Direction Fields

Consider the differential equation

$$\frac{dy}{dt} = f(t, y).$$

The derivative, $y' = \dfrac{dy}{dt}$, gives the slopes of tangent lines at points on the graph of the function $y = y(t)$.

A **direction field** for the above equation can be constructed by evaluating f at each point of a rectangular grid. At each point of the grid, a short line segment is drawn whose slope is the value of f at that point. Thus each line segment is tangent to the graph of the solution passing through that point.

The direction field is useful in investigating the behavior of solutions of the equation.

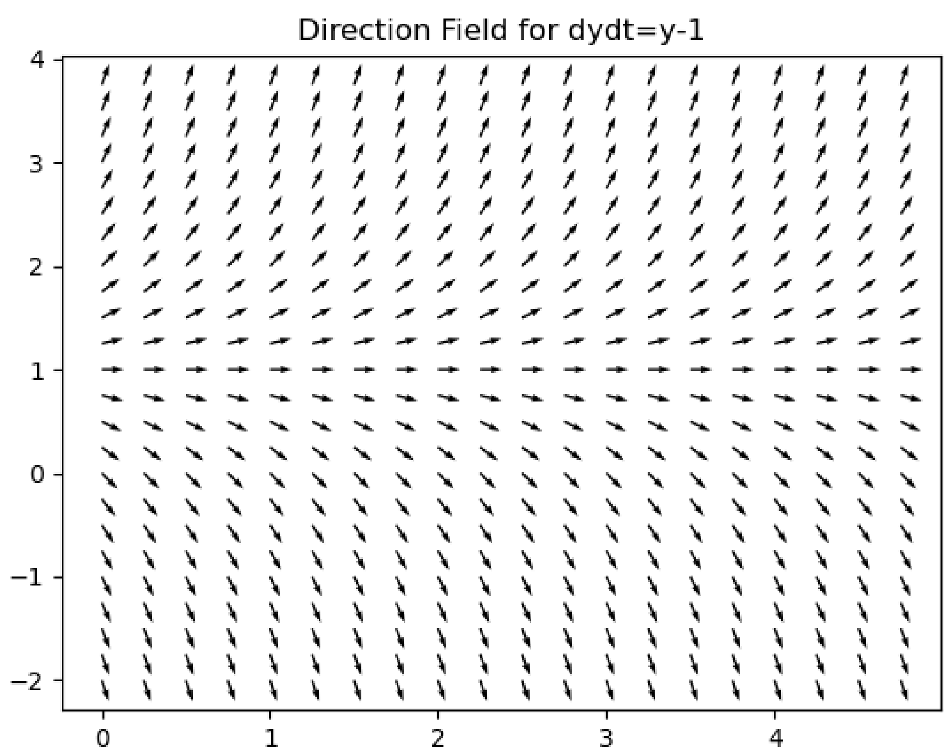
Definition. An **equilibrium solution** of a differential equation is a solution for which $\dfrac{dy}{dt} = y' = 0$.

Example 1. Draw a direction field for the differential equation

$$y' = y - 1.$$

Using the direction field, determine the behavior of y as $t \to \infty$.

Solution. Using Python (see Directionfield1 page) we get the following picture.



One solution of our equation is the constant function $y = 1$. This solution is an equilibrium solution.

The behavior of y as $t \to \infty$ depends on the initial value of y at $t = 0$. We can guess from the direction field that when $y(0) = y_0 > 1$,

