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ILLARIONOV,
ALEXANDER

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MAT 157 : PS 5

THE FOLLOWING PROPOSITIONS HAVE ALREADY BEEN PROVEN!

GIVEN f AND g ARE CONTINUOUS, AT a,

LP1. $f+g$ IS CONTINUOUS AT a

LP2. $f \cdot g$ IS CONTINUOUS AT a

LP3. $\frac{1}{g}$ IS CONTINUOUS AT a, IF $g(a) \neq 0$.

LP4. IF g IS CONTINUOUS AT a, AND f IS CONTINUOUS AT $g(a)$, THEN $f \circ g(a)$ IS CONTINUOUS AT a.

C1. WELL-DEFINED RATIONAL FUNCTIONS ARE CONTINUOUS IN THEIR DOMAIN.

B1. IF f IS CONTINUOUS
ON $[a,b]$, THEN THERE
IS SOME NUMBER y
IN $[a,b]$ SUCH THAT
$f(y) \geq f(x)$ FOR ALL
x IN $[a,b]$.

B2. IF f IS CONTINUOUS
ON $[a,b]$, THEN
THERE IS SOME NUMBER
$y \in [a,b]$ SUCH
THAT $f(y) \leq f(x)$
FOR ALL, $x \in [a,b]$.

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① LET $f(x)$ BE EQUAL TO

(i) $\cos\left(x^2-(\pi-x^2)\cos\left(\frac{x-3}{(x-\sqrt{2\pi})(x+\sqrt{2\pi})}\right)\right)=$

$$=\cos\left(x^2\left(1+\cos\left(\frac{x-3}{(x-\sqrt{2\pi})(x+\sqrt{2\pi})}\right)\right)-\right.$$

$$\left.-\pi\cos\left(\frac{x-3}{(x-\sqrt{2\pi})(x+\sqrt{2\pi})}\right)\right).$$

NOTE THAT $f(x)$ IS WELL-DEFINED FOR $x \in [0, \sqrt{\pi}]$, SINCE $|x| \neq \sqrt{2\pi}$

Let $h(x)= x^2-(\pi-x^2)\cos\left(\frac{x-3}{x^2-2\pi}\right)$

AND $g(x) = \cos x$.

THEREFORE, $g\circ h(x) = f(x)$.

NOTE THAT SINCE $\frac{x-3}{x^2-2\pi}$ IS A RATIONAL FUNCTION AND $|x| \neq 2\pi$, WHILE WE MAKE IT GIVEN THAT $\sin(x)$ IS CONTINUOUS, AND BECAUSE $\frac{\pi}{2}-x$ IS A POLYNOMIAL AND HENCE CONTINUOUS, $\sin\left(\frac{\pi}{2}-x\right)=\cos(x)$ IS CONTINUOUS, BY LP4

THEN $\cos\left(\frac{x-3}{x^2-2\pi}\right)$ IS CONTINUOUS AS WELL

BY LP4, $(\pi - x^2)$ IS CONTINUOUS, AND HENCE BY LP2, SINCE $\lambda :: x \mapsto -1$ IS CONTINUOUS,

$$-(\pi - x^2)\cos\left(\frac{x-3}{x^2-2\pi}\right)$$ IS CONTINUOUS.

BY C1, $\frac{x^2}{1}$ IS CONTINUOUS,

THEN $h(x) = x^2 - (\pi - x^2)\cos\left(\frac{x-3}{x^2-2\pi}\right)$ IS ALSO CONTINUOUS.

BY LP4, $g(h(x))$ IS CONTINUOUS,

THEN BY B1-2, $g \circ h(x) = f(x)$ IS BOUNDED ABOVE AND BELOW.

SO THAT $\exists$

SINCE $\forall x \in \mathbb{R}: \cos(x) \in [-1, 1]$,

$LB = -1$, $VR = 1$.

$3 < \pi < 4 < \frac{9}{2}$ AND

①(ii)

0

Let $f(x) = \frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$.

