Consider the set $V = \mathbb{R}$ with non-standard addition $\forall(x, y \in V) : x\tilde{+}y = x + y - 3$, non-standard scalar multiplication $\forall(x \in V, c \in \mathbb{R}) : c\tilde{\cdot}x = c(x - 3) + 3$ and non-standard neutral element $\tilde{0} = 3$.

Let $x \in V, y \in V, z \in V, a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{R}$.

1. Commutative Law :

$$
\begin{aligned}
x\tilde{+}y &= x + y - 3 && \text{Definition of } \tilde{+} && (1)\\
&= y + x - 3 && V = \mathbb{R} \text{ and Commutative Law for } \mathbb{R} && (2)\\
&= y\tilde{+}x && \text{Definition of } \tilde{+} && (3)
\end{aligned}
$$

2. Associative Law :

$$
\begin{aligned}
(x\tilde{+}y)\tilde{+}z &= (x + y - 3) + z - 3 && \text{Definition of } \tilde{+} && (4)\\
&= x + (y - 3 + z) - 3 && \text{Associative Law for } \mathbb{R} && (5)\\
&= x + (y + z - 3) - 3 && \text{Commutative Law for } \mathbb{R} && (6)\\
&= x\tilde{+}(y\tilde{+}z) && \text{Definition of } \tilde{+} && (7)
\end{aligned}
$$

3. Consider $x\tilde{+}\tilde{0}$:

$$
\begin{aligned}
x\tilde{+}\tilde{0} &= (x + \tilde{0} - 3) && \text{Definition of } \tilde{+} && (8)\\
&= x + 3 - 3 && \text{Definition of } \tilde{0} && (9)\\
&= x + 0 && \text{Existence of an Additive Inverse for } \mathbb{R} && (10)\\
&= x && \text{Existence of an Additive Identity for } \mathbb{R} && (11)
\end{aligned}
$$

4.

Theorem 0.1. *There exists an inverse element for all x in V.*

Proof. Consider $y = 3 - x$. Since $x \in V, V = \mathbb{R}, y \in V$.

Consider now $s = x\tilde{+}y$:

$$
\begin{aligned}
s &= x + y - 3 && \text{Definition of } \tilde{+} && (1)\\
&= x + 3 - x - 3 && \text{Definition of } y && (2)\\
&= x - x + 3 - 3 && \text{Commutative Law for } \mathbb{R} && (3)\\
&= 0 + 0 && \text{Existence of an Additive Inverse for } \mathbb{R} && (4)\\
&= 0 && \text{Existence of an Additive Identity for } \mathbb{R} && (5)\\
& && \Rightarrow y \text{ is the inverse element of } x && (6)
\end{aligned}
$$

□

5. Consider $1\tilde{\cdot}x$:

$$
\begin{array}{rll}
1\tilde{\cdot}x &= 1(x-3)+3 & \text{Definition of } \tilde{\cdot} \quad (7)\\
&= (x-3)1+3 & \text{Commutative Law for } \mathbb{R} \quad (8)\\
&= x-3+3 & \text{Existence of a Multiplicative Identity for } \mathbb{R} \quad (9)\\
&= x+3-3 & \text{Commutative Law for } \mathbb{R} \quad (10)\\
&= x+0 & \text{Existence of an Additive Inverse for } \mathbb{R} \quad (11)\\
&= x & \text{Existence of an Additive Identity for } \mathbb{R} \quad (12)
\end{array}
$$

6.

$$
\begin{array}{rll}
(a\cdot b)\tilde{\cdot}x &= (a\cdot b)(x-3)+3 & \text{Definition of } \tilde{\cdot} \quad (13)\\
&= a(b(x-3))+3 & \text{Associative Law for } \mathbb{R} \quad (14)\\
& & \quad (15)
\end{array}
$$

Since $\tilde{0}=3$, $a(b\tilde{\cdot}(x-3)) \in V$ and $b\tilde{\cdot}(x-3) \in V$ by Definition of $\tilde{\cdot}$, as well as by definition of $\tilde{0}$ and Existence of an Additive Identity for V it follows that $b\tilde{\cdot}(x-3)+\tilde{0} = b\tilde{\cdot}(x-3)$ and $a(b(x-3))+3 = a(b(x-3))+\tilde{0} = a(b(x-3))$, then $a(b(x-3))+3 = a(b(x-3)+\tilde{0})+\tilde{0} = a(b(x-3)+3) = a(b\tilde{\cdot}x)$, as required.

7. Consider $a\tilde{\cdot}x$ and $a\tilde{\cdot}y$.

$$
\begin{array}{rll}
a\tilde{\cdot}x &= a(x-3)+3 & \text{Definition of } \tilde{\cdot} \quad (16)\\
a\tilde{\cdot}y &= a(y-3)+3 & \text{Definition of } \tilde{\cdot} \quad (17)\\
\Rightarrow a\tilde{\cdot}x + a\tilde{\cdot}y &= a(x-3)+3+a(y-3)+3 & \quad (18)\\
&= a(x+y-3-3)+3+3 & \text{Commutative Law for } \mathbb{R} \quad (19)\\
& & \text{and Distributive Law for } \mathbb{R} \quad (20)\\
&= a(x+y-3-\tilde{0})+3+\tilde{0} & \text{Definition of } \tilde{0} \quad (21)\\
&= a(x+y-3-\tilde{0})+3 & \text{Existence of an Additive Identity for } V \quad (22)\\
&= a(x+y-3)+3 & \text{Lemma :: } -\tilde{0}=\tilde{0}\text{, since} \quad (23)\\
& & (-1)\tilde{\cdot}\tilde{0} = \tilde{0} = -\tilde{0} \text{ and} \quad (24)\\
& & \text{Existence of an Additive Identity for } V \quad (25)\\
&= a\tilde{\cdot}(x+y) & \text{Definition of } \tilde{\cdot} \quad (26)
\end{array}
$$

8. Consider $(a+b)\tilde{\cdot}x$.

$$
\begin{array}{rll}
(a+b)\tilde{\cdot}x &= (a+b)(x-3)+3 & \text{Definition of } \tilde{\cdot} \quad (27)\\
&= a(x-3)+b(x-3)+3 & \text{Distributive Lawfor } \mathbb{R} \quad (28)\\
&= \tilde{0}+a(x-3)+b(x-3)+3 & \text{Existence of an Additive Identity for } V \quad (29)\\
&= a(x-3)+\tilde{0}+b(x-3)+3 & \text{Commutative Law for } V \quad (30)\\
&= a(x-3)+3+b(x-3)+3 & \text{Definition of } \tilde{0} \quad (31)\\
&= a\tilde{\cdot}x + b\tilde{\cdot}x & \text{Definition of } \tilde{\cdot} \quad (32)
\end{array}
$$

Thus, V is a vector space.