Let $T \in \text{Hom}(V, V)$ be a linear transformation, where V is a finite-dimensional inner product space over $\mathbb{F}$.

Lemma 0.1

Suppose that $T = T^*$. Show that $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in V$.

Proof.

Take $x \in V$.

Since $\langle Tx, x\rangle = \langle x, T^*x\rangle$ by definition of an adjoint, then $\langle Tx, x\rangle = \langle x, Tx\rangle$ by assumption.

Moreover, $\langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$ from properties of an inner product, and thus $\langle Tx, x\rangle = \overline{\langle Tx, x\rangle}$, which means that $\langle Tx, x\rangle$ is real for all $x \in V$. $\square$

Lemma 0.2

If $\mathbb{F} = \mathbb{C}$ and $\langle Tx, x\rangle = 0$ for all $x \in V$, then $T = \mathbf{0}$.

Proof.

Let $x, y \in V$.

Note the following:

$$T(x + y) = T(x) + T(y) \tag{1}$$
$$T(x + iy) = T(x) + iT(y) \tag{2}$$

Since $\langle T(x + y), x + y\rangle = 0$, while $\langle Tx, x\rangle = 0$ and $\langle Ty, y\rangle = 0$ then

$$\begin{aligned}\langle T(x+y), x+y\rangle &= \langle Tx, x+y\rangle + \langle Ty, x+y\rangle && (3)\\ &= \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle && (4)\\ &= \langle Tx, y\rangle + \langle Ty, x\rangle = 0 && (5)\end{aligned}$$

Therefore, $\langle Tx, y\rangle = -\langle Ty, x\rangle$

Similarly,

$$\begin{aligned}\langle T(x+iy), x+iy\rangle &= \langle Tx, x+iy\rangle + i\langle Ty, x+iy\rangle && (6)\\ &= \langle Tx, x\rangle + \langle Tx, iy\rangle + i\langle Ty, x\rangle + i(-i)\langle Ty, y\rangle && (7)\\ &= \langle Tx, x\rangle - i\langle Tx, y\rangle + i\langle Ty, x\rangle + \langle Ty, y\rangle && (8)\\ &= -i\langle Tx, y\rangle + i\langle Ty, x\rangle = 0 && (9)\end{aligned}$$

Therefore, $\langle Tx, y\rangle = \langle Ty, x\rangle$.

Hence, by combining two equations above, we obtain that $\langle Tx, y\rangle = 0$, which holds for all $y \in V$. Note that $Tx \in V$, and thus $\langle Tx, Tx\rangle = 0$, which holds if and only if $Tx = 0$ for all $x \in V$. Therefore, $T = \mathbf{0}$. $\square$

Lemma 0.3

If $\mathbb{F} = \mathbb{C}$ and $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in V$, then $T = T^*$.

Proof.

Take $x \in V$.

Note that, by definition of an adjoint,

$$\langle Tx, x\rangle = \langle x, T^*x\rangle. \tag{10}$$

Moreover, since $\langle Tx, x\rangle \in \mathbb{R}$, then

$$\langle Tx, x\rangle = \overline{\langle x, Tx\rangle} = \langle x, Tx\rangle. \tag{11}$$

Therefore, subtracting (11) from (10), we obtain that

$$\langle x, T^*x - Tx\rangle = \langle x, (T^* - T)x\rangle = \langle (T^* - T)^*x, x\rangle = 0. \tag{12}$$

Therefore, since $\mathbb{F} = \mathbb{C}$, we get by 0.2, $(T^* - T)^* = \mathbf{0}$.

Then $T - T^* = \mathbf{0}$, and hence $T = T^*$. $\square$

Lemma 0.4

If $\mathbb{F} = \mathbb{R}$, then $\langle Tx, x\rangle = 0$ for all $x \in V$ if and only if $T^* = -T$.

Proof.

Since $\mathbb{F} = \mathbb{R}$, then $\langle Tx, x\rangle = \overline{\langle x, Tx\rangle} = \langle x, Tx\rangle$, and by definition of an adjoint, $\langle Tx, x\rangle = \langle x, T^*x\rangle$. Thus, $\langle x, Tx\rangle = \langle x, T^*x\rangle$.

Suppose first $\langle Tx, x\rangle = 0$. Then $\langle x, Tx\rangle = \langle x, T^*x\rangle = 0$, and thus, summing two equations, we obtain that, for all $x \in V$,

$$\langle x, (T + T^*)x\rangle = 0. \tag{13}$$

Note that this holds for an arbitrary $x \in V$. In particular, (13) holds for $x = (T + T^*x)$, and hence $(T + T^*)x = 0$ for all $x \in \mathbb{R}$, which means that $T + T^* = \mathbf{0}$, and thus $T^* = -T$.

Suppose now $T^* = -T$. Therefore, $T^* + T = \mathbf{0}$, which means that for all $x \in V$, $(T^* + T)x = 0$. Therefore, since $V^\perp = \{0\}$, then for all $x \in V$ $\langle (T^* + T)x, x\rangle = 0$. Therefore, $\langle T^*x, x\rangle = -\langle Tx, x\rangle$.

Note that $\langle Tx, x\rangle = \langle x, T^*x\rangle$ by the derivation above, and thus $\langle Tx, x\rangle = -\langle Tx, x\rangle$, which means that $2\langle Tx, x\rangle = 0$ and hence $\langle Tx, x\rangle = 0$. $\square$