Theorem. *Let $\mathbb{F} = \{a + i\sqrt{3}b \mid a, b \in \mathbb{Q}\}$ with the addition and multiplication defined as for $\mathbb{C}$. Then $\mathbb{F}$ is a field.*

Proof. Suppose $a, b, c, d \in \mathbb{Q}$.

Denote the element $a + i\sqrt{3}b$ of $\mathbb{F}$ as $\mathbb{F}(a, b)$.

Define $sm(a, b)$ as $a^2 + 3b^2$.

1. Since $a + c \in \mathbb{Q}$ and $b + d \in \mathbb{Q}$ by Multiplicative Closure of $\mathbb{Q}$, as well as $a + c = c + a$ and $b + d = d + b$ by Commutative Law for $\mathbb{Q}$, then $\mathbb{F}(a, b) + \mathbb{F}(c, d) = \mathbb{F}(c, d) + \mathbb{F}(a, b)$. Moreover, since $\mathbb{F} \subset \mathbb{C}$ by definition, $\mathbb{F}(a, b)\mathbb{F}(c, d) = \mathbb{F}(c, d)\mathbb{F}(a, b)$.

2. Since $\mathbb{F} \subset \mathbb{C}$, and $\mathbb{C}$ has an associative property, $\mathbb{F}$ obeys the Associative Law both for $+$ and $\cdot$.

3. Consider $\mathbb{F}(0, 0)$. Then $\forall a, b \in \mathbb{Q}$, since $\sqrt{3} \cdot 0 = 0$, then $\mathbb{F}(a, b) + \mathbb{F}(0, 0) = \mathbb{F}(a, b)$. Therefore, $\mathbb{F}(0, 0)$ is an additive neutral element. Similarly, since $\mathbb{F} \subset \mathbb{C}$ and $0 + i \cdot 0$ is a multiplicative neutral element for $\mathbb{C}$, $\mathbb{F}(0, 0)$ is also a multiplicative neutral element.

4. Consider $\mathbb{F}(a, b)$ and $\mathbb{F}(-a, -b)$. Since $a - a = 0$ and $b - b = 0$, $\mathbb{F}(a, b) + \mathbb{F}(-a, -b) = \mathbb{F}(0, 0)$. Therefore, there exists an additive inverse $\forall a, b$.

   Consider also $\mathbb{F}(a, b)$ and $\frac{\mathbb{F}(a,-b)}{sm(a,b)}$. Since $sm(a, b)$ is a sum of two rational numbers, then $sm(a, b) \in \mathbb{Q}$. Therefore, $\frac{\mathbb{F}(a,-b)}{sm(a,b)} = \mathbb{F}(\frac{a}{sm(a,b)}, -\frac{b}{sm(a,b)}) \in \mathbb{F}$. Since $\mathbb{F}(a, -b)$ is the complex conjugate of $\mathbb{F}(a, b)$ in $\mathbb{C}$, then $\mathbb{F}(a, b)\mathbb{F}(\frac{a}{sm(a,b)}, -\frac{b}{sm(a,b)}) = 1$ and $(\mathbb{F}(a, b))^{-1} = \frac{\mathbb{F}(a,-b)}{sm(a,b)}$.

5. Since $\mathbb{F} \subset \mathbb{C}$, elements of $\mathbb{F}$ obey the Distributive Law.

$\square$