1 Problem

Problem. Suppose a matrix A is given:

$$A = \begin{pmatrix} 6 & 3 & -4 \\ -1 & 1 & 1 \\ 3 & 2 & -1 \end{pmatrix}$$

Find a cycle basis (as defined in the lectures) and a Jordan canonical form of A.

Solution.

First we find a characteristic polynomial:

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{pmatrix} 6-\lambda & 3 & -4 \\ -1 & 1-\lambda & 1 \\ 3 & 2 & -1-\lambda \end{pmatrix} && (1)\\
&= (6-\lambda)((1-\lambda)(-1-\lambda) - 2) && (2)\\
&\quad + (3(-1-\lambda) + 8) && (3)\\
&\quad + 3(3 + 4(1-\lambda)) && (4)\\
&= (6-\lambda)((\lambda-1)(\lambda+1) - 2) && (5)\\
&+ \ (-3\lambda + 5) && (6)\\
&+ \ 3(-4\lambda + 7) && (7)\\
&= (6-\lambda)(\lambda^2 - 3) - 3\lambda + 5 - 12\lambda + 21 && (8)\\
&= -\lambda^3 + 6\lambda^2 + 3\lambda - 18 - 15\lambda + 26 && (9)\\
&= -\lambda^3 + 6\lambda^2 - 12\lambda + 8 && (10)\\
&= -\lambda^3 + 3 \cdot 2\lambda^2 - 3 \cdot 4\lambda + 2^3 && (11)\\
&= -(\lambda - 2)^3 && (12)
\end{aligned}$$

Thus,

$$\ker(A - 2I) = \ker \begin{pmatrix} 4 & 3 & -4 \\ -1 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix}. \tag{13}$$

Suppose that $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an eigenvector. Hence,

$$\begin{cases} 4x + 3y - 4z & = 0 \\ -x - y + z & = 0 \\ 3x + 2y - 3z & = 0 \end{cases}$$

Note that

$$\left[\begin{array}{ccc|c} 4 & 3 & -4 & 0 \\ -1 & -1 & 1 & 0 \\ 3 & 2 & -3 & 0 \end{array}\right] \tag{14}$$

$$R_3 - R_2 \to R_3 \rightsquigarrow \left[\begin{array}{ccc|c} 4 & 3 & -4 & 0 \\ -1 & -1 & 1 & 0 \\ 4 & 3 & -4 & 0 \end{array}\right] \tag{15}$$

$$R_1 - R_3 \to R_3 \rightsquigarrow \left[\begin{array}{ccc|c} 4 & 3 & -4 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{16}$$

$$R_1 + 4R_2 \to R_2 \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{17}$$

Hence, $y = 0$ and $x = z$, which means that $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ spans E_2. Since E_2 is one-dimensional, there is only one Jordan block corresponding to $\lambda = 2$ by Corollary to Theorem 7.9, and the dot diagram has only one column.

Note that

$$(A - 2I)^2 = \begin{pmatrix} 4 & 3 & -4 \\ -1 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix} \begin{pmatrix} 4 & 3 & -4 \\ -1 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \tag{19}$$

Hence, if $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an eigenvector in $\ker(A - 2I)^2$, then $x + y = z$.

We are trying to find x, y, z such that $(A - 2I)u = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Therefore, $4x + 3y - 4z = 1$, $-x - y + z = 0$, $3x + 2y - 3z = 1$.

Thus, $12x + 9y - 12z = 3$ nd $12x + 8y - 12z = 4$.

Therefore, $y = -1$, and thus $z = x - 1$.

Take $u = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Now take an orthogonal vector to u, for example, $w = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Let $\beta = \{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\}$. If β is linearly independent, then it is a cycle basis, because $\dim V = 3$. We prove now that it is linearly independent.

Suppose that there exist a_1, a_2 and a_3 such that the linear combination of the corresponding vectors in β is 0. Then

$$\begin{cases} a_1 + a_2 + 0a_3 & = 0 \\ 0a_1 + a_2 + a_3 & = 0 \\ a_1 + 0a_2 + a_3 & = 0 \end{cases}.$$

From the second and third equation we obtain that $a_1 = -a_3$. From the third and first equation we get that $a_2 = a_3$. From the second equation we get that $a_3 = 0$, hence $a_1 = a_2 = a_3 = 0$, and thus β is linearly independent.

Moreover,

$$(A - 2I)^3 = \begin{pmatrix} 4 & 3 & -4 \\ -1 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \tag{20}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{21}$$

and thus w is an generalised eigenvector of A.

Therefore, β is a cycle basis. $\square$