1 Problem

Problem.

Suppose that $\mathbb{F} = \mathbb{C}$ and $A = \begin{pmatrix} 3 & & & & & \\ 1 & 4 & & & & \\ -1 & 1 & 3 & & & \\ 2 & 0 & 1 & 3 & & \\ & & & & 4 & \\ & & & & 2 & 3 \end{pmatrix}$,

where all empty matrix entries are zeros.

Find a Jordan canonical form for A and find a basis of of K_3 that is a disjoint union of cycles of generalised eigenvectors of L_A.

Solution.

Since $(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$ and $\det(A - \lambda I) = \det(A - \lambda I)^T$, we have that A^T and A have the same eigenvalues.

Since A^T is upper-triangular while $\mathbb{F} = \mathbb{C}$, all the eigenvalues are on the eigenvalues.

Using the Laplacian expansion on A to obtain the characteristic polynomial of A^T and A, we obtain that $f(t) = (t-3)^4(t-4)^2$.

Note that

$$(A - 3I) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \tag{1}$$

$$\begin{matrix} R_3 \to \frac{1}{2}(R_3 + R_2) \\ R_6 \to R_6 - 2R_5 \end{matrix} \quad \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

$$R_2 \to R_2 - R_3 \quad \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

$$R_4 \to R_4 - R_2 \quad \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

Therefore, $\operatorname{rank}(A-3I)=4$, and thus $\operatorname{nullity}(A-3I)=2$, which means that there are 2 columns in the dot diagram.

Moreover,

$$(A-3I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \tag{6}$$

$$\begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to \frac{1}{2}(R_4 + R_2) \\ R_6 \to R_6 - 2R_5 \end{matrix} \quad \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

which means that $\operatorname{nullity}(A-3I)^2 = 3$.

Since $3-2=1$, there is only one dot in the second row of the dot diagram, which means that the second column must have only one dot.

Therefore, the blocks are 3×3 and 1×1.

Consider now

$$(A-4I) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \end{pmatrix} \tag{8}$$

$$\begin{matrix} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix} \quad \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \end{pmatrix} \tag{9}$$

$$\tag{10}$$

Thus, $\operatorname{nullity}(A-4I)=2$, and therefore there are two columns in the corresponding dot diagram.

$$(A-4I)^2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \end{pmatrix} \tag{11}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & 1 & 0 & 0 & 0 \\ -5 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{pmatrix} \tag{12}$$

$$\begin{matrix} R_2 \to R_1 + R_2 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 + 5R_1 \end{matrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{pmatrix} \tag{13}$$

$$R_4 \to R_4 + R_3 \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{pmatrix} \tag{14}$$

Thus, $\text{nullity}(A-4I)^2 = 2$, and there are two $2-2=0$ dots in the second row of the dot diagram, which means that there are two 1×1 Jordan blocks corresponding to $\lambda = 4$.

Hence, $[A]_\beta = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$.

To find a cycle basis for K_3, note that

$$(A-3I)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}. \tag{16}$$

By the equation (4) we have that $u = v = w = y = 0$, and $\ker(A - 3I)$ is spanned by $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

We know that there are two cycles, one of length 1 and the other of length 3.

Therefore, there exists $p = \begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \end{pmatrix} \in V$ such that $(T - 3I)^3 = 0$ but $(T - 3I)^2 \neq 0$.

In this way, from (6), at least one of $u + v$, $-u + v$ or y is nonzero, while from (15) $u + v = 0$ and $y = 0$. Therefore, $v \neq 0$, $y = 0$ and $u = -v$.

Hence, $p = \begin{pmatrix} u \\ -u \\ w \\ x \\ 0 \\ z \end{pmatrix}$.

Take $p = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$.

Then $(A - 3I)p = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, which is an eigenvalue of $(A - 3I)^2$, because $(A - 3I)^3 p = 0$.

$(A - 3I)^2 p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$, which is an eigenvalue of $(A - 3I)$, because $(A - 3I)^3 p = 0$.

If there exist a_1, a_2, a_3 such that

$$a_1 \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} = 0,$$

then from the first row we have that $a_1 = 0$, from the fourth we obtain that $a_3 = 0$, and thus $a_2 = 0$, which means that p generates a cycle basis of length 3.

Now, take $q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. We have already shown that it is an eigenvector of $(A - 3I)$. Since it is also not in the span of the cycle basis generated by p, because each element of such a basis has the last row equal to zero, while $\dim K_3 = 4$ from the characteristic polynomial, then we have an independent list of the right length, and thus a Jordan canonical basis β for $\dim K_3$:

$$\beta = \{ \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \}$$

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