1 Invariant Subspaces

Theorem 1.1

If $T \in \text{Hom}(V, V)$ and $W \subseteq V$ is T-invariant, then the characteristic polynomial of T_W, $f_w(t)$, divides the characteristic polynomial of T, $f(T)$.

Proof. Pick an ordered basis $\alpha = \{v_1, v_2, \ldots, v_d\}$ of W and extend it to an ordered basis $\beta = \{v_1, \ldots, v_n\}$ of V.

Then $[T]_\beta = \begin{pmatrix} [T_W]_\alpha & & * \\ & \ddots & \\ 0 & & A \end{pmatrix}$.

Note that

$$f(t) = \det([T]_\beta - tI) \tag{1}$$

$$= \det \begin{pmatrix} [T_W]_\alpha - tI_W & & * \\ & \ddots & \\ 0 & & A - tI_A \end{pmatrix} \tag{2}$$

$$= \det([T_W]_\alpha - tI)\det(A - tI) = f_W(t)\det(A - tI) \tag{3}$$

□

Theorem 1.2

Consider $T \in \text{Hom}(V, V)$ and non-zero $v \in V$, where V is a finite-dimensional vector space. Let W be a T-cyclic subspace generated by v.

Let $d \geq 1$ be the largest integer such that $v, T(v), \ldots, T^{d-1}(v)$ are linearly independent. Then $v, T(v), \ldots, T^{d-1}(v)$ is a basis of W and $d = \dim W$.

Proof. The largest d exists, since $\dim V$ is finite.

Let $U = \text{span}(v, T(v), \ldots, T^{d-1}(v)) \subseteq W$.

Claim. U is T-invariant.

Proof. Suppose $w = c_0 v + c_1 T(v) + \cdots + c_{d-1}T^{d-1}(v)$ and $T(w) = 0$.

Then $T(c_0 v + c_1 T(v) + \cdots + c_{d-1}T^{d-1}(v)) = c_0 Tv + c_1 T^2 v + \cdots + c_{d-1}T^d v = 0$

Since d is the largest integer such that $v, T(v), \ldots, T^{d-1}(v)$ are linearly independent, then c_{d-1} is non-zero, and thus $T^d(w) \in U$. □

U is T-invariant, and thus if $v \in U$, then $W \subseteq U$, since W is the smallest T-invariant subspace containing v. By definition of U, $U \subseteq W$, and thus $U = W$. Therefore, $\dim W = d$

□

Theorem 1.3

If $T^d v + a_{d-1}T^{d-1}v + \cdots + a_1 Tv + a_0 v = 0$, then the characteristic polynomial of T_W is

$$f_W(t) = (-1)^d(t^d + a_{d-1}t^{d-1} + \cdots + a_0)$$

Proof. Let $\beta = \{v, T(v), \ldots, T^{d-1}(v)\}$.
Then

$$[T]_\beta = \begin{pmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & & & -a_1 \\ 0 & 1 & & \vdots & -a_2 \\ & \ddots & \ddots & & \vdots \\ & & & 1 & -a_{d-1} \end{pmatrix}$$

Therefore,

$$\det([T]_\beta - tI) = \det \begin{pmatrix} -t & 0 & \ldots & 0 & -a_0 \\ 1 & -t & & & -a_1 \\ 0 & 1 & & \vdots & -a_2 \\ & \ddots & \ddots & & \vdots \\ & & & 1 & -a_{d-1} - t \end{pmatrix}$$

Now we use induction on d.
If $d = 1$, $\det(-a_0 - t) = -t - a_0 = (-1)(t + a_0)$.
Suppose that the claim is true fr $d - 1$. Consider the claim for d:

$$\begin{aligned} \det([T]_\beta - tI) &= \det \begin{pmatrix} -t & 0 & \ldots & 0 & -a_0 \\ 1 & -t & & & -a_1 \\ 0 & 1 & & \vdots & -a_2 \\ & \ddots & \ddots & & \vdots \\ & & & 1 & -a_{d-1} - t \end{pmatrix} \\ &= (-t)\det \begin{pmatrix} -t & & & -a_1 \\ 1 & & \vdots & -a_2 \\ \ddots & \ddots & & \vdots \\ & & 1 & -a_{d-1} - t \end{pmatrix} + (-a_0)(-1)^{1+d}\det \begin{pmatrix} 1 & -t & & \\ 0 & 1 & & \vdots \\ & \ddots & \ddots & \\ & & & 1 \end{pmatrix} \\ &= -(1)^d(t^d + a_{d-1}t^{d-1} + \cdots + a_1 t) + (-1)^d a_0, \end{aligned}$$

as required. $\square$

Example 1.4

$$T : \mathfrak{P}_3(\mathbb{R}) \to \mathfrak{P}_3(\mathbb{R}) \tag{4}$$
$$T(f(x)) = xf'(x) - f(x) \tag{5}$$

