1 Problem IV

Suppose that $T \in \operatorname{End}(V)$, where V is a finite-dimensional inner product space over $\mathbb{F}$. Suppose also that T is normal if $\mathbb{F} = \mathbb{C}$ and self-adjoint if $\mathbb{F} = \mathbb{R}$.

Let $U \in \operatorname{End}(V)$ be such that $TU = UT$.

For each $i \in [1, k] \cap \mathbb{N}$, let W_i be the eigenspace of T corresponding to the eigenvalue λ_i, and let T_i be the orthogonal projection of V on W_i.

Note that, by the Spectral Theorem, $V = \bigoplus_{i=1}^k W_i$, and $T = \sum_{i=1}^k \lambda_i T_i$.

Corollary 1 to Theorem 6.25 guarantees that $T^* = g(T)$ for some polynomial g. Suppose that $g(t) = a_0 + \sum_{i=1}^k a_i t^i$ such that $a_i \in \mathbb{F}$ for each $i \in [0, k] \cap \mathbb{Z}$.

Lemma 1.1

$UT^* = T^*U$

Proof.

Since $UT = TU$, then $UT^j = TUT^{j-1} = \cdots = T^jU$ for any $j \in \mathbb{Z}^+$. Therefore,

$$
\begin{aligned}
UT^* &= Ug(T) && (1)\\
&= U(a_0 I + \sum_{i=1}^k a_i T^i) && (2)\\
&= a_0 U + U(\sum_{i=1}^k a_i T^i) && (3)\\
&= a_0 U + \sum_{i=1}^k a_i T^i U && (4)\\
&= (a_0 + \sum_{i=1}^k a_i T^i)U && (5)\\
&= g(T)U && (6)\\
&= T^*U. && (7)
\end{aligned}
$$

Thus, $T^*U = UT^*$. □

Lemma 1.2

W_i is U-invariant.

Proof.

Suppose $x \in W_i$.

Let $\beta = \{v_1, \ldots, v_k\}$, $k = \dim W_i$, be the orthonormal basis of W_i consisting of eigenvectors. Since T is normal or self-adjoint, β is well-defined.

Let $v \in \beta$ be arbitrary.

Note that $UT(v) = \lambda_i Uv = TUv$, and thus $(T - \lambda_i I)Uv = 0$. Hence, $Uv \in \ker(T - \lambda_i I)$. By generalisation, since a linear transformation is defined by its action on a basis, for any $x \in W_i$, $Ux \in \ker(T - \lambda_i I) = W_i$, and thus W_i is U-invariant. □

Lemma 1.3

Suppose U is normal (if $\mathbb{F} = \mathbb{C}$) or self-adjoint (if $\mathbb{F} = \mathbb{R}$).

There exists an orthonormal basis of V that consists of vectors that are eigenvectors for both T and U.

Proof.

First we show that each W_i is $(U|_{W_i})^*$- and $U^*|_{W_i}$-invariant.

By Lemma 1.2, W_i is U-invariant.

Therefore, $U|_{W_i} : W_i \to W_i$, and thus $(U|_{W_i})^* : W_i \to W_i$ by definition of $(U|_{W_i})^*$, which means that W_i is $(U|_{W_i})^*$-invariant.

Since for any $x \in W_i$, $U^*x = \overline{\lambda_i}x \in W_i$, then W_i is $U^*|_{W_i}$-invariant.

Now we show that $U|_{W_i}$ is normal.

For any $x, y \in W_i$,

$$\langle U|_{W_i}x, y\rangle = \langle x, (U|_{W_i})^*y\rangle$$

and

$$\langle U|_{W_i}x, y\rangle = \langle Ux, y\rangle = \langle x, U^*y\rangle = \langle x, (U^*)|_{W_i}y\rangle,$$

which means that $\langle x, (U|_{W_i})^*y - (U^*)|_{W_i}y\rangle = 0$.

Since $(U|_{W_i})^*y - (U^*)|_{W_i}y \in W_i$ because W_i is both $(U|_{W_i})^*$- and $(U^*)|_{W_i}$-invariant, then taking $x = (U|_{W_i})^*y - (U^*)|_{W_i}y$ we obtain that $(U|_{W_i})^* = (U^*)|_{W_i}$ and thus W_i is U^*-invariant.

Since $UU^* = U^*U$ and thus $(UU^*)|_{W_i} = (U^*U)|_{W_i}$, while W_i is both U- and U^*-invariant, we obtain that $U|_{W_i}(U|_{W_i})^* = (U|_{W_i})^*U|_{W_i}$, and hence $U|_{W_i}$ is normal (if $\mathbb{F} = \mathbb{C}$ or self-adjoint (if $\mathbb{F} = \mathbb{R}$).

Therefore, there exists an orthonormal basis β_i of W_i consisting of eigenvectors of U. Since any vector in W_i is an eigenvector of T, while $V = \bigoplus_{i=1}^k W_i$, then $\gamma = \bigcup_{i=1}^k \beta_i$ is a basis of eigenvectors of U and T.

$\square$