Agenda: Chapter 5-7 in Friedberg et al

Marking Scheme: HW, Term Test (Thu, Feb 16), Final (13.3/26.7/60 OR 20/40/40)

Office Hours: Florian Herzig, Wednesday 3-4 pm (BA6186)

1 Review of Determinants

Let F be a field.

Let $A \in M_{n\times n}(F) \to \det(A) \in F$.

Note that for $n = 1$, $\det(a) = a$.

If $n = 2$, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

In general, compute the determinant by expanding along a row/column.

For example, expansion along row:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det \widetilde{A_{ij}}$$

For example,

$$\det \begin{pmatrix} 0 & 1 & 0 & 2 \\ 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ -1 & 2 & 3 & 4 \end{pmatrix} = -1 \cdot \det \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 3 \\ -1 & 3 & 4 \end{pmatrix} + 2 \det \begin{pmatrix} 0 & 0 & 2 \\ 3 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

Other properties:

- $\det A$ is zero if two rows are linearly dependent $\leftrightarrow$ $\operatorname{rank} A < n$
- if rows are interchanged, then det changes sign
- if a row is multiplied by k, then det is scaled by k
- if a multiple of a row i is added to row j, then det is unchanged
- det is linear along each row and column
- $\det AB = \det A \cdot \det B$
- $\det \begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix} = \det A' \det A''$ – a simiar result holds for any number of *blocks*
- $\det A^t = \det A$
- A is invertible $\Leftrightarrow$ $\det A \neq 0$
- If A is invertible, then $\det A^{-1} = (\det A)^{-1}$
- If A, B are similar, then $\det A = \det B$.
  Note. A, B are similar iff there exists an invertible Q such that $A = Q^{-1}BQ$
- if A is invertible, then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} \det \widetilde{A_{11}} & -\det \widetilde{A_{21}} & \dots \\ -\det \widetilde{A_{12}} & \ddots & \\ \det \widetilde{A_{13}} & \dots & \end{pmatrix}$$

2 Diagonalization

2.1 Eigenvalues, Eigenvectors

Motivation: simplification of the matrix form, decomposition of automorphisms (eg computation of A^{100})

Recall that A is diagonal if $A = \begin{pmatrix} A_{11} & 0 & 0 & \dots \\ 0 & A_{22} & 0 & \dots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & A_{nn} \end{pmatrix}$

Definition 2.1. For V a finite dimensional vector space, $T : V \to V$ a linear transformation, T is **diagonalisable** if there exists an ordered basis β such that $[T]_\beta$ is diagonal.

If $A \in M_{n\times n}(F)$, then A is **diagonalisable** if $L_A : F^n \to F^n$ is diagonalisable. Equivalently, A is similar to a diagonal matrix.

If T is diagonalisable and $[T]_\beta = \begin{pmatrix} D_{11} & 0 & \dots \\ & \ddots & \\ 0 & \dots & D_{nn} \end{pmatrix}$, where $\beta = (v_1, v_2, \dots, v_n)$, then $Tv_1 = D_{11}v_1, \dots, Tv_n = D_{nn}v_n$.

Definition 2.2. $Tv = \lambda v$ with $v \neq 0, \lambda \in F$, then v is an **eigenvector** of T with corresponding **eigenvalue** λ.

Similarly, an eigenvalue of A is an eigenvalue of L_A.

Example 2.3

If $T = \lambda I_v$ (ie $T(v) = \lambda v \ \forall v \in V$), then any nonzero $v \in V$ is an eigenvector of T.

Example 2.4

If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, then e_1, e_2, e_3 are eigenvectors with eigenvalues $1, 2, 3$.

Example 2.5

If T is arbitrary, then eigenvectors with the eigenvalue 0 are the nonzero elements of $\ker(T)$.

Example 2.6

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation by angle $\alpha \in (0, \pi)$, then there are no eigenvectors $\Rightarrow T$ is not diagonalisable.

Example 2.7

If $A = \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix}$, then $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector with the eigenvalue 1.

From above, if T is diagonalisable, then V has a basis consisting of eigenvectors of T.

Conversely, if $\beta = (v_1, v_2, v_3, \dots)$ i sa basis of eigenvectors $T(v_1) = \lambda v_1, \dots, T(v_2) = \lambda v_2$, then T is diagonalisable.

2.2 Finding Eigenvectors and Eigenvalues

If $T(v) = \lambda v, v \neq 0$, then $0 = T(v) - \lambda v = T(v) - \lambda I_v(v) = (T - \lambda I_v)(v) = 0 \Leftrightarrow v \in \ker(T - \lambda I_v)$.

