16/16

Name: ______________________

Student Number: ______________________

1. *[16 marks]* Write the following without absolute values; justify your answers.

(i) $|\sqrt{5}+\sqrt{6}-5| = 5-(\sqrt{5}+\sqrt{6})$ 4/4

1) $-5^2+(\sqrt{5}+\sqrt{6})^2 = (11+2\sqrt{30})-25 = 2\sqrt{30}-14$

2) $(2\sqrt{30})^2-14^2 = 120-196 = -76 < 0$

(ii) $\dfrac{|1+2\sqrt{2}-\sqrt{13}|}{|\sqrt{15}-4|} = \dfrac{1+2\sqrt{2}-\sqrt{13}}{4-\sqrt{15}}$ 4/4

1) $(1+2\sqrt{2})^2-(\sqrt{13})^2 = 9+4\sqrt{2}-13 = 4(\sqrt{2}-1) > 0$, since $2>1$.

2) $15-16=-1<0$

Find the least upper bound, if any, of each of the following sets; justify your answers.

NO LUB

(iii) $\left\{\frac{1}{x^2}\,\middle|\, x\in(3,4]\cap\mathbb{Q}\right\} = S$. Suppose there is $a \in (3,4]\cap\mathbb{Q}$ s.t. $\frac{1}{a^2} \geq \frac{1}{x^2} \mid x\in(3,4]\cap\mathbb{Q}$. By DEFINITION OF AN INTERVAL, $a>3$. But then there is ! $3<b<a$, $\Rightarrow \frac{1}{b^2} > \frac{1}{a^2}$, which is a contradiction 4/4

(iv) $S = \{x\sin(x) | x\in\mathbb{R}\}$. Consider $\ell \in \mathbb{R}$ s.t. $\ell\sin(\ell) \geq x\sin(x) \mid x\in\mathbb{R}$.

no LUB? THEREFORE, $\ell \geq \pi/2$, since $\frac{\pi}{2}\sin(\pi/2) = \pi/2 \in S$, AND $\ell(\sin\ell) \geq \pi/2$.

Consider also $\ell' = \ell + 2\pi > \ell$. $\Rightarrow \ell'\sin(\ell') - \ell\sin(\ell) =$

$= \sin(\ell)(\ell+2\pi-\ell) = 2\pi\sin(\ell) > 0$,

WHICH IS A CONTRADICTION. 4/4

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2. *[12 marks]* Consider the set

$$F = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}.$$

In fact, F is a field (but do not spend the time to prove it now).

Prove that F does not contain $\sqrt{3}$. *(Use the kind of argument we used to prove that $\sqrt{2}$ is irrational, not the kind relying on the construction of real numbers as sets of rational numbers...).*

SUPPOSE $\exists(a, b \in \mathbb{Q}): \quad a + b\sqrt{2} = \sqrt{3} \iff$

$\iff \quad a = \sqrt{3} - b\sqrt{2} \implies a^2 = 3 - 2b\sqrt{3}\sqrt{2} + 2b^2 \iff$

But then $a^2 + 2b\sqrt{3}\sqrt{2} = 3 + 2b^2$.

Now, LHS $= a^2 + 2b(a + b\sqrt{2})\sqrt{2}$

$= \underbrace{(a^2 + 4b^2)}_{\in \mathbb{Q}} + \underbrace{2ab}_{\in \mathbb{Q}}\sqrt{2} \in F.$

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On the other hand, RHS $\in F$ AND RHS $\in \mathbb{Q}$.

But then $3 + 2b^2 = c + 0\sqrt{2} \quad \exists c \in \mathbb{Q}$.

$\implies 2b\sqrt{3} = 0$. Since $2\sqrt{3} \neq 0$, $b = 0 \implies$

$\sqrt{3} = a \in \mathbb{Q}$. But $\sqrt{3}$ is IRRATIONAL. (suppose $\sqrt{3} = \frac{p}{q}$, $\gcd(p,q)$, $p, q \in \mathbb{Q} \implies$ $p^2 = 3q^2 \implies 3|p \implies 9|p^2 \implies 3|q^2$ #)

Hence, this is a contradiction.

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3. *[10 marks]* Suppose α is a real number satisfying $\alpha > 0$. Prove directly, i.e., from the definitions without using results proved in class or in the text or in the problem sets, that $\alpha < 2 \cdot \alpha$.

Consider $2 \cdot \alpha$. NOTE THAT:

$2 = \{x \in \mathbb{Q} : x < 2\}$. NOTE ALSO THAT

$2 \cdot \alpha = \{x \in \mathbb{Q} : x \leq 0$ OR $x < pq \; \forall (p \in \alpha, q \in 2, p > 0, q > 0)\}$

Since $\bar{1} \subset \bar{2}$ (since $\forall e \in \bar{1} : e < 1$ AND $1 \in \mathbb{Q} < 2 \in \mathbb{Q}$ THEN: $e \in \bar{2}$.)), TAKE $q \in \mathbb{Q} = 1$.

BY DEFINITION OF $2 \cdot \alpha$, $\exists x \in \mathbb{Q} : x \leq 0$ OR $x < p$ AND $x \in 2 \cdot \alpha$. But then $x \in \alpha$, since $x \leq 0$ OR $x < p \; \forall (p \in \alpha)$. BY THE NON-EXISTENCE OF THE LEAST ELEMENT Suppose now, $y \in \alpha$

$\Rightarrow$ y · (y) < pq ∀ ... ∈ 2·α, since $\alpha > 0$ (HENCE $y \leq 0$)

THEREFORE, $\alpha < 2 \cdot \alpha$

But why is $\alpha \neq 2\alpha$.

You're going a bit backwards.
If $x \in \alpha$, then since $1 < 2$ you have that [illegible]

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4. *[12 marks]* Consider the set

$$\alpha = \{x \in \mathbb{Q} | x < 0 \text{ or } x^2 < 2\}.$$

We showed in class that α is a real number. Prove that

$$\alpha \cdot \alpha = 2,$$

i.e., that α is a square-root of 2.

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BY DEFINITION,

$\alpha \cdot \alpha = \{x \in \mathbb{Q} : x \leq 0$ OR $x < \cancel{y^2}\ yz\ \forall (y > 0, y \in \alpha)\}$ $y, z \in \alpha$ $y, z > 0$

$2 = \{x \in \mathbb{Q} : x < 2\}$

Suppose $x \leq 0$ AND $x \in \alpha \Rightarrow x \in 2$.

Suppose $x > 0$ AND $x < y^2\ \forall (y > 0, y \in \alpha)$.

Since $\forall (y \in \alpha) : y < 2$, $x < 2 \Rightarrow x \in 2$.

Suppose now $z \in 2$, IF $z \leq 0$, THEN BY DEF $z \in \alpha \cdot \alpha$. IF $z < 2$, since $\forall (x \in \alpha \cdot \alpha, \cancel{x > 0}) : x < y^2)$ AND $y^2 < 2$, THEN $z \in \alpha \cdot \alpha$.

THEN $\alpha \cdot \alpha = 2$.