# 1 Structural Induction

Let $p : S \to \{T, F\}$ be a recursively defined predicate.

To prove $\forall s \in S.p(s)$ by structural induction, prove:

- $p(s)$ for all base cases of the definition
- $p(s)$ for the construct cases of the definition, assuming $s$ is true for the components.

We can define functions on recursively defined sets.

**Example 1.1**

For $f \in M$, let $n(f) =$ "the number of occurrences of propositional variables in $f$".

Then $n(P) = 1$ for any propositional variable $P$, and $n(f) = n(f') + n(f'')$ for $f = (f' \text{ OR } f'')$ and $f = (f' \text{ AND } f'')$.

Let $B$ be the set of all binary trees.

**Base Case**

The empty tree is in $B$.

**Constructor Case**

If $t_1, t_2 \in B$ and $r$ is a node, then we say $t_1 = \text{left}(t)$ and $t_2 = \text{right}(t)$.

**Example 1.2**

For $t \in B$, let $N(t)$ be the number of nodes in $t$

**Base Case**

$N(\text{empty tree}) = 0$

**Constructor Case**

$N(t) = 1 + N(\text{left}(t)) + N(\text{right}(t))$

**Example 1.3**

Let $L(t)$ be the number of leaves in $t$.

**Base Case**

Then $L(\text{empty tree}) = 0$ and $L(\text{one node tree }) = 1$.

**Constructor Case**

$L(t) = L(\text{left(t)} + L(\text{right}(t))$

**Theorem 1.4**

A binary three with $n$ nodes has at most $\lceil n/2 \rceil$ leaves. Thus,

$$\forall t \in B.L(t) \leq \lceil N(t)/2 \rceil ,$$

or, equivalently,

$$\forall n \in \mathbb{N}.\forall t \in B.(N(t) = n \text{ IMPLIES } L(t) \leq \lceil n/2 \rceil$$

### Example 1.5

For $t \in B$ and $n \in \mathbb{N}$, let $S(t,n) =$ ``$t$ has $n$ nodes'' and $AL(t,n) =$ `` $t$ has at most $n$ leaves''.

Then $\forall n \in N.\forall t \in B.(S(t,n) \text{ IMPLIES } A(t, \lceil n/2 \rceil))$.

Let $p(n) =$ `` $\forall t \in B.(S(t,n) \text{ IMPLIES } A(t, \lceil n/2 \rceil)$ ''.

Let $n \in \mathbb{N}$ be arbitrary.

Suppose $\forall i \in \mathbb{N}.(i < n \text{ IMPLIES } p(i))$. Let $t \in B$ be arbitrary.

Suppose $S(t,n)$.

To prove $A(t, \lceil n/2 \rceil)$.

**Case 1:**

$n = 0$.

Then $t$ has 0 nodes and 0 leaves. Since $0 = \lceil n/2 \rceil$, then $A(t, 0)$ is true.

Thus, $A(t, 0)$ is true.

**Case 2:**

$n > 0$.

Then $t$ has a root, a left subtree $t'$ and a right subtree $t''$.

Let $n' = N(t')$, $S(t', n')$, $n'' = N(t''), S(t'', n'')$.

Then $n = n' + n'' + 1$, so $n', n'' < n$.

By specialisation of inductive hypothesis, $A(t', \lceil n'/2 \rceil$, $A(t'', \lceil n''/2 \rceil)$ are true.

Then

$$L(t) = L(t') + L(t'') \leq \lceil n'/2 \rceil + \lceil n''/2 \rceil \leq \frac{n'+1}{2} + \frac{n''+1}{2} = \frac{n+1}{2}$$

If $n$ is odd, $\frac{n+1}{2} = \lceil n/2 \rceil$.
If $n$ is even, then $\frac{n+1}{2} = \frac{n}{2} + \frac{1}{2}$ and thus $\lceil n/2 \rceil = \frac{n}{2}$.
Since $L(t) \leq \frac{n}{2} + \frac{1}{2}$ and $L(t) \in \mathbb{N}$, then $L(t) \leq \frac{n}{2} = \lceil n/2 \rceil$.

Then $A(t, \lceil n/2 \rceil)$. $p(n)$ by generalisation. $\forall n \in \mathbb{N}.p(n)$ by strong induction.

### Theorem 1.6

Every integer greater than 1 can be written as a product of primes.

*Proof.*

Suppose the claim is false.

Let $n$ be the smallest integer that cannot be written as a product of primes. Then $n$ is not prime, since any prime can be written as a trivial product of itself.

Therefore, $n$ is composite, so there exists $k, m \in \mathbb{N}$ such that$k > 1$ and $m > 1$ and $n = k \cdot m$, while $k$ and $m$ are smaller than $n$. Since they can be written as a product of prmies, $n$ is also a product of primes, which is a contradiction. $\square$

**Definition 1.7** (Well Ordering Principle)**.** Every nonempty subset of $\mathbb{N}$ has the smallest element.

**Example 1.8**

Let $p(n) =$ "n cannot be written as aproduct of primes". Let $C = \{n \in \mathbb{N} \mid p(n) \text{ is false}\}$, $C \neq \emptyset$.

Assume $[\forall n \in \mathbb{N}.p(n)]$ is false. Then by the Well-Ordering Principle there exists the smallest element in $C$.