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① (i) HERE AND FURTHER $x \neq 1$.

Let $f(x) = \frac{x^2-1}{x^3-1} = \frac{x+1}{x^2+x+1}$, $x \neq 1$

$= 1 - \frac{x^2}{x^2+x+1}$, $x \neq 1$.

NOTE THAT $x^2+x+1 \neq 0$ $\forall x \in \mathbb{R}$ ($\Delta = 1-4 = -3 < 0$), AND SINCE ITS A POLYNOMIAL, IT IS EVERYWHERE-CONTINUOUS.

SIMILARLY, $x+1$ IS A POLYNOMIAL AND EVERYWHERE CONTINUOUS.

THUS, $\begin{cases} \lim_{x \to a} x^2+x+1 = a^2+a+1 \\ \lim_{x \to a} x+1 = a+1 . \end{cases}$

HENCE, BY PROPERTIES OF LIMITS ALREADY PROVEN, IF $a \neq 1$,

$$\lim_{x \to a} f(x) = \frac{a+1}{a^2+a+1}.$$

CONSIDER THE CASE WHEN $a = 1$.

LET $\varepsilon > 0$ BE GIVEN.

TAKE $\delta = \min\left\{1, \dfrac{\varepsilon(a^2+a+1)}{|2a+1|}\right\}$ SO THAT

$a < x < a+\delta$. THUS, $|x-a| = x-a$, AND

$$\begin{cases} a < x < a+1 \iff 0 < x-a < 1 \\ a < x < a+\dfrac{\varepsilon(a^2+a+1)}{|2a+1|} \implies (x-a)|2a+1| < \varepsilon(a^2+a+1). \end{cases}$$

SINCE $a < x < a+1$, $2a < x+a < 2a+1$

$$\implies (x-a)|x+a| < \varepsilon(a^2+a+1).$$

SINCE $a < x$, THEN $a^2+a+1 < x^2+x+1$ (*)

AND HENCE

$$|x-a|\,|x+a| < \varepsilon(x^2+x+1)$$

HENCE, $\left|\dfrac{x^2-a^2}{x^2+x+1}\right| < \varepsilon.$

NOTE THAT FROM (*), [illegible] $x > 1$

$$\frac{1}{x^2+x+1} < \frac{1}{a^2+a+1} \iff \frac{a^2}{x^2+x+1} < \frac{a^2}{a^2+a+1}.$$

$$\iff \frac{x^2}{x^2+x+1} - \frac{a^2}{x^2+x+1} > \frac{x^2}{x^2+x+1} - \frac{a^2}{a^2+a+1} \quad (\Delta)$$

1 (ε) (cont.)

Since $x > a \geq 1$, ... $= 1$,

$$\frac{x^2}{x^2+x+1} - \frac{a^2}{a^2+a+1} > \frac{x^2}{3x^2} - \frac{1}{3} = 0,$$

THEN $\dfrac{x^2}{x^2+x+1} - \dfrac{a^2}{a^2+a+1} > 0$ AND

HENCE FROM (Δ) $\left| \dfrac{x^2}{x^2+x+1} - \dfrac{a^2}{a^2+a+1} \right| < \varepsilon$.

Thus, since $x \neq 1$,

$x \to 1^+$

$$\left| \frac{(x-1)}{(x-1)}\left(1 - \frac{x^2}{x^2+x+1}\right) - \left(1 - \frac{a^2}{a^2+a+1}\right) \right| < \varepsilon,$$

Therefore, $\lim_{x \to 1^+} f(x) = \frac{2}{3}$. (II)

FOR THE SAME ε,

TAKE $\delta' = \min\left\{1, \dfrac{\varepsilon}{2a}\right\}$

for $a = 1$, $\delta' = \min\{1, \dfrac{\varepsilon}{2}\}$.

SUPPOSE $0 < a - x < \delta'$

Thus,

$$\begin{cases} 0 < x < 1 & ① \Leftrightarrow 1 > 1-x > 0 \\ 1-\varepsilon < x < 1 . & ② \end{cases}$$

FROM ②, $0 < 1 - x < \varepsilon$. (A)

① , $0 < 1 - x < 1$.

