MAT 157: CONTINUITY

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$f(x)$ IS CONTINUOUS ON $\mathbb{R}$ IF
$f(x)$ IS CONTINUOUS AT EVERY $x \in \mathbb{R}$.

SUPPOSE $a<b$. THEN $f(x)$ IS CONTINUOUS ON $[a,b]$ IFF

(i) $f(x)$ IS CONTINUOUS ON (a,b)

(ii) $\lim_{x \to a^+} f(x) = f(a)$, $\lim_{x \to b^-} f(x) = f(b)$.

CONTINUOUS FUNCTIONS ON CLOSED INTERVALS

THEOREM

IF $f(x)$ IS CONTINUOUS ON $[a,b]$ AND $f(a)<0$, $f(b)>0$ THEN $\exists c \in (a,b) : f(c)=0$

THEOREM

IF $f(x)$ IS CONTINUOUS ON $[a,b]$, THEN IT IS BOUNDED ABOVE ON $[a,b]$:

$\exists M \in \mathbb{R} : f(x) \leq M \quad \forall x \in [a,b]$.

THEOREM

IF $f(x)$ IS CONTINUOUS ON $[a,b]$, THEN $\exists c \in [a,b]$:

$f(c) \geq f(x) \quad \forall x \in [a,b]$,

$f(x)$ ACHIEVES ITS MAXIMUM VALUE.

LET $U = \{ x \mid f(x) < 0 \}$.

$a \in U \Rightarrow U \neq \emptyset$.

(?!): U CONTAINS A SMALL INTERVAL TO THE RIGHT OF a.

LET $c = \sup(U)$.

IF $f(c) > 0$, $\exists\, \delta > 0 : f(x) > 0$ ON $(c-\delta, c)$

$\Rightarrow (c-\delta, c) \notin U$, WHICH IS A CONTRADICTION.

$\Rightarrow f(c) \leq 0$

IF $f(c) < 0$, $\exists (\delta > 0): f(x) < 0$ FOR $x \in (c, c+\delta)$

SO $c \neq \sup(U)$, WHICH IS A CONTRADICTION.

$\therefore f(c) = 0$.

$f(a) > 0$, $f(b) < 0$.

Similar ARGUMENT!

THEOREM [INTERMEDIATE VALUE THEOREM]
CASE

IF f IS CONTINUOUS ON $[a,b]$,

$f(a) < r < f(b)$,

THEN $\exists\, c \in (a,b) : f(c = r)$.

$\longrightarrow$ CONSIDER $f(x) - r$

INTERMEDIATE VALUE THEOREM IF $f(x)$ IS CONTINUOUS ON $[a,b]$, IT ACHIEVES EVERY VALUE BETWEEN $f(a)$ AND $f(b)$

EXAMPLE $f(x) = x^2 - 2$ IS A POLYNOMIAL, SO IT IS CONTINUOUS ON $\mathbb{R}$. (EXERCISE.)

$f(0) = -2$, $f(2) = 2$

So $\exists\, a \in (0,2) : f(a) = 0$

$\Rightarrow a^2 - 2 = 0 \Rightarrow a = \sqrt{2}$

SUPPOSE $s > 0$,

CONSIDER $f(x) = x^2 - s$.

$f(0) = -s < 0$.

IF $s > 1$, $s^2 - s > 0 \Leftrightarrow f(s) > 0$.

$s \leq 1$, $s^2 \leq s \leq 1$, $f(2) = 4 - s \geq 3 > 0$