Algebras of operators on a Hilbert space

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Abstract

We survey the recent quest to extend key C*-algebra tools, perspectives, and results to general algebras of operators on a Hilbert space. A particular role is played by a new 'positive cone' we have introduced and studied with Charles Read and Matthew Neal. This gives a device/strategy to generalize results hitherto available only for C^*-algebras. In particular, to generalize C^*-algebraic results relying on positivity, and in particular on the existence of a positive cai. Much of this also is intimately related to a generalization of Akemann's noncommutative topology to such algebras.

I. Introduction

Operator algebra $=$ closed subalgebra of $B(H)$, for a Hilbert space H

unital: has an identity of norm 1

approximately unital: has a cai (contractive approx identity)

Think: 'noncomm. function algebra/uniform algebra'

or think: 'partial C^*-algebra'

- We often use C^*-algebras generated the algebra A
- or the diagonal $\Delta(A) = A \cap A^*$, a C^*-algebra
- Projection in A $=$ norm 1 (orthogonal) idempotent in A

Main theme in this talk:

generalize C^*-algebra tools and theories to operator algebras

using new ideas, such as the new 'positivity' alluded to

Why should a C^*/W^*-algebraist care?

Main answers: 1) Create a'bigger world' than C^*-algebras, a world totally compatible with C^*-algebra constructions...which can give you more freedom to work, and come and go between C^*-algebra-world as needed.

(By 'compatible', in particular we mean that constructions are the same (e.g. the important module tensor product of C^*-modules equals and is generalized by the Haagerup (module) tensor product).

2) Export your nice results, give them a second career!!

Example. Recent work of Bram Messner in KK-theory with spectral triples

• Spectral triple/unbounded KK-cycle (E, D) consists of a A-B-bimodule/Hilber space E, a selfadjoint unbounded $D : \mathrm{DomD} \subset E \to E$ satisfying some conditions, in particular that the set of $a \in A$ with $[D, a]$ bounded (and adjointable) is dense in A.

• C^*-algebra/C^*-module constructions were not appropriate for some of the computations in KK-theory with spectral triples he was interested in.

However if one considers 2×2 lower triangular matrices

$$\begin{bmatrix} a & 0 \\ [D, a] & \gamma a \end{bmatrix},$$

for a as above, γ coming from the grading. This forms a natural but nonselfadjoint operator algebra in $\mathbb{B}(E \oplus E)$. It lives in that 'bigger world', to which one may apply the nonselfadjoint generalizations of the C^*-theory and constructions, e.g. generalized C^*-modules over this, etc. He eventually gets the Kasparov products of such cycles defined by a direct algebraic formula that apparently was not previously possible.

Identities and approximate identities

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Theorem (Read, 2011) An operator algebra with a cai, has a cai (e_t) with positive real parts, and even with $\|1 - 2e_t\| \leq 1$.

(Recently found a shortish proof of this using 'noncommutative peak interpolation')

This result drew our attention to the set of operators x in an operator algebra A satisfying $\|1-2x\| \leq 1$. Indeed the positive scalar multiples of this set form a cone

Theorem For any operator algebra, the closure of this cone is the cone $\{a \in A : a + a^* \geq 0\}$

• Write $\mathfrak{S}_A$ for either of these cones. They will play a role for us very much akin to the role of the positive elements in (the ball of) a C^*-algebra. This surprising claim is justified at many points in our work. Eg. 'completely positive maps' in the usual sense are just the ones 'preserving' $\mathfrak{S}_A$

Moral: there is a convex set $\mathfrak{S}_A$ which replaces positivity for some purposes

This set is big for approximately unital algebras (densely spans the algebra, and contains a cai)

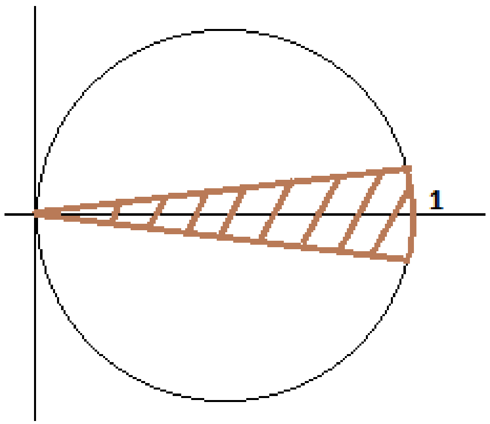
II. First consequences and lemmas

Some foundational lemmas:

Lemma If $x \in \mathfrak{S}_A$ then the operator algebra $\mathrm{oa}(x)$ generated by x has a cai

Proposition $\mathfrak{S}_A$ is closed under taking nth roots.

