1 Problem I

Lemma 1.1

$\ker T^m = \ker T^{m+1}$ for some $m \in \mathbb{Z}^+$ if and only if $\ker T^m = \ker T^{m+k}$ for any $k \in \mathbb{Z}^+$.

Proof.

Take $k \in \mathbb{Z}^+$.

From Problem III, we know that $\ker T^{m+k} \subseteq \ker T^{m+k+1}$.

Suppose now that $v \in \ker T^{m+k+1}$. Therefore, $T^{m+1}(T^k v) = 0$, and thus $T^k v \in \ker T^{m+1} = \ker T^m$.

Thus, $T^{m+k}v = 0$, and hence $v \in \ker T^{m+k}$.

The other direction is trivially obtained by setting $k = 1$. $\square$

Lemma 1.2

If $n = \dim V$, then $\ker T^n = \ker T^{n+1}$.

Proof.

Let $v \in V$ be arbitrary.

By way of contradiction, suppose that $\ker T^n \neq \ker T^{n+1}$.

Therefore, by Lemma 1.1, we get that $\ker T \neq \ker T^2 \neq \cdots \subseteq \ker T^{n+2}$. We also know that $\ker T \subseteq \ker T^2 \subseteq \cdots \subseteq T^{n+1}$, and thus there must be at least $n+1$ vectors spanning $\ker T^{n+1} \subseteq V$, which is a contradiction, since $n = \dim V$.

Thus, $\ker T^n = \ker T^{n+1}$. $\square$

Lemma 1.3

$K_\lambda(T) = \ker(T - \lambda I)^{\dim V}$.

Proof.

By definition of $K_\lambda(T)$, $\ker(T - \lambda I)^{\dim V} \subseteq K_\lambda(T)$.

Suppose now that $v \in K_\lambda(T)$. Thus, there exists a positive integer m such that $v \in (T - \lambda I)^m$.

By Lemma 1.2 we know that $\ker T^{\dim V} = \ker T^{\dim V + 1}$.

Therefore, by the Rank-Nullity Theorem, $\operatorname{rank} T^{\dim V} = \operatorname{rank} T^{\dim V + 1}$.

Thus, by Problem III, $K_\lambda T = \ker(T - \lambda I)^{\dim V}$. $\square$

Suppose that $\mathbb{F} = \mathbb{Z}_2$.

Consider $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ over $\mathbb{F}$.

Problem.

Find all eigenspaces and all generalised eigenspaces of the linear transformations L_A, L_B.

Solution.

We show that for L_A $E_0 = \text{span}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$ and $K_0 = \text{span}\{\begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}\}$,

while $E_1 = \text{span}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$ and $K_1 = \text{span}\{\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\}$.

For L_B, 0 is the only eigenvalue and $E_0 = K_0 = \text{span}\begin{pmatrix}1\\1\\1\end{pmatrix}$.

We calculate the characteristic polynomial of A by the repeated Laplacian expansion along the first column:

$$\begin{aligned}
\det(A - tI) &= (1-t)\det\begin{pmatrix}1-t & 1 & 1\\ 0 & 1-t & 1\\ 1 & 1 & 1-t\end{pmatrix} - \det\begin{pmatrix}1 & 1 & 1\\ 0 & 1-t & 1\\ 1 & 1 & 1-t\end{pmatrix} && (1)\\
&= (1-t)[(1-t)((1-t)^2 - 1) + t)] - [(1-t)^2 - 1 + t] && (2)\\
&= (1-t)[(1-t)((1-t)^2 - 1) + t)] + t(1-t) && (3)\\
&= (1-t)[(1-t)^3 - 1 + t + t] + t(1-t) && (4)\\
&= (1-t)[(1-t)^3 - 1 + 3t] && (5)\\
&= (1-t)[(1 - 3t + 3t^3 - t^3 - 1 + 3t] && (6)\\
&= (1-t)[(1 - 3t + 3t^2 - t^3 - 1 + 3t] && (7)\\
&= (1-t)[3t^2 - t^3] && (8)\\
&= -t^2(1-t)(t-3) && (9)\\
&= -t^2(1-t)(t-1) && (10)\\
&= t^2(t-1)^2 && (11)\\
& && (12)
\end{aligned}$$

