

# The real span of a dimension group

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Joint work with Greg Maloney

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# Dimension groups

A **dimension group** is a directed ordered group  $(G, G^+ := \{g \in G : g \geq 0\})$  satisfying:

- (i) Unperforation: if  $g + \cdots + g \geq 0$  then  $g \geq 0$
- (ii) Riesz interpolation: given  $a_1, a_2, c_1, c_2$  satisfying

$$\begin{array}{c} a_1 \\ a_2 \end{array} \leq \begin{array}{c} c_1 \\ c_2 \end{array},$$

$\exists b$  satisfying

$$\begin{array}{c} a_1 \\ a_2 \end{array} \leq b \leq \begin{array}{c} c_1 \\ c_2 \end{array}.$$

Examples: lattice ordered groups (use  $\max\{a_1, a_2\}$  or  $\min\{c_1, c_2\}$  as an interpolant),  
 $C(X, \mathbb{R})$  with strict order,  $f < g$  if  $f(x) < g(x) \forall x$ .

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# Dimension groups and AF algebras

Recall that an AF algebra is given by an inductive limit of finite-dimensional  $C^*$ -algebras.

$$A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A.$$

Ordered  $K_0$ -group computation:

Theorem (Elliott, Effros-Handelman-Shen)

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# A good embedding of a dimension group

Our goal: to understand dimension groups better by embedding them into real vector spaces.

Embed  $G$  into a real vector space  $V$  (s.t.  $G$  spans  $V$ ).

Set  $V^+ = \mathbb{R}^+ \cdot G^+$

= the real cone generated by  $G^+$ .

We want:

(i)  $(V, V^+)$  to be an ordered vector space  
(need  $V^+ \cap -V^+ = 0$ , ie. not just preordered)

(ii) to recover  $G^+$  from  $V^+$ :

$$G^+ = V^+ \cap G$$

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## Bad embeddings: example one

$(V, V^+)$  may not be an ordered vector space. Let  $G = \mathbb{Z}^2$ ,  
 $G^+ = \{(x, y) : x + \theta y \geq 0\}$  ( $\theta \notin \mathbb{Q}$ ).

This ordered group is denoted  $\mathbb{Z} + \theta\mathbb{Z}$ ; indeed, it embeds into  $V = \mathbb{R}$  by  $(x, y) \mapsto x + \theta y$ , which is a good embedding.

But  $(x, y) \mapsto x + \eta y$  (where  $\eta \neq \theta$ ) is bad, since  $V^+ = \mathbb{R}$ .  
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## Bad embeddings: example two

$(V, V^+)$  may not have Riesz interpolation.

Pick four  $\mathbb{Q}$ -linearly independent vectors  $v_1, \dots, v_4$  in one half-space of  $\mathbb{R}^3$ , such that none of them is in the cone generated by the other three.

Embed  $G = (\mathbb{Z}^4, \mathbb{N}^4)$  into  $V = \mathbb{R}^3$  by  
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# Good embeddings

There is always a good embedding:

## Theorem (Maloney-T)

If  $(G, G^+)$  is a finite-rank dimension group then there exists an embedding  $G \hookrightarrow V = \mathbb{R}^n$ , such that:

- (i) is an ordered vector space with Riesz interpolation; and
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In fact, we may use the canonical embedding

$$\begin{aligned} G &\hookrightarrow G \otimes \mathbb{Q} \\ &\cong \mathbb{Q}^n \hookrightarrow \mathbb{R}^n \end{aligned}$$

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The proof mainly looks at the positive functionals  $H \rightarrow \mathbb{R}$  for ideals  $H$  of  $G$ .

(An ideal is an order-convex, directed subgroup.)

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## Lemma: Restriction preserves extremeness

If  $K \subseteq H$ ,  $f : H \rightarrow \mathbb{R}$  is an extreme positive functional then  $f|_K$  is either zero or an extreme positive functional.

## Lemma: Restriction is one-to-one

If  $K \subseteq H$ ,  $f : K \rightarrow \mathbb{R}$  is an extreme positive functional which extends to a positive functional on  $H$ , then it has a unique extreme extension.

