MAT240: 20161006

LINEAR INDEPENDENCE & A BASIS OF A VECTOR SPACE

NOTATION

$\#A$ is CARDINALITY OF A	$\#\emptyset = 0$
$A \setminus B = \{x \in A \mid x \notin B\}$	$\#\{0\} = 1$
	$\#\mathbb{Z}_k = k$

$$\#(A \cup B) + \#(A \cap B) = \#A + \#B$$

$$\#(A \times B) = (\#A)(\#B)$$

FOR VECTORS $v_1, \dots, v_k$, CONSIDER

(*) $a_1 v_1 + \dots + a_k v_k = 0, \quad a_i \in \mathbb{F}$.

$v_1, \dots, v_k$ LINEARLY INDEPENDENT $\iff$ (*) HAS ONLY A TRIVIAL SOLUTION.

DEFINITION

A SUBSET β OF V IS A BASIS OF V IF

(1) β LINEARLY INDEPENDENT

(2) $\text{SPAN}(\beta) = V$

EXERCISE: (1) $V = F^n$, THEN $\beta = \{e_1, \dots, e_n\}$

WITH $e_1 = (1, 0, \dots, 0)$ "STANDARD BASIS"

$= (0, 1, \dots, 0)$

$\dots$

$= (0, 0, \dots, 0)$

(2) $V = \mathcal{P}(F)$, $F = \mathbb{R}, \mathbb{C}, \mathbb{Q}$

$\beta = \{1, x, x^2, \dots\}$ IS A BASIS

"STANDARD BASIS"

(3) VECTOR SPACE OF SEQUENCES

$(a_1, a_2, \dots)$ $a_i \in \mathbb{F}$

$e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, 0, \dots)$

$e_3 = (0, 0, 1, 0, \dots)$

$\beta = \{e_1, e_2, e_3, \ldots\}$ IS NOT A BASIS OF F^∞.

SPAN (β) IS THE SUBSPACE OF FINITE SEQUENCES.

NOTE F^∞ HAS A BASIS (NOT OBVIOUS) BUT CAN'T BE WRITTEN DOWN

(4) $V = \mathbb{R}$ AS A VECTOR SPACE OVER $\mathbb{Q}$, $\beta = ?$

THEOREM EVERY VECTOR SPACE V HAS A BASIS.

THEOREM LET V BE A VECTOR SPACE OVER F, AND $\beta \subseteq V$.

β IS A BASIS $\Leftrightarrow$ EVERY $v \in V$ IS UNIQUELY A LINEAR COMBINATION OF ELEMENTS IN β

NOTE UNIQUENESS IS UP TO RE-ARRANGEMENT OR ADDING ZEROS

PROOF ($\Leftarrow$) SUPPOSE EVERY v IS UNIQUELY A LINEAR COMBINATION OF ELEMENTS OF β.

IN PARTICULAR, SPAN $(\beta) = V$. SUPPOSE

$a_1 v_1 + \ldots + a_k v_k = 0$ WITH $\{v_i\}_{i=1}^{k} \in \beta$, $\{a_i\}_{i=1}^{k} \in F$.

BY UNIQUENESS OF 0

HENCE $a_1 = \ldots = a_k \Rightarrow$ LINEARLY INDEPENDENT

$\Rightarrow \beta$ IS A BASIS.

THEOREM

LET V BE A VECTOR SPACE OVER F. SUPPOSE THERE IS $S \subseteq V$, WITH $\# S$ FINITE, AND $\text{SPAN}(S) = V$. THEN THERE IS A SUBSET β OF S WHICH IS A BASIS OF V.

PROOF

IF $S = \emptyset$ OR $S = \{0\}$, THEN $V = \{0\}$ AND $\beta = \emptyset$ IS A BASIS.

IF NOT, THERE EXISTS $v_1 \in S$ WITH $v_1 \neq 0$. THEN $\{v_1\}$ IS LINEARLY INDEPENDENT.

IF $\text{SPAN}(v_1) = V$ WE'RE DONE.

IF NOT, THEN THERE MUST EXIST $v_2 \in S$ WITH $v_2 \notin \text{span}\{v_1\}$, FOR IF $S \subseteq \text{SPAN}\{v_1\}$, THEN $\text{SPAN}(S) \subseteq \text{SPAN}(v_1)$, CONTRADICTION.

THEN $\{v_1, v_2\}$ IS LINEARLY INDEPENDENT (BY THM FROM LAST TIME).

IF $\text{SPAN}\{v_1, v_2\} = V$, WE'RE DONE.

IF NOT, $\exists\, v_3 \in S$, $v_3 \notin \text{SPAN}\{v_1, v_2\}$, ETC.

SINCE S IS FINITE, THIS PROCESS EVENTUALLY STOPS AND WE GET $\{ \ldots v_5 \} \subseteq S$.

EXAMPLE

$V = \mathbb{R}^3$

$S = \{(1,0,0), (0,1,0), (2,2,0), (1,1,1)\}$

$\text{SPAN}(S) = \mathbb{R}^3$,

$\{v_1\} \to \{v_1, v_2\} \to \{v_1, v_2, v_4\}$

$(\Rightarrow)$ Suppose β is a basis.

Let $v \in V \Rightarrow v = a_1 v_1 + \dots a_k v_k$

with $v_i \in \beta$, $a_i \in F$,

because $V = \text{span}(\beta)$.

Suppose also $v = a_1' v_1' + \dots + a_l' v_l'$

with $v_i' \in \beta$, $a_i' \in F$.

$\Rightarrow$ By adding $0 \cdot v_j$ if necessary, it can be assumed that

$\{v_1, \dots, v_k\} = \{v_1', v_2', \dots, v_l'\}$.

WLOG, assume $v_1 = v_1'$,

$\dots$,

$v_k = v_k'$.

$$\Rightarrow \sum_{i=1}^{k} \left((a_k' - a_k) v_k \right) = 0$$

Since β is linearly independent,

$\bigodot_{i=1}^{k} \left(a_i = a_i' \right)$.