Note that $\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$ is an eigenvector of A with the corresponding eigenvalue of 0, since $1 + 1 = 0$. To find other possible eigenvectors $\begin{pmatrix}u\\v\\w\\x\end{pmatrix}$, we solve the system of equations:

$$\begin{cases} u + v + w + x &= 0\\ w + x &= 0\ .\\ v + w + x &= 0 \end{cases} \qquad (13)$$

Thus, $u+v=0$, $w+x=0$ and $v=0$. Therefore, $u=0$, and thus $w+x=0$, which is possible if either $w=1=x$ or $w=0=x$. Thus, $\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$ is the only eigenvector of E_0.

Suppose now that $\lambda=1$ is also an eigenvalue. Therefore,

$$\begin{cases} u+v+w+x & =u \\ u+v+w+x & =v \\ w+x & =w \\ v+w+x & =x \end{cases}. \tag{14}$$

Therefore, $u=v$, $x=0$ and $w+v=0$, and the eigenvector must be of the form $\begin{pmatrix}u\\u\\-u\\0\end{pmatrix}$.

Therefore, $u\neq 0$, and thus $u=1$, and $\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$ is the only eigenvector in E_1.

Note now that by Cayley-Hamilton Theorem $A^2(A-I)^2=A^4-2A^3+A^2=A^4+A^2=0$ (since $2=0$).

Therefore, since $-1=1$, we get that $A^2=A^4$.

Note also that $A^2=\begin{pmatrix}1+1 & 1+1+1 & 1+1+1+1 & 1+1+1+1 \\ 1+1 & 1+1+1 & 1+1+1+1 & 1+1+1+1 \\ 0 & 1 & 1+1 & 1+1 \\ 1 & 1+1 & 1+1+1 & 1+1+1\end{pmatrix}=\begin{pmatrix}0&1&0&0\\0&1&0&0\\0&1&0&0\\1&0&1&1\end{pmatrix}$.

If $\begin{pmatrix}u\\v\\w\\x\end{pmatrix}$ is an eigenvector of A^2 with the corresponding eigenvalue λ, we obtain the following system of equations:

$$\begin{cases} v & =\lambda u \\ v & =\lambda v \\ v & =\lambda w \\ u+w+x & =\lambda x \end{cases}. \tag{15}$$

Therefore, if the eigenvalue is 0, then $v=0$. Since then u,w,x are not all zero, there exists at least one nonzero entry. Since $u+w+x=0$, there must be exactly two nonzero entries, and there are three options: $\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$. Since $\begin{pmatrix}0\\0\\1\\1\end{pmatrix}+\begin{pmatrix}1\\0\\1\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$, any pair of these vectors spans $\ker A^2=\ker A^4$.

Consider now $(A-I)^2$:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1+1 & 1+1 \\ 0 & 1+1 & 1+1 & 1+1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1+1 \end{pmatrix} \tag{16}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \tag{17}$$

If $\begin{pmatrix} u \\ v \\ w \\ x \end{pmatrix}$ is an eigenvector of $(A-I)^2$ with the corresponding eigenvalue 0, we obtain the following system of equations:

$$\begin{cases} u+v &= 0 \\ v+w &= 0 \\ u+w &= 0 \end{cases} \tag{18}$$

Therefore, $u = v = w$ and there are four cases:

- $u = v = w = 1$, $x = 1$
- $u = v = w = 0$, $x = 1$
- $u = v = w = 1$, $x = 0$
- $u = v = w = 0$, $x = 0$

The fourth case gives us a zero vector, which is not an eigenvector.

The first case gives us $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ such that $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, and this this vector is in the span of the vectors given in the second and third case. Thus, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ span $\ker(T-I)^2$.

