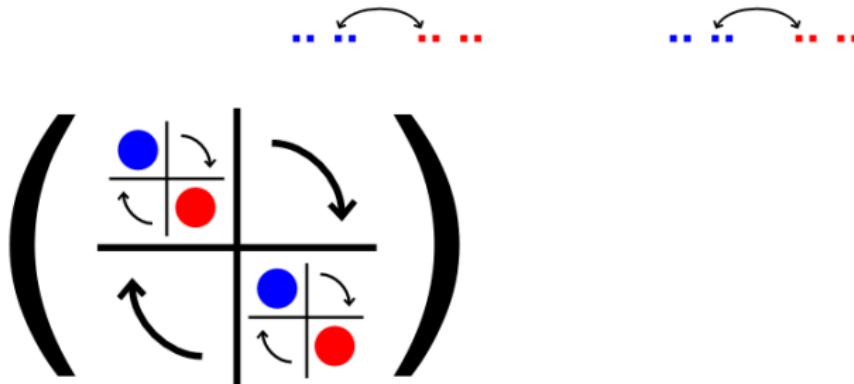
Idea: encode input data as continuous operators. Example:

The diagram illustrates the construction of a C^* -algebra. On the left, a 2x2 matrix is shown with a blue dot in the top-left position and a red dot in the bottom-right position. A black arrow points clockwise around the matrix. Above the matrix, three horizontal dashed lines (blue, blue, red) represent the rows, with arrows indicating a flow from left to right. To the right of an equals sign, the matrix is represented as $\begin{pmatrix} \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$, where each entry is a copy of the complex numbers \mathbb{C} .

$$M_2(\mathbb{C})$$

Constructions of C^* -algebras

Idea: encode input data as continuous operators. Example:



Constructions of C^* -algebras

Idea: encode input data as continuous operators. Example:

$$\left(\begin{array}{c|c} \text{blue circle} & \\ \hline \text{red circle} & \end{array} \right) = \left(\begin{array}{c|c} \text{C} & \text{C} \\ \hline \text{C} & \text{C} \end{array} \right)$$

$M_4(\mathbb{C})$

Constructions of C^* -algebras

Idea: encode input data as continuous operators. Example:

$$\left(\begin{array}{c|c} \text{Diagram 1} & \text{Diagram 2} \\ \hline \end{array} \right) = \left(\begin{array}{c|c} \text{Matrix 1} & \text{Matrix 2} \\ \hline \end{array} \right)$$

Diagram 1 consists of two 2x2 grids separated by a vertical line. Each grid has arrows indicating clockwise flow. Blue dots are at (1,1), (1,2), (2,1), and (2,2). Red dots are at (1,1), (1,2), (2,1), and (2,2). Arrows point from (1,1) to (1,2), (1,2) to (2,2), (2,2) to (2,1), and (2,1) to (1,1). Diagram 2 consists of two 4x4 grids separated by a vertical line. Each grid has arrows indicating clockwise flow. Blue dots are at (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), and (4,4). Red dots are at (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), and (4,4). Arrows point from (1,1) to (1,2), (1,2) to (1,3), (1,3) to (1,4), (1,4) to (2,4), (2,4) to (2,3), (2,3) to (2,2), (2,2) to (2,1), (2,1) to (3,1), (3,1) to (3,2), (3,2) to (3,3), (3,3) to (3,4), (3,4) to (4,4), (4,4) to (4,3), (4,3) to (4,2), (4,2) to (4,1), and (4,1) to (1,1).

Matrix 1 is an 8x8 matrix where each 2x2 block is labeled $\frac{\text{C}}{\text{C}\text{C}}$. Matrix 2 is an 8x8 matrix where each 2x2 block is labeled $\frac{\text{C}\text{C}}{\text{C}\text{C}\text{C}}$.

