Problem 4

Fix any matrix $A \in M_{n\times n}(F)$.

Let A^* denote a conjugate transpose of A.

Consider the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{F}^n$.

Lemma 0.1

For any $A, B \in M_{n\times n}(\mathbb{F})$, $(BA)^T = A^T B^T$.

Proof. Note that $(ba)^T_{ki} = (ba)_{ik} = \sum_{j=1}^n b_{ij}a_{jk}$, while
$(a^T b^T)_{ki} = \sum_{j=1}^n (a)^T_{kj}(b)^T_{ji} = \sum_{j=1}^n a_{jk}b_{ij} = \sum_{j=1}^n b_{ij}a_{jk}$. □

a)

Claim.

For any $x, y \in F^n$ (thought of as column vectors) we have $\langle x, Ay\rangle = \langle A^* x, y\rangle$.

Proof.

Note that, by definition, for all $x, y \in \mathbb{F}^n$, $\langle x, Ay\rangle = \sum_{i=1}^n (x)_i \overline{(Ay)_i}$, where $(x)_i$ and $(y)_i$ are ith entries in x and Ay respectively.

Therefore, by definition of matrix multiplication,

$$\begin{aligned} \langle x, Ay\rangle = x^T \times (\overline{A} \times \overline{y}) &= (x^T \times \overline{A}) \times \overline{y} && (1)\\ &= ((\overline{A})^T \times x)^T \times \overline{y} && (2)\\ &= \langle (\overline{A})^T \times x, y\rangle && (3)\\ &= \langle A^* x, y\rangle, && (4)\end{aligned}$$

where (2) comes from Lemma 0.1. □

b)

Claim.

Suppose that $B \in M_{n\times n}(\mathbb{F})$ and that $\langle x, Ay\rangle = \langle Bx, y\rangle$ for all $x, y \in \mathbb{F}^n$.

Then $B = A^*$.

Proof. From a), for any $x, y \in \mathbb{F}^n$ we have $\langle x, Ay\rangle = \langle A^* x, y\rangle$.

We are also given that $\langle x, Ay\rangle = \langle Bx, y\rangle$. Therefore,

$$\begin{aligned} \langle A^* x, y\rangle - \langle Bx, y\rangle &= \langle x, Ay\rangle - \langle x, Ay\rangle && (5)\\ \langle A^* x - Bx, y\rangle &= 0 && (6)\\ \langle (A^* - B)x, y\rangle &= 0 && (7)\end{aligned}$$

□

Since this holds for any $x, y \in M_{n\times n}(\mathbb{F})$, take $y = (A^* - B)x$.

Suppose, on the contrary, that $A^* - B \neq \mathbf{0}$. Therefore, neither input is zero, and thus $\langle (A^* - B)x, y\rangle$ is greater than zero, which is a contradiction. Therefore, $A^* = B$.