Suppose that V is a finite-dimensional inner product space and that $T : V \to V$ is a linear transformation.

Problem Show that $R(T^*)^\perp = \ker T$.

Solution

Note that for all $x, y \in V$, $\langle Tx, y\rangle = \langle x, T^* y\rangle$.

Therefore, if $x \in \ker T$, then $Tx = 0$. Therefore, from above, $\langle 0, y\rangle = 0 = \langle x, T^* y\rangle$.

Note that $T^* y \in \operatorname{Im} T^*$, and since y is arbitrary, it follows that $\ker T \subseteq (\operatorname{Im} T^*)^\perp$.

Suppose now $y \in (\operatorname{Im} T^*)^\perp$. Therefore, for all $u \in \operatorname{Im} T^*$, $\langle u, y\rangle = 0$.

Fix some u. Since $u \in \operatorname{Im} T^*$, there exists $x \in V$ such that $T^* x = u$.

Thus, $\langle T^* x, y\rangle = 0 = \langle x, Ty\rangle$.

Since u was chosen arbitrarily, then this holds for any u and corresponding x. Therefore $Ty = 0$, and hence $(\operatorname{Im} T^*)^\perp \subseteq \ker T$.

Therefore, $R(T^*)^\perp = \ker T$.

Lemma

For any finite dimensional vector space V, if $S \subseteq V$ and $S \neq \emptyset$, then $S = (S^\perp)^\perp$.

Proof.

Note that $S^\perp = \{x \in V : \forall y \in S. \langle x, y\rangle = 0\}$.

Therefore, $(S^\perp)^\perp = \{v \in V : \forall w \in S^\perp. \langle v, w\rangle = 0\}$.

Suppose $u \in S$ and $w \in S^\perp$. Since $\forall y \in S. \langle w, y\rangle = 0$, $\langle w, u\rangle = 0$.

Since $\langle w, u\rangle = \overline{\langle u, w\rangle}$ and $\overline{0} = 0$, then $\langle u, w\rangle = 0$.

Since w is arbitrary, $\forall w \in S^\perp. \langle u, w\rangle = 0$. Therefore, by definition, $u \in S^{\perp\perp}$, and thus $S \subseteq S^{\perp\perp}$.

Suppose now $s \in S^{\perp\perp}$.

Therefore, $\forall w \in S^\perp$. Take some $w \in S^\perp$.

Note that $\langle s, w\rangle = 0$. Similarly to the argument given above, we can deduce that $\langle w, s\rangle = 0$.

Since s is arbitrary, $\forall s \in S^{\perp\perp}. \langle s, w\rangle = 0$. Therefore, by definition of $S^\perp$, it follows that $s \in S$, and hence $S \subseteq S^{\perp\perp}$. □

From Lemma we obtain that $(\operatorname{Im} T^*)^{\perp\perp} = \operatorname{Im} T^* = (\ker T)^\perp$.

Moreover, since $T^{**} = T$, substituting T^*, we get that $(\operatorname{Im} T)^\perp = \ker T^*$.

By the rank-nullity theorem, $\operatorname{nullity} T^* + \operatorname{rank} T^* = \dim V$.

Since $V = \operatorname{Im} T \oplus (\operatorname{Im} T)^\perp$, we also obtain that $\dim V = \operatorname{rank} T + \dim \operatorname{Im} T^\perp$.

Thus,

$$\operatorname{rank} T + \dim \operatorname{Im} T^\perp = \operatorname{nullity} T^* + \operatorname{rank} T^*$$

Since $(\operatorname{Im} T)^\perp = \ker T^*$, we get $\dim(\operatorname{Im} T)^* = \operatorname{nullity} T^*$, and therefore, $\operatorname{rank} T = \operatorname{rank} T^*$.