Analytic properties of groups and their actions.

Summary: amenable groups (discrete, topological groups), random walks on groups (spectral radius, Liouville property, etc.), sofic groups, hyperlinear, Property (T), hyperbolic groups.

Amenable groups

$\boxed{\text{Def 1}}$ A group G is called amenable if it admits a fin additive measure μ s.t. $\mu(G) = 1$ and $\mu(gE) = \mu(E)$ $\forall E \subset G$ and $g \in G$.

$\boxed{\text{Def 2.}}$ G is amenable if G admits G-invariant mean.

μ is a mean on X if μ is a lin. functional on $\ell^\infty(X)$, $\mu(\chi_G) = 1$, $\mu(f) \geq 0$ $f \geq 0$ $f \in \ell^\infty(G)$

$\boxed{G \curvearrowright X}$

G-inv. : $\mu(g.f) = \mu(f)$ $\forall f \in \ell^\infty(X)$, $g \in G$.

Def 1. due to von Neumann

$\Updownarrow$

Def 2.

G has paradoxical decomp. if $\exists H_i, V_i$

$G = \bigsqcup_{i=1}^n H_i \cup \bigsqcup_{i=1}^m V_i$, $\exists g_i$ $\exists h_i$

$G = \bigsqcup^n g_i H_i = \bigsqcup^m h_i V_i$

$i=1$ $\qquad$ $i=1$

If G has parad. deco $\overset{\Leftarrow}{\Rightarrow}$ G is not amenable

$$1 = m(G) = \mu\left(\bigsqcup_{i=1}^{n} H_i\right) + \mu\left(\bigsqcup_{i=1}^{m} V_i\right) =$$

$$= \mu\left(\bigsqcup_{i=1}^{n} g_i H_i\right) + \mu\left(\bigsqcup_{i=1}^{m} h_i V_i\right) = 2$$

Remark: $G \curvearrowright X$ — amenable
X admits G-invariant mean.

Ex.: define paradoxical decomp. on X
prove that if $G \curvearrowright X$ amenable
then it is not paradoxical.

If G is amenable $\Rightarrow$ $G \curvearrowright X$ is amen.
$\forall X$.

Def 3 (Følner set):
G is amenable, if $\exists$ Følner sets
For any finite set $E \subseteq G$, $\varepsilon > 0$
$\exists F \subseteq_{fin} G$ s.t

$$|gF \triangle F| < \varepsilon |F| \quad \forall g \in E.$$

Def 3 $\Rightarrow$ Def 2

$E_i \nearrow G$ $\qquad$ sequence of fin sets

$\exists F_i \underset{fin}{\subseteq} G$ $\boxed{|gF_i \,\Delta\, F_i| < \frac{1}{i}|F_i| \quad \forall g \in E_i.}$

$$\mu_i = \frac{1}{|F_i|}\chi_{F_i}$$

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cluster point in ~~weak*~~-top.
gives a mean, which is G-invar.
Ozawa - Brown (on amenability).

Examples: (1) Fin. groups.

(2) Subexponential growth

$$\limsup_n |B(n)|^{1/n} = 1.$$

$B(n)$: ball of radius n over a given fin. gen. set.

Ex: $\exists$ a subset of $B(n)$ which is a Følner set.

(3) Abelian groups.

(4) Free group is not amenable.

$\mathbb{F}_2 = \langle a, b \rangle$.

$\underline{W(x)}$ = all words in $\mathbb{F}_2$ which start with x.

$$\mathbb{F}_2 = \{e\} \cup \omega(a) \cup \omega(b) \cup \omega(a^{-1}) \cup \omega(b^{-1})$$

$$\underline{\omega(x) = x\left(\mathbb{F}_2 \setminus \omega(x^{-1})\right)}$$

$$1 = m(\mathbb{F}_2) = m(\{e\}) + m(\omega(a)) + m(\omega(b)) + m(\omega(a^{-1})) + m(\omega(b^{-1})) =$$

$$= m(\{e\}) + [1 - m(\omega(a^{-1}))] + [1 - m(\omega(a))] + [1 - m(\omega(b))] + [1 - m(\omega(b^{-1}))] = 2.$$

<u>OPEN ISSUE</u>: Find non-amenable groups that don't contain $\underline{\mathbb{F}_2}$.

Examples of non-amenable groups without F_2

① Candidate: Thompson group F.

= group of all piece-wise linear homeo of $[0,1]$ with breaking points of the first derivative in $\mathbb{Z}[\frac{1}{2}]$ and slopes are powers of 2.

$F_2 \not\leq$ Thom. F.

(1965).

1980 Ol'shanskii: not amenable, all proper subgroups are torsion.

1983 Adyan: free Burnside group $B(n,m)$ is not amenable, n odd

$n \geq 665$

$m \geq 2$.

$B(n,m) = \langle x_1, \dots, x_n : x^m = e \quad \forall$ words x in $x_1, \dots, x_n \rangle$.

