Let $\mathbb{F}$ be any field.

Lemma 0.1. $\forall a \in \mathbb{F} : a \cdot 0 = 0$

Proof.

$$
\begin{aligned}
0 + 0 &= 0 && \text{Existence of an Additive Identity} && (1)\\
\Rightarrow a \cdot (0+0) &= a \cdot 0 + a \cdot 0 && \text{Distributive Law} && (2)\\
&= a \cdot 0 && \text{Definition of } = && (3)\\
(a \cdot 0 + a \cdot 0) - (a \cdot 0) &= a \cdot 0 - (a \cdot 0) && \text{Definition of } = && (4)\\
\Rightarrow a \cdot 0 + (a \cdot 0 - a \cdot 0) &= 0 && \text{Associative Law} && (5)\\
& && \text{and Existence of an Additive Inverse} &&\\
\Rightarrow a \cdot 0 + 0 &= 0 && \text{Existence of an Additive Inverse} && (6)\\
\Rightarrow a \cdot 0 &= 0 && \text{Existence of an Additive Identity} &&
\end{aligned}
$$

□

Lemma 0.2. $\forall a, b \in \mathbb{F} : ab = 0 \Leftrightarrow a = 0 \vee b = 0$

Proof. By Commutative Law and Lemma 0.1, $a = 0 \Rightarrow ab = ba = b \cdot 0 = 0$.

Similarly, $b = 0 \Rightarrow ab = a \cdot 0 = 0$. If $ab = 0$ and $b \neq 0$, by Existence of a Multiplicative Inverse

$\exists\ b^{-1} : abb^{-1} = 0 \cdot b^{-1}$, hence by Commutative Law $a \cdot 1 = b^{-1} \cdot 0$, then by Existence of a Multiplicative Identity and Lemma 0.1 $a = 0$.

If $ab = 0$ and $a \neq 0$, $\exists\ a^{-1} : a^{-1}ab = a^{-1} \cdot 0$, hence by Commutative Law and Lemma 0.1 $aa^{-1}b = 0$, then by Existence of a Multiplicative Inverse $1 \cdot b = 0$, and by Commutative Law and Existence of a Multiplicative Identity $b \cdot 1 = b = 0$.

If $a = 0 \wedge b = 0$, then by Lemma 0.1 $ab = 0 \cdot 0 = 0$ □

Lemma. *Cancellation Property*

$$
\begin{aligned}
&\forall a, b, c \in \mathbb{F} : a + c = b + c \Leftrightarrow a = b && (1)\\
&\forall a, b, c \in \mathbb{F}, c \neq 0 : ac = bc \Leftrightarrow a = b && (2)
\end{aligned}
$$

Proof. Suppose $a + c = b + c$.

$$
\begin{aligned}
\exists\ (-c) : c + (-c) &= 0 && \text{Existence of an Additive Inverse} && (3)\\
\Rightarrow\ (a + c) + (-c) &= (b + c) + (-c) && \text{Definition of } = && (4)\\
\Rightarrow\ a + (c + (-c)) &= b + (c + (-c)) && \text{Associative Law} && (5)\\
\Rightarrow\ a + 0 &= b + 0 && \text{Existence of an Additive Inverse} && (6)\\
\Rightarrow\ a &= b && \text{Existence of an Additive Identity} && (7)
\end{aligned}
$$

Suppose now $ac = bc$.

$$
\begin{aligned}
\exists\ c^{-1} : cc^{-1} &= 1 && \text{Existence of an Additive Inverse} && (8)\\
\Rightarrow\ (ac)c^{-1} &= (bc)c^{-1} && \text{Definition of } = && (9)\\
\Rightarrow\ a(cc^{-1}) &= b(cc^{-1}) && \text{Associative Law} && (10)\\
\Rightarrow\ a \cdot 1 &= b \cdot 1 && \text{Existence of a Multiplicative Inverse} && (11)\\
\Rightarrow\ a &= b && \text{Existence of an Additive Identity} && (12)
\end{aligned}
$$

□

Lemma 0.3. $\forall a, b \in \mathbb{F} : (-a)b = -ab$

Proof.

$$
\begin{aligned}
ab + (-a)b &= ba + b(-a) && \text{Commutative Law} && (1)\\
a + (-a) &= 0 && \text{Existence of an Additive Inverse} && (2)\\
\Rightarrow b(a + (-a)) &= b \cdot 0 && \text{Distributive Law} && (3)\\
& && \text{and Existence of an Additive Inverse} && (4)\\
&= 0 && \text{Lemma 0.1} && (5)\\
\Rightarrow ab + (-a)b &= 0 && \text{Definition of} = && (6)\\
\Rightarrow (-a)b + ab &= 0 && \text{Commutative Law} && (7)\\
(-a)b + ab - ab &= 0 - ab && \text{Definition of} = && (8)\\
\Rightarrow (-a)b + 0 &= -ab && \text{Existence of an Additive Inverse} && (9)\\
& && \text{and Existence of an Additive Identity} &&\\
&= (-a)b && \text{Existence of an Additive Identity} && (10)
\end{aligned}
$$