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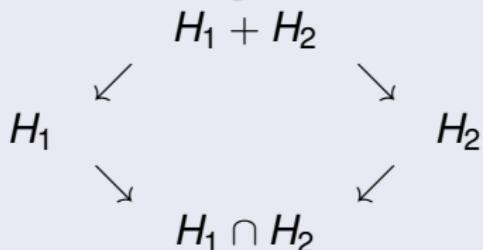
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there exists an extreme extension to  $H_1 + H_2$ .

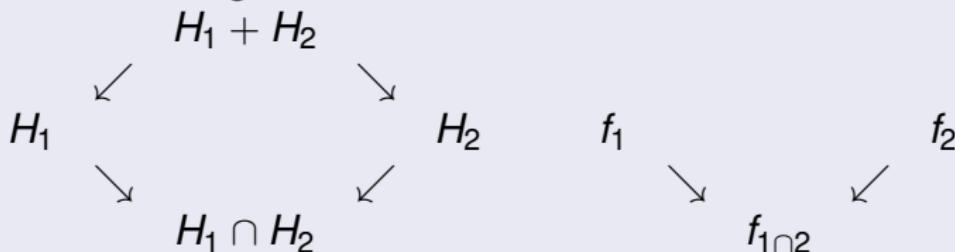
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Use uniqueness of extreme extensions to conclude that  $f$  is extreme.

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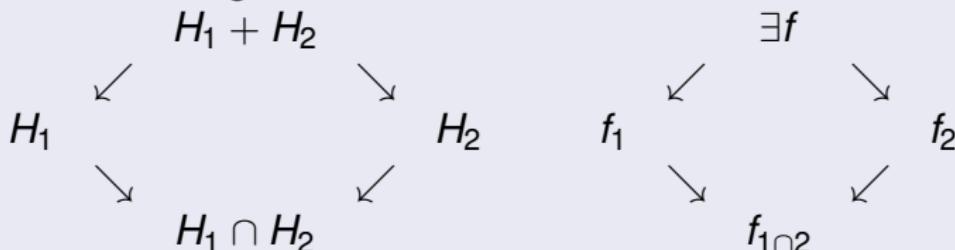
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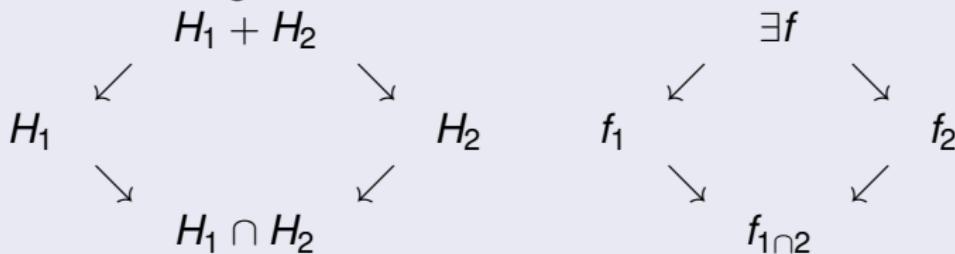
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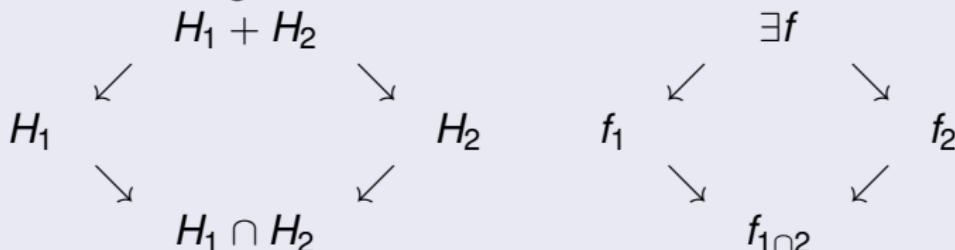
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Proof:  $f(x_1 + x_2) = f_1(x_1) + f_2(x_2)$  is the unique common extension of  $f_1, f_2$ , and it is positive.