$\not\supset$
$\mathbb{F}_2$

<u>Monod - Ozawa</u> (with restrict on n).
$n \geq 665$.

2013 Ol'shanski, Sapir: finitely presented not am. no $\mathbb{F}_2$.

Golod - Shafarevych, Ershov it has Property (T).

<u>Osin</u>: G without $\mathbb{F}_2$, with positive ℓ^2-Betti number.

<u>Monod</u>: piece-wise $PSL_2(A)$-homeo of projective line that fix ∞
A is a dense subring of $\mathbb{R}$.

<u>OPEN</u>: Dixmier similarity problem

G is amenable $\Leftrightarrow$ G is unitarizable.

$\forall$ bounded representation
$\pi : G \to B(H)$ $\exists$ $S \in B(H)$ - invert.

$S\pi(g)S^{-1}$ is unitary $\forall g \in G$.

$$\|\pi(g)\| \le \underline{C} \quad \forall g \in G.$$

Osin's is not unitarizable.

Pisier: Similarity problems & completely bdd maps.

Maria Gerasimova, Andreas Thom, Nicolas Monod.

<u>Combination of algebra & amenability</u>

① • groups of intermidiate growth.

topology + analysis.

② • fin. gen., ∞, simple, amenable

OPEN • fin presented, ∞, simple, amenable (Nekrashevych)

topol. + agebra

③ • fin gen. intermidiate growth, simple

OPEN • fin presented intermidiate growth simple.

<u>Intermidiate growth</u>:

Milnor 1968: a group not polynom not exponential.

Grigorchuk 1985.

Kate's book: all known example.

Algebra and amenability

- Topological full group.

X - Cantor set.

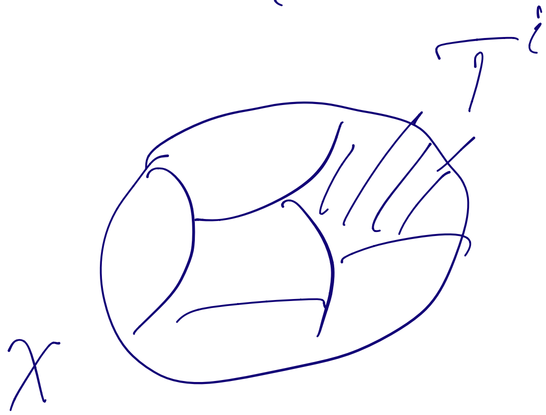
$Homeo(X) \ni T$

$[[T]]$ - topological full group of T, piece-wise powers of T.

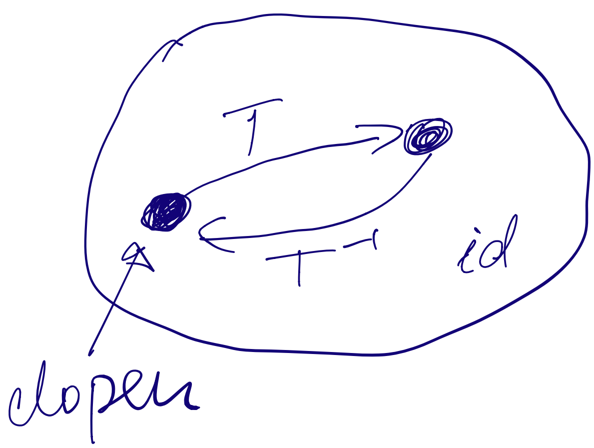
$\varphi \in [[T]]$ if $\varphi \in Homeo(X)$ $\exists$ clopen decomposition of X

$X = \bigsqcup_{i=1}^{n} C_i$, C_i - clopen

$$\varphi\big|_{C_i} = T^{k_i}$$



$\varphi \in [[T]]$
$\cap$
$Homeo(X)$.

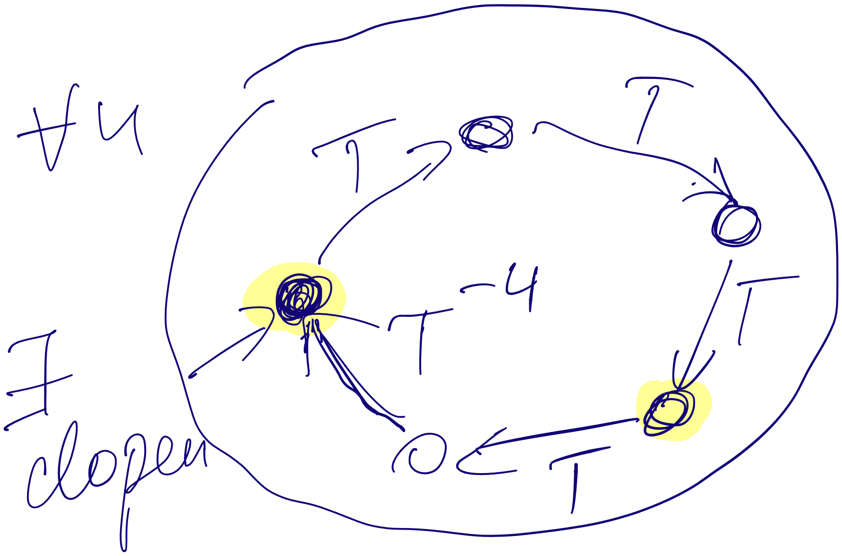


φ

$T \curvearrowright X$ minimally if

$\overline{\{T^i x : i \in \mathbb{Z}\}} = X \quad \forall x \in X$.

