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MAT240:

LINEAR COMBINATIONS, SPAN AND MORE

LINEAR COMBINATION :: LET V BE A VECTOR SPACE OVER F, AND $S \subseteq V$ BE A SUBSET. THEN

a) $v \in V$ IS A LINEAR COMBINATION OF VECTORS IN S IF $v = a_1 v_1 + \dots + a_k v_k$ WHERE $a_1, \dots, a_k \in F$, $v_1, \dots, v_k \in S$.

SPAN :: b) THE SET OF SUCH LINEAR COMBINATIONS IN S IS CALLED THE SPAN OF S, DENOTED $\mathrm{SPAN}(S)$. IF $S \neq \emptyset$, $\mathrm{SPAN}(S) = \{0\}$.

NOTE $\mathrm{SPAN}(S)$ IS A SUBSPACE OF V, BECAUSE IT'S NONEMPTY AND CLOSED UNDER ADDITION AND MULTIPLICATION. IT'S THE SMALLEST SUBSPACE CONTAINING S.

NOTE A ($\neq\emptyset$) SUBSET $W \subseteq V$ IS A SUBSPACE IF AND ONLY IF IT'S CLOSED UNDER LINEAR COMBINATIONS: $W = \mathrm{span}(W)$

EXAMPLES

EXAMPLE 1.

1) $V = \mathbb{R}^3$, $S = \{v_1, v_2, v_3\}$

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

A VECTOR $v = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$ LIES IN THE SPAN $\{v_1, v_2, v_3\}$ IF AND ONLY IF $t_1 + t_2 + t_3 = 0$. INDEED, THIS CONDITION IS NECESSARY BECAUSE v_1, v_2, v_3 HAVE THIS PROPERTY, AND SUFFICIENT BECAUSE FOR ANY SUCH v,

$$\begin{aligned} v &= t_1 v_1 - t_3 v_2 \\ &= t_2 v_2 - t_1 v_3 \\ &= -t_2 v_1 + t_3 v_3 \end{aligned}$$

IF $t_1 + t_2 + t_3 = 0$,

$$\operatorname{span}\{v_1, v_2, v_3\} = \left\{ v = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \middle| \, t_1 + t_2 + t_3 = 0 \right\}$$

$$= \operatorname{span}\{v_1, v_2\} = \operatorname{span}\{v_2, v_3\} =$$
$$= \operatorname{span}\{v_1, v_3\}$$

EXAMPLE

$V = \mathbb{R}^3$, $S = \{v_1, v_2, v_3\}$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

CLAIM: $\operatorname{span}(S) = \mathbb{R}^3$

LET $v = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$ BE GIVEN. CONSIDER THE EQUATION

$v = a_1 v_1 + a_2 v_2 + a_3 v_3$.

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 \\ a_2 + a_3 \end{pmatrix} \iff \begin{cases} a_1 + a_3 = t_1 \\ a_3 - a_2 = t_1 - t_2 \\ a_2 + a_3 = t_3 \end{cases} \iff \begin{cases} a_1 + a_3 = t_1 \\ a_3 - a_2 = t_1 - t_2 \\ a_3 = \dfrac{t_1 - t_2 + t_3}{2} \end{cases}$$

$$\Leftrightarrow \begin{cases} a_1 = \frac{1}{2}(t_1 + t_2 - t_3) \\ a_2 = \frac{1}{2}(-t_1 + t_2 + t_3) \\ a_3 = \frac{1}{2}(t_1 - t_2 + t_3) \end{cases}$$

$\Rightarrow$ EVERY $V \in \mathbb{R}^3$ IS UNIQUELY A LINEAR COMBINATION OF V_1, V_2, V_3

THE CLAIM IS TRUE FOR AN ARBITRARY FIELD NOT EQUAL TO $\mathbb{Z}_2$

REMARK

IN EXAMPLE 1, V_3 WAS A LINEAR COMBINATION OF V_1, V_2:

$$V_3 = -V_1 - V_2$$

I.E. $V_1 + V_2 + V_3 = 0$

"LINEAR DEPENDENCE"

IN EXAMPLE 2, V_1, V_2, V_3 ARE LINEARLY INDEPENDENT.

DEFINITION:

VECTORS $V_1, \ldots, V_k$ ARE <u>LINEARLY DEPENDENT</u> IF THERE EXIST SCALARS $a_1, \ldots, a_k$ NOT ALL ZERO SUCH THAT

$$\sum_{i=1}^{k} a_i V_i = 0.$$

NOTE

IF ONE OF THE V_i IS 0, THEN THEY ARE LINEARLY DEPENDENT, SINCE

$$0 \cdot V_1 + \ldots + 1 \cdot V_i + \ldots + 0 \cdot V_k = 0$$

NOTE

$V = V_1$ IS LINEARLY INDEPENDENT $\Leftrightarrow V \neq 0$
(b/c $a \cdot V = 0 \Rightarrow a = 0$ or $V = 0$)

NOTE

V_1, V_2 ARE LINEARLY DEPENDENT IF AND ONLY IF ONE IS A MULTIPLE OF THE OTHER.

INDEED, IF $a_1 V_1 + a_2 V_2 = 0$, WITH $a_1 \neq 0$,
THEN $V_1 = -\frac{a_2}{a_1}$

CONVERSELY, IF $V_1 = s \cdot V_2$.