$$M_8(\mathbb{C})$$

Constructions of C^* -algebras

Idea: encode input data as continuous operators. Example:

$$\left(\begin{array}{c|c} \text{...} & \text{...} \\ \text{...} & \text{...} \\ \hline \begin{array}{|c|c|} \hline \text{...} & \text{...} \\ \text{...} & \text{...} \\ \hline \end{array} & \begin{array}{|c|c|} \hline \text{...} & \text{...} \\ \text{...} & \text{...} \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline \text{...} & \text{...} \\ \text{...} & \text{...} \\ \hline \end{array} & \begin{array}{|c|c|} \hline \text{...} & \text{...} \\ \text{...} & \text{...} \\ \hline \end{array} \end{array} \right) = \overline{\left(\begin{array}{c|c} \text{...} & \text{...} \\ \text{...} & \text{...} \\ \hline \begin{array}{|c|c|} \hline \text{...} & \text{...} \\ \text{...} & \text{...} \\ \hline \end{array} & \begin{array}{|c|c|} \hline \text{...} & \text{...} \\ \text{...} & \text{...} \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline \text{...} & \text{...} \\ \text{...} & \text{...} \\ \hline \end{array} & \begin{array}{|c|c|} \hline \text{...} & \text{...} \\ \text{...} & \text{...} \\ \hline \end{array} \end{array} } \right)} \| \cdot \|$$

$$M_{2^\infty}$$

“Uniformly hyperfinite” (UHF) algebra.

Constructions of C^* -algebras

Idea: encode input data as continuous operators. Example:

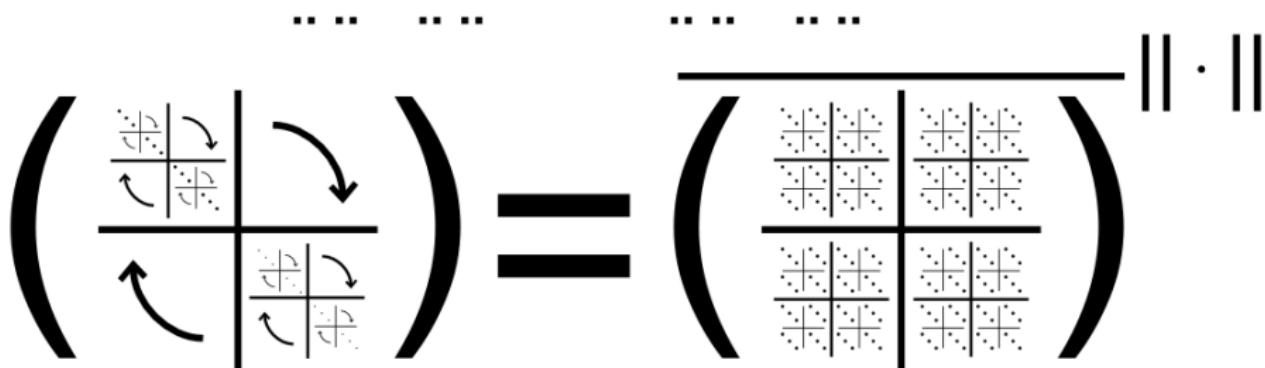
$$\left(\begin{array}{c|c} \text{dots} & \text{dots} \\ \text{dots} & \text{dots} \\ \hline \begin{array}{|c|c|} \hline \text{dots} & \text{dots} \\ \hline \text{dots} & \text{dots} \\ \hline \end{array} & \begin{array}{|c|c|} \hline \text{dots} & \text{dots} \\ \hline \text{dots} & \text{dots} \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline \text{dots} & \text{dots} \\ \hline \text{dots} & \text{dots} \\ \hline \end{array} & \begin{array}{|c|c|} \hline \text{dots} & \text{dots} \\ \hline \text{dots} & \text{dots} \\ \hline \end{array} \end{array} \right) = \overline{\left(\begin{array}{c|c} \text{dots} & \text{dots} \\ \text{dots} & \text{dots} \\ \hline \begin{array}{|c|c|} \hline \text{dots} & \text{dots} \\ \hline \text{dots} & \text{dots} \\ \hline \end{array} & \begin{array}{|c|c|} \hline \text{dots} & \text{dots} \\ \hline \text{dots} & \text{dots} \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline \text{dots} & \text{dots} \\ \hline \text{dots} & \text{dots} \\ \hline \end{array} & \begin{array}{|c|c|} \hline \text{dots} & \text{dots} \\ \hline \text{dots} & \text{dots} \\ \hline \end{array} \end{array} } \right)} \| \cdot \|$$

$$M_{2^\infty}$$

“Uniformly hyperfinite” (UHF) algebra.
Can likewise define M_{p^∞} for any $p \in \mathbb{N}$.

Constructions of C^* -algebras

Idea: encode input data as continuous operators. Example:

$$\left(\begin{array}{c|c} \text{dots} & \text{dots} \\ \hline \text{dots} & \text{dots} \end{array} \right) = \overline{\left(\begin{array}{c|c} \text{dots} & \text{dots} \\ \hline \text{dots} & \text{dots} \end{array} \right)} \| \cdot \|$$


“Uniformly hyperfinite” (UHF) algebra.

