1 A.P.Veselov (ULoughborough): Special Functions: From Gamma to Zeta

1.1 The Story of ζ

Integrable systems and special functions share at least one property – they have not been properly defined. In particular, special functions are those which were considered as special by (?) Watson and Whittaker.

Almost all the special functions arise from the study of differential equations, but Γ and ζ are important exceptions.

We will need two formulaes: the sum of consecutive $n-1$ integers, equal to $\frac{n(n-1)}{2}$, and the sum of the squares of consecutive $n-1$ integers, equal to $\frac{n(n-1)(2n-1)}{6}$. Moreover, $S_m(n) = \sum_{j=1}^{n-1} j^m = \frac{1}{m+1}(B_{m+1}(n) - B_{m+1})$, where B_{m+1} is Bernoulli's constant. (?)

It is worthwhile to note a function with a range of wonderful properties:

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k, \tag{1}$$

where $B_k(x) = \sum_{j=0}^{k} C_k^j B_k x^{kj}$ and $C = (B + x)^k$. From this equation we can derive that $B_m(x) = (-1)^m B_m(1 - x)$, which is symmetric around $x = \frac{1}{2}$.

In 1689 Jacob Bernoulli formulated Basel's problem, which, however, was already known to Wallis in 1668. The problem required to find what the limit of $\sum_{i=0}^{\infty} \frac{1}{i^2}$ is. Euler was obsessed with the problem, calculating the limit up to 20 significant figures, eventually finding the answer – $\frac{\pi^2}{6}$.

Bernoulli's numbers have a special significance. Let's list the first few of them – $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = B_5 = \cdots = 0$, $B_4 = -\frac{1}{30} = B_8$, $B_6 = \frac{1}{42}$, $B_{10} = \frac{5}{66}$, $B_{12} = -\frac{691}{2730}$.

The exciting fact is that we, knowing Bernoulli's numbers, can represent ζ in a particularly nice form:

$$\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}. \tag{2}$$

Euler knew that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, where $s > 1$ to guarantee the convergence. However, he was also adept at seeing connections between different representations, and derived that

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}} = \prod (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots). \tag{3}$$

What did Riemann add to the definition of the ζ function?

In 1859, Riemann analytically continued its definition to $s = \sigma + it \in \mathbb{C}$.

1.2 The Story of Γ

In January 1730 Goldbach wrote a letter to Euler stating his idea of generalising the factorial function for non-integer numbers. Euler, in turn, constructed such a function for $x > 0$:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t, \tag{4}$$

which has a nice property that $\Gamma(x+1) = x\Gamma(x)$.

We can analytically continue the function to negative real numbers by using this property. Moreover, there exists a *formula of completion*:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \tag{5}$$

which follows from the fact that $\frac{1}{\Gamma(x)\Gamma(1-x) = x\prod(1-\frac{x^2}{k^2}) = \frac{\sin \pi x}{\pi}}$.

From this, we can deduce that $(-\frac{1}{2})! = \sqrt(\pi)$, which helps us compute the Gaussian integral $I = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt(\pi)$.

Moreover, calculating the area of a unit ball, we obtain $VolS^{n-1} = \frac{2(\sqrt{\pi})^N}{\Gamma(\frac{N}{2})} = NVol(B^N)$.

1.3 Connection between ζ and Γ

How do ζ and Γ relate?

A beatiful connection can be derived – $\Gamma(s)\zeta(s) = \int_{x=0}^{\infty} \frac{x^{s-1}}{e^x-1}\,\mathrm{d}x$.

The idea for the proof is as follows. First, note that

$$\frac{1}{e^x - 1} = \frac{1}{e^x(1-e^{-x})} = \sum_{k=1}^{\infty} e^{-kx}, \tag{6}$$

which means that $\sum_{k=1}^{\infty}\int_{x=0}^{\infty} x^{s-1}e^{-kx}\,\mathrm{d}x$, and thus, if $t = kx$, then the equation can be rewritten as $(\infty_{t=0}^{\infty} t^{s-1}e^{-t}\,\mathrm{d}t)(\sum_{k=1})^{\infty}\frac{1}{k^s}$.

Moreover, $\zeta(s) = \frac{\Gamma(1-s)}{2\pi i}\int \frac{z^{s-1}}{e^{-z}-1}dz$. This analytically continued function is meromorphic, with the pole at $s = 1$.

The symmetry of $zeta(s)$ is striking, and allows us to infer immediately its trivial zeros:

$$\zeta(s) = (2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)). \tag{7}$$

Trivial zeros occur when $\sin(\frac{\pi s}{2}) = 0$, and thus $s = -2k$, $k \in \mathbb{N}$.

Moreover, we can deduce that $\zeta(-n) = (-1)^n\frac{B_{n+1}}{n+1}$.

1.4 Around the Riemann Hypothesis

Empirical evidence from direct calculation have shown that all the non-trivial zeros of the Riemann zeta function lie on the critical line with $\Re = \frac{1}{2}$.

If we look at the original work by Riemann, he noted that the first few dozen non-trivial zeros do not come in pairs.

One of the applications which a resolution of the Riemann Hypothesis would bring is the illumination of how primes are distributed. The Prime Number Theorem states that $\pi(x) \sim \frac{x}{\log x}$, where $\pi(x)$ is the prime-counting function, yielding the number of primes less than or equal to x, and $f(x)\tilde{g}(x)$ means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Chebyshev have shown that, for $\psi(x) = \sum \log p$, where $p^k \leq x$ and $p \in \mathbb{P}$, the Prime Number Theorem is equivalent to the theorem that $\psi(x)\tilde{x}$.

If the Riemann Hypothesis holds, then the approximation given by the PNT is the best possible.