$S(n)$.



$S(n) \subset [[T]]$.

<u>Matui</u>: $[[T]]'$ - simple.
(commutator)

fin. generated.

<u>Monod, J</u>: $[[T]]'$ amenable

<u>Nekrashevych ('19, Annals)</u>

$\exists$ subgroup $[[T]]$ s.t.

intermediate growth.

(every element is of fin

order), fin. generated,

<u>Simple</u>.

Analog $A(G)$

Construction of the group

$a \in Homeo(X)$, $a^2 = 1$.

A - finite group of $homeo(X)$

$\forall h \in A$, $z \in X$

$h(z) = z$ or $h(z) = a(z)$

and $\forall z \in X$ $\exists h \in A$

$h(z) = a(z)$.

A - fragmentation of a.

$D_\infty = \langle a, b : a^2 = b^2 = e \rangle$

A - fr. of a,

B - fragm. of b.

$\langle A, B \rangle \subset \mathrm{Homeo}(X)$.

<u>Rem.</u>: Grigorchuk is such.

<u>Nekrashevych</u>: Assume ξ is a fixed point of a and for every $h \in A$ s.t. $h(\xi) = \xi$ the interior of the set of fixed points of h is accumulated around ξ. Then $\langle A, B \rangle$ - period group.

$+$ on B then $\langle A, B\rangle$ is simple $+$ intermediate growth.

<u>Ex:</u> Show $\langle A, B\rangle < [[T]]$ for some $T \in \mathrm{Homeo}(X)$.

<u>Finitely presented</u> ?

<u>Random walks, Kesten's criteria, inverted orbits.</u>

<u>Thm (Kesten)</u>: A group Γ is gen. by a symmetric set S

$$S^{-1} = S$$

$\boxed{\operatorname{supp}\mu_1 = \operatorname{supp}\mu_2}$

$$\Longleftrightarrow \left\| \sum_{t\in S} \mu(t)\lambda(t) \right\| = 1$$

Γ is amenable $\Leftrightarrow$ $\dfrac{1}{|S|} \left\| \sum_{t\in S} \lambda(t) \right\|_{B(\ell^2(\Gamma))} = 1$

$\lambda : \Gamma \to B(\ell^2(\Gamma))$, $f \in \ell^2(\Gamma)$

$[\lambda(g) f](t) = f(g^{-1}t)$.

$\boxtimes$ Γ - amenable $\Rightarrow$

$\exists$ an almost invariant vector for λ. $\|\xi\|=1$ $\|\lambda(t)\xi - \xi\|_2 \to 0$

S - gen. $\exists F$ $\dfrac{|sF \triangle F|}{|F|} \to 0$.

$$\xi = \frac{\chi_F}{|F|^{\frac{1}{2}}} \longrightarrow$$

$\chi_F^{1/2}$

$\Downarrow$

$$\frac{1}{|S|} \left\| \sum_{t \in S} \lambda(t) \right\| = 1$$

Assume $\dfrac{1}{|S|} \left\| \sum_{t \in S} \lambda(t) \right\| = 1$

self adjoint

$\forall \varepsilon > 0$ $\exists \xi \in \ell^2(\Gamma)$

$$\frac{1}{|S|}\left|\left\langle \sum_{t\in S} \lambda(t)\xi, \xi\right\rangle\right| > 1-\varepsilon$$

$\xi \rightsquigarrow |\xi|$ ~~a~~ pointwise

$$\frac{1}{|S|}\left\langle \sum_{t\in S} \lambda(t)|\xi|, |\xi|\right\rangle \geq$$

$$\geq \frac{1}{S}\left\langle \sum_{t\in S} \lambda(t)\xi, \xi\right\rangle > 1-\varepsilon$$

$\Rightarrow$ $\| \lambda(t)|\xi_i| - |\xi_i| \| \to 0.$

$\Rightarrow$ λ has an almost inv. vector.

$\mu_i = \xi_i^2 \in \ell^1(\Gamma)$

cluster point in weak*-top

$\mu_i \Rightarrow \mu$ - mean.

μ is Γ-invariant. $\boxtimes$

Kesten's criteria in terms of random walks

(Woess & random walks)

S - symm. fin., $\langle S \rangle = \Gamma$,

$\mu(g^{-1}) = \mu(g)$, $g \in \Gamma$,

$\mu \in \ell^1(\Gamma)$ $\|\mu\|_1 = 1$

$\mu \geq 0$.

$$M(m) f(x) =$$

$$= \sum_{t \in \Gamma} f(t^{-1}x) \mu(t)$$

$f \in \ell^2(\Gamma)$.

$$M(\mu) \cdot M(\nu) = M(\mu * \nu)$$

$$\mu * \nu(x) = \sum_{t \in \Gamma} \mu(xt^{-1}) \nu(t)$$

$$\|M(\mu)\| \leq 1$$

$\Downarrow$

$M(\mu)$ is self-adjoint.

