1 Orthogonality

Definition 1.1. Two vectors $u, v \in V$ are called **orthogonal** if $\langle u, v\rangle = 0$.

Definition 1.2. A subset $S \subset V$ is called **orthogonal** if $\langle x, y\rangle = 0$ for all $x + y \in S$.

Definition 1.3. A vector $x \in V$ is called a **unit vector** if $\|x\| = 1$.

Definition 1.4. A subset $S \subset V$ is **orthonormal** if it is an orthogonal subset consisting of unit vectors. Thus, for all $x, y \in S$, $\langle x, y\rangle = \delta_{xy}$.

Remark 1.5. If $x \neq 0$, then $\frac{x}{\|x\|}$ is a unit vector.

Example 1.6

In $\mathbb{F}^n$ with the standard inner product the standard basis is orthonormal.

Example 1.7

See p. 335 in Friedberg et al.

In $\mathbb{H} = \mathfrak{C}[0, 2\pi]$ with $\langle f, q\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,\mathrm{d}t$, we have an orthonormal subset $\{f_n(t) = e^{\text{int}}\}$

Example 1.8

If $A \in M_{n\times n}(\mathbb{F})$, define $A^* = \overline{A^t} = \overline{A}^t$, i.e. $(A^*)_{ij} = \overline{A}_{ji}$. Therefore,

$$(AB)^* = B^*A^* \tag{1}$$
$$(A^*)^* = A \tag{2}$$

Define the $\langle\cdot,\cdot\rangle$: $\langle A, B\rangle = \operatorname{tr}(AB^*) = \operatorname{tr}(B^*A)$.

Check that it is a valid inner product (the last two properties are left as an exercise):

a)
$$\operatorname{tr}(cA_1 + A_2)B^* = \operatorname{tr} A_1B^* + A_2B^* \tag{3}$$
$$= c\langle A, B\rangle + \langle A_2, B\rangle \tag{4}$$

Note that we can see that this is a valid way to define an inner product on $M_{n\times n}(\mathbb{F})$ as follows:

$$\langle A, B\rangle = \operatorname{tr} AB^* \tag{5}$$
$$= \sum_{i=1}^{n}(AB^*)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(B^*)_{ji} \tag{6}$$
$$= \sum_{i,j=1}^{n} A_{ij}\overline{B}_{ij} \tag{7}$$

2 Gram-Schmidt and Orthogonal Complements

Let V be a vector space.

Definition 2.1. An **orthogonal basis** is a basis that is also orthogonal.

Theorem 2.2

If $S = \{v_1, \dots, v_n\}$ is an orthogonal subset of non-zero vectors, then for any x in $\operatorname{span} S$:

$$x = \sum_{i=1}^{n} \frac{\langle x, v_i \rangle}{\|v_i\|^2} v_i$$

In particular, if S is orthonormal, then $x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$.

Proof. If $x \in \operatorname{span} S$, then $x = \sum_{i=1}^{n} \lambda_i v_i$.

Take now $\langle \cdot, y \rangle$.

Then

$$\langle x, v_j \rangle = \langle \sum_{i=1}^{n} \lambda_i v_i, v_j \rangle \tag{8}$$

$$= \sum_{i=1}^{n} \langle v_i, v_j \rangle = \lambda_j \langle v_j, v_j \rangle = 0, \tag{9}$$

Note that $\sum_{i=1}^{n} \langle v_i, v_j \rangle = 0$ for $i \neq j$.

Therefore, $\lambda_j = \frac{\langle x, v_j \rangle}{\|v_j\|}$ $\square$

Corollary 2.3

If S is orthogonal and $0 \notin S$, then S is linearly independent.

Proof. If $\sum_{i=1}^{n} \lambda_i v_i = 0$ $(\lambda_i \in \mathbb{F}, v_i \in S]$, then $\langle \sum_{i=1}^{n} \lambda_i v_i, v_j \rangle = 0$ for all j. Since $v_j \neq 0$, then $\langle v_j, v_j \rangle \neq 0$, and hence $\lambda_j = 0$. $\square$

Question. Why do orthogonal bases exist? How do we find them?

Answer. An idea is to start with any basis $w_1, \dots, w_n$ of V.

Take $v_1 = w_1$.

Try to find $c \in \mathbb{F}$ such that $v_2 = w_2 + cv_1$ is orthogonal to v_1:

$$0 = \langle v_2, v_1 \rangle = \langle w_2 + cv_1, v_1 \rangle = \langle w_2, v_1 \rangle + c \langle v_1, v_1 \rangle = \|v_1\|^2 > 0 \tag{10}$$

Then $c = -\frac{\langle w_2, v_1 \rangle}{\|v_1\|^2}$.

Next try $v_3 = w_3 + dv_1 + ev_2$, with $d, e \in F$. Solve for d, e by using $\langle v_3, v_1 \rangle + \langle v_3, v_2 \rangle = 0$.

Theorem 2.4 (Gram-Schmidt)

Suppose $w_1, \ldots, w_k$ are linearly indepenedent in V.

Define $v_i = w_i - \sum_{j=1}^{i-1} \frac{\langle w_i, v_j \rangle}{\|v_j\|^2} v_j$ for all i inductively, with $v_1 = w_1$.

Then $\{v_1, \ldots, v_k\}$ is an orthogonal subset and $0 \notin S$, with the same span as $\{w_1, \ldots, w_k\}$.

Remark 2.5. Hence if $w_1, \ldots, w_k$ is a basis of V, then $v_1, \ldots, v_k$ is an orthogonal basis of V.

