1 Integrability of Continuous Functions

Definition 1.1. Note that $L(f,P)$, $U(f,P)$ are defined for any P, so that $L(f,P) \leq U(f,P)$. Then for any bounded f

$$L\int_a^b f(x)dx = \sup_P L(f,P)$$
$$U\int_a^b f(x)\,\mathrm{d}x = \inf_P U(f,P)$$

Lemma 1.2
$L\int_a^b f(x)\,\mathrm{d}x + L\int_c^b f(x)\,\mathrm{d}x = L\int_a^b f(x)\,\mathrm{d}x$

Lemma 1.3
$U\int_a^b f(x)\,\mathrm{d}x + U\int_c^b f(x)\,\mathrm{d}x = U\int_a^b f(x)\,\mathrm{d}x$

If $m < f(x) < M$, then $m(b-a) \leq L\int_a^b f(x)\,\mathrm{d}x \leq U\int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$.

Fix $x \in (a,b)$.

Define

$$L(x) = L\int_a^x f(x)\,\mathrm{d}x$$
$$U(x) = U\int_a^x f(x)\,\mathrm{d}x.$$

Observe that both always exist.

To fund $L'(x)$, we need to find

$$\lim_{h\to 0}\frac{L(x+h)-L(x)}{h} = \lim_{h\to 0}\frac{L\int_a^{x+h} f(t)dt}{h}$$

If $h > 0$, define

$$m_h = \inf\{f(t)\,|\,x \leq t \leq x+h\} \tag{1}$$
$$M_h = \sup\{f(t)\,|\,x \leq t \leq x+h\} \tag{2}$$

Fix some $x \in (a,b)$.

Therefore,

$$m_h(x+h-x) \leq L\int_x^{x+h} f(t)dt \leq U\int_x^{x+h} f(t)dt \leq M_h(x+h-x) \tag{3}$$
$$\Leftrightarrow m_h \leq \frac{L\int_x^{x+h} f(t)dt}{h} \leq \frac{U\int_x^{x+h} f(t)dt}{h} \leq M_h \tag{4}$$
$$\Leftrightarrow m_h \leq \frac{L(x+h)-L(x)}{h} \leq \frac{U(x+h)-U(x)}{h} \leq M_h \tag{5}$$

If $h < 0$, the inequalities are similar.

If f is continuous at x, then

$$\lim_{h\to 0} m_h = \lim_{h\to 0} M_h = f(x),$$

so $L'(x) = f(x) = U'(x)$ and thus there exists a cotstant $c \in \mathbb{R}$ such that $U(x) = L(x)+c$.

But $U(a) = L(a) = 0$, so $c = 0$ and hence $U(x) = L(x)$. In particular, $L(b) = U(b)$, so f is integrable.

Question. If f is integrable and $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$. What about $G(x) = \int_x^b f(t)dt$?

Answer. Let $H(x) = \int_a^x f(t)dt + \int_x^b f(t)dt = \int_a^b f(t)dt$.

So $H'(x) = 0$. Therefore,

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x + \frac{\mathrm{d}}{\mathrm{d}x}\int_x^b f = f(x) + G'(x),$$

and thus $G'(x) = -f(x)$.

Example 1.4

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\cos x}^{\sin x}\frac{t}{1+t^2}\,\mathrm{d}t = \cos x \sin x\left(\frac{1}{1+\cos^2 x} + \frac{1}{1+\sin^2 x}\right) \tag{6}$$