1 Introduction to Representation Theory

1.1 Definitions

Let G be a group.

Definition 1.1. A *representation* of the group G is fully defined by a homomorphism $\rho : G \to GL(V)$, where $GL(V)$ is the set of automorphisms.

Let V be a representation of G. If $g \in G$, $v \in V$, then we write $gv = \rho(g)v$ and $g_V = \rho(g)$.

A *subrepresentation* $U \subset V$is a subspace $gu \in U$ for $u \in U$.

If V_1 and V_2 are representations of G, then $V_1 \oplus V_2$ is a representation $g(v_1, v_2) = (gv_1, gv_2)$.

A homomorphism of representations $V_{1 \to V_2}$ is a linear transformation $\phi : V_1 \to V_2$ such that $\phi(gv_1) = g\phi(v_1)$.

Denote the space of linear transformations $V_1 \to V_2$ as $\mathrm{Hom}(V_1, V_2)$, and the space of homomorphisms as $\mathrm{Hom}_G(V_1, V_2)$.

Example 1.2

If $G = S_n$, then $V = \mathbb{C}^n$ is a representation of G.

$V_1 = \{x, \cdots, x \,|\, x \in \mathbb{C}\}$ is a subrepresentation.

1.2 Group Algebra

Definition 1.3. Let G be a finite group. A group ring $G \subset \mathbb{Z}G = \{\sum_{g \in G} x_g g \,|\, x_g \in \mathbb{Z}\}$ is a set of additions and multiplications of the basis elements – as in G.

Suppose a group algebra $\mathbb{C}G$ is given, so that $\mathbb{Z}G \subset \mathbb{C}G$. Then the representation of the algebra $\mathbb{C}G$ is a homomorphism $\rho : \mathbb{C}G \to \mathrm{Hom}(V, V)$.

1.3 Complete Representability (?)

Let G be a finite group, and suppose that V denotes a finite representation.

A representation V is irreducible, if any subrepresentation is $\{0\}, V$.

A representation V is almost reducible, if for all subrepresentations $U \subset V$ there exists a subrepresentation $U' \subset V$ such that $V = U \oplus U'$.

Note that if V is almost reducible, then V is isomorphic to the direct sum of irreducible representations.

Theorem 1.4 (Maschke's Theorem)

All representations mod $\mathbb{C}$ are almost reducible.

Proof.

Suppose that the inner product is Hermitian, and let $U \ suV$ be a subspace. Then $U \oplus U^{\perp} = V$.

We say that $\langle \cdot, \cdot \rangle$ is G-invariant, if for all $g \in G$ and $u, v \in V$ we have $(gu, gv) = (u, v)$.

If $\langle \cdot, \cdot \rangle$ is G-invariant and U is a subrepresntation, then $U^{\perp}$ is a subrepresentation.

It is enough to show that there exists a G-invariant Hermitian dot product.

Let $\langle \cdot, \cdot \rangle$ be an arbitrary dot product.

Then $\langle \cdot, \cdot \rangle_{inv} : (u, v)_{inv} = \sum_{h \in G}(hu, hv)$.

We check the invariance: $(gu, gv)_{inv} = \sum_{h \in G}(hgu, hgv) = \sum_{h \in G}(hu, hv) = (u, v)_{inv}$.

□

Lemma 1.5 (Schur's Lemma)

If G is a finite group, and U, V are irreducible representations. If U is not isomorphic to V, then $\mathrm{Hom}_G(U, V) = 0$. Moreover, $\mathrm{Hom}_G(V, V) = \{x \cdot \mathrm{id}V \mid x \in \mathbb{C}\}$.

Proof.

Let $\phi \in \mathrm{Hom}_G(U, V)$. Then $\ker \phi \subset U$, $\mathrm{im}\, \phi \subset V$ are subrepresentations.

Note that $u \in \ker \phi$ if and only if $\phi(u) = 0$, which is equivalent to $\phi(gu) = 0$ and $gu \in \ker \phi$.

