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MAT 402

Classical Geometries

is so large that $1/(90 + 360n) < \delta$.

As an illustration of the use of the definition of a function approaching a limit, we have reserved the function shown in Figure 14, a standard example, but one of the most complicated:

$$f(x) = \begin{cases} 0, & x \text{ irrational}, \ 0 < x < 1 \\ 1/q, & x = p/q \text{ in lowest terms}, \ 0 < x < 1. \end{cases}$$

(Recall that p/q is in lowest terms if p and q are integers with no common factor and $q > 0$.)

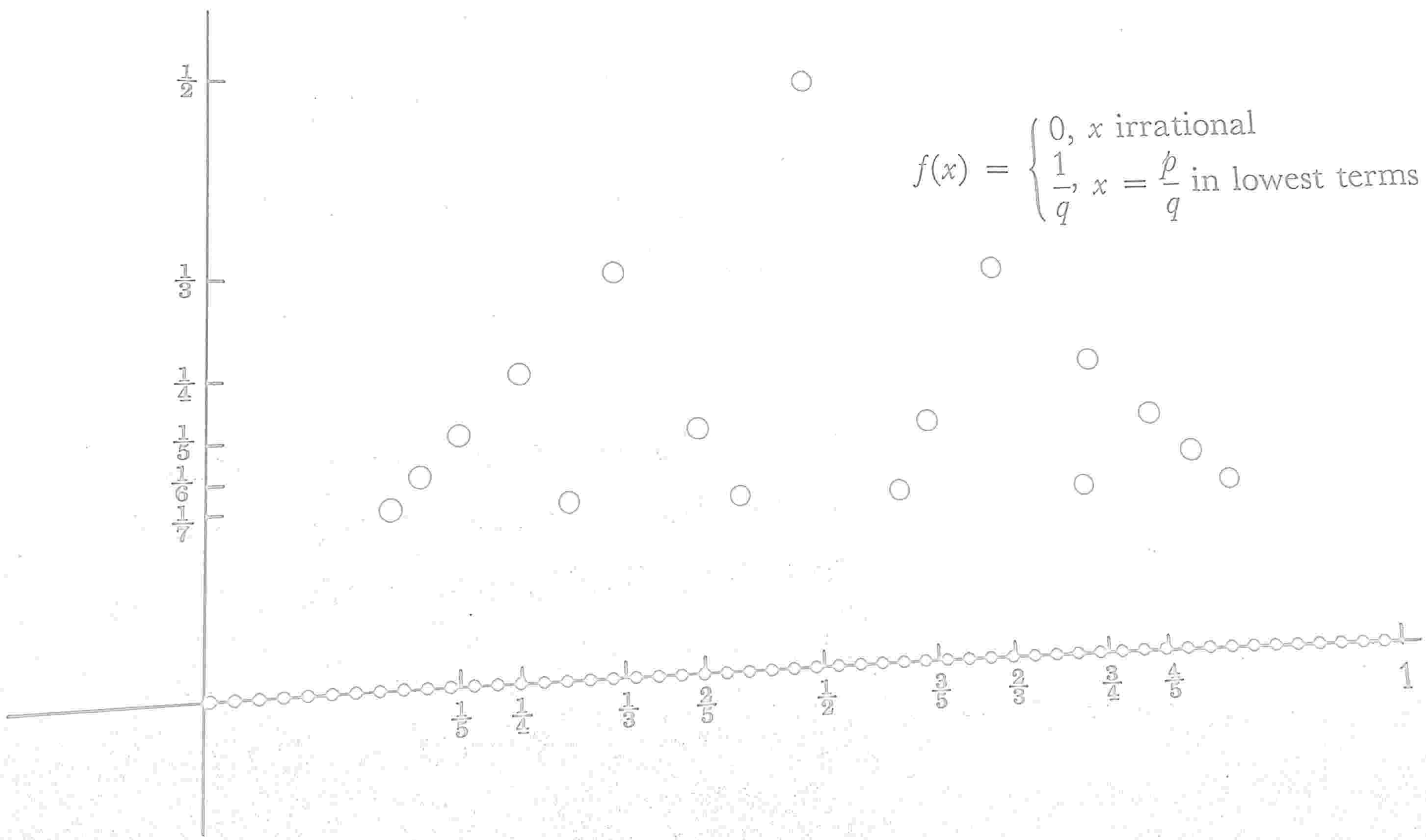

FIGURE 14

For any number a, with $0 < a < 1$, the function f approaches 0 at a. To prove this, consider any number $\varepsilon > 0$. Let n be a natural number so large that $1/n \leq \varepsilon$. Notice that the only numbers x for which $|f(x) - 0| < \varepsilon$ could be false are:

$$\frac{1}{2}; \frac{1}{3}, \frac{2}{3}; \frac{1}{4}, \frac{3}{4}; \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \quad \ldots \quad ; \frac{1}{n}, \quad \ldots \quad , \frac{n-1}{n}.$$