FROM ①, $1 < 2x+1 < 3 \Rightarrow \frac{1}{3} < \frac{1}{2x+1}$

Moreover, $1 < x^2+x+1 \Rightarrow$

FROM (A), $0 < (1-x) < \frac{3\varepsilon}{(2x+1)}(x^2+x+1)$

$$\Rightarrow 0 < \frac{|2x+1||-1+x|}{3(x^2+x+1)} < \varepsilon$$

$$\Rightarrow \left| \frac{3x^2-(x^2+x+1)}{3(x^2+x+1)} \right| =$$

$$= \left| \frac{x^2}{x^2+x+1} - \frac{1}{3} \right|$$

$$= \left| \left(1 - \frac{x+1}{x^2+x+1}\right) - \left(1 - 2/3\right) \right|$$

$$= \left| \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)} - 2/3 \right| < \varepsilon \Rightarrow \lim_{x \to 1} f(x) = 2/3 . \text{ (II)}$$

SINCE (I) = (II), $\lim_{x \to 1} f(x) = 2/3$

(ii)

$$\lim_{x \to a} \frac{x+3}{|x|-3} \; \begin{cases} = \dfrac{a+3}{|a|-3}, & x \notin \{-3, 3\} \\ = -1, & a = -3 \\ \in \emptyset, & a = 3. \end{cases}$$

① $x+3$ IS A POLYNOMIAL AND THUS EVERYWHERE-CONTINUOUS.

2/2

$|x|$ IS CONTINUOUS EVERYWHERE BY LEMMA I, PROVED LATER. BY THE LIMIT LAWS ALREADY PROVEN EARLIER, $|x|-3$ IS EVERYWHERE CONTINUOUS. THUS, IF $x \notin \{-3, 3\}$, BY THE LIMIT LAWS ALREADY PROVEN,

$$\lim_{x \to a} \frac{x+3}{|x|-3} = \frac{a+3}{|a|-3}.$$

② IF $a = -3$, LET $f(x) = -1$ AND $h(x) = -1$. LET $g(x) = \frac{x+3}{|x|-3}$

TAKE $b, c \in \mathbb{R}$ SUCH THAT $b < a < c < 0$.

THUS, WE HAVE $\lim_{x \to b} f(x) = -1 = \lim_{x \to c} h(x)$.

FOR $(b, -3) \cup (-3, c)$, SINCE $c < 0$ $\frac{x+3}{-(x+3)} = -1$.

THUS, BY THE SQUEEZE THEOREM PROVEN LATER, SINCE $f(x) \leq g(x) \leq h(x)$,

$\lim_{x \to -3} g(x) = -1$.

(3) CONSIDER THE CASE WHEN $a = 3$.

SUPPOSE $\varepsilon > 0$ IS GIVEN.

IF $x > 0$, $x \neq 3$,

$$g(x) = \frac{x+3}{x-3} = 1 + \frac{6}{x-3}.$$

SUPPOSE ℓ IS A LIMIT OF $g(x)$ AT $x = 3$, i.e.

$$\exists \delta > 0 : |x-3| < \delta \Rightarrow |g(x) - \ell| < \varepsilon.$$

THUS, $\frac{1}{\delta} < \frac{1}{|x-3|}$

$$\Longrightarrow 1 + \frac{6}{\delta} < \left|1 + \frac{6}{|x-3|}\right| \text{ (since } \delta > 0)$$

$|g(x) - \ell| \geq |g(x)| - |\ell|$, $\Rightarrow$

$$|g(x)| \leq \varepsilon + |\ell| \Rightarrow \frac{|x+3|}{|x-3|} < \varepsilon + |\ell|$$

CONSIDER $x_0 = \min\left(3 + 3\left(\varepsilon + |\ell| - \frac{1}{2}\right)^{-1}, \; 3 + \frac{\delta}{2}\right)$

$\Rightarrow |x_0 - 3| < \delta$. Moreover,

$$|g(x)| \geq |2(\varepsilon + |\ell|)| > \varepsilon + |\ell| \;\#.$$

Hence $\forall \varepsilon > 0$ there is no limit of $g(x)$ as $x \to 3$ $\square$

