Notes on Real Analysis

1 Foundations

1.1 Postulates

1.1.1 Numbers

1. **Real Numbers as a Field**

    a) **Associativity**

    $\forall a, b, c \in \mathbb{R} : a + (b + c) = (a + b) + c$

    Exercise 1:

    Prove that the sums of an arbitrary number of equivalent variables in an immutable sequence are equal up to the placement of parentheses.

    Exercise 2:

    Let the immutable sequence written in such a form that there are no two elements not parenthesised be called a *nested* sequence.

    For example, $((a + b) + c) + d$ and $(a + b) + (c + d)$ are both nested sequences.

    How many different nested sequences can be written from a sequence of n letters?

    b) **Commutativity of Addition**

    $\forall\ a, b \in \mathbb{R} : a + b = b + a$

    c) **Commutativity of Multiplication**

    $\forall\ a, b \in \mathbb{R} : a \times b = b \times a$

    d) **Existence of an Additive Identity**

    $\exists\ 0 \in \mathbb{R}\ \forall a \in \mathbb{R} : a + 0 = a$

    e) **Existence of a Multiplicative Identity**

    $\exists\ 1 \in \mathbb{R}\ \forall a \in \mathbb{R} : a \times 1 = a$

    f) **Existence of an Additive Inverse**

    $\forall\ a \in \mathbb{R}\ \exists - a : a + (-a) = 0$

    g) **Existence of a Multiplicative Inverse**

    $\forall\ a \in \mathbb{R}\ \exists\ a^{-1} : a \times a^{-1} = 1$

    h) **Distributivity**

    $\forall\ a, b, c \in \mathbb{R} : a \times (b + c) = a \times b + a \times c$

2. **Real Numbers as an Ordered Field**

   Let P be the set of positive numbers.

   Let the binary operator $>$ be defined so that $\forall\ a, b \in \mathbb{R} : a > b \iff a - b \in P$.

   Similarly, $\forall\ a, b \in \mathbb{R} : a < b \iff b - a \in P$.

   a) **Trichotomy Law**

      $\forall\ a \in \mathbb{R}$ one and only one of the following holds:

      - $a = 0$
      - $a \in P$
      - $a \notin P$

   b)