(μ before - uniform measure.)

$\sigma(\mu)$ - spectrum of $M(\mu)$

$$\rho(\mu) = \max \{|t| : t \in \sigma(\mu)\}$$

$$\|M(\mu)\| = \rho(\mu).$$

<u>Thm</u>: let Γ be fin. gen.

μ - symmetric measure

with fin support.

$$\|M(\mu)\| = \lim_n \mu^{*2n}(e)^{1/2n}.$$

Corollary: Γ is amenable

$$\iff \lim_n \mu^{*2n}(e)^{1/2n} = 1.$$

$\boxtimes$ $$M(\mu^{*n}) = M(\mu)^n$$

$$\mu^{*n}(e) = M(\mu)^n \delta_e$$

$$= \langle M(\mu)^n \delta_e, \delta_e \rangle$$

$$\mu^{*n}(e) \leq \|M(\mu)^n\|.$$

$$\langle M(\mu)^n \delta_e, \delta_e \rangle = \int_{[-1,1]} t^n \nu(dt)$$

$$\mu^{*2n}(e)^{1/2n} = \langle M(\mu)^{2n}\delta_e, \delta_e\rangle^{1/2n}$$

$$= \left(\int_{[-1,1]} t^{2n}\, \nu(dt)\right)^{1/2n}$$

$$\lim_n \mu^{*2n}(e)^{1/2n} = \lim_n$$

$$= \|t\|_\infty = \max\left(|t| : t \in \sigma(M(\mu))\right)$$

$$= \rho(M(\mu)) = \|M(\mu)\| \quad \boxtimes$$

$\boxed{\text{Inverted orbit}}$

Schreier graphs

if $\Gamma \curvearrowright X$ is amenable

$\Rightarrow \Gamma$ is amenable

Liouville property.

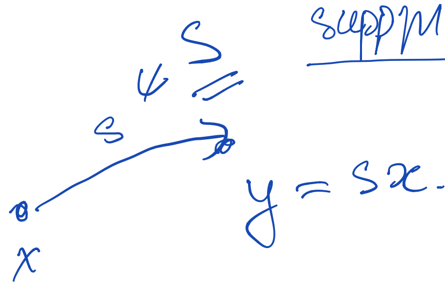
<u>Inverted orbits</u> $\Leftrightarrow$ amenability

$G \curvearrowright G, \quad G \curvearrowright X$

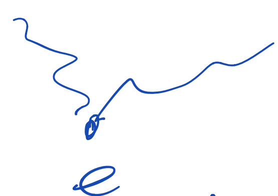
$\langle S \rangle = G, \quad |S| < \infty.$

$G \curvearrowright X \longrightarrow$ Schreier graph

$\underline{\operatorname{supp} \mu = S},$



$G \longrightarrow$ Cayley graph

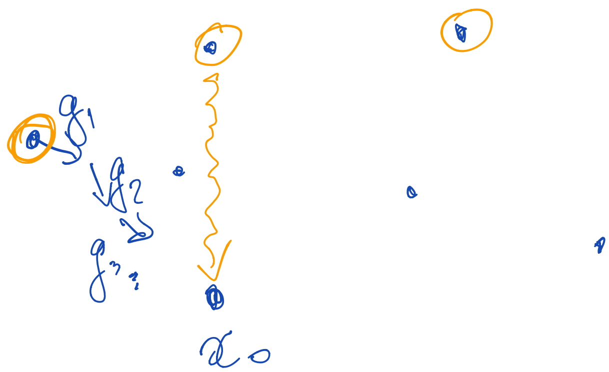


<u>Inverted orbit</u>: fix $x_0 \in X$.

$O_n = \{x_0, \; g_1^{-1} x_0, \; g_1^{-1} g_2^{-1} x_0,$

$$\mu(g_i) = \mu(g_i^{-1})$$

$$g_1^{-1}g_2^{-1}g_3^{-1}x_0, \dots, g_1^{-1}\cdots g_n^{-1}x_0\}$$



Def: $G \curvearrowright X$ extensive amenable:

$$\lim_{n\to\infty} -\frac{1}{n}\log \mathbb{E}\left(2^{-|O_n|}\right) = 0.$$

$$G = \langle \operatorname{supp} m\rangle = S$$

no dependency on m

Thompson group F $\longleftrightarrow$ the Cayley graph is hard.

$$F \curvearrowright [0,1]\cap \mathbb{Z}[\tfrac{1}{2}] \Rightarrow F\text{-amenable}$$

amenable

$G \curvearrowright X$ - amenable $\Longleftrightarrow$

$$\forall \varepsilon > 0 \ \forall V \subset_{fin} G \ \exists X_\varepsilon \subset X$$

