Corollary 0.1

$T \in \mathrm{Hom}(V, V)$ is diagonalisable if and only if $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$, where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues.

Proof. If T is diagonalisable, then there exists an ordered basis of eigenvectors β. Note that $\beta \subseteq E_{\lambda_1} + \cdots + E_{\lambda_k}$, and therefore $\operatorname{span} \beta \subseteq E_{\lambda_1} + \cdots + E_{\lambda_k}$, showing that $V = E_{\lambda_1} + \cdots + E_{\lambda_k}$. Since eigenvectors are distinct, they are linearly independent. Thus, $w_1 + \cdots + w_k = 0$ for $w_i \in E_{\lambda_i}$ implies $w_i = 0$, and hence $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$.

Conversely, suppose $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$. Pick a basis β_i of E_{λ_i} for all $i \in [1, k] \cap \mathbb{N}$. Note that $\beta_1 \cup \cdots \cup \beta_k$ is a basis of V, which implies that T is diagonalisable. □

We can also give a better argument for the theorem from last lecture.

Theorem 0.2

If T is diagonalisable, then characteristic polynomial splits and $\dim E_\lambda = m_\lambda$.

Proof. Suppose $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis of eigenvectors. Suppose also $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of T. Renumbering the basis if necessary, the first d_1 basis elements elements have a corresponding eigenvalue λ_1, the next d_2 basis elements have an eigenvalue λ_2, and so on.

Therefore, since T is diagonalisable,

$$[T]_\beta = \begin{pmatrix} \lambda_1 &&&&&& \\ & \ddots &&&&& \\ && \lambda_1 &&&& \\ &&& \lambda_2 &&& \\ &&&& \ddots && \\ &&&&& \lambda_2 & \\ &&&&&& \ddots \end{pmatrix}$$

Moreover, $d_i = m_{\lambda_i}$. Note that $d_i \leq \dim E_{\lambda_i} \leq m_{\lambda_i}$, which implies that

$\dim E_{\lambda_i} = m_{\lambda_i}$. □

Theorem 0.3

If $A \in M_{n\times n(F)}$, then $f(A) = 0$, where $f(t)$ is the characteristic polynomial of A.

Example 0.4

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $f(t) = t^2 - 5t - 2$ and

$$f(A) = f(A) = A^2 - 5A - 2I = \begin{pmatrix} 7 - 5 - 2 & 10 - 10 - 0 \\ 15 - 15 - 0 & 22 - 20 - 2 \end{pmatrix} = \mathbf{0}$$

Remark 0.5. Note that $f(t) = g(t)h(t)$, which implies that $f(A) = g(A)h(A)$.

Remark 0.6. Similarly $f(T)$ can be defined, where T is a linear transformation.

Example 0.7 (Wrong 'Proof')
$f(t) = \det(A - tI) \Rightarrow f(A) = \det(A - AI) = 0$.

Remark 0.8. If A is diagonal, the proof is easy.

Definition 0.9. A subspace $W \subseteq V$ is T-invariant if $T(W) \subseteq W$.

e.g. 0, V, $\ker(T)$, $\operatorname{Im}(T)$, E_λ

If $W \subseteq V$ is T-invariant, we can define

$$T_W \in \operatorname{Hom}(W, W) \text{ by restriction } \forall x \in W : T_W(x) = T(x)$$

Definition 0.10. If $v \in V$, the T-cyclic subspace generated by v is $\operatorname{span}\{v, T(v), T^2(v), \ldots\}$.

Claim. T-cyclic subspace generated by v is T-invariant.

Proof. $T(a_0 + a_1T(v_1) + \cdots + a_nT(v_n)) \in \operatorname{span}\{v, T(v), T^2(v), \ldots\}$ □

Claim. T-cyclic subspace is the smallest T-invariant subspace containing v.

Proof. If $v \in W$, where W is a W-invariant subspace, by definition of of W,
Tv, T^2v, T^3v, $T^{n-1}v$ must also be in W, and thus a T-cyclic subspace generated by v is in W. □