NOTE.

IF TWO OF VECTORS $V_1, \ldots, V_k$ ARE MULTIPLES OF EACH OTHER, THEN $V_1, \ldots, V_k$ LINEARLY DEPENDENT.

IF $V_i = V_j$ WITH $i \neq j$, THEN

$1 \cdot V_i - 1 \cdot V_j = 0$

DEFINITION

LET V BE A VECTOR SPACE OVER F. A SUBSET $S \subseteq V$ IS <u>LINEARLY DEPENDENT</u> IF THERE EXIST DISTINCT $v_1, \ldots, v_k \in S$ AND SCALARS $a_1, \ldots, a_k$, NOT ALL ZERO, SUCH THAT

$$a_1 v_1 + \ldots + a_k v_k = 0.$$

EXAMPLE

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

LINEARLY DEPENDENT.

NOTE

THUS, S IS LINEARLY DEPENDENT $\iff$ <u>SOME</u> DISTINCT VECTORS $\displaystyle\bigcup_{i=1}^{k} v_i \in S$ ARE LINEARLY DEPENDENT.

NOTE

IF $(S \subseteq V)$ IS LINEARLY DEPENDENT, AND $S \subseteq S' \subseteq V$, THEN S' IS LINEARLY DEPENDENT

$\iff$

NOTE

IF S IS LINEARLY INDEPENDENT AND $S' \subseteq S$, THEN S' IS LINEARLY ~~DEPENDENT~~

Suppose $S \subseteq T$

$k > 1$

NOTE

$V_1, \dots, V_k$ ARE LINEARLY DEPENDENT IFF ONE OF THE V_i'S IS A LINEAR COMBINATION OF THE OTHERS.

$$0 = \sum_{i=1}^{k} a_j v_i \quad \text{WITH } a_i = 0, \text{ Suppose } a_i \neq 0,$$

$$\Leftrightarrow \quad -a_i v_i = \sum_{i=1}^{k} a_i v_i - a_j v_i$$

$$\Leftrightarrow \quad v_i = v_i - \sum_{j=1}^{k} \frac{a_j v_j}{a_i}$$

NOTE

$\bigcirc_{i=1}^{k} v_i$ ARE LINEARLY INDEPENDENT

IFF $\sum_{i=1}^{k} a_i v_i = 0$ WITH $a_i \in F$

$\Longleftrightarrow a_1 = a_2 = \dots = a_k = 0$

EXAMPLE

CONSIDER:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

SUPPOSE

$$\bigoplus_{i=1}^{4} a_i A_i = 0, \quad \exists j \in [1;4]; \; a_j \neq 0.$$

$$\Leftrightarrow \begin{pmatrix} a_1 + a_2 + a_3 + a_4 & a_1 - a_2 + a_3 + a_4 \\ a_1 + a_2 - a_3 + a_4 & a_1 + a_2 + a_3 - a_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} a_1 + a_2 + a_3 + a_4 = 0 \\ a_1 + a_2 - a_3 + a_4 = 0 \\ a_1 - a_2 + a_3 + a_4 = 0 \\ a_1 + a_2 + a_3 - a_4 = 0 \end{cases}$$

$a_3 = 0 \Rightarrow a_1 = 0, \; a_1 = 0$

$\Rightarrow a_3 = 0 \quad \#$

UNLESS $2 = 0$.

THEOREM IF S IS LINEARLY INDEPENDENT SUBSET OF V, $v \notin S$, $v \in V$, THEN

$S \cup \{v\}$ IS LINEARLY INDEPENDENT

$\iff v \notin \mathrm{SPAN}(S)$

PROOF

$S \cup \{v\}$ LINEARLY DEPENDENT

$\iff v \in \mathrm{SPAN}(S)$.

$\Leftarrow$ a) SUPPOSE $v \in \mathrm{SPAN}(S)$. THEN

$$v = \sum_{i=1}^{k} a_i v_i \quad \text{WITH } \{v_i\}_{i=1}^{k},\ v_i \in S \text{ DISTINCT}$$

THUS $0 = (-1)v + a_1 v_1 + \dots + a_k v_k$.

$\Rightarrow$ b) SUPPOSE THAT $S \cup \{v\}$ IS LINEARLY DEPENDENT.

$$\Rightarrow \sum_{j=1}^{k} a_j v_j + a_{k+1} v = 0 \quad \text{WITH}$$

AT LEAST ONE a_j NOT ZERO.

AND $a_{k+1} \neq 0$.

THEN $$v = -\frac{1}{a_{k+1}} \left(\sum_{i=1}^{k} a_i v_i \right)$$

$\Rightarrow v \in \mathrm{SPAN}(S)$

DEFINITION

LET V BE A VECTOR SPACE OVER $\mathbb{F}$.

A SUBSET β OF V IS CALLED A BASIS OF V IF:

- β IS LINEARLY INDEPENDENT
- $\operatorname{SPAN}(\beta) = V$

EXAMPLE

$\{A_1, A_2, A_3, A_4\}$ IS THE BASIS OF $M_{2\times 2}(\mathbb{F})$ IF $1+1 \neq 0$.

EXAMPLE

$V = \mathbb{R}^n$, $\beta = \{e_1, \dots, e_n\}$ WITH $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\dots$, $e_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$

IS A BASIS