Can likewise define M_{p^∞} for any $p \in \mathbb{N}$.

Note: closure in a weaker topology gives a famous von Neumann algebra \mathcal{R} , the “hyperfinite II_1 -factor.”

There is only one Cantor set, but:

Glimm (1959)

For p, q prime, $M_{p^\infty} \cong M_{q^\infty}$ if and only if $p = q$.

Note for later: $M_{p^\infty} \otimes M_{p^\infty} \cong M_{p^\infty}$.

There is only one Cantor set, but:

Glimm (1959)

For p, q prime, $M_{p^\infty} \cong M_{q^\infty}$ if and only if $p = q$.

Note for later: $M_{p^\infty} \otimes M_{p^\infty} \cong M_{p^\infty}$.

Amenability

“Amenability” for C^* -algebras is characterised by a finite approximation property:

Definition

A C^* -algebra A is **amenable** if there is a sequence (or net) of **approximately $^\triangle$** commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\quad id_A \quad} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

where F_n is **finite dimensional**.

Requiring $*$ -homomorphisms would be too strong.

Instead, we require ϕ_n, ψ_n to be “completely positive contractions” (maps that order structure).

\triangle : approximately means $\|\psi_n(\phi_n(a)) - a\| \xrightarrow{n \rightarrow \infty} 0 \quad \forall a \in A$.

“Amenability” for C^* -algebras is characterised by a finite approximation property:

Definition

A C^* -algebra A is **amenable** if there is a sequence (or net) of **approximately $^\triangle$** commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\quad id_A \quad} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

where F_n is **finite dimensional**.

Requiring $*$ -homomorphisms would be too strong.

Instead, we require ϕ_n, ψ_n to be “completely positive contractions” (maps that order structure).

\triangle : approximately means $\|\psi_n(\phi_n(a)) - a\| \xrightarrow{n \rightarrow \infty} 0 \quad \forall a \in A$.

“Amenability” for C^* -algebras is characterised by a finite approximation property:

Definition

A C^* -algebra A is **amenable** if there is a sequence (or net) of **approximately $^\triangle$** commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\quad id_A \quad} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

where F_n is **finite dimensional**.

Requiring $*$ -homomorphisms would be too strong.

Instead, we require ϕ_n, ψ_n to be “completely positive contractions” (maps that order structure).

\triangle : approximately means $\|\psi_n(\phi_n(a)) - a\| \xrightarrow{n \rightarrow \infty} 0 \quad \forall a \in A$.

Amenability

“Amenability” for C^* -algebras is characterised by a finite approximation property:

Definition

A C^* -algebra A is **amenable** if there is a sequence (or net) of **approximately[△]** commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\quad id_A \quad} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

where F_n is **finite dimensional**.

Requiring $*$ -homomorphisms would be too strong.

Instead, we require ϕ_n, ψ_n to be “completely positive contractions” (maps that order structure).

\triangle : approximately means $\|\psi_n(\phi_n(a)) - a\| \xrightarrow{n \rightarrow \infty} 0 \quad \forall a \in A$.

Amenability

“Amenability” for C^* -algebras is characterised by a finite approximation property:

Definition

A C^* -algebra A is **amenable** if there is a sequence (or net) of **approximately $^\triangle$** commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\quad id_A \quad} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

where F_n is **finite dimensional**.

Requiring $*$ -homomorphisms would be too strong.

Instead, we require ϕ_n, ψ_n to be “completely positive contractions” (maps that order structure).

\triangle : approximately means $\|\psi_n(\phi_n(a)) - a\| \xrightarrow{n \rightarrow \infty} 0 \quad \forall a \in A$.

Covering dimension

From Lebesgue, we have the following fruitful notion.

Definition

The **covering dimension** of a normal topological space X is the least number d such that, every finite open cover \mathcal{U} of X has finite refinement \mathcal{V} that can be coloured with $(d + 1)$ colours.

Covering dimension

From Lebesgue, we have the following fruitful notion.

