The ideal intersection property for essential groupoid C*-algebras

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Theorem (Murray-von Neumann 1936)

For a discrete group G, the group von Neumann algebra $L(G)$ is simple (i.e. factorial) iff G is ICC.

Reduced discrete group C*-algebras

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Note: $H \leq G$ is confined if it is non-trivial and "almost normal" in the sense that

$$1 \notin \overline{\{gHg^{-1}\}}.$$

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Theorem (Elliott 1980, ..., Archbold-Spielberg 1993)

If G is amenable then $\mathrm{C}(X) \times_\lambda G$ is simple if and only if $G \curvearrowright X$ is minimal and topologically free.

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Note: Key idea is to consider the inclusion $\mathrm{C}_0(X) \times_\lambda G \subseteq \mathrm{C}(X^*) \times_\lambda G$, where $X^* = X \sqcup \{\infty\}$.

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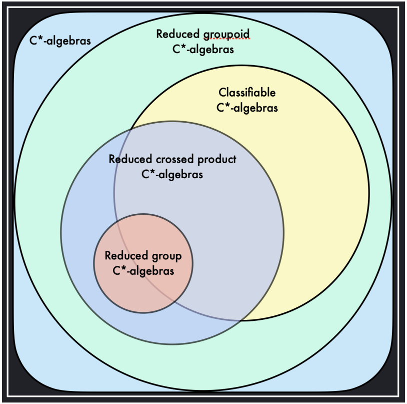

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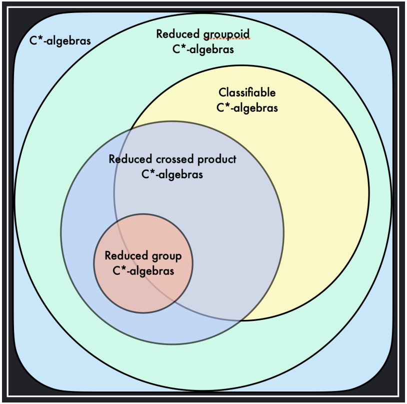

For example, Xin Li showed that every separable simple nuclear Z-stable UCT (i.e. classifiable within Elliott's program) C*-algebra arises as the reduced C*-algebra of a twisted étale Hausdorff groupoid.

Étale groupoids

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Connes and Renault showed that **étale** groupoids give rise to an extremely rich class of C*-algebras.

Reduced C*-algebra of a (Hausdorff) étale groupoid

Let $\mathcal{G}$ be a Hausdorff étale groupoid. For $x \in \mathcal{G}^{(0)}$, obtain a *-representation $\lambda_x : C_c(\mathcal{G}) \to \mathcal{B}(\ell^2(\mathcal{G}_x)$ defined by

$$\lambda_x(f)\delta_g = \sum_{h \in \mathcal{G}_{r(g)}} f(h)\delta_{hg}.$$

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The regular *-representation $\lambda : C_c(\mathcal{G}) \to \oplus_{x \in \mathcal{G}^{(0)}} \mathcal{B}(\ell^2(\mathcal{G})x)$ of $\mathcal{G}$ is

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The **reduced C*-algebra** $\mathrm{C}^*_\lambda(\mathcal{G})$ of $\mathcal{G}$ is the C*-algebra generated by $\lambda(C_c(\mathcal{G}))$.

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If $\mathcal{G}$ is amenable, then $\mathrm{C}^*_\lambda(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is minimal and topologically principal.

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Note: One direction only. We will return to this point.

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This is the quotient of $\mathrm{C}^*_\lambda(\mathcal{G})$ by ideal of "singular" elements

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where $E : \mathrm{C}^*_\lambda(\mathcal{G}) \to B^\infty(\mathcal{G}^{(0)})/M^\infty(\mathcal{G}^{(0)})$ is a conditional expectation onto the Dixmier algebra of bounded Borel functions modulo the functions with meager support.

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Sufficient conditions for simplicity of $\mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ established in work of Clark-Exel-Pardo-Sims-Starling from 2019 and Kwaśniewski-Meyer from 2021.

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2. Replace $\mathcal{G}$ by the "Furstenberg groupoid" $\partial_F \mathcal{G} \rtimes \mathcal{G}$, which is more tractable. Essentiality of the inclusion $\mathrm{C}^*_{\mathrm{ess}}(\mathcal{G}) \subseteq \mathrm{C}^*_{\mathrm{ess}}(\partial_F \mathcal{G} \rtimes \mathcal{G})$ implies simplicity of smaller C*-algebra equivalent to simplicity of larger C*-algebra.

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Note: Traditionally, a a $\mathcal{G}$-C*-algebra is fibered over $\mathcal{G}^{(0)}$ and so contains a central copy of $\mathrm{C}(\mathcal{G}^{(0)})$.

