(i) HERE AND FURTHER IN (i) ASSUME $x \neq 1$.

1. SUPPOSE $\varepsilon > 0$ IS GIVEN.
2. LET $\delta = \min\{1, \frac{\varepsilon}{4}\}$

3.1. Suppose $\delta = 1 \leq \frac{\varepsilon}{4} \Rightarrow \varepsilon \geq 4$.

$\Rightarrow \quad |x-1| < 1 \iff 0 < x < 2 \quad (*)$

$\Rightarrow \quad |x-1||x+2| < |x+2| \quad (x \neq -2)$

$\Rightarrow \quad |x^2 + x + 1 - 3| < |x+2|$

$\iff \quad \left| \frac{x^3-1}{x-1} - 3 \right| < |x+2| \quad$ (SINCE $x \neq 1$)

But FROM $(*)$ $2 < x+2 < 4 \Rightarrow |x+2| < 4 \leq \varepsilon$

$\Rightarrow \quad \left| \frac{x^3-1}{x-1} - 3 \right| < \varepsilon.$

2/2.

3.2.

SUPPOSE $\delta = \frac{\varepsilon}{4} \leq 1 \Rightarrow \varepsilon \leq 4$.

$\Rightarrow \quad |x-1| < \frac{\varepsilon}{4}$. SINCE $\delta < 1$, $2 < x+2 < 4 \Rightarrow |x+2| < 4$.

$\Rightarrow \quad |x+2||x-1| < \varepsilon, \ x \neq 1$

$\iff \quad \left| \frac{x^3-1}{x-1} - 3 \right| < \varepsilon$

$\Rightarrow \quad \lim_{x \to 1} \frac{x^3-1}{x-1} = 3 \quad \square$

1. SUPPOSE $\varepsilon > 0$ IS GIVEN.

2. LET $\delta = \min\{1, (\sqrt{3}+2)\varepsilon\}$

3.1. SUPPOSE $\delta = 1 \leq (\sqrt{3}+2)\varepsilon \implies \varepsilon \geq \frac{1}{2+\sqrt{3}}$.

$$\implies |x-4| = |(\sqrt{x}-2)(\sqrt{x}+2)| = |\sqrt{x}-2||\sqrt{x}+2| < 1.$$

$$\implies |\sqrt{x}-2| < \frac{1}{\sqrt{x}+2}.$$ MOREOVER,

$$|x-4| < 1 \implies 3 < x < 5.$$

$$\sup_{3<x<5}\left(\frac{1}{\sqrt{x}+2}\right) = \frac{1}{\sqrt{3}+2}$$

$$\implies |\sqrt{x}-2| < \frac{1}{\sqrt{x}+2} < \frac{1}{\sqrt{3}+2} \leq \varepsilon.$$

3.2. SUPPOSE $\delta = (\sqrt{3}+2)\varepsilon \leq 1 \implies \varepsilon \leq \frac{1}{\sqrt{3}+2}$.

2/2

$$\implies |x-4| < (\sqrt{3}+2)\varepsilon \leq 1 \implies 3 < x < 5 \;(*)$$

$$\implies |\sqrt{x}-2||\sqrt{x}+2| < (\sqrt{3}+2)\varepsilon.$$ FROM $(*)$,

$$\sqrt{3}+2 < \sqrt{x}+2 \iff \frac{1}{\sqrt{x}+2} < \frac{1}{\sqrt{3}+2} \quad (\sqrt{x}+2 > 0 \;\forall x \in \mathbb{R})$$

$$\implies |\sqrt{x}-2| < \varepsilon.$$

$$\implies \lim_{x\to 4}\sqrt{x} = 2$$

$\square$

2 (iii)

CLAIM (B)

$$\lim_{x \to a} \left(\frac{x^2 - 2}{x^2 + a^2} \right) = \frac{a^2 - 2}{2a^2}, \quad a \neq 0.$$

$\iff$ (C2) $\lim_{x \to a} (x^2 - 2) \cdot \lim_{x \to a} (x^2 + a^2)^{-1} = \frac{a^2 - 2}{2a^2}$ BY MULT. LAW FOR LIMITS.