Definition

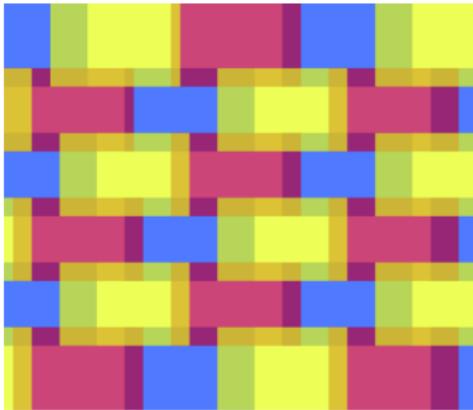
The **covering dimension** of a normal topological space X is the least number d such that, every finite open cover \mathcal{U} of X has finite refinement \mathcal{V} that can be coloured with $(d + 1)$ colours.

Covering dimension

From Lebesgue, we have the following fruitful notion.

Definition

The **covering dimension** of a normal topological space X is the least number d such that, every finite open cover \mathcal{U} of X has finite refinement \mathcal{V} that can be coloured with $(d + 1)$ colours.



Covering dimension

From Lebesgue, we have the following fruitful notion.

Definition

The **covering dimension** of a normal topological space X is the least number d such that, every finite open cover \mathcal{U} of X has finite refinement \mathcal{V} that can be coloured with $(d + 1)$ colours.

(Warning: this is a minor, but equivalent, variation on the standard definition.)

Covering dimension

From Lebesgue, we have the following fruitful notion.

Definition

The **covering dimension** of a normal topological space X is the least number d such that, every finite open cover \mathcal{U} of X has finite refinement \mathcal{V} that can be coloured with $(d + 1)$ colours.

(Warning: this is a minor, but equivalent, variation on the standard definition.)

Highly desirable to extend dimension to all C^* -algebras (from the class $\{C_0(X, \mathbb{C})\}$ of abelian C^* -algebras).

Nuclear dimension

Nuclear dimension marries the two previous concepts: amenability and covering dimension.

Definition (Kirchberg–Winter '04, Winter–Zacharias '10)

The **nuclear dimension** of a C^* -algebra A is the least integer d such that there is a sequence (or net) of approximately commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\quad \text{id}_A \quad} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

such that ψ_n is a completely positive contraction **and** (F_n, ϕ_n) **can be coloured with $d + 1$ colours**.

$d + 1$ -colourable: F_n decomposes as a direct sum

$F_n^{(0)} \oplus \cdots \oplus F_n^{(d)}$ and $\phi_n|_{F_n^{(d)}}$ preserves orthogonality.

Nuclear dimension

Nuclear dimension marries the two previous concepts: amenability and covering dimension.

Definition (Kirchberg–Winter '04, Winter–Zacharias '10)

The **nuclear dimension** of a C^* -algebra A is the least integer d such that there is a sequence (or net) of approximately commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\quad \text{id}_A \quad} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

such that ψ_n is a completely positive contraction and (F_n, ϕ_n) can be coloured with $d + 1$ colours.

$d + 1$ -colourable: F_n decomposes as a direct sum

$F_n^{(0)} \oplus \cdots \oplus F_n^{(d)}$ and $\phi_n|_{F_n^{(d)}}$ preserves orthogonality.

Nuclear dimension

Nuclear dimension marries the two previous concepts: amenability and covering dimension.

Definition (Kirchberg–Winter '04, Winter–Zacharias '10)

The **nuclear dimension** of a C^* -algebra A is the least integer d such that there is a sequence (or net) of approximately commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\quad \text{id}_A \quad} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

such that ψ_n is a completely positive contraction **and** (F_n, ϕ_n) **can be coloured with $d + 1$ colours.**

$d + 1$ -colourable: F_n decomposes as a direct sum

$F_n^{(0)} \oplus \cdots \oplus F_n^{(d)}$ and $\phi_n|_{F_n^{(d)}}$ preserves orthogonality.

Nuclear dimension

Nuclear dimension marries the two previous concepts: amenability and covering dimension.

Definition (Kirchberg–Winter '04, Winter–Zacharias '10)

The **nuclear dimension** of a C^* -algebra A is the least integer d such that there is a sequence (or net) of approximately commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\quad \text{id}_A \quad} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

such that ψ_n is a completely positive contraction **and** (F_n, ϕ_n) **can be coloured with $d + 1$ colours.**

$d + 1$ -colourable: F_n decomposes as a direct sum

$F_n^{(0)} \oplus \cdots \oplus F_n^{(d)}$ and $\phi_n|_{F_n^{(d)}}$ preserves orthogonality.

