1 Problem I

Suppose that V is a finite-dimensional inner product space over $\mathbb{F}$. Suppose that $u \in V$ satisfies $\|u\| = 1$. Define the linear transformation $T \in \operatorname{End}(V)$ by $T(x) = x - 2\langle x, u\rangle u$.

Lemma 1.1

$Tx = x$ if and only if x is orthogonal to u.

Proof.

Suppose first $Tx = x$. Therefore, $Tx = x - 2\langle x, u\rangle u = x$, and thus $2\langle x, u\rangle u = 0$.

Since $\|u\| = 1$, $u \neq 0$. Therefore, $\langle x, u\rangle = 0$, and thus x is orthogonal to u.

Suppose now that x is orthogonal to u.

Therefore, $\langle x, u\rangle = 0$, and thus $Tx = x - 2\langle x, u\rangle u = x$. □

Lemma 1.2

$Tx = -x$ if and only if $x \in \operatorname{span} u$.

Proof.

Suppose first $Tx = -x$.

Therefore, $Tx = x - 2\langle x, u\rangle u = -x$, and thus $2\langle x, u\rangle u = 2x$.

Since $\langle x, u\rangle \in \mathbb{F}$, $x = \langle x, u\rangle u \in \operatorname{span} u$.

Suppose now $x \in \operatorname{span} u$, so that there exists $k \in \mathbb{F}$ such that $x = ku$.

Hence, $\langle x, u\rangle = \langle ku, u\rangle = k\langle u, u\rangle = k$, since $\|u\| = 1$.

Therefore, $Tx = x - 2\langle x, u\rangle u = x - 2ku = x - 2x = -x$. □

Lemma 1.3

$T^2 = I, T^* = T$ and T is unitary/orthogonal.

Proof.

For any $x \in V$, note that

$$\begin{aligned}
T^2 x &= T(Tx) = T(x - 2\langle x, u\rangle u) && (1)\\
&= Tx - 2\langle x, u\rangle Tu && (2)\\
&= x - 2\langle x, u\rangle u - 2\langle x, u\rangle Tu && (3)\\
&= x - 2\langle x, u\rangle (u + Tu) && (4)\\
&= x - 2\langle x, u\rangle (u + u - 2\langle u, u\rangle u) && (5)\\
&= x - 2\langle x, u\rangle (2u - 2\|u\|^2 u) && (6)\\
&= x - 2\langle x, u\rangle (2u - 2u) && (7)\\
&= x. && (8)
\end{aligned}$$

Thus, $T^2 = I$.

For any $v, w \in W$, by definition of T^*, $\langle Tv, w\rangle = \langle v, T^*w\rangle$.

Note the following:

$$\begin{aligned}
\langle Tv, w\rangle &= \langle v - 2\langle v, u\rangle u, w\rangle && (9)\\
&= \langle v, w\rangle - 2\langle v, u\rangle\langle u, w\rangle && (10)\\
&= \langle v, w\rangle - \langle v, \overline{2\langle u, w\rangle} u\rangle && (11)\\
&= \langle v, w - 2\overline{\langle u, w\rangle} u\rangle && (12)\\
&= \langle v, w - 2\langle w, u\rangle u\rangle && (13)\\
&= \langle v, Tw\rangle && (14)
\end{aligned}$$

Now, $\langle v, T^*w\rangle = \langle v, Tw\rangle$ for any $v, w \in V$. Taking $v = T^*w - Tw$, we obtain that $T^* = T$.

Moreover,

$$\begin{aligned}
\langle Tx, Tx\rangle &= \langle x, T^*Tx\rangle && (15)\\
&= \langle x, T^2x\rangle && (16)\\
&= \langle x, Ix\rangle && (17)\\
&= \langle x, x\rangle && (18)
\end{aligned}$$

Since $\langle x, x\rangle \geq 0$ for any $x \in V$, then taking square roots of both sides we obtain that $\|Tx\| = \|x\|$, and thus T is unitary/orthogonal.

□

Problem. Find the characteristic polynomial of T.

Solution.

Since T is both self-adjoint and orthogonal, all eigenvalues of T have an absolute value of 1.

By Lemma 1.1, if $\dim V \geq 2$, 1 is an eigenvalue of T (since T is self-adjoint, there is an orthonormal basis of eigenvectors).

By Lemma 1.2, if $\dim V \geq 1$, -1 is an eigenvalue of T.

Since the coefficient corresponding to a monomial of the highest degree in the characteristic polynomial is $(-1)^n$, where $n = \dim V$, the characteristic polynomial of T is as follows:

$$p(\lambda) = (-1)^n(\lambda - 1)(\lambda + 1).$$

□