LEMMA (L1) ☆ SPIVAK

$$\lim_{x \to a} f(x) g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

SUBLEMMA (SL)

$$\left(|x - x_0| < \min\left\{ 1, \frac{\varepsilon}{2(|y_0| + 1)} \right\} \right) \wedge \left(|y - y_0| < \frac{\varepsilon}{2(|x_0| + 1)} \right)$$

$$\Rightarrow \quad |xy - x_0 y_0| < \varepsilon$$

PROOF (SL):

$$|x - x_0| < 1 \Rightarrow |x| - |x_0| \leq |x - x_0| < 1,$$

$$\Rightarrow |x| < 1 + |x_0|$$

$$\begin{aligned} \Rightarrow |xy - x_0 y_0| &= |x(y - y_0) + y_0 (x - x_0)| \\ &\leq |x| |y - y_0| + |y_0| \cdot |x - x_0| \\ &< (1 + |x_0|) \cdot \frac{\varepsilon}{2(|x_0| + 1)} + |y_0| \cdot \frac{\varepsilon}{2(|y_0| + 1)} \\ &< \varepsilon \end{aligned}$$

PROOF (L). DENOTE $\ell = \lim_{x \to a} f(x)$, $m = \lim_{x \to a} g(x)$

SUPPOSE WE ARE GIVEN $\varepsilon > 0$.

$\Rightarrow \exists \delta_1, \delta_2 > 0 : \left(0 < |x-a| < \delta_1 \Rightarrow |f(x) - \ell| < \min\left\{1, \frac{\varepsilon}{2(|m|+1)}\right\}\right)$

$\wedge \left(0 < |x-a| < \delta_2 \Rightarrow |g(x) - m| < \frac{\varepsilon}{2|\ell|+1}\right)$.

Let $\delta = \min(\delta_1, \delta_2)$.

By (SL),

$|f(x)\, g(x) - \ell m| < \varepsilon$ $\square$

SUBLEMMA II (SL2)

IF $y_0 \neq 0$ AND $|y - y_0| < \min\left\{\frac{|y_0|}{2}, \frac{\varepsilon |y_0|^2}{2}\right\}$,
THEN $y \neq 0$ AND $\left|\frac{1}{y} - \frac{1}{y_0}\right| < \varepsilon$.

PROOF (SL2)

$|y_0| - |y| \leq |y - y_0| < \frac{|y_0|}{2} \Rightarrow |y| > \frac{|y_0|}{2}$.

IF $y \neq 0$, THEN $\frac{1}{|y|} < \frac{2}{|y_0|} \Rightarrow \left|\frac{1}{y} - \frac{1}{y_0}\right| = \frac{|y_0 - y|}{|y||y_0|} < \frac{2}{|y_0|} \cdot \frac{1}{|y_0|} \cdot \frac{\varepsilon |y_0|^2}{2} = \varepsilon$ $\square$

LEMMA II (L2)

$\lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{m}$

PROOF (L2)

PROOF (L2)

SUPPOSE ... GIVEN $m \neq 0$.

SUPPOSE $\varepsilon > 0$ IS GIVEN SUCH THAT THERE IS $\delta > 0$:

$$0 < |x - a| < \delta \Rightarrow |g(x) - m| < \min\left(\frac{|m|}{2}, \frac{\varepsilon |m|^2}{2}\right).$$

$\Rightarrow$ $g(x) \neq 0$ AND $\left|\frac{1}{g(x)} - \frac{1}{m}\right| < \varepsilon$ BY (SL2) $\square$

BY (L1) AND (L2), (C2) IFF (C1).

CONSIDER $f(x) = x^2 + c$, $c \in \mathbb{R}$.

SUPPOSE $\varepsilon > 0$ IS GIVEN.

Let $\delta = \min\left\{1, \frac{\varepsilon}{2|a|+1}\right\}$.

