Consider $V = \mathfrak{P}_2(\mathbb{C})$ and the linear transformation $T \in \operatorname{Hom}(V, V)$ given by

$$T(f(x)) = xf'(x) + xf(1) + f(2).$$

Problem. Find the eigenvalues of T.

Solution. Note that $\gamma = \{1, x, x^2\}$ is an ordered basis of $\mathfrak{P}(\mathbb{C})$. Note that a linear transformation is completely determined by its action on a basis. Thus,

$$\begin{aligned} T(1) &= x + 1 && (1)\\ T(x) &= x + x + 2 = 2x + 2 && (2)\\ T(x^2) &= 2x^2 + x + 4 && (3) \end{aligned}$$

Therefore, $[T]_\gamma = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

Consider $\det([T]_\gamma - \lambda I) = 0$.

$$\begin{aligned} \det([T]_\gamma - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 2 & 4 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix} \\ &= \det \begin{pmatrix} -\lambda & \lambda & 3 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & 3\lambda - \lambda^2 & 3+\lambda \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix}, \end{aligned}$$

which, expanding along the first column, becomes

$$\det([T]_\gamma - \lambda I) = -\lambda(3-\lambda)(2-\lambda) = 0 \qquad (4)$$

Therefore, the possible eigenvalues are $\lambda = 0, \lambda = 3, \lambda = 2$. □

Problem. Find a basis β for which $[T]_\beta$ is a diagonal matrix.

Solution. First, we find eigenvectors corresponding to the eigenvalues found above.

For $\lambda = 0$, $\begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$ if and only if

$$\begin{aligned} x + 2y + 4z &= 0 && (5)\\ x + 2y + z &= 0 && (6)\\ 0 + 0 + 2z &= 0. && (7) \end{aligned}$$

Therefore, $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ spans E_0.

For $\lambda = 2$, $\begin{pmatrix} -1 & 2 & 4 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$ if and only if

$$\begin{aligned} -x + 2y + 4z &= 0 && (8) \\ x + 0 + z &= 0 && (9) \\ 0 + 0 + 0 &= 0. && (10) \end{aligned}$$

Therefore, $\begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$ spans E_2.

For $\lambda = 3$, $\begin{pmatrix} -2 & 2 & 4 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$ if and only if

$$\begin{aligned} -2x + 2y + 4z &= 0 && (11) \\ x - y + z &= 0 && (12) \\ 0 + 0 + -z &= 0. && (13) \end{aligned}$$

Therefore, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ spans E_3.

Since the eigenvalues found are distinct, the corresponding eigenvalues are linearly independent, and thus, since there are three of them and the dimension of $\mathfrak{P}(\mathbb{C})$ is 3, they form a basis.

Take $\beta = \{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \}$. Then the corresponding diagonal matrix is

$$[T]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

as required. $\square$