• Such n-th roots have spectrum and numerical radius within a cigar which is as thin as we like:



Theorem An operator algebra A with cai, has, for every $\epsilon > 0$, a cai (e_t) with the spectrum and numerical range of every e_t contained in a cigar which is as thin as we like (we say "nearly positive").

- An operator with numerical range contained in $[0, 1] \times [-\epsilon, \epsilon]$, in fact is near to a positive operator.

Corollary An algebra A of operators has a cai iff the linear span of $\mathfrak{S}_A$ is dense in A

• Moreover, in any case, the closure D of the linear span of $\mathfrak{S}_A$ is the biggest approximately unital operator algebra inside A, and is a hereditary subalgebra (HSA, that is $DAD \subset D$).

• One may use this subalgebra to extend some part of the theory of operator algebras with cai, to arbitrary operator algebras.

Application: theory of HSA's

• No time to describe this here, but our results enable the HSA (hereditary subalgebra, $DAD \subset D$), as a viable notion for general algebras of Hilbert space operators

(It is not a viable notion for Banach algebras.)

... and to generalize the basic facts about HSA's in C^*-algebras (B-Hay-Neal, B-Read, and other coauthors).

Sample result: Semisimplicity descends to HSA's

Application: generalization of Hilbert C^*-modules

- In several older papers, we have generalized the notion of Hilbert C^*-modules, and their theory, to the general operator algebra situation.

Foundational question: What is it?

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Foundational question: What is it?

- The (more recent) simplest definition, for an operator algebra A, is an operator space X which is also a right A-module, such that there exists completely contractive A-module maps φ_t, ψ_t, going between X and the space of columns with entries in A, such that $\psi_t(\varphi_t(x)) \to x$ for all $x \in X$.

- For this to get off the ground one needs the theory mentioned on previous pages!

Applications to noncommutative topology

- Akemann has a theory of noncommutative topology for C^*-algebras, where one generalizes the basic theorems of topology to this noncommutative setting. This has been developed further by him, L. G. Brown, G. Pedersen, and others, into a powerful tool.

- We have generalized Akemann's noncommutative topology to general operator algebras in joint work with Hay, Neal, and Read. We define open, closed, compact projections in A^{**}, and develop their theory analogously to the C^*-algebra case (that is, reprise the noncommutative variants of the facts and theorems from topology). The recent theory uses our 'positivity', for example in [B-Neal, B-Read, 2011–2013].

- So e.g. p is open in A^{**} iff $p \in A^{**}$ and p is open in B^{**}. It turns out to be not so relative; nothing depends for example on which particular containing C^*-algebra you use.

• So e.g. p is open in A^{**} iff there is a net of contractions in A with $x_t = px_t \to p$ weak*.

• A cool thing about open projections: they are in one-to-one correspondence with a nice class of right ideals in A. (Or left ideals.) This is Hay's theorem below. So this relative noncommutative topology is intimately related to the structure of this class of one-sided ideals in A

Lemma For any operator algebra A, if $x \in \mathfrak{S}_A$, with $x \neq 0$, then the left support projection of x equals the right support projection. In fact this projection $s(x)$ is the weak* limit of $(x^{\frac{1}{n}})$, and is an open projection in the sense of Akemann.

Recent nc-topology work: Compact projections

- Sayl q is closed iff $1 - q$ is open

Definition. A closed projection is compact in A^{**} iff there is an $a \in A$ such that $aq = q$.

Theorem A projection $q \in A^{**}$ is compact iff q is closed in $(A^1)^{**}$, and hence iff q is compact in B^{**}. If A is approximately unital then we may rechoose a above in $\mathfrak{S}_A$.

We have several noncommutative Urysohn lemmata. The following is perhaps the best:

Theorem (A noncommutative Urysohn lemma) If q is a compact projection and r is a closed projection in A^{**}, with $qr = rq = 0$, then there exists $a \in \mathfrak{S}_A$, and $qa = aq = q, ra = ar = 0$.

III Building r-ideals

An r-ideal is a right ideal with a left cai

An ℓ-ideal is a left ideal with a right cai

We now discuss how r-ideals/HSA's/cais are built for an operator algebra A

- There are bijective correspondence between the r-ideals and ℓ-ideals (and with HSA's). But this is nontrivial.

