1 Sandpile Model and Divisors in Graphs

1.1 Introduction

Definition 1.1. Let Γ be a finite connected subgraph.

Define a **boundary** $\delta\Gamma$ as a set of vertices which neighbours do not belong to Γ.

Define a **state** as $\phi : \Gamma \to \mathbb{Z}_{\geq 0}$ over a set of vertices.

If $v \in \Gamma \setminus \delta\Gamma, \phi(v) \geq 4$, then we can make a **toppling** $\phi \to \phi'$ such that:

$$\phi'(v) = \phi(v) - 4 \tag{1}$$
$$\phi'(w) = \phi(w) + 1, \text{ if } W \sim V, \text{ where } \sim \text{ means that } W \text{ is a neighbouring subgraph} \tag{2}$$
$$\phi'(w) = \phi(w) \text{ otherwise} \tag{3}$$

Relaxation of ϕ is a sequence of topplings while they are possible.

Exercise 1.2. Relaxation always ends.

Exercise 1.3. Order of topplings does not matter, so that the relaxation is unique.

Denote the result of the relaxation as ϕ^0.

Definition 1.4. The function of topplings is a function $F : \Gamma \to \mathbb{Z}_{\geq 0}$ mapping ϕ to ϕ^0.

Let $F(v)$ denote the number of topplings in v.

Exercise 1.5. Prove that F is well-defined.

Exercise 1.6. $\phi^0 = \phi + \Delta F$.

Definition 1.7. $(\Delta F)(i, j) = F(i+1, j) + F(i-1, j) + F(i, j-1) + F(i, j+1) - 4F(i, j)$.

Exercise 1.8. The toppling function F is a dot-wise minimal function among functions $G : \Gamma \to \mathbb{Z}_{\geq 0}$ such that $\phi + \Delta G \leq 3$.

Note. A dot-wise minimal function F is such that $F(v) = \min(G(v)$ for all $G)$ and $\phi + \Delta G \leq 3$ is satisfied.

Definition 1.9. Let $\langle k \rangle$ denote the state in which each node has k sand grains. Denote the maximally stable state as $\langle 3 \rangle$.

Definition 1.10. The state ϕ is **revertible** if there exists $\psi \geq 0$ such that $\phi = (\langle 3 \rangle + \psi)^0$, where $+$ is defined dot-wise.

Definition 1.11. $\phi \oplus \psi = (\phi + \psi)^0$.

Theorem 1.12

Revertible states with the operation $\oplus$ form a group, so that $(\phi \oplus \psi) + \beta = \phi \oplus (\psi + \beta)$, $\psi + \phi = \phi + \psi$, there exists $\phi \oplus e = \phi$, and there exist inverse elements.

Note. $\langle 0 \rangle$ is not the unit element in the sand group.

1.2 Model on a Random Graph

Suppose that Γ is a random graph, where I is a special vertex of Γ called a *dump*.

The dump is $\delta\Gamma$ pulled into one node.

Suppose that Γ is in relaxation, and there are no topplings in a dump.

Let's measure the frequency of change over the area of the sand avalanche. The graph would be linear and declining with the growing area. This phenomenon is called *self-organised criticality*.

1.3 Forbidden Configurations

Example 1.13

In a revertible state there cannot be a graph with two states without any topplings, which can be seen as follows.

Look at the node of the last toppling, which can be either in the vertex 1 or vertex 2 to obtain a contradiction.

Definition 1.14. Suppose that $D \subset \Gamma$. The state ϕ on D i called forbidden, if $\phi(v)$ is less than the number of neighbours of $v \in D$.

Exercise 1.15. Prove that a revertible state does not contain forbidden configurations.

Exercise 1.16. The unit in the sand group is equal to $(\langle 8\rangle - \langle 8\rangle^0)^0$.

To prove the theorem, take a state ϕ, add itself and relax. Repeat the procedure for all the nodes. This procedure will eventually cycle over some nodes. Now, take another state ψ, and add it to one of the cycle nodes. How can you proceed?