Suppose a field $\mathbb{F}$ is given.

Let V be the vector space of all finite sequences $(a_1, a_2, \dots)$ with $a_i \in \mathbb{F}$.

Finite means only finitely many a_i are non-zero.

For all $v \in V$ we define the *length of the sequence* v as the index of the element in the ordered sequence for which all the elements with greater index are $0 \in \mathbb{F}$.

Let V^* be the dual space of V, $V^* = \mathscr{L}(V, F)$.

Claim. V^* is isomorphic to the space F^∞ of all sequences.
Thus, there exists an invertible linear map from V^* onto $\mathbb{F}^\infty$.

Proof. Denote an arbitrary sequence $(a_1, a_2, \dots) \in \mathbb{F}^\infty$ as α.

Consider the map $\Phi : \mathbb{F}^\infty \to V^*$ such that $\alpha \mapsto l_\alpha$, where l_α is a linear functional defined for $\beta = (b_1, b_2, \dots, b_n, 0, \dots) \in V$ as follows:

$$l_\alpha(\beta) = \sum_{i=1}^{n} a_i b_i + \sum_{j=n+1} a_j 0 \tag{1}$$

First we show that l_α is linear.

Suppose $\boldsymbol{x} = (x_1, x_2, \dots, x_n, 0, \dots) \in V$, and $\boldsymbol{y} = (y_1, y_2, \dots, y_m, 0, \dots) \in V$.

Consider $l_\alpha(\boldsymbol{x} + \boldsymbol{y})$.

If $m = n$,

$$\begin{align}
l_\alpha(\boldsymbol{x} + \boldsymbol{y}) &= l_\alpha((x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_m)) \tag{2}\\
&= l_\alpha(x_1 + y_1, x_2 + y_2, \dots, x_n + y_m, 0, \dots) \tag{3}\\
&= \sum_{i=1}^{n} a_i(x_i + y_i) \tag{4}\\
&= \sum_{i=1}^{n} (a_i x_i) + \sum_{i=1}^{m} (a_i y_i) \tag{5}\\
&= l_\alpha(\boldsymbol{x}) + l_\alpha(\boldsymbol{y}) \tag{6}
\end{align}$$

Without loss of generality, suppose now $m > n$.

$$\begin{align}
l_\alpha(\boldsymbol{x} + \boldsymbol{y}) &= l_\alpha((x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_m)) \tag{7}\\
&= l_\alpha(x_1 + x_1, x_2 + x_2, \dots, x_n + y_n, \dots, 0 + y_{n+1}, \dots, 0 + y_m, 0, \dots) \tag{8}\\
&= \sum_{i=1}^{n} a_i(x_i + y_i) + \sum_{i=n+1}^{m} (a_i y_i) \tag{9}\\
&= \sum_{i=1}^{n} (a_i x_i) + \sum_{i=1}^{m} (a_i y_i) \tag{10}\\
&= l_\alpha(\boldsymbol{x}) + l_\alpha(\boldsymbol{y}) \tag{11}
\end{align}$$

Now, consider $l_\alpha(c\boldsymbol{x})$ for some $c \in \mathbb{F}$.

$$\begin{aligned}
l_\alpha(c\boldsymbol{x}) &= \sum_{i=1}^{n}(a_i(cx_i)) && (12)\\
&= c\sum_{i=1}^{n}(a_i x_i) && (13)\\
&= c l_\alpha(\boldsymbol{x}) && (14)
\end{aligned}$$

Thus, l_α is additive and homogeneous, and thus linear.
Then we show that Φ is linear.

Consider $\Phi(c\alpha+\beta)$, for $c \in \mathbb{F}, \alpha, \beta \in \mathbb{F}^\infty$, with $\alpha = (a_1, a_2, \dots) \in V^*$, and $\beta = (b_1, b_2, \dots) \in V^*$.

