1 Simple Curves

- simple loop with exponentially increasing complexity can be generated by introducing three pins on the plane and then *blender* them by interchanging continuously two points (the second with the third, then the new first with the second, etc). Note that
- Space of simple closed curves on the surface is the space of integral points in the larger space isomorphic to $\mathbb{R}^{log-6}$. Thus $\mathbb{Z}^{log-6} \subseteq \mathbb{R}^{log-6}$
- The number of geodesics grows like a polynomial (cf. Mariyam Mirzakhany)

2 Random Paths in the Plane

- Take a very fine grid and follow a random walk, which gives a combinatorial path in a canonical way such that beyond some point the properties of the curve depend only on the underlying surface
- Brownian motion and picking curves at random give the same asymptotic results.
- Selberg trace formula can give, however, something more.
- A random walk is Markov, while construction of simple curves requires avoidance of *cornering.*
- There is a way around it. Take an ordinary random walk on a lattice and, if the walk comes to the place where it was, make the loop and erase it.
- cf. Sheffield and Viklund: Random Simple Curves

3 Three Punctured Sphere

The plane has no topology, however it is not an obstacle – we can, for example, puncture it. Consider the triply punctured Riemann sphere with 0, 1 and ∞ removed. How does the hyperbolic geometry of this object look like? There is a metric corresponding to it. What do we want to study? We may study geodesics – but there are no simple loops on the triply punctured spheres!

We can change the rules of the game and still count the nonexisting geodesics.

Consider non-simple closed geodesics. There are two invariants which can be associated with them. One of them is the combinatorial length, number of times it passes through the upper half-plane, which is half the number of times the curve crosses $\mathbb{R}$. Another one is the number of times it crosses itself.

For any loop the self-intersection number is at least the combinatorial length minus 1.

Define a *defect* as the difference of the self-intersection number and combinatorial length.

Almost simple loops with a fixed defect have quadratic growth.

The number of loops of a given type can be given using binomial coefficients and by describing the motifs of a curve.

This formula can be generalized for a greater number of punctures, but the sum in the formula becomes infinite.

To prove the theorem, geodesics must be described in the topological and combinatorial terms. Choose a base point in either half-plane and draw loops around the infinity, which defines a free groop generated by three terms such that their product is identity.

Key terms: Coxeter group, orbifold fundamental group, mirrors, generators, orientation preserving subgroup of the orientation group

Combinatorial itinerary can be described by words, which tie nicely with the describing groups. This, in turns, can help us prove the theorem on the defect bounds.