Nuclear dimension

Nuclear dimension marries the two previous concepts: amenability and covering dimension.

Definition (Kirchberg–Winter '04, Winter–Zacharias '10)

The **nuclear dimension** of a C^* -algebra A is the least integer d such that there is a sequence (or net) of approximately commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\quad \text{id}_A \quad} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

such that ψ_n is a completely positive contraction **and** (F_n, ϕ_n) **can be coloured with $d + 1$ colours.**

$d + 1$ -colourable: F_n decomposes as a direct sum

$F_n^{(0)} \oplus \cdots \oplus F_n^{(d)}$ and $\phi_n|_{F_n^{(d)}}$ preserves orthogonality.

Nuclear dimension: properties

Definition

$\dim_{nuc} A \leq d$ if there exist approximately commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

such that ψ_n is a completely positive contraction and (F_n, ϕ_n) can be coloured with $(d + 1)$ colours.

For a compact metrisable space X ,

$$\dim_{nuc} C(X, \mathbb{C}) = \dim X = \dim_{nuc} C(X, M_p(\mathbb{C}))$$

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

Nuclear dimension: properties

Definition

$\dim_{nuc} A \leq d$ if there exist approximately commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

such that ψ_n is a completely positive contraction and (F_n, ϕ_n) can be coloured with $(d + 1)$ colours.

For a compact metrisable space X ,

$$\dim_{nuc} C(X, \mathbb{C}) = \dim X = \dim_{nuc} C(X, M_p(\mathbb{C}))$$

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

Nuclear dimension: properties

Definition

$\dim_{nuc} A \leq d$ if there exist approximately commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

such that ψ_n is a completely positive contraction and (F_n, ϕ_n) can be coloured with $(d + 1)$ colours.

For a compact metrisable space X ,

$$\dim_{nuc} C(X, \mathbb{C}) = \dim X = \dim_{nuc} C(X, M_p(\mathbb{C}))$$

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

Nuclear dimension: properties

Definition

$\dim_{nuc} A \leq d$ if there exist approximately commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

such that ψ_n is a completely positive contraction and (F_n, ϕ_n) can be coloured with $(d + 1)$ colours.

For a compact metrisable space X ,

$$\dim_{nuc} C(X, \mathbb{C}) = \dim X = \dim_{nuc} C(X, M_p(\mathbb{C}))$$

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

Nuclear dimension: properties

Definition

$\dim_{nuc} A \leq d$ if there exist approximately commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

such that ψ_n is a completely positive contraction and (F_n, ϕ_n) can be coloured with $(d + 1)$ colours.

For a compact metrisable space X ,

$$\dim_{nuc} C(X, \mathbb{C}) = \dim X = \dim_{nuc} C(X, M_p(\mathbb{C}))$$

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

Nuclear dimension: properties

Definition

$\dim_{nuc} A \leq d$ if there exist approximately commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ & \searrow \psi_n & \nearrow \phi_n \\ & F_n & \end{array}$$

such that ψ_n is a completely positive contraction and (F_n, ϕ_n) can be coloured with $(d + 1)$ colours.

For a compact metrisable space X ,

$$\dim_{nuc} C(X, \mathbb{C}) = \dim X = \dim_{nuc} C(X, M_p(\mathbb{C}))$$

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

The Jiang-Su algebra

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

What is \mathcal{Z} ?

It is a lot like M_{p^∞} except no nontrivial projections; in fact, it is more like an interpolation between M_{p^∞} and M_{q^∞} where p, q are coprime.

$$\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z};$$

$$M_{p^\infty} \cong \mathcal{Z} \otimes \mathcal{Z}.$$

The Jiang-Su algebra

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

What is \mathcal{Z} ?

It is a lot like M_{p^∞} except no nontrivial projections; in fact, it is more like an interpolation between M_{p^∞} and M_{q^∞} where p, q are coprime.

$$\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z};$$

$$M_{p^\infty} \cong \mathcal{Z} \otimes \mathcal{Z}.$$

The Jiang-Su algebra

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

What is \mathcal{Z} ?

It is a lot like M_{p^∞} except no nontrivial projections; in fact, it is more like an interpolation between M_{p^∞} and M_{q^∞} where p, q are coprime.

$$\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z};$$

$$M_{p^\infty} \cong \mathcal{Z} \otimes \mathcal{Z}.$$

The Jiang-Su algebra

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

What is \mathcal{Z} ?