- The basic theory of these ideals ([B-Hay-Neal], etc) generalizes the basic C^*-algebra case, but some things are much harder.

- r-ideals are in an order preserving, bijective correspondence with the open projections $p \in A^{**}$

- Indeed the weak* limit of a left cai for an r-ideal is an open projection, and is called the *support projection* of the ideal.

Conversely, if p is an open projection in A^{**}, then $\{a \in A : a = pa\}$ is an r-ideal in A.

- If $a \in \mathfrak{S}_A$ then $\overline{aA}$ is an r-ideal of A, and the support projection of this r-ideal is $s(a)$. So $\overline{aAa} = \{x \in A : x = s(a)xs(a)\}$

Similarly, e.g. $\overline{Aa} = \{x \in A : x = xs(a))\}$. These are the 'principal' ones

Theorem Let A be any operator algebra

(1) Every separable r-ideal in A, is equal to $\overline{xA}$, for some $x \in \mathfrak{S}_A$.

(2) The closure of a countable sum of r-ideals of the form at the end of (1), is of the same form.

Theorem Let A be any operator algebra (not necessarily with an identity or approximate identity). The r-ideals in A, are precisely the closures of increasing unions of ideals of the form $\overline{xA}$, for $x \in \mathfrak{S}_A$.

Corollary If A is a separable operator algebra, generating a C^*-algebra B, then the open projections in $A^{\perp\perp}$ are precisely the $s(x)$ for $x \in \mathfrak{S}_A$.

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Corollary If A is a separable operator algebra with cai, then there exists an $x \in \mathfrak{S}_A$ with $A = \overline{xA} = \overline{Ax} = \overline{xAx}$.

Corollary Any separable operator algebra with cai has a countable cai consisting of mutually commuting elements, indeed of form $(x^{\frac{1}{k}})$ for an $x \in \mathfrak{S}_A$.

These all generalize well-known C^*-algebra results, and were unobtainable previously

Another generalization of a well-known result for C^*-algebras:

Theorem Let A be any operator algebra with cai. The following are equivalent:

(i) A has a countable cai.

(ii) A has an element in $\mathfrak{S}_A$ whose real part is strictly positive.

(iii) There is an element x in $\mathfrak{S}_A$ with $s(x) = 1_{A^{**}}$.

IV. When xA and Ax are closed/pseudoinvertibility

The C^*-algebra result:

We recall that 'well supported' operators are those operators x that have a 'spectral gap' for $|x|$ at 0, that is 0 is absent from, or is isolated in, the spectrum of $|x|$.

Theorem (Harte-Mbekhta) An element x of a C^*-algebra A is well supported iff xA is closed, and iff there exists $y \in A$ with $xyx = x$.

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Theorem (Harte-Mbekhta) An element x of a C^*-algebra A is well supported iff xA is closed, and iff there exists $y \in A$ with $xyx = x$.

Such a y is called a *generalized inverse* or *pseudoinverse*.

We get a similar result, about pseudoinvertibility in nonselfadjoint operator algebras, and with 'spectral gap' for x not $|x|$, for our cone.

Theorem For any operator algebra A, if $x \in \mathfrak{S}_A$, then the following are equivalent:

(i) $\mathrm{oa}(x)$ is unital and x is invertible in $\mathrm{oa}(x)$.

(ii) xAx is closed.

(iii) xA and Ax are closed.

(iv) There exists $y \in A$ with $xyx = x$.

Also, the latter conditions imply

(v) 0 is isolated in, or absent from, $\mathrm{Sp}_A(x)$.

The y may be taken to be in $\mathrm{oa}(x)$. Finally, if further $\mathrm{oa}(x)$ is semisimple, then conditions (i)–(v) are all equivalent.

- We do not know if xA is closed iff Ax is closed

- In a nonsemisimple setting, 0 being an isolated point in $\mathrm{Sp}(x)$ need not imply that xA is closed.