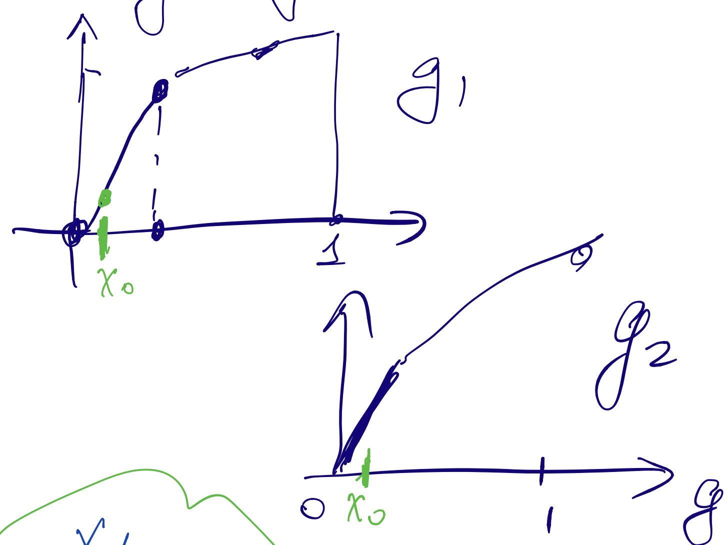
$$|g X_\varepsilon \triangle X_\varepsilon| < \varepsilon \cdot |X_\varepsilon| \quad \forall g \in V.$$

(Suffice V as gen set)

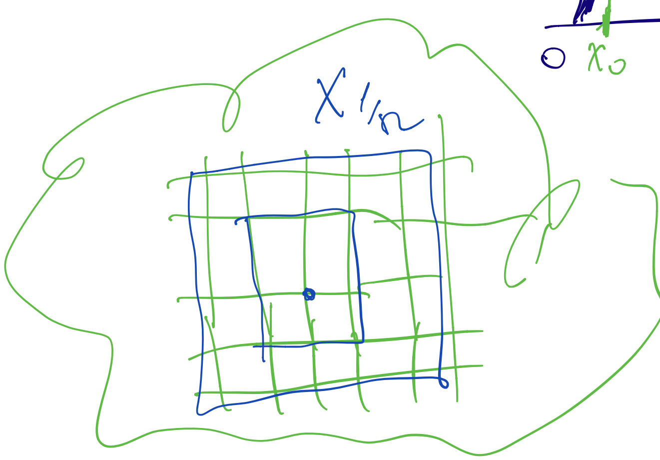
$\mathbb{Z} \curvearrowright \mathbb{Z}$, $\{\pm 1\}$

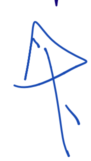
$$X_{\frac{1}{n}} = [-n, n] \leftarrow \text{Følner}.$$

$$F = \langle g_1, g_2 \rangle$$

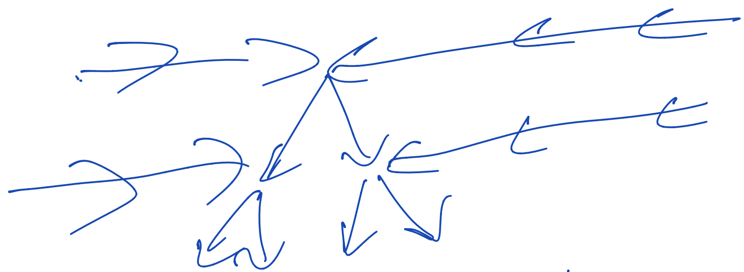


$X_{\frac{1}{n}}$





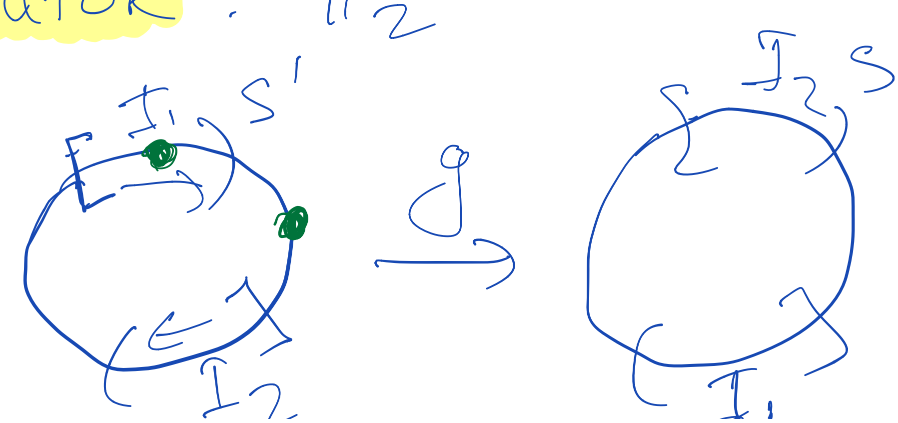
Thm: $F \curvearrowright [0,1] \cap \mathbb{Z}[\tfrac{1}{2}]$ is extensively amenable $\iff$ F is amenable.



