1 Existence and Uniqueness of Jordan Canonical Form

Theorem 1.1

If a characteristic polynomial $f(t)$ of T splits, then

a) $V = \bigoplus_{i=1}^{r} K_{\lambda_i}$, where λ_i for $i \in [1, r] \cap \mathbb{N}$ are distinct eigenvalues.

b) $\dim K_\lambda = m_\lambda$ for any eigenvalue λ.

Proof.

First we show that $V = \sum_{i=1}^{r} K_{\lambda_i}$ by induction on r.

Let $W = \operatorname{im}(T - \lambda I)^{m_\lambda}$.

Since $\dim V = \dim K_{\lambda_1} + \dim W$, while $K_{\lambda_1} \cap W = \{0\}$,

then $\dim(K_{\lambda_1} + W) = \dim(K_{\lambda_1}) + \dim(W) = \dim V$, and thus $V = K_{\lambda_1} \oplus W$.

Note hat W is T-invariant and T_W has eigenvalues $\lambda_2, \ldots, \lambda_r$.

If $x \in W$, $x = (T - \lambda I)^{m_\lambda}(y)$ for some $y \in V$.

Therefore, $Tx = T(T - \lambda I)^{m_\lambda}(y) = (T - \lambda I)^{m_\lambda} T(y) \in W$.

Thus, W is T-invariant.

If $Tv = T|_W(v) = \mu v$ and $v \neq 0$, then μ is an eigenvalue of T, then μ is an eigenvalue of T, and hence $\mu \in \{\lambda_1, \ldots, \lambda_r\}$. If $\mu = \lambda_1$, then $v \in W \cap K_{\lambda_1} = \{0\}$ by the previous remark.

The generalised eigenspaces of $T|_W$ are $K_{\lambda_2}, \ldots, K_{\lambda_r}$. Notice that $K_\lambda^W = K_\lambda \cap W$, where K_λ^W is

By Theorem 7.1(b), since $(T - \lambda_1 I)^{m_{\lambda_1}} : K_{\lambda_i} \to K_{\lambda_i}$ for all $i \neq 1$, then $K_{\lambda_i} \subseteq \operatorname{im}((T - \lambda_1 I)) = W$ for all $i \neq 1$, and therefore $K_{\lambda_i}^W = K_{\lambda_i} \cap W = K_{\lambda_i}$ for all $i \neq 1$.

Thus, $K_{\lambda_1}^W = \{0\}$.

Now we apply the inductive hypothesis to $T_W : W \to W$ (the characteristic polynomial splits and it has $r - 1$ eigenvalues).

Then $W = \sum_{i=2}^{r} K_{\lambda_i}^W = \sum_{i=2}^{r} K_{\lambda_i}$, and hence $V = K_{\lambda_1} + W = \sum_{i=1}^{r} K_{\lambda_i}$.

Since $V = \sum_{i=1}^{r} K_{\lambda_i}$ and $\dim V \leq \sum_{i=1}^{r} \dim K_{\lambda_i} \leq \sum_{i=1}^{r} m_i = \dim V$,

and thus $\dim V = \sum_{i=1}^{r} \dim K_{\lambda_i}$ and $\dim K_{\lambda_i} = m_i$, which means that $V = \bigoplus_{i=1}^{r} K_{\lambda_i}$. $\square$

Now we can fin a nice basis for each K_λ separately.

If $x \in K_\lambda$ and $x \neq 0$, there is a smallest $l \geq 1$ such that $(T - \lambda I)^l x = 0$.

We call a set $\{(T - \lambda I)^{l-1} x, (T - \lambda I)^{l-2} x, \ldots, (T - \lambda I)x, x\}$ a **cycle of generalised eigenvectors** corresponding to λ of length l. Let's call $(T - \lambda I)^{l-1}$ an *initial vector* and x an *end vector*.

The initial vector is in $N(T - \lambda I) = E_\lambda$, and hence it is an eigenvector for λ.