$$\lim_{t\to\infty} y(t) = \infty$$

and when $y(0) = y_0 < 1$,

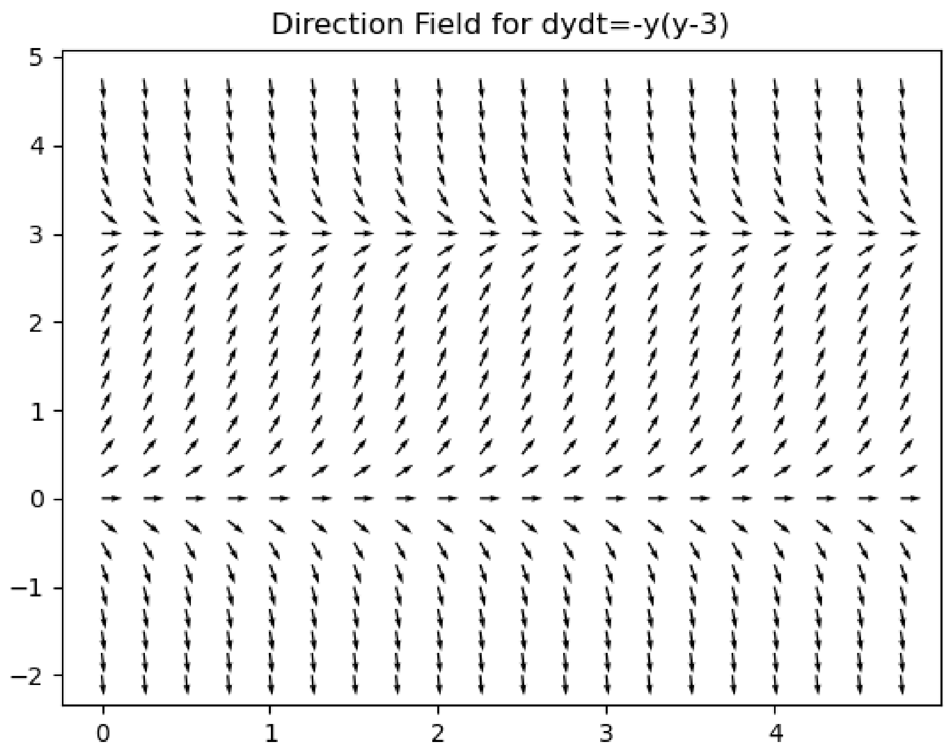
$$\lim_{t\to\infty} y(t) = -\infty.$$

Example 2. Draw a direction field for the differential equation

$$y' = -y(y - 3).$$

Using the direction field, guess the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.

Solution. Using Python (see Directionfield2 page) we get the following picture.



There are two equilibrium solutions: $y = 0$ and $y = 3$. We can guess from the direction field that when $y(0) = y_0 > 3$,

$$\lim_{t\to\infty} y(t) = 3,$$

when $0 < y(0) = y_0 < 3$,

$$\lim_{t\to\infty} y(t) = 3,$$

and when $y(0) = y_0 < 0$,

$$\lim_{t\to\infty} y(t) = -\infty.$$

The first two limits are correct (one can solve the equation and prove that). The third one is wrong. It turns out that if $y(0) = y_0 < 0$ then the solution has a vertical asymptote at $t = T > 0$ and the solution is not defined for $t \geq T$.

Computer assignment 1

1. Draw a direction field for the differential equation

$$\frac{dy}{dt} = (\sin t)(\cos y).$$

Use a $[0, 5]$ by $[-3, 3]$ grid.

2. Draw a direction field for the differential equation

$$\frac{dy}{dt} = y^2 - 1.$$

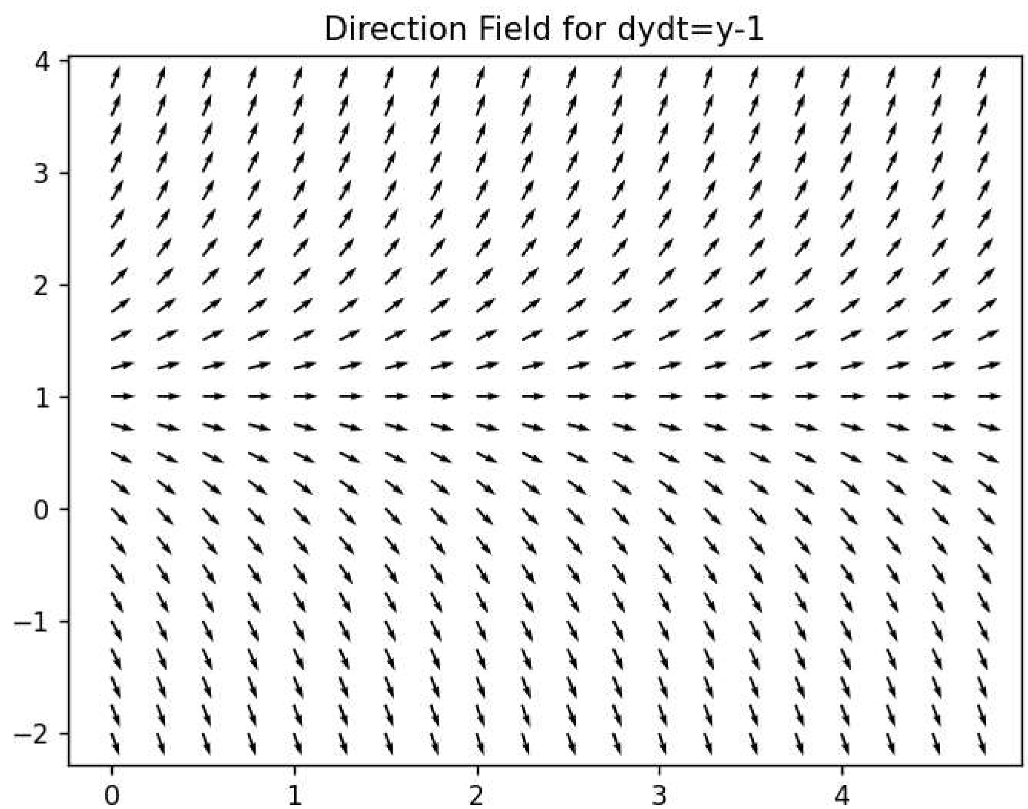
Use a $[0, 5]$ by $[-3, 3]$ grid. Using the direction field, guess the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.