It is a lot like M_{p^∞} except no nontrivial projections; in fact, it is more like an interpolation between M_{p^∞} and M_{q^∞} where p, q are coprime.

$$\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z};$$

$$M_{p^\infty} \cong \mathcal{Z} \otimes \mathcal{Z}.$$

The Jiang-Su algebra

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

What is \mathcal{Z} ?

It is a lot like M_{p^∞} except no nontrivial projections; in fact, it is more like an interpolation between M_{p^∞} and M_{q^∞} where p, q are coprime.

$$\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z};$$

$$M_{p^\infty} \cong \mathcal{Z} \otimes \mathcal{Z}.$$

The Jiang-Su algebra

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

What is \mathcal{Z} ?

It is a lot like M_{p^∞} except no nontrivial projections; in fact, it is more like an interpolation between M_{p^∞} and M_{q^∞} where p, q are coprime.

$\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$;

$M_{p^\infty} \cong \mathcal{Z} \otimes \mathcal{Z}$.

The Jiang-Su algebra

If A is simple and $\dim_{nuc} A < \infty$ then either $A = M_k(\mathbb{C})$ (some k) or $A \cong A \otimes \mathcal{Z}$.

What is \mathcal{Z} ?

It is a lot like M_{p^∞} except no nontrivial projections; in fact, it is more like an interpolation between M_{p^∞} and M_{q^∞} where p, q are coprime.

$$\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z};$$

$$M_{p^\infty} \cong \mathcal{Z} \otimes \mathcal{Z}.$$

Dimension reduction

What is $\dim_{nuc} C(X, \mathbb{C}) \otimes M_{p^\infty}$?

There is tension between the constant dimension of $C(X, M_{p^n}) = C(X, \mathbb{C}) \otimes M_{p^n}$ and the extra space afforded by the regularity of M_{p^∞} .

Theorem (T-Winter '14)

$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{p^\infty}) \leq 2$ for any compact space X .
(Also, $\dim_{nuc} (C(X, \mathbb{C}) \otimes \mathcal{Z}) \leq 2$.

This is a bit surprising, since

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes C(Y, \mathbb{C})) = \dim(X \times Y) \geq \dim X = \dim_{nuc} C(X, \mathbb{C}),$$

so dimension reduction, where

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{2^\infty}) < \dim_{nuc} C(X, \mathbb{C})$$

is a purely noncommutative phenomenon.

Dimension reduction

What is $\dim_{nuc} C(X, \mathbb{C}) \otimes M_{p^\infty}$?

There is tension between the constant dimension of $C(X, M_{p^n}) = C(X, \mathbb{C}) \otimes M_{p^n}$ and the extra space afforded by the regularity of M_{p^∞} .

Theorem (T-Winter '14)

$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{p^\infty}) \leq 2$ for any compact space X .
(Also, $\dim_{nuc} (C(X, \mathbb{C}) \otimes \mathcal{Z}) \leq 2$.

This is a bit surprising, since

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes C(Y, \mathbb{C})) = \dim(X \times Y) \geq \dim X = \dim_{nuc} C(X, \mathbb{C}),$$

so dimension reduction, where

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{2^\infty}) < \dim_{nuc} C(X, \mathbb{C})$$

is a purely noncommutative phenomenon.

Dimension reduction

What is $\dim_{nuc} C(X, \mathbb{C}) \otimes M_{p^\infty}$?

There is tension between the constant dimension of $C(X, M_{p^n}) = C(X, \mathbb{C}) \otimes M_{p^n}$ and the extra space afforded by the regularity of M_{p^∞} .

Theorem (T-Winter '14)

$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{p^\infty}) \leq 2$ for any compact space X .

(Also, $\dim_{nuc} (C(X, \mathbb{C}) \otimes \mathcal{Z}) \leq 2$.

This is a bit surprising, since

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes C(Y, \mathbb{C})) = \dim(X \times Y) \geq \dim X = \dim_{nuc} C(X, \mathbb{C}),$$

so dimension reduction, where

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{2^\infty}) < \dim_{nuc} C(X, \mathbb{C})$$

is a purely noncommutative phenomenon.

Dimension reduction

What is $\dim_{nuc} C(X, \mathbb{C}) \otimes M_{p^\infty}$?

There is tension between the constant dimension of $C(X, M_{p^n}) = C(X, \mathbb{C}) \otimes M_{p^n}$ and the extra space afforded by the regularity of M_{p^∞} .