THEN
$$\begin{cases} -1 < x - a < 1 \iff a - 1 < x < a + 1. \\ |x - a| < \frac{\varepsilon}{2|a|+1} \Rightarrow \end{cases}$$

Since $|x+a| \leq |x| + |a| \leq 1 + |a| \cdot 2$ (FROM $|x| - |a| \leq |x - a| < 1$),

THEN $|x - a||x + a| < \varepsilon$

2/2.

$\iff$ $|x^2 - a^2| < \varepsilon$

$\iff$ $|x^2 + c - (a^2 + c)| < \varepsilon$.

→ try to learn how to do this w/out the lemmas!

THEREFORE,
$$\begin{cases} \lim_{x \to a} x^2 - 2 = a^2 - 2 \\ \lim_{x \to a} x^2 + a^2 = a^2 + a^2 = 2a^2 \end{cases} \Rightarrow \lim_{x \to a} \left(\frac{x^2 - 2}{x^2 + a^2}\right) = \frac{a^2 - 2}{2a^2}$$

(iv)

1. SUPPOSE $\varepsilon > 0$ IS GIVEN.

TAKE $\delta = \min\left(1, \frac{4\varepsilon}{7}\right)$.

THUS, $|x-3| < \delta \implies 2 < x < 4$.

MOREOVER, $|x-3| < \frac{4\varepsilon}{7}$,

$\iff \frac{7}{4}|x-3| < \varepsilon$.

Since $|x+3| < 7$,

AND $5 - \sin(7x) \geq 4 \quad \forall (x \in \mathbb{R})$ (since $\sin(7x) \in [-1, 1]$),

THEN $\frac{1}{4} \geq \frac{1}{|5-\sin(7x)|}$.

THEREFORE, $\frac{7}{4} > \frac{|x+3|}{|5-\sin(7x)|}$

$$\implies \frac{|x+3|}{|5-\sin(7x)|}|x-3| < \frac{7}{4}|x-3| < \varepsilon$$

2/2.

$$\iff \left|\frac{x^2-9}{5-\sin(7x)}\right| < \varepsilon.$$

THUS, $\lim_{x \to 3} \frac{x^2-9}{5-\sin(7x)} = 0$

(2)

CLAIM:

(i) $\lim_{x \to 0} \left(\frac{x^3-1}{x-1} \right) = 1$

PROOF.

HERE AND FURTHER $x \neq 1$.

1. SUPPOSE $\varepsilon > 0$ IS GIVEN.
2. TAKE $\delta = \min\{1, \frac{\varepsilon}{2}\}$.

THEREFORE, $|x| < 1 \Leftrightarrow -1 < x < 1 \Rightarrow \quad 0 < x+1 < 2$

MOREOVER, $|x| < \frac{\varepsilon}{2}$.

SINCE $|x+1| < 2$, $\quad |x+1||x| < 2|x| < \varepsilon$

$\Leftrightarrow \quad |x^2+x+1-1| < 2|x| < \varepsilon$.

2|2

Since $x \neq 1$, $\left| \frac{x^3-1}{x-1} - 1 \right| < 2|x| < \varepsilon$.

$\Rightarrow \left| \frac{x^3-1}{x-1} - 1 \right| < \varepsilon \Rightarrow \lim_{x \to 0} \frac{x^3-1}{x-1} = 1$.

□

(ii) CLAIM:

$\lim_{x \to -3} \left(\frac{x+3}{x^2-9} \right) = -\frac{1}{6}$

PROOF:

SUPPOSE $x \neq 3$ AND $x \neq -3$.

2|2

LET $\varepsilon > 0$ BE GIVEN.

TAKE $\delta = \min\{1, 30\varepsilon\}$.

$\Rightarrow |x+3| < 1$, $(-4 < x < -3) \vee (-3 < x < -2)$ (*)

MOREOVER, $|x+3| < 30\varepsilon \Leftrightarrow \frac{|x+3|}{6} \cdot \frac{1}{5} < \varepsilon$.

FROM (*), $|x-3| > 5 \Leftrightarrow \frac{1}{5} > \frac{1}{|x-3|}$

$\Rightarrow \frac{|x+3|}{6|x-3|} < \frac{1}{5} \frac{|x+3|}{6} < \varepsilon$.

