1 Review

We have seen two types of multiplicity:

algebraic $m_\lambda : (t-\lambda)^{m_\lambda} | f(t)$, $(t-\lambda)^{m_\lambda+1} \not| f(t)$, where $f(t)$ is a characteristic polynomial

geometric $\dim(E_\lambda) = \dim\ker(T - \lambda I_v)$

We have also proved that $1 \leq$ geom $\leq$ alg

Example 1.1

For $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $m_1 = 2$ and $\dim(E_1 = 2$

For $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $m_1 = 2$ and $\dim E_1 = 1$.

2 Further Tests of Diagonalisability

Definition 2.1. A polynomial $f(t) \in \mathfrak{P}(F)$ splits over F if $f(t)$ is a product of linear factors in $\mathfrak{P}(F)$, i.e. $f(t) = c(t - a_1)\cdots(t - a_n), c, a_i \in F$

Example 2.2

$t^2 + 1$ does not split over $\mathbb{R}$, but it splits over $\mathbb{C}$ (any polynomial over $\mathbb{C}$ splits)

Claim 2.3. T is diagonalisable $\Leftrightarrow$ the characteristic polynomial splits over F and $\dim E_\lambda = m_\lambda$ for all eigenvalues λ.

We now prove $\Rightarrow$.

Theorem 2.4

If T is diagonalisable, then the characteristic polynomial splits over F.

Proof. Take an ordered basis β such that $[T]_\beta = \begin{pmatrix} a_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & a_n \end{pmatrix}$. Then the characteristic polynomial is $f(t) = (a_1 - t)\cdots(a_n - t)$. □

Theorem 2.5

If T is diagonalisable, then $\dim(E_\lambda) = m_\lambda$ for all eigenvalues λ.

Proof. T is diagonalisable $\Rightarrow$ there exists an ordered basis $\beta = (v_1, \ldots, v_n)$ of eigenvectors.

Consider $\lambda_1, \ldots, \lambda_n$, the distinct eigenvalues of T.

Suppose d_i is the number of basis elements with the eigenvalue λ_i.

$d_i \leq \dim(E_{\lambda_i}) \leq m_{\lambda_1}$, since we have d_i linearly independent vectors in E_{λ_i}.

Note that $\sum_{i=1}^n d_i \leq \sum_{i=1}^n \dim(E_{\lambda_i}) \leq \sum_{i=1}^n m_i \leq n$. Therefore,

$\dim E_{\lambda_i} = m_{\lambda_i}$ □

Now we prove $\Leftarrow$ of the claim.

Lemma 2.6

Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T.

If $v_i \in E_{\lambda_i}$ and $\sum_i v_i = 0$, then $v_i = 0$.

Proof. Suppose not. Renumber these vectors in such a way that $v_1, \ldots, v_s$ are nonzero and $v_{s+1} = \cdots = v_k = 0$. By assumpton, $\sum_{i=1}^s v_i = 0$, which is a contradiction, since $v_1, \ldots, v_s$ must be linearly independent. □

Theorem 2.7

If the characteristic polynomial splits over F and $\dim E_\lambda = m_\lambda$ for all eigenvalues, then T is diagonalisable.

Proof. The characteristic polynomial $f(t)$ splits, so

$$f(t) = c(t-\lambda_1)^{m_1} \cdots (t-\lambda_k)^{m_k} = c(t-a_1)\cdots(t-a_n),$$

where $\lambda_i \in F$ are distinct and $m_i \geq 1$. □

Therefore, T has eigenvalues λ_i with algebraic multiplicities m_i.

By assumption, $\dim(E_{\lambda_i}) = m_i$. Let $\beta_1 \cup \cdots \cup \beta_k$ is a basis of V.

Claim. $\beta_1 \cup \cdots \cup \beta_k$ is a basis of V.

Proof of the Claim. Note that $\sum_{i=1}^n \dim(E_{\lambda_i}) = \sum_{i=1}^k m_i = \deg f = n$. We need to check that $\beta_1 \cup \cdots \cup \beta_k$ is linearly independent.

Write $\beta_i = (v_{i,1}, \ldots, v_{i,m_i})$.

Suppose $\sum_{i=1}^k \sum_{j=1}^m a_{ij} v_{ij} = 0$ for $a_{ij} \in F$.

Therefore, for all i, $\sum_{j=1}^{m_i} a_{ij} v_{ij} = 0$ by Lemma.

Hence, $a_{ij} = 0$ for all i, j, since β_i is a basis.

Thus, $\beta_1 \cup \cdots \cup \beta_k$ is a basis of eigenvectors (since each $v_{ij} \neq 0$), so T is diagonalisable. □

An equivalent version of Claim 2.3 can be stated:

Claim. T is diagonalisable $\Leftrightarrow \begin{cases} \dim E_\lambda = m_\lambda \text{ for all eigenvalues } \lambda \\ \sum m_\lambda = \dim V \end{cases}$

Proof. Note that $f(t) = c\prod_{i=1}^k (t-\lambda_i)^{m_i} g(t)$, where $g(t)$ does not have linear factors.

