1 Analytic Perspectives in Arithmetic Statistics

Let $V = V(\mathbb{Z})$ be a latics of binary quadratic forms $au^2 + buv + cv^2$ such that $a, b, c \in \mathbb{Z}$.

Note that there is an action of $\mathrm{SL}_2(\mathbb{Z})$.

We also define a discriminant $\mathrm{Disc}(f) = b^2 - 4ac$.

We know that $\mathrm{Disc}(g \circ f) = \mathrm{Disc}(f)$ for all $g \in \mathrm{GL}_2(\mathbb{Z})$.

Let $V' = \{v \in V \,|\, \mathrm{Disc}(v) \neq 0\}$.

We also know that the action of $\mathrm{GL}_2(\mathbb{C})$ on $V'(\mathbb{C})$ has one orbit, $\mathrm{GL}_2(\mathbb{R})$ on $V(\mathbb{F})$ has two orbits, while actions of $\mathrm{GL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$ on $V'(\mathbb{Z})$ have infinitely many orbits!

Therefore, there is *more information stored* in $\mathbb{Z}$.

Let $h(D)$ be the number of orbits of discriminant D. It can be shown that it is always finite and greater than one for $D \equiv 0, 1 \mod 4$.

Gauss' Conjecture (now Theorem): If $D < 0$ and $h(D) = 1$, then we have that $D \in \{-3, -4, -7, -8, -11, -19, -43, -67, -163\}$.

Theorem 1.1

For each D, the set of equivalence classes of BQFs of Disc D forms an abelian group.

Remark 1.2. $\mathrm{Cl}(D) \to \mathrm{Cl}(\theta)$ is an isomorphism such that $\theta = [1, \frac{D+\sqrt{D}}{2}] \subseteq \theta_Q(\sqrt{\theta_Q}(\sqrt{D})$.