# 1 Probabilistic Proofs of Existence

## 1.1 Introduction

Constructions can have different nature.

Sometimes explicit construction of mathematical objects is very challenging. In these cases, we can construct an approximating object, which may not have all the *good* properties we want, but which at least lacks all the *bad* ones we want to avoid.

## 1.2 Introduction to the Ramsey Theory

Suppose that a complete graph is given with $n$ edges. Let's choose a subgraph with $k$ nodes and colour its nodes red or blue. If we choose a subgraph randomly, then the probability that it is coloured is $2^{\frac{k(k-1)}{2}}2$. If we look at the total probability, without any regard for which subgraph we choose, we get $\binom{n}{k} 2^{1-\frac{k(k-1)}{2}}$. The upper bound is thus $n^k 2^{1-\frac{k(k-1)}{2}}$. Can we do better?

## 1.3 Markov's Inequality

Suppose that a random variable $X$ is given. We define the expected value of $X$ as $\mathbb{E}(X) = \sum x_i \Pr(x_i)$.

**Theorem 1.1** (Markov's Inequality)
Suppose $X \geq 0$. Then $\Pr(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$.

Why does it hold? It is easier to see if we substitute $X$ for $Y$, where $Y \leq X$, and $Y = \begin{cases} 0, X < a \\ a, X \geq a \end{cases}$.

We define variance as $\mathrm{Var}(X) = \mathbb{E}(X - \mathbb{E}(X))^2 = \sigma^2$.

**Theorem 1.2** (Chebyshev's Inequality)

$$\Pr\left(\frac{|X - \mathbb{E}(X)|}{\sigma} \geq a\right) \leq \frac{1}{a^2},$$

**Theorem 1.3** (Chernov's Inequality)
Suppose that $X_1, \ldots, X_n$ are independent random variables such that $X_1, \ldots, X_n \in \{0, 1\}$. Let $X = \sum_j X_j$, and take $\delta > 0$. Then

$$\Pr(X > (1+\delta)\mathbb{E}(X)) \leq e^{\frac{-\delta \log(1+\delta)\mathbb{E}(X)}{2}}$$

## 1.4 Error-Correcting Codes

Suppose that we code an $n$-bit word with $m$ bits which is transmitted via a channel introducing $d$ errors.

Note that in this case $\sum_{j=0}^{d} \binom{m}{j} \geq \binom{m}{d} \geq \frac{(m-d)^d}{d!}$, and thus we can write $2^n \frac{(m-d)^d}{d!} \leq 2^m$.