1 Problem III

Suppose $T \in \operatorname{End}(V)$.

Lemma 1.1

$\ker T \subseteq \ker T^2 \subseteq \cdots \subseteq \ker T^k \subseteq \ker T^{k+1} \subseteq \cdots$

Proof.

Suppose $v \in \ker T$. Therefore, $Tv = 0$, and hence $T^2 v = T(0) = 0$, an thus $Tv \in \ker T^2$.

Now, assume $\ker T^{k-1} \subseteq \ker T^k$ for some $k \in \mathbb{Z}^+$, and let $v \in \ker T^k$ be arbitrary. Thus, $T^{k+1}v = TT^k v = T(0) = 0$, and hence $v \in \ker T^{k+1}$. Therefore, $T^k \subseteq T^{k+1}$.

Hence, $\ker T \subseteq \ker T^2 \subseteq \cdots \subseteq \ker T^k \subseteq \ker T^{k+1} \subseteq \cdots$ by induction.

□

Lemma 1.2

If $\operatorname{rank} T^m = \operatorname{rank} T^{m+1}$ for some $m \geq 0$,

then $\operatorname{rank} T^m = \operatorname{rank} T^k$ and $\ker T^m = \ker T^k$ for any $k \geq m$.

Proof.

Suppose $\operatorname{rank} T^m = \operatorname{rank} T^{m+1}$ for some $m \geq 0$.

We want to prove that for any $k \in \mathbb{Z}^+$, $\operatorname{rank} T^{m+k} = \operatorname{rank} T^{m+k+1}$.

Since $\operatorname{im} T^{m+k}$ is T-invariant, because $\operatorname{im} T$ is T-invariant and $\operatorname{im} T^{m+k} \subseteq \operatorname{im} T$, then $\operatorname{im} T^{m+k+1} \subseteq \operatorname{im} T^{m+k}$.

Similarly, since $\operatorname{im} T^{m+1} \subseteq \operatorname{im} T^m$ and $\operatorname{rank} T^m = \operatorname{rank} T^{m+1}$, then $\operatorname{im} T^m = \operatorname{im} T^{m+1}$.

Suppose now that $u \in \operatorname{im} T^{m+k}$. Therefore, there exists $x \in V$ such that $T^{m+k}x = u$.

Hence, $T^m(T^k x) = u$, and then $u \in \operatorname{im} T^m = \operatorname{im} T^{m+1}$.

Thus, there exists $w \in V$ such that $T^{m+1}w = u = T^{m+k}x$, so that $T^m(T^k x - w) = 0$ and $T^k x - w \in \ker T^m$.

From Lemma 1.1, we have $\ker T^m \subseteq \ker T^{m+1}$, and thus $T^{m+1}(T^k x - w) = 0$. Hence, $T^{m+k+1}x = T^{m+1}w = u$. Therefore, $u \in \operatorname{im} T^{m+k+1}$, and thus $\operatorname{im} T^{m+k+1} = \operatorname{im} T^{m+k}$. Since $\operatorname{im} T^m = \operatorname{im} T^{m+1}$, by transitive law we obtain that for any $n \geq m$ we have $\operatorname{im} T^m = \operatorname{im} T^n$ and $\operatorname{rank} T^m = \operatorname{rank} T^n$.

Now we prove that for any $k \in \mathbb{Z}^+$, $\ker T^{m+k+1} = \ker T^{m+k}$.

Take arbitrary $k \in \mathbb{Z}^+$.

By the rank-nullity theorem, $\dim V = \operatorname{rank} T^{m+k} + \operatorname{nullity} T^{m+k}$. Since we have already shown that $\operatorname{im} T^{m+k+1} = \operatorname{im} T^{m+k}$, while $\dim V = \operatorname{rank} T^{m+k+1} + \operatorname{nullity} T^{m+k+1}$, we see that $\operatorname{nullity} T^{m+k} = \operatorname{nullity} T^{m+k+1}$. Since also $\ker T^{m+k} \subseteq \ker T^{m+k+1}$ by Lemma 1.1, we see that $\ker T^{m+k} = \ker T^{m+k+1}$, which means that for any $n \geq m$ we have $\ker T^m = \ker T^n$. □

Lemma 1.3

$\operatorname{rank}(T-\lambda I)^m = \operatorname{rank}(T-\lambda I)^{m+1}$ for some $m \geq 0$ if and only if $K_\lambda = \ker(T-\lambda I)^m$.

Proof.

Note that, by definition of K_λ, for any $m \in \mathbb{Z}^+$, $\ker(T-\lambda I)^m \subseteq K_\lambda$.

Suppose first that $\operatorname{rank}(T-\lambda I)^m = \operatorname{rank}(T-\lambda I)^{m+1}$.

Let $v \in K_\lambda$ be arbitrary. Therefore, there exists $k \in \mathbb{Z}^+$ such that $x \in \ker(T-\lambda I)^k$.

If $k \leq m$, by Lemma 1.1 we have that $\ker(T-\lambda I)^k \subseteq \ker(T-\lambda I)^m$, and therefore $v \in \ker(T-\lambda I)^m$.

If $k > m$, by Lemma 1.2 we have that $\ker(T-\lambda I)^m = \ker(T-\lambda I)^k$, and thus $v \in \ker(T-\lambda I)^m$.

Therefore, $\ker(T-\lambda I)^m = K_\lambda$.

Suppose now that $K_\lambda = \ker(T-\lambda I)^m$.

From Lemma 1.1, $\ker(T-\lambda I)^m = K_\lambda \subseteq \ker(T-\lambda I)^{m+1}$. By definition of K_λ, we have $\ker(T\lambda I)^{m+1} \subseteq K_\lambda$. Therefore, $\ker(T-\lambda I)^m = \ker(T-\lambda I)^{m+1}$, and thus we obtain $\operatorname{nullity}(T-\lambda I)^m = \operatorname{nullity}(T-\lambda I)^{m+1}$.

By the rank-nullity theorem, we also know that $\operatorname{rank}(T-\lambda I)^m = \dim V - \operatorname{nullity}(T-\lambda I)^m$ and $\operatorname{rank}(T-\lambda I)^{m+1} = \dim V - \operatorname{nullity}(T-\lambda I)^{m+1}$.

Thus, $\operatorname{rank}(T-\lambda I)^m = \operatorname{rank}(T-\lambda I)^{m+1}$, proving the implication to the left. $\square$

Lemma 1.4

T is diagonalisable if and only if the characteristic polynomial of T splits and $\operatorname{rank}(T-\lambda I) = \operatorname{rank}(T-\lambda I)^2$ for all eigenvalues λ.

Proof.

Suppose first that T is diagonalisable. Therefore, we know by Theorem 5.6 that the characteristic polynomial of T splits.

Moreover, by Corollary to Theorem 7.4, $E_\lambda = K_\lambda$.

Thus, $K_\lambda = E_\lambda$, and hence $K_\lambda = \ker(T-\lambda)$ for any eigenvalue λ.

By Lemma 1.3, we therefore obtain that $\operatorname{rank}(T-\lambda I) = \operatorname{rank}(T-\lambda I)^2$ for any eigenvalue λ.

Suppose now that the characteristic polynomial of T splits and $\operatorname{rank}(T-\lambda I) = \operatorname{rank}(T-\lambda I)^2$ for all eigenvalues λ.

By Lemma 1.3, $K_\lambda = \ker(T-\lambda I) = E_\lambda$ for any eigenvalue λ. Also, since the characteristic polynomial splits, by Corollary to Theorem 7.4 we obtain that T is diagonalisable. $\square$