Dimension and Non-commutativity

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1. Introduction

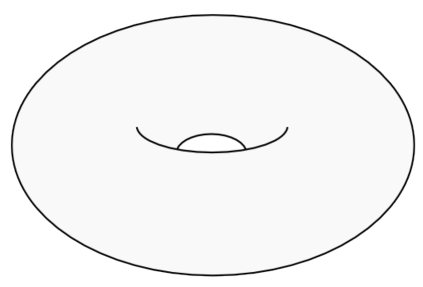
Mathematics provides the means to model the real world with abstract objects. A **topological space** is an abstract object which can model such things as:

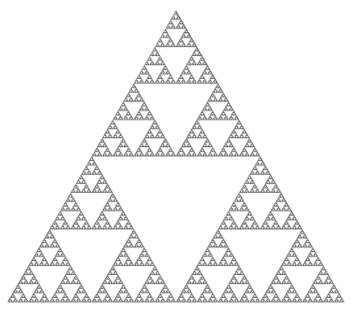
- a shape in two, three, or more dimensions;
- the possible states of a physical system;
- the points in a large set of data.

The concept of a topological space not only handles these different notions under a common framework, but also includes intrinsically mathematical (non-real-world) cases.

More examples:

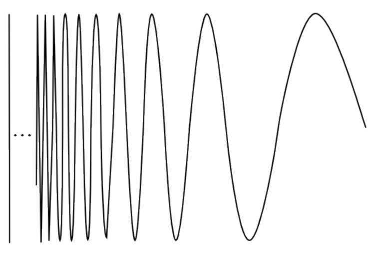
Torus (doughnut surface)

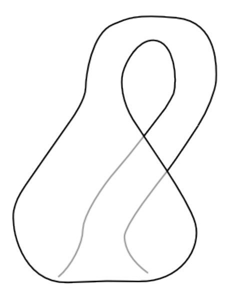




Sierpinski triangle

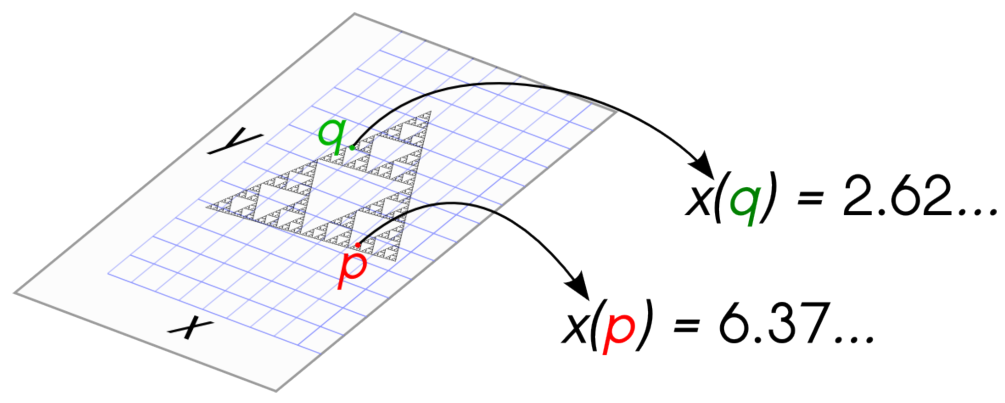
Topologists' sine curve





Klein bottle

It is often useful to understand a space in terms of parametrisations of the space with **continuous functions**: ways of continuously assigning a number to each point of the space. If our space can be drawn on a piece of paper, then we may impose a grid of (x, y)-coordinates, and assigning each point its x-coordinate (say) is an example of such a continuous function.

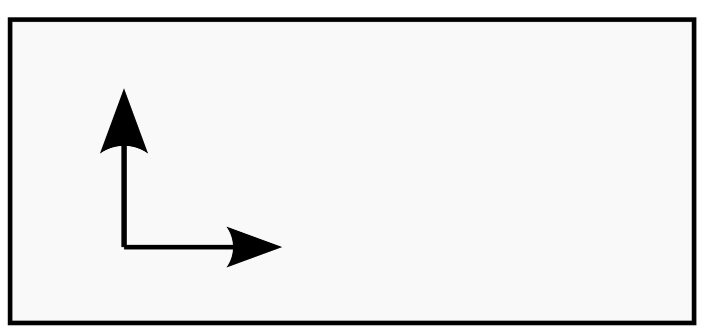


Another example of a continuous function is: assigning each point the value $x \cdot y$.

