Suppose an operator T, $T : \mathbb{R}^4 \to \mathbb{R}^4$, is given such that

$$T : (a_1, a_2, a_3, a_4) \mapsto (0, a_1, 2a_2, 3a_3)$$

Since the linear map is uniquely determined by its action on the ordered basis of its domain, while the standard ordered basis is

$$\beta(\mathbb{R}^4) = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\},$$

then the following holds:

$$\begin{aligned} T(1,0,0,0) &= (0,1,0,0) && (1)\\ T(0,1,0,0) &= (0,0,2,0) && (2)\\ T(0,0,1,0) &= (0,0,0,3) && (3)\\ T(0,0,0,1) &= (0,0,0,0) && (4) \end{aligned}$$

and thus:

$$[T]_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \tag{5}$$

Note that the set consisting of $\text{span}\{T(1,0,0,0), T(0,1,0,0), T(0,0,1,0), T(0,0,0,1)\}$ is spanned by $\beta' = (0,1,0,0), (0,0,1,0), (0,0,0,1) \in \beta(\mathbb{R}^4)$, which is thus linearly independent. Therefore, since $\#\beta' = 3$, $\text{rank}(T) = 3$ and $\dim \mathbb{R}^4 = 4$, and hence $\text{nullity}(T) = 1$ by the rank-nullity theorem.

From Equations 1 to 4, it follows that

$$\begin{aligned} T \circ T(1,0,0,0) &= T(0,1,0,0) = (0,0,2,0) && (6)\\ T \circ T(0,1,0,0) &= T(0,0,2,0) = (0,0,0,6) && (7)\\ T \circ T(0,0,1,0) &= T(0,0,0,3) = (0,0,0,0) && (8)\\ T \circ T(0,0,0,1) &= T(0,0,0,0) = (0,0,0,0) && (9) \end{aligned}$$

and thus:

$$[T^2]_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix} \tag{10}$$

Note that the set consisting of $\text{span}\{T^2(1,0,0,0), T^2(0,1,0,0), T^2(0,0,1,0), T^2(0,0,0,1)\}$ is spanned by $\beta'' = (0,0,1,0), (0,0,0,1) \in \beta(\mathbb{R}^4)$, which is thus linearly independent. Therefore, since $\#\beta'' = 2$ and $\dim \mathbb{R}^4 = 4$, $\text{rank}(T^2) = 2$, and hence $\text{nullity}(T^2) = 2$ by the rank-nullity theorem.

From Equations 6 and 9 it follows that

$$T \circ T \circ T(1,0,0,0) = T(0,0,2,0) = (0,0,0,6) \tag{11}$$
$$T \circ T \circ T(0,1,0,0) = T(0,0,0,6) = (0,0,0,0) \tag{12}$$
$$T \circ T \circ T(0,0,1,0) = T(0,0,0,0) = (0,0,0,0) \tag{13}$$
$$T \circ T \circ T(0,0,0,1) = T(0,0,0,0) = (0,0,0,0) \tag{14}$$

and thus:

$$[T^3]_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

Note that the set consisting of $\operatorname{span}\{T^3(1,0,0,0), T^3(0,1,0,0), T^3(0,0,1,0), T^3(0,0,0,1)\}$ is spanned by $\beta''' = (0,0,0,1) \in \beta(\mathbb{R}^4)$, which is linearly independent. Therefore, since $\#\beta''' = 1$ and $\dim \mathbb{R}^4 = 4$, $\operatorname{rank}(T^3) = 1$, and hence $\operatorname{nullity}(T^3) = 3$ by the rank-nullity theorem.

From Equations 11 and 14 it follows that

$$T \circ T \circ T \circ T(1,0,0,0) = T(0,0,0,6) = (0,0,0,0) \tag{16}$$
$$T \circ T \circ T \circ T(0,1,0,0) = T(0,0,0,0) = (0,0,0,0) \tag{17}$$
$$T \circ T \circ T \circ T(0,0,1,0) = T(0,0,0,0) = (0,0,0,0) \tag{18}$$
$$T \circ T \circ T \circ T(0,0,0,1) = T(0,0,0,0) = (0,0,0,0) \tag{19}$$

and thus:

$$[T^4]_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

Note that the set consisting of $\operatorname{span}\{T^4(1,0,0,0), T^4(0,1,0,0), T^4(0,0,1,0), T^4(0,0,0,1)\}$ is spanned by $\beta'''' = \emptyset \in \beta(\mathbb{R}^4)$, which is thus linearly independent. Therefore, since $\#\beta'''' = 0$ and $\dim \mathbb{R}^4 = 4$, $\operatorname{rank}(T^4) = 0$, and hence $\operatorname{nullity}(T^4) = 4$ by the rank-nullity theorem.

Since $T\{\mathbf{0}\} = \mathbf{0}$ for all linear maps, it follows that for $k > 4$, $\forall (v \in \mathbb{R}^4) : T^k(v) = \mathbf{0}$. Therefore, since $\#\beta_k = 0$ and $\dim \mathbb{R}^4 = 4$, $\operatorname{rank}(T^k) = 0$, and hence $\operatorname{nullity}(T^4) = 4$ by the rank-nullity theorem.