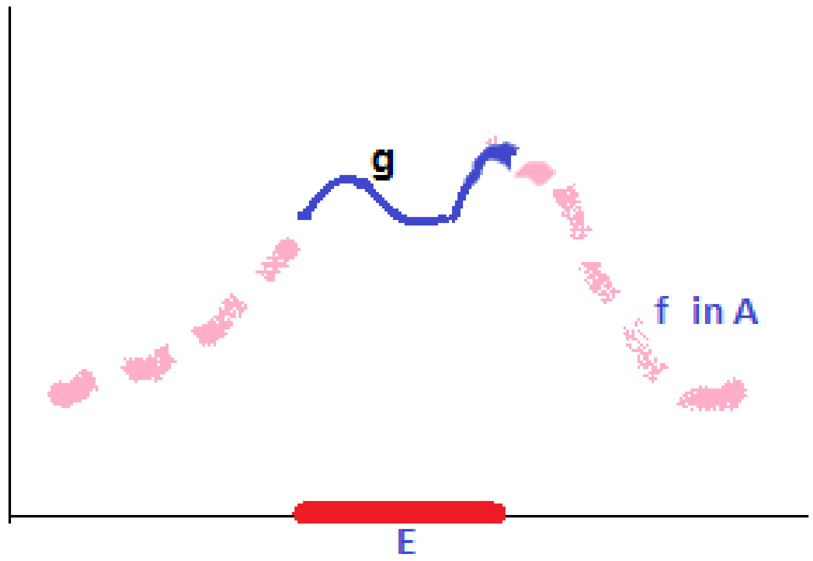
V. Interpolation in operator algebras

- Here the goal is to simultaneously generalize the C^*-algebraic interpolation of Akemann, Pedersen, Brown, etc; and classical function theoretic peak interpolation, which we will describe on the next slides

Setting for classical peak interpolation:

Given: a fixed algebra A of continuous scalar functions on a compact (for convenience in this talk) Hausdorff space K, ...

... and one tries to build functions in A which have prescribed values or behaviour on a fixed closed subset E of K (or on several disjoint subsets), with $\|f\| = \|f_{|E}\|$.



• The sets E that 'work' for this are the p-sets, namely the closed sets whose characteristic functions are in the 'second annihilator' $A^{\perp\perp}$ (or weak* closure) of A in $C(K)^{**}$

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Glicksberg's peak set theorem characterizes these sets as the intersections of peak sets, i.e. sets $f^{-1}(\{1\})$ for a norm 1 function f in A.

• In the separable case, they are just the peak sets (one doesnt need intersections)

Peak set: $E = f^{-1}(\{1\})$ for a norm 1 function f in A. One may rechoose f such that $|f| < 1$ on E^c, in which case $f^n \to \chi_F$.

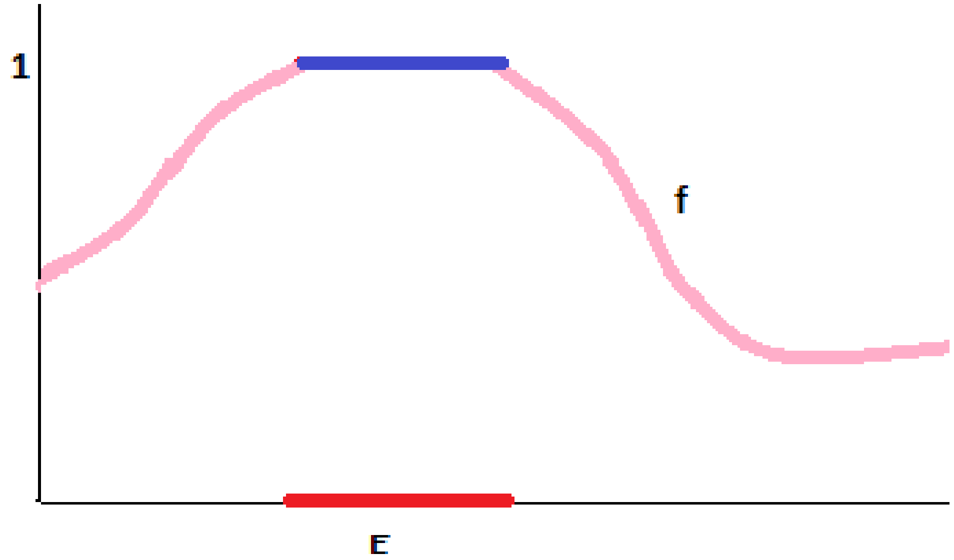


Figure 1: A peak set E

A primary example of a peak interpolation result, which originated in results of Errett Bishop, says:

Theorem If h is a continuous strictly positive scalar valued function on K, then the continuous functions on E which are restrictions of functions in A, and which are dominated in modulus by the 'control function' h on E, have extensions f in A with $|f(x)| \leq h(x)$ for all $x \in K$.

