Administrativia: discussed problem 2 with Ming Feng Wan, no extra material consulted

Problem I

(1) $\boxed{\text{Let } f \in \mathcal{F} \text{ and be arbitrary.}}$

(2) $\boxed{\text{Let } g \in \mathcal{F} \text{ and be arbitrary.}}$

(3) $\boxed{\forall n \in \mathbb{Z}^+.(f(n) = 2^{\frac{1}{n}} \text{ AND } g(n) = 2f(n))}$ — Assumption

(4) $\boxed{\text{Let } n \in \mathbb{Z}^+ \text{ be arbitrary.}}$

(5) $\boxed{f(n) = 2^{\frac{1}{n}} \text{ AND } g(n) = 2f(n))}$ — Specialization, (3)-(4)

(6) $f(n) = 2^{\frac{1}{n}}$ — Use of Conjunction, (5)

(7) $\log_2 f(n) = \frac{1}{n}$ — Definition of log, (6)

(8) $g(n) = 2f(n)$ — Use of Conjunction,(5)

(9) $\log_2 g(n) = 1 + \frac{1}{n}$ — Definition of log, (7),(6)

(10) $\boxed{\text{Let } c \in \mathbb{R}^+ \text{ be arbitrary.}}$

(11) $\boxed{\text{Let } b \in \mathbb{Z}^+ \text{ be arbitrary.}}$

(12) $\boxed{\text{Let } m \in \mathbb{Z}^+ \text{ be such that } m \geq \max\{\frac{1}{c}, b\}}$

(13) $m \geq \frac{1}{c}$ AND $m \geq b$ — Construction, (12)

(14) $\boxed{m \geq \frac{1}{c}}$ — Use of Conjunction, (13)

(15) $m > \frac{1-c}{c}$ — Well-Ordering of $\mathbb{R}$, (14)

(16) $c > \frac{1}{m} - \frac{c}{m}$ — Multiplication by $\frac{c}{m} \in \mathbb{R}^+$, (15)

(17) $c + \frac{c}{m} > \frac{1}{m}$ — Addition of $\frac{c}{m} \in \mathbb{R}^+$, (16)

(18) $c(1 + \frac{1}{m}) > \frac{1}{m}$ — Distributive Law of $\mathbb{R}$, (17)

(19) $\log_2 f(m) = \frac{1}{m}$ — Specialisation, (7)

(20) $\log_2 g(m) = 1 + \frac{1}{m}$ — Specialisation, (9)

(21) $\log_2 f(m) < c \cdot \log_2 g(m)$ — Substitution, (18), (19), (20)

(22) $\boxed{m \geq b}$ — Use of Conjunction, (13)

(23) $m \geq b$ AND $\log_2 f(m) < c \cdot \log_2 g(m)$ — Proof of Conjunction, (21), (22)

(24) $\exists n \in \mathbb{Z}^+.$
$(n \geq b \text{ AND } \log_2 f(n) < c \cdot \log_2 g(n))$ — Proof by Construction, (12)-(23)

(25) $\forall b \in \mathbb{Z}^+.\exists n \in \mathbb{Z}^+.$
$(n \geq b \text{ AND } \log_2 f(n) < c \cdot \log_2 g(n))$ — Generalisation, (11), (24)

(26) $\forall c \in \mathbb{R}^+ . \forall b \in \mathbb{Z}^+ . \exists n \in \mathbb{Z}^+ .$ Generalisation, (10), (25)
$(n \geq b \text{ AND } \log_2 f(n) < c \cdot \log_2 g(n))$

(27) NOT $[\exists c \in \mathbb{R}^+ . \exists b \in \mathbb{Z}^+ . \forall n \in \mathbb{Z}^+ .$ Tautology, (26)
$(n \geq b \text{ IMPLIES } \log_2 f(n) \geq c \cdot \log_2 g(n))]$

(28) $\log_2 f \notin \Omega(\log_2 g)$ Definition of Ω, (27)

(29) $[\forall n \in \mathbb{Z}^+ . (f(n) = 2^{\frac{1}{n}} \text{ AND } g(n) = 2f(n))]$ Proof of Implication, (3), (28)
IMPLIES $\log_2 f \notin \Omega(\log_2 g)$

(30) $\forall g \in \mathcal{F} . \big([\forall n \in \mathbb{Z}^+ . (f(n) = 2^{\frac{1}{n}} \text{ AND } g(n) = 2f(n))]$ Generalisation, (2)
IMPLIES $\log_2 f \notin \Omega(\log_2 g)\big)$

(31) $\forall f \in \mathcal{F} . \forall g \in \mathcal{F} .$ Generalisation, (1)
$\big([\forall n \in \mathbb{Z}^+ . (f(n) = 2^{\frac{1}{n}} \text{ AND } g(n) = 2f(n))]$
IMPLIES $\log_2 f \notin \Omega(\log_2 g)\big)$

Problem II

(1) Let $f \in \mathcal{F}$ and be arbitrary.

(2) Let $g \in \mathcal{F}$ and be arbitrary.