CLAIM

$$\lim_{x \to +\infty} \frac{1}{x^2+1} = 0.$$

PROOF

SUPPOSE $\varepsilon > 0$ IS GIVEN.

TAKE $N = \sqrt{\frac{1}{\varepsilon}}$.

SUPPOSE $x > \sqrt{\frac{1}{\varepsilon}}$.

Therefore, $x^2 + 1 > \frac{1}{\varepsilon} + 1$

AND HENCE $\frac{1}{x^2+1} < \frac{\varepsilon}{\varepsilon+1} < \varepsilon$,

AND THUS $\left| \frac{1}{x^2+1} - 0 \right| < \varepsilon$.

HENCE, $\lim_{x \to +\infty} \frac{1}{x^2+1} = 0$. □

NOTE THAT $f(x)$ IS WELL-DEFINED FOR all $x \in \mathbb{R}$, SINCE $1+x^2 \neq 0$,

BY THE CLAIM,

$$\lim_{x \to +\infty} f(x) = 1 \quad , \text{ AND}$$

THUS $\left| f(x) - 1 \right| < \varepsilon \quad \forall (\varepsilon > 0)$.

BY Δ INEQUALITY,

$$\left| f(x) \right| - 1 \leq \left| f(x) - \varepsilon \right| < \varepsilon$$

NO

THUS, $UB = 1 + \varepsilon$, $LB = -1 - \varepsilon$

what's ε!

$\square$

④

① (iii)

LET $f(x) = \frac{2+x}{3-x}$.

CLAIM 1.

$\forall(\varepsilon > 0)$ there EXISTS $N \in \mathbb{R}$: $N < x < 3 \Rightarrow |f(x)| > \varepsilon$.

PROOF.

LET $\varepsilon > 0$ BE GIVEN.

TAKE $N = 3 - \frac{5}{\varepsilon + 1} < 3$

SUPPOSE $N < x < 3$.

THEREFORE, $3 - x < \frac{5}{\varepsilon + 1}$.

$\Leftrightarrow$ $\varepsilon + 1 < \frac{5}{3-x}$ (SINCE [illegible] 2)

$\Leftrightarrow$ $\varepsilon < \frac{2+x}{3-x} = f(x)$.

SINCE $\varepsilon > 0$ $3 > N > -2$ AND THUS

$\frac{2+x}{3-x} = |f(x)| > \varepsilon$ AND $f(x) > 0$.

$\square$

THUS, $f(x)$ IS NOT BOUNDED FROM ABOVE.

CLAIM 2.

$\forall \varepsilon > 0, \exists N \in \mathbb{R}: 3 < x < N \Rightarrow |f(x)| > \varepsilon.$

PROOF

SUPPOSE $\varepsilon > 0$ IS GIVEN.

TAKE $N = 3 + \frac{4}{\varepsilon}$.

THEREFORE, $3 < x < 3 + \frac{4}{\varepsilon}$

$\Leftrightarrow 0 < x - 3 < \frac{4}{\varepsilon} \Rightarrow \varepsilon < \frac{4}{x-3}$.

SINCE $4 < 5 < x + 2$,

THEN $\varepsilon < \frac{x+2}{x-3}$. SINCE $x - 3 > 0$,

$\frac{x+2}{x-3} = |f(x)|$, AND THUS $|f(x)| > \varepsilon$, $\square$.

BUT $f(x) < 0$

THUS, $f(x)$ IS NOT BOUNDED FROM BELOW.

(iv) LET $f(x) = \frac{1}{2+|x|}$.

NOTE THAT

$|x| \geq 0 \quad \forall x \in \mathbb{R}$.

THUS, $2 + |x| \geq 2$

②

$\Longleftrightarrow \quad \frac{1}{2} \geq \frac{1}{2+|x|}$.

FOR $x = 0$, $\frac{1}{2+|x|} = 1/2 \Longrightarrow$

$\frac{1}{2}$ IS AN UPPER BOUND FOR $f(x)$

CLAIM

$\forall \varepsilon > 0 \; \exists N > 0 : \; x > N \Longrightarrow 0 < f(x) < \varepsilon.$

PROOF

SUPPOSE $\varepsilon > 0$ IS GIVEN.