(If a is rational, then a might be one of these numbers.) However many of these numbers there may be, there are, at any rate, only finitely many. Therefore, of all these numbers, one is closest to a; that is, $|p/q - a|$ is smallest for one p/q among these numbers. (If a happens to be one of these numbers, then consider only the values $|p/q - a|$ for $p/q \neq a$.) This closest distance may be chosen as the δ. For if $0 < |x - a| < \delta$, then x is *not* one of

$$\frac{1}{2}, \ldots, \frac{n-1}{n}$$

and therefore $|f(x) - 0| < \varepsilon$ *is* true. This completes the proof. Note that our description of the δ which works for a given ε is completely adequate—there is no reason why we must give a formula for δ in terms of ε.

Armed with our definition, we are now prepared to prove our first theorem; you have probably assumed the result all along, which is a very reasonable thing to do. This theorem is really a test case for our definition: if the theorem

THEOREM T0. $\forall c, a \in \mathbb{R}: f(x) = c \Rightarrow f'(a) = 0$

THEOREM T1.

$$f(x) = x^n \ \exists (x \in \mathbb{N}) \Rightarrow \forall a \in \mathbb{R}: f'(a) = na^{n-1}$$

T2: LET f, g BE DIFFERENTIABLE AT a.

P1. $(f+g)'(a) = f'(a) + g'(a)$

P2. $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

LET f BE DIFFERENTIABLE AT a,

P3. $g(x) = cf(x)$

$\Rightarrow g'(a) = c \cdot f'(a)$

P4. $g(x) \neq 0$

$\Rightarrow \left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{[g(a)]^2}$

T3. S1. $\sin'(a) = \cos(a) \quad \forall a \in \mathbb{R}$

S2. $\cos'(a) = -\sin(a) \ \forall a \in \mathbb{R}.$

CR. IF g IS DIFFERENTIABLE AT a,

f IS DIFFERENTIABLE AT $g(a)$,

THEN $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$

TD. $f(x) = \tan(x)$

$\Rightarrow f'(x) = \sec^2(x).$

DC. IF f IS DIFFERENTIABLE AT a, then f IS CONTINUOUS AT a.

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(i) LET $f(x) = 3x^4 - \pi x^3 + \sqrt{3}x^2 - 2x + 7$

SINCE
$$\left.\begin{aligned} (3x^4)' &= 12x^3, \\ (-\pi x^3)' &= -3\pi x^2, \\ (\sqrt{3}x^2)' &= 2\sqrt{3}x, \\ (-2x)' &= -2 \end{aligned}\right\} \text{FROM T1.}$$
$$(7)' = 0 \quad \} \text{ FROM T0.}$$

FROM P1,

$$f'(x) = 12x^3 - 3\pi x^2 + 2\sqrt{3}x - 2.$$

(ii) LET $h(\theta) = \cos^2(\theta) + \sin^2(\theta)$

BY CR AND T3,

$$(\cos^2(\theta))' = -2\cos\theta \sin\theta,$$
$$(\sin^2(\theta))' = 2\sin\cos\theta$$

BY P1, $h'(\theta) = 0$

(iii) LET $g(x) = \dfrac{x^2}{x^2+1} = 1 - \dfrac{1}{x^2+1}$.

BY CR AND P4, $\left(\dfrac{-1}{x^2+1}\right)' = \dfrac{2x}{(x^2+1)^2}$.

By T0, $(1)' = 0.$

By P1, $g'(x) = \dfrac{2x}{(x^2+1)^2}.$

(iv) LET $k(x) = (x^3 - 2x + 3)\sin(x)$

BY T1, $(x^3)' = 3x^2$

$(-2x)' = -2$

BY T0, $(3)' = 0.$

BY P1, $(x^3-2x+3)' = 3x^2 - 2.$

BY P2, AND S1 $k'(x) = (3x^2-2)\sin x + (x^3-2x+3)\cos x.$

(V) LET $p(\theta) = \sec\theta.$

$= \dfrac{1}{\cos\theta}$

BY S2, $(\cos\theta)' = -\sin\theta.$

By P4, AND T4 $p'(\theta) = \dfrac{-1}{\cos^2\theta} \cdot (-\sin\theta) = \tan\theta \sec\theta.$

(vi) LET $q(x) = \frac{d\sin(x)}{dx}$.