1. (iii) HERE AND FURTHER $x \neq 2.5$.

$$f(x) = \frac{\lfloor x \rfloor}{(5x-2)^2} \doteq \begin{cases} = \frac{n}{(5x-2)^2}, \exists n \in \mathbb{Z}: n \leq x < n+1, x \neq 2.5 \\ \in \emptyset, x = 0.4 \text{ XOR } x \in \mathbb{Z}. \end{cases}$$

CLAIM: $\lim_{x \to a} f(x)$: ① $\begin{cases} = \frac{n}{(5a-2)^2}, \exists n \in \mathbb{Z}: n \leq a < n+1, a \neq 2.5 \\ \in \emptyset, a = 2.5 \text{ XOR } x \in \mathbb{Z} \end{cases}$ ③ ②

PROOF:

① NOTE THAT $f(x)$ DEFINED ON THE INTERVAL $x \in (n, n+1)$ $\exists n \in \mathbb{Z}$; $x \neq 2.5$, IS CONTINUOUS, SINCE $\frac{(5x-2)^2}{n} = \frac{25x^2}{n} - \frac{10x}{n} + \frac{4}{n}$ IS A POLYNOMIAL AND THUS EVERYWHERE-CONTINUOUS, AND HENCE, BECAUSE $x \neq 2.5$,

$\frac{n}{(5x-2)^2}$ IS CONTINUOUS AND THUS

$$\lim_{\substack{x \to a \in (n, n+1), \\ n \in \mathbb{Z}}} f(x) = \frac{n}{(5a-2)^2}.$$

② FROM ①, IF $x \in (n, n+1)$, $\lim_{x \to a < n+1} f(x) = \frac{n}{(5a-2)^2}$,

~~ALSO~~ IF $x \in (n+1, n+2)$, $\lim_{x \to a > n+1} f(x) = \frac{n+1}{(5a-2)^2}$.

THUS, $\lim_{x \to (n+1)^+} f(x) \neq \lim_{x \to (n+1)^-} f(x)$,

AND ② FOLLOWS.

③ LET $y = 5x - 2$,

SINCE $\lim_{x \to 0.4} 5x - 2 = 5 \cdot 0.4 - 2 = 0$ (POLYNOMIAL),

$\nexists L$ (BECAUSE $5x-2$ IS A POLYNOMIAL)

$\nexists L \in \mathbb{R} : \lim_{y \to 0} \frac{1}{y} \cdot \lim_{y \to 0} \frac{1}{y} \cdot \lim_{y \to 0} n = L$,

SINCE $\forall \varepsilon > 0 : \exists N \in \mathbb{R} : x > N \Rightarrow \left|\frac{1}{y}\right| > \varepsilon$,

WHICH HAS BEEN PROVEN EARLIER.

$\square$

1 (w). LET $D = (-\infty, 0) \cup (0, 2) \cup (2, +\infty)$

$$\forall (x \in D): \quad -1 \leq \sin\left(\frac{1}{x} - \frac{1}{x-2}\right) \leq 1$$

$$\iff \quad -|x^2-2x| \leq |x^2-2x| \sin\left(\frac{1}{x} - \frac{1}{x-2}\right) \leq |x^2-2x|.$$

WLOG, LET $f(x) = -|x^2-2x|$,

$g(x) = |x^2-2x| \sin\left(\frac{1}{x} - \frac{1}{x-2}\right)$,

$h(x) = |x^2-2x|$.

NOTE THAT $f(0) = f(2) = 0 = h(2) = h(0)$. ①

(*) SINCE x^2-2x IS A POLYNOMIAL, IT IS CONTINUOUS EVERYWHERE, AND FROM THE COROLLARY TO LEMMA I, $|x^2-2x|$ IS CONTINUOUS EVERYWHERE

$\Rightarrow$ FROM THE PROVEN PROPERTIES OF LIMITS, $-|x^2-2x|$ IS ALSO CONTINUOUS EVERYWHERE. (SINCE $\lim\limits_{x\to a} 0 = 0, \forall a \in \mathbb{R}$).