Thus, f splits if and only if $\sum m_i = \dim V = n$. □

To find a basis of eigenvectors, the claim in the previous proof shows that we only need to find basis of each eigenspace and then take its union.

Problem. Consider $\begin{pmatrix} 4 & 0 & -3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $F = \mathbb{Q}$. Find a matrix Q such that $Q^{-1}AQ$ is diagonal.

Solution.

1. Note that $f(t) = -(t-1)^2(t-3)$, which splits over $\mathbb{Q}$ with $m_1 = 2$ and $m_3 = 1$.
2. For $\lambda = 1$, $E_1 = \ker(A - I) = \operatorname{span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$.
3. For $\lambda = 3$, $E_3 = \ker(A - 3I) = \operatorname{span}\left(\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}\right)$.

   Therefore, the basis of eigenvectors of $\mathbb{Q}^3$ is $\beta = \{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}\}$, so $[L_A]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$\square$

Example 2.8

Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ for any F. Thus, $f(t) = t^2$, $m_0 = 2$, $\dim E_0 = 1$, and thus A is not diagonalisable.

Example 2.9

Take $A = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix}$, $F = \mathbb{R}$.

Note that $f(t) = t^2 - 2t + 2$ does not split over $\mathbb{R}$, and thus A is not diagonalisable.

3 Direct Sums

Definition 3.1. If $W_1, \ldots, W_k$ are subspaces of V, then V is the direct sum of $W_1, \ldots, W_k$ if for every $v \in V$ there exist a unque $w_i \in W_i$ for $i \in \{1, \ldots, k\}$ such that $v = \sum_{i=1}^k w_i$. We denote it as $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.

Example 3.2

If $v_1, \ldots, v_k$ is a basis of V, then $V = Fv_1 \oplus Fv_2 \oplus \cdots \oplus Fv_n$.

Example 3.3

Note that $\mathfrak{P}_n(F) = \mathfrak{P}_{n-1}(F) \oplus Fx^n$.

Definition 3.4. If $W_1, \ldots, W_k$ are subspaces of V, then the sum $W_1 + \cdots + W_k$ is the subspace given by $\operatorname{span} \bigcup_{i=1}^k W_i$.

Example 3.5

Note that $\mathfrak{P}(F) = \mathfrak{P}_1(F) + \{f \in \mathfrak{P}_3(F) : f(0) = 0\}$, but it is not a direct sum.

Theorem 3.6

The following are equivalent:

(a) $V = W_1 \oplus \cdots \oplus W_k$

(b) $V = W_1 + \cdots + W_k$ and, if $w_1 + \cdots + w_k = 0$, where $w_i \in W_i$, then $v_i = 0$

(c) $V = W_1 + \cdots + W_k$ and $W_i \cap (\sum_{j \neq i} W_j) = \{0\}$

Remark 3.7. When $k = 2$, (c) says that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

Proof. We prove first that (a) implies (c).

Suppose $w_i \in W_i \cap (\sum_{j \neq 1} W_j)$. Then $w_i = w_1 + \cdots + w_{i-1} + w_{i+1} + \cdots w_k$, and since 0 can be represented uniquely by the definition of a direct sum, then all $w_j = 0$ and thus $w_i = 0$.

If $v \in V$, then by the definition of a direct sum, $v = \sum_{i=1}^k w_i$ for $w_i \in W_i$.

Thus, $V = W_1 + \cdots + W_k$.

Then we prove that (c) implies (b).

Note that $V = W_1 + \cdots + W_k$.

If $v_1 + \cdots v_k = 0$ $(v_i \in V)$, then $v_j = -\sum_{i=1, i \neq j}^k v_i$. But $v_i = \sum_{i=1}^k w_i$ for $w_i \in W_i$ and $W_i \cap (\sum_{j \neq i} W_j) = \{0\}$, and thus v_i for all i are 0.

Finally, we prove that (b) implies (a).

Any $v \in V = W_1 + \cdots + W_k$ can be represented as $v = \sum_{i=1}^k w_i$ for $w_i \in W_i$.

We prove that this representation is unique.

Suppose that v is also $v = \sum_{i=1}^k w_i'$. Then $0 = \sum (w_i - w_i')$ and hence $w_i = w_i'$. □

Corollary 3.8

If $V = W_1 \oplus \cdots \oplus W_k$ and β_i is a basis of W_i, then $\beta = \bigcup_{i=1}^k \beta_i$ is a basis of V. In particular, $\dim V = \sum_{i=1}^k \dim W_i$.

Proof. The span of β contains each W_i, hence it also contains $W_1 + \cdots + W_k = V$. By (c) of Theorem 3.6, $\operatorname{span} \beta_i \cap \bigcup_{i \neq j} (\operatorname{span} \beta_j) = \{0\}$, and thus, since all β_i are linearly independent, then the union of the bases is linearly independent as well, while

$\dim V = \sum_{i=1}^k \dim W_i$, as required. □