2. Dimension

One way of measuring the complexity of a topological space is the notion of **dimension**. Intuitively, the dimension of a space can be interpreted as the number of independent directions that one may travel within the space. The plane is two-dimensional, since every direction in the plane is a combination of "up/down" and "left/right".

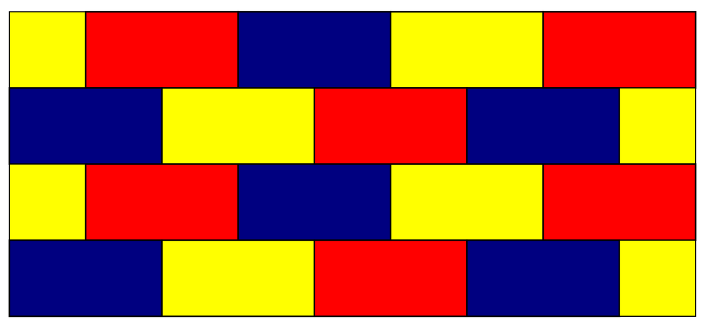
1 direction = 1 dimension



2 directions = 2 dimensions

It has been known for a long time that this notion of dimension is, in fact, equivalent to the following **coloured-tiles version**: a topological space has dimension n if you can cover the space with arbitrarily small tiles with $n+1$ colours, such that tiles that touch have different colours.

2 colours = 1 dimension



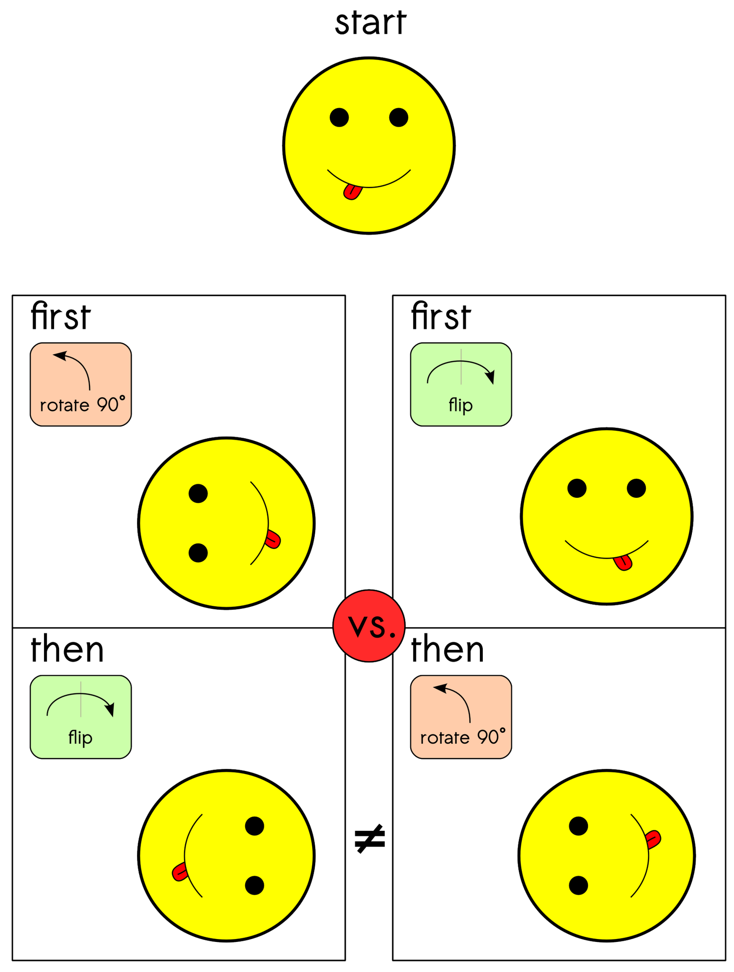
3 colours = 2 dimensions

3. Non-commutativity

For most of us, the ***order of multiplying numbers doesn't matter***. After all, 4×5 and 5×4 both equal 20. In mathematics, this irrelevance of the order of multiplication is called **commutativity**. Numbers are commutative, and therefore so are things that can be described by numbers, such as the continuous functions described earlier.