NOTE THAT SINCE $-1 \leq \sin x \leq 1 \quad \forall x \in \mathbb{R}$,

THEN $f(x) \leq g(x) \leq h(x)$.

FROM (*), $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = 0$ FOR $a \in \{0, 2\}$,

THEN $\lim\limits_{x\to a} g(x) = 0$ FOR $a \in \{0, 2\}$ BY THE SQUEEZE THEOREM.

CONSIDER THE CASE WHEN X DOES NOT APPROACH 0 OR 2:

IT IS GIVEN AS AN ASSUMPTION THAT $\sin(x)$ IS CONTINUOUS.

SINCE X AND X − 2 ARE POLYNOMIALS WITH $(x \neq 0 \wedge x \neq 2)$, THEY ARE EVERYWHERE-CONTINUOUS, SO THAT BY THE LIMIT LAWS PROVEN BEFORE

$g(x) = \left(\frac{1}{x} - \frac{1}{x-2}\right)$ IS CONTINUOUS.

$\Rightarrow$ $\sin(g(x))$ IS CONTINUOUS.

ALSO, $x^2 - 2x$ IS A POLYNOMIAL $\Rightarrow$ BY THE LIMIT LAWS PROVEN BEFORE

$(x^2 - 2x)\sin(g(x))$ IS CONTINUOUS AND HENCE

$$\lim_{\substack{x \to a, \\ a \neq 0, \\ a \neq 2}} (x^2 - 2x)\sin\left(\frac{1}{x} - \frac{1}{x-2}\right) = (a^2 - 2a)\sin\left(\frac{1}{a} - \frac{1}{a-2}\right)$$

$\square$

END OF EX 1.

②

(i) LET $\varepsilon > 0$ BE GIVEN.

LET $f(x) = \dfrac{|x-1|}{x-1}$, $x \neq 1$.

THUS, $f(x) = \begin{cases} -1, & x < 1 \\ 1, & x > 1 \end{cases}$

LET $\delta = \varepsilon$ SO THAT $0 < 1 - x < \delta$.

HENCE, $|1-x| < \delta = \varepsilon$.

CONSIDER $\Delta = |f(x) + 1|$.

SINCE $0 < 1-x$, $x < 1$ AND HENCE $f(x) = -1$ $\forall (x < 1)$.

THUS, $\Delta = 0 < \varepsilon$.

(ii) CONSIDER $f(x) = \dfrac{|x^2 - 9|}{x-3}$, $x \neq 3$.

$= \dfrac{|x-3|\,|x+3|}{x-3}$, $x \neq 3$

THUS, $f(x) = \begin{cases} |x+3|, & x > 3 \\ -|x+3|, & x < 3. \end{cases}$

LET $\varepsilon > 0$ BE GIVEN.

TAKE $\delta = \varepsilon$ SO THAT

$0 < x - 3 < \delta$. CONSIDER $\Delta = |f(x) - 6|$.

SINCE $x > 3$, $f(x) = x+3 \Rightarrow$

$\Delta = |x-3|$

SINCE $0 < x-3 < \delta = \varepsilon$, $|x-3| < \varepsilon$

$$\Rightarrow \lim_{x \to 3^+} \frac{|9-x^2|}{x-3} = 6$$

2/2.

2, (iii) LET $\varepsilon > 0$ BE GIVEN.

TAKE $N = \frac{|6\varepsilon+1|+1}{2\varepsilon} > 0$. LET'S MAKE

$x > N > 0$

~~SUPPOSE THAT~~ THEN $\exists \varepsilon > 0$

$\left|\frac{x+3}{x^2+5}\right| \geq \varepsilon$. THEREFORE,

$|x+3| \geq \varepsilon |x^2+5|$

SINCE $x > N \geq 0$, $x+3 \geq \varepsilon x^2 + 5\varepsilon$,

I.E. $\varepsilon x^2 - x + 5\varepsilon - 3 \leq 0$.

