1 Structural Stability

1.1 Introduction

Suppose that a differential equation $\dot{x} = v(x)$ is given, where $x \in \mathbb{R}^n$. A solution of this equation is a vector function $\phi : I \to \mathbb{R}^n$ such that $\dot{\phi}(t) = v(\phi(t))$. This is an analytic description.

Now, consider $\Omega \subset \mathbb{R}^n$, a vector field on a real manifold. Imagine putting a toy boat on this manifold. The boat would move according to the analytic solution we have discussed above. The path according to which the boat moves is called an *orbit*, which is a phase curve $\operatorname{im} \phi$.

1.2 Equilibrium Points and Limit Cycles

Orbits of any differential equation can be classified into three types: topologically equivalent to a point, circle or a line.

Definition 1.1. If $v(a) = 0$, we say that a is a equilibrium point.

In turn, closed orbits may contain other closed orbits.

Definition 1.2. An isolated closed orbit is called a *limit cycle*.

Example 1.3

Consider a plane with polar coordinates, and suppose that $\dot{\phi} = 1$ and $\dot{r} = r(1 - r)$. In this way, all the boats move from 0 to 1 and from the point at infinity to one.

There is also a limit cycle corresponding to a circle with the radius 1.

A vector field is called *typical*, if a small perturbation of vector field does not destroy the key properties of a vector field. Now, let's consider equilibrium points and limit cycles of a *typical* vector field.

Equilibrium points in $\mathbb{R}^2$ come in a wide variety. For example, a saddle point is, informally, a repelling equilibrium point, unless the boat moves straight to its center. There are also knots, for which a phase diagram looks like a plot for a point charge, and foci, which is spiral.

Theorem 1.4

Open, everywhere dense set in $Vect^1(S^2)$ consists of vector fields with a finite number of foci.

What is the asymptotic behaviour of other trajectories?

Trajectories of a vector field with a finite number of equilibrium points on a two-dimensional sphere asymptotically approach either a equilibrium point, a cycle or a polycycle. A polycycle is a polyhedron made of separatrices, which can make up a complex picture.

Theorem 1.5

Trajectories of a vector field from an open, everywhere dense set in $Vect^1(S^2)$ can revolve around either equilibrium points or limit cycles, if there is only a finite number of special points or cycles.

Smale have significantly advanced the study of differential equations on manifolds, proving that the previous theorem holds for all vector fields in $Vect^k(M^n)$, where M is compact and n arbitrary.

1.3 Smale Horseshoe

Suppose a unit square is given, which we divide into 5 equal parts, both in the horizontal and vertical direction. Colour the second and fourth rectangles thus obtained. The Smale horseshoe map f then transforms the horizontal rectangles into the vertical, and the vertical rectangles into the horizontal. At first sight, this map does not immediately seem to be useful, but Smale's idea have revolutionised the study of dynamical systems.

1.4 Symbolic Dynamics

Let $\Lambda = \{p \,|\, f^n p \text{ is defined for all } n \in \mathbb{Z}\}$.

Definition 1.6. A destiny for a point $p \in \Lambda$ is $\omega = \ldots\omega_{-n}\ldots\omega_0\ldots\omega_n\ldots$ such that $\omega_n = j$ if and only if $f^n(p) \in D_j$.

Theorem 1.7

Each sequence of 0 and 1 is realisable as a unique destiny of some point.

Let $p \in \Lambda$ and $\omega = \omega(p)$. What is $\omega(f(p))$? If σ_w is a shift of ω to the left once, then the answer is $\sigma_w\omega$. As a result, f has a countable number of periodic orbits.

Note. Suppose that a point p is periodic. Then her destiny is a periodic sequence.

Now, let ω be a periodic sequence. Then by Theorem 1.7 we know that there exists $p \in \Lambda$ such that $\omega = \omega(p)$, which means that p is also periodic.

Now, assume that the period of ω is n. Then $\sigma^n\omega = \omega$. But we know that $\omega(f^n p) = \sigma^n\omega = \omega = \omega^p$, and thus $p = f^n(p)$.