Let $V = \mathscr{P}_5(\mathbb{R})$ with the standard basis $\{q_0, q_1, \ldots, q_5\}$,
where $q_0 = 1$, $q_1 = x$, …, $q_5 = x^5$.
Consider $\phi : V \to \mathbb{R}$ such that $\phi(p) = p(3)$.
Note that for $\alpha \in \mathbb{R}, r \in \mathscr{P}_5(\mathbb{R})$:

$$\phi(\alpha p + r) = (\alpha p + r)(3) = \alpha p(3) + r(3) = \alpha\phi(p) + \phi(r),$$

and hence ϕ is a linear functional. Since $\phi : V \to \mathbb{R}$ and ϕ is linear, then ϕ is an element of a dual space.
Let $\{q_0^*, q_1^*, \ldots, q_5^*\}$ be a dual basis.
Note that, for i, j in $\{0, 1, \ldots, 5\}$, $q_i^*(q_j) = \delta_{ij}$.
Consider the following:

$$\phi(q_0) = 1 \tag{1}$$
$$\phi(q_1) = 3 \tag{2}$$
$$\phi(q_2) = 9 \tag{3}$$
$$\phi(q_3) = 27 \tag{4}$$
$$\phi(q_4) = 81 \tag{5}$$
$$\phi(q_5) = 243 \tag{6}$$

Therefore, ϕ can be represented in the following form:

$$\phi = \sum_{i=0}^{5} \phi(q_i) q_i^* = \sum_{i=0}^{5} 3^i q_i^*,$$

since the left and right sides coincide on application of any linear combination $\sum_{i=0}^5 a_i q_i$, because $q_i^*(\sum_{i=0}^5 a_i q_i) = a_i$ by linearity, and thus

$$\begin{aligned} \phi(\sum_{i=0}^{5} a_i q_i) &= \sum_{i=0}^{5} \phi(q_i) q_i^* \sum_{i=0}^{5} a_i q_i && (7)\\ &= \sum_{i=0}^{5} \phi(q_i) a_i && (8) \end{aligned}$$

Suppose now $\phi(p) = \int_0^1 t^2 p(t)dt$.
Note that for $\alpha \in \mathbb{R}, r \in \mathscr{P}_5(\mathbb{R})$:

$$\begin{aligned} \phi(\alpha p + r) &= \int_0^1 t^2(\alpha p + r)(t)dt && (9)\\ &= \int_0^1 t^2(\alpha p(t) + r(t))dt && (10)\\ &= \int_0^1 t^2 \alpha p(t)dt + \int_0^1 r(t)dt && (11)\\ &= \alpha \int_0^1 t^2 p(t)dt + \int_0^1 r(t)dt && (12)\\ &= \alpha\phi(p) + \phi(r) && (13) \end{aligned}$$

and hence ϕ is a linear functional. Since $\phi : V \to \mathbb{R}$ and ϕ is linear, then ϕ is an element of a dual space.
Consider the following:

$$\phi(q_0) = \int_0^1 t^2 = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3} \tag{14}$$

$$\phi(q_1) = \int_0^1 t^3 = \left[\frac{t^4}{4}\right]_0^1 = \frac{1}{4} \tag{15}$$

$$\phi(q_2) = \frac{1}{5} \tag{16}$$

$$\phi(q_3) = \frac{1}{6} \tag{17}$$

$$\phi(q_4) = \frac{1}{7} \tag{18}$$

$$\phi(q_5) = \frac{1}{8} \tag{19}$$

Therefore, ϕ can be represented in the following form:

$$\phi = \sum_{i=0}^{5} \phi(q_i) q_i^* = \sum_{i=0}^{5} \frac{1}{i+3} q_i^*,$$

since the left and right sides coincide on application of any linear combination $\sum_{i=0}^{5} a_i q_i$, because $q_i^*(\sum_{i=0}^{5} a_i q_i) = a_i$ by linearity, and thus

$$\phi(\sum_{i=0}^{5} a_i q_i) = \sum_{i=0}^{5} \phi(q_i) q_i^* \sum_{i=0}^{5} a_i q_i \tag{20}$$

$$= \sum_{i=0}^{5} \phi(q_i) a_i \tag{21}$$