Problem.

Suppose that $A \in M_{n\times n}(\mathbb{F})$ is such that its characteristic polynomial splits.

Then A and A^t have the same Jordan canonical form and $A \sim A^t$.

Solution.

Let $A \in M_{n\times n}(\mathbb{F})$ be such that its characteristic polynomial $f(t)$ splits.

Note that for any $\lambda \in \mathbb{F}$, $(A - \lambda I)^t = A^t - \lambda I^t = A^t - \lambda I$.

Therefore, $\det(A^t - \lambda I) = \det(A - \lambda I)$, and thus the characteristic polynomial of A^t is $f(t)$.

Thus, A^t has the same eigenvalues as A.

Let K_λ be a generalised eigenspace of A corresponding to an eigenvalue λ and K'_{λ_i} be a generalised eigenspace of A^t corresponding to λ.

Note that $((A - \lambda I)^r)^t = ((A - \lambda I)^t)^r = (A^t - \lambda I)^r$, where the first equality holds since for any P and Q in $M_{n\times n}(\mathbb{F})$ we know that $(PQ)^t = Q^tP^t$ and the second follows from the discussion above.

For any matrix $C \in M_{n\times n}(\mathbb{F})$, we know that $\operatorname{rank} C = \operatorname{rank} C^t$.

Therefore, $\operatorname{rank}(A - \lambda I)^r = \operatorname{rank}((A - \lambda)^r)^t = \operatorname{rank}(A^t - \lambda I)^r$ for any $r \in \mathbb{N}$.

In particular, $\operatorname{rank}(A^t - \lambda I) = \operatorname{rank}(A - \lambda I)$, and thus from the rank-nullity theorem we obtain that $\operatorname{nullity}(A^t - \lambda I) = \operatorname{nullity}(A - \lambda I)$, which means that the dot diagrams of $A^t - \lambda I$ and $A - \lambda I$ have the same number of columns. Moreover, we can deduce that for $r \geq 2$, we have

$$\operatorname{rank}(A - \lambda I)^{r-1} - \operatorname{rank}(A - \lambda I)^r = \operatorname{rank}(A^t - \lambda I)^{r-1} - \operatorname{rank}(A^t - \lambda I)^r,$$

and thus each row in the dot diagrams of $(A - \lambda I)$ and $(A^t - \lambda I)$ have the same number of dots, which means that $(A - \lambda I)$ and $(A^t - \lambda I)$ have the same dot diagram.

Since this dot diagram corresponds to a unique Jordan canonical form (up to reordering of blocks), we infer that A and A^t have the same JCF.

Denote this JCF as J.

Since by the change of basis formula there exist invertible matrices B and B' such that $A = B^{-1}JB$ and $A^t = B'^{-1}JB'$, we deduce that $J = BAB^{-1} = B'A^tB'^{-1}$.

Thus, $A = B^{-1}B'A^tB'^{-1}B$. Since $(B'^{-1}B)^{-1} = B^{-1}B'$, taking $Q = B'^{-1}B$ we see that $A = Q^{-1}A^tQ$, and thus $A \sim A^t$. $\square$