1 Applications of the Fundamental Theorem of Calculus

Corollary 1.1

If f is continuous on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a).$$

Proof. Let $F(x) = \int_a^x f$. Then $F' = f = g'$ on $[a, b]$. Thus, there is a number $c \in \mathbb{R}$ such that

$$F = g + c$$

Note that $F(a) = g(a) + c = 0$ by definition of F, and thus $c = -g(a)$ $\square$

Theorem 1.2 (The Second Fundamental Theorem of Calculus)

If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a).$$

Proof. Let $P = \{t_0, t_1, \ldots, t_n\}$ be any partition of $[a, b]$.
By the Mean Value Theorem, there is a point x_i in $[t_{i-1}, t_i]$ such that

$$g(t_i) - g(t_{i-1}) = g'(x_i)(t_i - t_{i-1}) = f(x_i)(t_i - t_{i-1})$$

If $m_i = \inf\{f(x) \,|\, t_{i-1} \leq x_i \leq t_i\}$ and $M_i = \sup\{f(x) \,|\, t_{i-1} \leq x_i \leq t_i\}$, then

$$m_i(t_i - t_{i-1}) \leq f(x_i)(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1}),$$

and thus

$$m_i(t_i - t_{i-1}) \leq g(t_{i-1}) - g(t_i) \leq M_i(t_i - t_{i-1}),$$

giving

$$\sum_{i=1}^n m_i(t_i - t_{i-1}) \leq g(b) - g(a) \leq \sum_{i=1}^n M_i(t_i - t_{i-1}).$$

Therefore, $L(f, P) \leq g(b) - g(a) \leq U(f, P)$ for every partition P, which means that

$$\int_a^b f = g(b) - g(a)$$

$\square$