1 Sequences

Consider a function $f(x) = e^{-x}\cos(2x+1)$.

We can use the Squeeze Theorem to show that $\lim_{x\to\infty} f(x) = 0$.

Therefore, $\lim_{n\to\infty} a_n = 0$.

Theorem 1.1

If $a_n = f(n)$, for some function $f(x)$ and $lim_{x\to\infty}f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

Example 1.2

Consider $a_n = a^n$ for some real constant a.

If $a > 0$, define $f(x) = a^x = e^{x\log a}$.

Since $\lim_{x\to\infty} e^{x\log a} = \exp(\lim_{x\to\infty} x\log a) = \begin{cases} 0, & \text{if } \log a < 0 \\ \infty, & \text{if } \log a > 0 \end{cases}$, then

$$\lim_{n\to\infty} a_n = \begin{cases} 0, & \text{if } 0 < a \le 1 \\ \infty, & \text{if } a > 1 \end{cases}$$

.

If $a < 0$,then $a_n = (-1)^n|a|^n$, and hence 0 if $|a| < 1$. If $|a| > 1$ or $a = -1$, then the sequence diverges.

Example 1.3

Suppose that $f(x) = \sin(\pi x)$. Then $a_n = f(n) = 0$, and thus $\lim_{x\to\infty} a_n = 0$, but $\lim_{x\to\infty} \sin x$ does not exists.

A sequence is said to be bounded if $|a_n| < M$ and bounded above if $a_n < M$.

A sequence is said to be strictly increasing if $a_n < a_{n+1}$ and nondecreasing if $a_n \le a_{n+1}$.

Theorem 1.4

If $\{a_n\}$ is bounded and nondecreasing, then $\{a_n\}$ converges.

Proof.

Let $L = \sup a_n$.

Suppose some $\epsilon > 0$ is given.

Therefore, there exists M such that $L - a_M < \epsilon$.

But for any $n \ge M$, we have that $a_n \ge M$. So $L - a_n \le L - a_m < \epsilon$, which means that $\lim_{n\to\infty} a_n = L$. $\square$

Suppose now that $\lim_{x\to\alpha} f(x) = L$.

If $\{a_n\}$ is a sequence, then the domain of $f(x)$ so that $a_n \neq \alpha$ for any $n \in \mathbb{R}$ and $\lim_{x\to\infty} a_n = \alpha$, then $\lim_{n\to\alpha} f(a_n) = L$.

More is true. If $\lim_{n\to\infty} f(a_n) = L$ for all sequences of the stated type, then $\lim_{x\to\alpha} f(x) = L$.

Note that we can always find a nondecreasing or a nonincreasing subsequence.

Theorem 1.5

A bounded sequence always has a convergent subsequence.

Proof.

Pick a nondecrueasing or nonincreasing subsequence. Since it is bounded, we know that it has a limit. □

2 Series

In our current framework, $\sum_{i=1}^{\infty}$ does not make sense.

We define the nth partial sumas $S_n = \sum_{i=1}^{n} a_n$.

If the sequence $\{s_n\}$ converges, say to S, then we say that $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n = S$.

2.1 Geometric Series

Remember that $\sum_{n=0}^{\infty} r^n = \frac{1-r^{n+1}}{1-r}$.

Do these partial sums tend to a limit?

If $|r| < 1$, then $r^{n+1} \to 0$.

So $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ converges.

When $r = 1$ and $r = -1$, it diverges.