

## Integral Calculus

Motivation: finding areas

Although triangulation provides a viable approach to compute areas of geometric shapes, it is not obvious that the non-unique partitions add up to the same area.

Problems also arise in the calculation of the area of a circle. What's  $\pi$ ?

We introduce the notion of the *area under the graph between  $a$  and  $b$* , corresponding to the signed area under the graph of  $y = f(x)$  above the  $x$ -axis and between the lines  $x = a$  and  $x = b$ .

This area can be approximated by computing  $f$  on a *partition*  $P$  of  $[a, b]$ , a set of points  $a = t_0 < t_1 < t_2 < \dots < t_n = b$ .

For each  $i = 1, \dots, n$  define  $m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$ ,  $M_i = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$ .

**Note.** Since  $f$  is not assumed to be continuous, note that  $\inf$  and  $\sup$  are used instead of  $\min$  and  $\max$ .

Assume  $f$  is bounded, so that  $m_i$  and  $M_i$  exist.

Define the *lower sum*  $L(f, P)$  of the partition as follows:

$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1})$$

Define the *upper sum* similarly:

$$U(f, P) = \sum_{i=1}^n M_i(t_i - t_{i-1})$$

**e.g.** for a constant function  $f(x) = c > 0$ ,  $U(f, P) = L(f, P)$  for any partition  $P$ .

**Note.** For a partition  $P$ ,

$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1}) \leq \sum_{i=1}^n M_i(t_i - t_{i-1}) = U(f, P),$$

since  $m_i \leq M_i$  for all  $i$  by definition.