If $f(x) = x^3 - 1$, then

$$T(f(x)) = x(3x^2) - (x^3 - 1) = 2x^3 + 1 \tag{6}$$
$$T^2(f(x)) = T(2x^3 + 1) = x(6x^2) - (2x^3 + 1) = 4x^3 - 1 \tag{7}$$

Note that the first two are linearly independent, while all of three are linearly dependent.

Therefore, the T-cyclic subspace W generated by $f(x)$ has a basis $\{x^3 - 1, 2x^3 + 1\}$.

So $T^2(f(x)) = 4x^3 - 1 = 4x^3 - 1 = 2f + 1T(f)$, giving the characteristic polynomial of T_W as $t^2 - t - 2$.

2 Cayley-Hamilton Theorem

Theorem 2.1 (Cayley-Hamilton Theorem)
Consider $T \in \mathrm{Hom}(V, V)$ with the characteristic polynomial $f(t)$. Then $f(T) = 0$.

e.g. For the linear transfomation above, the Cayley-Hamilton Theorem says that

$$T_W^2 - T_W - 2I_W = 0$$

.

Proof. We need to show that $f(T)v = 0$ for all $v \in V$.

Note that $f(T)$ is a linear transformation.

If $v = 0$, $f(T)(0) = 0$.

If $v \neq 0$, let W be a T-cyclic subspace generated by v with the dimension $d = \dim W$.

By Theorem 1.3, if $v, Tv, \dots, T^{d-1}v$ is a basis of W, then

$$T^d v + a_{d-1}T^{d-1}v + \cdots + a_0 v = 0 \tag{8}$$

and the characteristic polynomial $f_W(t)$ of T_W is such as

$$f_W(t) = (-1)^d(t^d + a_{d-1}t^{d-1} + \cdots + a_0)$$

.

By Equation (8) we see that $f_W(T)(v) = 0$.

By Theorem 1.1, $f_W(T)|f(t)$, and thus $f(t) = g(t)f_W(t)$ for some polynomial $g(t)$.

Therefore, $f(T) = g(T)f_W(T)$, which gives

$$f(T)(v) = (g(T)f_W(T))(v) = g(T)(f_W(T)(v)) = 0.$$

□

Remark 2.2. The Cayley-Hamilton Theorem can also be applied to matrices $A \in M_{n\times n}(\mathbb{F})$, which can be obtained by considering $T = L_A : \mathbb{F}^n \to \mathbb{F}^n$.

Theorem 2.3

Let $T \in \mathrm{Hom}(V, V)$ and $V = W_1 \oplus \cdots \oplus W_k$, each subspace W_i being T-invariant. Then $f(T) = f_1(t) \cdots f_k(t)$, where $f(T)$ is a characteristic polynomial of T and $f_i(T)$ is a characteristic polynomial of T_{W_i}.

Proof. Pick an ordered basis β_i of W_i for $i = 1, \ldots, k$, and let $\beta = \beta_1 \cup \cdots \cup \beta_k$. Since the sum of W_i is direct, β is a basis of V.

Order β canonically.

Then

$$[T]_\beta = \begin{pmatrix} [T_{W_1}]_{\beta_1} & 0 & & \cdots & 0 \\ 0 & [T_{W_2}]_{\beta_2} & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & & \cdots & & [T_{W_k}]_{\beta_k} \end{pmatrix} \tag{9}$$

Therefore,

$$\det[T]_\beta = \det \begin{pmatrix} [T_{W_1}]_{\beta_1} - tI_{\beta_1} & 0 & & \cdots & 0 \\ 0 & [T_{W_2}]_{\beta_2} - tI_{\beta_2} & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & \cdots & & & [T_{W_k}]_{\beta_k} - tI_{\beta_k} \end{pmatrix} \tag{10}$$

$$= \prod_{i=1}^{k} \det([T_{W_i}]_{\beta_i} - tI_{\beta_i}) \tag{11}$$

$$= \prod_{i=1}^{k} f_i(t) \tag{12}$$

□