Theorem (T-Winter '14)

$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{p^\infty}) \leq 2$ for any compact space X .
(Also, $\dim_{nuc} (C(X, \mathbb{C}) \otimes \mathcal{Z}) \leq 2$.

This is a bit surprising, since

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes C(Y, \mathbb{C})) = \dim(X \times Y) \geq \dim X = \dim_{nuc} C(X, \mathbb{C}),$$

so dimension reduction, where

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{2^\infty}) < \dim_{nuc} C(X, \mathbb{C})$$

is a purely noncommutative phenomenon.

Dimension reduction

What is $\dim_{nuc} C(X, \mathbb{C}) \otimes M_{p^\infty}$?

There is tension between the constant dimension of $C(X, M_{p^n}) = C(X, \mathbb{C}) \otimes M_{p^n}$ and the extra space afforded by the regularity of M_{p^∞} .

Theorem (T-Winter '14)

$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{p^\infty}) \leq 2$ for any compact space X .
(Also, $\dim_{nuc} (C(X, \mathbb{C}) \otimes \mathcal{Z}) \leq 2$.

This is a bit surprising, since

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes C(Y, \mathbb{C})) = \dim(X \times Y) \geq \dim X = \dim_{nuc} C(X, \mathbb{C}),$$

so dimension reduction, where

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{2^\infty}) < \dim_{nuc} C(X, \mathbb{C})$$

is a purely noncommutative phenomenon.

Dimension reduction

What is $\dim_{nuc} C(X, \mathbb{C}) \otimes M_{p^\infty}$?

There is tension between the constant dimension of $C(X, M_{p^n}) = C(X, \mathbb{C}) \otimes M_{p^n}$ and the extra space afforded by the regularity of M_{p^∞} .

Theorem (T-Winter '14)

$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{p^\infty}) \leq 2$ for any compact space X .
(Also, $\dim_{nuc} (C(X, \mathbb{C}) \otimes \mathcal{Z}) \leq 2$.

This is a bit surprising, since

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes C(Y, \mathbb{C})) = \dim(X \times Y) \geq \dim X = \dim_{nuc} C(X, \mathbb{C}),$$

so dimension reduction, where

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{2^\infty}) < \dim_{nuc} C(X, \mathbb{C})$$

is a purely noncommutative phenomenon.

Dimension reduction

What is $\dim_{nuc} C(X, \mathbb{C}) \otimes M_{p^\infty}$?

There is tension between the constant dimension of $C(X, M_{p^n}) = C(X, \mathbb{C}) \otimes M_{p^n}$ and the extra space afforded by the regularity of M_{p^∞} .

Theorem (T-Winter '14)

$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{p^\infty}) \leq 2$ for any compact space X .
(Also, $\dim_{nuc} (C(X, \mathbb{C}) \otimes \mathcal{Z}) \leq 2$.

This is a bit surprising, since

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes C(Y, \mathbb{C})) = \dim(X \times Y) \geq \dim X = \dim_{nuc} C(X, \mathbb{C}),$$

so dimension reduction, where

$$\dim_{nuc} (C(X, \mathbb{C}) \otimes M_{2^\infty}) < \dim_{nuc} C(X, \mathbb{C})$$

is a purely noncommutative phenomenon.

Dimension reduction

Theorem (T-Winter '12)

$\dim_{nuc} C(X, \mathbb{C}) \otimes M_{p^\infty} \leq 2$ for any compact space X .

More generally, inspired by what is known (and conjectured) for classifiable C^* -algebras:

Conjecture

If A is simple and nuclear then $A \otimes M_{p^\infty}$ has finite nuclear dimension (for any p).

(Equivalently, $A \otimes \mathcal{Z}$ has finite nuclear dimension.)

Theorem (T-Winter '12)

$\dim_{nuc} C(X, \mathbb{C}) \otimes M_{p^\infty} \leq 2$ for any compact space X .

More generally, inspired by what is known (and conjectured) for classifiable C^* -algebras:

Conjecture

If A is simple and nuclear then $A \otimes M_{p^\infty}$ has finite nuclear dimension (for any p).

(Equivalently, $A \otimes \mathcal{Z}$ has finite nuclear dimension.)

Dimension reduction

Theorem (T-Winter '12)

$\dim_{nuc} C(X, \mathbb{C}) \otimes M_{p^\infty} \leq 2$ for any compact space X .

