1 Introduction to Markov Processes

Suppose we have a two-state Markov chain, and there is a rabbit in one of the states waiting and hopping with the time distributed exponentially. Thus, $\mathbb{P}(\text{waiting time} > t) = e^{-\beta t}$.

What is the probability that by the time t the rabbit will hop from the state 0 to the state 1?

By the law of total probability we obtain:

$$p_{01}(t) = \int_0^t \beta e^{-\beta s} p_{11}(t-s)\,\mathrm{d}s.$$

Similarly,

$$p_{10}(t) = \int_0^t \delta e^{-\delta s} p_{00}(t-s)\,\mathrm{d}s.$$

To proceed, we recall the definition of the Laplacian transformation.

Suppose $\phi : [0, +\infty) \to \mathbb{R}$. Then:

$$(\mathcal{L}\phi)(\lambda) = \int_0^{+\infty} e^{-\lambda s}\phi(s)\,\mathrm{d}s.$$

We state that $\mathcal{L}\phi(\lambda)$ uniquely defines ϕ.

Thus, we can write:

$$\Phi(t) = \int_0^t A(s)B(t-s)\,\mathrm{d}s.$$

How can we write $\mathcal{L}\Phi$ in terms of $\mathcal{L}A$ and $\mathcal{L}B$?

To do this, we introduce a substitution $u = t - s$:

$$\mathcal{L}\Phi = \int_0^{+\infty}\int_0^{+\infty} -e^{-\lambda u} e^{\lambda t} A(s)B(u)dsdu.$$

Note that $\mathcal{L}(\beta e^{-\beta s}) = \frac{\beta}{\beta+\lambda}$.

Therefore, taking the Laplacian transformation of the system we have seen before,

$$\begin{cases} \frac{1}{\lambda}\mathcal{L}p_{00}(\lambda) = \frac{\beta}{\beta+\lambda}\mathcal{L}p_{11}(\lambda) \\ \frac{1}{\lambda}(\lambda) = \frac{\delta}{\delta+\lambda}p_{00}(\lambda). \end{cases}$$

1.1 Derivation of Kolmogorov's Equations

See also the Kolmogorov-Chapman inequality.

Let $p_{ij}(t)$ be a Markov semigroup which is continuous at 0.

Note that $p_{ij}(t+s) = \sum_{k\in S} p_{ik}(t)p_{kj}(s)$.

Moreover,

$$p_{ii}(t+s) \geq p_{ii}(t)p_{ii}(s),$$

and $\lim_{t\to 0}\frac{1-p_{ii}(t)}{t} = C(i) = -q_{ii}$, which means that the chain is stable.

We know that $p_{ii}(t) + \sum_{j\in S\setminus i} p_{ij}(t) = 1$.

Suppose that i is a stable state. We can deduce that $\forall j \limsup_{t\to 0}\frac{p_{ij}(t)}{t} < +\infty$.

Note also that $p_{ii}(t) \geq e^{-c(i)t}$, and

$$c(i) = -q_{ii} = \lim_{t\to 0}\frac{1-p_{ii}(t)}{t}.$$

Moreover,

$$p_{ij}(n\delta) \geq \sum_{k=0}^{n-1}(p_{ii}(\delta))^k p_{ij}(\delta)p_{jj}((n-k-1)\delta),$$

and

$$\frac{p_{ij}(n\delta)}{n\delta} \geq \frac{p_{ij}(\delta)}{\delta}e^{-c(i)n\delta}.$$

Therefore, $p_{ii}(k\delta) \geq (p_{ii}(\delta))^k$.

Take $n\delta = t$. Thus, $q_{ij} = \limsup_{t\to 0}\frac{p_{ij}(t)}{t}$.

Furthermore, $\frac{p_{ij}(t)}{t} \geq q_{ij}e^{-c(i)t}\inf p_{jj}(\tau)$.

Hence, $\liminf_{t\downarrow 0}\frac{p_{ij}(t)}{t} = q_{ij}$.

Since $1-p_{ii}(t) = \sum_{j\in S\setminus i} p_{ij}(t)$, we obtain that $-q_{ii} \geq \sum_{j\in S\setminus i} q_{ij}$. The proof is left as an exercise.

The state is called regular if $q_{ii} = -\sum_{j\in S\setminus i} q_{ij}$.

Now we are ready to prove the Kolmogorov backward equation.

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}p_{ij}(t) = \sum_{k\in S} q_{ik}p_{kj}(t).$$

It is worthwhile to note that the state is non-istantaneous and regular.

Now,

$$\frac{p_{ij}(t+h)-p_{ij}(t)}{h} = \sum_{k\in S}\frac{p_{ik}(h)-\delta_{ik}}{h}p_{kj}(t),$$

which follows directly from the Kolmogorov-Chapman Equations.

Also,

$$\frac{p_{ij}(t+h)-p_{ij}(t)}{h} - \sum q_{ik}p_{kj}(t) = \sum_{k\in S}\frac{p_{ik}(h)-q_{ik}}{h}p_{kj}(t),$$

and there exists such N that $N\subset S$ is finite and

$$\sum_{k\in S\setminus\{N\}}\left(\frac{p_{ik}(h)}{h} - q_{ik}\right)p_{kj}(t) = \sum_{k\in S}\frac{p_{ik}(h)-q_{ik}}{h}p_{kj}(t).$$

Now, we compute

$$\frac{\sum_{k\in S\setminus N} p_{ik}(h)}{h} + \frac{\sum_{l\in N\setminus\{i\}} p_{il}(h)}{h} + \frac{1-p_{ii}(h)}{h} = 0.$$

Moreover, $\frac{\sum_{l\in N\setminus\{i\}} p_{il}(h)}{h} = \sum_{l\in N\setminus\{i\}} q_{il} + d_2(h) + q_{ii} + d_1(h)$.

Note that $\left|\frac{\sum_{l\in N\setminus\{i\}} p_{il}(h)}{h}\right| < \epsilon$ if h is small enough.

1.2 Forward Kolmogorov Equation

It can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t) = \sum_{k\in S} p_{ik}(t) q_{kj}.$$

Thus,

$$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \sum_{k\in S} p_{ik}(t) \left(\frac{p_{kj}(h) - \delta_{kj}}{h}\right).$$

Differentiating both sides term by term, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t) = \sum_{k\in S} p_{ik}(t) q_{kj}.$$