Theorem. *Arbitrary monic polynomials $p_k \in \mathcal{P}_n(\mathbb{R})$ of degree $k \in [0:n]_{\mathbb{Z}}$ form a basis β of $\mathcal{P}_n(\mathbb{R})$:*

Proof. By definition, $p_i = a_i x^i$ for some i in $[0:n]_{\mathbb{Z}}$ and $a_i \in \mathbb{R}$.

Let $\alpha = \{p_0, p_1, \dots, p_n\}$.

Let S be the set of all the possible linear combinations of monic polynomials. Thus, $S = \{\sum_{i=0}^{n} a_i x^i | a_i, x_i \in \mathbb{R}\}$, which is exactly the definition of $\mathcal{P}_n(\mathbb{R})$.

Thus, $\operatorname{span}(\alpha) = \mathcal{P}_n(\mathbb{R})$.

Moreover, α is linearly independent.

To obtain a contradiction, suppose it is not, and hence

$$\exists (p_0, p_1, p_2, \dots, p_n \in \mathcal{P}_n(\mathbb{R}),\ b_0, b_1, b_2, \dots, b_n \in \mathbb{R}, \prod_{i=1}^{n} b_0 b_1 b_2 \cdot \ldots \cdot b_n \neq 0) : \sum_{i=0}^{n} b_i p_i = 0.$$

Consider the sum $s = \sum_{i=0}^{n} b_i p_i$ for $x = 1$:

$$s(1) = \sum_{i=0}^{n} a_i b_i = 0.$$

Since monomials p_i are arbitrary, choose a_i such that $a_i = 1$ for all i in $[0:n]_{\mathbb{Z}}$.

Therefore, from the equation above, $\sum_{i=0}^{n} b_i = 0$. Therefore, $b_0 = -(\sum_{i=1}^{n} b_i)$.
Consider now $s(2)$:

$$s(2) = \sum_{i=0}^{n} 2^i b_i = 0.$$

Thus, $s(2) - b_0 = \sum_{i=1}^{n} (2^i - 1) b_i = 0$.

Continue the procedure by evaluating s for the coefficient of b_i with the least i in the sum and eliminate the corresponding b_i by subtraction. When all the b_i are eliminated but for the last one, we obtain that $b_n = 0$. In the next iteration, obtain that $b_{n-1} = 0$. By similar argument we get that $b_0 = b_1 = \cdots = b_n = 0$, which is a contradiction.

Thus, $\operatorname{span}(\alpha) = \mathcal{P}_n(\mathbb{R})$ and α is linearly independent. Hence, $\beta = \alpha$.

$\square$