More generally, inspired by what is known (and conjectured) for classifiable C^* -algebras:

Conjecture

If A is simple and nuclear then $A \otimes M_{p^\infty}$ has finite nuclear dimension (for any p).

(Equivalently, $A \otimes \mathbb{Z}$ has finite nuclear dimension.)

Dimension reduction

Theorem (T-Winter '12)

$\dim_{nuc} C(X, \mathbb{C}) \otimes M_{p^\infty} \leq 2$ for any compact space X .

More generally, inspired by what is known (and conjectured) for classifiable C^* -algebras:

Conjecture

If A is simple and nuclear then $A \otimes M_{p^\infty}$ has finite nuclear dimension (for any p).

(Equivalently, $A \otimes \mathcal{Z}$ has finite nuclear dimension.)

Conjecture

If A is simple and nuclear then $A \otimes M_{p^\infty}$ has finite nuclear dimension (for any p).

Theorem (Elliott-Niu-Santiago-T-Winter '14)

$\dim_{nuc}(A \otimes M_{p^\infty}) \leq 2$ if A is subhomogeneous, i.e., a subalgebra of $C(X, M_k)$.

Theorem (Matui-Sato '13, Sato-White-Winter '14, Bosa-Brown-Sato-T-White-Winter '14)

$\dim_{nuc}(A \otimes M_{p^\infty}) \leq 1$ if A is simple, unital, amenable, and not too many extreme traces. (Specifically, if $\partial_e T(A)$ is compact.)

Dimension reduction

Conjecture

If A is simple and nuclear then $A \otimes M_{p^\infty}$ has finite nuclear dimension (for any p).

Theorem (Elliott-Niu-Santiago-T-Winter '14)

$\dim_{nuc}(A \otimes M_{p^\infty}) \leq 2$ if A is subhomogeneous, i.e., a subalgebra of $C(X, M_k)$.

Theorem (Matui-Sato '13, Sato-White-Winter '14, Bosa-Brown-Sato-T-White-Winter '14)

$\dim_{nuc}(A \otimes M_{p^\infty}) \leq 1$ if A is simple, unital, amenable, and not too many extreme traces. (Specifically, if $\partial_e T(A)$ is compact.)

Dimension reduction

Conjecture

If A is simple and nuclear then $A \otimes M_{p^\infty}$ has finite nuclear dimension (for any p).

Theorem (Elliott-Niu-Santiago-T-Winter '14)

$\dim_{nuc} (A \otimes M_{p^\infty}) \leq 2$ if A is subhomogeneous, i.e., a subalgebra of $C(X, M_k)$.

Theorem (Matui-Sato '13, Sato-White-Winter '14, Bosa-Brown-Sato-T-White-Winter '14)

$\dim_{nuc} (A \otimes M_{p^\infty}) \leq 1$ if A is simple, unital, amenable, and not too many extreme traces. (Specifically, if $\partial_e T(A)$ is compact.)

Dimension reduction

Conjecture

If A is simple and nuclear then $A \otimes M_{p^\infty}$ has finite nuclear dimension (for any p).

Theorem (Elliott-Niu-Santiago-T-Winter '14)

$\dim_{nuc}(A \otimes M_{p^\infty}) \leq 2$ if A is subhomogeneous, i.e., a subalgebra of $C(X, M_k)$.

Theorem (Matui-Sato '13, Sato-White-Winter '14, Bosa-Brown-Sato-T-White-Winter '14)

$\dim_{nuc}(A \otimes M_{p^\infty}) \leq 1$ if A is simple, unital, amenable, and not too many extreme traces. (Specifically, if $\partial_e T(A)$ is compact.)

Dimension reduction

Conjecture

If A is simple and nuclear then $A \otimes M_{p^\infty}$ has finite nuclear dimension (for any p).

Theorem (Elliott-Niu-Santiago-T-Winter '14)

$\dim_{nuc}(A \otimes M_{p^\infty}) \leq 2$ if A is subhomogeneous, i.e., a subalgebra of $C(X, M_k)$.

Theorem (Matui-Sato '13, Sato-White-Winter '14, Bosa-Brown-Sato-T-White-Winter '14)

$\dim_{nuc}(A \otimes M_{p^\infty}) \leq 1$ if A is simple, unital, amenable, and not too many extreme traces. (Specifically, if $\partial_e T(A)$ is compact.)