Figure 2: Errett Bishop

We will refer to this result as the Bishop-type theorem

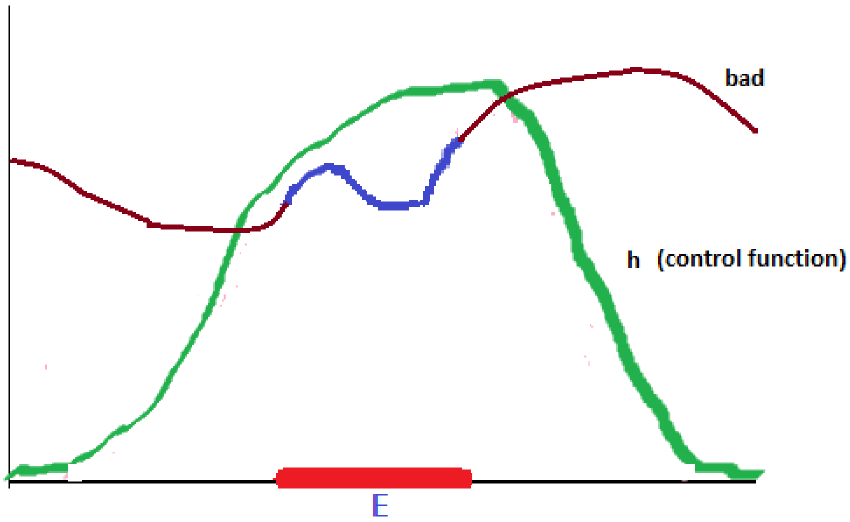


Figure 3: Extension dominated by control function

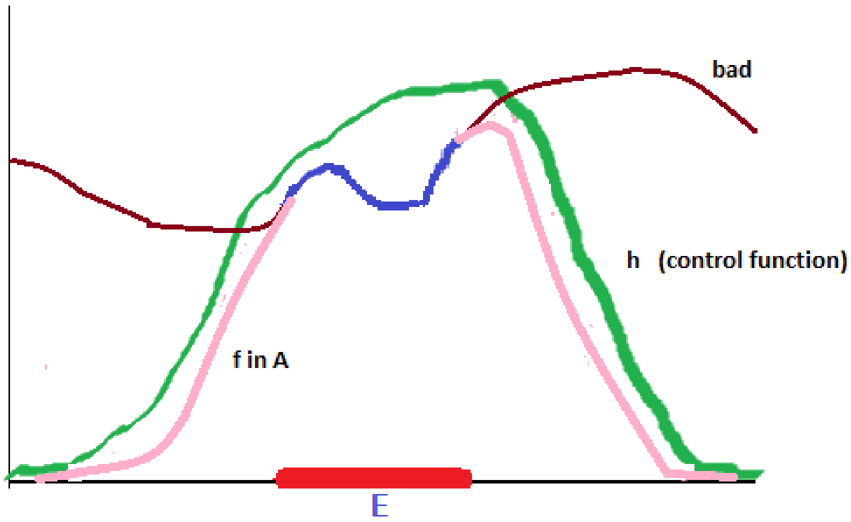


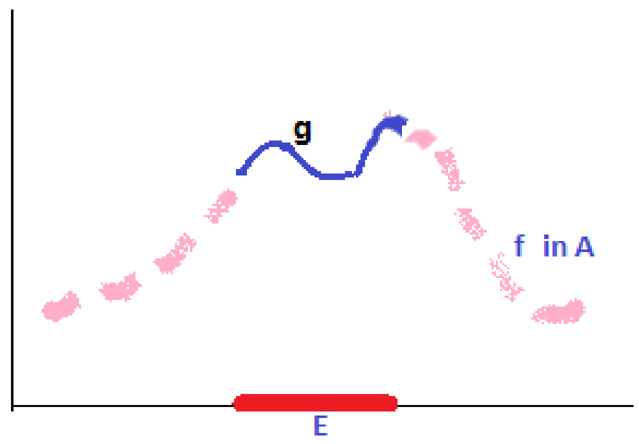
Figure 4: Extension dominated by control function

• A special case of interest is when $h = 1$; for example when this is applied to the disk algebra one obtains the well known Rudin-Carleson theorem which tells you exactly when one can extend a continuous function on a subset of the circle, to a function in the disk algebra (so continuous on the circle and analytic inside the circle), without increasing the supremum norm of the function.

