Consider $V = \mathcal{P}_2(\mathbb{F})$ with the inner product $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)\overline{g(t)}dt$.

Let $\beta = \{1+x, x^2\}$ and let W be the subspace of V spanned by β.

Problem. Find $W^{\perp}$.

Solution

Let $v_1 = 1+x$ and $v_2 = x^2$. Therefore, $\langle 1+x, 1+x\rangle = \int_{-1}^{1}(1+x)^2 = [x + x^2 + {}^1\!/_3 x^3]_{-1}^{1}$, and thus $\|1+x\|^2 = 2 + {}^2\!/_3 = \frac{8}{3}$.

i	v_i	$\sum_{j=1}^{i-1} \frac{\langle v_i, u_j\rangle}{\langle u_j, u_j\rangle} u_j$	u_i	$\|u_i\|^2$	e_i
1	$1+x$	–	$1+x$	$\int_{-1}^{1}(1+x)^2 = [x+x^2+{}^1\!/_3x^3]_{-1}^{1}$ $\|1+x\|^2 = 2+{}^2\!/_3 = \frac{8}{3}$.	$\sqrt{6}/4(1+x)$
2	x^2	$\frac{\langle v_2,u_1\rangle}{\langle u_1,u_1\rangle}u_1 =$ $\frac{\int_{-1}^{1} x^2+x^3dx}{8/3}(1+x) =$ $\frac{[1/3x^3+1/4x^4dx]_{-1}^{1}}{8/3}(1+x) =$ $\frac{1}{4}(1+x)$	$x^2 - \frac{1}{4}(1+x)$ $x^2 - \frac{1}{4}x - {}^1\!/_4$	$\int_{-1}^{1}(x^2 - \frac{1}{4}x - {}^1\!/_4)^2$ $= \frac{7}{30}$	$\sqrt{\frac{30}{7}}(x^2 - \frac{1}{4}x - {}^1\!/_4)$

Note that 1 is not in the span of β. Let $v_3 = 1$.

Therefore,

$$\sum_{j=1}^{2} \frac{\langle v_3, u_j\rangle}{\langle u_j, u_j\rangle}u_j = \frac{\langle 1, u_1\rangle}{\langle u_1, u_1\rangle}u_1 + \frac{\langle 1, u_2\rangle}{\langle u_2, u_2\rangle}u_2 \tag{1}$$

$$= \frac{\int_{-1}^{1}(1+x)dx}{8/3}(1+x) + \frac{\int_{-1}^{1}(x^2 - 1/4\ x - 1/4)dx}{7/30}(x^2 - 1/4\ x - 1/4) \tag{2}$$

$$= 3/4 + 3/4\ x + 5/7\ x^2 - 5/28\ x - 5/28 \tag{3}$$

$$= 5/7\ x^2 + 4/7\ x + 4/7 \tag{4}$$

Thus,

$$u_3 = 1 - (5/7\ x^2 + 4/7\ x + 4/7) = -5/7\ x^2 - 4/7\ x + 3/7 \tag{5}$$

Since $\beta' = \{u_1, u_2\}$ is a basis for W, while $\beta' \cup \{u_3\}$ is a basis for V by construction, since $V = W \oplus W^{\perp}$, then $W^{\perp}$ is spanned by u_3.

Note that $\|u_3\|^2 = \int_{-1}^{1}(-5/7\ x^2 - 4/7\ x + 3/7)^2dx = 8/21$, and thus

$$e_3 = \frac{\sqrt{42}}{4}(-5/7\ x^2 - 4/7\ x + 3/7)$$

Problem. Fix $u \in \mathbb{F}$ and suppose that $\theta_u : V \to F$ is the linear function given by $\theta_u(f(x)) = f(u)$. Find $g_u(x) \in V$ such that $\theta_u(f(x)) = \langle f(x), g_u(x)\rangle$ for all $f(x) \in V$.

Solution

Let $y = \sum_{i=1}^{n} \overline{\theta_u(e_i)}e_i$, where e_i are orthonormal vectors given in the tables above. Then by Theorem 6.8 in Friedberg *et al*, we obtain that $g_u(x) = y$.

The coefficients before each e_i in the expression for y are given below:

- $\overline{\theta_u(e_1)} = \frac{\sqrt{6}}{4}(1+u)$

- $\overline{\theta_u(e_2)} = \sqrt{\frac{30}{7}}(u^2 - \frac{1}{4}u - 1/4)$
- $\overline{\theta_u(e_3)} = \frac{\sqrt{42}}{4}(-5/7\ u^2 - 4/7\ u + 3/7)$

Problem. Suppose that $T : V \to V$ is defined by $T(f(x)) = f'(x)$. Find $T^*(1 + x)$.

Solution

Note that $\beta = \{1, x, x^2\}$ is an orthonormal basis of $\mathcal{P}_2(\mathbb{F})$.

Note the following:

- $T(1) = 0$
- $T(x) = 1$
- $T(x^2) = 2x$

Therefore,

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Since $[T^*]_\beta = [T_\beta]^*$, it follows that $T^*(1 + x)$ can be represented in the matrix form as

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Therefore, $T^* = x + 2x^2$.