Let $\gamma = (ca_1 + b_1, ca_2 + b_2, \dots) \in V^*$
Note the following:

$$\begin{aligned}
\Phi(c\alpha+\beta) &= \Phi(c(a_1, a_2, \dots) + (b_1, b_2, \dots)) && (15)\\
&= \Phi((ca_1, ca_2, \dots) + (b_1, b_2, \dots)) && (16)\\
&= \Phi(ca_1 + b_1, ca_2 + b_2, \dots) && (17)\\
&= \Phi(\gamma) && (18)\\
&= l_\gamma && (19)
\end{aligned}$$

Note also the following:

$$c\Phi(\alpha) + \Phi(\beta) = cl_\alpha + l_\beta \tag{20}$$

Therefore, for $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in V$,

$$\begin{aligned}
\Phi(c\alpha+\beta)(\boldsymbol{x}) &= l_\gamma(\boldsymbol{x}) && (21)\\
&= \sum_{i=1}^{n}(ca_i + b_i)x_i + \sum_{j=n+1}(ca_j + b_j)0 && (22)\\
&= c\sum_{i=1}^{n}(a_i x_i) + c\sum_{j=n+1}(a_j 0) + \sum_{i=1}^{n} b_i x_i + \sum_{j=n+1}(b_j 0) && (23)\\
&= cl_\alpha + l_\beta && (24)\\
&= c\Phi(\alpha) + \Phi(\beta) && (25)
\end{aligned}$$

Thus, Φ is additive and homogeneous. Hence, Φ is linear.

We prove now that Φ is injective.

Suppose $\chi = (x_1, x_2, \dots) \in \ker(\Phi)$. Thus, $\Phi(\chi)$ is the zero function, and hence

$$\forall(\boldsymbol{x} \in V)\colon l_\chi(\boldsymbol{x}) = 0.$$

Consider $\eta = \{e_1, e_2, \dots, e_n\}$, where n is the length of $\boldsymbol{x} \in V$, a set of all sequences with all elements equal to 0 but the i^{th}, which is equal to 1. Then, by definition, for all i from 1 to n, $l_\chi(e_i) = 0$, and hence $x_i = 0$. Since $\boldsymbol{x}$ is arbitrary, for all $i \in \mathbb{N}$ $x_i = 0$. Therefore, χ is a zero vector in $\mathbb{F}^\infty$, which is an infinite sequence of zeroes. Since the choice of χ was also arbitrary, then $\ker(\Phi) = \{0_{\mathbb{F}^\infty}\}$, and hence Φ is injective.

Finally, we prove that Φ is surjective.

Let $\beta_{\mathbb{F}^\infty} = \{\epsilon_1, \epsilon_2, \dots\}$ be the standard basis of $\mathbb{F}^\infty$. Note that ϵ_i is an infinite sequence of zeroes but for the i^{th} coordinate, where it is equal to 1.

For any $\chi = (x_1, x_2, \dots) \in \mathbb{F}^\infty$, consider $\Phi(\chi) = l_\chi$.

Since

$$\chi = \sum_{i=1}^{\infty} x_i e_1,$$

while Φ is linear, then

$$l_\chi = \Phi(\chi) = \Phi(\sum_{i=1}^{\infty} x_i e_1) = \sum_{i=1}^{\infty} x_i \Phi(e_i) = \sum_{i=1}^{\infty} x_i l_{e_i}.$$

By definition of l_{e_i}, for all $v = (v_1, v_2, \dots, v_n, 0, \dots) \in V$

$$l_{e_i} : v \mapsto v_i,$$

where v_i is the i^{th} coordinate of v.

Thus, $l_{e_i}(e_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta, for $i, j \leq n$,
and $l_{e_i}(v) = 0$ for $i > n$ and $v \in V$.

Therefore, $l(e_i) = x_i$ for $i \leq n$ and $l(e_i) = 0$ for $i > n$.

Suppose now that some l exists in V^*.

For some $n \in \mathbb{N}$, evaluate l at e_i for all $e_i \in \beta_{\mathbb{F}^\infty}$ such that $0 < i \leq n$.

From the argument above, since $l(e_i) = x_i$ for $i \leq n$ and $l(e_i) = 0$ for $i > n$, while $e_i \in \beta_{\mathbb{F}^\infty}$, then l is a uniquely defined map Φ from V to $\mathbb{F}$ which maps the sequence $(x_1, x_2, \dots, x_n, 0, \dots)$ to l. Since l has been chosen arbitriarily, Φ is surjective.

Since Φ is linear, injective and surjective, then Φ is an isomorphism from F^∞ to V^*. Therefore, V^* is isomorphic to the space F^∞ of all sequences. $\square$