Let V be a vector space over the field $F = \mathbb{R}$, and $W \subset V$ a subspace.

Let $S \subset V$ be a subset with finitely many elements, $S \neq \emptyset$.

(a)

Theorem. *$W \cup S$ is a subspace of $V \Leftrightarrow S \subset W$.*

Proof.

*

$(\Leftarrow)$

Suppose that $S \subset W$. Therefore, $\forall x \in S : x \in W$. But then $W \cup S$=W by definition of $\cup$. Since W is a subspace, then $W \cup S$ is a subspace.

*

$(\Rightarrow)$

Suppose that $W \cup S$ is a subspace of V.

To obtain a contradiction, assume that $\exists x \in S : x \notin W$.

Since $x \in S$, then $x \in W \cup S$.

Since S is finite and $F = \mathbb{R}$, by $\mathbb{Q} \subset \mathbb{R}$ and Archimedean Property of Rational Numbers there exists some $p \in \mathbb{Q}, p \neq 0 : px \notin S$.

Since $W \cup S$ is a subspace, $px \in (W \cup S)$ by the definition of a subspace.

Since $px \notin S$, then $px \in W$.

But W is a subspace, and $\exists (p^{-1} \in \mathbb{Q}) : (p^{-1}p = 1 \in \mathbb{Q})$, then $pp^{-1} \cdot x = 1 \cdot x \in W$ by the definition of a subspace. But then by (VS5) in Friedberg *et al*, $x \in W$, which is a contradiction. Thus, $S \subset W$.

□

(b)

Let V be a vector space over the field $\mathbb{F} = \mathbb{Z}_2$, and W a subspace of V.

Let $S \subset V$ so that S has finitely many elements, $S \neq \emptyset$.

Let $V = \{0, 1\}$ with addition of $\mathbb{Z}_2$ and scalar multiplication such that for $c \in \mathbb{F}, a \in V : c \cdot a \in V$. V is a vector space since $\mathbb{Z}_2$ is closed under addition and multiplication and all elements in $\mathbb{Z}_2$ are also in V.

Let $S = \{1\}, W = \{0\}$. Consider W with vector addition defined as $0 + 0 = 0$ and scalar multiplication $\forall c \in \mathbb{F} : c \cdot 0 = 0$. We now prove that W is a subspace of V.

Since $0 \in V$ and $\forall(a, b \in V) : a + b \in V$, then, by Existence of an Additive Identity for $\mathbb{Z}_2$, if $a = b = 0$, then $0 + 0 = 0 \in V$. Since $0 + 0 = 0 + 0$ and $0 \in W$, vector addition of W is closed.

Similarly, since $\forall e \in \mathbb{F} : e \cdot 0 = 0$ and all elements in $\mathbb{F}$ are also in V, the product of the scalar multiplication of a scalar in $\mathbb{F}$ and $0 \in V$ is also $0 \in V$. But by definition of W, $0 \in W$ is also in V. Hence, scalar multiplication of W is closed.

Since W is also nonempty, W is a subspace of V.

Since $(W \cup S) \subseteq V$ and $S \not\subset W$, then V, S, W constitute a counterexample.