MAT 240: 20161110

ROW OPERATIONS

THE FOLLOWING MOVES DO NOT CHANGE THE SOLUTION SET OF THE SYSTEM OF LINEAR EQUATIONS:

- INTERCHANGING TWO EQUATIONS
- MULTIPLY AN EQUATION BY A NON-ZERO SCALAR
- ADDING SCALAR MULTIPLES OF ONE EQUATION TO ANOTHER EQUATION.

(*) (**)

$$\begin{cases} \sum_{j=1}^{n} A_{1j} = b_1 \\ \cdots \\ \sum_{j=1}^{n} A_{mj} = b_m \end{cases} \qquad \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

CONVENIENT TO WORK WITH

$$\left(\begin{array}{ccc|c} A_{11} & \cdots & A_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ A_{m1} & \cdots & A_{mn} & b_m \end{array}\right)$$

EXAMPLE

$$\left(\begin{array}{ccc|c} 1 & -4 & 6 & 10 \\ 0 & -2 & 1 & 5 \\ 4 & -11 & 11 & 12 \end{array}\right) =$$

$$= \left(\begin{array}{ccc|c} 1 & -4 & 6 & 10 \\ 0 & -2 & 1 & 5 \\ 0 & 5 & -13 & -28 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{124}{21} \\ 0 & 1 & 0 & -37/21 \\ 0 & 0 & 1 & -\frac{37}{21} \end{array}\right)$$

MORE ABOUT GAUSS ELIMINATIONS LATER

THEOREM $A \in M_{n \times n}(\mathbb{F})$, AND THAT A' IS OBTAINED FROM A BY ELEMENTARY ROW OPERATIONS. THEN $A' = PA$ WHERE $P \in M_{m \times m}(\mathbb{F})$ IS AN INVERTIBLE MATRIX.

IN PART, P IS OBTAINED BY APPLYING THE SAME ROW OPERATIONS TO THE IDENTITY.

EXAMPLES.

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\end{pmatrix} \rightsquigarrow \begin{pmatrix}A_{21}&A_{22}&A_{23}\\A_{11}&A_{12}&A_{13}\end{pmatrix}$$

$$\begin{pmatrix}a&0\\0&1\end{pmatrix}\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\end{pmatrix} = \begin{pmatrix}aA_{11}&aA_{12}&aA_{13}\\A_{21}&A_{22}&A_{23}\end{pmatrix}$$

$$\begin{pmatrix}1&0\\c&1\end{pmatrix}\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\end{pmatrix} = \begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}+cA_{11}&A_{22}+cA_{12}&A_{23}+cA_{13}\end{pmatrix}$$

IDEA OF PROOF. ROW OPERATIONS CAN BE REGARDED AS A CHANGE OF BASIS IN F^m.

Consider the linear map $T: F^n \to F^m$.

$$\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} \longmapsto A\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$$

Then $A = [T]_\beta^\gamma$ for standard basis of F^n, F^m. $A' = [T]_\beta^{\gamma'}$ with new basis γ'
β_E F^m.

$$I = [I]_\gamma^{\gamma'}\,[T]_\beta^\gamma$$

ELEMENTARY COLUMN OPERATIONS

- INTERCHANGING THE COLUMNS
- MULTIPLYING COLUMN BY NON-ZERO SCALAR
- ADDING MULTIPLES OF ONE COLUMNS TO OTHERS

WARNING

FOR LINEAR SYSTEM OF EQUATIONS, THESE DO CHANGE THE SOLUTION

REMARK

A VIEWED AS A COLUMN OPERATOR, CHANGING THE BASIS OF THE COLUMN, WHICH MEANS THAT THE BASES OF THE DOMAIN IS ALSO CHANGED.

THEOREM

SUPPOSE $A' \in M_{m \times n}(F)$ IS OBTAINED FROM A BY AN ELEMENTARY COLUMN VECTOR.

THEN

$M = MQ$, WHERE $Q \in M_{n \times n}(F)$ IS INVERTIBLE, AND IS OBTAINED BY APPLYING SOME COLUMN OPERATION TO THE IDENTITY MATRIX

RECALL. RANK $(T) = \dim(\mathcal{R}(T))$

FOR $A \in M_{m \times n}(F)$, DEFINE

RANK (A) = RANK OF THE CORRESPONDING

LINEAR MAP $F^n \longrightarrow F^m$.

THM: ROW AND COLUMN OPERATIONS DO NOT CHANGE THE RANK OF A AND SO CAN BE USED TO FIND THE RANK OF A.