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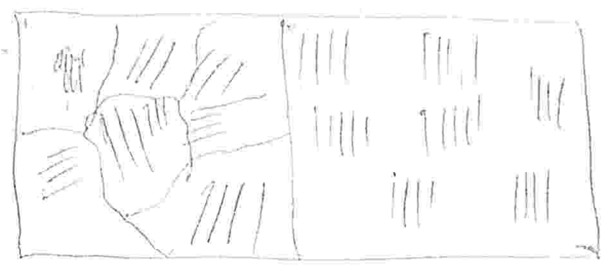
GEOMETRIC SCHRÖDINGER EQUATIONS AND TOPOLOGICAL SOLITONS



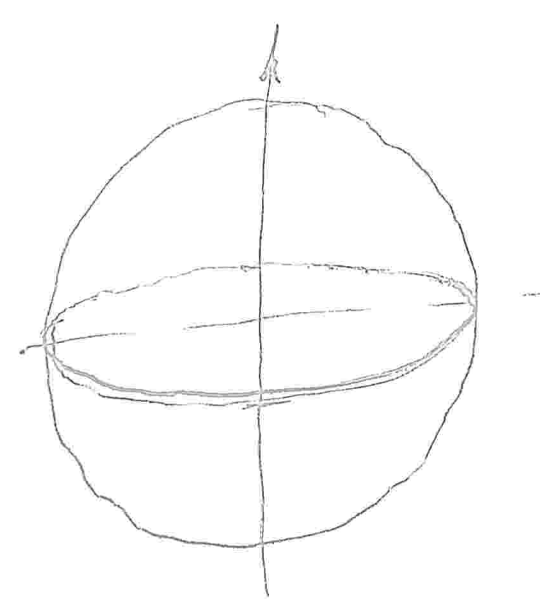
- MAGNETIZATION VECTOR $\vec{m} = \begin{bmatrix} m_1(x) \\ m_2(x) \\ m_3(x) \end{bmatrix} \in \mathbb{R}^3$

$$|\vec{m}(x,t)| = 1.$$

- STATIC CONFIGURATIONS MINIMIZE ENERGY

DYNAMICS: $\vec{m}(x,t)$ PRECESSES IN EFF. FIELD

- $$\frac{\partial \vec{m}}{\partial t} = -\vec{m} \times \Delta \vec{m} + \dots$$



$$\mathcal{E}(\vec{m}) = \mathcal{E}_{ex}(\vec{m}) + \mathcal{E}_{ani}(\vec{m}) + \mathcal{E}_{DM}(\vec{m})$$

$$\left(+ \mathcal{E}_{ext}(\vec{m}) + \mathcal{E}_{stray}(\vec{m}) + \dots \right)$$

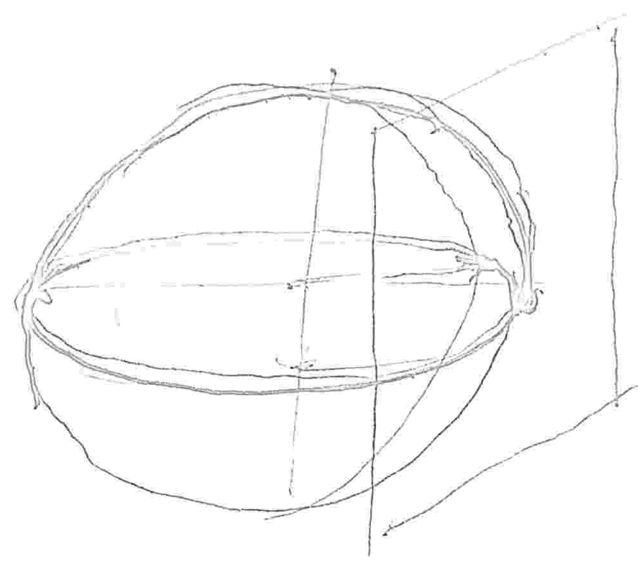
GEOMETRIC VIEW: EVOLUTION PDE FOR MAPS

- TIME-DEVELOPMENT MAPS BETWEEN RIEMANNIAN MANIFOLDS:

$$\vec{m}(\cdot, t) : M (= \mathbb{R}^2) \longrightarrow N (= S^2)$$

$$\Longleftrightarrow \quad \vec{m}(x,t) \in \mathbb{R}^3 \,\big|\, |\vec{m}(x,t)| = 1.$$