(3) $f \in \Omega(g)$ AND $\forall n \in Z^+ . (f(n) \geq 2 \text{ AND } g(n) \geq 1)$ Assumption

(4) $f \in \Omega(g)$ Use of Conjunction, (3)

(5) $\exists c \in \mathbb{R}^+ . \exists b \in \mathbb{Z}^+ . \forall n \in \mathbb{Z}^+ .$ Definition of Ω, (4)
$(n \geq b \text{ IMPLIES } f(n) \geq c \cdot g(n))$

(6) Let $s \in \mathbb{R}^+$ and $z \in \mathbb{Z}^+$ be such that $\forall n \in \mathbb{Z}^+ .$ Instantiation, (5)
$(n \geq z \text{ IMPLIES } f(n) \geq s \cdot g(n))$

(7) Let $m \in \mathbb{Z}^+$ be such that $m \geq z$.

(8) $\forall n \in \mathbb{Z}^+ (n \geq z \text{ IMPLIES } f(n) \geq s \cdot g(n))$ Construction, (6)

(9) $m \geq z \text{ IMPLIES } f(m) \geq s \cdot g(m))$ Specialisation, (7), (8)

(10) $m \geq z$ Construction, (7)

(11) $f(m) \geq s \cdot g(m)$ Modus Ponens, (9), (10)

(12) $\log_2 f(m) \geq \log_2 s + \log_2 g(m)$ Increasing log, (11)

(13) $\forall n \in Z^+ . (f(n) \geq 2 \text{ AND } g(n) \geq 1)$ Use of Conjunction, (3)

(14) $f(m) \geq 2 \text{ AND } g(m) \geq 1$ Specialisation, (7), (13)

(15) $f(m) \geq 2$ Use of Conjunction, (14)

(16)	$\log_2 f(m) \geq 1$	Increasing log, (15)
(17)	$g(m) \geq 1$	Use of Conjunction, (14)
(18)	$\log_2 g(m) \geq 0$	Increasing log, (15)
(19)	$m \geq z$ IMPLIES $\log_2 f(m) \geq \log_2 s + \log_2 g(m)$	Proof of Implication, (10), (12)
(20)	$\forall n \in \mathbb{Z}^+.(n \geq z$ IMPLIES $\log_2 f(n) \geq \log_2 s + \log_2 g(n))$	Generalisation, (6), (7), (19)
(21)	$\boxed{\forall c \in \mathbb{R}^+.\forall b \in \mathbb{Z}^+.\exists n \in \mathbb{Z}^+.(n \geq b \text{ AND } \log_2 f(n) < c\log_2 g(n))}$	Assumption
(22)	$\forall b \in \mathbb{Z}^+.\forall c \in \mathbb{R}^+.\exists n \in \mathbb{Z}^+.(n \geq b$ AND $\log_2 f(n) < c\log_2 g(n))$	Tautology, (21)
(23)	$\forall n \in \mathbb{Z}^+. \log_2 g(n) = 0$	Assumption
(24)	$\forall c \in \mathbb{R}^+.\forall b \in \mathbb{Z}^+.\exists n \in \mathbb{Z}^+.(n \geq b$ AND $\log_2 f(n) < 0)$	Logical Substitution,(21),(23)
(25)	$\forall c \in \mathbb{R}^+.\forall b \in \mathbb{Z}^+.\exists n \in \mathbb{Z}^+. \log_2 f(n) < 0$	Use of Conjunction,(24)
(26)	$\forall n \in \mathbb{Z}^+. \log_2 f(n) \geq 1$	Generalisation, (16)
(27)	$\forall \log x \in \mathbb{R}.x \in \mathbb{R}^+$	Nonnegativity of log (informal)
(28)	$\log_2 g(n) > 0$	Proof by Contradiction,(23)-(27)
(29)	$\exists r' \in \mathbb{R}^+.r' \cdot \log_2 g(n) > \log_2 s$	Archimidean Property of $\mathbb{R}$
(30)	Let r be a number such that $r \cdot \log_2 g(n) > \log_2 s$	Instantiation, (29)
(31)	$r > \frac{\log_2 s}{\log_2 g(n)}$	Algebra of $\mathbb{R}$, (27)
(32)	$\exists n \in \mathbb{Z}^+.(n \geq z$ AND $\log_2 f(n) < (r+1)\log_2 g(n))$	Specialization, (21)
(33)	$\exists n \in \mathbb{Z}^+.(n \geq z$ AND $\log_2 f(n) < \log_2 s + \log_2 g(n))$	Definition of r, (32), (31)
(34)	$\exists c \in \mathbb{R}^+.\exists b \in \mathbb{Z}^+.\forall n \in \mathbb{Z}^+.(n \geq b$ IMPLIES $\log_2 f(n) \geq c\log_2 g(n))$	Proof by Contradiction,(21)-(33)
(35)	$\log_2 f \in \Omega(\log_2 g)$	Definition of Ω, (34)
(36)	$\big(f \in \Omega(g)$ AND $\forall n \in Z^+.(f(n) \geq 2$ AND $g(n) \geq 1)\big)$ IMPLIES $\log_2 f \in \Omega(\log_2 g)$	Proof of Implication, (3), (35)
(37)	$\forall f \in F.\forall g \in \mathcal{F}.\big(f \in \Omega(g)$ AND $\forall n \in Z^+.(f(n) \geq 2$ AND $g(n) \geq 1)\big)$ IMPLIES $\log_2 f \in \Omega(\log_2 g)$	Generalisation, (1), (2)