Theorem 0.1

Suppose f is integrable and $m \leq f(x) \leq M$ for all x in $[a, b]$. Then, $m(b-a) \leq \int_a^b f \leq M(b-a)$.

Proof. Note that $m(b-a) \leq L(f, P)$ and $U(f, P) \leq M(b-a)$ for any partition P. Then $m(b-a) \leq L(f, P) \leq \int_a^b f \leq U(f, P) \leq M(b-a)$. □

1 Fundamental Theorem of Calculus

Theorem 1.1

Let f be integrable on $[a, b]$ and define F on $[a, b]$ by $F(x) = \int_a^x f$.

Then F is continuous on $[a, b]$.

Proof. Take c in $[a, b]$. Since f is integrable on $[a, b]$, it is bounded. Define M as a number such that $\left|f(x)\right| \leq M$ for all $x \in [a, b]$. If $h > 0$, then

$$F(c+h) - F(c) = \int_a^{c+h} f - \int_a^c f = \int_c^{c+h} f.$$

Since $-M \leq f(x) \leq M$ for all x, it follows from Theorem 0.1 that

$$-M \cdot h \leq \int_c^{c+h} f \leq M \cdot h$$

Thus,

$$-M \cdot h \leq F(c+h) - F(h) \leq M \cdot h$$

If $h < 0$, a similar inequalitiy can be derived, since $F(c+h) - f(h) = -\int_{c+h}^c f$. Therefore, for $[c+h, c]$,

$$Mh \leq \int_{c+h}^c f \leq -Mh,$$

which gives

$$M \cdot h \leq F(c+h) - F(h) \leq -M \cdot h$$

Thus,

$$\left|F(c+h) - F(h)\right| \leq M|h|$$

Therefore, for any $\epsilon > 0$, if $|h| < \frac{\epsilon}{M}$,

$$\left|F(c+h) - F(h)\right| < \epsilon$$

□

Theorem 1.2

Let f be integrable on $[a, b]$ and define F on $[a, b]$ by $F(x) = \int_a^x f$.

If f is continuous at c in $[a, b]$, then F is differentiable at c, and $F'(c) = f(c)$.

Proof. Suppose first that c is in (a, b). By definition,

$$F'(c) = \lim_{h \to 0} \frac{F(c+h) - F(c)}{h}.$$

For each h, define m_h and M_h as follows:

$$m_h = \inf\{f(x) \,|\, c \leq x \leq c + h\} \tag{1}$$

$$M_h = \sup\{f(x) \,|\, c \leq x \leq c + h\} \tag{2}$$

From Theorem 0.1,

$$m_h \cdot h \leq \int_c^{c+h} f \leq M_h \cdot h,$$

and thus

$$m_h \leq \frac{F(c+h) - F(c)}{h} \leq M_h$$

If $h \leq 0$, then

$$m_h = \inf\{f(x) \,|\, c + h \leq x \leq c\} \tag{3}$$

$$M_h = \sup\{f(x) \,|\, c + h \leq x \leq c\} \tag{4}$$

From Theorem 0.1,

$$m_h \cdot (-h) \leq \int_{c+h}^{c} f \leq M_h \cdot (-h),$$

and thus

$$m_h \cdot (-h) \geq F(c+h) - F(c) \geq M_h \cdot (-h).$$

For $h < 0$,

$$m_h \leq \frac{F(c+h) - F(c)}{h} \leq M_h.$$

This inequality holds for any integrable function. However, since f is continuous,

$$\lim_{h \to 0} m_h = \lim_{h \to 0} M_h = f(c)$$

and thus

$F'(c) = \lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = f(c)$. ☐

Note. If G is defined by $G(x) = \int_x^b f$, then $G(x) = \int_a^b f - \int_a^x f$. Therefore,

$$G'(c) = -f(c)$$