BY S1, $q(x) = \cos x$,

BY S2, $q'(x) = -\sin x$.

(2)

2

IN GENERAL, THE SLOPE OF THE TANGENT TO THE GRAPH OF $f(x)$ AT a IS $f'(a)$.

SUPPOSE $y = f'(a)x + C$ IS THE EQUATION OF THE TANGENT, $C \in \mathbb{R}$.

THUS

$$y\big|_{x=a} = f'(a)a + C = f(a)$$

$$\Rightarrow \quad y = f'(a)(x-a) + f(a).$$

(i) LET $f(x) = \frac{1}{x-1}$.

FROM P4, T0 AND T1,

$$f'(x) = \frac{-1}{(x-1)^2}.$$

THUS, $f'(2) = -1$, AND HENCE

$$y = -(x-2) + \frac{1}{2-1}$$

$$= -x+3.$$

(ii) LET $f(x) = \sin(x)$.

FROM S1, $f(\sin(x))' = \cos x$

THUS, AT $x = \pi/4$, SINCE $\cos(\pi/4) = \frac{\sqrt{2}}{2} = \sin(\pi/4)$,

THEN $y = \frac{\sqrt{2}}{2}(x - \pi/4) + \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right).$

(iii) LET $f(x) = x^3 - 3x^2 + 3x - 2$. AT

FROM P1, T0, T1,

$f'(x) = 3x^2 - 6x + 3$

THUS, AT $x = 1$, $f'(1) = 0$.

HENCE, $y = 0(x-1) + 1$

$= 1$.

(iv) LET $f(x) = \dfrac{x^2}{5 + \sin(5x^7 - 4\tan(\pi x) + 3x^2 - 2)}$.

FROM P2,

$$f'(x) = \frac{(x^2)'}{5+\sin(5x^7-4\tan(\pi x)+3x^2-2)} + x^2 \cdot \left(\frac{1}{5+\sin(5x^7-4\tan(\pi x)+3x^2-2)}\right)'.$$

LET $g(x) = 5+\sin(5x^7-4\tan(\pi x)+3x^2-2)$.

FROM T1, P1, T2, T0, TD, CR,

$$g'(x) = \cos(5x^7-4\tan(\pi x)+3x^2-2) \cdot (35x^6 - 4\sec^2(\pi x)\cdot\pi + 6x)$$

NOTE THAT $g(0) = 5+\sin(-2)$,

$$g'(0) = \cos(-2)\cdot(-4\pi) = -4\pi\cos(-2) \in \mathbb{R}.$$

THUS, $f'(x) = \dfrac{2x}{g(x)} + x^2 \left(\dfrac{-g'(x)}{g^2(x)}\right)$ BY T1, P4,

THEREFORE, $f'(0) = 0$.

HENCE, $y = 0$.

③ LET $f(x) = x^3 - 6x^2$ on $\mathbb{R}$.

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(i) THEN BY P1, T1, P3,

$$f'(x) = 3x^2 - 12x.$$

FROM ②, $y = f'(a)(x-a) + f(a)$

IS THE EQUATION OF THE TANGENT LINE TO THE GRAPH OF $f(x)$ AT $x=a$.

AT $x=a$,

THUS, $y = (3a^2 - 12a)(x-a) + a^3 - 6a^2$

$$= 3a(a-4)x - 3a^3 + 12a^2 + a^3 - 6a^2$$

$$= 3a(a-4)x - 2a^2(a-3)$$

(ii) IF $y(x)$ AND $f(x)$ INTERSECT,

$y(x) = f(x)$. Suppose they intersect at $x \neq a$.

HENCE $f'(a)(x-a) = f(x) - f(a)$.