Use uniqueness of extreme extensions to conclude that  $f$  is extreme.

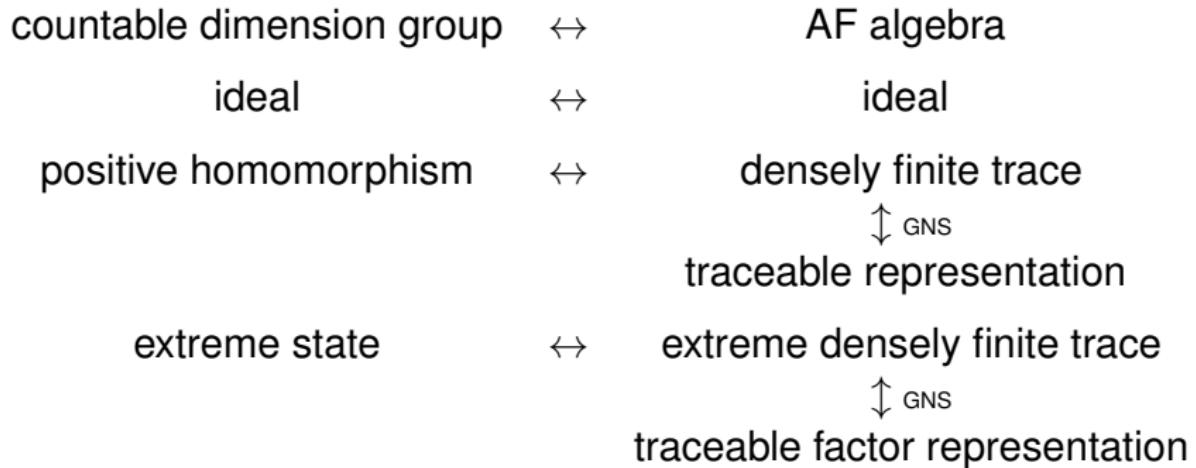
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Can prove the two lemmas for  $K_0(\text{AF})$  using operator theory.



