

MATRIX MULTIPLICATION

RECALL:

V IS A VECTOR SPACE,
 $\beta = \{v_1, \dots, v_n\}$ ORDERED
 BASIS, $\dim V = n \in \mathbb{N}$.

$$V \longrightarrow F^n, v \mapsto [v]_\beta$$

WITH INVERSE $F^n \longrightarrow V,$

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \longmapsto \sum_{j=1}^n a_j v_j.$$

SUPPOSE V, W ARE GIVEN WITH BASES

$$\{v_1, \dots, v_n\}, \{w_1, \dots, w_m\}.$$

DEFINE AN ISOMORPHISM $\mathcal{L}(V, W) \rightarrow M_{m \times n}(F)$
 WITH INVERSE $M_{m \times n}(F) \rightarrow \mathcal{L}(V, W)$

$$A = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn} \end{pmatrix} \mapsto T$$

WHERE T IS GIVEN AS

$$T \left(\sum_{j=1}^n a_j v_j \right) = \sum_{i=1}^m \left(\sum_{j=1}^n A_{ij} a_j \right) w_i$$

THUS, THE j^{th} COLUMNS OF $[T]_{\beta}^{\gamma} = A$
 ARE THE COEFFICIENTS OF $T(v_j)$
 IN BASIS $\{w_1, \dots, w_m\}$.

EXAMPLE

FIND THE COORDINATE
 REPRESENTATION FOR

$$T: \mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R}),$$

$$p \mapsto p''' + p'$$

FOR THE STANDARD BASES

$$\beta = \{1, x, x^2, x^3\}, \gamma = \{1, x, x^2\}:$$

$$T(1) = 0$$

$$T(x) = 1$$

$$T(x^2) = 2x$$

$$T(x^3) = 6x + 3x^2$$

$$\Rightarrow [T]_{\beta}^{\gamma} = \begin{pmatrix} 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

REMARK

$[T]_{\beta}^{\delta}$ DEPENDS ON THE
CHOICE OF ORDERED BASES.

INTERCHANGING w_{j_1}, w_{j_2} OF W
INTERCHANGES THE j_1, j_2 -TH ROW
OF $[T]_{\beta}^{\delta}$.

INTERCHANGING v_{j_1}, v_{j_2} OF V
INTERCHANGE THE j_1, j_2 -TH
COLUMN OF $[T]_{\beta}^{\delta}$.

NOTATION

IF $W = V$, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

IF $\delta = \beta$, $[T]_{\beta}^{\delta} = [T]_{\beta}^{\beta}$

EXAMPLE

THE MATRIX OF $I_V \in \mathcal{L}(V)$

$V \mapsto v$ HAS A MATRIX

$$[T]_{\beta}^{\beta} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & 1 \end{pmatrix} = I$$

MORE GENERALLY, GIVEN AN ISOMORPHISM

$$T: V \rightarrow W, \text{ AND } \beta = \{v_1, \dots, v_n\}, \\ \gamma = \{w_1, \dots, w_m\}.$$

ARE RELATED BY $w_i = T(v_i)$, THEN

$$[T]_{\gamma}^{\beta} = I_{n \times m}$$

QUESTION:

HOW TO DESCRIBE $w = T(v)$
IN TERMS OF OUR BASES?

$$\text{IF } [v]_{\beta} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, [w]_{\gamma} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$b = \sum_{j=1}^n A_{ij} a_j$$

IN SHORT,

$$[T]_{\gamma}^{\beta} [v]_{\beta} = [T(v)]_{\gamma}$$

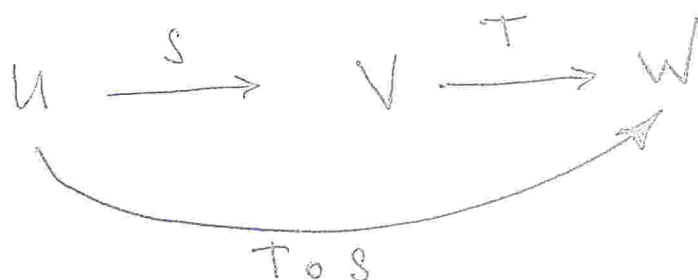
THE i -TH ENTRY OF $[T(V)]_\gamma$

IS THE "PRODUCT" OF THE i -TH

ROW OF $[T]_\gamma^\beta$ WITH $[V]_\beta$.

QUESTION:

How to DESCRIBE
 $W = T(V)$, $T \circ S$?



$\alpha = \{u_1, \dots, u_n\}$: A BASIS FOR U

$\beta = \{v_1, \dots, v_n\}$: A BASIS FOR V

$\gamma = \{w_1, \dots, w_m\}$: A BASIS FOR W

$$[T \circ S]_\alpha^\gamma : A = [T]_\gamma^\beta, B = [S]_\beta^\alpha$$

$$T \circ S(u_k) = T\left(\sum_{j=1}^n B_{jk} v_j\right)$$

$$= \sum_{j=1}^n B_{jk} \sum_{i=1}^m A_{ij} w_i$$

$$= \sum_{i=1}^m \left(\sum_{j=1}^n A_{ij} B_{jk} \right) w_i$$

$$[T \circ S]_x^y = C,$$

$$C_{ik} = \sum_{j=1}^n A_{ij} B_{jk}$$

→ DEFINITION OF A MATRIX PRODUCT