$\mathbb{E}\left[2^{-|O_u|}\right]$?

Thm: IET $\curvearrowright S^1$ = ext. amen. $\iff$ IET is amenable

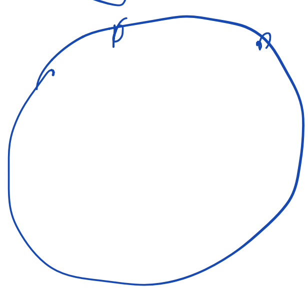
Katok: $\mathbb{F}_2 <$ IET.



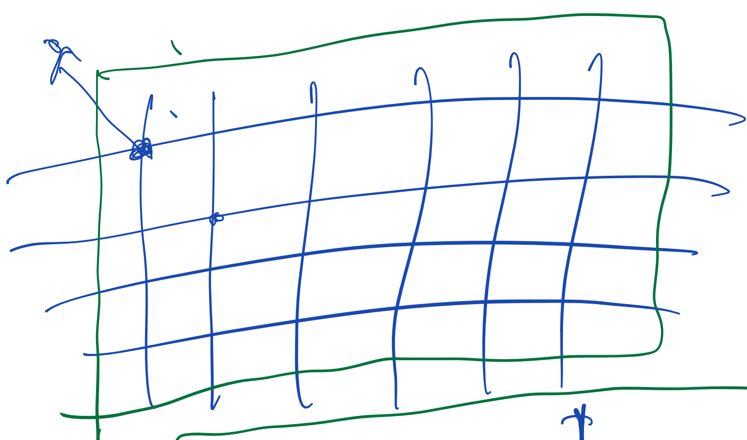
preserves, orientation + the length.

IET $\curvearrowright S^1$ – amenable.

$g_1, g_2 \in IET$



g_1, g_2 piecewise rotations.



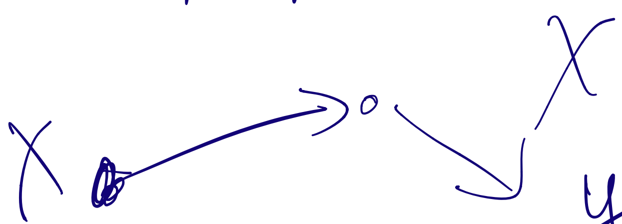
$\exists c$ $\boxed{|g_i x - x| < c}$ $\forall i$

$g_1, g_2 \Rightarrow R_{\theta_1}, \dots, R_{\theta_n}$

Liouville property

$G \curvearrowright X$, μ - proba on G,

$\langle \operatorname{supp} \mu \rangle = G$, $\mu(g) = \mu(g^{-1})$

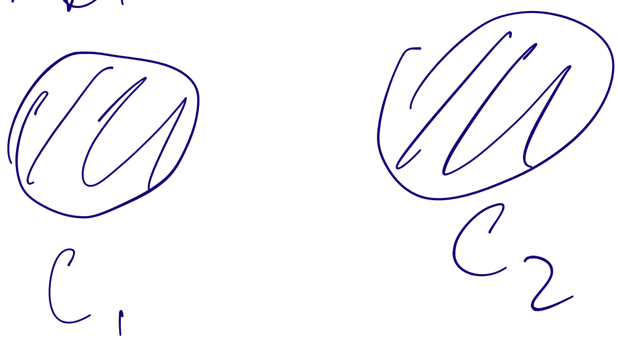


$$P_\mu(x,y) = \sum_{\substack{g \in G \\ gx = y}} \mu(g)$$

$f : X \to \mathbb{R}$ is μ-harmonic

$$f(x) = \sum_{y \in X} f(y)\, P_\mu(x,y)$$

An action of G on X is μ-Liouville if every bdd μ-harmonic function is constant.
($\Rightarrow G \curvearrowright X$ transitive)



$G \curvearrowright X$ is <u>Liouville</u> if $\exists$ generating measure μ s.t. $G \curvearrowright X$ is μ-Liouville.

<u>Thm</u> (Kaimanovich–Vershik): G is

[illegible]

amenable $\Leftrightarrow$ $G \wr G$ is Liouville.

<u>Fact:</u> if $G \curvearrowright X$ is <u>not</u> Liouville $\Rightarrow$ $G \curvearrowright G$ is not Liouville.

$G \curvearrowright X$ - not L. $\Rightarrow$ G is not amen.

$\Rightarrow$ if $f: X \to \mathbb{R}$ μ-harm

$\Rightarrow$ $f_x : G \to \mathbb{R}$

$f_x = f(gx)$ - μ-harm.

<u>Thm (J):</u> $G \curvearrowright X$ trans.

TFAE:

(1) $G \curvearrowright X$ is Liouville.

(2) (analog of Følner)

$\forall \varepsilon > 0 \ \forall F \subseteq_{fin} X \ \exists E \subset_{fin} G$

$$|Ex \triangle Ey| \leq \varepsilon |E|$$

$\forall x, y \in F.$

?

