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MAT 240:
LINEAR TRANSFORMATIONS

REVIEW DEFINE $\circ(x) : \mathcal{S} \longrightarrow \{F, T\}$ ST $\circ(x) = T$ IFF x IS FINITE.

IF $\circ(\text{DIM } V)$:

S IS LINEARLY INDEPENDENT $\Rightarrow$ $\# S \leq \text{DIM } V$.

WE PROVED THIS FOR $\circ(\# S)$.

IF $\circ(\text{DIM } V)$, EVERY LINEARLY INDEPENDENT SET S IS FINITE.

IF $\neg(\circ(S))$:

CHOOSE ANY SUBSET $T \subseteq S$

WITH $\# T = \dim V + 1$.

SINCE $\perp(T)$, $\# T \leq \dim V$

$\#$

LINEAR TRANSFORMATION

LET V, W BE VECTOR SPACES OVER F.

A FUNCTION $T: V \to W$ IS CALLED A <u>LINEAR TRANSFORMATION</u> IF:

- $T(v_1 + v_2) = T(v_1) + T(v_2) \ \forall \ v_1, v_2 \in V$
- $T(av) = a\,T(v) \ \forall \ v \in V, a \in F.$

NOTE

- $T(0) = 0$
- $T(-v) = -T(v)$
- $T\left(\bigoplus_{i=1}^{m} a_i v_i\right) = \bigoplus_{i=1}^{m} a_i T(v_i)$

EXAMPLES

- $T: \mathbb{R}^3 \to \mathbb{R}^2, \ (t_1, t_2, t_3) \mapsto (t_1, t_2)$
- $T: \mathbb{R}^2 \to \mathbb{R}^3, \ (t_1, t_2) \mapsto (t_1, t_2, 0)$
- $T: \mathbb{R} \to \mathbb{R}^3, \ t \mapsto (t, t, t)$
- $T: V \to V, \ v \mapsto -v$

NON-EXAMPLES:

- $S: \mathbb{R} \to \mathbb{R}, \ x \mapsto x^2$
- $S: \mathbb{R} \to \mathbb{R}, \ x \mapsto mx + b, \ b \neq 0$

EXAMPLES

- $T: V \to V,\ v \mapsto v$

  IDENTITY TRANSFORMATION

- $T: V \to W,\ v \mapsto 0$

  ZERO TRANSFORMATION

- $T: M_{m\times n}(F) \to M_{n\times m}(F),$

  $$A \mapsto A^t$$

  IS LINEAR SINCE

  $$(A_1 + A_2)^t = A_1^t + A_2^t$$

  $$(aA)^t = aA^t$$

- $T: \mathcal{F}(\mathbb{R}, \mathbb{R}) \to \mathbb{R},$

  $$f \mapsto f(0)$$

  $$(f_1 + f_2)(0) = f_1(0) + f_2(0)$$

  $$(af)(0) = af(0)$$

- $\mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}),$

  $$p \mapsto p'$$

$$a_0 + \bigoplus_{i=1}^{n} a_i x_i \longmapsto a_1 + \bigoplus_{i=2}^{n-1} i\, a_i x^{i-1}$$

* : TRUE FOR ANY FIELD WITH $k = \underbrace{1 + \dots + 1}_{k \text{ times}} \in F$

- $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$,

$$p \longmapsto \int_0^x p(t)\,dt$$

- $T : F^\infty \longrightarrow F^\infty$,

$$\left(\bigodot_{i=1}^{\infty} a_i \right) \longmapsto \left(0, \bigodot_{i=1}^{\infty} a_i \right)$$

DEFINITION

LET $T : V \to W$ BE LINEAR.

WE DEFINE $N(T) = \{ v \in W \mid T(v) = 0 \}$

NULL SPACE (OR KERNEL)

$R(T) = \{ w \in W \mid w = T(v), v \in W \}$

IS CALLED A RANGE

LET $T : M_{n \times n}(F) \longrightarrow M_{n \times n}(F)$.

$$A \longmapsto A + A^t$$

$N(T) = \{$ SKEW-SYMMETRIC MATRICES $\}$

$R(T) = \{$ SYMMETRIC MATRICES $\}$

CONVERSELY IF $B = A + A^t$ GIVEN $B = B^t$, THEN $A = \frac{1}{2} B$

EXAMPLE:

$$T : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}(\mathbb{R}),$$

$$p \longrightarrow p^{(k)},$$

$$N(T) = \mathcal{P}_{k-1}(\mathbb{R})$$

$$R(T) = \mathcal{P}_{n-k}(\mathbb{R})$$

THEOREM

FOR $T : V \to W$ LINEAR, THE SUBSETS $N(T)$, $R(T)$ ARE SUBSPACES OF V, resp. W.

PROOF

$N(T)$, $R(T)$ ARE NON-EMPTY SINCE $T(0) = 0$, AND BOTH ARE CLOSED UNDER ADDITION AND MULTIPLICATION.