MAT157: LIMITS II   20161004

LIMIT $\forall \varepsilon > 0 \; \exists \delta > 0 : \; 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

EXAMPLE $f(x) = c, \; \forall x \; \Big| \; \lim_{x \to a} f(x) = c$

PROOF   GIVEN $\varepsilon > 0$, CHOOSE ANY $\delta > 0$

$|f(x) - c| = |c - c| = 0 < \varepsilon, \; \forall x$

EXAMPLE

$g(x) = x$

$\lim_{x \to a} g(x) = \lim_{x \to a} x = a$

PROOF

$\varepsilon > 0$, LET $\delta = \frac{\varepsilon}{2}$.

IF $0 < |x-a| < \frac{\varepsilon}{2}$.

THEN $|g(x) - a| = |x - a| < \frac{\varepsilon}{2} < \varepsilon$

EXAMPLE $h(x) = x^2$; $\lim_{x \to a} h(x) = \lim_{x \to a} x^2 = a^2$

PROOF

$\varepsilon > 0$.

$$|x+a| = |(x-a) + 2a| \leq |x-a| + |2a| < \delta + |2a|$$

INSIST THAT $\delta \leq 1 \Rightarrow$

$|x+a| < 1 + |2a|$.

NEED $\delta \leq \frac{\varepsilon}{1+|2a|}$. CHOOSE $\min\left(1, \frac{\varepsilon}{1+|2a|}\right)$.

$\square$

EXAMPLE

LET $f(x) = \begin{cases} 0, & \text{IF } x \text{ IS IRRATIONAL} \\ \frac{1}{n}, & \text{IF } x = \frac{m}{n} \text{ IN LOWEST TERMS, WITH } n > 0 \end{cases}$

$\lim_{x \to a} f(x) = 0 \quad \forall a \in \mathbb{R}.$

PROOF

GIVEN $\varepsilon > 0$

CHOOSE n SO THAT $n > \frac{1}{\varepsilon}$ $\left(\Leftrightarrow \frac{1}{n} < \varepsilon\right)$.

$\forall\, m \geq n,\ \frac{1}{m} < \varepsilon$

SUPPOSE $a \in (0,1)$. THE ONLY POINTS IN $(0,1)$ WHERE $f(x) > \varepsilon$ WILL BE $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}$

LET δ = min DISTANCE FROM a TO ANY OF THESE NUMBERS (EXCEPT ITSELF IF IT'S ONE OF THEM)

IF $0 < |x-a| < \delta$, THEN

$f(x)$ will EITHER BE 0 OR $\frac{1}{m}$, with $m \geq n$, SO $|f(x)| = f(x) < \varepsilon$