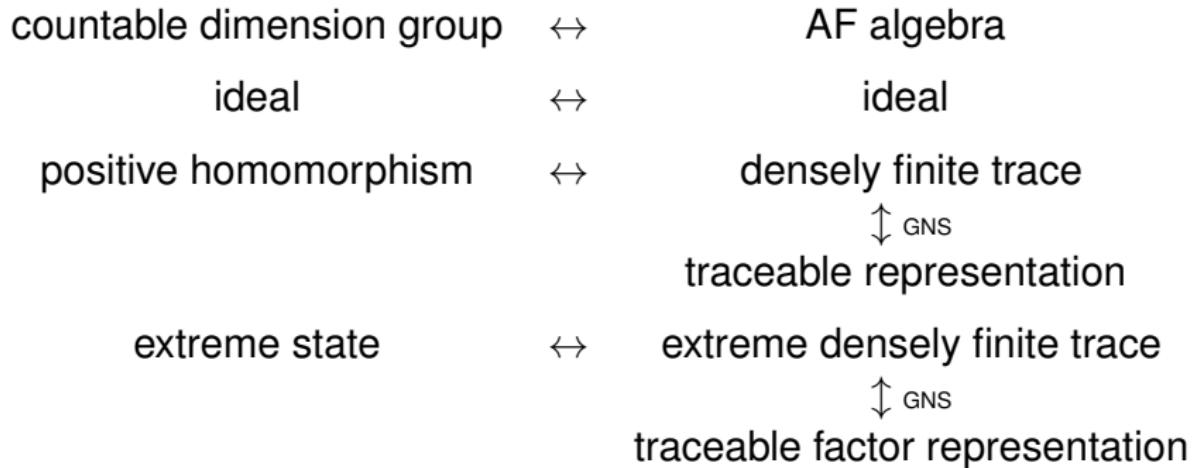
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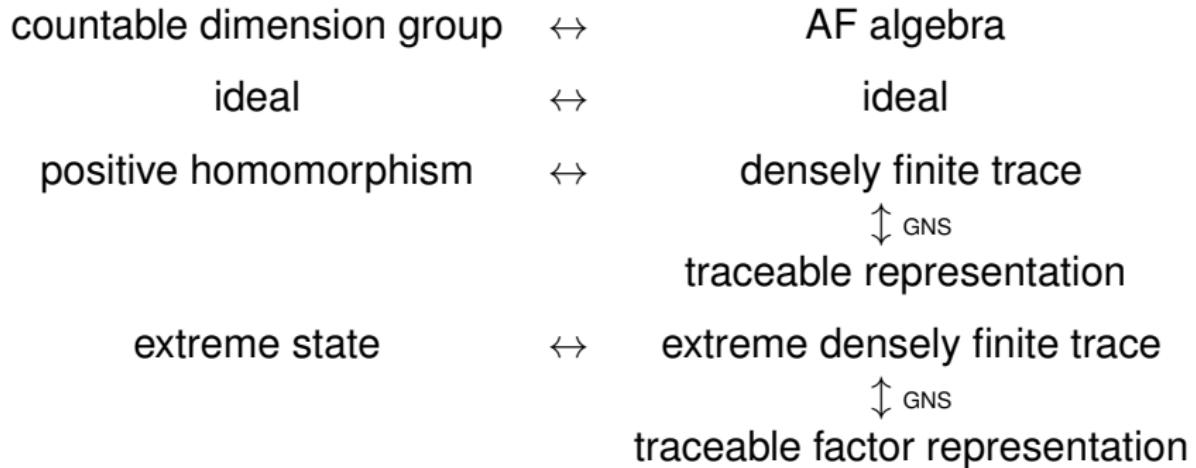
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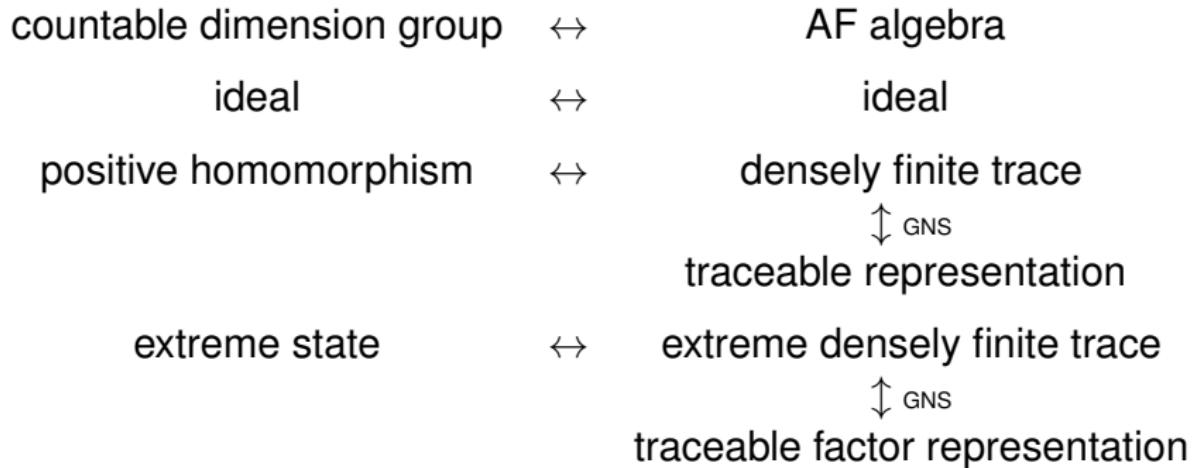
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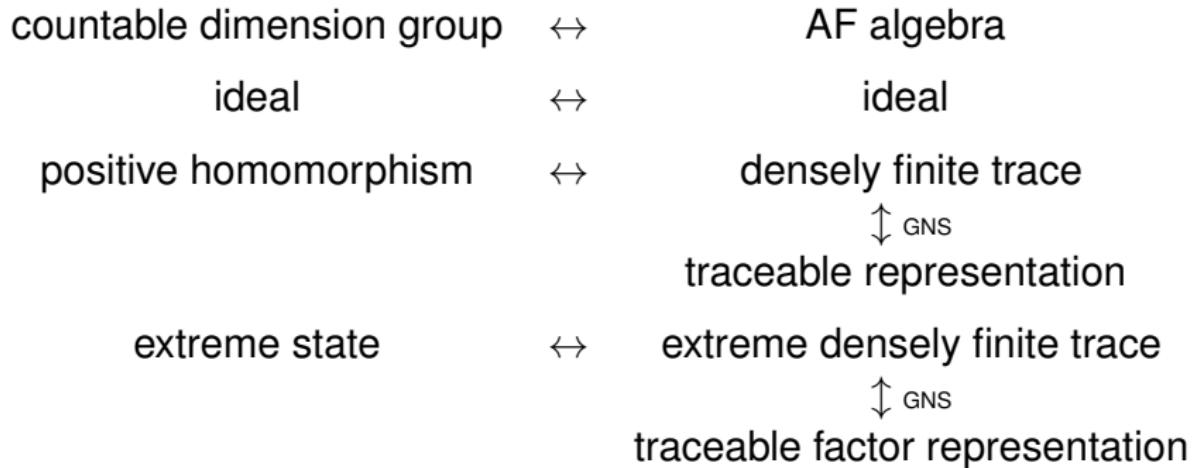