The direction fields must be done using Python **not** by hand. You must submit on gradescope only one pdf file containing the 2 direction fields and the answer for the second problem. This is **not** a group assignment.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [2]:
```
matplotlib notebook
```

In [3]:
```
T,Y=np.meshgrid(np.arange(0,5,.25),np.arange(-2,4,.25))
dYdT=Y-1# put f(t,y) here to find slope
U=1/(1+dYdT**2)**0.5*np.ones(T.shape) # Normalizes the arrows to see near-zero slopes.
V=1/(1+dYdT**2)**0.5*dYdT
plt.figure()
plt.title('Direction Field for dydt=y-1')
Q=plt.quiver(T,Y,U,V) # draws the arrows at (X,Y) with slope dYdX
```

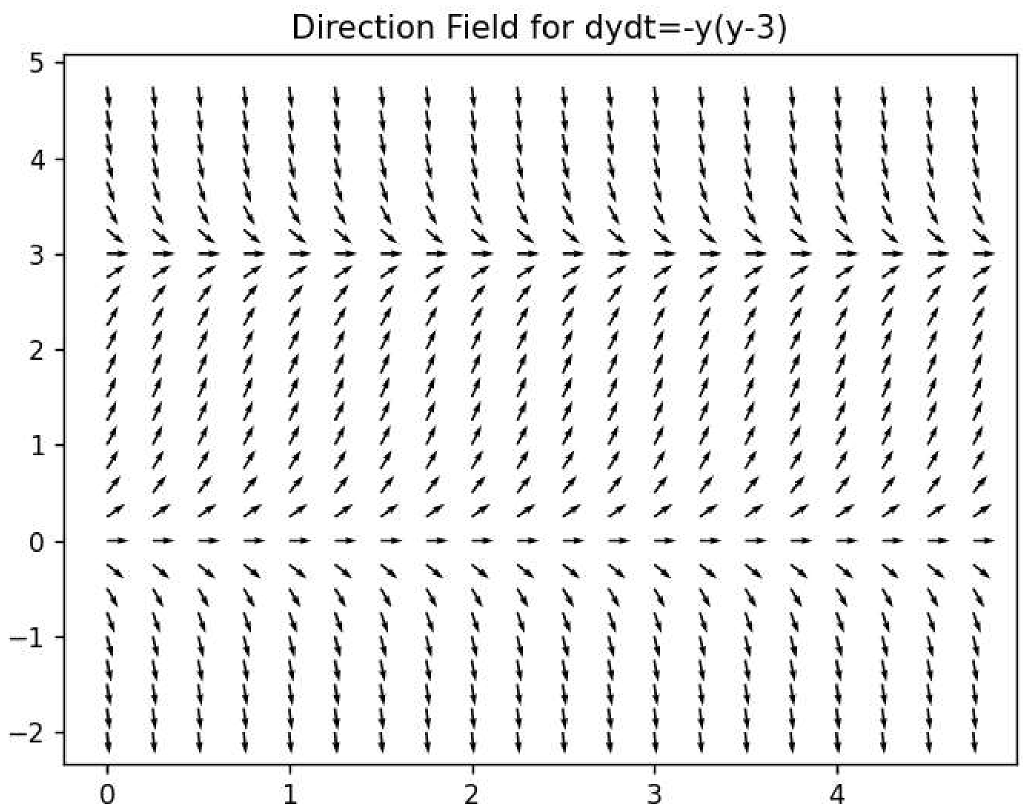


In [ ]:

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [2]:
```
matplotlib notebook
```

In [3]:
```
T,Y=np.meshgrid(np.arange(0,5,.25),np.arange(-2,5,.25))
dYdT=-Y*(Y-3)# put f(t,y) here to find slope
U=1/(1+dYdT**2)**0.5*np.ones(T.shape) # Normalizes the arrows to see near-zero slopes.
V=1/(1+dYdT**2)**0.5*dYdT
fig=plt.figure()
plt.title('Direction Field for dydt=-y(y-3)')
Q=plt.quiver(T,Y,U,V) # draws the arrows at (X,Y) with slope dYdX
fig.savefig('directionfield2.png')
```



In [ ]: