C*-algebras: structure and classification

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Running theme: approximately inner flip

Let A be a unital C*-algebra.

Definition

A *-homomorphism $\phi : A \to A$ is *inner* if there exists a unitary $u \in A$ such that

$$\phi(a) = u^* a u.$$

It is *approximately inner* if there exists a net (u_i) of unitaries in A such that

$$\lim_i \|\phi(a) - u_i^* a u_i\| = 0.$$

E.g., There are no nontrivial inner *-homomorphisms on $C(X)$.

E.g., All *-homomorphisms $M_n \to M_n$ are inner (exercise).

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$$M_2 = \left(\begin{array}{c|c} \mathbb{C} & \mathbb{C} \\ \hline \mathbb{C} & \mathbb{C} \end{array}\right)$$

All *-homomorphisms $M_{2^\infty} \to M_{2^\infty}$ are approximately inner.

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$$M_4 = \left(\begin{array}{cc|cc} \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \hline \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \end{array} \right)$$

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$$M_8 = \left(\begin{array}{cc|cc|cc|cc}
\mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\
\mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\
\hline
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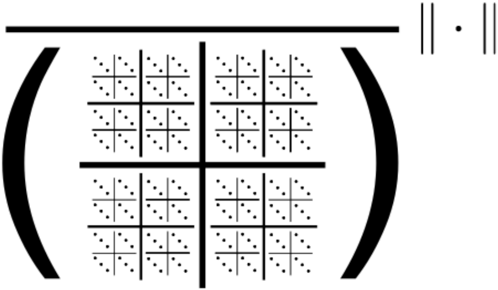
$$M_{2^\infty} =$$



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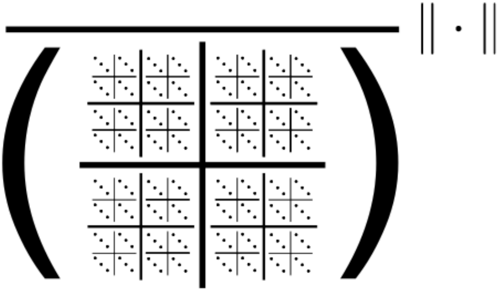

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Definition

$A \otimes B$ denotes the *minimal,* a.k.a. *spatial* tensor product of A and B (which contains $A \otimes_{\mathrm{alg}} B$ as a dense *-subalgebra). The *flip* map is the *-homomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$ determined by

$$\sigma_{A,B}(a \otimes b) := b \otimes a.$$

Question

When is the flip $\sigma_{B,B} : B \otimes B \to B \otimes B$ approximately inner?

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E.g., M_n has inner flip, since $M_n \otimes M_n \cong M_{n^2}$.

E.g., M_{2^∞} has approximately inner flip, since $M_{2^\infty} \otimes M_{2^\infty} \cong M_{2^\infty}$.

E.g., $C(X)$ has approximately inner flip if and only if $X = \{*\}$.

More generally, approximately inner flip $\Rightarrow$ simple.

For this question, we'll stick to the case that B is separable.

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Some history

Connes '76: the only separable II_1 factor with (von Neumann algebraic) approximately inner flip is $\mathcal{R}$.

This was a key step in Connes's proof that $\mathcal{R}$ is the only injective II_1-factor.

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A C*-algebra A is nuclear if it satisfies the following equivalent conditions:

(i) $A \otimes B$ is the only way of completing $A \otimes_{\text{alg}} B$ to a C*-algebra;

(ii) A has the completely positive approximation property: there exists a net $((F_i, \phi_i, \psi_i))_i$ of finite dimensional C*-algebras F_i and c.p.c. maps

$$\begin{tikzcd} A \arrow[dr, "\psi_i"'] & & A \\ & F_i \arrow[ur, "\phi_i"'] & \end{tikzcd}$$

such that for all $a \in A$, $\lim_i \|\phi_i(\psi_i(a)) - a\| = 0$.

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E.g., $C^*_r(G)$ is nuclear if and only if G is amenable.

E.g., $C(X) \rtimes G$ is nuclear whenever G is amenable.

Theorem (Effros–Rosenberg)

If B has approximately inner flip then B is nuclear.

Non-nuclear C*-algebras are too big (intractable) to hope to classify.

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Let $p, q \in A$ be projections. We write $p \sim q$ if there exists $v \in A$ such that

$$p = v^* v \quad \text{and} \quad vv^* = q.$$

Fact

If $\phi : A \to A$ is approximately inner, then $\phi(p) \sim p$ for all projections $p \in A$.

