1 Surface Gluing, Wigner's Circle Law and Free Convolutions

1.1 Revision

Definition 1.1. kth moment of a random variable X is $m_k = \mathbb{E}(X^k)$.

Definition 1.2. The *Laplace transform of a function* ρ is a function $\Phi(t) = \int_{\mathbb{R}} e^{tx}\rho(x)\,\mathrm{d}x = \mathbb{E}(e^{tX})$.

Note. $\Phi(0) = 1$.

What happens if we differentiate Φ?

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} e^{tx}\rho(x)\,\mathrm{d}x \tag{1}$$
$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}e^{tx}\rho(x)\,\mathrm{d}x \tag{2}$$
$$= \int_{-\infty}^{\infty} xe^{tx}\rho(x)\,\mathrm{d}x \tag{3}$$
$$= \mathbb{E}(Xe^{tX}). \tag{4}$$

Therefore, $(\frac{\mathrm{d}}{\mathrm{d}t})^n\Phi(t) = \mathbb{E}(X^n e^{tX})$, which means that $(\frac{\mathrm{d}}{\mathrm{d}t})^k|_{t=0}\Phi(t) = \mathbb{E}(X^k) = m_k$.

Now,

$$\int_{-\infty}^{\infty} e^{tx}\rho(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx}e^{-\frac{x^2}{2}}\,\mathrm{d}x \tag{5}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2}\,\mathrm{d}(x-t)\cdot e^{\frac{t^2}{2}} \tag{6}$$
$$= e^{\frac{t^2}{2}}. \tag{7}$$

Therefore, $\Phi(t) = e^{\frac{t^2}{2}} = 1 + \frac{t^2}{2} + \frac{(\frac{t^2}{2})^2}{2!} + \ldots$, and hence $m_2 = 1$. Similarly, by looking at the Taylor series for $\Phi(t)$, we obtain that $m_4 = 3$. In general, $m_{2k} = (\frac{\mathrm{d}}{\mathrm{d}t})^{2k}[\frac{(\frac{t^2}{2})^k}{k!}]$.

It is worthwhile to note that $(\frac{\mathrm{d}}{\mathrm{d}t})^{2k}|_{t=0}(e^{\frac{t^2}{2}})$ is the number of unique ways to pair up $2k$ numbers into k pairs, which can be deduced to be $\frac{(2k)!}{k!2^k}$, and thus $m_{2k} = \frac{(2k)!}{k!2^k}$.

Theorem 1.3

Let $X_1, \ldots, X_{2k}$ be random variables forming a Gaussian vector with the mean 0. Then

$$\mathbb{E}(X_1 \ldots X_{2k}) = \sum_{\text{the number of ways to pair up } 2k \text{ elements into } k \text{ pairs } (j_i, j_i')} \prod_{i=1}^{u} \mathbb{E}(X_{j_i}X_{j_i'}).$$

Theorem 1.4 (Wigner Semicircle Law)

Eigenvalues of random matrices fall into a compact semicircle region, provided a matrix is big enough.

Now, let $f : S_1^N \to \mathbb{R}$, such that $Lip(f) \leq C$, if $\left|f(x) - f(y)\right| < C|x-y|$ for all x and y in S_1^N.

Theorem 1.5 (Concentration of Measure)

If f is dependent on many independent variables in a Lipschitz way, then f is essentially constant.