MAT240 Problem Set IV: Problem 3 Wednesday 12^{th} October, 2016

Suppose on the contrary that S is linearly dependent.

Thus, $\exists(\alpha, \beta, \frac{\gamma}{2} \in \mathbb{R}, \alpha\beta\gamma \neq 0) : \alpha \sin(x) + \beta \cos(x) + \frac{\gamma}{2} \sin(2x) = 0 \; \forall(x \in \mathbb{R})$.

Hence, $\alpha \sin(x) + \beta \cos(x) + \gamma \sin(x) \cos(x) = 0$.

Take $x = \frac{\pi}{2}$. Then by the equation above $\alpha = 0$, since $\sin(\frac{\pi}{2}) = 1, \; \cos(\frac{\pi}{2}) = 0$.

Take $x = 0$. Then again by the equation above $\beta = 0$, since $\sin(0) = 0, \; \cos(0) = 1$.

Take now $x = \frac{\pi}{4}$. Since $\alpha = \beta = 0$ and $\cos(\frac{\pi}{4}) \sin(\frac{\pi}{4}) = \frac{1}{2}$, then $\frac{\gamma}{2} = 0$.

Hence, $\alpha = \beta = \gamma = 0$, which is a contradiction.