1 Review

Let V be a finite dimensional inner product space, and let $W \subseteq V$ be a subspace.

Suppose $x \in V$, and let $w \in W, w' \in W^{\perp}$ be unique vectors such that $v = w + w'$.

Then $P_W \in \text{End}(V)$ is an *orthogonal projection* such that $P_W(x) = w$.

We have already proven that $\operatorname{im} P_W = W$ and $\ker P_W = W^{\perp}$.

Moreover, $T \in \text{End}(V)$ is an orthogonal projection if and only if $T^2 = T = T^*$.

2 Spectral Theorem Revisited

Theorem 2.1

Suppose $T \in \text{End}(V)$ is normal (if $\mathbb{F} = \mathbb{C}$) or self-adjoint (if $\mathbb{F} = \mathbb{R}$).

Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T.

Let W_i be a λ_i-eigenspace.

Let $T_i = P_{W_i}$ be the orthogonal projection onto W_i.

Then the following properties hold:

a) $V = W_1 \oplus \cdots \oplus W_k$

b) $W_i^{\perp} = \sum_{j \neq i} W_j = \bigoplus_{j+1} W_j$.

c) $T_1 + \cdots + T_k = I$

d) $T_i T_j = \delta_{ij} T_i$

e) $T = \sum_{i=1}^{k} \lambda_i T_i$.

Proof.

a) By Theorem 6.16/6.17, T has an orthonormal basis of eigenvectors. Therefore, T is diagonalisable, which is by Corollary to Theorem 5.10 equivalent to the statement $V = W_1 \oplus \cdots \oplus W_k$.

b) Since T is normal, eigenvectors for distinct eigenvalues are orthogonal by Theorem 6.15. Hence, $W_j \subseteq W_i^{\perp}$ for all $j \neq i$, and thus $\sum_{j \neq i} W_j \subseteq W_i^{\perp}$.

   We also know that $\sum_{j \neq i} W_j = \bigoplus_{j \neq i} W_j$ because by part (a) if $\sum_{j \neq i} w_j = 0$ for $w_j \in W_j$, then $w_j = 0$.

   Therefore, $\dim(\bigoplus_{j \neq i} W_j) = \sum_{j \neq i} \dim W_j$.

   Since $\dim W_i^{\perp} = \dim V - \dim W_i = \sum_{j \neq i} \dim W_j$, then $\bigoplus_{j \neq i} W_j = W_i^{\perp}$.

c) By (a), any $v \in V$ can be uniquely written as $v = \sum_{i=1}^{k} w_i$ for $w_i \in W_i$.

   Therefore, $T_i(\sum_{i=1}^{k} w_i = \sum_{j=1}^{k} T_i(w_j) = T_i(w_i) = w_i$, since by (b), $w_j \in \ker T_i$ for $j \neq i$.

   Therefore, $(\sum_{i=1}^{k} T_i)(\sum_{i=1}^{k} w_i) = \sum_{i=1}^{k} w_i = I$.

d) Note that $T_i T_j(\sum_{i=1}^{k} w_i = T_i(w_j) = \delta_{ij} w_j = \delta_{ij} T_j(\sum_{i=1}^{k} w_i)$.

e) Note that $T(\sum_{i=1}^{k} w_i) = \sum \lambda_i w_i$.

   Moreover, $(\sum_{i=1}^{k} \lambda_i T_i)(\sum_{i=1}^{k} v_i) = \sum_{i=1}^{k} \lambda_i w_i$.

$\square$

Lemma 2.2

Suppose $T \in \text{End}(V)$ is normal (if $\mathbb{F} = \mathbb{C}$) or self-adjoint (if $\mathbb{F} = \mathbb{R}$).

Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T.

Let W_i be a λ_i-eigenspace.

Let $T_i = P_{W_i}$ be the orthogonal projection onto W_i.

a) If $g(t)$ is a polynomial, then $g(T)) = \sum_{i=1}^{k} g(\lambda_i)T_i$.

b) $T^* = \sum_{i=1}^{k} \overline{\lambda_i}T_i$.

Proof.

a) Write $g(t) = \sum_{i=0}^{n} a_i t^i$, where $a_i \in \mathbb{F}$. Then

$$g(T)(\sum_{i=1}^{k} w_i) = (\sum_{i=1}^{n} a_i T^i)(\sum_{i=1}^{k} w_i) \tag{1}$$

$$= \sum_{i=1}^{k}(\sum_{j=1}^{n} a_j T^j)(w_i) \tag{2}$$

$$= \sum_{i=1}^{k}(\sum_{j=1}^{n} a_j \lambda_i^j w_i) \tag{3}$$

$$= \sum_{i=1}^{k} g(\lambda_i) w_i \tag{4}$$

$$= \sum_{i=1}^{k} g(\lambda_i) T_i(\sum_{j=1}^{k} w_j). \tag{5}$$

Therefore, $g(T) = \sum_{i=1}^{k} g(\lambda_i)T_i$.

b) Note that

$$T^* = (\sum_{i=1}^{k} \lambda_i T_i)* \tag{6}$$

$$= \sum_{i=1}^{k} \overline{\lambda_i} T_i^* \tag{7}$$

$$= \sum_{i=1}^{k} \overline{\lambda_i} T_i, \tag{8}$$

since $T_i^* = T_i$.

$\square$

Corollary 2.3

Suppose $\mathbb{F} = \mathbb{C}$. Then T is normal if and only if $T^* = g(T)$ for some polynomial $g(t)$.

Proof.

If $T^* = g(T)$, then $TT^* = Tg(T) = g(T)T = T^*T$.

If T is normal, then $T = \sum_{i=1}^{k} \lambda_i T_i$ by Theorem 2.1.

Lemma 2.2 tells us that $T^* = \sum_{i=1}^{k} \overline{\lambda_i} T_i$ and $g(T)) = \sum_{i=1}^{k} g(\lambda_i) T_i$.

By Lagrange interpolation (see section 1.6), we can find a polynomial $g(t)$ such that $g(\lambda_i) = \overline{\lambda_i}$. Then $T^* = g(T)$.

□