1 Additive Actions and Hassett-Tschinkel Correspondence

1.1 Compactification of Affine Space

Let $A^n = \{(x_1, \dots, x_n) \,|\, x \in \mathbb{R} \vee x \in \mathbb{C}\}$.

Note that such a space is not compact. How can we devise its compact representation?

For example, representing A^n as a plane, we can project it onto a sphere. Choose a point on the plane, and construct a sphere touching the plane at the chosen point. Then build a correspondence between the plane and the sphere by throwing lines from the north pole at the points on the plane.

Another method is to construct a *projective space*. Lines passing through a selected point $p \in A^n$ in in this space do not intersect, and thus each point in A^n is represented uniquely. Moreover, our projective space $\mathbb{P}^n = \{[z_0 : z_1 : \cdots : z_n]\}$ is such that:

- $\{z_0, \dots, z_n\} \neq (0, \dots, 0)$
- $(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)$ for all $\lambda \neq 0$

The third method embeds A^n in a torus.

There is a wealth of materials on the problem of describing the compactifications of A^n fully. See, for example, Hirzebruch (1954).

1.2 Actions

Suppose that a group G acts on a set X, $G \times X \to X$ in such a way that $(g, x) \mapsto gx$. Thus we obtain:

1. $ex = x$ for all $x \in X$
2. $(g_1 g_2)x = g_1(g_2 x)$

Example 1.1

Let $X = A^n$, $G = (A^n, +) = \mathbb{G}_a^n$. A parallel translation can be the action:

$$(a_1, \dots, a_n)(x_1, \dots, x_n) = (x_1 + a_1, \dots, x_n + a_n).$$

Definition 1.2. An orbit of $x \in X$, denoted as Gx, is a set $\{gx \,|\, g \in G\}$.

The action of a group is called *transitive* if $X = Gx$.

Problem.

Describe all the equivariant completions of a space A^n, i.e. open embeddings $A^n \hookrightarrow X$, such that the action of parallel translations $\mathbb{G}_a^n \times A^n \to A^n$ is extended to a group action $\mathbb{G}_a^n \times X \to X$. Note that, in the context of algebraic geometry, all the functions defining the action are polynomial.

Example 1.3

Suppose that we are given an action $\mathbb{G}_a^n \times \mathbb{P}^n \to \mathbb{P}^n$ such that

$$(a_1, \ldots, a_n) \circ [z_0 : z_1 : \cdots : z_n] = [z_0 : z_1 + a_1 z_0 : \cdots : z_n + a_n z_0].$$

If $z_0 = 1$, the action is that of a parallel translation.

If $\mathrm{z}_0 = 0$, points are fixed.

Example 1.4

Suppose that we have an action $\mathbb{G}_a^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ such that

$$(a_1, a_2) \circ [z_0 : z_1 : z_2] = [z_0 : z_1 + a_1 z_0 : z_2 + a_2 z_0].$$

Exercise 1.5. Check that $(a_1, a_2)[z_0 : z_1 : z_2] = [z_0 : z_1 + a_1 z_0 : z_2 + a_1 z_1 + (\frac{a_1^2}{2} + a_2) z_0]$ is also an action, but different from the action above.

1.3 Finite-Dimensional Algebras

Suppose that A is a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ and that bilinear *multiplication* $A \times A \to A$ is defined as $(a, b) \mapsto ab$.

We require the multiplication to be asociative, commutative, and to have a unit element 1 in A such that $1 \cdot a = a \cdot 1 = a$.

Vector spaces $\mathbb{R}, \mathbb{C}$ with standard multiplication operators and $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$ with a term-by-term standard multiplication are several examples of such a vector space.

Definition 1.6. An *ideal* $I \subseteq A$ is a subspace such that for all $a \in A$ and $b \in I$ we have $ab \in I$.

1.4 Quotient Algebra

We define a quotient space as $A/I = \{a + I \,|\, a \in A\}$ with the operation of multiplication defined so that $(a + I)(b + I) = ab + I$.

For example, $\mathbb{C}[x, y]/(x^3, xy, y^2) = \{\alpha_0 1 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2\}$.

Problem.

Classify all finite-dimensional algebras over $\mathbb{C}$.

Exercise 1.7 (Hilbert's Basis Theorem)**.** All such algebras are in the form $\mathbb{C}[x_1, \ldots, x_n]/I$.

Definition 1.8. An ideal $I \subset A$ is called *maximal*, if $I \subseteq J \subseteq A$ implies that $I = J$ or $J = A$.

Definition 1.9. Algebra is defined as *local* if in A there exists a unique maximal ideal.

Example 1.10

$\mathbb{C}[x, y]/(x^3, xy, y^2)$ defined earlier is local.

Definition 1.11. $a \in A$ is called revertible, if there exists $b \in A$ such that $ab = 1$.

Definition 1.12. $a \in A$ is called nilpotent, if there exists $m > 0$ such that $a^m = 0$.

Problem.

For algebras over $\mathbb{C}$, prove that

- if a is nilpotent, then $1 + a$ is revertible
- if A is local, then it is representable in the form $\langle 1 \rangle \oplus \mathfrak{M}$, where $\mathfrak{M}$ is a maximal ideal in A, all $a \in \mathfrak{M}$ is nilpotent and all $a \in A/\mathfrak{M}$ are revertible.
- Show that all finite-dimensional algebras can be uniquely decomposed into the direct sum of local algebras.

If we look at the number of algebras of particular dimension, we get the following picture:

- for $\dim A = 1$, the only algebra is $\mathbb{C}$.
- for 2, the only algebra is $\mathbb{C}[x]/(x^2)$
- for 3, there are two algebras: $\mathbb{C}[x]/(x^3)$ and $\mathbb{C}[x, y]/(x^2, xy, y^2)$
- for 4, there are 4 algebras
- for 5, we get 9 algebras
- for 6, there are 25 numbers
- for ≥ 7, there is an infinite number of algebras

1.5 Hassett-Tschinkel Correspondence

Hassett and Tschinkel (1999) have shown that, over $\mathbb{C}$ the set of equivariant compactifications $A^n \hookrightarrow \mathbb{P}^n$ is equivalent to the set of local associative commutative algebras with unity of dimension $n + 1$.

Define $\exp(a)$ for $a \in A$ as

$$\exp(a) = 1 + \frac{a}{1!} + \frac{a^2}{2!} + \dots$$

If a is nilpotent, then we obtain a polynomial.

Exercise 1.13. $\exp(a)\exp(b) = \exp(a + b)$.

Proof ($\Leftarrow$).

Suppose that A is a local algebra with an action $\mathbb{G}_a^n \times \mathbb{P}^n \to \mathbb{P}^n$ such that $\mathbb{P}^n = \mathbb{P}(A)$ and $\mathbb{G}_a^n = \exp(\mathfrak{M}) = 1 + \mathfrak{M}$.

Exercise 1.14. Continue the proof.

$\square$

The proof in the other direction requires the notion of cyclic modules, representation theory and Lie algebras.