Remark. T is diagonalisable if $[T]_\beta$ is diagonal for some ordered basis β.

Matrix A is diagonalisable if L_A is diagonalisable (i.e. A is similar to a diagonal matrix).

v is an eigenvector of T with an eigenvalue λ if $T(v) = \lambda v$.

If v is an eigenvector of T with an eigenvalue λ, $v \in \ker(T - \lambda I_v)$. So λ is an eigenvalue of T if and only if $\ker(T - \lambda I_v) \neq 0$, or, equivalently, $\det(T - \lambda I_v) = 0$, which is called a characteristic polynomial $f(\lambda)$:

$$f(\lambda) = (-1)^n \lambda^n + \cdots \mid n = \dim V$$

Note. For $T \in \mathrm{Hom}(V, V)$ and ordered basis β, v is an eigenvector of T if and only if $[v]_\beta$ is an eigenvector of $[T]_\beta$.

Reason: $(\Rightarrow)[T]_\beta[v]_\beta = [Tv]_\beta = [\lambda v]_\beta = \lambda[v]_\beta$.

Example 0.1

Let $V = \mathfrak{P}_1(\mathbb{R}), T(p(x)) = p'(x)$. Compute eigenvectors/eigenvalues.

Solution. Consider the basis $\beta = (1, x)$.

$$T(1) = 0 \tag{1}$$
$$T(x) = 1 \tag{2}$$
$$[T]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3}$$

The characteristic polynomial is $f(t) = \det([T]_\beta - tI) = \det(\begin{pmatrix} -t & 1 \\ 0 & -t \end{pmatrix}) = t^2 \Rightarrow$ the only eigenvalue is 0.

$\ker(T)$ = all constant polynomials (1-dim) $\Rightarrow \nexists$ basis of eigenvectors, so T is not diagonalisable. $\square$

Example 0.2

$A = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix}, F = \mathbb{R}$.

The characteristic polynomial is $f(t) = \det(\begin{pmatrix} 2-t & -2 \\ 1 & -t \end{pmatrix}) = t^2 - 2t + 2 = (t-1)^2 + 1 \Rightarrow$ no eigenvectors/eigenvalues (no roots) $\Rightarrow$ A is not diagonalisable (however, it *is* if $F = \mathbb{C}$).

1 Diagonalisability

1.1 Tests for Diagonalisability

- If T has $n = \dim V$ distinct eigenvalues in F, then T is diagonalisable.

  To see this is true, consider the following.

Theorem 1.1

If $v_1, \dots, v_r$ are eigenvectors of T with distinct eigenvalues $\lambda_1, \dots, \lambda_r$, then $v_1, \dots, v_r$ are linearly independent.

Proof. We are using induction on r.

If $n = 1$, since $v_1 \neq 0$, the claim holds.

Suppose the claim holds for $r - 1$.

Suppose also $a_1 v_1 + a_2 v_2 + \cdots + a_r v_r = 0$.

Applying T on both sides, we obtain

$$a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 + \cdots + a_r \lambda_r v_r = 0.$$

From the equation above, $a_1 \lambda_r v_1 + a_2 \lambda_r v_2 + \cdots + a_r \lambda_r v_r = 0$, and thus

$$a_1(\lambda_1 - \lambda_r)v_1 + \cdots + a_{r-1}(\lambda_{r-1} - \lambda_r)v_{r-1} = 0$$

By inductive hypothesis, $v_1, \dots, v_{r-1}$ are linearly independent, and since λ_i are distinct, $a_i = 0$. Since $v_r \neq 0$, then $a_r = 0$. □

Corollary 1.2

The test works.

Proof. Take $v_1, \dots, v_n$ eigenvectors corresponding to the n distinct eigenvalues $\lambda_1, \dots, \lambda_n \Rightarrow$ by the theorem 1.1, they form a basis of eigenvectors. □

Remark 1.3. T can be diagonalisable with fewer than n distinct eigenvalues.

Example 1.4

Take T with the matrix $\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$

Then eigenvalues are $\lambda_1, \dots, \lambda_n$, since the characteristic polynomial is $f(t) = (\lambda_1 - t) \cdots (\lambda_n - t)$ – they are not necessarily distinct.

- We need a better test.

Definition 1.5. If λ is an eigenvalue of T, the λ-eigenspace of T is $\ker(T - \lambda I_v) = \{v \in V \,|\, T(v) = \lambda v\} = E_\lambda$.

Definition 1.6. If λ is an eigenvalue of T (or A), the algebraic multiplicity of λ is the multiplicity m with which λ is a root of the characteristic polynomial $f(t)$, i.e. m is the largest integer such that $(t - \lambda)^m$ divides $f(t)$. Note that $1 \leq m \leq n$.

The multiplicity is sometimes denoted by m_λ.

Theorem 1.7

If λ is an eigenvalue of T, then $1 \leq \dim(E_\lambda) \leq m$.

Proof. Let $d = \dim(E_\lambda)$. Pick the basis $v_1, \dots, v_d$ of E_λ and extend it to the basis $\beta = \{v_1, v_2, \dots, v_n\}$ of V.

Then

$$[T]_\beta = \begin{pmatrix} \lambda & \cdots & 0 & * \\ \vdots & \ddots & \vdots & \\ 0 & \dots & \lambda & \\ 0 & \cdots & 0 & \\ \vdots & \ddots & 0 & A \\ 0 & \cdots & 0 & \end{pmatrix} \Rightarrow \det([T]_\beta - tI) = \det \begin{pmatrix} \lambda - t & \cdots & 0 & * \\ \vdots & \ddots & \vdots & \\ 0 & \dots & \lambda - t & \\ 0 & \cdots & 0 & \\ \vdots & \ddots & 0 & A - tI_A \\ 0 & \cdots & 0 & \end{pmatrix},$$

which simplifies to

$$\det([T]_\beta - tI) = \det \begin{pmatrix} \lambda - t & \cdots & 0 & * \\ \vdots & \ddots & \vdots & \\ 0 & \dots & \lambda - t & \\ 0 & \cdots & 0 & \\ \vdots & \ddots & 0 & A - tI_A \\ 0 & \cdots & 0 & \end{pmatrix} = (\lambda - t)^d \det(A - tI_A)$$

$\square$