$\Delta = 1 - 4\varepsilon(5\varepsilon - 3)$

$= -20\varepsilon^2 + 12\varepsilon + 1$

$$\Rightarrow x \in \left(\frac{1-\sqrt{\Delta}}{2\varepsilon}, \frac{1+\sqrt{\Delta}}{2\varepsilon}\right).$$

SINCE $N = \frac{1+\sqrt{(6\varepsilon+1)^2}}{2\varepsilon} > \frac{1+\sqrt{-20\varepsilon^2+12\varepsilon+1}}{2\varepsilon}$

$(\because 36\varepsilon^2 > -20\varepsilon^2 \ \forall \varepsilon \in \mathbb{R})$, WE GET A CONTRADICTION. #

$$\Rightarrow \lim_{x \to \infty} \frac{x+3}{x^2+5} = 0$$

(iv)

LET $f(x) = \dfrac{4 + \cos(13\pi x)}{x - 43}$.

SUPPOSE $\varepsilon > 0$ IS GIVEN.

LET $N = 42 + \dfrac{5}{\varepsilon}$, AND

SUPPOSE $x < N$.

SUPPOSE ALSO THAT $|f(x)| \geq \varepsilon$

for ALL SUCH x.

THEREFORE,

$$5 \geq 4 + \cos(13\pi x) \geq \varepsilon x - 43\varepsilon$$

HENCE $\dfrac{5}{\varepsilon} + 43 \geq x$. BUT

$x < \dfrac{5}{\varepsilon} + 42$, WHICH IS A CONTRADICTION.

HENCE $\lim\limits_{x \to -\infty} \dfrac{4 + \cos(13\pi x)}{x - 43} = 0$

1/2.

* END OF EX 2.

Notice that $4 + \cos(13\pi x) \in [3, 5]$

$\therefore$ You will see that $\lim\limits_{x \to -\infty} f(x) = 0$.

③ THEOREM: GIVEN $f(x)$, $g(u)$, IF $\lim_{x \to a} f(x) = L$ AND $\lim_{u \to L} g(u) = M$, THEN $\lim_{x \to a} g \circ f(x) = M$

PROOF:

LET $\varepsilon > 0$

BY DEFINITION OF A LIMIT,

① $\exists \delta > 0 : |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

② $\exists \delta' > 0 : |u - L| < \delta' \Rightarrow |g(u) - M| < \varepsilon$.

FROM ②, IF $u = f(x)$, THEN

③ $\exists \delta' > 0 : |f(x) - L| < \delta' \Rightarrow |g \circ f(x) - M| < \varepsilon$.

TAKE $\delta' = \varepsilon$.

FROM ①, $\exists \delta > 0 : |x - a| < \delta \Rightarrow |f(x) - L| < \delta'$. ④

THUS, FROM ③ AND ④

$\exists \delta > 0 : |x - a| < \delta \Rightarrow |g \circ f(x) - M| < \varepsilon$.

HENCE, $\lim_{x \to a} g \circ f(x) = M$.

The statement is false as its only strictly true for continuous functions.

0/5.

④ THEOREM: ① $\forall (a, b, c \in \mathbb{R} : b < a < c)$

$$\forall \left(\begin{array}{l} f(x), g(x), h(x): \\ \text{②} \; [g(x) \leq f(x) \leq h(x) \mid \forall x \in (b,a) \cup (a,c)] \\ \text{③} \; \lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L \end{array} \right) :$$

$$\exists L \in \mathbb{R} : \lim_{x \to a} f(x) = L.$$

PROOF:

LET $\varepsilon > 0$.

FROM ③,

④ $\exists \delta_1 > 0 : |x - a| < \delta_1 \Rightarrow |g(x) - L| < \varepsilon$,

⑤ $\exists \delta_2 > 0 : |x - a| < \delta_2 \Rightarrow |h(x) - L| < \varepsilon$.

LET $\delta = \min\{\delta_1, \delta_2\}$.

FROM ④,

$|x - a| < \delta \Rightarrow -\varepsilon < g(x) - L < \varepsilon$

FROM ⑤,

$|x - a| < \delta \Rightarrow -\varepsilon < h(x) - L < \varepsilon$,

$$\Rightarrow \begin{cases} h(x) < \varepsilon + L \\ g(x) > L - \varepsilon \end{cases}$$

5/5.

