MAT240:

APPLICATIONS OF ELEMENTARY ROW OPERATIONS

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IF A' IS OBTAINED FROM A BY SEQUENCE OF ROW AND COLUMN OPERATIONS, THEN

$$A' = PAQ,$$

WHERE P, Q ARE INVERTIBLE SQUARE MATRICES.

IN FACT, P IS OBTAINED BY APPLYING ROW OPERATIONS TO $I_{m \times m}$ (SKIPPING COLUMN OPERATIONS), Q OBTAINED BY COLUMN OPERATIONS TO $I_{n \times n}$

ANY $A \in M_{m \times n}(F)$ GIVES A LINEAR MAP

$$L_A : F^n \longrightarrow F^m, \quad x \longmapsto Ax,$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} A_{11} & \cdots \\ \vdots & \\ \vdots & \cdots \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

THEN, DEFINE $\text{rank}(A) = \text{rank}(L_A)$, $\text{nullity}(A) = \text{nullity}(L_A)$.

NOTE: A IS THE MATRIX OF $T = L_A$

WITH RESPECT TO STANDARD BASES

$R(L_A) \leq F^m$ IS THE

SPAN OF THE COLUMNS OF A,

BECAUSE THE j-TH COLUMN OF A

IS THE IMAGE OF THE j-TH

STANDARD BASIS VECTOR.

THE RANGE $R(L_A)$ DOES NOT

CHANGE UNDER ~~ROW~~ COLUMN OPERATIONS C1, C2, C3.

THE NULL SPACE $N(L_A)$ DOES NOT

CHANGE UNDER ROW OPERATIONS.

$$N(L_A) = \{ x \in F^n \mid Ax = 0 \}$$

SOLUTION SET OF A LINEAR SYSTEM

OF EQUATIONS.

BY DEFINITION, RANK(A) = RANK(L_A).

THEOREM ① RANK(A) = MAX NUMBER OF LINEARLY INDEPENDENT COLUMNS OF A

= MAX —— " —— ROWS OF A

② RANK$(AB) \leq$ RANK(A) WITH EQUALITY IF B IS AN INVERTIBLE SQUARE MATRIX

③ SIMILARLY,

RANK$(AB) \leq$ RANK(B) WITH EQUALITY IF A IS AN INVERTIBLE SQUARE MATRIX

③

IF A' IS OBTAINED FROM A BY ROW AN COLUMN OPERATIONS THEN

$A' = PAQ$, WHERE P, Q ARE INVERTIBLE.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

ASSUMING $\mathbb{F} = \mathbb{R}$

$$\begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 4 & 0 \\ 4 & 1 & 0 & 5 & 1 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 \\ 4 & 1 & 0 & 5 & 1 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 & 1 \\ 6 & 3 & 2 & 3 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 \\ 6 & 3 & 2 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(3) RANK (A) DOES NOT CHANGE UNDER ROW AND COLUMN OPERATIONS.

(4) RANK (A^t) = RANK (A)

PROOF

(1) $R(L_A)$ IS SPANNED BY COLUMN VECTORS AND SINCE FOR ANY SUBSET OF S SUCH THAT $\dim(\mathrm{span}(\beta))$ = MAXIMUM # OF LINEARLY INDEPENDENT VECTORS IN S.

SECOND PART WILL FOLLOW FROM (4).

(2) Let $T \in \mathcal{L}(V, W)$, $S \in \mathcal{L}(U, V)$ WITH U, V, W FINITE DIMENSIONAL.

$R(T \circ S) \leq$ RANK (T), WITH EQUALITY IF S IS AN ISOMORPHISM.