- These constitute the first steps in building more complicated functions in A with prescribed values or behaviors on given closed subsets of K.

- We see next how the above theory 'goes noncommutative'.

What is noncommutative peak interpolation?:



$f = g$ on E becomes $fq = gq$

where q is the closed projection playing the role of (the characteristic function of) g

Our idea: So we want to take the classical interpolation results (like the Bishop-type theorem), and replace $A \subset C(K)$ by a subalgebra A of a C^*-algebra, replace closed sets E by closed projections q, and replace 'set statements' with 'algebra statements' like $f = g$ on E by $fq = gq$.

Exercise: What does $|f| \leq h$ on E become?

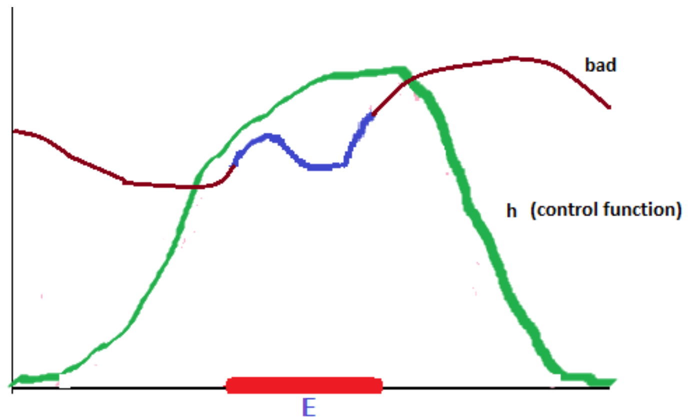
Answer: $f^*qf \leq h^*qh$, or ...

Noncommutative peak interpolation started in the PhD thesis of student Damon Hay

- Recently we have essentially completed the peak interpolation theory.

We have a good idea of what works and what does not. What remains is further applications.

Theorem (Noncommutative Bishop type) Suppose that A is a unital (resp. not necessarily unital) operator algebra, a subalgebra of a unital C^*-algebra B. Suppose that q is a closed (resp. compact) projection in A^{**}. If $b \in A$ with $bq = qb$, and $qb^*bq \leq qh$ for an invertible positive $h \in B$ which commutes with q, then there exists an element $g \in A$ with $gq = qg = bq$, and $g^*g \leq h$.



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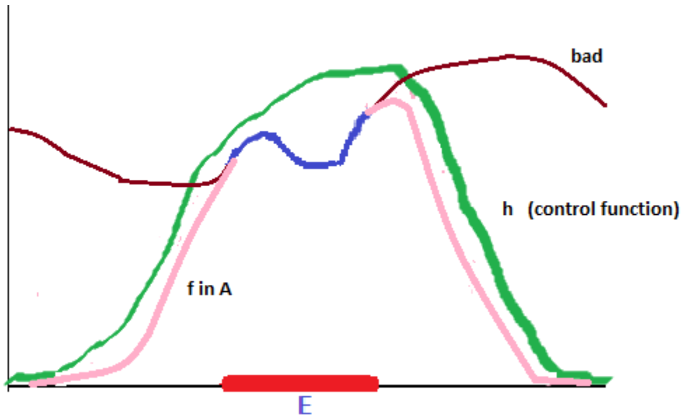

Reminder: The sets E that 'work' for classical peak interpolation are the p-sets, namely the closed sets whose characteristic functions are in the 'second annihilator' (or weak* closure) of A in $C(K)^{**}$

Glicksberg's peak set theorem characterizes these sets as the intersections of peak sets, i.e. sets $F = f^{-1}(\{1\})$ for a norm 1 function f in A.

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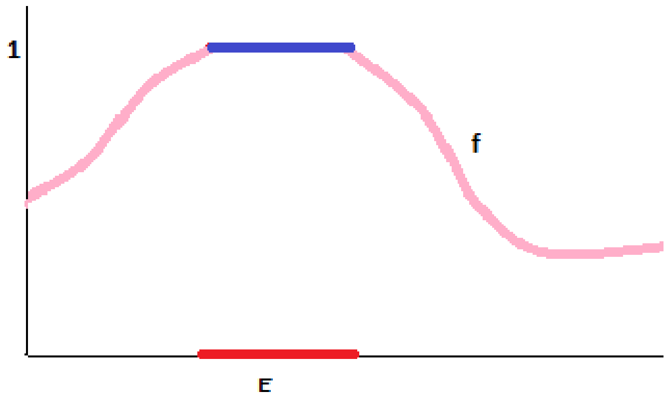
Theorem (Noncommutative Glicksberg peak set theorem of B-Read) The closed (resp. compact) projections in $A^{\perp\perp}$ are precisely the decreasing limits (or infima) of peak projections. If A is separable, they are just the peak projections.

