Administrativia: no discussions, no extra material consulted

Problem I

Consider the following algorithm:

ISUM(A)

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(1)  best ← 0
(2)  if A[1] > best then best ← A[1] fi
(3)  for i ← 1 to n − 1 do
(4)      b ← A[i + 1]
(5)      if b > best then best ← b fi
(6)      for j ← i down to 1 do
(7)          b ← b + A[j]
(8)          if b > best then best ← b fi
(9)      od
(10) od
(11) return(best)
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What is the worst number of assignments performed by this algorithm?

Solution

Let $T : \mathbb{Z}^+ \to \mathbb{N}$ be the map such that for any $n \in \mathbb{Z}^+$ $T(n)$ is the maximum number of assignments performed by the algorithm above on arrays A of length n.

Let n be arbitrary positive integer.

Let A be an arbitrary array of length n.

From the algorithm, there is one assignment in the line (1), and at most one assignment in the line (2).

From the line (3), there are $n-1$ iterations of the for-loop, and thus there are $n-1$ assignments performed in the line (4).

There is at most one assignment for each iteration of the for-loop in the line (5), and thus at most $n-1$ assignments executed on line (5) in total.

For each (i)th iteration initiated in the line (3), there are i iterations initiated by the line (6). Thus, for each (i)th iteration there are i assignments given by the line (7). There is at most 1 assignment on line (8), and hence for each (i)th iteration there are at most i assignments executed on line (8).

Therefore,

$$T(n) \leq 1 + 1 + (n-1) + (n-1) + \sum_{i=1}^{n-1} \big(i + i\big) \tag{1}$$

$$= 2n + n(n-1) = n^2 + n \tag{2}$$

Thus, $T(n) \leq n^2 + n$.

Consider now an array G of length $n \in \mathbb{N}$ such that, for all $i \in [1, n] \cap \mathbb{N}$, $G[i] = F[i]$, where $F(1) = 1$ and $F(2) = 2$, and, for $j \geq 3$, $F(j) = F(j-1) + F(j-2)$.

> **Lemma 0.1**
> For all $j \in \mathbb{Z}^+$, $0 < F(j) < F(j+1)$.

Proof:

We proceed by induction.

For any $i \in \mathbb{Z}^+$, let $P(i)$ denote the predicate "$F(i) < F(i+1)$ and $F(i) > 0$".

Base Case

Note that by definition $F(1) = 1 > 0$, $F(2) = 2 > 0$, and thus $0 < F(1) < F(2)$. Thus, $P(1)$ holds.

Moreover, $F(3) = F(1) + F(2) = 3 > 0$, and thus $0 < F(2) < F(3)$.

Inductive Step

Assume the hypothesis holds for some $k \in \mathbb{Z}^+$ such that $k \geq 3$.

Thus, $0 < F(k) < F(k+1)$. By definition of F, since $k \geq 3$, $F(k+2) = F(k) + F(k+1)$.

By inductive hypothesis, $F(k) > 0$, and thus $F(k+2) > F(k+1)$.

Since $F(k+1) > 0$, then $F(k+2) > F(k+1) > 0$.

Therefore, $P(k+1)$.

Conclusion

For all $j \in \mathbb{Z}^+$, $0 < F(j) < F(j+1)$ by induction. ■

For the array G, ISUM(G) executes one assignment in the line (1), and since $F(1) = 1 > 0$, there is a second assignment on line (2).

Since there are $n-1$ iterations of the for-loop initiated in the line (3), there are $n-1$ unconditional assignments in the line (4). Since for each i from 1 to $n-1$ $G[i+1] = F(i+1) > F(i) = G[i]$ by Lemma 0.1, then there are $n-1$ assignments in line (5).

For each (i)th iteration initiated in the line (3), there are i iterations initiated by the line (6).

For each (i)th iteration there are i unconditional assignments given by the line (7).

Since for each (i)th iteration $G[i]$ is positive by Lemma 0.1, then the if-condition in the line (8) is satisfied, and hence there are i assignments executed from the line (8).

