1 Domino Tilings

1.1 Introduction

There are several key questions we may ask about domino tilings:

- Does a tiling exist?
- How many tilings are there?
- How does a random tiling look like?

There are 4 types of domino parts, depending on the colour configuration a part would have if the shape was coloured like a chessboard. Shape of the base element significantly affects the patterns emerging with the increasing number of copies. For example, some tilings possess regions of *freezing*, where only one colour is predominant.

A domino tiling can be represented as a covering of a corresponding dual graph with dimers.

Theorem 1.1 (Kusteleyn, Temperdey-Fisher (1961))

The number of tiling of a rectangle n by m is

$$\sqrt{\prod_{i=1}^{m}\prod_{k=1}^{n} 2(\cos\frac{\pi i}{m+1} + i\cos\frac{\pi k}{n+1})}.$$

Proof. The number of tilings is $\sqrt{\operatorname{perm} A}$, where A is the adjacency matrix for a dual graph and perm is a permanent, which can be calculated similarly to a determinant, but without any considerations for the sign of the terms.

The formula for the number of tesselations can be obtained as follows.

Take, for instance, two tiled squares, and put one over the other. Traverse the combined squares by following their patterns depending on the corresponding chessboard colouring until you return to the initial position, which yields a cycle.

Lemma 1.2

Any tesselation can be obtained from the other by applying one transformation, which depends on the shape of a base unit.

We can add signs to the terms in A to obtain a new matrix K such that $\operatorname{perm} A = |\det K|$.

The following method of building K is due to R.Kenyon(2000). Assign i to all the vertical edges and 1 to the horizontal. The other way would include a chessboard-like assignment of signs to the edges.

Note that $\det K = \prod^{nm} \lambda_i$.

Exercise 1.3. Continue the proof.

$\square$

Theorem 1.4 (Kuperberg et al)

The number of tilings of an aztec diamond with a side of size n is $2^{\frac{n(n+1)}{2}}$.

Definition 1.5. Take a rectangle, a choose two internal squares denoted as x and y. Then a *coupling function* $C(x, y) = K^{-1}_{x,y}$.

Note. $C(x, y)$ corresponds to the number of tesselations with a sign. For instance, $C(x, x + 1)$ is the number of domino tilings with $x, x + 1$ as entries.

Thus we obtain $\sum_{\alpha=\pm 1,\pm i} \alpha C(x, y + \alpha) = \begin{cases} 0, x \neq y \\ 1, x = y \end{cases}$.

There exists an operator D such that $DC(x, x) = DC(x, y) = 0$.

1.2 Measuring Order

One of the methods to predict the pattern is to compute the average height function. See, for example, Cohn, Kenyon and Propp for an in-depth study.