Problem.

Suppose that the characteristic polynomial $f(t)$ of $A \in M_{7\times 7}(\mathbb{R})$ splits, with the only zeroes being 1 and 2.

Assume that the eigenspaces E_1 and E_2 of L_A have dimension 2 and 3, respectively.

a) How many such matrices A are there, up to similarity?

b) If we also know that the algebraic multiplicity m_1 equals 3, determine the Jordan canonical form of A.

Solution.

Let V be a vector space such that $L_A \in \mathrm{End}(V)$. Note that $\dim V = 7$.

Let d_1 be the multiplicity of the eigenvalue 1, and let d_2 be the multiplicity of the eigenvalue 2.

Since 1 and 2 are the only eigenvalues, $d_1 + d_2 = \dim V = 7$.

Since E_1 has the dimension of 2, there are two columns in the dot diagram of K_1. Therefore, $d_1 \geq 2$

Similarly, since E_2 has the dimension of 3, there are three columns in the dot diagram of K_2. Thus, $d_2 \geq 3$ and thus $7 - d_2 = d_1 \leq 4$.

We now look at each case.

If $d_1 = 2$, then $d_2 = 5$, and there is only one possibility for the dot diagram corresponding to K_1 (because $2 = 1 + 1$). The number of possible dot diagrams corresponding to K_2 in this case is equal to the number of partitions of 5 with 3 terms, which is equal to 2 $(1 + 2 + 2,\ 1 + 1 + 3)$.

If $d_1 = 3$, then $d_2 = 4$, and there is only one possibility for the dot diagram corresponding to K_1 (the only partition of 3 up to the order of elements, consisting of two terms, is $1 + 2$). Similarly, there is only one possibility for the dot diagram of K_2 (the partition of 4 with 3 elements is $1 + 1 + 2$).

If $d_1 = 4$, then $d_2 = 3$. The number of partitions of 4 with 2 terms is 2 $(1 + 3,\ 2 + 2)$, and thus there are two possibilities for the dot diagram of K_1. There is only one possible dot diagram of K_2, because the only partition of 3 with three terms is $1 + 1 + 1$.

Therefore, there are $1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5$ possible matrices A, equivalent up to similarity.

If $d_1 = 3$, from the discussion above we already know that JCF can be inferred:

$$[A]_\beta = \begin{pmatrix} 1 & 1 & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 2 & 1 & & \\ & & & & 2 & & \\ & & & & & 2 & \\ & & & & & & 2 \end{pmatrix}$$

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