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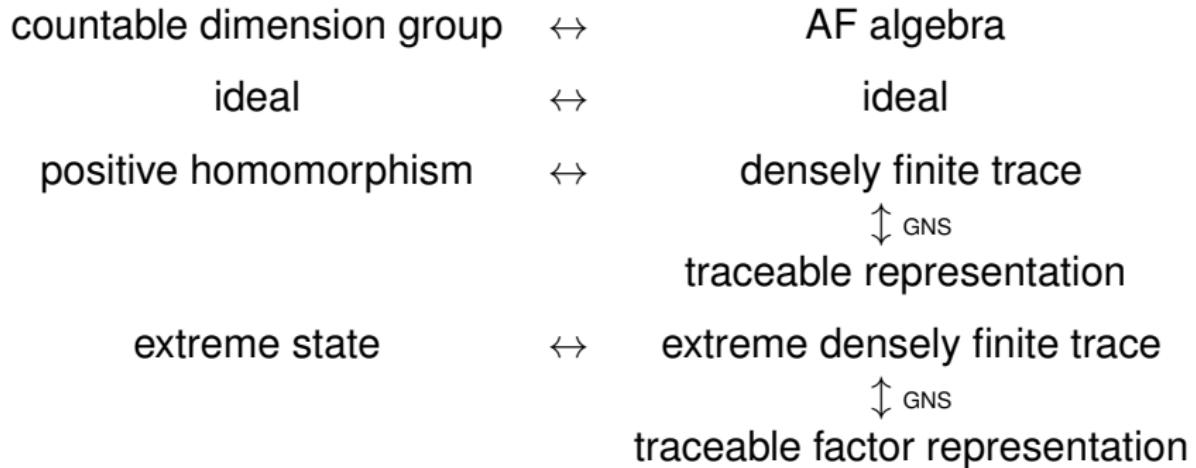
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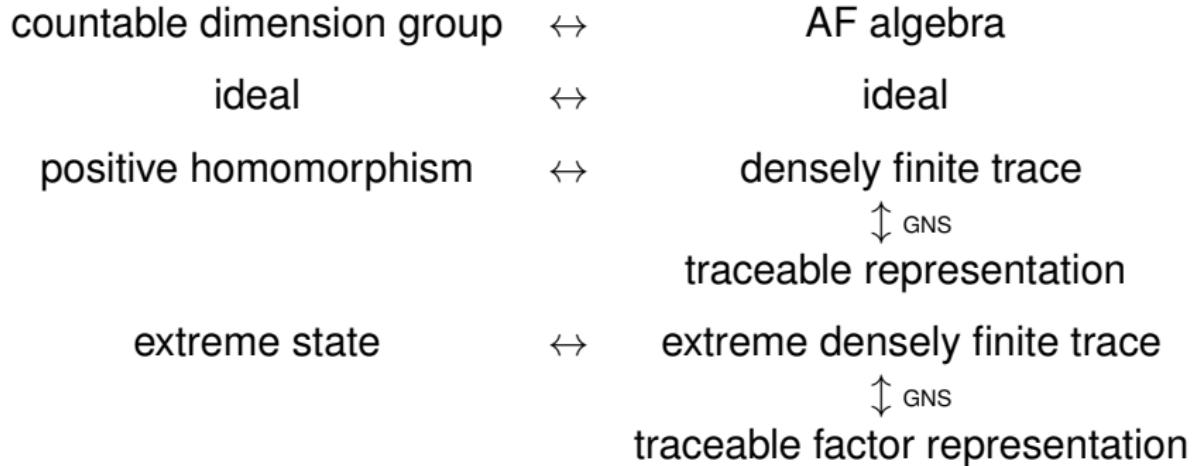
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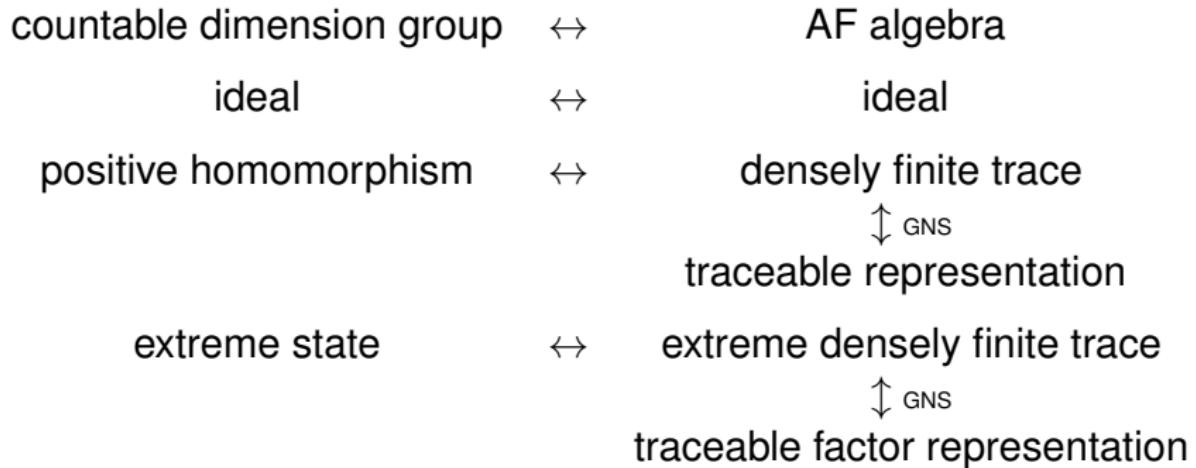
