1 Geometry of Discrete Painleve Equations

1.1 Revision

We have discussed the QRT mapping, which is ϕ^2, where ϕ looks as follows:

$$\begin{cases} \overline{x} = \frac{(x-a)(x-a^{-1})}{y(x+a)(x+a^{-1})} \\ \overline{y} = x \end{cases}$$

The inverse of ϕ is then:

$$\begin{cases} \underline{x} = y \\ \underline{y} = \frac{(y-a)(y-a^{-1})}{x(y+a)(y+a^{-1})} \end{cases}$$

We can draw the picture for the dynamics as a map. For this, introduce the coordinates induced by $x, y, X = \frac{1}{x}$ and $Y = \frac{1}{y}$. Then we can draw points with the respective coordinates cycling over a, a^{-1}, $-a$ and $-a^{-1}$.

Now, take $x = u + a$, $y = u \cdot v$. This gives us the following:

$$\overline{x} = \frac{u(u + a - a^{-1})}{uv(u + 2a)(u + a + a^{-1})}. \tag{1}$$

Note that, if we account for $u = 0$, we can eliminate u from both sides.

1.2 Blowing Up

Suppose that $(0, 0)$ is a centre.

Draw a graph of y over x. Note that a point (a, b) lies on the linkes $x = a$ and $y = b$.

Let $k = \frac{b}{a}$. We can fully and redundantly describe the system by a triple (a, b, k).

Thinking in a projective space, $k = [\zeta_0 : \zeta_1]$, we obtain that x corresponds to $[0 : 1]$ and y to $[1 : 0]$.

In this way, we can talk in terms of triples $x, y; \zeta$.

We can blow-up the point in such a way that $\frac{y}{x} = \frac{\zeta_0}{\zeta_1}$, with $S = \{x\zeta_0 = y\zeta_1\}$.

The blow-up makes a point correspond to a line, with all the variables having a respective counterpart.

We can also make a transformation $x = u = UV$ and $y = uv = V$ to make the picture easier.

Now, let's look at the plane with a point and its blown-up counterpart, which can be described by the *exceptional divisor*.

A bijection of line L through the point on the plane crossing the blown-up counterpart is descibed by a divisor $L - E$. Now, let's look at a line with the divisor $M - E$, making $L \cdot M = 1$. What can we say about the lines $(L - E)(M - E)$? They are stretched, $(L - E)(M - E) = 0$, which means that $L \cdot M - E \cdot M - L \cdot E + E \cdot E = 0$. We already know that $L \cdot M = 1$. From the intersections, we can also eliminate $E \cdot M$ and $L \cdot E$, which are equal to 0, and thus $E \cdot E = 0$.

Let's denote the points of intersections as P_1 to P_8, starting on the line $x = 0$ and going in the clockwise direction. Their blown-up counterparts are denoted in a similar way with $E_1 \to E_8$, but with the corresponding coordinate surfaces of $H_x - E_1 - E_2$, X, and $H_y - E_5 - E_6$, where X is a QRT surface.

Now, define $Pic(X) = \text{span}_{\mathbb{Z}}\{\mathcal{H}_X, \mathcal{H}_y, \varepsilon_1, \ldots, \varepsilon_\gamma\}$, where $\mathcal{H}_X \cdot \mathcal{H}_Y = 1$ and $\varepsilon_i \cdot \varepsilon_i = -1$.

The coordinate surfaces have indices of intersection of -2.

We introduce mappings on the Picard lattice $\phi_* : Pic(X) \to Pic(\overline{X})$ and $\phi^* : Pic(\overline{X}) \to Pic(X)$.

A vertical line, by the formula above, goes to the horizontal image.

If, however, y is constant, then we obtain $k(\overline{y} + a)(\overline{y} + a^{-1})\overline{x} - (\overline{y} - a)(\overline{y} - a^{-1}) = 0$.

Thus, $\mathcal{H}_x \mapsto \overline{\mathcal{H}_y}$ and $\mathcal{H}_y \mapsto \overline{\mathcal{H}_x} + 2\overline{\mathcal{H}_y} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4$.

Exercise 1.1.

$$\phi_0 : \mathcal{H}_x \mapsto \overline{\mathcal{H}_y} \tag{2}$$

$$\overline{\mathcal{H}_x} + 2\overline{\mathcal{H}_y} - \overline{\varepsilon_{1234}} \tag{3}$$

$$\varepsilon \mapsto \overline{\varepsilon_0} \tag{4}$$

$$\varepsilon_2 \mapsto \overline{\varepsilon_5} \tag{5}$$

$$\varepsilon_3 \mapsto \overline{\varepsilon_8} \tag{6}$$

$$\varepsilon_4 \mapsto \overline{\varepsilon_7} \tag{7}$$

$$\varepsilon_5 \mapsto \overline{\mathcal{H}_y} - \overline{\varepsilon_1} \tag{8}$$

$$\varepsilon_6 \mapsto \overline{\mathcal{H}_y} - \overline{\varepsilon_2} \tag{9}$$

$$\varepsilon_7 \mapsto \overline{\mathcal{H}_y} - \overline{\varepsilon_3} \tag{10}$$

$$\varepsilon_8 \mapsto \overline{\mathcal{H}_y} - \overline{\varepsilon_4} \tag{11}$$

ϕ is not QRT. Let's write QRT:

$$\begin{cases} \overline{\overline{x}} = \frac{(\overline{x}-\overline{a})(\overline{x}\overline{a}^{-1})}{\overline{y}(\overline{x}+\overline{a}(\overline{x}+\overline{a}^{-1}))} \\ \overline{\overline{y}} = \overline{x} = \frac{(x-a)(x-a^{-1})}{y(x+a)(x+a^{-1})} \end{cases}$$

Let's turn to the Painleve (?) equations:

$$q - P_{VI} = q - P(\frac{A_3^{(1)}}{D_5^{(1)}}),$$

where $q = \frac{b_2 b_4 b_5 b_6}{b_1 b_2 b_7 b_8}$. For some f and g,

$$\begin{cases} \overline{f} = \frac{\overline{b_7 b_8}}{f} \frac{(\overline{g}-\overline{b_1})(\overline{g}-\overline{b_2})}{(\overline{g}-\overline{b_3})(\overline{g}-\overline{b_4})} \\ \overline{g} = \frac{b_3 b_4}{g} \frac{(f-b_5)(f-b_6)}{(f-b_7)(f-b_8)}. \end{cases}$$