1 Irrationality of e

Note that $e = 1 + \sum_{i=1}^{n} \frac{1}{i!} + R_n(1) = 1 + \sum_{i=1}^{n} \frac{1}{i!} + \frac{e^t}{(n+1)!}$ for some $t \in [0, 1]$.

Note that e^x is increasing, and thus $e^t \leq e < 3$.

Therefore, $R_n(1) \leq \frac{3}{(n+1)!}$.

Theorem 1.1

e is irrational.

Proof.

Suppose that $e = \frac{a}{b} \in \mathbb{Q}$.

Choose $n > b$, $n \geq 4$. Then

$$\frac{a}{b} = 1 + \sum_{i=1}^{n} 1i! + R_n(1)$$

Therefore, $n!\frac{a}{b} = n! + n! \sum_{i=1}^{n} \frac{1}{i!} + n!R_n$.

Therefore, $n!R_n \in \mathbb{Z}$.

But $R_n \leq \frac{3}{(n+1)!}$, and thus $n!R_n \leq \frac{3}{n+1} \leq \frac{3}{5}$ and hence it is not an integer.

Therefore, by contradiction, e is irrational. □

Theorem 1.2

e is not algebraic.

Proof.

Exercise. □