1 Rational Functions

The key in integrating a function in the form $\frac{p(x)}{q(x)}$ is factoring the denominator. The following theorem can be proven:

Theorem 1.1

Any real polynomial can be factored as a unique product up to the order of linear terms in the form $ax+b$ and irreducible quadratic terms Ax^2+Bx+C with $B^2-4AC<0$.

Given a rational function $\frac{p(x)}{q(x)}$, if $\deg p \geq \deg q$, then a polynomial division can be performed:

$$\frac{p(x)}{q(x)} = P(x) + \frac{r(x)}{q(x)},$$

with $\deg(r) < \deg(q)$ and P a polynomial. Thus, it suffices to show how to integrate $\frac{p}{q}$ when $\deg p < \deg q$.

The following result can be shown:

Theorem 1.2

Any rational function in the form $\frac{p}{q}$ when $\deg p \leq \deg q$ can be expressed as a sum of terms, each of which has only one term in the denominator, possibly repeated.

e.g. $\frac{1}{x^2-1} = -\frac{1}{2(x+1)} + \frac{1}{2(x-1)}$