• There are many equivalent ways to define peak projections (see Hay's thesis, etc). In fact they are the weak* limits of f^n for $f \in \mathrm{Ball}(A)$ in the cases that such limit exists.



VI. Equivalence and comparison in operator algebras

• Since the beginning of the subject of C^*-algebras and von Neumann algebras, equivalence and comparison of elements has been central to the theory.

• Murray-von Neumann equivalence of projections p, q in a von Neumann algebra: $p \sim q$ iff $p = uu^*, \;\; q = u^*u$

$\cdots$ and there is a matching notion of subequivalence:

$p \precsim q$ iff $p \sim q' \leq q$

- In comparison theory one considers various equivalence relations for elements in a C^*-algebra, and coarser notions of ordering of elements in a C^*-algebra than the usual $\leq$, generalizing in some sense the comparison of projections in a von Neumann algebra above

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• Cuntz equivalence and subequivalence has recently been reformulated by Ortega-Rørdam-Thiel using some simpler equivalence and subequivalence relations, and Akemann's noncommutative topology (namely open, closed and compact projections in the second dual of the C^*-algebra)

- We will describe these simpler equivalence and subequivalence relations, the most important one being Pedersen equivalence, which we write as $a \sim_A b$. A slight coarsening of this gives Blackadar equivalence and subequivalence. Then we will generalize these to general operator algebras.

Our main goal in this project was to begin to transfer some portion of the tools, results, and perspectives of comparison theory to more general operator algebras than C^*-algebras

Equivalence and comparison in C^*-algebras

To introduce the ideas, consider a relation between two elements a and b which one may define in any monoid or algebra A: namely that there exists $x, y \in A$ with $a = xy, b = yx$

If A is a group, then this defines an equivalence relation. In an algebra this is not an equivalence relation in general.

In fact this fails to be an equivalence relation even for the case $A = M_n$, the n by n matrices. (Even in this simple case this relation is quite subtle, and is not what we want)

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In fact this fails to be an equivalence relation even for the case $A = M_n$, the n by n matrices. (Even in this simple case this relation is quite subtle, and is not what we want)

How to fix this problem in an operator algebra or C^*-algebra?

In a C^*-algebra A, the 'fix' is to insist that $y = x^*$ above; and then this defines an equivalence relation $\sim$ on the positive cone A_+ of A.

This is sometimes called *Pedersen equivalence*.

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We will expand this 'fix' to a larger set than A_+, and to general operator algebras

- The role of the positive cone in our generalization is played by $\mathfrak{S}_A = \{a \in A : ||1 - 2a|| \leq 1\}$

Throughout A is a fixed operator algebra, with cai for simplicity, and B is a C^*-algebra containing A, and $a, b \in \mathfrak{S}_A$

We write $a \sim_c b$ in A if $a = xy, b = yx$ for contractions $x, y \in A$

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This is wrong too. But a tweak of it works:

Theorem Suppose that $a, b \in \mathfrak{S}_A$, and let $c = a^{\frac{1}{2}}$ and $d = b^{\frac{1}{2}}$. TFAE:

(i) $a \sim_c b$ as above but with $|y| = |c|$.

(ii) $a \sim_c b$ as above but with $|y| = |c|, |y^*| = |d^*|, |x| = |d|, |x^*| = |c^*|$.

(iii) $a \sim_c b$ as above but with $x = cR$ and $y = Sc$ for some contractions R, S.

(iv) For all $n \in \mathbb{N}$, there exist $x_n, y_n \in \mathrm{Ball}(A)$ with $a^{\frac{1}{n}} = x_n y_n, b^{\frac{1}{n}} = y_n x_n$, and the sequence $(y_n a)$ has a norm convergent subsequence.

(v) There exists $v \in \Delta(A^{**})$ with $s(a) = v^* v$, and $b = vav^*$, and $va \in A$.

Remarks. 1) $s(a)$ in (v) above is the support projection of a (the smallest projection p with $a = ap$ (or pa))

2) None of these conditions depend the particular C^*-algebra B containing A.