$\Rightarrow$ FROM ②,

$L - \varepsilon < f(x) < L + \varepsilon$

$\Leftrightarrow -\varepsilon < f(x) - L < \varepsilon$

$\Leftrightarrow |f(x) - L| < \varepsilon \;\; \forall x \mid |x - a| < \delta$.

$\Leftrightarrow \lim_{x \to a} f(x) = L$ □

⑤

LET $g(x) = (x^4 + 2x^2)\sin\left(\frac{1}{x^2+1}\right)$.

(i) NOTE THAT $\forall x \in \mathbb{R}: -1 \leq \sin x \leq 1$. ①

CONSIDER $f(x) = -(x^4 + 2x^2) = -x^4 - 2x^2$

AND $h(x) = x^4 + 2x^2$.

SINCE $f(x)$ and $h(x)$ ARE POLYNOMIALS, THEY ARE CONTINUOUS EVERYWHERE.

THUS, $\lim_{x \to 0} f(x) = f(0) = 0$ AND $\lim_{x \to 0} h(x) = h(0) = 0$.

NOTE THAT $f(x) \leq g(x) \leq h(x)$ FROM ①,
$(x^4 + 2x^2 \geq 0 \ \forall x \in \mathbb{R})$
Since $\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0$,

$\Rightarrow$ BY THE SQUEEZE THEOREM,

$\lim_{x \to 0} g(x) = 0$.

(ii) LET $a = 0, b = \frac{4}{3} \Rightarrow a < 1 < b$.

THEN $\tan\left(\frac{\pi}{4}a\right) = 0$, $\tan\left(\frac{\pi}{4}b\right) = \sqrt{3}$

~~Let $h(x) = x - 1$, $f(x) = \frac{x-1}{4}$.~~

~~Then for $x \in \left(0, \frac{4}{3}\right)$: $f(x) \leq \frac{x-1}{1+\tan^2\left(\frac{\pi x}{4}\right)} \leq h(x)$,~~

~~since $\forall x \in (a,b)$: $\tan^2\left(\frac{\pi}{4}x\right) \in (0,3)$~~

~~AND $\frac{1}{\tan^2\left(\frac{\pi}{4}x\right)+1} \in \left(\frac{1}{4}, 1\right)$.~~

NOTE THAT SINCE $f(x)$ AND $h(x)$ ARE POLYNOMIALS, THEY ARE CONTINUOUS EVERYWHERE, AND HENCE

$$\lim_{x\to 1} f(x) = f(1) = 0 = h(1) = \lim_{x\to 1} h(x) \quad ①$$

But then by THE SQUEEZE THEOREM, SINCE $f(x) \le \dfrac{x-1}{1+\tan^2\left(\frac{\pi x}{4}\right)} \le h(x)$ AND ①,

$$\lim_{x\to 1} \frac{x-1}{1+\tan^2\left(\frac{\pi x}{4}\right)} = 0.$$

(iii) LET $k(x)$ BE A FUNCTION DEFINED ON (b, c) $\exists (b, c \in \mathbb{R})$ SUCH THAT $a \in (b, c)$.

TAKE $b', c' \in (b, c)$. LET $\Delta = k(c') - k(b')$

BY TRICHOTOMY LAW FOR $\mathbb{R}$,

$\Delta > 0 \lor \Delta < 0 \lor \Delta = 0$.

WLOG, TAKE $k(c') \le k(b')$.

THEREFORE,

$$\frac{1}{4+(k(b'))^2} \le \frac{1}{4+(k(c'))^2}.$$

WLOG TAKE

THEN BY

$$\frac{|x^2-a^2|}{4+(k(b'))^2} \le \frac{|x^2-a^2|}{4+(k(c'))^2}.$$

What if $k(x)$ has some kind of discontinuity or undesirable behaviour in (b, c)? $\therefore$ this pf. doesn't work.

0/2.

LET L, G BE TWO SETS: $a \in (b, c)$.

$$L = \{ x \in (b, a) \cup (a, c) : k(x) < k(b') \},$$

$$G = \{ x \in (b, a) \cup (a, c) : k(x) > k(c') \}.$$

CONSIDER $L \cap G$.