TAKE $N = \frac{1}{\varepsilon} > 0$.

SUPPOSE $x > N$.

$\Longleftrightarrow \quad x > \frac{1}{\varepsilon} \Longrightarrow |x| > \frac{1}{\varepsilon}$

$\Longleftrightarrow \quad 2 + |x| > 2 + \frac{1}{\varepsilon} > 0$

$\Longleftrightarrow \quad \varepsilon > \frac{\varepsilon}{2\varepsilon + 1} > \frac{1}{2+|x|} > 0$

THUS $0 < \left| \frac{1}{2+|x|} \right| < \varepsilon.$ ☐

HENCE, 0 IS A LOWER BOUND FOR $f(x)$.

② ⓘ LET $f(x) = \frac{1}{x-1}$.

CLAIM 1

$$\forall(\varepsilon > 0)\ \exists(N > 0):$$

$$N < x < 1 \Rightarrow |f(x)| > \varepsilon.$$

PROOF.

LET $\varepsilon > 0$ BE GIVEN.

TAKE $N = 1 - \frac{1}{\varepsilon}$.

SUPPOSE $1 - \frac{1}{\varepsilon} < x < 1$. THUS, $1 - x > 0$.

HENCE, $1 - x < \frac{1}{\varepsilon} \Leftrightarrow \frac{1}{1-x} > \varepsilon$.

SINCE $1 - x > 0$, $\frac{1}{|x-1|} > \varepsilon$ □

THUS, THERE IS NO LIMIT AS $x \to 1^-$.

CLAIM 2.

$$\forall(\varepsilon > 0)\ \exists(N > 0):$$

$$1 < x < N \Rightarrow |f(x)| > \varepsilon.$$

PROOF.

LET $\varepsilon > 0$ BE GIVEN.

TAKE $N = 1 + \frac{1}{\varepsilon}$.

SUPPOSE $1 < x < 1 + \frac{1}{\varepsilon}$. THUS, $x - 1 > 0$.

HENCE, $x - 1 < \frac{1}{\varepsilon} \Leftrightarrow \varepsilon < \frac{1}{x-1}$.

SINCE $x - 1 > 0$, $\left|\frac{1}{x-1}\right| > \varepsilon$. □

THUS, THERE IS NO LIMIT AS $x \longrightarrow 1^{+}$.

HENCE, THERE IS NO LIMIT AS $x \longrightarrow 1$.

SINCE FOR $x = 1$ $\frac{1}{x-1}$ IS UNDEFINED

ALSO, THEN THE DISCONTINUITY IS NOT REMOVABLE.

□

② ②(ii)

LET $f(x) = \dfrac{x^{2} - x - 6}{|x| - 3}$

$= \dfrac{(x-3)(x+2)}{|x| - 3}$

FOR $x \in (0,3) \cup (3, +\infty)$:

$f(x) = x + 2$.

CLAIM.

$$\lim_{x \to 3} f(x) = 5$$

PROOF

LET $\varepsilon > 0$ BE GIVEN.

TAKE $\delta = \min\{1, \varepsilon\}$.

SUPPOSE $0 < |x-3| < \delta$.

②(ii) (cont.)

THUS, $-1 < x-3 < 1 \iff 2 < x < 4$ ①

AND $-\varepsilon < x-3 < \varepsilon \iff 3-\varepsilon < x < \varepsilon+3$ ②

FROM ①, $|x| = x$ AND

FROM ② $-\varepsilon < (x+2) - 5 < \varepsilon$.