Consider now $(A-I)^3$:

$$(A-I)^2(A-I) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{19}$$

$$= \begin{pmatrix} 1 & 1 & 1+1 & 1+1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1+1 \\ 0 & 1 & 1 & 1+1 \end{pmatrix} \tag{20}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{21}$$

Note that the associated system of equations for an eigenvector with an eigenvalue 0 is exactly the same as for $(A-I)^2$, and hence $\ker(A-I)^2 = \ker(A-I)^3$, which by Lemma 1.1 means that $\ker(A-I)^{\dim V} = \ker(A-I)^2 = 0$. Therefore, by Lemma 1.3 and previous remarks, we have $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ span K_1.

Since $\dim V = 4$, by Lemma 1.3 we can be sure that there are no eigenvectors in K_λ other than those already found.

Now we consider B, calculating its characteristic polynomial:

$$\det(B-tI) = (1-t)\det\begin{pmatrix} 1-t & 1 \\ 0 & 1-t \end{pmatrix} + \det\begin{pmatrix} 1 & 0 \\ 1-t & 1 \end{pmatrix} \tag{22}$$

$$= (1-t)^3 + 1 \tag{23}$$

$$= (2-t)((1-t)^2 - (1-t) + 1) \tag{24}$$

$$= -t(t^2 - t + 1) \tag{25}$$

Suppose $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ is an eigenvector of B with a corresponding eigenvalue λ. From the characteristic polynomial we see that $\lambda = 0$ is the only eigenvalue of T. Therefore,

$$\begin{cases} u + v = 0 \\ v + w = 0 \\ u + w = 0 \end{cases} . \tag{26}$$

Hence, $v = u = w$, and since the vector is an eigenvector, than $v = u = w \neq 0$, and hence $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is the vector spanning E_0 of B.

Consider B^2:

$$B^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \tag{27}$$

$$= \begin{pmatrix} 1 & 1+1 & 1 \\ 1 & 1 & 1+1 \\ 1+1 & 1 & 1 \end{pmatrix} \tag{28}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{29}$$

Because B^2 can be optained by first swapping the first and the third row and then by swapping the second and the new third row, we see that they have the same characteristic polynomial. Solving the system of equations as above, we obtain that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is its eigenvector.

Therefore, $\ker B = \ker B^2$, and by Lemma 1.1 we have that $\ker B = \ker B^{\dim W}$, where W is such that $L_B \in \mathrm{End}(W)$, which by Lemma 1.3 means that K_0 is spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\square$

Lemma 1.4

For $L_A \in \mathrm{End}(V)$, $V = K_0 \oplus K_1$.

Proof.

From the computations above, we see that for the found bases of K_0 and K_1, denoted by β_0 and β_1 respectively, $\beta = \beta_0 \cup \beta_1$ is a basis of V, since β has the length of 4 and is linearly independent. To prove the latter claim, suppose that there exist $a_1, \ldots, a_4 \in \mathbb{Z}_2$ such that

$$a_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0. \tag{30}$$

Therefore,

$$\begin{cases} a_2 + a_3 & = 0 \\ a_3 & = 0 \\ a_1 + a_2 + a_3 & = 0 \\ a_1 + a_4 & = 0 \end{cases},$$

and thus $a_3 = a_2 = a_1 = a_4 = 0$, which means that β is linearly independent.

Therefore, β is a basis of V. Since $\beta_0 \cap \beta_1 = \emptyset$ and thus $K_0 \cap K_1 = \{0\}$, we can see that $V = K_0 \oplus K_1$. $\square$

Note that for the given $L_B \in \operatorname{End}(W)$, we cannot decompose W into the eigenspaces of L_B, because there is only eigenspace with the dimension 1, while $\dim W = 3$.

Note also that, by the calculations above, for L_A the characteristic polynomial is $t^2(t-1)^2$, and thus $\dim K_0(L_A) = 2$ and $\dim K_1(L_A) = 2$ match the algebraic multiplicities. Similarly, for L_B, the characteristic polynomial is $-t(t^2-t+1) = t(t^2+t+1)$, which does not split (the determinant of the second factor is$\delta = -3 < 0$), and thus $\dim K_0(L_B) = 1$ also matches the algebraic multiplicity.