1 Review

$T \in \text{Hom}(V,V)$ is unitary/orthogonal if

- $\|Tx\| = \|x\|$
- $\langle Tx, Ty\rangle = \langle x, y\rangle$
- $TT^* = T^*T = I$
- T sends an orthonormal basis to an orthonormal basis.

Definition 1.1. For $A, B \in M_{n\times n}(\mathbb{F})$, A and B are called unitarily/orthogonally equivalent if there exists Q unitary/orthogonalsuch that $A = Q^{-1}BQ$.

Remark 1.2. If β is an orthonormal basis, then T is unitary/orthogonal if and only if $[T]_\beta$ is unitary/orthogonal.

Remark 1.3. If β, β' are orthonormal bases, then the change-of-basis matrix $[I]_\beta^{\beta'}$ is unitary/orthogonal.

e.g. If β is a standard basis, which is orthonormal, then $[I]_\beta^{\beta'}$ is unitary/orthogonal, since colums are an orthonormal basis of $\mathbb{F}^n$. Thus, if β, β' are orthonormal bases, then $[T]_\beta$, $[T]_{\beta'}$ are unitarily/orthogonally equivalent.

Lemma 1.4

Let $T \in \text{Hom}(V,V)$. If T has an eigenvalue λ, then T^* has an eigenvalue $\overline{\lambda}$.

Theorem 1.5

If the characteristic polynomial of T splits, then there exists an orthonormal basis β such that $[T]_\beta$ is upper-triangular.

Proof.

We proceed by induction on $\dim V$.

If $n = 1$, then the change-of-basis matrix is one by one, and hence upper-triangular.

If $n > 1$, assume the claim holds for $n - 1$.

Since the characteristic polynomial $f(t)$ of T splits, then T has an eigenvalue, say $\lambda \in \mathbb{F}$.

Then by Lemma 1.4, T^* has an eigenvalue $\overline{\lambda}$, say $T^*v = \overline{\lambda}v$, with $\|v\| = 1$.

Thus, $W = \operatorname{span} v$ is T^*-invariant, and thus $W^\perp$ is T-invariant, with $\dim W^\perp = n - 1$.

By Theorem 5.21, the characteristic polynomial of $T_{W^\perp}$ divides $f(t)$, so $f_{W^\perp}(t)$ splits.

By inductive hypothesis, hdece exits an orthonormal basis γ of $W^\perp$ such that $[T_{W^\perp}]_\gamma$ is upper-triangular. Therefore, $\beta = \gamma \cup \{v\}$ is an orthonormal basis of V.

Since $[T_{W^\perp}]_\gamma$ is upper-triangular, then $[T]_\gamma$ is upper-triangular.

Corollary 1.6

If $A \in M_{n\times n}(\mathbb{F})$ and the characteristic polynomial of A splits, then $A$is unitarily/orthogonally equivalent to an upper-triangular matrix.

$\square$

2 Orthogonal Projections and The Spectral Theorem

Let V be a finite-dimensional inner product space.

Suppose $W \subset V$. Thus, $V = W \oplus W^\perp$, and thus we can define a linear map, called an orthogonal projection onto W, $P_W \in \operatorname{Hom}(V, V)$ by $P_W(w + w^\perp) = w$.

Theorem 2.1 a) $\operatorname{im} P_W = W$, $\ker P_W = W^\perp$.

b) $P_W^2 = P_W$ and $P_W^* = P_W$

Proof.

We prove (b).

By definiton, $P'(w + w^\perp) = P_W P_W(w + w^\perp) = P_w w = w$.

To check that $P_W^* = P_W$, observe the following, for $v, w \in W$:

$$\langle P_W(w + w^\perp), v + v^\perp\rangle = \langle w, v + v^\perp\rangle = \langle w, v\rangle \tag{1}$$
$$\langle w + w^\perp, P_W(v + v^\perp)\rangle = \langle w + w^\perp, v\rangle = \langle w, v\rangle \tag{2}$$

□

Theorem 2.2 (6.24)

Suppose $T \in \operatorname{Hom}(V, V)$. Then T is an orthogonal projection if and only if $T^2 = T$ and $T^* = T$.

Proof.

We have proved the first direction in Theorem 2.1.

Suppose now $T^2 = T$ and $T^* = T$.

Let $W = \operatorname{im}(T)$. We want to show that $T = P_W$.

Since $\operatorname{im}(T^*)^\perp = \ker T$, then $W^\perp = \ker(T)$.

Since $T = T^*$, $\operatorname{im} T^* = \operatorname{im} T$.

Therefore, $V = \ker T \oplus \operatorname{im} T$.

Fix $v \in V$. Since $V = \ker T \oplus \operatorname{im} T$, there exists $w \in \operatorname{im} T$, $w' \in \ker T$ such that $v = w + w'$.

Then $Tv = T(w + w') = Tw + Tw'$, and since $w' \in \ker T$, $Tv = Tw$.

Note that $x = Tx + (x - Tx)$. Observe that $Tx = T^2x + T(x - Tx)$, and since $T = T^2$, then $x - Tx \in \ker T$. Therefore, $w + w' - Tw - Tw' = w - Tw + w' \in \ker T$, which means that $w - Tw \in \ker T$. But $w - Tw \in \operatorname{im} T$, because $\operatorname{im} T$ is T-invariant, while $V = \operatorname{im} T \oplus \ker T$, and thus $w - Tw = 0$.

Hence, $Tv = Tw = w$, and hence T is an orthogonal projection. □