20160927

MAT 240: VECTOR SPACES

DEFINITION

A VECTOR SPACE V OVER A FIELD F IS A SET WITH TWO BINARY OPERATIONS:

$+ : V \times V \longrightarrow V$ :: ADDITION

$\circ : F \times V \longrightarrow V$ :: SCALAR MULTIPLICATION

WITH A DISTINGUISHED ELEMENT $0 \in V$ SUCH THAT:

PROPERTIES

V1: $\forall x, y \in V: x+y = y+x$

V2: $\forall x, y, z \in V: (x+y)+z = x+(y+z)$

V3: $\forall x \in V: x+0 = x$

V4: $\forall x \in V \; \exists y \in V: x+y=0$

V5: $\forall a, b \in F, x \in V: a \circ (bx) = (a \cdot b)x$

V6: $\forall x \in V: 1 \circ x = x$

V7: $\forall a \in F \; \forall x, y \in V: a \cdot (x+y) = a \cdot x + a \cdot y$

V8: $\forall a, b \in F, x \in V: (a+b) \cdot x = a \cdot x + b \cdot x$

NOTE

$0 \in F \neq 0 \in V$, OR

$0_F \neq 0_V$

EXAMPLES

EXAMPLE 1

$F^n = \underbrace{F \times \dots \times F}_{n \text{ times}}$ IS A VECTOR SPACE OVER F. $0_{F^n} = \underbrace{(0, 0, \dots, 0)}_{n\text{-tuples}}$

ELEMENTS ARE IN TUPLES $(a_1, \dots, a_n)$, $a_1, \dots, a_n \in F$.

$+ : (a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots$

$a \cdot (a_1, \dots, a_n) = (a a_1, \dots, a a_n)$

NOTE

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = (a_1, \dots, a_n)$$

EXAMPLE 2

$M_{m \times n}(F)$: $m \times n$ ARE MATRICES WITH COEFFICIENTS IN F.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & & & \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

ARRAY OF NUMBERS, m ROWS, n COLUMNS

$$0_V = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & & & 0 \end{pmatrix}$$

EXAMPLE 3

LET S BE A SET. THEN

$\mathcal{F}(S,F)$ = SET OF FUNCTIONS

$f, g \longrightarrow F$ IS A VECTOR SPACE, WITH + AND $\cdot$ DEFINED AS:

$(f+g)(s) = f(s) + g(s)$

$(af)s = a \cdot f(s)$

$0_V = f(s)$

EXAMPLE 4

$\mathcal{P}(F)$ POLYNOMIALS WITH COEFFICIENTS IN

A SET OF FUNCTIONS $p(x) = \sum_1^n a_i x^i + a_0$

FOR SOME n, WITH $a_0, \ldots, a_n \in F$

$+, \cdot, 0$ ARE THOSE FROM $\mathcal{F}(F,F)$ (cf. Ex 3)

EXAMPLE 5

SEQUENCES OF ELEMENTS IN F.

$V = (a_1, a_2, \ldots) \mid a_1, a_2, \ldots, a_i \in F$

CONSIDER SEQUENCES SUCH THAT ONLY FINITELY MANY $a_1, a_2, \ldots a_i$ ARE NON-ZERO. GET VECTOR SPACE $V' \subset V$

EXAMPLE 6

$\mathbb{C}$ IS A VECTOR SPACE OVER $\mathbb{R}$

IN FACT, $\mathbb{C} = \mathbb{R}$

$\mathbb{R}$ IS A VECTOR SPACE OVER $\mathbb{Q}$

$\mathbb{C}$ IS A VECTOR SPACE OVER $\mathbb{Q}$

NOTION

IN GENERAL, IF $\mathbb{R}$ IS A FIELD CONTAINING F AS A SUBFIELD, THEN $\mathbb{R}$ IS A VECTOR SPACE OVER F.

DEFINITION

$F < K$ IS A SUBFIELD OF THE FIELD K IF F IS A SUBSET AND ALSO A FIELD, BUT WITH ADDITION, MULTIPLICATION, 0, 1 THOSE OF K.

EXERCISE

THE FIELD WITH 4 ELEMENTS CONTAINS $\mathbb{Z}_2$ AS A SUBFIELD.