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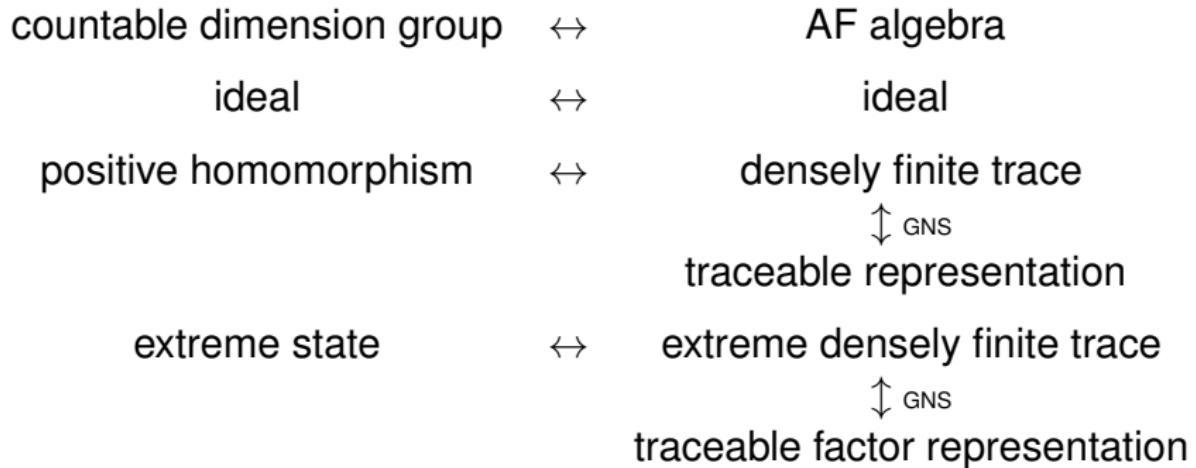
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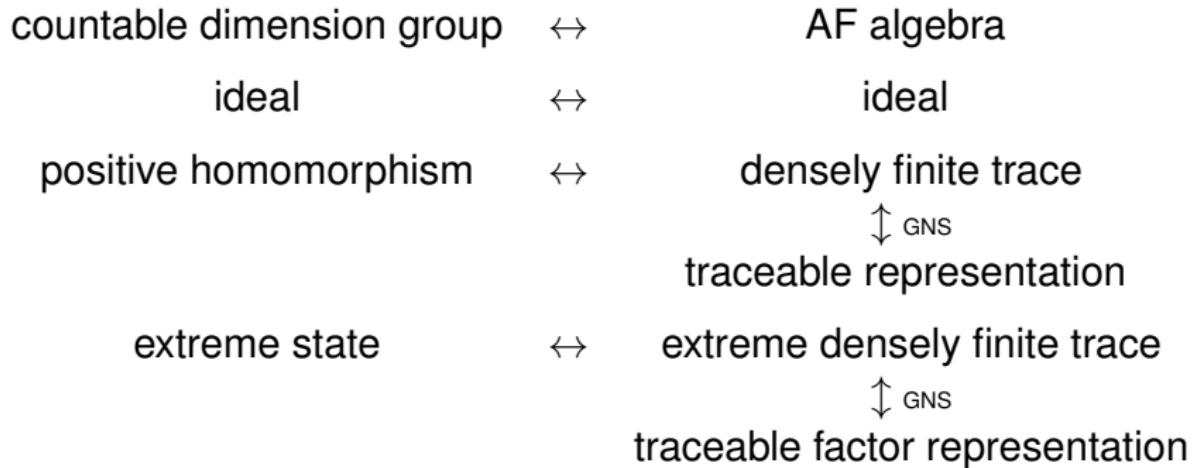
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# $K_0$ -groups of AF algebras

