1 Nonregular Languages

Consider $L = \{a^i b^i \mid a \geq 1\}$.

Suppose that L is regular.

Therefore, there exists a DFA $(Q, \{a, b\}, \delta, q_0, F)$ which accepts L.

Let $n = |Q|$.

Consider the states $q_i = \delta^*(q_0, a^i)$ for $i = 0, \ldots, n$.

By the pigeonhole principle, there exist $0 \leq i < j \leq n$ such that $q_i = q_j$.

Then $\delta^*(q_0, a^i b^j) = \delta^*(\delta^*(q_0, a^i), b^j)$, and since $q_i = q_j$, then $\delta^*(q_0, a^i b^j) = \delta^*(\delta^*(q_0, a^j), b^j) = \delta^*(q_0, a^j b^j) \in F$, since $a^j b^j \in L$, which means that $a^i b^j \in L$. But $a^i b^j \notin L$, since $i \neq j$, which is a contradiction. Thus, L is not regular.

Now we prove that $A = \{a^m b^r \mid m \neq r\}$ is not regular.

Suppose A is regular.

Then $(\{a, b\}^* - A) \cap \mathfrak{L}(aa^*bb^*)$ is also regular, because $\mathfrak{L}(aa^*bb^*)$ is regular and $\{a, b\}^* - A$ is regular by closure under complementation.

But $(\{a, b\}^* - A) \cap \mathfrak{L}(aa^*bb^*) = \{a^i b^i \mid i \geq 1\}$, which is not regular, and thus we obtain a contradiction.

Consider now $B = \{a^i b^j c^h \mid i, j, h \geq 0 \text{ AND } i = 1 \text{ IMPLIES } j = h\}$.

Suppose that B is regular.

Then $B \cap \mathfrak{L}(abb^*cc^*) = \{ab^j c^h \mid j = h \geq 1\} = C$ is regular.

Consider the homomorphism $T : \{a, b, c\} \to \{a, b\}^*$ such that $T(a) = \epsilon$, $T(b) = a$ and $T(c) = b$.

Then $T(C) = \{a^i b^i \mid i \geq 1\}$ is regular since regular languages are close under homomorphism. But $T(C) = L$ is not regular, which is a contradiction. Thus, B is not regular.

1.1 Pumping Lemma

Lemma 1.1

For every regular language $L \subseteq \Sigma^*$ there exists $n \in \mathbb{Z}^+$ such that, for all $x \in L$, if $|x| \geq n$, then there exist u, v and w in Σ^* such that $v \neq \epsilon$, $|uv| \leq n$, $x = uvw$ and for all $k \in \mathbb{N}$ we have $uv^k w \in L$.

Proof.

Since L is regular, it is accepted by some DFA $M = (Q, \epsilon, \delta, q_0, F)$.

Let $n = |Q|$, and suppose that $x = x_1 \cdots x_m \in L$, where $m \geq n$, is arbitrary.

Now we construct the states $q_i = \delta^*(q_0, x_1 \cdots x_i)$ for $i \in [1, n] \cap \mathbb{N}$.

By the pigeonhole principle, there exist $0 \leq i < j \leq n$ such that $q_i = q_j$.

Let $u = x_1 \cdots x_i$, $v = x_{i+1} \cdots x_j$ and $w = x_{j+1} \cdots x_m$.

Note that $\delta^*(q_0, uv^k) = \delta^*(q_0, uv)$ for all $k \geq 0$.

So $\delta^*(q_0, uv^k w) = \delta^*(q_j, w) = \delta^*(q_0, uvw) = \delta^*(q_0, x) \in F$.

Hence $uv^k w \in L = \mathfrak{L}(M)$. $\square$

Example 1.2

Let $P = \{z \in \{0,1\}^* \mid z = z^R\}$ be a language of palindromes.

We prove that P is not regular.

Suppose P is regular.

Then by the pumping lemma, there exists $n \in \mathbb{Z}^+$ such that for all $x \in P$, $\{x\} \geq n$ implies that there exist $u \in \{0,1\}^*$, $v \in \{0,1\}^*$ and $w \in \{0,1\}^*$ such that $v \neq \epsilon$ and $|uv| \leq n$, $x = uvw$ and for all $k \in \mathbb{N}$ we have $uv^kw \in P$.

Consider $x = 0^n10^n \in P$.

There exist $u, v, w \in \{0,1\}^*$ such that $v \neq \epsilon$ and $|uv| \leq n$, $x = uvw$ and for all $k \in \mathbb{N}$ we have $uv^kw \in P$.

So $u = 0^i$, $v = 0^j$ for some $i \geq 0$ and $j \geq 0$ such that $i + j \leq n$. Thus, $w = 0^{n-i-j}10^n$.

Consider $k = 2$.

Then $uv^2w = 0^{i+2j+n-i-j}10^n = 0^{n+j}10^n \notin P$, which is a contradiction.

Example 1.3

We prove that $L = \{a^mb^r \mid m \neq r\}$ is not regular.

Suppose that L is regular.

Then it satisfies the pumping lemma.

Let $n \in \mathbb{Z}^+$ be the constant rom the pumping Lemma.

Consider $x = a^{n!}b^{(n+1)!} \in L$.

Then there exist $u, v, w \in \{a,b\}^*$ that satisfy the conditions of the pumping lemma.

Let $u = a^i$, $v = a^j$ for some $i \geq 0, j > 0$ such that $i + j \leq n$.

Let $k = 1 + \frac{n \cdot n!}{j}$.

Then $uv^kw = a^mb^{(n+1)!}$, where $m = n! + (k-1)j = (n+1)!$.

Hence,$uv^kw \notin L$.