$\Rightarrow \left| \frac{x+3}{(x+3)(x-3)} - \left(-\frac{1}{6}\right) \right| = \left| \frac{6+x-3}{6(x-3)} \right| < \varepsilon$

THUS, $\lim_{x \to -3} \left(\frac{x+3}{x^2-9} \right) = -\frac{1}{6}$ □

2. (iii) CLAIM:

$$\lim_{x \to -5} \frac{1}{(x+5)^2} = l \text{ is FALSE } \forall l \in \mathbb{R}.$$

PROOF:

SUPPOSE SUCH AN l EXISTS.

$$\Rightarrow \text{"} \forall \varepsilon > 0 \ \exists \delta > 0 : |x+5| < \delta \Rightarrow \left| \frac{1}{(x+5)^2} - l \right| < \varepsilon. \quad (1)$$

Take δ' SUCH THAT $|x+5| < \delta' < \delta$.

$$\Rightarrow \left| \frac{1}{(x+5)^2} - l \right| < \varepsilon, \text{ WHENEVER } |x+5| < \delta'.$$

$$\Rightarrow \frac{1}{\delta^2} < \frac{1}{|x+5|^2}, \quad \frac{1}{(\delta')^2} < \frac{1}{|x+5|^2} \quad \text{AND} \quad \frac{1}{(\delta')^2} > \frac{1}{\delta^2}.$$

RESTRICT δ' FURTHER SO THAT

$$\frac{1}{(\delta')^2} \geq \frac{1}{\delta^2} + l + \varepsilon$$

2/2.

$$\Leftrightarrow (\delta')^2 \leq \frac{1}{\frac{1}{\delta^2} + l + \varepsilon} = \frac{\delta^2}{1 + (l+\varepsilon)\delta^2}$$

(since $\left| \frac{1}{|x+5|^2} - l \right| < \varepsilon$, $0 < \frac{1}{|x+5|^2} < \varepsilon + l$, AND

HENCE $0 < \frac{\delta^2}{(x+5)^2} < \delta^2(\varepsilon + l)$ AND $1 < 1 + \delta^2(\varepsilon + l)$,

THEN $(\delta')^2 \leq \frac{\delta^2}{1+(l+\varepsilon)\delta^2} < \delta^2$ AND (1) STILL HOLDS,

since $\frac{\delta}{\sqrt{1+(l+\varepsilon)\delta^2}} < \delta$ ($l + \varepsilon > 0$, $\delta^2 > 0 \Rightarrow$ THE EXPR. INSIDE $\sqrt{\cdot}$ IS > 0)).

Thus, $\frac{1}{(\delta')^2} > \frac{1}{(x+5)^2}$, WHICH IS A CONTRADICTION.

#

(iv) CLAIM:

$$\lim_{x\to 0}\left(x(x^2-5)\,(2+\cos((x+1)^2-2))\right)=0.$$

PROOF:

SUPPOSE $\varepsilon>0$.

TAKE $\delta=\min\{1,\frac{\varepsilon}{15}\}$.

SINCE $|x|<\delta$, $-1<x<1$. (*)

MOREOVER, $|x|<\frac{\varepsilon}{15} \iff 15|x|<\varepsilon$.

FROM (*), $|x^2-5|\leq 5$

SINCE $\cos(A)\leq 1 \ \forall A\in\mathbb{R}$,

$$|2+\cos(A)|\leq 3$$

$$\Rightarrow \quad |2+\cos(x^2+2x-1)|\,|x^2-5|\leq 15$$

2/2.

$$\Rightarrow \quad |x|\,|2+\cos(x^2+2x-1)|\,|x^2-5|\leq 15|x|<\varepsilon$$

THUS, $$\lim_{x\to 0}\left(x(x^2-5)(2+\cos((x+1)^2-2))\right)=0$$

□

③ LET $f(x)=\frac{1}{x^n}$, $n\in\mathbb{Z}$.

CLAIM:

$\lim_{x\to 0} f(x)=\ell$ IS FALSE $\forall \ell\in\mathbb{R}$

PROOF:

SUPPOSE $\varepsilon>0$ IS GIVEN, AND A LIMIT EXISTS:

$\Rightarrow \exists\,\delta>0: |f(x)-\ell|<\varepsilon$.

