1 Integrability Condition

1.1 Review

Theorem 1.1

If f is bounded on $[a, b]$, then f is also integrable on $[a, b]$ if and only if for all $\epsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon$$

Proof. Assume that f is given such that f is bounded.

Suppose the condition $U(f, P) - L(f, P) < \epsilon$ is true for any P.

Since $L(f, P) \leq \sup\{L(f, P')\} \leq \inf\{U(f, P')\} \leq U(f, P)$, it follows that $\inf\{U(f, P')\} - \sup\{L(f, P')\} < \epsilon$.

Since this is true for all $\epsilon > 0$, $\inf\{U(f, P')\} = \sup\{L(f, P')\}$. Thus, f is integrable.

Conversely, suppose that f is integrable. Thus, $\inf\{U(f, P)\} = \sup\{L(f, P)\}$ for any P.

Therefore, there exist partitions P', P'' for any $\epsilon > 0$ such that $\inf\{U(f, P'')\} - \sup\{L(f, P')\} < \epsilon$.

Let P be the partition which contains both P', P''. According to the lemma,

$L(f, P') \leq L(f, P)$ and $U(f, P) \leq U(f, P'')$. Therefore, $U(f, P) - L(f, P) < \epsilon$, as required.

For any P, □

1.2 Continuity and Integrability

Theorem 1.2

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Proof. Since f is continuous on $[a, b]$, it is also bounded on $[a, b]$.

It has been shown that f is uniformly continuous on $[a, b]$. Thus, there is some $\delta > 0$ for all x and y in $[a, b]$ such that if $|x - y| < \delta$, then $\left|f(x) - f(y)\right| < \frac{\epsilon}{2(b-a)}$.

Choose a partition $P = \{t_0, t_1, \ldots, t_n\}$ such that $|t_i - t_{i-1}| < \delta$. Then for each i we obtain

$$\left|f(x) - f(y)\right| < \epsilon \text{ for all } x, y \text{ in } [t_{i-1}, t_i].$$

Therefore, $M_i - m_i \leq \frac{\epsilon}{2(b-a)} < \frac{\epsilon}{b-a}$.

This holds for any i, and thus

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) < \frac{\epsilon}{b-a} \sum_{i=1}^{n} (t_i - t_{i-1}) = \epsilon$$

□