$G \curvearrowright X$ has Følner

$\forall \varepsilon > 0 \ \forall F \subseteq_{fin} G$

$\exists E \subset_{fin} X$

$$|gE \triangle hE| < \varepsilon |E|$$

$\forall g, h \in F.$

$F \curvearrowright [0,1] \cap \mathbb{Z}[\tfrac{1}{2}] = X$

yes to 2

Liouville

<u>n-Liouville</u> $G \curvearrowright X^n$ Liouville

completely-Liouville $G \curvearrowright X^n$ is L $\forall n$.

$(n+1)$-Liouville $\Rightarrow$ n-Liouville

<u>strong transitivity</u>.

$G \curvearrowright X^n$ - transitive.

$\forall n.$

$F \curvearrowright X$ is completely Liouville

(if is not c.L. $\Rightarrow$ not amenable)

Prop(I):

3-Liouville

$\Updownarrow$

(2-Liouville
1-Liou
is satisfied)

$$\forall \varepsilon > 0 \quad \exists V \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$$

$$\Big| \{ (p_1(x), p_2(x)) : x \in V \}$$
$$\cap \{ (p_2(x), p_3(x)) : x \in V \}$$
$$\cap \{ (p_1(x) + p_2(x), p_3(x)) : x \in V \}$$

$\#\{(p_1(x),\ p_2(x)+p_3(x))$

$x\in V\}| \geq (1-\varepsilon)|V|$

<u>Thm</u> (Schneider-Thom):

Yes, $F \cap [0,1] \cap \mathbb{Z}[\frac{1}{2}]$

is compl. Liouville.

$X, \quad X^n$

<u>Ring</u> <u>conjectures</u>

(due Kaplanski).

K - a field, G - a group.

$K[G]$ - a group ring.

$\sum_{i=1}^{n} \lambda_i g_i \qquad \lambda_i \in K,\ g_i \in G.$

① Zero divisor conjecture.

If a group ring $K[G]$ has a zero divisor then G has torsion i.e, $\exists\, a,b \in K[G]$ $ab=0$

torsion $\exists\, g\in G$ $g^n = e$ $g\neq e$.

Ex: $\mathbb{Z}_2[G]$ $G\ni g$ $g^2=e$. Find zero divisor.

$a=b=e-g$

(2) Idempotent conjecture

If G has no torsion then $K[G]$ does not have non-trivial idempotents, ($a^2=a \Rightarrow a=1$ or $a=0$)

(3) Units conjecture

If G has no torsion $\Rightarrow$ there are no non-trivial units ($ab=1 \Rightarrow a=kg$, $k\in K, g\in G$).

$(k^{-1}g^{-1})(kg)=1$

(3) $\Rightarrow$ (1) $\Rightarrow$ (2)

disproved

Giles Gardam: $\mathbb{Z}_2[G]$ that has a non-trivial unit.

G - no torsion

ZDC $\Rightarrow$ (2) $x^2=x \Rightarrow x(x-1)=0$.

(3) $\Rightarrow$ (1) $ab=0$ $K[G]$ - prime

$\exists\, c\in K[G]$ $bca\neq 0$

$$(bca)^2 = (bca)(bca) = 0$$

$$(\underset{\neq 0}{1+bca})(1-bca) = 1$$

Atiyah conj. $\Rightarrow$ ZDC

Baum-Connes $\Rightarrow$ idempotent conj.

Kaplanski direct finiteness

$$ab = e \Rightarrow ba = e$$

$\forall a, b \in K[G]$ $\forall K$ $\forall G$.

(no assumption on torsion)

Holds for $\mathbb{C}$

Valette + Berger: $\mathbb{C}[G] \hookrightarrow vn(G)$ $\cap$ $B(\ell^2 G)$

$\exists \tau$ on $\mathbb{C}[G]$

$x \in \mathbb{C}[G]$ $\tau(x) = \langle x\delta_e, \delta_e \rangle$

$$\boxed{\text{if } p \in vN(G) \text{ is idemp., then } \tau(p)=1 \Rightarrow p=1.}$$

$ab=1 \Rightarrow ba$ is idempot.

$$(ba)(ba) = ba$$

$$\tau(ba) = \tau(1) = 1 \Rightarrow ba = 1.$$

Proof of the box:

$$z = 1 + (p^* - p)^*(p^* - p) =$$

$= 1 - p^* - p + p^*p + pp^*$ - invertible in $vN(G)$

$q = pp^* z^{-1}$ then

$$pp^* z = (pp^*)^2 \Rightarrow q^2 = q$$

z commutes with p, $q^* = q$.

$pq = q$, $qp = p$.

$$\tau(q) = \tau(pq) = \tau(qp) = \tau(p) = 1$$

$$\tau(\underbrace{(1-q)^*(1-q)}_{\Downarrow}) = 1 - \tau(q) = 0$$

$$1 - q = 0 \Rightarrow q = 1 \Rightarrow p = 1 \quad \boxtimes$$

? Kaplanski is open for finite fields.

