1 Kepler's Laws and Planetary Motion

Achievements in observing the planetary motion was brought by:

- Ptolemy
- Tycho Brahe
- Kepler

1.1 Kepler's Laws

1. Planets move in ellipses (not quite true, cf. *conic sections*), with the sun at a focus.
2. Equal areas are swept in equal times.
3. If a is the length of the major axis of an ellipse and T is the period of one revolution of a planet around the sun, then $\frac{a^3}{T^2} = \text{const.}$

Kepler inferred these laws from empirical data. Newton developed a *theory of fluxes* and proved Kepler's laws formally.

Consider a two-body system with one of them acting as the sun (a fixed point) and another acting as a planet and having a velocity which is a sum of two linearly independent vectors, with one of the components pointed to the sun, so that all the motion lies in a plane.

Introduce polar coordinates with the origin at the sun and the vector $r(t)$ corresponding to the position of the orbiting planet at time t.

Let's write

$$c(t) = r(t)(\cos(\theta(t)), \sin(\theta(t))).$$

Write $e(t) = (\cos(\theta(t)), \sin(\theta(t)))$ so that $c(t) = r(t)e(t)$.

Consider a derivative of $c(t)$:

$$c'(t) = (r'(t)\cos(\theta(t)) - r(t)\sin(\theta(t))\theta'(t), r'(t)\sin(\theta(t)) + r(t)\cos(\theta(t))\theta'(t)). \tag{1}$$

If $u = (a, b)$ and $v = (c, d)$, write $\det(u, v) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Therefore,

$$\det(c, c') = r^2(t)\theta'(t) \det \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{2}$$

$$= r^2(t)\theta'(t) \tag{3}$$

Note that the lower sum for the section area of any curve in polar coordinates is

$$\sum_i r_i^2(\theta_i - \theta_{i-1}),$$

and the upper sum is $\sum_i R_i^2(\theta_i - \theta_{i-1})$, and thus $\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2(\theta(t))\theta'(t)\,\mathrm{d}t$.

By the Fundamental Theorem of Calculus,

$A'(t) = \frac{1}{2}r^2(t)\theta'(t) = \frac{1}{2}\det(c, c')$.

Note that Kepler's Second Law amounts to $A'' = 0$, i.e. A' is a constant.

$$A'' = \frac{1}{2}\frac{\mathrm{d}\det(c, c')}{\mathrm{d}t} \tag{4}$$
$$= \frac{1}{2}(\det(c', c') + \det(c, c'')) \tag{5}$$
$$= \frac{1}{2}\det(c, c'') \tag{6}$$

Observe that $A'' = 0$ if and only if c'' as a multiple of c, i.e. the acceleration is always c and the corresponding force is along c, which makes it a central force.

Theorem 1.1 (Kepler's Second Law)

Equal areas are swept out in equal times if and only if the force is central.

Newton hypothesized that gravity is an "inverse-square" force, with the strength proportional to $\frac{1}{r^2}$.

This means that

$$c''(t) = -\frac{K}{r^2(t)}e(t).$$

Since we know that $r^2\theta' = H$, where H is some constant, so $\frac{c''(t)}{\theta'(t)} = -\frac{K}{H}e(t)$.

Note that $\frac{c''(t)}{\theta'(t)}$ is the derivative of $c'(\theta^{-1})(\theta(t))$.

Write $D(\theta) = c'(\theta^{-1}(\theta))$.

So $D' = -\frac{K}{H}e(\theta) = -\frac{K}{H}(\cos\theta, \sin\theta)$.

We can solve the following:

$$D = -\frac{K}{H}(\sin\theta + A, -\cos\theta + B)$$

Putting this into $\det(c, c') = r^2\theta'(t)$ and dealing with the associated algebra and geometry, it is possible to show that

$$r = L(1 + \epsilon\cos\theta),$$

which is the equation of a conic with one focus at the origin and eccentricity of ϵ:

- if $0 < \epsilon < 1$, then the trajectory is an ellipse
- if $\epsilon = 1$, then the trajectory is a parabola
- if $\epsilon > 1$, then the trajectory is a hyperbola

A similar argument also shows that to get orbits that are conic sections, a central force has to be an inverse-square force.