- ENERGY: $\mathcal{E}(\vec{m}(\cdot,t)) := \frac{1}{2} \int_M |\nabla \vec{m}|^2 \, dx$



$$-\mathcal{E}'(\vec{m}) = \mathrm{PROJ}_{T_{\vec{m}} S^2} \Delta \vec{m}$$

$$= D_j \partial_j \vec{m}$$

$$= \Delta \vec{m} + |\nabla \vec{m}|^2 \vec{m}$$

DIRICHLET'S ENERGY?

- GEOMETRIC /PHYSICAL EVOLUTION PDE ARISING FROM $\mathcal{E}$:

- HEAT-FLOW

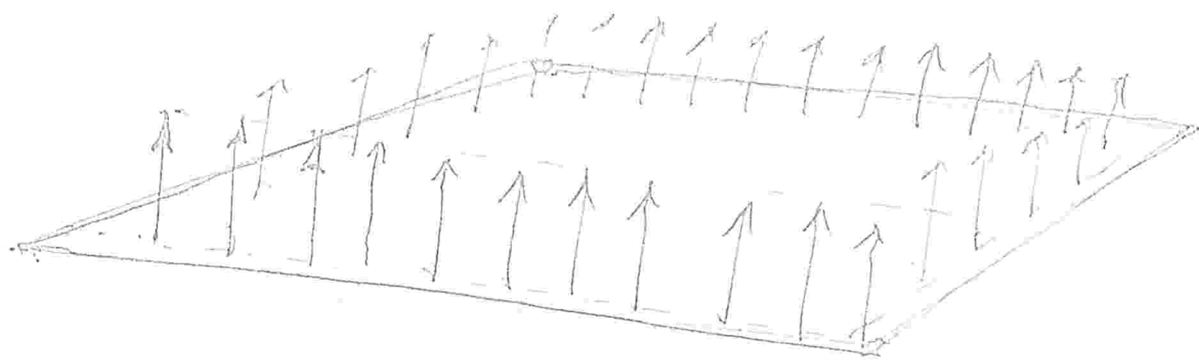
$$\partial_t \vec{m} = \Delta \vec{m} + |\nabla \vec{m}|^2 \vec{m}$$

- WAVE MAP

- SCHRÖDINGER MAP; COMPLEX-STRUCTURE

$$\partial_t \vec{m} = \vec{m} \times \Delta \vec{m}$$

o MINIMIZE $\mathcal{E}(\vec{m})$ ⟶ TOPOLOGICAL MAGNETIC SOLITONS



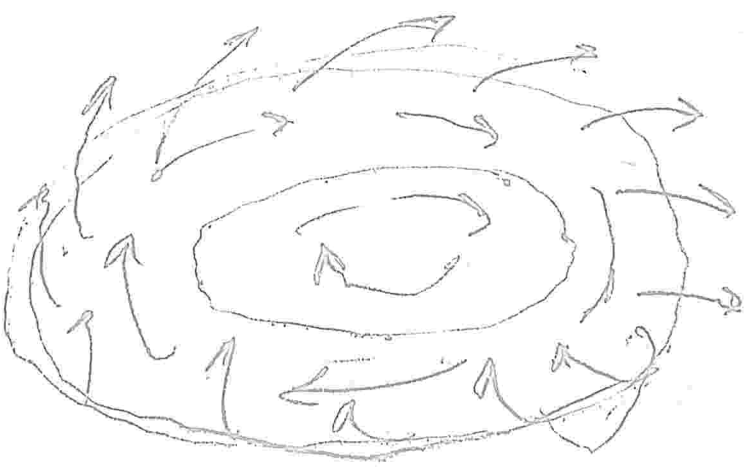
CONSIDER A LOCAL DEVIATION FROM
A PURE FERROMAGNETIC STATE

