Claim. Suppose that $M \in M_{n\times n}(\mathbb{F})$ has three distinct eigenvalues λ, μ, ν and that $\dim E_\lambda = n-2$. Then M is diagonalizable.

Proof. Let $T = \mathfrak{L}_M$. Note that T is diagonalizable if and only if

$$\dim V = \dim E(\lambda, T) + \dim E(\mu, T) + \dim E(\nu, T).$$

We prove that this condition indeed holds.

Since μ and ν are distinct eigenvalues, $\dim E(\mu, T) \geq 1$ and $\dim E(\nu, T) \geq 1$. Given that $\dim E_\lambda = n-2$, we obtain

$$\dim E(\lambda, T) + \dim E(\mu, T) + \dim E(\nu, T) \geq n = \dim V.$$

Since $V = E(\lambda, T) \oplus E(\mu, T) \oplus E(\nu, T)$,

$$\dim E(\lambda, T) + \dim E(\mu, T) + \dim E(\nu, T) \leq n = \dim V.$$

Therefore,

$$\dim V = \dim E(\lambda, T) + \dim E(\mu, T) + \dim E(\nu, T),$$

and thus T is diagonalizable. $\square$

Problem. Give an example of a matrix with precisely three distinct eigenvalues that is not diagonalizable.

Solution. By the Claim above, if M is not diagonalizable but has three distinct eigenvalues, neither of them has $\dim E_{\lambda_i} = n-2$.

Moreover, if M is not diagonalizable, then

$$\dim V > \dim E(\lambda, T) + \dim E(\mu, T) + \dim E(\nu, T).$$

Suppose $M \in M_{4\times 4}(\mathbb{Q})$ is defined over $\mathbb{Q}$.

Take $M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 5 & 0 \end{pmatrix}$.

Consider $\det(M - \lambda I) = 0$.

$$\det(M - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & -1-\lambda & 0 \\ 0 & 0 & 5 & -\lambda \end{pmatrix} = 0 \tag{1}$$

Expanding along the first column and using the fact that $M - \lambda I$ has a normal Jordan form, we obtain

$$\det(M - \lambda I) = -\lambda \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 5 & -\lambda \end{pmatrix} = -\lambda^2(1-\lambda)(1+\lambda), \tag{2}$$

which gives possible eigenvalues of $0, 1, -1$. Note that they are distinct.

For $\lambda = 0$,

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \mathbf{0}, \tag{3}$$

if $x = 0$, $y = 0, z = 0$. Thus, $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ spans E_0.

For $\lambda = 1$,

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 5 & -1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \mathbf{0}, \tag{4}$$

if $w = z$, $y = 0$ and $z = 5y = 0 = w$. Thus, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ spans E_1.

For $\lambda = -1$,

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \mathbf{0}, \tag{5}$$

if $w = -z$, $x = 0$ and $z = -5y$. Thus, $\begin{pmatrix} 5 \\ 0 \\ 1 \\ -5 \end{pmatrix}$ spans E_{-1}.

Since there are only three eigenvectors, while the domain of $T = \mathfrak{L}_M$ has the dimension 4, there is no basis for the domain consisting of eigenvectors, and thus M is not diagonalizable, while there are three distinct eigenvalues corresponding to M. $\square$