TAKE δ' s.t. $|x|<\delta'<\delta \Rightarrow$ For δ', $|f(x)-\ell|<\varepsilon$.

RESTRICT δ' FURTHER: $(\delta')^n\leq \dfrac{1}{\frac{1}{\delta^n}+|\ell|+\varepsilon}=\dfrac{\delta^n}{1+(|\ell|+\varepsilon)\delta^n}$ ①

3 (CONT.)

Since $0 < |x| < \delta$, $0 < |x|^n < \delta^n \iff 0 < \left|\frac{1}{f(x)}\right|^n < \delta^n$.

Similarly, $0 < \left|\frac{1}{f(x)}\right|^n < (\delta')^n$.

Moreover, $|f(x) - \ell| + |\ell| \geq |f(x)|$

$\Rightarrow$ FROM $|f(x) - \ell| < \varepsilon$, $|\ell| + \varepsilon > |f(x)|$.

$$\Rightarrow (|\ell| + \varepsilon)\delta^n > \frac{1}{|x|^n}\delta^n$$

$$\Rightarrow \frac{1}{1 + (|\ell| + \varepsilon)\delta^n} < \frac{1}{1 + \frac{\delta^n}{|x|^n}} = \frac{|x|^n}{|x|^n + \delta^n}$$

$$\Rightarrow \text{FROM } ①, \quad (\delta')^n < \frac{|x|^n \delta^n}{|x|^n + \delta^n} < \frac{\delta^n \cdot \delta^n}{|x|^n}.$$

IF $\delta' > 1$, CONSTRUCT δ'' SUCH THAT $(\delta'') < \delta'$ FROM THE PROCESS ABOVE. BY THE ABOVE INEQUALITIES, δ'' SUCH THAT $(\delta'')^n < \frac{(\delta')^n}{2}$ CAN BE FOUND. CONTINUE UNTIL $\delta^{[k+1]} < \delta^{[k]}$ ARE FOUND SUCH THAT $\delta^{[k]} < 1$.

THUS, TWO DELTAS, Δ' AND Δ, CAN BE FOUND, SUCH THAT $\Delta' < \Delta$.

IF $\delta < 1$, SET $\Delta' = \delta'$ AND $\Delta = \delta$.

$$\Rightarrow (\Delta')^n < \frac{\Delta^{2n}}{|x|^n} < \frac{1}{|x|^n},$$ WHICH IS A CONTRADICTION.

SINCE $\lim_{x \to c} c = c$, SETTING $c = 0$ WE GET 5/5.

$\lim_{x \to 0} 0 = 0$. But FROM THE SCALAR AND SUM LAWS

FOR THE LIMITS, WE GET $\lim_{x \to 0} \left(\frac{1}{x^n} - \frac{1}{x^n} \right) = \lim_{x \to 0} (0) = 0$,

WHICH SATISFIES THE CONDITIONS.

(4) ONE OF THE DOWNSIDES OF DEFINING A LIMIT ON AN OPEN INTERVAL ONLY IS THAT

$f(x) = \sqrt{x}$, which is real FOR $x \geq 0$,

HAS NO LIMIT BY DEFINITION AT $x = 0$,

SINCE THERE IS NO REAL NUMBER

TO WHICH $f: \mathbb{R} \to \mathbb{R}$ MAPS $|x| < \varepsilon$ 5/5

FOR $-\varepsilon < x < 0$ AND $\varepsilon > 0$.

ON THE OTHER HAND,

$f(x^2) = \sqrt{|x^2|} = |x|$, which has the limit of 0:

Suppose $\varepsilon > 0$.

Consider $\delta = \varepsilon \Rightarrow$ FOR $x \geq 0$, $x < \varepsilon \Leftrightarrow f(x) < \varepsilon$.

THUS, $-x > \varepsilon < -x$, i.e. $-\varepsilon < f(x)$ for $x < 0$.

$\Rightarrow |x| < \varepsilon \Rightarrow f(x) < \varepsilon$

$\square$