MAT 157: REAL NUMBERS

DECIMAL EXPANSIONS

EXAMPLE

$$0.\bar{1} = \frac{1}{9}$$

$$0.\bar{1} + 0.\bar{1} = 0.\bar{2}$$

$$0.\bar{1} + 0.\bar{9} = 1.\bar{1}$$

$1 = 0.\bar{9} \Rightarrow$ DECIMAL EXPANSIONS ARE NOT UNIQUE.

CONSIDER A SET OF RATIONAL NUMBERS α WIT THE FOLLOWING PROPERTIES:

1. $x \in \alpha \wedge y < x \Rightarrow y \in \alpha$
2. $\alpha \notin \emptyset$
3. $\alpha \subset \mathbb{Q}$
4. $\forall y \in \alpha \; \exists x \in \alpha : x > y$ :: NO UPPER BOUND.

EXAMPLE

$$\alpha = \{ x \in \mathbb{Q} : x < 0 \vee x^2 < 2 \}$$

THE SET α CORRESPONDS TO $\sqrt{2} \notin \mathbb{Q}$

CHECK:

1. 1) $x \leq 0$ OK

   2) $x > 0 \wedge y < 0$ OK

   3) SUPPOSE $x, y > 0$

   $\Rightarrow x^2 < 2$

   $0 < y < x$

   $\Rightarrow y^2 < x^2 < 2$

   $\Rightarrow y \in \alpha$

2. $1 \in \alpha$

3. SUPPOSE $b \in \alpha$ IS THE LARGEST ELEMENT.

   IN PARTICULAR, $b > 1$.

   $b^2 < 2$

   $\therefore r = 2 - b^2 > 0$.

   CONSIDER

   $(b+a)^2 < 2 \;;\; a > 0$

   $\Leftrightarrow b^2 + 2ab + a^2 < 2$

   $\Rightarrow a^2 + 2ab < r$

   $b^2 < 2 \Rightarrow b < 2$

   $\Rightarrow 2ab + a^2 < 4a + a^2 \Leftrightarrow ab < 2a$

   ASSUME: 1. $4a < \frac{r}{2}$

   2. $a < 1 \Rightarrow a^2 < a$

   $\Rightarrow a^2 < a < \frac{r}{8} \Rightarrow 2ab + a^2$.

   SINCE $2ab < 4a < \frac{r}{2}$ AND $a^2 < \frac{r}{8} < \frac{r}{2}$,

   $2ab + a^2 < r$ #

IF $\alpha, \beta \in \mathbb{R}$,
WE SAY $\alpha < \beta$
IF $\alpha \subset \beta$.

A SET $A \subseteq R$ IS
SAID TO BE BOUNDED
IF $\exists \beta \forall \alpha : \alpha \leq \beta$,

A LEAST UPPER BOUND
OF A IS A NUMBER λ
SUCH THAT IT IS
THE SMALLEST UPPER BOUND.

THEOREM:

ANY BOUNDED SET
OF REAL NUMBERS
HAS A UNIQUE LUB

;; FALSE FOR $\mathbb{Q}$.

eg $\{x \in \mathbb{Q} \mid x < 0 \text{ or } x^2 < 2\}$,
LUB "SHOULD BE" $\sqrt{2}$,
but $\sqrt{2} \notin \mathbb{Q}$.

PROOF

SUPPOSE A IS BOUNDED.

$$:: \lambda = \bigcup_{\alpha \in A} \alpha$$

1. SUPPOSE THAT $a, b \in \mathbb{R} \wedge a \leq b \wedge b \in \lambda$
$\Rightarrow \exists \alpha : b \in \alpha$. Moreover, $\alpha \in A$ AND $a \in \alpha < b$.

EXERCISE. CONTINUE THE PROOF.