New category of groupoid C*-algebras

Solution: Instead of acting by elements of $\mathcal{G}$, act by elements of the "pseudogroup" $\Gamma(\mathcal{G})$ consisting of open bisections of $\mathcal{G}$, i.e. open subsets of $\mathcal{G}$ on which the range and source maps restrict to homeomorphisms.

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A $\mathcal{G}$-C*-algebra A is a C*-algebra containing a (not necessarily central) copy of $C(\mathcal{G}^{(0)})$ along with compatible families of hereditary subalgebras (A_U) and *-isomorphisms $(\alpha_\gamma)_{\gamma \in \Gamma(\mathcal{G})}$ satisfying

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There is a minimal injective $\mathcal{G}$-C-algebra $\mathrm{C}(\partial_F \mathcal{G})$ in the category of $\mathcal{G}$-C*-algebras.*

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Solution: Instead of acting by elements of $\mathcal{G}$, act by elements of the pseudogroup $\Gamma(\mathcal{G})$ consisting of open bisections of $\mathcal{G}$, i.e. open subsets of $\mathcal{G}$ upon which the range and source maps restrict to homeomorphisms.

Definition (sketch)

A $\mathcal{G}$-C*-algebra A is a C*-algebra containing a copy of $C(\mathcal{G}^{(0)})$ along with compatible families of hereditary sub-C*-algebras (A_U) and *-isomorphisms $(\alpha_\gamma)_{\gamma \in \Gamma(\mathcal{G})}$ satisfying

$$\alpha_\gamma : A_{\mathrm{supp}(\gamma)} \to A_{\mathrm{im}(\gamma)}.$$

Theorem (KKLRU 2021)

There is a minimal injective $\mathcal{G}$-C-algebra $\mathrm{C}(\partial_F \mathcal{G})$ in the category of $\mathcal{G}$-C*-algebras.*

We refer to $\partial_F \mathcal{G}$ as the **Furstenberg boundary** of $\mathcal{G}$. For Hausdorff $\mathcal{G}$, coincides with Furstenberg boundary constructed by Borys.

Ideal intersection property

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The C*-algebra $\mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ has the **ideal intersection property** if every nonzero ideal of $\mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ has nonzero intersection with $\mathrm{C}_0(\mathcal{G}^{(0)})$.

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Invariant subspaces of $\mathcal{G}^{(0)}$ clearly give rise to ideals of $\mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$. If $\mathcal{G}$ is Hausdorff and inner exact, then the ideal intersection property is equivalent to the statement that every ideal in $\mathrm{C}^*_{\lambda}(\mathcal{G})$ arises in this way. Note: inner exactness is necessary.

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Theorem (KKLRU 2021)

The following are equivalent:

1. *The C*-algebra $\mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property.*
2. *The C*-algebra $\mathrm{C}^*_{\mathrm{ess}}(\partial_F \mathcal{G} \rtimes \mathcal{G})$ has the ideal intersection property.*
3. *The Furstenberg boundary $\partial_F \mathcal{G}$ is free.*
4. *There is a unique $\mathcal{G}$-pseudoexpectation from $\mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ to $\mathrm{C}(\partial_F \mathcal{G})$.*

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Corollary

The C-algebra $\mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ is simple if and only if $\partial_F \mathcal{G}$ is minimal and free.*

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Corollary

The C-algebra $\mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ is simple if and only if $\partial_F \mathcal{G}$ is minimal and free.*

Note: We can apply this result when $\mathcal{G}^{(0)}$ is not compact by replacing $\mathcal{G}$ with a suitable one-point compactification.

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Theorem (KKLRU 2021)

The C-algebra $\mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is minimal and has no amenable confined isotropy subgroups.*

Intrinsic characterization of the ideal intersection property

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A **section** of isotropy subgroups a collection

$$\Lambda = \{(x, H_x) : x \in \mathcal{G}^{(0)}, H_x \leq \mathcal{G}_x^x\}.$$

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Theorem (KKLRU 2021)

The C-algebra $\mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property if and only if $\mathcal{G}$ has no amenable essentially confined sections of isotropy subgroups.*

Relative Powers Averaging Property

We generalize K-Haagerup and Amrutan-Ursu's characterization of simplicity in terms of Powers-type averaging properties.

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Note: The terminology "generalized" probability measure is used in the sense of Amrutan-Ursu.

Related open problems

1. In the non-Hausdorff case, what is the relationship between the ideal structure of $\mathrm{C}^*_{ess}(\mathcal{G})$ and $\mathrm{C}^*_\lambda(G)$?

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2. How does the triviality of the Furstenberg boundary $\partial_F \mathcal{G}$ relate to the amenability of $\mathcal{G}$?

3. When is the reduced C*-algebra of a twisted group simple?

Thanks!