Let $p_0, \dots, p_3 \in \mathcal{P}_3(\mathbb{R})$ and

$$f(t) = \det(A_4) = \det \begin{vmatrix} p_0(t) & p_1(t) & p_2(t) & p_3(t) \\ p_0'(t) & p_1'(t) & p_2'(t) & p_3'(t) \\ p_0''(t) & p_1''(t) & p_2''(t) & p_3''(t) \\ p_0'''(t) & p_1'''(t) & p_2'''(t) & p_3'''(t) \end{vmatrix}.$$

.

Claim. $f(t) = f(0)$ for all $t \in \mathbb{R}$.

Proof. Let the given polynomials be represented as follows:

$$p_0(t) = a_{0,0} + a_{1,0}t + a_{2,0}t^2 + a_{3,0}t^3 \tag{1}$$
$$p_1(t) = a_{0,1} + a_{1,1}t + a_{2,1}t^2 + a_{3,1}t^3 \tag{2}$$
$$p_2(t) = a_{0,2} + a_{1,2}t + a_{2,2}t^2 + a_{3,2}t^3 \tag{3}$$
$$p_3(t) = a_{0,3} + a_{1,3}t + a_{2,3}t^2 + a_{3,3}t^3 \tag{4}$$

Therefore, row-reducing repeatedly without changing the value of the determinant and assuming first that $t \neq 0$,

$$\begin{vmatrix} p_0(t) & p_1(t) & p_2(t) & p_3(t) \\ p_0'(t) & p_1'(t) & p_2'(t) & p_3'(t) \\ p_0''(t) & p_1''(t) & p_2''(t) & p_3''(t) \\ p_0'''(t) & p_1'''(t) & p_2'''(t) & p_3'''(t) \end{vmatrix} = \tag{5}$$

$$\begin{vmatrix} p_0(t) & p_1(t) & p_2(t) & p_3(t) \\ p_0'(t) & p_1'(t) & p_2'(t) & p_3'(t) \\ 6a_{3,0}t + 2a_{2,0} & 6a_{3,1}t + 2a_{2,1} & 6a_{3,2}t + 2a_{2,2} & 6a_{3,3}t + 2a_{2,3} \\ 6a_{3,0} & 6a_{3,1} & 6a_{3,2} & 6a_{3,3} \end{vmatrix} = \tag{6}$$

$$\begin{vmatrix} p_0(t) & p_1(t) & p_2(t) & p_3(t) \\ 3a_{3,0}t^2 + 2a_{2,0}t + a_{1,0} & 3a_{3,1}t^2 + 2a_{2,1}t + a_{1,1} & 3a_{3,2}t^2 + 2a_{2,2}t + a_{1,2} & 3a_{3,3}t^2 + 2a_{2,3}t + a_{1,3} \\ 2a_{2,0} & 2a_{2,1} & 2a_{2,2} & 2a_{2,3} \\ 6a_{3,0} & 6a_{3,1} & 6a_{3,2} & 6a_{3,3} \end{vmatrix} = \tag{7}$$

$$\begin{vmatrix} p_0(t) & p_1(t) & p_2(t) & p_3(t) \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} \\ 2a_{2,0} & 2a_{2,1} & 2a_{2,2} & 2a_{2,3} \\ 6a_{3,0} & 6a_{3,1} & 6a_{3,2} & 6a_{3,3} \end{vmatrix} = \begin{vmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} \\ 2a_{2,0} & 2a_{2,1} & 2a_{2,2} & 2a_{2,3} \\ 6a_{3,0} & 6a_{3,1} & 6a_{3,2} & 6a_{3,3} \end{vmatrix} \tag{8}$$

Note that by direct substitution the same result follows in case $t = 0$.

From the last expression, note that the determinant is not dependent on t, and thus $f(0) = f(t)$ for all t.

If $f \neq 0$, then none of any two rows are multiples of each other, since otherwise we obtain the contradiction that the determinant is zero. Therefore, the column vectors form a linearly independent set of cardinality 4. Since $\mathscr{P}_3(\mathbb{R})$ has the dimension of 4, then this set of column vectors must be a basis.

If, on the other hand, the column vectors form a basis of $\mathscr{P}_3(\mathbb{R})$, then a linear combination of these vectors is not zero, provided that not all scalar coefficients of the corresponding vectors are zero.

Note that

$$f(t) = \sum_{\sigma} (-1)^{|\sigma|} (A_4)_{\sigma(1)1} \dots (A_4)_{\sigma(4)4}, \tag{9}$$

where the summation goes over all permutations σ, and $|\sigma|$ is the parity function, which is equal to 0 if the permutation σ is even and to 1 if it is odd.

Note also that since the first row of A_4 consists of polynomials of degree 4 with non-zero leading coefficients, then none of the entries below the entries in the first row are zero.

Therefore, none of $(A_4)_{\sigma(i)i}$ are zero, and hence each term in equation (9) is not zero.

Moreover, equation (9) yields $4! = 24$ terms, in which each corresponding $3! = 6$ terms contain exactly one element from the corresponding column of the first row. Group them accordingly in such a way that the linear combination of the entries in the first row is obtained. Since the set of the vectors in the first row is linearly independent, and scalar coefficients (not containing the product of the entries themselves) before each term are equal to -1 or 1, then the resultant sum is not zero, and hence $f(t) \neq 0$. $\square$