WARNING: RANK$(T \circ S)$ = RANK T DOES NOT IMPLY THAT S IS AN ISOMORPHISM

$N(T \circ S) \supseteq N(S) \Rightarrow \mathrm{nullity}(T \circ S) \geq \mathrm{nullity}(S)$

$\Rightarrow$ rank $(T \circ S) \leq \mathrm{rank}(S)$ WITH EQUALITY IF T IS INVERTABLE.

$$\begin{pmatrix} 0 & 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 3 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 2 & 4 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 4 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

THEOREM Every matrix $A \in M_{m\times n}(F)$ of rank r can be brought into "block form".

$$\left(\begin{array}{c|c} I_{r\times r} & O_{r(n-r)} \\ \hline O_{(m-r)\times r} & O_{(m-r)(n-r)} \end{array}\right)$$

COROLLARY rank (A^t) = rank (A)

PROOF If $A=0$, we are done. Otherwise, there is at least one non-zero entry a, which can be moved by R1, C1 to $(1,1)$ position.

$$\left(\begin{array}{c|ccc} a & * & * & * \\ \hline * & & & \\ * & & & \\ * & & & \end{array}\right) \xrightarrow{R_2} \left(\begin{array}{c|ccc} 1 & * & * & * \\ \hline * & & & \\ * & & & \\ * & & & \end{array}\right)$$

Subtract multiples of 1st row off other rows, likewise with columns.

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & * & \\ \vdots & & & \end{array}\right)$$

Thus, A is replaced with

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & A_1 & \\ 0 & & & \end{array}\right)$$

If $A_1 = 0$, we are done. Otherwise, repeat.

PROBLEM Let $\mathcal{X}$ be a set with n elements, $n \geq 3$.
Let $S_1, \dots, S_m$ be proper subsets of $\mathcal{X}$, s.t.
any pair of distinct elements of $\mathcal{X}$
lies in the unique S_j.

Prove that $m \geq n$.

INVERSE OF A MATRIX

Let $A \in M_{n \times n}(F)$ be a square matrix.
If rank $(A) = n$, then L_A is invertible,
A is invertible.

That is, there exists B with $AB = I_{n \times n} = BA$.

The jth column of A^{-1} is $v_j = A^{-1} e_j$,
where e_j is the j-th standard basis vector.

Thus, the j-th column is the solution of the system of equations $A v_j = e_j$, which is the special case of $Ax = b$.

To solve the system $Ax = b$, row operations on $(A|b)$ are performed.

Given another equation $Ax = c$, consider
$(A|\underline{bc})$.

In our case, the r.h.s is $e_1, \dots, e_n$

PROBLEM IN A TOWN WITH n INHABITANTS, THERE ARE N CLUBS.
EACH CLUB HAS AN ODD NUMBER OF MEMBERS.
ANY TWO DISTINCT CLUBS HAVE AN EVEN NUMBER OF JOINT MEMBERS.
PROVE THAT $n \geqslant N$.

SOLUTION.

INTRODUCE THE MATRIX $A \in M_{n \times N}(\mathbb{Z}_2)$.
WHERE $A_{ij} = \delta_{ij}$.
Every club has an odd number of members $\Rightarrow \sum_{i=1}^{n} A_{ij} = 1$.

TWO DISTINCT CLUBS HAVE AN EVEN NUMBER OF COMMON MEMBERS

$$\Rightarrow \begin{cases} \sum_{i=1}^{n} A_{ij} A_{ik} = 0 \\ \sum_{i=1}^{n} A_{ij}^2 = 1. \end{cases}$$

$$\Leftrightarrow \quad A^T A = I_{N \times N}$$

Thus, $N = \operatorname{rank}(I_{N \times N}) \leq \operatorname{rank} A \leq n$.

Thus, the augmented matrix in the form $(A|e_1,\dots,e_n)=(A|I)$ is obtained.

Perform row operations to make left side an identity matrix

$$(A|I) \rightsquigarrow (I|B) \Rightarrow B = A^{-1}.$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) =$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & -1 & 2 \end{array}\right) =$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1/3 & 2/3 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 0 & 0 & 2 & 0 & 4/3 & -2/3 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1/3 & 2/3 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 2/3 & -1/3 \\ 0 & 1 & 0 & 1 & -4/3 & 2/3 \\ 1 & 0 & 0 & 0 & -1/3 & 2/3 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/3 & 2/3 \\ 0 & 1 & 0 & 1 & -4/3 & 2/3 \\ 0 & 0 & 1 & 0 & 2/3 & -1/3 \end{array}\right)$$