MAT 157

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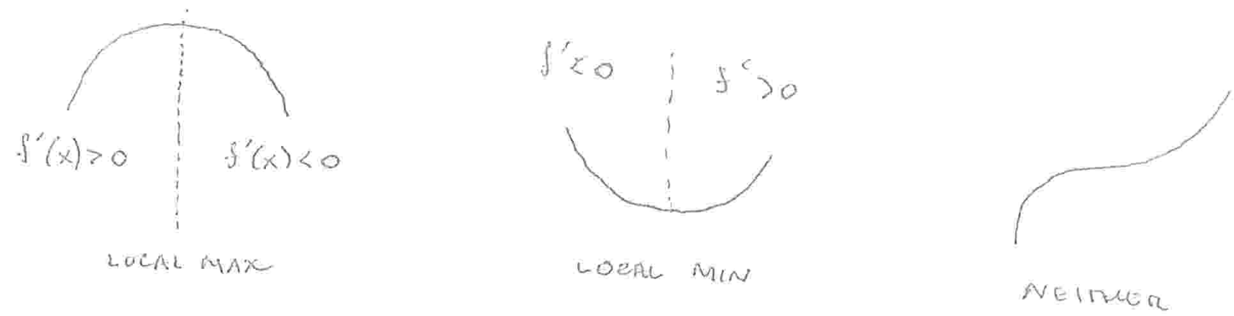
APPLICATIONS OF MVT



POLYNOMIALS

$$f(x) = \sum_{i}^{n} a_i x^i + a_0$$

$$f'(x) = \sum_{i=1}^{n-1} (i+1) a_{(i+1)} x^i + a_1$$

EXERCISE:

A POLYNOMIAL OF DEGREE n HAS AT MOST n roots.

PROOF:

A POLYNOMIAL OF DEGREE $\underline{1}$ IS $ax - b$, WHICH HAS EXACTLY ONE ROOT.

SUPPOSE THE CLAIM IS TRUE FOR POLYNOMIALS UP TO DEGREE $n-1$.

Suppose $\deg(f)$ AND SUPPOSE THAT f HAS MORE THAN n ROOTS.

By ROLLE'S THEOREM, THERE IS A ROOT OF $f'(x)$ BETWEEN ANY TWO ADJACENT ROOTS.

IF f HAS MORE THAN n ROOTS, THEN f' HAS MORE THAN $n-1$ ROOTS.

But $\deg f' = n-1$, WHICH IS A CONTRADICTION.

GRAPHING

1. CRITICAL POINTS x
2. EVALUATE f AT CRITICAL POINT.
3. FIND THE SIGN OF f' BETWEEN CRITICAL POINTS.
4. FIND ROOTS OF f.
5. FIND $\lim_{x \to \pm\infty} f(x)$.

THEOREM

IF $f'(a)=0$ AND $f''(a)>0$, THEN $f(x)$ HAS A LOCAL MIN AT $x=a$.

PROOF. SUPPOSE $f'(x)=0$, $f''(x) > 0$

$$f''(x) = \lim_{x \to a} \frac{f'(x)-f'(a)}{x-a}$$

$\exists \delta > 0$ IF $\frac{f'(x)}{x-a} > 0$ for $|x-a| < \delta$. Then $f'(x) > 0$ for $x \in (a, a+\delta)$, $f'(x) < 0$ for $(a-\delta, a)$.

THEOREM IF $f(x)$ IS CONTINUOUS ON $[a,b]$ AND $f'(x)$ IS DEFINED ON (a,b), EXCEPT POSSIBLY AT $c \in (a,b)$, BUT $\lim_{x \to c} f'(x)$ EXISTS, then f is DIFFERENTIABLE AT $x = c$ AND

$$f'(c) = \lim_{x \to a} f'(x).$$

PROOF $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ ← Consider

BY MVT, FOR ANY x near c, $\exists$ a_x s.t. $\frac{f(x)-f(c)}{x-c} = f'(a_x)$

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c} f'(a_x) = \lim_{x \to c} f'(x)$$

$\lim_{x \to c} f(x)$ EXISTS, CALL IT L.

GIVEN $\varepsilon > 0$; $\exists \delta > 0$ SUCH THAT

$|f'(x) - L|$ whenever $|x-c| < \delta$.

IF $|x-c| < \delta$, THEN

$|a_x - c| < |x-c| < \delta$

$\Rightarrow |f'(a_x) - L| < \varepsilon$.

THM CAUCHY'S MEAN VALUE THEOREM

$f(x), g(x)$ ARE CONTINUOUS ON $[a, b]$, DIFFERENTIABLE ON (a, b).

THEN $\exists\ x \in (a, b)$ s.t.

$$f'(x)\left[g(b) - g(a)\right] = g'(x)\left[f(b) - f(a)\right]$$

PROOF DEFINE $h(x) = f(x)\left(g(b) - g(a)\right) - g(x)\left(f(b) - f(a)\right)$.

$h(a) = f(a)g(b) - g(a)f(b)$.

$h(b) = g(b)f(a) - f(b)g(a) = h(a)$.

BY ROLLE'S THEOREM, $\exists\ x :\ h'(x) = 0$, i.e.

$f'(x)\left(g(b) - g(a)\right) - g'(x)\left(f(b) - f(a)\right) = 0$,

AS REQUIRED

L'Hôpital's Rule

IF $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$

AND IF f, g ARE DIFFERENTIABLE ON SOME INTERVAL AROUND a, EXCEPT POSSIBLY AT $x = a$, AND IF $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ EXISTS.

THEN $\lim_{x \to a} \frac{f(x)}{g(x)}$ EXISTS AND

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$