1 Heights

Definition 1.1. A basic height is a mapping $h : \overline{\mathbb{R}} \to \mathbb{R}_{\geq 0}$, which has a natural correspondence with a degree of rational or algebraic functions.

The concept of a height was first introduced by Weil, and then developed further by Arakelov.

1.1 Origins

- Transcendental numbers
- Diophantine equations $f_\alpha(x_1, \dots, x_n) = 0$, where $x_i \in \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \mathbb{R}, \mathbb{Q}_p, \dots$

The concept of a height is similar to the concept of an absolute value, but with greater generality.

For example, the statement

If $\in \mathbb{Z} \setminus \{0\}$, then $|m| \geq 1$ describes the discreteness of $\mathbb{Z}$.

1.2 Applications

Suppose $\alpha \in \overline{\mathbb{Q}}$ and $f(x) = \sum_{i=0}^{d} a_i x^{d-i}$ are such that $f(\alpha) = 0$, and $d \geq 1$, $a_i \in \mathbb{Z}$, while $a_0 > 0$.

Then the **Weil height** can be defined as $R(t) = \frac{A(t)}{B(t)}$, where $A, B \in \mathbb{C}[t]$ are coprime.

Then we can define

$$\deg R = \sum_{x_0 \in \mathbb{C}} \max(0, -\operatorname{ord}_{x_0}(R) + max(0, -\operatorname{ord}_\infty(R) \tag{1}$$

$$= \max(\deg A, \deg B). \tag{2}$$

Then we can define a dictionary, which identifies the correspondence between $\mathbb{C}[t]$ and $\mathbb{Z}$, $\mathbb{C}(t)$ and $\mathbb{Q}$, $t - x_0$ for $x_0 \in \mathbb{C}$ and prime numbers p, ord_{x_0} and ord_p, ∞ and $|r|_\infty = |r|$, where $r = \frac{a}{b}$, where a, b are corresponding integers.

Armed with this dictionary, we then define a height as follows:

$$h(r) = \sum_{p \in \mathbb{P}} \max(0, -\operatorname{ord}_p(r)) \log p + \log \max(1, |r|) \tag{3}$$

$$= \sum \log \max(1, |r|_p) = \log \max(|a|, |b|) \tag{4}$$

We can also define the *exponential height*, given by $H(r) = \exp(h(r))$.

If $\alpha \in \overline{\mathbb{Q}}$, and $\alpha \in K$ is a number field, with ν an arbitrary value of K, then

$$h(\alpha) = \sum_{\nu \in M_K} \log \max(1, |\alpha|_{\nu,K} \geq 0$$

The following formula can also be proven.

Theorem 1.2

If $\alpha_1, \ldots, \alpha_d$ are roots of $f(x)$ then $H^d(\alpha) = a_0 \prod_{i=1}^{d} \max(1,|\alpha_i|)$.

For example, $H(1+\sqrt{2} = \sqrt{1+\sqrt{2}}$.

1.3 Properties

- $h(\sum_{i=1}^{r} \alpha_i) \leq \sum_{i=1}^{r} h(\alpha_i) + \log r$
- $h(\prod_{i=1}^{r} \alpha_i \leq \sum_{i=1}^{r} h(\alpha_i)$
- $h(\alpha^n) = |n|\, h(\alpha) \Rightarrow h(\xi) = 0$ for a root of unity ξ, and $h(\xi\alpha) = h(\alpha)$.

  Moreover, $h(p(\alpha)) = \deg p \cdot h(\alpha) + O(1)$, whenever p is a fixed polynomial.
- $h(\alpha^r) = h(\alpha)$

1.4 Applications

Applications of heights include the proofs of Northcott's and Kronecker's Theorems.

Moreover, heights are useful in the study of multiplicative algebraic curves $\mathbb{G}(\overline{QQ}) = \overline{\mathbb{Q}}^K$ and elliptic curves $y^2 = x^3 + ax + b$.

For example, let p be a point on an elliptic curve with rational coordinates, $p \in E(\mathbb{Q})$. Then $h(mp) = m^2 h(p) + O(1)$, and $h(p) = h(x(p))$. If some point p_0 is some fixed point, then $h(p + p_0) = h(p) + O(1)$. Tate-Neron have show that height in this case behaves as a quadratic form.

In turn, Mordell-Weil have shown that the group $E(\mathbb{Q})$ is finitely generated. The proof of the theorem is in two steps. The first, arithmetical, step, is to observe that there is a finite set of points on $E(\mathbb{Q})$ such that if $\mathbb{Q}$ is any point on $E(\mathbb{Q})$, then $\mathbb{Q} = 2\mathbb{Q}' + R$, for $Q' \in E(\mathbb{Q})$ and $R \in S$.

The second step is to show that $h(\mathbb{Q}) = h(z\mathbb{Q}') + O(1) = 14h(\mathbb{Q}') + O(1)$.