Theorem 1.2

If γ is a basis of V which is a disjoint union of cycles γ_i for $1 \le i \le r$ of generalised eigenvectors, let $W_i = \text{span}(\gamma_i)$.

a) W_i is T-invariant and $[T_{W_i}]_{\gamma_i}$ is a Jordan block.

b) $[T]_\gamma$ is in JCF.

Proof.

a) Fix i.

Suppose
$$\gamma_i = \{(T - \lambda I)^{l-1}(x), \ldots, (T - \lambda I)x, x\}.$$

Note that $W_i = \text{span}(\gamma_i)$, but $\gamma_i \subseteq \gamma$, so γ_i is linearly independent and thus a basis of W_i.

Let $v_j = (T - \lambda I)^{l-j}x$ for $1 \le j \le l$.

We know that

$$(T - \lambda J)v_j = (T - \lambda J)^{l-j+1}(x) \tag{1}$$
$$= (T - \lambda J)^{l-(j-1)}x \tag{2}$$
$$= \begin{cases} v_{j-1} \text{ if } j > 1 \\ 0 \text{ if } j = 1 \end{cases} \tag{3}$$

Therefore,
$$Tv_j = \begin{cases} \lambda v_j + v_{j-1}, \text{ if } j > 1 \\ \lambda v_j, \text{ if } j = 1 \end{cases}.$$

So $Tv_j \in W_i$ for all j, and thus W_i is T-invariant and $[T_{W_i}]_{\gamma_i}$ is a Jordan block.

b) Note that, by definition, $\gamma = \bigcup_{i=1}^r \gamma_i$. The matrix representation $[T]_\gamma$ has Jordan blocks on a diagonal, and thus $[T]_\gamma$ is in a Jordan Canonical Form.

$\square$

Theorem 1.3

Suppose $\gamma_1, \ldots, \gamma_r$ are cycles of generalised eigenvalues corresponding to the **same** eigenvalue λ.

If the initial vectors are linearly independent, then the sets γ_i are disjoint and $\gamma = \bigcup_{i=1}^r \gamma_i$ is linearly independent.

Proof.

Let $W = \text{span}\,\gamma$. From Theorem 1.2, W is T-invariant.

Let $U = T - \lambda I : W \to W$.

Note that $\gamma_i = \{U^{l_i - 1}x_i, \ldots, Ux_i, x_i\}$.

We proceed by induction on the number of vectors $l_1 + \cdots + l_r$.

If $\sum_{i=1}^r l_i = 1$, there is a one-dimensional cycle which is linearly independent trivially.

Suppose $U^{l_1-1}(x_1), \ldots, U^{l_r-1}x_r$, which are all in $E_\lambda = \ker(U)$, are linearly independent, and then $\dim\ker(U) \geq r$.

On the other hand, $\gamma_i' = \{U^{l_i-1}x_i, U^2x_i, Ux_i\}$ is a cycle of length $\lambda_i - 1$ contained in $\operatorname{im} U$.

The total number of vectors is r fewer than before, so we can apply induction to $\gamma_1', \ldots, \gamma_r'$. Therefore,

$\bigcup_{i=1}^r \gamma_r'$ is a linearly independent disjoint union.

Therefore, $\dim\operatorname{im}(U) \geq \sum_{i=1}^r (l_i - 1) = -r + \sum_{i=1}^r l_i$.

Hence, by the dimension theorem,

$$d = \dim\operatorname{im} U + \dim\ker U \geq (\sum_{i=1}^r \lambda_i - r) + r \tag{4}$$

$$= \sum_{i=1}^r l_i \geq |\gamma| \geq \dim W = d. \tag{5}$$

Thus, the equality holds, so $|\gamma| = \sum l_i$, and thus γ is a disjoint union.

Therefore, $|\gamma| = \dim W$ and thus γ is a basis of W, and thus it is linearly independent. $\square$