THUS, $\frac{x-3}{|x|-3} = 1$ (SINCE $0 < |x-3|$),

AND HENCE

$$-\varepsilon < \frac{(x-3)}{|x|-3}(x+2) - 5 < \varepsilon.$$

THUS,

$$\left| \frac{x-3}{|x|-3}(x+2) - 5 \right| < \varepsilon$$

AND HENCE $\lim_{x \to 3} f(x) = 5$. □

THEREFORE, SINCE $f(x)$ IS ALSO UNDEFINED FOR $x=3$, THEN THE DISCONTINUITY AT $x=3$ IS REMOVABLE

IF $x < 0,\ x \neq -3,$

$$f(x) = \frac{x^2 - x - 6}{-x - 3} = \frac{x^2 - 3}{-x - 3} + 1$$

$$= \frac{(x^2 - 9)}{-(x+3)} - \frac{6}{x+3} + 1$$

$$= -(x - 3) + 1 - \frac{6}{x+3}$$

$$= 4 - x - \frac{6}{x+3}.$$

SUPPOSE $\lim\limits_{x \to -3} \frac{-1}{x+3} = L \in \mathbb{R}.$

BY THE PROPERTIES OF LIMITS ALREADY PROVEN, WHERE FROM C1 $\lim\limits_{x \to -3} \frac{(4-x)}{1} = 7$,

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} \left[7 - \frac{6}{x+3} \right] = 7 - 6L$$

SUPPOSE $6\varepsilon > 0$ IS GIVEN.

BY THE ASSUMPTION ABOVE,

$\left| 7 - \frac{6}{x+3} - 7 + 6L \right| < 6\varepsilon$ for SOME $\delta > 0$ SUCH THAT $|x+3| < \delta$

$\Leftrightarrow \left| \frac{1}{x+3} - L \right| < \varepsilon$ FOR SOME $\delta > 0$, SUCH THAT $|x+3| < \delta$

2 (ii) (CONT.).

TAKE $\delta' = \min\left\{\delta, \frac{1}{|L|+\varepsilon}\right\}$

SUPPOSE, $|x+3| < \delta'$, i.e.

$$\begin{cases} |x+3| < \delta & ① \\ |x+3| < \dfrac{1}{|L|+\varepsilon} & ② \end{cases}$$

FROM ②,

$$\left|\frac{1}{x+3}\right| > |L| + \varepsilon. \iff \left|\frac{1}{x+3}\right| - |L| > \varepsilon$$

FROM Δ INEQ.,

$$\left|\frac{1}{x+3} - L\right| \geq \left|\frac{1}{x+3}\right| - |L|$$

$$\Rightarrow \quad \left|\frac{1}{x+3} - L\right| > \varepsilon.$$

BUT ALSO $|x+3| < \delta$, AND HENCE $\left|\frac{1}{x+3} - L\right| < \varepsilon$, WHICH IS A CONTRADICTION $\Rightarrow L \in \emptyset$

$\#$

THEREFORE, THE DISCONTINUITY OF $f(x)$ AT $x = -3$ IS NOT REMOVABLE.

2(iii)

LET $f(x) = \dfrac{(x+3)^2}{x+3}$.

IF $x = -3$, $f(x) = x+3$.

LET $\varepsilon > 0$ BE GIVEN.

TAKE $\delta = \varepsilon$ AND SUPPOSE
$0 < x+3 < \delta$.

THUS, $0 < x+3 < \varepsilon \Rightarrow |x+3| < \varepsilon$.

$$\Rightarrow \lim_{x \to (-3)^+} f(x) = 0. \quad ①$$

SUPPOSE NOW $-\delta < x+3 < 0$.

THUS, $-\varepsilon < x+3 < 0 \Rightarrow |-x-3| = |x+3| < \varepsilon$.

$$\Rightarrow \lim_{x \to (-3)^-} f(x) = 0 \quad ②.$$

SINCE ① = ②, $\lim_{x \to -3} f(x) = 0$

AND HENCE, SINCE $f(x)$ IS ALSO UNDEFINED AT $x = -3$, THE DISCONTINUITY OF $f(x)$ AT $x = -3$ IS REMOVABLE.

②. (iv)

LET $f(x) = \dfrac{x-2}{x^2-9}$.

$= \dfrac{x-3+1}{x^2-9}$

IF $|x| \neq 3$, $f(x) = \dfrac{1}{x+3} + \dfrac{1}{x^2-9}$.

