1 Approximating Functions

What do we mean when we say that a polynomial approximates $f(x)$?

Can we say what we definitely do not mean by polynomial approximation? Yes! For example, in case of discontinuous functions, we cannot find a suitable continuous function *close enough* to be similar.

We have already shown that $f(x) = e^x$, a polynomial $P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ is a *good* approximation. What we mean by *good* in a special case is, for instance, that for greater n, $P_n(1)$ gets closer to e.

Suppose now $g(x) = \log x$.

Note that, for $n > 0$, $g^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$.

Therefore, $Q_n(x) = f(a) + \sum_{k=1}^{n} \frac{-1^{k-1}}{ka^k}(x-a)^k$. However, this is a *very slow* approximation, which is not *good*.

For another example, take $h(x) = \arctan x$. Since $h'(x) = \frac{1}{1+x^2}$, to find $h^{(k)}(x)$ is a nontrivial task.

Suppose now $f(x)$ is n-time differentiable at $x = a$.

Let $P_n(x) = \sum_{k=0}^{n} \frac{f^k(x)}{k!}(x-a)^k$.

To make an approximation *good*, one of the methods is to minimise $\left|P_n(x) - f(x)\right|$. However, it does not account for where the approximation is centered. Therefore, it makes intuitive sense to consider $\frac{P_n(x)-f(x)}{(x-a)^n}$. Thus, for polynomials,

$$\lim_{x\to a} \frac{P_n(x) - f(x)}{(x-a)^n} = 0.$$

The great news is that it is also true for any function f.

Theorem 1.1

$\lim_{x\to a} \frac{P_n(x)-f(x)}{(x-a)^n} = 0$ for all f and corresponding Taylor polynomials $P_n(x)$.

Proof.

Note the following:

$$\frac{P_n(x) - f(x)}{(x-a)^n} = \frac{\sum_{k=0}^{n-1} \frac{f^k(x)}{k!}(x-a)^k - f(x) + \frac{f^n(x)}{n!}(x-a)^n}{(x-a)^n} \tag{1}$$

$$= \frac{\sum_{k=0}^{n-1} \frac{f^k(x)}{k!}(x-a)^k - f(x)}{(x-a)^n} + \frac{f^n(x)}{n!}. \tag{2}$$

We know that the consecutive derivatives of $\sum_{k=0}^{n-1} \frac{f^k(x)}{k!}(x-a)^k$ at a are equal to the derivatives of f at a up to the $(n-1)$-degree.

Therefore, by recursive application of the l'Hospital rule,

$$\lim_{x\to a} \frac{\sum_{k=0}^{n-1} \frac{f^k(x)}{k!}(x-a)^k - f(x)}{(x-a)^n} = \lim_{x\to a} \frac{f^{n-1}(a) - f^{n-1}(x)}{n!(x-a)} \tag{3}$$

$$= -\frac{f^n(a)}{n!}. \tag{4}$$

Therefore, $\lim_{x\to a} \frac{P_n(x)-f(x)}{(x-a)^n} = 0$, and thus close to $x = a$, $f(x)$ behaves like $P_n(x)$. $\square$