$L \cap G$ IS NOT EMPTY, SINCE

$k(c') \leq k(c')^2 \leq k(b')$ AND

$k(c') \leq k(b') \leq k(b')$.

THUS, THERE ARE AT LEAST TWO DISTINCT ELEMENTS IN $L \cap G$. (*)

DENOTE $\alpha = GLB(L \cap G)$,

$\beta = LUB(L \cap G)$.

FROM (*), $\alpha < \beta$.

THUS, CONSIDER α', β' ($\alpha' > \beta'$) $\in (\alpha, \beta)$.

WLOG, ASSUME $k(\beta') \leq k(x) \leq k(\alpha')$.

THEN

$$\frac{1}{4+(k(\alpha'))^2} \leq \frac{1}{4+(k(x))^2} \leq \frac{1}{4+(k(\beta'))^2} \quad (A)$$

WITHOUT SQUEEZE THM.!

(iv) LET $\varepsilon > 0$ BE GIVEN.

TAKE $\delta = \min\left\{1, \dfrac{3\varepsilon}{2\sqrt{\pi}+1}\right\}$

SO THAT

$$\begin{cases} 0 < x - \sqrt{\pi} < 1 & ① \\ 0 < x - \sqrt{\pi} < \dfrac{3\varepsilon}{2\sqrt{\pi}+1}. & ② \end{cases}$$

FROM ①,

$$\sqrt{\pi} < x < \sqrt{\pi}+1$$

$$\iff \quad 2\sqrt{\pi} < x + \sqrt{\pi} < 2\sqrt{\pi}+1 \quad ①$$

FROM ②, $\dfrac{(x-\sqrt{\pi})(2\sqrt{\pi}+1)}{3} < \varepsilon.$

THUS FROM ①, $\dfrac{(x-\sqrt{\pi})(x+\sqrt{\pi})}{3} < \varepsilon.$

NOTE THAT $-1 \le \cos(|x-2| + 3\tan(x^2)) \le 1$,

AND HENCE $3 \le \cos(|x-2|+3\tan(x^2)) + 4 \le 5.$

HENCE, $0 < \dfrac{x^2 - \pi}{4 + \cos(|x-2| + 3\tan(x^2))} < \varepsilon.$

NOTE THAT $\forall\, y \in \mathbb{R}: k^2(y) + 4 \geq 0.$

SINCE $\dfrac{x^2 - a^2}{(k(y))^2 + 4}$, $\forall\, y \in \{\alpha', \beta'\}$ IS A

POLYNOMIAL, IT IS CONTINUOUS EVERYWHERE.

THUS, for $y \in \{\alpha', \beta'\}$:

$$\lim_{x \to a} \frac{x^2 - a^2}{4 + (k(y))^2} = \frac{a^2 - a^2}{4 + (k(y))^2} = 0. \quad (*')$$

(IF $x^2 - a^2 > 0$,

FROM $(*)$, $$\frac{x^2 - a^2}{4 + (k(\alpha'))^2} \leq \frac{x^2 - a^2}{4 + (k(x))^2} \leq \frac{x^2 - a^2}{4 + (k(\beta'))^2},$$

AND FROM $(*')$, BY THE SQUEEZE THEOREM,

$$\lim_{x \to a} \frac{x^2 - a^2}{4 + (k(x))^2} = 0.$$

IF $x^2 < a^2$, THE ROLES OF $\dfrac{x^2 - a^2}{4 + (k(\beta'))^2}$ AND $\dfrac{x^2 - a^2}{4 + (k(\alpha'))^2}$

ARE REVERSED, BUT WITH $(*')$ STILL HOLDING

THE SAME RESULT FOLLOWS.

$\square$

5(IV)(CONT.)

HENCE,

$$\lim_{x \to \sqrt{\pi}^+} \frac{x^2-\pi}{4+\cos(|x-2|+3\tan(x^2))} = 0$$

WITH SQ. THM.:

TAKE $f(x) = \frac{x^2-\pi}{3}$, $g(x) = \frac{x^2-\pi}{4+\cos(|x-2|+3\tan(x^2))}$,

$h(x) = \frac{x^2-\pi}{5}$.