Thus, if $\phi \neq 0$, then $\ker \phi \neq U = \{0\}, \mathrm{im}\, \phi = V$. Hence, ϕ is an isomorphism.

Let x be an eigenvalue of ϕ, so that $\phi - x\mathrm{id}_V$ is irrevertible, and thus equal to 0. □

Therefore, if U, V are representations and U is irreducible, then the multiplicity of U in V is $\dim \mathrm{Hom}_G(V, U)$.

Note that the proof of Schur's lemma holds for all irreducible V, and thus $\mathrm{Hom}_G(V_1 \oplus V_2, U) = \mathrm{Hom}_G(V_1, U) \oplus \mathrm{Hom}_G(V_2, U)$, which means that it holds for all V.

Example 1.6

Let $V = \mathbb{C}G$. Then the multiplicity of U in $\mathbb{C}G$ is equal to $\dim \mathrm{Hom}_G(\mathbb{C}G, U)$.

Let $F \in \mathbb{C}G$ and $F = \sum_g x_g g$. Suppose $x_g = x_{hgh^{-1}}$ for all $h, g \in G$. Then for all irreducible V F_V is a constant operator.

Proof.

For all $h \in G$, $Fh = hF$. Thus, $F = hFh^{-1} = \sum_{g \in G} x_g(hgh^{-1}) = \sum_g x_{hgh^{-1}}(hgh^{-1})$.

Note that F_v is a homomorphism of representations. Thus, $Fh = hF \Rightarrow V \to V$. $F_V h_v = h_V F_V$, which happens if and only if $F_V \in \mathrm{Hom}_G(V, V)$. □

1.4 Characters

Suppose that G is a finite group and V is a finite-dimensional representation. A *character* $\mathrm{char}_V : G \to \mathbb{C}$ is such that $\mathrm{char}_V(g) = \mathrm{tr}(g_V)$.

Example 1.7

Let $\mathbb{C}$ be a trivial representation.

Then $\mathrm{char}_{\mathbb{C}}(g) = 1$, $\mathrm{char}_{\mathbb{C}G}(g) = \begin{cases} |G|, g = e \\ 0, \text{ otherwise} \end{cases}$, where the last case is justified by the fact that there are no diagonal matrix elements of $h \mapsto gh$.

Note that $\mathrm{char}_{V_1 \oplus V_2} = \mathrm{char}_{V_1} + \mathrm{char}_{V_2}$.

Lemma 1.8

$\mathrm{char}_V(hgh^{-1}) = \mathrm{char}_V(g)$.

1.5 Orthogonality of Characters

Let $Cl(G) = \{f : G \to \mathbb{C} | f(hgh^{-1}) = f(g)\}$. Note that $\mathrm{char}_V \in Cl(G)$.

Recall that the Hermitian dot product $(F_1, F_2) = \frac{1}{|G|} \sum_{g \in G} \overline{F_1(g)} F_2(g)$.

Theorem 1.9

Characters of irreducible representations is an orthonormal basis of $Cl(G)$.

For representations U, V, we have $(ch_U, ch_V) = \dim \mathrm{Hom}_G(U, V)$.

If $F \in Cl(G)$, then $F = \sum_{g \in G} F(g) g \in \mathbb{C}G$. Moreover, if V is irreducible, then we know that $F_V = x \cdot \mathrm{id}_V$.

Thus, $x = \frac{|G|}{\dim V}(\overline{F}, \mathrm{char}_V)$.

To prove this, note that

$$
\begin{aligned}
x &= \frac{\mathrm{tr}(F_V)}{\dim V} && (1)\\
&= \frac{1}{\dim V} \sum_{g \in G} F(g) \,\mathrm{tr}(g_v) && (2)\\
&= \frac{1}{\dim V} \sum F(g) \,\mathrm{char}_V(g) && (3)\\
&= \frac{|G|}{\dim V} (\overline{F}, \mathrm{char}_V). && (4)
\end{aligned}
$$

Proving the theorem, we can use Schur's lemma and previous comments to show that a basis is orthonormal.