SINCE $|x| \neq 3$, $= \dfrac{1}{x+3}\left(1 + \dfrac{1}{x-3}\right)$. (★)

SUPPOSE $\lim\limits_{\cancel{|x|} \to -3} f(x) = L \in \mathbb{R}$.

THUS, $\forall \varepsilon > 0\ \exists \delta > 0 : |x+3| < \delta \Rightarrow |f(x) - L| < \varepsilon$

TAKE $\delta' = \min\left\{\delta, \dfrac{4}{5}, \dfrac{1}{|L|+\varepsilon}\right\}$.

SUPPOSE $|x+3| < \delta'$.

$$\Leftrightarrow \begin{cases} ① & -4 < x < -2 \Leftrightarrow -1 < x+3 < 1 \Leftrightarrow -7 < x-3 < -5 \\ ② & |x+3| < \dfrac{4/5}{|L|+\varepsilon} \\ ③ & |x+3| < \delta \end{cases}$$

FROM ①, $-\dfrac{1}{5} < \dfrac{1}{x-3} < -\dfrac{1}{7} \Rightarrow \dfrac{4}{5} < \left|\dfrac{1}{x-3} + 1\right| < 6/7$

$\Rightarrow$ FROM ②, $|x+3| < \dfrac{\left|1 + \dfrac{1}{x-3}\right|}{|L|+\varepsilon}$

② iv (CONT.)

THUS, $\dfrac{|x-2|}{|x^2-9|} > |L| + \varepsilon$

$$\Longleftrightarrow \quad (\star) = \left|\frac{x-2}{x^2-9}\right| - |L| > \varepsilon$$

FROM $\triangle$ INEQUALITY,

$$(\star) \leq \left|\frac{x-2}{x^2-9} - L\right|,$$

AND THUS $\left|\dfrac{x-2}{x^2-9} - L\right| > \varepsilon$

BUT FROM ③ AND ASSUMPTION, SINCE $|x-3| < \delta$,

$|f(x) - L| < \varepsilon$, WHICH IS A CONTRADICTION. THUS, $L \in \emptyset$.

SUPPOSE NOW,

$$\lim_{x \to 3} f(x) = L \in \mathbb{R}.$$

NOTE THAT $|L| > \frac{1}{7}$, SINCE $f(-2.9) > 0$
FROM $(\overline{\underline{\star}})$

② iv (CONT) HENCE, $7|L|-1 > 0$.

BY DEFINITION OF THE LIMIT,

$$\forall \varepsilon > 0\ \exists \delta > 0 : |x-3| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

SUPPOSE $\varepsilon > 0$ IS GIVEN.

LET $\delta' = \min \left\{ \delta, 1, \dfrac{1}{7(|L|+\varepsilon) - 1} \right\}$

SUPPOSE $|x-3| < \delta'$.

$$\Leftrightarrow \begin{cases} -1 < x-3 < 1 \Leftrightarrow 2 < x < 4 & ① \\ |x-3| < \dfrac{1}{7(|L|+\varepsilon) - 1} & ② \\ |x-3| < \delta. & ③ \end{cases}$$

FROM ①, $5 < x+3 < 7 \Rightarrow \dfrac{1}{7} < \dfrac{1}{|x+3|} < \dfrac{1}{5}$

$\Rightarrow$ FROM ②, $\Rightarrow |x-3| < \dfrac{1}{\left(|L|+\varepsilon - \frac{1}{7}\right) 7}$

$$< \frac{1}{|x+3|\left(|L|+\varepsilon - \frac{1}{7}\right)}$$

$$< \frac{1}{|x+3|\left(|L|+\varepsilon - \frac{1}{|x+3|}\right)}$$

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② iv (cont)

THUS,

$$|L| + \varepsilon - \frac{1}{|x+3|} < \frac{1}{|x+3||x-3|}.$$

[illegible] 1,

$$\overset{\text{THEN}}{\iff} \quad \varepsilon < \frac{1}{|x+3||x-3|} + \frac{1}{x+3} - |L|$$

BY Δ INEQ. $\left| \quad \right| < \left| \frac{1}{x^2-9} + \frac{1}{x+3} - L \right|$

BUT BY ③ AND THE ASSUMPTION,

$|f(x) - L| < \varepsilon$, WHICH IS A CONTRADICTION. THUS, $L \in \emptyset$.