Properties τ:

$\rightarrow$ (1) unital & tracial

$\tau(e) = 1$, $\tau(ab) = \tau(ba)$

vN $\rightarrow$ (2) $\tau(a^*a) \geq 0$

$\rightarrow$ (3) $\tau(a^*a) = 0 \Rightarrow a = 0$.

<u>Elek</u>: group rings of sofic groups admit an analog of trace.

if G is sofic $\Rightarrow$ $K[G]$ satisfies Kaplanski's conjecture for K.

Def: Γ is sofic if $\forall F \overset{\text{fin}}{\subseteq} \Gamma$, $e \in F$ $\forall \varepsilon > 0$ $\exists n \in \mathbb{N}$

$\varphi: F \to S(n)$ $\mathcal{U}(n)$

(1) $\varphi(e) = 1_n$

(2) $d(\varphi(gh), \varphi(g)\varphi(h)) < \varepsilon$

$\forall g, h \in F$

(3) $\varphi(g)$ does not have fixed points, $g \neq e$

$d(\varphi(g), e) = 1.$

$\forall g \in F.$

$$d(\sigma_1, \sigma_2) = \frac{1}{n} |\{i : \sigma_1(i) \neq \sigma_2(i)\}|$$

d is trace distance

Remarks: finite groups, residually finite.

Γ is sofic $\Leftrightarrow$ all fin gen. subgroups are sofic.

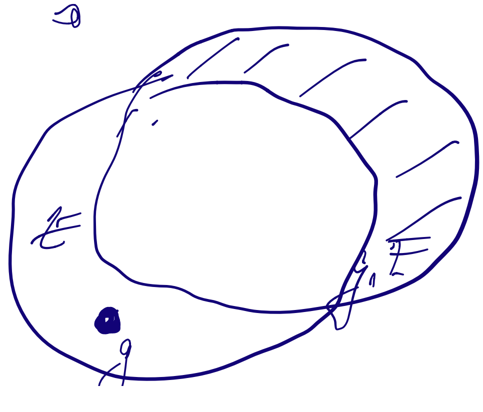
If Γ is amenable $\Rightarrow$ sofic.

Følner: $\forall F \subset_{fin} \Gamma$ $\exists E \subset_{fin} \Gamma$

$\Rightarrow$ $|gE \Delta E| < \varepsilon |E|$ $\forall g \in F.$

$$S(n) \Rightarrow S\left(\bigcup_{g \in F} gE\right)$$

$\{g_1, \dots, g_n\} = F$



define φ on $\bigcup gE = X$

$g \in F$

$\varphi(g) = gx$ if both $x, gx \in X$

Then complete to a permutation on the rest of the points

+ no fixed points.

<u>Thm (Elek)</u>: sofic groups are hyperlinear

<u>Exercise</u>: view $S(n)$ as a subgroup of $U(n)$.

<u>Thm (Elek)</u>: sofic $\Rightarrow \forall K$ $\forall a, b \in K[G]$

$ab = e \Rightarrow ba = e$.

<u>Idea</u>:

$$R_\infty = \prod_\alpha M_{n_\alpha}(K)$$

Pseudo-rank function:

$$\longrightarrow N(\{r_\alpha\}) = \lim_{\omega} \frac{\dim_K \operatorname{rank} r_\alpha}{n_\alpha}$$

(1) $N(0) = 0 \quad N(1) = 1$

$N(x) \in [0,1] \quad \forall x \in R_\infty$

(2) $N(x+y) \leq N(x) + N(y)$

(3) $N(xy) \leq \min\{N(x), N(y)\}$

$\forall x, y \in R_\infty$.

Giles Gardam

$K[G]$ Kaplanski ?

$\mathbb{Z}_2[G]$ - Kapl. is open.

we want to disprove.

$a, b \in \mathbb{Z}_2[G] \quad ab = 1 \not\Rightarrow ba = 1.$

$a = g_1 + g_2 + \dots + g_n \qquad b = h_1 + \dots + h_m$

$$\left(\sum g_i\right)\left(\sum h_j\right) = 1$$

$$\sum_{i,j} g_i h_j = 1 \qquad \boxed{g_0 h_0 = 1}$$

must have many cancellations

$$\boxed{g_i h_j = g_{i'} h_{j'}}$$

G has a lot of relations

G' to be a group generated by $g_0, \dots, g_n, h_0 \dots h_m$ and green relations

$$\boxed{ba = 1 \qquad \begin{array}{l} h_0 g_0 = 1 \\ h_i g_j = h_{i'} g_{j'} \end{array}}$$

GAP $\rightarrow$ can define group with relations, check if $g = e$

Pascal: ZDC treat cases of groups up to $n \leq 5$, $m \leq 5$.

Ken Dykema: Kapl. $n \leq 5$

$m \leq 7$.

any n, $m = 3$ ~~+~~ by hand.