LEMMA I: $\lim_{x \to a} |x| = |a|$

PROOF:

BY DEFINITION OF $|x|$,

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0. \\ 0, & x = 0 \end{cases}$$

CLAIM I: THE INTERVAL $(0, +\infty)$ DOES NOT CONTAIN ITS GLB $= 0$.

PROOF: SUPPOSE $\lambda \in (0, +\infty)$ AND $\lambda = GLB(0, +\infty)$.

$\Rightarrow 0 < \lambda \Leftrightarrow 0 < \frac{\lambda}{2} < \lambda \Rightarrow \frac{\lambda}{2} \in (0, +\infty)$. #

CLAIM II: THE INTERVAL $(-\infty, 0)$ DOES NOT CONTAIN ITS LUB.

PROOF: SUPPOSE $\gamma \in (-\infty, 0)$ AND $\gamma = LUB(-\infty, 0)$.

$\gamma < 0 \Leftrightarrow \gamma < \frac{\gamma}{2} < 0 \Rightarrow \frac{\gamma}{2} \in (-\infty, 0)$ #.

FROM CLAIM I II ABOVE, IF $x > 0$, $a > 0$,

$\forall \varepsilon > 0 \; \exists \delta > 0 : |x - a| < \delta \Rightarrow |x - a| < \varepsilon$,

SINCE $a > 0$ ($|a| > 0$), AND ε CAN BE SET EQUAL TO δ. THUS

SIMILARLY,
FROM CLAIM 2 ABOVE, IF $x<0$, $a<0$, THEN

$$\forall(\varepsilon>0)\exists(\delta>0): 0<|a-x|<\delta \Rightarrow |a-(-x)|<\varepsilon,$$

since $a<0$ ($\Rightarrow -|a|<0$) AND
ε CAN BE SET EQUAL TO δ SO THAT

$$|(-x)-|a||<\varepsilon.$$

THUS, IF $x<0$, $a<0$, THEN $\lim_{x\to a}|x|=|a|$, AND

IF $x>0$, $a>0$, THEN $\lim_{x\to a}|x|=|a|$.

CONSIDER $f(x)=-x$, $g(x)=|x|$, $h(x)=x$

since $f(x)$ AND $h(x)$ are polynomials,

$\lim_{x\to 0} f(x)=f(0)=0=h(0)=\lim_{x\to 0}h(x)$. ① NOTE

THAT BY DEFINITION OF $|x|$, $\forall x\in\mathbb{R}$

$f(x)\le g(x)\le h(x)$. Thus by the

Squeeze theorem, AND ①, $\lim_{x\to 0} g(x)=0$.

Hence, $|x|$ is CONTINUOUS EVERYWHERE ON ITS DOMAIN. □

COROLLARY TO LEMMA I : IF $f(x)$ IS CONTINUOUS ON SOME INTERVAL, THEN $|f(x)|$ IS ALSO CONTINUOUS ON THIS INTERVAL.

PROOF : BY THE PROPERTIES OF LIMITS ALREADY PROVEN, AND FROM LEMMA I, IF $g(x) = |x|$, $\therefore g \circ f(x)$ IS CONTINUOUS ON THE GIVEN INTERVAL.