Let V be a vector space over $\mathbb{F}$, and $T \in \mathscr{L}(V)$.

Let W, Z be subspaces of V so that $V = W \oplus Z$.

Definition. A function $T : V \to V$ is called the projection from V onto W along Z if, for $x = w + z$ with $w \in W$ and $z \in Z$, we have $T(x) = w$.

Claim. For $\exists(W,\ Z \subseteq V) : V = W \oplus Z$ there exists $T \in \mathscr{L}(V, V)$ so that T is the projection from V onto W along $Z \Leftrightarrow T \circ T = T$

First, we prove the following lemma.

Lemma (Direct Sum Subspace Disjunction | DSSD)
$V = W \oplus Z \Rightarrow U \cap W = \{0\}$

Proof of DSSD. Suppose that $V = W \oplus Z$.

Consider some $v \in W \cap Z$. Note that by definition of a subspace, $0 \in W \cap Z$, and hence $W \cap Z$ is not empty. Therefore, since $v \in W$, then $\exists(-v) \in W : v + (-v) = 0$ by the existence of the additive inverse in a vector space. Similarly, such an element must exist in Z. By the uniqueness of the additive inverse, $(-v) \in W \cap Z$.

Thus, $0 = v + (-v)$, where $v \in U, (-v) \in Z$. By the uniqueness of the representation of 0 as the sum of a vector in W and a vector in Z, since $0 = 0 + 0$ by the existence of an additive identity, then $v = 0$. Thus, $W \cap Z = \{0\}$. □

Proof of the Claim. $(\Rightarrow)$: Suppose first that such $W,\ Z,\ T$ exist.

By DSSD, $W \cap Z = \{0\}$.

Since T is a linear map, then $T(0) = 0$. Therefore, $T \circ T(0) = 0$.

Take $v \in V$ so that $v \neq 0$. Therefore, from line 1,

$$\exists(w \in W, z \in Z) : v = w + z.$$

By definition of T, $T(v) = w$.

Since $w = w + 0$ and $T(0) = 0$, then $T \circ T(v) = T(w + 0) = w = T(v)$.

Therefore, $T \circ T = T$.

$(\Leftarrow)$: Suppose now that $T \circ T = T$.

Definition. A **space partition** P o f V is a set of subspaces V_i, $i \in I$, where I is some index set, such that:

1. $$\bigcup_{i \in I} V_i = V$$

2. $$\forall(i, j \in I, i \neq j) : A_i \cap A_j = \{0\}$$

Let $\overline{V}$ be the set of all space partitions of V.

We claim that $\overline{V}$ is not empty.

Consider a basis of V, $\beta(V)$. Since $\beta(V)$ is linearly independent, any subset of $\beta(V)$ is also linearly independent. Take any subset $\sigma \subset \beta(V)$. Therefore, $\operatorname{span}(\sigma) \cup \operatorname{span}(\beta(V) \setminus \sigma)$ is a space partition, since $\sigma \cup (\beta(V) \setminus \sigma) = \beta(V)$ and $\sigma \cap (\beta(V) \setminus \sigma) = \emptyset$, while $\operatorname{span}(\sigma)$ and $\operatorname{span}(\beta(V) \setminus \sigma)$ are subspaces of V. Therefore, by definition of a space partition and DSSD, $V = \operatorname{span}(\sigma) \oplus \operatorname{span}(\beta(V) \setminus \sigma)$, and thus there exist subspaces W, Z so that $V = W \oplus Z$.

Suppose then that some W, Z are given satisfying $V = W \oplus Z$. Therefore,

$$\forall (v \in V) \exists (w \in W, z \in Z) : v = w + z.$$

Suppose T is defined with the following further set of restrictions R:

1. $$\forall (z \in Z) : T(z) = 0$$
2. $$\forall (w \in W) : T(w) = w$$

Note that the definition above is equivalent to saying that T is a zero linear map for all z in Z and an identity map for all w in W. We claim that the condition $T \circ T = T$ and $T \in \mathscr{L}(V, V)$ also holds.

Since $V = W \oplus Z$, then $T \in \mathscr{L}(V, V)$, because for all such w, z $T(w + 0z) = T(w) = w$ and $T(0w + z) = T(z) = 0$.

Consider now $T(v)$, $v = w + z$ for some $w \in W, z \in Z$.

Since T is a linear map, $T(v) = T(w + z) = T(w) + T(z)$.

Suppose that the restrictions R hold.

Therefore, since $T(w) = w$ and $T(z) = 0$, $T(v) = w$, and thus

$$T \circ T(v) = T(w) = w = T(v).$$

Since $T \circ T = T$ by assumption, suppose that the restrictions R hold. Since

$$\forall (v \in V) \exists (w \in W) : T(v) = w,$$

then T is a projection of V onto W along Z, and we are done. $\square$