1 Introduction to Inner Product Spaces

Assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

1.1 Inner Products

Definition 1.1. Suppose V is a vector space over $\mathbb{R}$ or $\mathbb{C}$.

An **inner product** on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that

a) $\langle \cdot, \cdot \rangle$ is linear in the first component, i.e. $\langle cx + y, z \rangle = c\langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in V$ and $c \in \mathbb{F}$.

b) $\langle x, y \rangle = \langle \overline{y}, x \rangle$ for all $x, y \in V$, where $\overline{y}$ is a complex conjugate.

c) $\langle x, x \rangle > 0$ for all non-zero x in V

Remark 1.2.

- If $\mathbb{F} = \mathbb{R}$, then by (b) we obtain $\langle x, y \rangle = \langle y, x \rangle$.
- By (b), $\langle x, x \rangle = \overline{\langle x, x \rangle} \Rightarrow \langle x, x \rangle \in \mathbb{R}$
- By (a), $\langle 0, x \rangle = 0$ for all x and thus by (b) $\langle x, 0 \rangle = 0$ for all x
- By (b), we obtain that $\langle \cdot, \cdot \rangle$ is **conjugate linear** in the second component, since

$$\begin{aligned} \text{LHS} &= \overline{\langle cy + z, x \rangle} && (1) \\ &= \overline{c\langle y, x \rangle + \langle z, x \rangle} && (2) \\ &= \overline{c} \cdot \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} && (3) \\ &= \overline{c}\langle x, y \rangle + \langle x, z \rangle = \text{RHS} && (4) \end{aligned}$$

Example 1.3

If $V = \mathbb{F}^n$, define $\langle a, b \rangle = \sum_{i=1}^{n} a_i \cdot \overline{b_i}$, also known as a **standard inner product on** $\mathbb{F}^n$.

Check it!