Thus, T has an eigenvalue $\lambda \Leftrightarrow \ker(T - \lambda I_v) \neq 0 \Leftrightarrow T - \lambda I_v$ is not injective $\Leftrightarrow T - \lambda I_v$ is not invertible.

In that case, the eigenvectors of an eigenvalue λ are the non-zero elements of $\ker(T - \lambda I_v)$.

Similarly for $A \in M_{n\times n}(F)$.

Example 2.8

Let $A = \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix}$, $\lambda \in F$.

Then $\det(A - \lambda I) = (4 - \lambda)(-1 - \lambda) + 6 = (\lambda - 1)(\lambda - 2)$.

Thus $\lambda_1 = 1, \lambda_2 = 2$.

Find the corresponding eigenvectors to obtain $(1, -1)$ corresponding to $\lambda = 1$ and $(3, -2)$ corresponding to $\lambda = 2$.

Since they span F^2, A is diagonalisable.

Definition 2.9. The **characteristic polynomial** of A is the polynomial

$$f(\lambda) = \det(A - \lambda I)$$

Example 2.10

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $f(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc)$.

Remark 2.11. If A, B are similar, they have the same characteristic polynomial. Hence if $T \in \mathrm{Hom}(V, V)$, then the characteristic polynomial can be defined for T as the characteristic polynomial of $[T]_\beta$ for any β.

Theorem 2.12

For $A \in M_{n\times n}(F)$,

a) The characteristic polynomial of A has degree n, with the leading coefficient $(-1)^n$.

b) The number of distinct eigenvalues is less than or equal to n.

Proof.

Claim. For $B \in M_{n\times n}(F)$, with entries that are linear functions in λ, then $\det B$ is a polynomial in λ of degree at most n.

Proof. If $n = 1$, the claim follows from the fact that $\det(a) = a$ for any $a \in F$. Suppose the claim is true for some $n = k - 1 \in \mathbb{N}$.

Note that $\det B = \sum_{j=1}^{n}(-1)^{i+j} B_{ij} \det \widetilde{B_{ij}} \Rightarrow \deg(\det B) \leq k$ by induction, since $\deg(\det \widetilde{B_{ij}}) \leq k-1$ by inductive hypothesis. □

a) To prove (a), we use induction on n again.

   The statement is true for $n = 1$.

   Suppose it is also true for some $n - 1$. Then

$$\begin{aligned} f(\lambda) &= \det(A - \lambda I) \\ &= (A_{11} - \lambda) \det \begin{pmatrix} a_{22} - \lambda & A_{23} & \dots \\ \vdots & & \vdots \\ A_{n2} & \dots & A_{nn} - \lambda \end{pmatrix} - A_{12} \det \begin{pmatrix} A_{21} & A_{23} & \dots \\ \vdots & A_{33} - \lambda & \dots \\ & \ddots & \end{pmatrix} \pm \dots \end{aligned}$$

   Note that the determinants of the above expressions correspond to $(n-1) \cdot (n-1)$ matrices, and thus by inductive hypothesis their leading coefficient is $(-1)^{n-1}$. Expanding the first brackets, we can see that the highest power of λ in the expression has the coefficient $(-1)^n$, as required.

   Thus, the claim is true by induction.

b)

   Note. See App E, Cor 2 Ex 5.1/21

   Claim. Let $T \in \mathfrak{L}(V)$. Suppose $\lambda_1, \lambda_2, \dots, \lambda_m$ are distinct eigenvalues of T and $v_1, v_2, \dots, v_m$ are corresponding eigenvectors. Then $v_1, v_2, \dots, v_m$ are linearly independent.

   Proof. By way of contradiction, suppose that $v_1, v_2, \dots, v_m$ are linearly dependent. Since they are linearly dependent, there exists $k \in \mathbb{Z}$ such that

$$v_k \in \operatorname{span}(v_1, v_2, \dots, v_{k-1}).$$

   Thus there exist $a_1, a_2, \dots, a_{k-1}$ such that

$$v_k = \sum_{i=1}^{k-1} a_i v_i. \tag{1}$$

   Applying T to both sides, we obtain

$$T(v_k) = \lambda_k v_k = \sum_{i=1}^{k-1} a_i \lambda_i v_i.$$

   From (1),

$$0 = \sum_{i=1}^{k-1} a_i (\lambda_k - \lambda_i) v_i$$

   Since all λ_i are distinct and all v_i are linearly independent, then a_i are all zero, and thus v_k is zero, which is a contradiction to the hypothesis that v_k is an eigenvector.

   In conclusion, the assumption that $v_1, v_2, \dots, v_m$ are linearly dependent is false. □

The claim above implies that $m \leq \dim V = n$, where m is the number of distinct eigenvalues.

$\square$