**Lemma: Restriction preserves extremeness**

If  $K \subseteq H$ ,  $f : H \rightarrow \mathbb{R}$  is an extreme positive functional then  $f|_K$  is either zero or an extreme positive functional.

**Proof:**  $H = K_0(A)$ ,  $K = K_0(I)$ ,  $f$  corresponds to traceable factor representation  $\pi : A \rightarrow B(\mathcal{H})$ .

Then  $\pi(A)'' = \pi(I)'' \oplus M$ , so  $\pi|_I$  is either 0 or a factor rep.

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# $K_0$ -groups of AF algebras

Lemma: Restriction is one-to-one

If  $K \subseteq H$ ,  $f : K \rightarrow \mathbb{R}$  is an extreme positive functional which extends to a positive functional on  $H$ , then it has a unique extreme extension.

Proof:  $H = K_0(A)$ ,  $K = K_0(I)$ , and let  $f$  correspond to the trace  $\tau$  on  $I$ .

There exists a rep.  $\pi$  of  $A$  such that  $\pi(I)''$  is a factor with a faithful trace  $\rho$ , and  $\tau = \rho \circ \pi|_I$ .

Then  $\tilde{\pi}(a) = \pi(a)1_{\pi(I)''} = \text{WOT-lim } \pi(ae_\alpha)$  is a factor rep. of  $A$ .

And, if  $\pi$  is a factor rep. then  $1_{\pi(I)''} = 1_{\pi(A)''}$  so  $\pi = \tilde{\pi}$ .

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