However, there are mathematical objects for which the order of multiplication ***does matter***. Multiplication, where the answer depends on the order, is called **non-commutative**. Some examples:

Rotations and flips:



Matrix multiplication: Matrices — arrays of numbers — are used often in solving real-world problems. Multiplication of matrices is a crucial operation, which is non-commutative, even though the numbers from which they are assembled are commutative.

$$\overbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}^{x}\overbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}^{y} = \begin{pmatrix} 0\cdot 0+1\cdot 1 & 0\cdot 0+1\cdot 0 \\ 0\cdot 0+0\cdot 1 & 0\cdot 0+0\cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

whereas

$$\overbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}^{y}\overbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}^{x} = \begin{pmatrix} 0\cdot 0+0\cdot 0 & 0\cdot 1+0\cdot 0 \\ 1\cdot 0+0\cdot 0 & 1\cdot 1+0\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Heisenberg's uncertainty principle: This principle is one of the most fundamental laws in physics, making precise how the position ($\hat{x}$) and momentum ($\hat{p}$) of a particle cannot both be accurately measured. It is described by the non-commuting of the operations of measuring $\hat{x}$ and $\hat{p}$:

$$\hat{x} \cdot \hat{p} = \hat{p} \cdot \hat{x} + i\hbar.$$

A **non-commutative space** or **C^*-algebra** is a mathematical objects which are like the continuous function on a topological space, except that multiplication may be non-commutative. ***We can use dimension to measure the complexity of non-commutative spaces.*** This is done using the coloured-tile version of dimension.

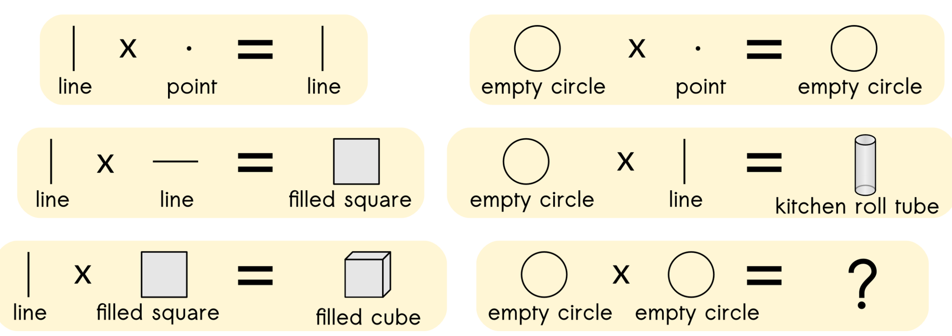
4. Objectives

We are interested in understanding:

- How to compute the dimension of non-commutative spaces; and
- What the dimension of a non-commutative space tells us about it.

5. Combining spaces

Two spaces, A and B, may be combined to form a **product**, $A \times B$.



What happens with the dimension? In most cases,

dimension of A + dimension of B = dimension of $A \times B$.

Some abstract, non-intuitive examples don't satisfy this formula. But in all cases of topological spaces (including these funny examples):

dimension of $A \leq$ dimension of $A \times B$

and

dimension of $B \leq$ dimension of $A \times B$.

For non-commutative spaces, we have shown that a curious and unexpected phenomenon arises:

Theorem. *(Tikuisis-Winter) For a certain non-commutative space, $\mathcal{Z}$, the dimension of $A \times \mathcal{Z}$ is always at most 2. In particular,*

dimension of $A \gg$ dimension of $A \times \mathcal{Z}$

when A is a high-dimensional space.

Non-commutative spaces of smaller dimension are less complex, and more amenable to powerful techniques and analysis. This theorem, therefore, can be interpreted as ***good news***.

The non-commutative space $\mathcal{Z}$ is one that has been studied in many other contexts, and arises naturally and frequently in the study of non-commutative spaces. One important feature of $\mathcal{Z}$ is that it is the same as its product with itself, just as the topological space consisting of a point is equal to its product with itself. Many results show that very many highly non-commutative spaces are automatically equal to a product of one non-commutative space and $\mathcal{Z}$, just as ***every space A is equal to a product of A and a point***.

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