Problem 3

Let V be a finite-dimensional vector space over $\mathbb{F}$. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. Fix scalars $c_1, c_2, \ldots, c_n \in \mathbb{F}$ and define for any $a_i, b_i \in \mathbb{F}$:

$$\langle \sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} b_i v_i \rangle = \sum_{i=1}^{n} c_i a_i \overline{b_i}$$

Let $\varepsilon = \{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis given by the Gram-Schmidt Procedure for the standard inner product $[\cdot, \cdot]$.

a) Suppose first that $\langle \cdot, \cdot \rangle$ is an inner product.

By definition of $\langle \cdot, \cdot \rangle$,

$$\begin{aligned}
\langle \sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} b_i v_i \rangle &= \sum_{i=1}^{n} a_i \langle v_i, \sum_{i=1}^{n} b_i v_i \rangle && (1)\\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \overline{b_j} \langle v_i, v_j \rangle && (2)\\
&= \sum_{i=1}^{n} c_i a_i \overline{b_i} && (3)
\end{aligned}$$

Since $\langle v_i, v_j \rangle \geq 0$ and the final sum contains only products of $a_i \overline{b_i}$ multiplied by some scalar, we can infer that $\langle v_i, v_j \rangle = \delta_{ij} \langle v_i, v_j \rangle$. Moreover, since for any $v, w \in V$ $\langle v, w \rangle = \overline{\langle w, v \rangle}$, c_i must be real and thus equal to its complex conjugate. Therefore, $c_i = \langle v_i, v_i \rangle \in \mathbb{R}^+$, where $\mathbb{R}^+$ denotes positive reals, which is a necessary condition.

Note that inequality is strict, since there is no zero v_i.

To prove that this condition is sufficient, suppose $\langle v_i, v_i \rangle = c_i \in \mathbb{R}^+$. Therefore,

$$\langle \sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} b_i v_i \rangle = \sum_{i=1}^{n} a_i \overline{b_i} \langle v_i, v_i \rangle = \sum_{i=1}^{n} a_i \overline{b_i} c_i$$

We prove that this is an inner product.

Positivity

For all $v = \sum_{i=1}^{n} a_i v_i \in V$, $\langle v, v \rangle = \sum_{i=1}^{n} a_i \overline{a_i} c_i = \sum_{i=1}^{n} |a_i|^2 c_i \geq 0$, since $|\cdot| \geq 0$ and $c_i > 0$.

Definiteness

Let $v = \sum_{i=1}^{n} a_i v_i \in V$.

Suppose $\langle v, v \rangle = 0$. Therefore, $\sum_{i=1}^{n} |a_i|^2 c_i = 0$. Therefore, $\langle v, v \rangle$ is 0 if and only if $a_i = 0$ for all i, and therefore $v = 0$.

Additivity in the First Slot

Note that for all $u = \sum_{i=1}^{n} x_i v_i, v = \sum_{i=1}^{n} y_i v_i, w = \sum_{i=1}^{n} z_i v_i \in V$

$$\langle u + v, w \rangle = \sum_{i=1}^{n} (x_i + y_i) \overline{z_i} c_i = \sum_{i=1}^{n} x_i \overline{z_i} c_i + \sum_{i=1}^{n} y_i \overline{z_i} c_i = \langle u, w \rangle + \langle v, w \rangle.$$

Homogeneity in the First Slot

For all $v = \sum_{i=1}^{n} a_i v_i, w = \sum_{i=1}^{n} b_i v_i \in V$ and $\lambda \in \mathbb{F}$,

$$\langle \lambda v, w \rangle = \sum_{i=1}^{n} \lambda a_i \overline{b_i} c_i = \lambda \sum_{i=1}^{n} a_i \overline{b_i} c_i = \lambda \langle v, w \rangle$$

Conjugate Symmetry

For all $v = \sum_{i=1}^{n} a_i v_i, w = \sum_{i=1}^{n} b_i v_i \in V$,

$$\langle v, w \rangle = \sum_{i=1}^{n} a_i \overline{b_i} c_i = \sum_{i=1}^{n} \overline{\overline{a_i} b_i \overline{c_i}} = \sum_{i=1}^{n} \overline{\overline{a_i} b_i c_i} = \overline{\langle w, v \rangle}$$

Therefore, $\langle \cdot, \cdot \rangle$ is an inner product.

b) Let ϕ be the linear transformation $\phi \in \mathrm{Hom}(V, V)$ such that $\phi(v_i) = e_i$. Note that such a transformation is well-defined, since every homomorphism is defined by its action on the basis of the domain.

Moreover, since ε is a basis, the range of ϕ is the whole V, which means, from the rank-nullity theorem, that ϕ is an isomorphism. Therefore, ϕ is invertible.

Define an inner product $\rangle \cdot, \cdot \langle$ such that for all $v, w \in V$ $\rangle v, w \langle = \langle \phi v, \phi w \rangle$, where $\langle \cdot, \cdot \rangle$ is a standard inner product.

We prove now that $\rangle \cdot, \cdot \langle$ is indeed an inner product:

Positivity

For all $v \in V$, $\rangle v, v \langle = \langle \phi v, \phi v \rangle \geq 0$ by definition of $\langle \cdot, \cdot \rangle$.

Definiteness

Suppose $\rangle v, v \langle = 0$. Therefore, $\langle \phi v, \phi v \rangle = 0$. By definition of $\langle \cdot, \cdot \rangle$, $\phi v = 0$, and since ϕ is an isomorphism and hence injective, then $v = 0$.

Additivity in the First Slot

Note that for all $u, v, w \in V$

$$\rangle u + v, w \langle = \langle \phi(u + v), \phi w \rangle = \langle \phi u, \phi w \rangle + \langle \phi v, \phi w \rangle = \rangle u, w \langle + \rangle v, w \langle$$

Homogeneity in the First Slot

For all $v, w \in V$ and $\lambda \in \mathbb{F}$,

$$\rangle \lambda v, w \langle = \langle \phi(\lambda v), \phi w \rangle = \langle \lambda \phi v, \phi w \rangle = \lambda \langle \phi v, \phi w \rangle = \lambda \rangle v, w \langle$$

Conjugate Symmetry

For all $v, w \in V$,

$$\rangle v, w \langle = \langle \phi v, \phi w \rangle = \overline{\langle \phi w, \phi v \rangle} = \overline{\rangle w, v \langle}$$

Therefore, $\rangle \cdot, \cdot \langle$ is an inner product.

Observe that for any $i, j \in [1, n] \cap \mathbb{N}$:

$$\rangle v_i, v_j \langle = \langle \phi v_i, \phi v_j \rangle = \langle e_i, e_j \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} = \delta_{ij},$$

and thus β is an orthonormal basis, if $\rangle\cdot,\cdot\langle$ is an inner product.

c)

Claim. $\rangle\cdot,\cdot\langle$ is unique.

Proof. Suppose, on the other hand, there exists an inner product $[\cdot,\cdot]$ such that β is orthonormal under $[\cdot,\cdot]$.

Therefore, by definition, for all $i, j \in [1, n] \cap \mathbb{N}$,

$$[v_i, v_j] = \delta_{ij} = \langle e_i, e_j \rangle = \langle \phi v_i, \phi v_j \rangle = \rangle v_i, v_j \langle.$$

Thus, $\rangle\cdot,\cdot\langle$ is unique. $\square$