Problem. Find an orthogonal matrix in $M_{3\times 3}(\mathbb{R})$ with first row $(2/3, 1/3, 2/3)$.

Solution.

Suppose $A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ a & b & c \\ d & e & f \end{pmatrix}$, with $a, b, c, d, e, f \in \mathbb{R}$, is such that A is an orthogonal matrix. Therefore, $A^*A = AA^* = I$.

Note that $A^* = \frac{1}{3}\begin{pmatrix} 2 & a & d \\ -1 & b & e \\ -2 & c & f \end{pmatrix}$

Therefore,

$$AA^* = \frac{1}{9}\begin{pmatrix} 9 & 2a-b-2c & 2d-e-2f \\ 2a-b-2c & a^2+b^2+c^2 & ad+be+cf \\ 2d-e-2f & ad+be+cf & d^2+e^2+f^2 \end{pmatrix} \tag{1}$$

Moreover,

$$A^*A = \frac{1}{9}\begin{pmatrix} 4+a^2+d^2 & -2+ab+de & -4+ac+df \\ -2+ab+de & 1+b^2+e^2 & 2+bc+ef \\ -4+ac+df & 2+bc+ef & 4+c^2+f^2 \end{pmatrix} \tag{2}$$

Since $A^*A = AA^* = I$, then, from the diagonals of A and A^* we obtain

$$\begin{cases} 4+a^2+d^2 &= 9 \\ a^2+b^2+c^2 &= 1+b^2+e^2 \\ d^2+e^2+f^2 &= 4+c^2+f^2 \end{cases}, \tag{3}$$

and therefore

$$\begin{cases} a^2+d^2 &= 5 \\ a^2+c^2 &= 1+e^2 \\ d^2+e^2 &= 4+c^2 \end{cases} \tag{4}$$

Suppose $a = 1$, $b = -2$, $c = 2$, $d = 2$, $e = 2$, $f = 1$.

Therefore,

$$AA^* = \frac{1}{9}\begin{pmatrix} 9 & 2+2-4 & 4-2-2 \\ 2+2-4 & 1+4+4 & 2-4+2 \\ 4-2-2 & 2-4+2 & 4+4+1 \end{pmatrix} \tag{5}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} \tag{6}$$

$$= I \tag{7}$$

$$= \frac{1}{9}\begin{pmatrix} 4+1+4 & -2-2+4 & -4+2+2 \\ -2-2+4 & 1+4+4 & 2-4+2 \\ -4+2+2 & 2-4+2 & 4+4+1 \end{pmatrix} \tag{8}$$

$$= A^*A \tag{9}$$

Therefore, the matrix A is orthogonal if

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

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$\square$