3) It should be admitted that these happen much much more rarely than in the C^*-algebra case. However it does arise quite naturally in interesting contexts, as can be seen from some of our results below.

We say that a is Pedersen equivalent to b in A, and write $a \sim_A b$, if the equivalent conditions in the last theorem hold.

- If $a, b \in \mathrm{Ball}(A_+)$ in the case A is a C^*-algebra, then $a \sim_A b$ iff there exists $x \in A$ with $a = xx^*, b = x^*x$ (the usual Pedersen equivalence)

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- In a finite dimensional operator algebra, $a \sim_A b$ iff $a^{\frac{1}{n}} \sim_c b^{\frac{1}{n}}$ for all $n \in \mathbb{N}$.

- The relation $\sim_A$ in the case that $A = M_n$ is exactly unitary equivalence of elements of $\mathfrak{S}_{M_n}$

Proposition Suppose that $r > 0$. Then $a \sim_A b$ iff $a^r \sim_A b^r$.

Corollary $\sim_A$ is an equivalence relation on $\mathfrak{S}_A$

- If $a \in \mathfrak{S}_A$, then $\overline{aAa}$ is a HSA, the associated open (support) projection is $s(a)$

- We define $a \cong b$ if $s(a) = s(b)$, or equivalently if $\overline{aA} = \overline{bA}$, or equivalently if $\overline{aAa} = \overline{bAb}$.

- Any comparison relation between elements in A which is invariant under this relation $\cong$, can be translated into a comparison between open projections in A^{**}, and vice versa

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Blackadar type equivalence: We define $a \sim_s b$ if there exist $a', b' \in \mathfrak{S}_A$, with $a \cong a', a' \sim_A b'$, and $b' \cong b$.

- It will follow from a result below that this is an equivalence relation.

• Define $p \sim_{A,PZ} q$, for open projections $p, q \in A^{**}$, if there is a $*$-open tripotent u with $u^*u = p, uu^* = q$. (A $*$-open tripotent is a generalization of 'open projections' to partial isometries, discussed in more detail in a later section).

In the C^*-algebra case this coincides with an equivalence of open projections introduced by Peligrad and Zsidó.

• Parts of the following generalizes results of Lin and Ortega-Rordam-Thiel:

Theorem: TFAE:

(i) $a \sim_s b$.

(ii) $\overline{aA} \cong \overline{bA}$, completely isometrically via an A-module map.

(iii) $s(a) \sim_{A,PZ} s(b)$.

(iv) There exists $b' \in \mathfrak{c}_A$, with $a \sim_A b'$ and $b' \cong b$.

(v) There exists $a' \in \mathfrak{c}_A$, with $a \cong a' \sim_A b'$.

(vi) $\overline{aAb}$ is a 'principal hereditary bimodule'

Subequivalence: We define $a \precsim_s b$ if there exists $b' \in \overline{bAb}$ such that $a \sim_s b'$. This is clearly equivalent to: there exists $b' \in \overline{bAb}$ such that $a \sim_A b'$. We call this *Blackadar comparison in* A.

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• If p, q are open projections in A^{**} we say that $p \precsim_{A,PZ} q$ if there is an open projection $q' \leq q$ in A^{**} with $p \sim_{A,PZ} q'$. We call this *Peligrad-Zsidó subequivalence in* A^{**}.

The following is the version of a result in [Ortega-Rordam-Thiel] in our setting:

Corollary If $a, b \in \mathfrak{c}_A$, TFAE:

(i) $a \precsim_s b$.

(ii) $p_a \precsim_{A,PZ} p_b$.

(iii) There exist a pair of completely isometric A-module maps $\Phi : \overline{aA} \to \overline{bA}$ and $\Psi : \overline{Aa} \to \overline{Ab}$, such that $\Psi(x)\Phi(y) = xy$ for all $x \in \overline{Aa}, y \in \overline{aA}$.

Section VIII. Where can this lead?

• Main answer at this point: Generalizing results and theories that are only known for C^*-algebras, to more general algebras of operators on a Hilbert space. This expands the theory of such algebras in useful ways, and strengthens the 'bigger world' that C^*-algebraists can feel free to use. I call this C^*-theory for operator algebras, and we already have several examples of this.

• Also, generalizing more of the classical applications from function algebra theory/function theory of peak sets, to operator algebras.