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$$p \sim q \text{ iff } \quad \exists v \in A,\ p = v^*v \text{ and } vv^* = q.$$

The K_0-group packages $\sim$-equivalence classes into a group.

$K_0(A) :=$ abelian group generated by equivalence classes of projections from $\bigcup_n M_n(A)$, with

$$[p] + [q] := \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right].$$

$K_0(A)_+ := \{[p] \mid p \in M_n(A)\}$.

K_0 is a functor, since a *-homomorphism $\phi : A \to B$ induces a map $\phi_* : K_0(A) \to K_0(B)$ by

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K-theory also includes $K_1(A)$, built out of unitaries.

Isomorphic C*-algebras have the same K-theory.

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A *trace* on a unital C*-algebra is a self-adjoint linear function $\tau : A \to \mathbb{C}$ such that $\tau(xy) = \tau(yx)$ and $\tau(1) = 1$.

$T(A)$ denotes the set of traces on A.

A C*-algebra with approximately inner flip has at most one trace.

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Classification

Conjecture (Elliott, '90s)

If A, B are simple separable nuclear unital infinite dimensional C*-algebras then $A \cong B$ if and only if they have the same *Elliott invariant*, consisting of K-theory paired with traces.

One hopes to also "classify" homomorphisms, and in particular determine from K-theory whether two isomorphisms are approximately unitarily equivalent.

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If we know the K-theory of A, can we tell whether the flip map induces the identity on K-theory?

Künneth formula for K-theory

If A satisfies the Universal Coefficient Theorem, then there is an exact sequence relating the K-theory of $A \otimes B$ to the K-theory of A and B.

The *Universal Coefficient Theorem* is a mysterious property.

Open question

Does every nuclear C*-algebra satisfy the Universal Coefficient Theorem?

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Does every nuclear C*-algebra satisfy the Universal Coefficient Theorem?

K-theory computations

If we know the K-theory of A, can we tell whether the flip map induces the identity on K-theory?

Künneth formula for K-theory

If A satisfies the Universal Coefficient Theorem, then there is an exact sequence relating the K-theory of $A \otimes B$ to the K-theory of A and B.

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Does every nuclear C*-algebra satisfy the Universal Coefficient Theorem?

With the Künneth formula, one can say precisely when the flip map induces the identity on K-theory.

Theorem (T)

If A satisfies the Universal Coefficient Theorem, then the flip induces the identity on the K-theory of $A \otimes A$ if and only if $K_0(A) \oplus K_1(A)$ is one of the following groups:

(i) 0;
(ii) $\mathbb{Z}$;
(iii) $\mathbb{Q}_n$;
(iv) $\mathbb{Q}_m/\mathbb{Z}$;
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where m, n are supernatural numbers (infinite products of primes), m has infinite type, m divides n, and

$$\mathbb{Q}_n := \left\{ \frac{p}{q} \,\middle|\, q \text{ divides } n \right\}.$$

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Theorem (Villadsen, Rørdam, Toms)

The Elliott conjecture is false.

That is, there exist infinite dimensional simple separable nuclear unital C*-algebras A, B with the same K-theory and traces, but $A \not\cong B$.

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Regularity

There exists a C*-algebra $\mathcal{Z}$ (the *Jiang–Su* algebra) such that:
(i) $\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$, and
(ii) A has the same K-theory and traces as $A \otimes \mathcal{Z}$, for all A.

Hence if the Elliott conjecture were true, then $A \cong A \otimes \mathcal{Z}$ for every simple separable nuclear unital infinite dimensional C*-algebra A.

The right class of C*-algebras for classification are ones satisfying $A \cong A \otimes \mathcal{Z}$, a condition called *$\mathcal{Z}$-stability*.

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Conjecture (Toms–Winter)

If A is a simple, separable, nuclear, infinite dimensional, unital C*-algebra, then the following are equivalent:

(i) A is $\mathcal{Z}$-stable ($A \cong A \otimes \mathcal{Z}$);

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Theorem

The Elliott conjecture is true for:

$\mathcal{Z}$-stable C*-algebras with no traces (Kirchberg, Phillips '00);

$\mathcal{Z}$-stable C*-algebras with one trace (Winter '06 & '07, Lin–Niu '08, Matui–Sato '12 & '14, T–White–Winter '15);

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Approximately inner flip

Corollary (T)

Let A be a $\mathcal{Z}$-stable C*-algebra which satisfies the Universal Coefficient Theorem. Then A has approximately inner flip iff A has at most one trace and $K_0(A) \oplus K_1(A)$ is one of the following groups:

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Question

Does there exist a non-$\mathcal{Z}$-stable C*-algebra with approximately inner flip?

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