Let $\mathbb{F}$ be any field.

Lemma 0.1. $\forall a \in \mathbb{F} : a \cdot 0 = 0$

Proof.

$$\begin{aligned}
0 + 0 &= 0 && \text{Existence of an Additive Identity} && (1)\\
\Rightarrow a \cdot (0+0) &= a \cdot 0 + a \cdot 0 && \text{Distributive Law} && (2)\\
&= a \cdot 0 && \text{Definition of } = && (3)\\
(a \cdot 0 + a \cdot 0) - (a \cdot 0) &= a \cdot 0 - (a \cdot 0) && \text{Definition of } = && (4)\\
\Rightarrow a \cdot 0 + (a \cdot 0 - a \cdot 0) &= 0 && \text{Associative Law} && (5)\\
& && \text{and Existence of an Additive Inverse} && \\
\Rightarrow a \cdot 0 + 0 &= 0 && \text{Existence of an Additive Inverse} && (6)\\
\Rightarrow a \cdot 0 &= 0 && \text{Existence of an Additive Identity} &&
\end{aligned}$$

□

Lemma 0.2. $\forall a, b \in \mathbb{F} : ab = 0 \Leftrightarrow a = 0 \vee b = 0$

Proof. By Commutative Law and Lemma 0.1, $a = 0 \Rightarrow ab = ba = b \cdot 0 = 0$.

Similarly, $b = 0 \Rightarrow ab = a \cdot 0 = 0$. If $ab = 0$ and $b \neq 0$, $\exists\, b^{-1} : abb^{-1} = 0 \cdot b^{-1}$, hence by Commutative Law and Existence of a Multiplicative Inverse $a \cdot 1 = b^{-1} \cdot 0$, then by Existence of a Multiplicative Identity and Lemma 0.1 $a = 0$.

If $ab = 0$ and $a \neq 0$, $\exists\, a^{-1} : a^{-1}ab = a^{-1} \cdot 0$, hence by Commutative Law and Lemma 0.1 $aa^{-1}b = 0$, then by Existence of a Multiplicative Inverse $1 \cdot b = 0$, and by Commutative Law and Existence of a Multiplicative Identity $b \cdot 1 = b = 0$.

If $a = 0 \wedge b = 0$, then by Lemma 0.1 $ab = 0 \cdot 0 = 0$ □

Theorem 0.3. $1 + 1 + 1 + 1 = 0 \in \mathbb{F} \Rightarrow 1 + 1 = 0$

Proof. Consider $(1 + 1) \cdot (1 + 1)$.

$$\begin{aligned}
(1+1)(1+1) &= (1+1) \cdot 1 + (1+1) \cdot 1 && \text{Distributive Law} && (1)\\
1 \cdot (1+1) + 1 \cdot (1+1) &= 1 + 1 + 1 + 1 && \text{Distributive Law and Commutative Law} && (2)
\end{aligned}$$

Therefore, $1 + 1 + 1 + 1 = (1 + 1)(1 + 1) = 0$ by assumption.

Now, since $(1 + 1)(1 + 1) = 0$ and $1 + 1 = 1 + 1$, by Lemma 0.2 $1 + 1 = 0$ as required. □