We can make it into an orthonormal basis by normalisation:

$$\frac{v_1}{\|v_1\|}, \ldots, \frac{v_k}{\|v_k\|}$$

Corollary 2.6

Any finite-dimensional inner product space has an orthonormal basis.

Proof. We use induction on k.

Let $k = 1$. Then $v_1 = w_1$ is non-zero, and the rest follows.

If the claim is true for $k - 1$, then we know that

- $\{v_1, \ldots, v_{k-1}\}$ is orthogonal, $v_i \neq 0$ for all $i < k$.
- $\operatorname{span} v_1, \ldots, v_{k-1} = \operatorname{span} w_1, \ldots, w_{k-1}$.

We need to check that

1. $\langle v_k, v_i \rangle = 0$ for all $i < k$, which means that $\{v_1, \ldots, v_k\}$ is orthogonal.
2. $v_k \neq 0$
3. $\operatorname{span} v_1, \ldots, v_k = \operatorname{span} w_1, \ldots, w_k$

For (1),

$$\langle v_k, v_i \rangle = \langle w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j, v_i \rangle \tag{11}$$

$$= \langle w_k, v_i \rangle - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} \langle v_j, v_i \rangle = 0 \text{ by (a), if } i \neq j \tag{12}$$

$$= \langle w_k, v_i \rangle - \frac{\langle w_k, v_i \rangle}{\|v_i\|^2} \|v_i\|^2 \tag{13}$$

For (2), if $v_k = 0$, then

$$w_k = \sum_{i=1}^{k-1} (\ldots) v_i \in \operatorname{span}\{v_1, \ldots, v_{k-1}\} = \operatorname{span} w_1, \ldots, w_{k-1},$$

which contradicts the assumption that $(w_i)_1^k$ are linearly independent. Therefore, $v_k \neq 0$.

For (3), note that

$$\operatorname{span}\{v_1,\dots,v_k\} = \operatorname{span}\operatorname{span} v_1,\dots,v_{k-1},w_k = \operatorname{span}\{w_1,\dots,w_{k-1},w_k\}.$$

□

Example 2.7

In $\mathbb{R}^3$, if the basis is $w_1 = (1,1,1)$, $w_2 = (1,1,0)$, $w_3 = (1,0,0)$, then $v_1 = w_1 = (1,1,1)$,

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}),$$

while

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2 = (\frac{1}{2}, \frac{1}{2}, 0)$$

Example 2.8

See p. 345 in Friedberg et al.

Let $V = \mathfrak{P}(\mathbb{R})$ and $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,\mathrm{d}t$.

If $w_1 = 1$, $w_2 = x$, $w_3 = x^3$, then we obtain the orthogonal subset $v_1 = 1$, $v_2 = x$, $v_3 = x^2 - \frac{1}{3}$, and $v_4 = x^3 - \frac{3}{5}x, \dots$, which are called *Legendre polynomials*.

Remark 2.9.

From Theorem 2.2, if $\beta = \{v_1,\dots,v_n\}$ is an orthonormal basis, then $x = \sum_{i=1}^{n} \langle x, v_i\rangle v_i$. Moreover, $T \in \operatorname{Hom}(V,V)$ and β is an orthonormal basis, then $A = [T]_\beta$ has entries $A_{ij} = \langle T(v_i), v_i\rangle$.

Definition 2.10. If $S \subseteq V$, then let an **orthogonal complement of S** be $S^\perp = \{x \in V \mid \langle x, y\rangle = 0. \forall y \in S\}$.

Remark 2.11. If $S^\perp$ is a subspace of V, then $0 \in S^\perp$. If $x_1, x_2 \in S^\perp$, then for $y \in S$

$$\langle cx_1 + x_2, y\rangle = c\langle x_1, y\rangle + \langle x_2, y\rangle = 0$$

Thus, $cx_1 + x_2 \in S^\perp$.

Example 2.12

If $V = \{0\}^\perp$, then $V^\perp = \{0\}$, since $\langle x, x\rangle > 0$ for $x \neq 0$.

Remark 2.13. $S^\perp = \operatorname{span} S^\perp$, because if x is orthonormal to vectors in S, then x is orthonormal to any linear combination of vectors in S.

Theorem 2.14

If $W \subseteq V$, then $V = W \oplus W^\perp$. Moreover, if $\{v_1,\dots,v_k\}$ is an orthonormal basis of W and $x = w + z$ for $w \in W, z \in W^\perp$, then $w = \sum_{i=1}^{k} \langle x, v_i\rangle v_i$.

Proof. Pick an orthonormal basis $v_1, \ldots, v_k$ of W.

Suppose $x = w + z$ with $w \in W$ and $z \in W^{\perp}$. Therefore, $w = \sum_{i=1}^{k} \langle w, v_i \rangle v_i = \sum_{i=1}^{k} \langle x, v_i \rangle v_i$ since $z \in W^{\perp}$.

To check that $V = W \oplus W^{\perp}$, first we prove that $V = W + W^{\perp}$.

Take $x \in V$. Define $w = \sum_{i=1}^{k} v_i \in W$. Then $z_i = x - w$.

Note that

$$\langle z, v_i \rangle = \langle x - w, v_j \rangle = \langle x, v_j \rangle = \sum_{i=1}^{k} \langle x, v_i \rangle \langle v_i, v_j \rangle = \delta_{ij} \tag{14}$$

$$= \langle x, v_j \rangle - \langle x, v_j \rangle = 0 \tag{15}$$

Therefore, $z \in W^{\perp}$, since $\{v_1, \ldots, v_k\}$ is a basis.

To check that $W \cap W^{\perp} = \{0\}$, suppose that $x \in W \cap W^{\perp}$. Then $\langle x, x \rangle = 0$, and hence $x = 0$. $\square$