1 More on Orthogonal Complements

Remark 1.1. If $x = w + z$ $(w \in W,\ z \in W^{\perp}$, then w is the **orthogonal projection of x on W**. In fact, any orthogonal projection is a linear map.

> **Corollary 1.2**
>
> If V is finite dimensional, then $\dim V = \dim W + \dim W^{\perp}$ and $(W^{\perp})^{\perp} = W$.

Proof. Since $V = W \oplus W^{\perp}$, then $\dim V = \dim W + \dim W^{\perp}$.

Moreover, $\dim(W^{\perp}) = \dim V - \dim W^{\perp} = \dim W$.

We prove now that $W \subset (W^{\perp})^{\perp}$.

If $x \in W$, then $\langle x, y\rangle = 0$ for all $y \in W^{\perp}$. Therefore, $x \in (W^{\perp})^{\perp}$. ☐

> **Corollary 1.3**
>
> If $x = w + z$ for $w \in W, z \in W^{\perp}$, then w is the unique vector closest to x in W. Thus, for all $u \in W$ such that $u \neq w$:
>
> $$\|x - u\| < \|x - w\|$$

Proof. If $a, b \in V$ and $\langle a, b\rangle = 0$, then $\|a + b\|^2 = \|a\|^2 + \|b\|^2$.

Therefore, $\left\|(w - u) + z\right\|^2 = \|w - u\|^2 + \|z\|^2 \geq \|z\|^2$. ☐

> **Theorem 1.4**
>
> Let $S = \{v_1, \ldots, v_k\}$ be an orthonormal subset such that $\dim V = n$:
>
> a) Then S can be extended to an orthonormal basis of V
>
> b) If $W = \text{span}\{v_1, \ldots, v_k\}$, then $W^{\perp} = \text{span}\{v_{k+1}, \ldots, v_n\}$.

Proof.

a) First, extend $\{v_1, \ldots, v_k\}$. Use the Gram-Schmidt procedure to make it orthonormal. Note that any orthonormal subset is linearly independent.

   The first k elements of the new basis are unchanged. In this way, $v_1, \ldots, v_n$ is an orthonormal basis of V after normalisation.

b) If $V = \text{span}\{v_k, \ldots, v_n\}$, then $\langle x, y\rangle = 0$ for all $x \in W$ and $g \in V$, which is logically equivalent to $V \subset W^{\perp}$. Since $\dim W^{\perp} = \dim U$, then $V = W^{\perp}$.

☐

2 Adjoints

If V is a finite dimensional inner product space and $T \in \text{Hom}(V, V)$, then there exists a unique $T^* \in \text{Hom}(V, V)$, called an **adjoint** such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$.

Note that $[T^*]_\beta = ([T]_\beta)^*$. Tho show this, note that if $y \in V$, the function $f_y : V \to \mathbb{F}$ is linear.

Theorem 2.1

Let V be a finite dimensional vector space. If $f : V \to \mathbb{F}$ is linear, then there exists a unique $y \in V$ such that $f = f_y$, i.e. $f(x) = \langle x, y \rangle$ for all $x \in V$.

Proof. Pick an orthonormal basis $v_1, \ldots, v_n$.

Suppose $f = f_y$.

Therefore, $f(v_1) = \langle v_1, y \rangle$. Hence, $y = \sum_{i=1}^{n} \langle y, v_i \rangle v_i = \sum_{i=1}^{n} \overline{f(v_i)} v_i$.

Define $y = \sum_{i=1}^{n} \overline{f(v_i)} v_i$.

To prove that y is unique, it's enough to show that $f(v_i) = f_y(v_i)$.

Observe that $f_y(v_i) = \langle v_i, y \rangle = \langle v_i, \sum_{j=1}^{n} \overline{f(v_j)} v_j \rangle$.

Therefore, $f_y(v_i) = \sum_{j=1}^{n} f(v_j) \langle v_i, v_j \rangle = f(v_i)$. $\square$