1 Review

Suppose that $a_n \geq 0$, and let $s_n = \sum_{k=1}^{n} a_k$.

Remark 1.1. Note that $a_n = s_n - s_{n-1}$.

So if the series converges, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = 0$.

Theorem 1.2

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} a_n = 0$.

Remark 1.3. Note that the condition $\lim_{n\to\infty} a_n = 0$ is necessary but not sufficient.

2 Limit Comparison Test

We have already shown that if $a_i \leq b_i$ for all $i \in I$, then $\sum_{i\in I} a_i$ converges whenever $\sum_{i\in I} b_n$ converges.

Theorem 2.1

Suppose there exists a nonzero constant $c \in \mathbb{R}$ and $N > 0$ such that $\forall n \in N.(a_n \leq cb_n)$, then if $\sum_{n\in I} b_n$ converges, then $\sum_{i\in I} a_n$ converges.

Example 2.2

Consider $\sum_{n=1}^{\infty} \frac{1}{2n+1}$.

Since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{1}{2n+1}$, because $\frac{1}{2n+1} > 3 \cdot \frac{1}{n}$ for $n > 1$.

Corollary 2.3

If $\lim_{in\to\infty} \frac{a_n}{b_n} = c \neq 0$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Proof.

If $\frac{a_n}{b_n} \to c$, then there exists $N \in \mathbb{N}$ such that $\forall n \in N. \frac{a_n}{b_n} < c + 1$ and hence the theorem applies. $\square$

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We can use the inerval test. Since $f'(x) = -\frac{2}{x^2} < 0$, then f is decreasing for $x > 0$.

Note that $\int_{i=1}^{\infty} \frac{\mathrm{d}x}{x^2} = \lim_{R\to\infty} \int_1^R \frac{\mathrm{d}x}{x^2}$, which is then equal to 1. Since the integral converges, then the series converges.

We call series in the form $\sum_{i=1}^{\infty} \frac{1}{n^p}$ a p-**series**.

For $p > 1$, almost exactly the same calulation show that $\int_{i=1}^{\infty} \frac{\mathrm{d}x}{x^p}$ converges, then $\sum \frac{1}{n^p}$ converges.

Example 2.4

Consider $a_n = \frac{1}{3^n +}$.

Then $\frac{a_{n+1}}{a_n} = \frac{1+\frac{1}{3^n}}{3+\frac{1}{7+\frac{1}{3^{n+1}}}}$.

3 Ratio Test

Theorem 3.1

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$ and

- $r < 1$, then $\sum a_n$ converges
- $r > 1$, then $\sum a_n$ diverges
- $r = 1$, then the test is inconclusive.

In the example above, $r = 1$, and thus the sequence converges.

We have seen that $\log x = \sum_{i=1}^{n} (-1)^{i+1} \frac{x^i}{i!}$.