Let U, V, W be vector spaces over $\mathbb{F}$, and let

$$T \in \mathscr{L}(V, W), S \in \mathscr{L}(U, V).$$

Let $T \circ S \in \mathscr{L}(U, W)$ be the composition.

Claim. $\ker(S) = \ker(T \circ S) \Leftrightarrow \operatorname{Im}(S) \cap \ker(T) = 0$.

Proof. 1. Suppose that $\ker(S) = \ker(T \circ S)$. Thus,

$$\forall u \in U : S(u) = 0 \Leftrightarrow T(S(u)) = 0.$$

Note that $S(0) = 0$ and $T(0) = 0$, since S, T are linear transformations, and hence

$$(0 \in \operatorname{Im}(S)) \wedge (0 \in \ker(T)) \Rightarrow \{0\} \subseteq \operatorname{Im}(S) \cap \ker(T).$$

Now, suppose that $x \in \operatorname{Im}(S) \cap \ker(T)$.

Therefore, $\exists (u' \in U) : S(u') = x$. Since $T(x) = 0$, then $T(S(u')) = 0$, and hence $(T \circ S)(u') = 0$, which means that $u' \in \ker(T \circ S)$. But by assumption $\ker(S) = \ker(T \circ S)$ and thus $S(u') = 0$, hence $S(u') = x = 0$. Therefore, $\forall x \in (\operatorname{Im}(S) \cap \ker(T)) : x = 0$, and hence $\operatorname{Im}(S) \cap \ker(T) = \{0\}$.

2. Suppose now that $\operatorname{Im}(S) \cap \ker(T) = \{0\}$.

Consider $x \in \ker(S)$. By definition, $S(x) = 0$. Since T is a linear transformation, $T(S(x)) = T \circ S(x) = T(0) = 0$. Therefore, $\ker(S) \subseteq \ker(T \circ S)$.

Consider now $y \in \ker(T \circ S)$. By definition, $T(S(y)) = 0$. Therefore, $S(y) \in \ker(T)$. Moreover, $S(y) \in \operatorname{Im}(S)$, and hence $S(y) \in \operatorname{Im}(S) \cap \ker(T)$. By the assumption, $\operatorname{Im}(S) \cap \ker(T) = \{0\}$, and hence $S(y) = 0$, which means that $y \in \ker(S)$. Therefore, $\ker(T \circ S) \subseteq \ker(S)$, and since also $\ker(S) \subseteq \ker(T \circ S)$, then $\ker(S) = \ker(T \circ S)$.

From 1 and 2, it follows that $\ker(S) = \ker(T \circ S) \Leftrightarrow \operatorname{Im}(S) \cap \ker(T) = 0$. $\square$