BUT $f(x)$ IS ALSO UNDEFINED FOR $|x| = 3$. THUS, THE DISCONTINUITIES OF $f(x)$ AT $|x| = 3$ ARE NOT REMOVABLE.

③ (i)

LET $f(x) = \dfrac{1}{x-1}$ BE DEFINED OVER $[2,3]$.

SINCE $1 \notin [2,3]$ AND $f(x)$ IS A RATIONAL FUNCTION, BY C1 IT IS CONTINUOUS, AND HENCE BY B1 IT ACHIEVES ITS MAXIMUM.

(ii) LET $f(x) = \dfrac{x^3-5x^2+2x-4}{x^2-x-2}$ BE DEFINED OVER $[0,4]$.

THUS, $f(x) = \dfrac{x^3-5x^2+2x-4}{(x+1)(x-2)}$.

REMARK

SINCE $2 \in [0,4]$, IT IS NECESSARY TO CHECK WHETHER AS $x \to 2^+$ $f(x)$ GROWS POSITIVE INDEFINITELY.

NOTICE THAT $f(2,1) = \dfrac{2,1^2(2,1-5)+4,2-4}{3,1 \times 0,1}$

< 0, WHILE

$f(0) = \dfrac{-4}{1\cdot(-2)} = 2$.

CLAIM.

$$\lim_{x \to 2^-} f(x) = L \in \emptyset.$$

3(ii) (cont.)

PROOF

SUPPOSE $\lim_{x \to 2^-} f(x) = -12L + 2 \in \mathbb{R}$.

NOTE THAT

$$f(x) = x - 4 - \frac{12}{(x-2)(x+1)}.$$

SINCE $x-4$ IS A RATIONAL FUNCTION, BY C1 IT IS CONTINUOUS, AND HENCE

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(2 - \frac{12}{(x-2)(x+1)} \right)$$

$= -12L+2$, SUPPOSE $\varepsilon > 0$ IS GIVEN

THEREFORE, $\forall 12\varepsilon > 0 : \exists \delta > 0 :$

$$\left| \frac{1}{(x-2)(x+1)} - L \right| < \varepsilon.$$

TAKE $\delta' = \min\left\{\delta, 1, \frac{1}{4(\varepsilon + |L|)}\right\}$,

AND SUPPOSE THAT

$0 < x-2 < \delta'$.

3(ii) CONT

THEREFORE,

$$\left\{\begin{array}{ll} 0 < x-2 < 1 \iff 2 < x < 3 & \textcircled{1} \\ 0 < x-2 < \dfrac{1}{4(\varepsilon + |L|)} & \textcircled{2} \\ 0 < x-2 < \delta . & \textcircled{3} \end{array}\right.$$

FROM ①, $3 < x+1 < 4$

$$\implies \frac{1}{4} < \frac{1}{x+1} < \frac{1}{3}$$

$\implies$ FROM ②, $0 < x-2 < \dfrac{1}{(x+1)(\varepsilon + |L|)}$

$$\implies \frac{1}{(x+1)(x-2)} > \varepsilon + |L|$$

$$\iff \frac{1}{(x+1)(x-2)} - |L| > \varepsilon$$

$\iff$ BY Δ INEQUALITY,

$$\left| \frac{1}{(x+1)(x-2)} - L \right| > \varepsilon .$$

3(ii) (cont)

BUT FROM ③ AND THE ASSUMPTION,

$$\left| \frac{1}{(x-2)(x+1} - L \right| < \varepsilon,$$

WHICH IS A CONTRADICTION.

THUS, $L \in \emptyset$ □

FROM THE CLAIM IT FOLLOWS THAT $f(x)$ IS NOT BOUNDED ABOVE OR BELOW. THUS, IT DOES NOT ACHIEVE ITS MAXIMUM.