NOTE $f'(x) = 3x(x-4)$

$= 3(x-2)^2 - 12$

SINCE $f'(a) = 3a(a-4)$,

$f(a) = a^2(a-6)$,

THEN

$$x^3 - 6x^2 = 3a(a-4)(x-a) + a^2(a-6)$$

$$\iff (x-a)(x^2+ax+a^2) - 6(x-a)(x+a) - 3a(a-4)(x-a) = 0$$

$$\iff (x-a)(x^2+ax+a^2-6x-6a-3a^2+12a)=0$$

$$\iff (x-a)(x^2+(a-6)x-2a^2+6a)=0$$

$$\iff (x-a)(\underbrace{x^2+(a-6)x-2a(a-3)})=0$$

$$\Delta = (a-6)^2 + 8a(a-3)$$

$$= 9a^2-36a+36$$

$$= 9(a^2-4a+4)$$

$$= 9(a-2)^2$$

THUS, SINCE $x \neq a$,

$$x = \frac{-(a-6) \pm 3\sqrt{(a-2)^2 + 1}}{2}$$

$$x = \frac{-(a-6) \pm 3|a-2|}{2},$$

IF $a \geq 2$, $\left[\begin{array}{l} x = \dfrac{-a+6+3a-6}{2} = a \\ x = \dfrac{-a+6-3a+6}{2} \\ \quad = -2a+6 \end{array}\right.$

IF $a < 2$, $\left[\begin{array}{l} x = \dfrac{-a+6+3(2-a)}{2} = 6-2a \\ x = a. \end{array}\right.$

SINCE $x \neq a$, THEN $x = 6-2a \neq a \Rightarrow a \neq 2$.

NOTE THAT IF $a = 2$, $x = 2$ IS THE ONLY POINT OF INTERSECTION WHICH IS THE ONLY POINT OF TANGENCY.

(iii) FROM (ii), IF $a \neq 2$, THERE ARE 2 POINTS OF INTERSECTION.

IF $a = 1$, THERE IS ONLY ONE POINT OF INTERSECTION, WHICH IS THE POINT OF TANGENCY.

(4) PROBLEM:

SUPPOSE $f(x)$ AND $g(x)$ ARE TWO FUNCTIONS and $a \in D(f)$, $a \in D(g)$.

SUPPOSE $\exists\ \delta > 0 : (a-\delta, a+\delta) \in D(f) \cap D(g)$,

AND $f(x) = g(x)\ \forall x \in (a-\delta, a+\delta)$.

(i). WLOG, Suppose $\lim_{x \to a} f(x) = L \mid \exists (L \in \mathbb{R})$.

same delta?

0

Thus, $\forall \varepsilon > 0\ \exists\ \delta' > 0 : |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

SUPPOSE $\varepsilon > 0$ IS GIVEN. Therefore, there EXISTS δ' SUCH THAT

$x \in \cancel{(a-\delta, a+\delta)}$ AND $|f(x) - L| < \varepsilon$.

$(a-\delta', a+\delta')$

X

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NOTE THAT $\forall\ x \in (a-\delta, a+\delta')$

$x \in D(f) \cap D(g)$, thus $g(x)$ IS WELL-DEFINED AND

$f(x) = g(x)$, hence

$|g(x) - L| < \varepsilon.$

$\boxed{\text{Consider } \delta'' = \min(\delta, \delta')\,!}$

SINCE ε WAS TAKEN ARBITRARILY, IT FOLLOWS THAT $\forall(x \in D(f) \cap D(g))$

$$\forall(\varepsilon > 0)\ \exists(\delta > 0) : |x-a| < \delta$$

$$\Rightarrow |g(x) - L| < \varepsilon.$$

THUS, BY DEFINITION,

$$\lim_{x \to a} g(x) = L.$$

The argument is similar, IF $\lim_{x \to a} g(x) = L$ IS ASSUMED FIRST.

(ii) WLOG SUPPOSE THAT f IS DIFFERENTIABLE AT a.

Thus, $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ EXISTS.

Therefore, $\exists L \in \mathbb{R}$:

$\forall \varepsilon > 0\ \exists \delta' > 0 : 0 < |h| < \delta' \Rightarrow$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = L.$$

By assumption, $\exists \delta > 0$:

$x \in (a-\delta, a+\delta)$

$\Rightarrow x \in D(f) \cap D(g)$.

Consider $\delta'' = \min(\delta', \delta)$

~~Take $\delta' = \delta \Rightarrow f(a+h)$ is defined~~ (as well as $f(a)$ is defined).

Why?

Since $f(a+h) = g(a+h)$ and $f(a) = g(a)$ by assumption, it follows from above

$\forall \varepsilon > 0\ \exists \delta' > 0 : h < \delta' \Rightarrow$

$$\frac{g(a+h) - g(a)}{h} = L.$$

Does it work

Can't δ' to ~~set~~ arbitrary values!

Therefore, g is differentiable at a. The argument is similar if it is assumed that g is differentiable. $\square$

①

5.

