1 Series and Absolute Convergence

Definition 1.1. A series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |a_n|$ converges.

A series which converges but does not converge absolutely is said to **converge conditionally**.

> **Theorem 1.2**
>
> If $s = \sum_{n=1}^{\infty} a_n$ converges absolutely, then s converges.

Proof.

Since s converges absolutely, then $\sum_{n=1}^{\infty} |a_n|$ converges.

Therefore, $\sum_{i=0}^{m-n} |n+i| \to 0$ a $m, n \to \infty$.

But $\left| \sum_{i=1}^{m-n} a_{n+i} \right| \leq \sum_{i=1}^{m-n} |a_{n+i}|$, which means that $\sum_{i=1}^{m-n} a_{n+i}$ goes to 0.

Therefore, $\sum a_n$ converges. $\qquad\square$

Write $a_n^+ = \begin{cases} a_n \text{ if } a_n \geq 0 \\ 0 \text{ otherwise} \end{cases}$ and $a_n^- = \begin{cases} a_n \text{ if } a_n \leq 0 \\ 0 \text{ otherwise} \end{cases}$.

Then $\sum a_n = \sum a_n^+ + \sum a_n^-$.

> **Theorem 1.3**
>
> $\sum a_n$ converges absolutely if $\sum a_n^+$ and $\sum a_n^-$ converge absolutely.
>
> In this case, $\sum a_n = \sum a_n^+ + \sum a_n^-$ and $\sum |a_n| = \sum a_n^+ - \sum a_n^-$.

> **Example 1.4**
>
> Consider $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$.
>
> We can *rearrange* the series so that it converges to 17.
>
> Note that
>
> $$1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n+1} - \frac{1}{2} - \frac{1}{4} - \dots + \frac{1}{2n+5} + \dots$$
>
> Every term of the original series appears eventually in the rearranged series, and it converges to 17.
>
> Note that there is nothing special about the choice of 17 in this cas. Given any A, we can rearrange any conditionally convergent series to converge to A.

> **Theorem 1.5**
>
> If $\sum a_n converges$ absolutely, then any rearrangement converges absolutely and the sums are equal.

Remark 1.6. The addition symbol in series is a misnomer. A series is a limit of a sequence, not the sum in the usual sense.

Theorem 1.7 (Leibniz Theorem)

If $a_1 > a_2 > a_3 > \cdots > a_n > 0$ and $\lim_{n\to\infty} a_n = 0$, then $\sum_{i=1}^{\infty}(-1)^n a_n$ converges.

Example 1.8

Take $\sum(-1)^{n+1}\frac{1}{n}$ and $\sum(-1)^{n+1}\frac{1}{n^2}$.

It can be shown that the first series converges unconditionally, while the second converges conditionally.

1.1 Power Series

Series in the form $\sum_{n=0}^{\infty} a_n(x-a)^n$ are called **power series**.

For example, all Taylor series are power series.

Take $s = \sum_{i=1}^{\infty} \frac{|x|^2}{n!}$ and apply the Ratio Test:

$$\frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}} = \frac{|x|}{n+1} \to 0.$$

Therefore, $s = \sum_{i=1}^{\infty} \frac{|x|^2}{n!}$ converges absolutely and therefore converges.

Consider now $\log(x+1) = \sum_{i=1}^{n}(-1)^{n+1}\frac{x^n}{n}$.

Using the ratio test we obtain that

$$\frac{(-1)^{n+2}\frac{x^{n+1}}{n+1}}{(-1)^n \frac{x^n}{n}}.$$

The sums convergewhen $|x > 1|$, and diverges if $|x < 1|$.

What happens when $|x| = 1$?

Take $x = 1$.

Then $\sum(-1)^{n+1}\frac{1}{n}$ converges to $\log 2$, at least conditionally, and $\sum_{i=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n}$ diverges (harmonic series are obtained).

Theorem 1.9

A power series $\sum a_n(x-a)^n$ converges absolutely on some interval $(a-r, a+r)$ and diverges if $|x-a| > r$

It may diverge, converge conditionall or converge absolutely at each endpoint.

Remark 1.10. Note that r might denote ∞, which would mean that it converges absolutely on $\mathbb{R}$.

Example 1.11

Consider $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$.

Note that it converges on $(-1, 1)$ but not at ± 1.