③ (iii). LET $f(x)$ BE EQUAL TO $\frac{1}{x^2+3}$ ON $\mathbb{R}$. NOTE THAT $x^2 \geq 0 \quad \forall x \in \mathbb{R}$.

THUS, $x^2+3 \geq 3$

$$\iff \frac{1}{x^2+3} \leq 1/3.$$

HENCE, SINCE $f(0)=1/3$, f ACHIEVES ITS MAXIMUM.

3.(iv) LET $f(x) = \frac{x-2}{x^2-10x+12}$

$$= \frac{x-2}{(x-5)^2-13}$$

②

BE DEFINED ON $[3,5]$.

THUS, $3 \leq x \leq 5$.

HENCE, $-2 \leq x-5 \leq 0$.

$\implies 0 \leq (x-5)^2 \leq 4$

$\iff -13 \leq (x-5)^2-13 \leq -9$.

THUS, SINCE THE DENOMINATOR IS NOT EQUAL TO ZERO FOR ALL $x \in [3,5]$, BY C1, SINCE $f(x)$ ALSO IS A RATIONAL FUNCTION THEN f IS CONTINUOUS,

HENCE BY B1 IT ACHIEVES ITS MAXIMUM.

④ SUPPOSE $f(x)$ IS CONTINUOUS ON $[a,b]$ AND $f(x) > 0 \ \forall x \in [a,b]$.

CLAIM.

$\frac{1}{f(x)}$ IS BOUNDED ON $[a,b]$.

PROOF.

⑤

BY LP3, SINCE $f(x)$ IS CONTINUOUS ON $[a,b]$ AND $\forall x \in [a,b]$. $f(x) \neq 0$, THEN $\frac{1}{f(x)}$ IS CONTINUOUS,

$\Rightarrow$ BY B1, IT IS BOUNDED ABOVE,

ALSO, BY B2, IT IS BOUNDED BELOW,

THUS, $f(x)$ IS BOUNDED

□

⑤

SUPPOSE $f(x)$ IS DEFINED ON $\mathbb{R}$ AND IS CONTINUOUS AT $x = c$.

CLAIM:

$$f(x) \text{ IS BOUNDED ON } [a,b]$$

$$\iff (x-c)\,f(x) \text{ IS BOUNDED ON } [a,b].$$

PROOF.

($\Rightarrow$) SUPPOSE $f(x)$ IS BOUNDED ON $[a,b]$,

f-LEMMA

NOTE THAT $g(x) = x - c$

A RATIONAL FUNCTION AND THUS CONTINUOUS OVER $\mathbb{R}$ BY C1.

SINCE $f(x)$ IS CONTINUOUS AT $x = c$, THEN BY LP2 $g(x) \cdot f(x)$ IS CONTINUOUS AT $x = c$ AND

THUS $\lim_{x \to c} g(x) \cdot f(x) = g(c) \cdot f(c)$

$= 0$.

SINCE $f(x)$ IS BOUNDED ON $[a,b]$, $\exists\, N, N' \in \mathbb{R} : N \leq f(x) \leq N'$.

LEMMA: $g(x)$ IS BOUNDED ON $[a, b]$.

PROOF. SUPPOSE $a \le x \le b$.

THEREFORE, $a - c \le x - c \le b - c$.

THUS, $g(x)$ IS BOUNDED $\square$

IF $x \geq c$,

THUS, $N(x-c) \le (x-c) f(x) \le N'(x-c)$

FROM LEMMA, $(a-c)N \le (x-c)N$,

$(x-c)N' \le (b-c)N'$.

THUS, $(a-c)N \le (x-c) f(x) \le (b-c)N'$.

HENCE, $(x-c) f(x)$ IS BOUNDED.

IF $x < c$,

$N(x-c) \geq (x-c) f(x) \geq N'(x-c)$,

FROM LEMMA, $(a-c)N' \le (x-c)N'$,

$(x-c)N \le (b-c)N$.

THUS, $N'(x-c) \le (x-c) f(x) \le (b-c)N$.

HENCE, $(x-c) f(x)$ IS AGAIN BOUNDED.