1 Ellipsoids

Note that the general equation for an ellipse is $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$.

Thus, the equation of an upper half of an ellipse is $y = b\sqrt{1 - \frac{x^2}{a^2}}$.

Thus, the volume of an ellipse is given by the following integral:

$$\int_{-a}^{a} \pi b^2 (1 - \frac{x^2}{a^2}) \, \mathrm{d}x = \frac{4}{3}\pi a b^2$$

2 Method of Shells

Another method of evaluating a volume is the *method of slices.*

Suppose a circle of radius r is drawn on the Cartesian plane such that its centre is on the y-axis at the distance of r from the origin. Revolve this circle around the x-axis to obtain a torus, also called an *annulus.*

Although the standard method of computing its volume can be applied, another way is available.

Note that the equation of the given circleis $x^2 + (y - R)^2 = r^2$.

Thus, the upper and lower semicircle is given by $x = \pm\sqrt{r^2 - (y - R)^2}$.

Slice the circle in horizontal stripes and revolve this stripes around the x-axis. In such a way, a set of cylindrical shells is obtained. We may easily apply the standard method to evaluate the volume of each shell of thickness $\mathrm{d}y$, radius r and width of $2\sqrt{r^2 - (y - R)^2}$.

The volume of a shell is thus given by $2\pi y \cdot 2\sqrt{r^2 - (y - R)^2} \, \mathrm{d}y$.

Therefore, the total volume of a torus is $\int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2 - (y - R)^2} \, \mathrm{d}y$, and hence

$$\begin{aligned}
V &= \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2 - (y - R)^2} \, \mathrm{d}y && (1)\\
&= 4\pi \int_{R-r}^{R+r} y\sqrt{r^2 - (y - R)^2} \, \mathrm{d}y && (2)\\
&= 4\pi(\int_{-r}^{r} u\sqrt{r^2 - u^2} \, \mathrm{d}u + R \int_{-r}^{r} \sqrt{r^2 - u^2} \, \mathrm{d}u && (3)\\
&= 2\pi^2 R r^2 && (4)
\end{aligned}$$

3 Surface Area

The surface area of a solid of revolution can also be computed by considering thin slices of its surface.

Suppose a graph of a function $f(x)$ is drawn in the first quart of the Cartesian plane, and the corresponding solid of revolution is drawn.

Thenthe radius of a slice is $f(x)$ and the circumference is $2\pi f(x)$.

Note that the width of a strip is given by

$$\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{1 + f'(x)^2} \, \mathrm{d}x. \tag{5}$$

Therefore, the surface area of a solid is given by $2\pi \int_a^b f(x)\sqrt{1+f(x)^2}\,\mathrm{d}x$. For a sphere, $f(x) = \sqrt{R^2 - x^2}$ and hence $f'(x) = \frac{-x}{\sqrt{R^2-x^2}}$, which, subbing into the obtained equation (5), gives us the surface area of $4\pi R^2$.

Consider now a horn obtained by rotating the graph of $f(x) = \frac{1}{x}$ from 1 to ∞ around the x-axis.

Thus,

$$V = \int_1^\infty \pi \frac{1}{x^2}\,\mathrm{d}x = \lim_{R\to 0} \int_1^R \pi \frac{1}{x^2}\,\mathrm{d}x = \pi$$