1 Brain Maths III

1.1 Revision

Some neurons persist in a sleeping state until they are acted upon by electric signals or chemicals. This can be represented by stationary points of vector fields in $\mathbb{R}^n$, which can also be perturbed until they fall back into the stable state.

Some neurons can also turn into an excited state, when they give off spikes periodically or quasiperiodically. This behaviour, in turn, can be modelled by closed trajectories, called *attracting cycles* in the parameter space, which are resistant to perturbations.

We will study how stable states can turn into attracting cycles.

1.2

Informally, a vector field $\dot{x} = f(x)$, where $f(x)$ is a multidimensional function, is called structurally stable if there exists $\epsilon > 0$ such that for all g with $d_{C'}(f, g) < \epsilon$ the behaviour of f and g is similar.

1.3 Smooth Classification

We say that $f \sim g$ if there exists a diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $gh(x) = (Dh)_x(f(x))$.

Example 1.1

Suppose that dim $= 1$, and let $f(0) = 0$ and $h(0) = 0$, which means that $g(0) = 0$. If h is a diffeomorphism, we thus have $Df(0) = Dg(0)$. Now, if $f = h(x)$, then $g(y) = \frac{\partial h(x)}{\partial x} \cdot f(x) = \frac{\partial}{\partial} h(h^{-1}y) \cdot f(h^{-1}(y))$. Thus, $\frac{\partial g(0)}{\partial y} = \frac{\partial h(0)}{\partial x} \cdot \frac{\partial f(0)}{\partial x} \cdot \frac{\partial h^{-1}(0)}{\partial y}$, and hence $\frac{\partial g(0)}{\partial y} = \frac{\partial f(0)}{\partial x}$.

1.4 Topological Classification

Let h be a homeomorphism. For all x and time t, we want the flux to be in the form $\Phi_t^g(h(x)) = h(\Phi_t^f(x))$. Note that Φ_t is smooth.

Now, there exists $x \in \mathbb{R}^n$ and $T > 0$ such that $\Phi_T^f(x) = x$. From the condition above, we also have $\Phi_T^g(h(x)) = h(x)$.

1.5 Orbital Topological Equivalence

If there exists a homeomorphism transforming oriented trajectories of f into the oriented trajectories of g, then we can classify our vector fields up to orbital topological equivalence.

Example 1.2

Let dim $= 1$. For $\epsilon > 0$, if $\|f - g\| < \epsilon$ and $\left\|f' - g'\right\| < \epsilon$, then $d_{C'}(f, g) < \epsilon$.

Theorem 1.3

Suppose $\dot{x} = f(x)$ is given. It is structurally stable, if

- all stationary points are isolated
- for all stationary points we have $f'(p) \neq 0$.

Note. Note that structurally stable vector fields form an open set in the space of all real vector spaces.

1.6 Bifurcations

Let f_α be a family of vector fields dependent on the parameter α, where $\alpha \in (-\alpha_0, \alpha_0)$. We say that α is *regular*, if f_α is structurally stable. Moreover, we say that α is a *bifurcation point*, if f_α is not structurally stable.

Theorem 1.4

For $\dim = 1$, a typical family of vector fields f_α has only a saddle-knot bifurcation point.

The two-dimensional case is more complex.