MAT157 20161128

L'Hôpitals RULE

∞-VERSION:

IF $\displaystyle\lim_{x \to \infty} \frac{f'(x)}{g'(x)}$ EXISTS,

THEN $\displaystyle\lim_{x \to \infty} \frac{f(x)}{g(x)}$ EXISTS AND

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} \stackrel{?}{=} \lim_{t \to 0^+} \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})}$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{t \to 0^+} \frac{f(\frac{1}{t})}{g(\frac{1}{t})}$$

BY CHAIN RULE, $\displaystyle\frac{d}{dt}\left(f\left(\frac{1}{t}\right)\right) = f'\left(\frac{1}{t}\right)\cdot\left(-\frac{1}{t^2}\right)$;

$$\frac{d}{dt}\left(g\left(\frac{1}{t}\right)\right) = g'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)$$

$$\Rightarrow \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{t \to 0^+} \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \lim_{t \to 0^+} \frac{f(\frac{1}{t})}{g(\frac{1}{t})} =$$

$$= \lim_{x \to \infty} \frac{f(x)}{g(x)}$$

REMARK: NEED $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$

OR $\begin{cases} \lim_{x \to a} f(x) \to \infty \\ \lim_{x \to a} g(x) \to \infty \end{cases}$

PROOF OF L'HÔPITAL'S RULE.

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$$

ALTHOUGH WE DON'T NEED TO ASSUME f, g ARE DEFINED AT $x = a$, LET'S EXTEND THEM TO BE CONTINUOUS THERE:

$$f(a) = g(a) = 0.$$

By CMVT, we can find $g'(u_x)(f(x) - f(a)) =$
$= f'(u_x)[g(x) - g(a)]$, u_x is between a and x

GIVEN $\varepsilon > 0$, $\exists\, \delta > 0$: $|b-a| < \delta \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon$

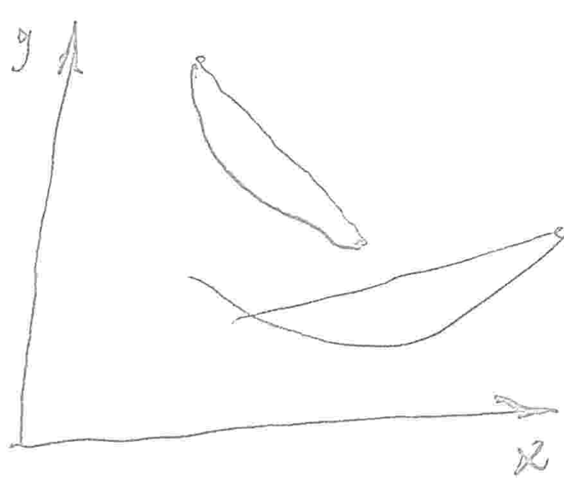
IF $|x-a| < \delta$, THEN $|u_x - a| < \delta$ AND

$$\left| \frac{f'(u_x)}{g'(u_x)} - L \right| < \varepsilon, \text{ so } \frac{f'(u_x)}{g'(u_x)} \longrightarrow \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\frac{f'(u_x)}{g'(u_x)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

$\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ EXISTS $\Rightarrow g' \neq 0$ near a.

CONVEXITY



DEFINITION

$f(x)$ IS <u>CONVEX</u> ON AN INTERVAL IF FOR ANY $a < x < b$ IN THE INTERVAL,

$$f(x) < \frac{f(b) - f(a)}{b-a}(x-a) + f(a)$$

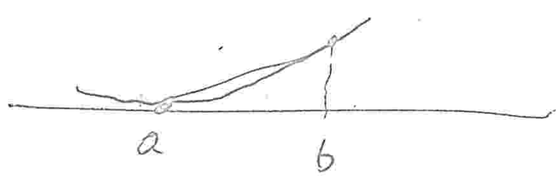
IF $f(x) > \frac{f(b) - f(a)}{b-a}(x-a) + f(a)$, f IS <u>CONCAVE</u> ON AN INTERVAL

CONCAVE DOWN

CONCAVE UP

THEOREM IF f IS CONVEX AND DIFFERENTIABLE AT $x = a$, THEN THE GRAPH OF $f(x)$ LIES ABOVE THE TANGENT LINE AT $x = a$, EXCEPT RIGHT AT THE POINT OF CONTACT $(a, f(a))$

PROOF



$$\lim_{y \to a} \frac{f(y) - f(a)}{y - a} = f'(a)$$

EXERCISE