Theorem (1)

Let V, W be vector spaces over $\mathbb{Q}$, and let $T : V \to W$ be a function.
Then the following holds:

$$\forall(v_1, v_2 \in V) : T(v_1 + v_2) = T(v_1) + T(v_2) \Rightarrow \text{ T is linear.}$$

Proof. Suppose $v \in V$. Since $0 \in V$, then $v + 0 = v$, and, by definition of T,

$$T(v + 0) = T(v) = T(v) + T(0).$$

By the cancellation law for V, it follows that $T(0) = 0$.

Now, since $\exists((-v) \in V) : v + (-v) = 0$, then it follows that

$$T(0) = T(v + (-v)) = T(v) + T(-v) = 0.$$

Thus, by the cancellation law for V, $T(v) = -T(-v)$, and hence T is odd.

Suppose $v' = (n+1)v,\ n \in \mathbb{N} \subset \mathbb{Q}$. Then by definition of T, $T(v + v') = T(v) + T(v')$, and thus

$$T((n+1)v) = T(v + nv) = T(v) + T(nv). \tag{1}$$

Since T is additive by definition, to prove that T is linear, it is necessary to show that T is homogeneous. Since $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$, the condition will be shown to hold first for $\mathbb{N}$, then for $\mathbb{Z}$ and finally for $\mathbb{Q}$.

Consider the statement $S(k) : T(kv) = kT(v)$.

Since $v \in V, T(v) \in W$, then $T(1 \cdot v) = 1 \cdot T(v)$. Thus, $S(1)$ is true.

Suppose $\exists(k \in \mathbb{N}) : S(k)$ holds.

Consider $S(k+1) : T((k+1)v) = (k+1)T(v)$.

From the equation 1, setting $n = k$,

$$T((k+1)v) = T(v) + T(kv).$$

By the hypothesis, $T(kv) = kT(v)$, and thus

$$T((k+1)v) = T(v) + kT(v) = kT(v) + T(v) = (k+1)T(v),$$

which is exactly $S(k+1)$.

Since $S(1)$ is true, then $S(k)$ holds for all $k \in \mathbb{N}$ by induction.

Since T is odd, while $k \in \mathbb{N}$, then it follows that

$$T((-k)v) = -T(-(-k)v) = -(T(kv)) = -(kT(v)) = (-k)T(v).$$

Also, $T(0) = 0$, and thus $S(k)$ holds for all $k \in \mathbb{Z}$.

Note that for all $m \in \mathbb{Z}$ and $k \in \mathbb{Z} \setminus \{0\}$, since $\frac{1}{k} \in \mathbb{Q}$,

$$T(v) = T(\frac{k}{k}v) = kT(\frac{1}{k}v),$$

and hence $T(\frac{1}{k}v) = \frac{1}{k}T(v)$. Therefore,

$$T(\frac{m}{k}v) = \frac{1}{k}T(mv) = \frac{m}{k}T(v),$$

and hence $S(n)$ holds for all $n \in \mathbb{Q}$.

Thus, T is homogeneous, and since it is also additive by definition, then T is linear. □

Claim. Theorem (1) does not hold for all V, W defined over $\mathbb{C}$.

Proof. Suppose Theorem (1) holds for all V, W defined over $\mathbb{C}$.

Let V, W be vector spaces over $\mathbb{C}$ with elements from $\mathbb{C}$.

Let $z \in V$. Consider a map $T : V \to W$ such that $z \mapsto z + \overline{z}$.
Note that by definition of a complex conjugate, $z + \overline{z} \in \mathbb{R} \subset \mathbb{C}$.

Note also that T is additive, since for all $u, v \in \mathbb{C}$:

$$(u + v) + (\overline{u} + \overline{v}) = (u + \overline{u}) + (v + \overline{v}).$$

Consider $a = i \in \mathbb{C}$ and $z = 1 \in V$.

Note that $T(1) = 2 \in W,\ T(i \cdot 1) = 0 \in W,\ iT(1) = 2i \in W$.

Thus, $T(i \cdot 1) \neq iT(1)$, which is a contradiction. □

Let $\mathbb{P}$ denote the set of prime numbers.

Claim. Theorem (1) holds for V, W defined over $\mathbb{F} = \mathbb{Z}_p$ for $p \in \mathbb{P}$.

Proof. The proof is similar to the proof of Theorem (1).

Suppose $v \in V$. Since $0 \in V$, then $v + 0 = v$, and, by definition of T,

$$T(v + 0) = T(v) = T(v) + T(0).$$

By the cancellation law for V, it follows that $T(0) = 0$.

Suppose $v' = (n+1)v,\ n \in \mathbb{F}, n \neq p-1$. Then by definition of T, $T(v+v') = T(v)+T(v')$, and thus

$$T((n + 1)v) = T(v + nv) = T(v) + T(nv). \tag{2}$$

Since T is additive by definition, to prove that T is linear, it is necessary to show that T is homogeneous.

Consider the statement $S(k) : T(kv) = kT(v)$.

First, note that $T(0v) = T(0) = 0 = 0T(v)$. Thus, $S(0)$ is true.
Suppose $\exists(k \in \mathbb{F}, k \neq (p - 1)) : S(k)$ holds.

Consider $S(k + 1) : T((k + 1)v) = (k + 1)T(v)$.

From the equation 2, setting $n = k$,

$$T((k + 1)v) = T(v) + T(kv).$$

By the hypothesis, $T(kv) = kT(v)$, and thus

$$T((k+1)v) = T(v) + kT(v) = kT(v) + T(v) = (k+1)T(v),$$

which is exactly $S(k+1)$.

Since $S(0)$ is true, by the above argument it follows that $S(1)$ holds. Thus, $S(2)$ holds also. Repeating the argument for all $k \in [2 : p-2]$ (since $\mathbb{F}$ is finite, this procedure stops, and in the last step it is shown that $S(p-1)$ holds since $S(p-2)$ is true), it follows that T is homogeneous for all $k \in \mathbb{F}$, and thus the claim holds. $\square$