SINCE $f(x)$ AND $h(x)$ ARE POLYNOMIALS, THEY ARE CONTINUOUS EVERYWHERE, AND HENCE

$$\lim_{x \to \sqrt{\pi}} f(x) = \lim_{x \to \sqrt{\pi}} h(x) =$$

$$= f(\sqrt{\pi}) = \frac{\pi-\pi}{3} = h(\sqrt{\pi}) = \frac{\pi-\pi}{5} = 0.$$

SINCE $-1 \leq \cos(|x-2|+3\tan(x^2)) \leq 1$,

$f(x) \leq g(x) \leq h(x) \quad \forall x \in \mathbb{R}$,

AND HENCE $\lim_{x \to \sqrt{\pi}} g(x) = 0$ BY

SQ. THM. $\Rightarrow \lim_{x \to \sqrt{\pi}^+} g(x) = 0$.

LEMMA I:

$$\lim_{x \to a} |x| = |a|.$$

PROOF:

BY DEFINITION OF $|x|$,

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 0, & x = 0 \end{cases}$$

CLAIM I: THE INTERVAL $(0, +\infty)$ DOES NOT CONTAIN ITS $GLB = 0$.

PROOF:

SUPPOSE $\lambda \in (0, +\infty)$ AND $\lambda = GLB(0, +\infty)$.

$$\Rightarrow 0 < \lambda \Leftrightarrow 0 < \frac{\lambda}{2} < \lambda \Rightarrow \frac{\lambda}{2} \in (0, +\infty). \#$$

CLAIM II: THE INTERVAL $(-\infty, 0)$ DOES NOT CONTAIN ITS LUB.

PROOF:

SUPPOSE $\gamma \in (-\infty, 0)$ AND $\gamma = LUB(-\infty, 0)$.

$$\gamma < 0 \Leftrightarrow \gamma < \frac{\gamma}{2} < 0 \Rightarrow \frac{\gamma}{2} \in (-\infty, 0) \#.$$

FROM CLAIM I AND II ABOVE, IF $x > 0$, $a > 0$,

$$\forall \varepsilon > 0 \; \exists \delta > 0 : |x - a| < \delta \Rightarrow |x - a| < \varepsilon,$$

SINCE $a > 0$ ($|a| > 0$), AND ε CAN BE SET EQUAL TO δ. THUS

SIMILARLY,
FROM CLAIM 2 ABOVE, IF $x<0$, $a<0$, THEN

$$\forall(\varepsilon>0)\exists(\delta>0): 0<|a-x|<\delta \Rightarrow |a-(x)|<\varepsilon,$$

since $a<0$ ($\Rightarrow -|a|<0$) AND ε CAN BE SET EQUAL TO δ SO THAT

$$|(-x)-|a||<\varepsilon.$$

THUS, IF $x<0$, $a<0$, THEN $\lim_{x\to a}|x|=|a|$, AND

IF $x>0$, $a>0$, THEN $\lim_{x\to a}|x|=|a|$.

CONSIDER $f(x)=-x$, $g(x)=|x|$, $h(x)=x$

Since $f(x)$ and $h(x)$ are polynomials,

$$\lim_{x\to 0} f(x)=f(0)=0=h(0)=\lim_{x\to 0} h(x). \text{ (1)}$$

NOTE THAT BY DEFINITION OF $|x|$, $\forall x\in\mathbb{R}$

$f(x)\le g(x)\le h(x)$. Thus by the Squeeze theorem AND (1), $\lim_{x\to 0} g(x)=0$.

Hence, $|x|$ is CONTINUOUS EVERYWHERE ON ITS DOMAIN. $\square$

COROLLARY TO LEMMA I : IF $f(x)$ IS CONTINUOUS ON SOME INTERVAL, THEN $|f(x)|$ IS ALSO CONTINUOUS ON THIS INTERVAL.

PROOF : BY THE PROPERTIES OF LIMITS ALREADY PROVEN AND FROM LEMMA I, IF $g(x) = |x|$, $\therefore g \circ f(x)$ IS CONTINUOUS ON THE GIVEN INTERVAL.