□

Corollary 0.3.1. $\forall a \in \mathbb{F} : -b = (-1)b$

Proof. From Lemma 0.3, if $a = 1$, then $(-1)b = -1 \cdot b$

$$
\begin{aligned}
-1 \cdot b &= -b \cdot 1 && \text{Commutative Law} && (1)\\
\Rightarrow (-1)b &= -b && \text{Definition of} = && (2)\\
& && \text{and Existence of a Multiplicative Identity} &&
\end{aligned}
$$

□

Lemma 0.4. $-(-a) = a$

Proof.

$$
\begin{aligned}
a + (-a) &= 0 && \text{Existence of an Additive Inverse} && (1)\\
(-1)(a + (-a) &= (-1)0 && \text{Definition of} = && (2)\\
(-1)a + (-1)(-a) &= 0 && \text{Distributive Law} && (3)\\
& && \text{and Lemma 0.1} &&\\
\Leftrightarrow -a - (-a) &= 0 && \text{Corollary 0.3.1} && (4)\\
a + (-a - (-a)) &= a + 0 && \text{Definition of} = && (5)\\
(a - a) - (-a) &= a && \text{Associative Law} && (6)\\
& && \text{and Existence of an Additive Identity} && (7)\\
0 - (-a) &= a && \text{Existence of an Additive Inverse} && (8)\\
-(-a) &= a && \text{Existence of an Additive Identity} && (9)
\end{aligned}
$$

□

Theorem 0.5. $x \cdot x = y \cdot y \Leftrightarrow x = y \vee x = -y$

Lemma 0.5.1. $x = y \vee x = -y \Rightarrow x \cdot x = y \cdot y$

Proof. Suppose $x = y$.

$$
\begin{aligned}
x \cdot x &= x \cdot y && \text{Definition of } = && (1)\\
y \cdot y &= x \cdot y && \text{Definition of } = && (2)\\
\Rightarrow x \cdot x &= y \cdot y && \text{Transitive Law} && (3)
\end{aligned}
$$

Suppose $x = -y$.

$$
\begin{aligned}
x \cdot x &= x \cdot (-y) && \text{Definition of } = && (4)\\
(-y)(-y) &= x \cdot (-y) && \text{Definition of } = && (5)\\
(-y)(-1)y &= (-1)(-y)y && \text{Corollary 0.3.1 and Commutative Law} && (6)\\
&= -(-y)y && \text{Corollary 0.3.1} && (7)\\
&= y \cdot y && \text{Lemma 0.4} && (8)\\
\Rightarrow x \cdot x &= y \cdot y && \text{Transitive Law} && (9)
\end{aligned}
$$

Lemma 0.5.2. $\forall a, b \in \mathbb{F} : ab = 0 \Leftrightarrow a = 0 \vee b = 0$

Proof. By Commutative Law and Lemma 0.1, $a = 0 \Rightarrow ab = ba = b \cdot 0 = 0$.

Similarly, $b = 0 \Rightarrow ab = a \cdot 0 = 0$. If $ab = 0$ and $b \neq 0$, $\exists\ b^{-1} : abb^{-1} = 0 \cdot b^{-1}$, hence by Commutative Law and Existence of a Multiplicative Inverse $a \cdot 1 = b^{-1} \cdot 0$, then by Existence of a Multiplicative Identity and Lemma 0.1 $a = 0$.

If $ab = 0$ and $a \neq 0$, $\exists\ a^{-1} : a^{-1}ab = a^{-1} \cdot 0$, hence by Commutative Law and Lemma 0.1 $aa^{-1}b = 0$, then by Existence of a Multiplicative Inverse $1 \cdot b = 0$, and by Commutative Law and Existence of a Multiplicative Identity $b \cdot 1 = b = 0$.

If $a = 0 \wedge b = 0$, then by Lemma 0.1 $ab = 0 \cdot 0 = 0$ $\square$

Lemma 0.5.3. $\forall a, b \in \mathbb{F} : (a + b)(a - b) = a \cdot a - b \cdot b$

Proof.

$$
\begin{aligned}
(a+b)(a-b) &= (a+b)a + (a+b)(-b) && \text{Distributive Law} && (1)\\
&= (a(a+b)) - (b(a+b)) && \text{Commutative Law} && (2)\\
&= (a \cdot a + ab) - (ba - b \cdot b) && \text{Distributive Law} && (3)\\
&= a \cdot a + (ab - (ba - b \cdot b)) && \text{Associative Law} && (4)\\
&= a \cdot a + ((ab - ba) - b \cdot b) && \text{Associative Law} && (5)\\
&= a \cdot a + ((ab - ab) - b \cdot b) && \text{Commutative Law} && (6)\\
&= a \cdot a + (0 - b \cdot b) && \text{Existence of an Additive Inverse} && (7)\\
&= a \cdot a - b \cdot b && \text{Existence of an Additive Identity} && (8)
\end{aligned}
$$

$\square$

If $x \cdot x = y \cdot y$, by Cancellation Property and Lemma 0.5.3 $(x - y)(x + y) = 0$.

Therefore, by Lemma 0.5.2 and Cancellation Property $x = y \vee x = -y$. $\square$

Theorem 0.6. $a \cdot a = 1 \Rightarrow a = 1 \vee a = -1$

Proof. Let $x = a,\ y = 1$. Then by Theorem 0.5 $a = 1 \vee a = -1$. $\square$