PROPERTIES OF VECTOR SPACES V OVER F

a) CANCELLATION LAW

$\forall\, x, x', y \in V: x+y = x'+y \Rightarrow x = x'$

$\forall\, a \in F, x, x' \in V: ax = ax' \Rightarrow x = x'.$
$a \neq 0_F$

$\forall\, a, a' \in F, x \in V: ax = a'x \Rightarrow a = a'.$
$x \neq 0_V$

b) UNIQUENESS OF NEUTRAL ELEMENT

IF $0' \in V$ WITH $x + 0' = x$ FOR SOME $x \in V \Rightarrow 0' = 0$.

c) " "

$0_F\, x = 0_V \;\forall\; x \in V$

$a \cdot 0_V = 0_V \;\forall\, a \in F$

d) $(-a)x = -(a \cdot x) = a \cdot (-x) \; \forall \; a \in F \; x \in V$

e) $a \cdot x = 0_V \iff a = 0$ OR $x = 0_V$

$\longrightarrow$ LOOK INTO GROUPS

SUBSPACES OF VECTOR SPACES

A SUBSPACE W OF A VECTOR SPACE V (OVER FIELD F) IS A SUBSET $W \subset V, W \neq \emptyset$, WHICH IS ALSO A VECTOR SPACE OVER F WITH ADDITION AND SCALAR MULTIPLICATION AND NEUTRAL ELEMENT THOSE OF V.

THEOREM

A SUBSET $W \subset V$ OF A VECTOR SPACE IS A SUBSPACE IF AND ONLY IF, $W \neq 0$,

1. $x, y \in W \Rightarrow x + y \in W$
2. $a \in F, x \in W \Rightarrow ax \in W$.

PROOF

IF $W \subset V$ IS A SUBSPACE THEN BY DEFINITION OF SUBSPACE 1 AND 2 HOLD. CONVERSELY, IF 1 AND 2 HOLD, WTS THAT W WITH THOSE OPERATIONS OF $+$ AND $\cdot$ IS A VECTOR SPACE.

IF $x \in W$, THEN $y = -x$ SATISFIE $x + y = 0$, AND $y \in W$, SINCE $y = (-1)x \in W$ FROM 2.

FOR i IN $[1, 8] \setminus [4]$ HOLD FOR ELEMENTS OF W BECAUSE THEY HOLD FOR ELEMENTS OF V.

EXAMPLES

1) IF V IS ANY VECTOR SPACE, $\{0\}$, V ARE SUBSPACES.

2) $\mathcal{P}(\mathbb{F})$ POLYNOMIALS WITH COEFFICIENTS IN $\mathbb{F}$ IS A SUBSPACE OF $\mathcal{F}(F, F)$.

$\mathcal{P}_n(F)$: POLYNOMIALS OF DEGREE $\leq n$ IS A SUBSPACE OF $\mathcal{P}(F)$.

$\mathcal{P}_m(F) \subseteq \mathcal{P}_n(F)$ IS A SUBSPACE FOR $m \leq n$.

3) SEQUENCES OF FINITE LENGTH ARE SUBSPACE OF VECTOR SPACE OF ALL SEQUENCES $(a_1, a_2, a_{\ldots})$

4) FOR A $m \times n$ MATRIX $A \in M_{m \times n}(F)$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

DENOTE BY $A^T \in M_{n \times m}(F)$ SUCH THAT THERE IS A ONE-TO-ONE CORRESPONDENCE BETWEEN $a_{ij} \in A$ AND $a_{ji} \in A^T$ FOR ALL $i \in [1, m]$ AND $j \in [1, n]$.

SQUARE MATRICES: $V = M_{n \times n}(F)$ — HAVE SUBSPACES.

- SYMMETRIC MATRICES $(A^T = A)$
- SKEW-SYMMETRIC MATRICES $(A^T = -A)$
- DIAGONAL MATRICES (IE ONLY DIAGONAL ENTRIES ARE NON-ZERO)

* UPPER TRIANGULAR MATRICES

FOR $j \geq i$, $a_{ij} \neq 0$.

5) FOR $\mathbb{C}$ AS A VECTOR SPACE OVER $\mathbb{Q}$:

$R \subseteq \mathbb{C}$