Problem. Consider the matrix $A = \begin{pmatrix} -5 & 8 \\ -4 & 7 \end{pmatrix}$. Compute A^n for any $n > 0$.

Solution. Calculating the characteristic polynomial of A, we obtain

$$f(\lambda) = (-5 - \lambda)(7 - \lambda) + 32.$$

Therefore,

$$\begin{aligned} & (\lambda - 7)(\lambda + 5) + 32 = 0 && (1) \\ \Leftrightarrow \quad & \lambda^2 - 2\lambda - 3 = 0 && (2) \\ \Leftrightarrow \quad & (\lambda = -1) \vee (\lambda = 3) && (3) \end{aligned}$$

Thus, for $\lambda = -1$, the corresponding eigenvalues in the form $\begin{pmatrix} x \\ y \end{pmatrix}$ is such that:

$$\begin{pmatrix} -4 & 8 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4x + 8y \\ -4x + 8y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore, $x = 2y$ and thus $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ spans E_{-1}.

For $\lambda = 3$, the corresponding eigenvalues in the form $\begin{pmatrix} x \\ y \end{pmatrix}$ is such that:

$$\begin{pmatrix} -8 & 8 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8x + 8y \\ -4x + 4y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore, $x = y$ and thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ spans E_3.

Suppose $T \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ is a linear transformation corresponding to the matrix A.

Since $\mathbb{R}^2 = E_1 \oplus E_2$ (vectors spanning E_1 and E_2 are linearly independent, then T is diagonalisable. Thus, taking $\gamma = \{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$, we obtain

$$[T]_\beta = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Denote $\beta = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$.

Note that
$[I]_\beta^\gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = U$. Moreover,

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= 1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} && (4) \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= (-1) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} && (5) \end{aligned}$$

11 and hence $[I]_\gamma^\beta = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = U^{-1}$.

Therefore,

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Since $UU^{-1} = I$, it follows that

$$A^n = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

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