1 More on Logarithms

Theorem 1.1

If $x, y > 0$, then $\log(xy) = \log(x) + \log(y)$

Proof. For some $y > 0$, let $f(x) = \log(yx)$. Therefore, $f'(x) = \frac{1}{x} = \log'(x)$.

Thus, there exists a number c such that $f(x) = \log(x) + c$. When $x = 1$, $f(1) = \log y$ by definition of f and $f(1) = c$ by the obtained relation.

Hence, $\log xy = \log x + \log y$. ☐

Corollary 1.2

For all $n \in \mathbb{N}$, $\log(x^n) = n \log(x)$.

Corollary 1.3

$\log(\frac{x}{y}) = \log x - \log y$.

Since for any $n \in \mathbb{N}$, $\log 2^n = n \log 2$. Moreover, $\log(\frac{1}{2^n}) = -n \log 2$, and thus log is neither bounded above nor below. Since log is continuous, it follows that it takes all the values in $\mathbb{R}$.

Definition 1.4. $\forall x \in \mathbb{R}. \exp x = \log^{-1} x$

Theorem 1.5

$\forall x \in \mathbb{R}. \exp' x = \exp x$

Proof. Observe that

$$\begin{aligned} \exp'(x) = (\log^{-1} x)' &= \frac{1}{\log(\log^{-1} x)} && (1) \\ &= \frac{1}{\frac{1}{\log^{-1} x}} && (2) \\ &= \log^{-1} x = \exp x && (3) \end{aligned}$$

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Theorem 1.6

For any $x, y \in \mathbb{R}$, $\exp(x + y) = \exp(x)\exp(y)$

Proof. Let $x' = \exp x$ and $y' = \exp y$.

Then $x + y = \log x' + \log y' = \log x'y'$, and thus $\exp(x + y) = \exp(x)\exp(y)$. ☐

Definition 1.7. $e = \exp(1)$

From (1.6) we obtain that for any $x \in \mathbb{Q}$, $\exp(1)^x = \exp(x)$. It is consistent with our earlier use of the exponential notation to define e^x as $\exp x$ for all $x \in \mathbb{R}$.

Definition 1.8. $e^x = \exp x$.

Definition 1.9. If $a > 0$, for any real number $a^x = e^{x \log a}$.