The intuitive understanding of angles can be made more precise by considering a unit circle.

In this way, an angle $0 \leq \theta \leq 2\pi$ corresponds to a unique point on the unit circle.

Since the equation of a semicircle on the axes xOy is $y = \sqrt{1-x^2}$ and the area of the unit circle is π, we thus define $\pi = 2\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x$. The area of the sector corresponding to an angle θ with $\cos\theta = x$, if $-1 \leq x \leq 1$, is

$$A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\,\mathrm{d}t.$$

Consider now $A'(x)$:

$$A'(x) = \frac{1}{2}(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}) - \sqrt{1-x^2} = \frac{-1}{2\sqrt{1-x^2}}$$

Note that A' is well-defined for $x \in (-1, 1)$.

We also note that $A(-1) = \frac{\pi}{2}$ and $A(0) = 0$. The graph of A is such that it is:

- decreasing (since $A' < 0$)
- injective

Therefore, it has an inverse.

Question. How to express $\cos\theta$ as a function of $2A$?

Answer. Note that $\cos\theta$ is the inverse function of $2A$. Given $2A$, which determines an angle θ, define $\cos\theta$ to be the unique x such that $2A = 2A(x)$, thus $\theta = 2A(cos\theta)$.

We define $\sin\theta = \sqrt{1-\cos^2\theta}$ and $B = 2A$. B is the inverse of $\cos\theta$, and thus, from above,

$$\frac{\mathrm{d}\cos\theta}{\mathrm{d}\theta} = \frac{1}{B'(\cos\theta)} = -\sin\theta$$

Similarly,

$$\frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta} = \frac{\mathrm{d}\sqrt{1-\cos^2\theta}}{\mathrm{d}\theta} = \cos\theta$$