1 Introduction to Markov Processes III

We have already seen that $\frac{\mathrm{d}p_{ij}}{\mathrm{d}t} = \sum_{k\in S} q_{ik}p_{kj}(t)$.

Moreover, $p_{ij}(t) = (\exp(tQ))_{ij} = \sum_{k=0}^{\infty} \frac{t^kQ^k}{k!}$.

Thus, $\frac{\mathrm{d}}{\mathrm{d}t}P = QP$.

We can rewrite the first equation above as follows:

$$\frac{\mathrm{d}p_{ij}}{\mathrm{d}t} = -c(i)p_{ij}(t) + \sum_{k\neq i} q_{ik}p_{kj(tQ)}.$$

Let's solve the equation in the form $\frac{\mathrm{d}}{\mathrm{d}t}p = -cp(t) + g$. We obtain $p = Re^{-ct}$, where $\frac{\mathrm{d}R}{\mathrm{d}S} = ge^{cS}$. Thus, $R = \int_0^t ge^{cs}\,\mathrm{d}s$, where $p = \int_0^t ge^{c(s-t)}\,\mathrm{d}s$. Hence,

$$p_{ij}(t) = \delta_{ij}e^{-ct} + \int_0^t e^{c(s-t)}(\sum_{k\neq i} q_{ik}p_{kj}(s))\,\mathrm{d}s.$$

Thus, we can write

$$p_{ij}(t) = \delta_{ij}e^{-c(i)t} + \int_0^t e^{c(i)(s-t)}(\sum_{k\neq i} q_{ik}p_{kj}(s))\,\mathrm{d}s$$

Let's construct explicitly a solution of this equation with a specific probabilistic meaning. Then we will check that the solution satisfies the Kolmogorov-Chapman equation, and discuss the uniqueness of solutions.

The method of finding a solution is that of sequential approximations:

$$p_{ij}^{(0)}(t) = \delta_{ij}e^{-c(i)t} \tag{1}$$

$$p_{ij}^{(n+1)}(t) = \delta_{ij}e^{-c(i)t} + \int_0^t e^{c(i)(s-t)}\sum_{k\neq i} q_{ik}p_{kj}^{(n)}(s)\,\mathrm{d}s \tag{2}$$

Note that $p_{ij}^{(n+1)}(t) \geq p_{ij}^{(n)}(t)$, which can be shown by induction.

Thus, $p_{ij}^{(n+1)}(t) - p_{ij}^{(n)}(t) = \overline{p_{ij}}(t) = \lim_{n\to\infty} p_{ij}^{(n)}(t)$, which means that

$$p_{ij}^{(n+1)}(t) - p_{ij}^{(n)}(t) = \int_0^t e^{c(i)(s-t)}\sum_{k\neq i} q_{ik}(p_{kj}^{(n)}(s) - p_{kj}^{(n-1)}(s))\,\mathrm{d}s.$$

For all i, we have $\sum_j p_{ij}^{(n)}(t) \leq 1$ and $\sum_j \overline{p_{kj}}(t) \leq 1$.

Let $\widehat{p_{ij}}$ be a solution. We know that $\widehat{p_{ij}} \geq \overline{p_{ij}}$, given that $\widehat{p_{ij}} \geq p_{ij}^{(n)}$.

The constructed solution is therefore minimal. Therefore, in the non-explosive case, when $\sum_j \overline{p_{ij}} = 1$, the solution is unique.

Remember that $p_{ij}^{(0)} = \delta_{ij}e^{-c(i)t}$, and thus $p_{ij}^{(1)}(t) = \delta_{ij}e^{-c(i)t} + \int_0^t e^{c(i)(s-t)}q_{ij}e^{-c(j)s}\,\mathrm{d}s$.

Hence, $p_{ij}^{(2)}(t) = \delta_{ij}e^{-c(i)t} + \int_0^t e^{c(i)(s-t)-c(j)s}q_{ij}$.

In general,

$$p_{ij}^n(t) = \delta_{ij} e^{-c(i)t} + \sum_{i \neq k_1, k_1 \neq k_2, \dots, k_{r-2} \neq k_{r-1}} q_{ik_1} q_{ik_2} \dots q_{r-1} \times I,$$

where $I = \int \cdots \int \exp(-c(i)s_1 - c(k_1)s_2 - \{s_1 + \cdots + s_r < t\} - \cdots - c(k_{r-1})s_r - c(j)(t - s_1 - \cdots - s_r))ds_1 \dots ds_r$.

Let $\Lambda = \delta_{ij} c(i)$, $\Pi = \{p_{ij}\}, \pi_{ij} = \frac{q_{ij}}{c(i)}$.

Then $P(t) = e^{-t\Lambda} + \sum_{r=1+s_1+\cdots+s_r<t}^{\infty} \int e^{-s_1\Lambda} \Lambda \Pi \dots e^{-(t-s_1-\cdots-s_r)\Lambda} ds_1 \dots ds_r$.