The maximum number of assignments in the worst case is no better than the number of assignments for the array G, and hence

$$T(n) \geq 1 + 1 + (n-1) + (n-1) + \sum_{i=1}^{n-1} (i + i) \tag{3}$$

$$= 2n + n(n-1) = n^2 + n \tag{4}$$

.

Since $T(n)$ is also less than or equal to $n^2 + n$, $T(n) = n^2 + n$.

Thus, $\forall n \in \mathbb{Z}^+ . T(n) = n^2 + n$ by generalisation.

Problem II

For any given A, define $R : \mathbb{Z}^+ \to \mathbb{N}$ as $R(n) =$ "the worst case number of assignments of sums of elements to $best, best'$ and b performed by RSUM$(A, 1, n)$".

If $n = 1$, then there is one assignment on line 1 and at worst one assignment on line 4.

If $n > 1$, there is one assignment on line 1, and there is one assignment on line 6.

There are $n - \left\lfloor \frac{n+1}{2} \right\rfloor$ iterations initiated in the line 7, and thus there are $n - \left\lfloor \frac{n+1}{2} \right\rfloor$ assignments on line 8 and at worst $n - \left\lfloor \frac{n+1}{2} \right\rfloor$ assignments on line 9.

Then there are 2 assignments from lines 10 and 11.

In the line 12, a for-loop is initiated with $\left\lfloor \frac{n+1}{2} \right\rfloor$ iterations, and hence there are $\left\lfloor \frac{n+1}{2} \right\rfloor$ assignments on line 13 and at most $\left\lfloor \frac{n+1}{2} \right\rfloor$ assignments on line 14.

Then there is one assignment on line 15.

On line 16, there are $R(\left\lfloor \frac{n+1}{2} \right\rfloor)$ assignments.

On line 17, there is at most one assignment.

On line 18, there are $R(n - \left\lfloor \frac{n+1}{2} \right\rfloor)$ assignments.

On line 19, there is at most one assignment.

Therefore, in the worst case,

$$R(n) = \begin{cases} 2, \text{if } n = 1 \\ 2n + 7 + R(\left\lfloor \frac{n+1}{2} \right\rfloor) + R(n - \left\lfloor \frac{n+1}{2} \right\rfloor), \text{if } n > 1 \end{cases}$$

Note that, by definition of a floor function, if $z \in \mathbb{Z}^+$, then $\lfloor z + 1/2 \rfloor = z$.

Consider $R(m)$ for $m = 2^k$ such that $k \in \mathbb{Z}^+$:

$$\begin{aligned} R(2^k) &= \begin{cases} 2, \text{if } k = 0 \\ 2^{k+1} + 7 + R(2^{k-1}) + R(2^{k-1}), \text{if } k > 0 \end{cases} \\ &= \begin{cases} 2, \text{if } k = 0 \\ 7 + 2^{k+1} + 2R(2^{k-1}), \text{if } k > 0 \end{cases} \end{aligned}$$

Claim. For any $k \in \mathbb{N}^+$, $R(2^k) = (2^k - 1)7 + (k+1)2^{k+1}$.

Proof:

For any $n \in \mathbb{N}^+$, let $Q(n) =$"$R(2^n) = (2^n - 1)7 + (n+1)2^{n+1}$".

We proceed by induction on n.

Base Case

If $n = 0$, then $R(2^0) = R(1) = 2 = (2^0 - 1)7 + 2^{0+1}$ and thus $Q(0)$ holds.

Inductive Step

For $n > 0$, suppose $Q(n-1)$ holds.

By the recurrent definition of $R(n)$, we obtain that $R(n) = 7 + 2^{n+1} + 2R(2^{n-1})$, which is, by inductive hypothesis, equivalent to

$$\begin{aligned} R(n) &= 7 + 2^{n+1} + 2((2^{n-1} - 1)7 + n2^n) && (5)\\ &= 7 + (n+1)2^{n+1} + (2^n - 2)7 && (6)\\ &= (2^n - 1)7 + (n+1)2^{n+1}. && (7) \end{aligned}$$

Thus, $Q(n)$ holds, and hence $\forall n \in \mathbb{N}. P(n)$ by induction.