Problem. Let $T_4 : P(\mathbb{R}) \to P(\mathbb{R}), p \mapsto T(p)$.
Suppose that $(T(p))(x) = p''(x) + 2xp'(x) - p(x)$.
Find the matrix of T relative to the standard ordered basis $1, x, \ldots, x_4$ of $P4(\mathbb{R})$.

Solution. Note the following:

$$\begin{aligned}
(x^4)' &= 4x^3 & (x^4)'' &= 12x^2 \quad (1)\\
(x^3)' &= 3x^2 & (x^3)'' &= 6x \quad (2)\\
(x^2)' &= 2x & (x^2)'' &= 2 \quad (3)\\
x' &= 1 & x'' &= 0 \quad (4)\\
(1)' &= 0 & (1)'' &= 0 \quad (5)
\end{aligned}$$

Since a linear transformation can be determined uniquely by its action on a basis of the domain, its action on each individual vector in the basis can be considered to find the matrix representation of the map:

$$(T(1))(x) = 0 + 2x \cdot 0 - 1 = -1 \tag{6}$$
$$(T(x))(x) = 0 + 2x - x = x \tag{7}$$
$$(T(x^2))(x) = 2 + 4x^2 - x^2 = 2 + 3x^2 \tag{8}$$
$$(T(x^3))(x) = 6x + 6x^3 - x^3 = 6x + 5x^3 \tag{9}$$
$$(T(x^4))(x) = 12x^2 + 8x^4 - x^4 = 12x^2 + 7x^4 \tag{10}$$

Thus,

$$[T]_\beta = \begin{bmatrix} -1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 3 & 0 & 12 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

$\square$