The values of $\sin x$ and $\cos x$ not in $[0, \pi]$ can be defined by a two-step piecing process:

1. If $\pi \leq x \leq 2\pi$, then
$$\sin x = -\sin(2\pi - x) \tag{1}$$
$$\cos x = \cos(2\pi - x) \tag{2}$$

2. If $x = 2\pi k + x'$ for some $k \in \mathbb{Z}$ and some $x' \in [0, 2\pi]$, then
$$\sin x = \sin(x') \tag{3}$$
$$\cos x = \cos(x') \tag{4}$$

This extended definition is consistent with all the usual properties we expect from the trigonometric functions:

1. $\sin^2 x + \cos^2 x = 1$, by the geometric argument

2. $$\sin'(x) = \cos x \tag{5}$$
$$\cos'(x) = -\sin x \tag{6}$$

   For example, if $\pi < x < 2\pi$, then $\sin x = -\sin(2\pi - x)$, and thus
$$\sin'(x) = -\sin'(2\pi - x)(-1) = \cos x.$$

   If x is a multiple of π, then considering the fact that sin is continuous in the ϵ-neighbourhood of x will give us a similar conclusion.

Theorem 0.1

If $-1 < x < 1$, then
$$\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}} \tag{7}$$
$$\arccos'(x) = \frac{-1}{\sqrt{1 - x^2}} \tag{8}$$

If $x \in \mathbb{R}$, then
$$\arctan'(x) = \frac{1}{1 + x^2}$$

Proof.
$$\arcsin'(x) = (f^{-1})'(x) \tag{9}$$
$$= \frac{1}{f'(f^{-1}(x))} \tag{10}$$
$$= \frac{1}{\sin'(\arcsin x)} \tag{11}$$
$$= \frac{1}{\cos(\arcsin x)} \tag{12}$$

Note that $\sin^2(\arcsin x) + \cos^2(\arcsin x) = 1$, and thus

$$\cos(\arcsin x)^2 = \sqrt{1 - x^2}$$

The second formula can be derived from the fact that

$$A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\,\mathrm{d}t.$$

and that $2A(\cos x) = x$.

Finally, by Pythagoras's identity,

$$\begin{aligned}
\arctan'(x) &= (h^{-1})'(x) && (13)\\
&= \frac{1}{h'(h^{-1}(x))} && (14)\\
&= \frac{1}{\tan'(\arctan x)} && (15)\\
&= \frac{1}{\sec^2(\arcsin x)} && (16)\\
&= \frac{1}{x^1 + 1} && (17)
\end{aligned}$$

□

Lemma 0.2

Suppose f has a second derivative everywhere and that the following conditions are satisfied:

$$\begin{aligned}
f'' + f &= 0 && (18)\\
f(0) &= 0 && (19)\\
f'(0) &= 0 && (20)
\end{aligned}$$

Then f is a zero function.

Proof. From the equation (18) given we obtain that

$$f'f'' + ff' = 0.$$

Therefore, $(f')^2 + f^2$ is a constant function, which by the other two conditions is equal to 0. Therefore, $f(x) = 0$ for all x.

□

Lemma 0.3

Suppose f has a second derivative everywhere and that the following conditions are satisfied:

$$f'' + f = 0 \tag{21}$$
$$f(0) = a \tag{22}$$
$$f'(0) = b \tag{23}$$

Then f is in the form $b \cdot \sin + a \cdot \cos$.

Proof. We use the result given by the previous lemma.

Let $g(x) = f(x) - b\sin x - a\cos x$.

Then

$$g'(x) = f'(x) - b\cos x + a\sin x \tag{24}$$
$$g''(x) = f''(x) + b\sin x + a\cos x \tag{25}$$

Note that $g'' + g = 0$, $g(0) = 0$, and $g'(0) = 0$, which shows that $g = 0$. □

Theorem 0.4

If $x, y \in \mathbb{R}$, then

$$\sin(x+y) = \sin x \cos x + \cos x \sin y \tag{26}$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{27}$$

Proof. For any particular number y we can shift sin so that $f(x) = \sin(x+y)$. Then $f'(x) = \cos(x+y)$ and $f''(x) = -\sin(x+y)$.

Therefore,

$$f'' + f = 0 \tag{28}$$
$$f(0) = \sin y \tag{29}$$
$$f'(0) = \cos y, \tag{30}$$

which by Lemma 0.3 implies that

$$f(x) = \sin x \cos y + \cos y \sin x,$$

and thus $\sin(x+y) = \sin x \cos x + \cos x \sin y$.

The second formula can be proven similarly.

□