1 Brain Maths IV

1.1 Revision

Let $\dot{x} = f(x)$ for $x \in \mathbb{R}^2$ be such that $f(0)$, $Df(0) = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$, and $\operatorname{Spec} Df = \{\lambda_1, \lambda_2\}$.

If $\lambda_1, \lambda_2 < 0$, then the equilibrium points of f are stable knots.

If $\lambda_1 < 0 < \lambda_2$, then the equilibrium points are saddles.

If the eigenvalues are complex conjugates of each other with the nonzero imaginary parts, then the equilibrium points of f are foci.

Theorem 1.1 (Grobman-Hartman)

If an equilibrium point is hyperbolic (i.e., $\Re(\lambda_i) \neq 0$), then, locally, f is topologically dual to Df.

As a consequence, a vector field in the neighbourhood of a hyperbolic equilibrium point is stable.

Thus, if we also have that $d_C(f, g) < \epsilon$, then $f \sim Df \sim Dg \sim g$.

1.2 Hadamard-Perron Theorem

Theorem 1.2

Suppose that $Df(0)$ is a saddle point.

Then there exist unique w^s and w^u such that for all $x \in w^s$ we have $f^n(x) \to 0$ as $n \to +\infty$ and $w^u \to 0$ as $n \to -\infty$.

Theorem 1.3

Stable equilibrium of a typical 1-parametric family f_α can degenerate in either of two cases:

- $\lambda_1 = 0$, $\lambda_2 \neq 0$ In this case, we obtain a saddle-knot bifurcation.
- $\lambda_{1,2} = \pm i w$, $w \neq 0$ This is an Andronov-Hopf bifurcation.

Now, if $\alpha = 0$ and one of the eigenvalues is also equal to 0, then another theorem applies:

Theorem 1.4 (Central Manifold Theorem)

There exists $W^c \in C^0$ and unique W^s, W^u such that $T_0 W^i = E^i$, where $i \in \{s, u, c\}$.

1.3 Reduction Principle

It is worthwhile to note that we can make up a system which is topologically equivalent to our saddle-knot:

$$\begin{cases} \dot{x} = f_\alpha(x) \\ \dot{\alpha} = 0 \end{cases} \sim \begin{cases} \dot{x_1} = v_\alpha(x) \\ \dot{x_2} = x_2 \\ \dot{x_3} = -x_3 \end{cases},$$

where x_2 and x_3 are called *saddle extensions*. This allows us to classify our equilibrium points and predict how stable they are.

1.4 Attractive Cycles

Cycles can degenerate due to bifurcations (eg homoclinic orbits of saddle-knots can form). The way by which the cycle is tranformed depends on the nature of the eigenvalues of Df.

If the parameter of bifurcation changes, for example, for neurons, if the current increases, then a cycle can form at a particular frequency. For example, a phenomenon of *ghost attractors* can arise, which usually means that a cycle is very slow..

The classification into integrators and resonators originates exactly from the differences in the way attractor cycles behave under bifurcations.