$\Rightarrow$ N = "Skyrmion #" $\in \mathbb{Z}$

o STEREOGRAPHIC PROJECTION $\varphi = \dfrac{m_1 + i m_2}{1 + m_3}$

$$\partial_1 \vec{m} = \pm \int \partial_2 \vec{m} \Leftrightarrow (\partial_1 \pm i\partial_2)\varphi = 0$$

Cauchy - Riemman



IF $\kappa = 0$, SCALING PRECLUDES A MINIMIZER

∘ $\vec{m}_\delta(x) = \vec{m}\left(\frac{x}{\delta}\right)$

∘ [Bogdanov et al 94-]

$\kappa \neq 0$: CHIRAL MAGNETIC SKYRMIONS

∘ NO RIGOROUS RESULTS ON CHIRAL MAGNETIC SKYRMIONS

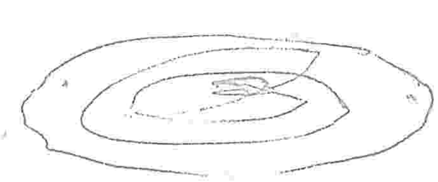
→ EASY-PLANE ANISOTROPY

WINDING:

$$J = \deg\left(\vec{m}(x)\Big|_{|x| = R \gg 1} : S^1 \to S^1\right)$$

$$\in \mathbb{Z} :$$

$J \neq 0 \longrightarrow$ a vortex configurations



PROBLEM: INFINITE ENERGY.

DYNAMICS
- HEAT-FLOW
- SCHRÖDINGER MAP
- LANDAU-LIFSHITZ

WITH GIVEN INITIAL DAT

$$\vec{m}_\delta(x,t) = \vec{m}\left(\frac{x}{\delta}, \frac{t}{\delta^2}\right)$$

HAVE ENERGY CRITICAL 2D

HARMONIC HEAT FLOW

- THE HEAT-FLOW HAS AT MOST FINITELY MANY SINGULAR POINTS, AT WHICH NON-TRIVIAL HARMONIC MAPS BUBBLE OFF

EQUIVARIANT HEAT FLOW

$n=1$: the solution is a
→ bubbling of a harmonic map.
→ DEPENDENT ON the winding

$n \geq 3$: no blobbing in finite time, asymptotic stability

$n=2$: SINGULARITIES CAN FORM WITHIN INFINITE TIME

SCHRÖDINGER Maps : below- Threshold :

$\longrightarrow$ GAPS

ABOVE - THRESHOLD:
Behaviour of near-harmonic , degree n equivariant solutions

- TOPOLOGICAL SOLITONS
- SKYRMION

→ CHIRAL MAGNETIC SKYRMIONS
  - STABILITY
  - DYNAMICS
  - EXISTENCE

- STEREOGRAPHIC PROJECTION $\left| \varphi = \frac{m_1 + i m_2}{1 + m_3} \right.$ smaller

$$2 E(\vec{m}) \approx 4 \int |\nabla \varphi|^2 + \int (1 - |\varphi|^2)^2$$

ie $|\varphi(x)| \approx 1$ ("GINZBURG-LANDAU")

→ - VORTEX PAIRS HAVE FINITE ENERGIES

- HEAT FLOW / GRADIENT FLOW | HARMONIC EQUIVARIANT
- Cauchy initial value problem
- PDE : COMPARISON CR.?
- SCHRÖDINGER MAP
- easy-plane vortex-pair