CLAIM

FOR $n \geq 2$,

$$\frac{d^n}{dx^n}\left[\sin x\right] = (-1)^{\left\lfloor \frac{2n+3}{4} \right\rfloor} \sin\left(x + \pi\left(\frac{n}{2}(n+2)\right)\right)$$

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FOR $n = 1$,

$$\frac{d}{dx} \sin x = \cos x,$$

PROOF.

(1) FIRST, NOTE THE FOLLOWING:

$n=1$, $\sin'(x) = \cos x$

$n=2$, $\sin''(x) = -\sin x$

$n=3$, $\sin'''(x) = -\cos x$

$n=4$, $\sin^{IV}(x) = \sin x$.

Therefore, $\frac{d^n}{dx^n}\left[\sin x\right]$ IS PERIODIC

FOR $n \in \mathbb{N}$ WITH THE PERIOD 4.

Let $d(n) = \frac{d^n}{dx^n}\left[\sin x\right]$

Thus, for $n \in \mathbb{N}$,

$$(*)\quad \begin{cases} \text{IF } n \equiv 0 \bmod 4, & d(n) = \sin x \\ n \equiv 1 \bmod 4, & d(n) = \cos x = -\sin\left(x + \frac{3\pi}{2} + 2\pi k\right) \ \forall k \in \mathbb{Z} \\ n \equiv 2 \bmod 4, & d(n) = -\sin x = -\sin\left(x + 2\pi k\right) \ \forall k \in \mathbb{Z} \\ n \equiv 3 \bmod 4, & d(n) = -\cos x = \sin\left(x + \frac{3\pi}{2} + 2\pi k\right) \ \forall k \in \mathbb{Z}. \end{cases}$$

NOTE THAT $\frac{d(n)}{|d(n)|}$ IS ALSO PERIODIC

WITH PERIOD 4.

CONSIDER $(-1)^{\left\lfloor \frac{2n+3}{4} \right\rfloor} = \operatorname{sign}(n)$

IF $n \equiv 0 \bmod 4$, THEN $n = 4k \ \exists k \in \mathbb{N}$ $\Rightarrow \left\lfloor 2k + \frac{3}{4} \right\rfloor = \underset{\text{EVEN}}{2k} \Rightarrow \operatorname{sign}(n) = 1$.

IF $n \equiv 1 \bmod 4$, $n = 4k+1 \ \exists k \in \mathbb{N}$ $\Rightarrow \left\lfloor 2k+1 \right\rfloor = 2k+1 \Rightarrow \operatorname{sign}(n) = -1$

IF $n \equiv 2 \bmod 4$, $n = 4k+2 \ \exists k \in \mathbb{N}$ $\Rightarrow \left\lfloor 2k+1+\frac{3}{4} \right\rfloor = 2k+1 \Rightarrow \operatorname{sign}(n) = -1$

IF $n \equiv 3 \bmod 4$, $n = 4k+3 \ \exists k \in \mathbb{N}$ $\Rightarrow \left\lfloor 2k + \frac{9}{4} \right\rfloor = 2(k+1) \Rightarrow \operatorname{sign}(n) = 1$.

NOTE THAT $\operatorname{sign}(n) \ \forall n \in \mathbb{N}$ IS ONE-TO-ONE CORRESPONDENCE WITH THE SIGN OF $\sin(\cdot)$ IN $(*)$.

NOW CONSIDER $\pi\left(\frac{n}{2}(n+2)\right) = \tau(n)$.

IF $n = 4k \; \exists k \in \mathbb{N} \Rightarrow \tau(n) = 2[k(4k+2)]\pi$

EVEN-MULTIPLE OF π
(E.M. π)

$n = 4k+1 \; \exists k \in \mathbb{N} \Rightarrow \tau(n) = \pi\left((4k+3)\left(2k+\frac{1}{2}\right)\right) =$

$= \underbrace{\pi\left(2\left[(4k+3)k + 2k\right]\right)}_{E.M.\,\pi} + \frac{3\pi}{2}$

$n = 4k+2$, $\tau(n) = \underbrace{\pi(2k+1)4(k+1)}_{E.M.\,\pi.}$

$n = 4k+3$, $\tau(n) = \frac{3\pi}{2} + \underbrace{2k(4k+5)\pi}_{E.M.\,\pi}$

NOTE THAT $\tau(n)$ is in one-to-one correspondence WITH THE INPUT OF $\sin(\cdot)$ OTHER THAN x.

SINCE $\frac{d}{dx}[\sin x] = \cos x$

$\Rightarrow$ THE CLAIM ALSO HOLDS $\forall (n \in \mathbb{N})$

$\square$