MAT157 DIFFERENTIAL 20161021

MATHPLUS. MATH. UTORONTO.CA

THEOREM [T1] IF $F(x)$ IS CONTINUOUS ON $[a,b]$, THEN IT IS UNIFORMLY CONTINUOUS.

LEMMA [L1]

ASSUME $f(x)$ IS CONTINUOUS ON $[a,c]$.

SUPPOSE $a<b<c$ AND FOR SOME $\varepsilon>0$

$\exists \delta>0$: $x,y \in [a,b]$ IF $|x-y|<\delta$, THEN $|f(x)-f(y)|<\varepsilon$ AND FOR $x,y \in [b,c]$, IF

$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$.

THEN $\exists \delta_1>0$: IF $x,y \in [a,c]$ AND $|x-y|<\delta_1$, THEN

$|f(x)-f(y)|<\varepsilon$.

PROOF [L1]

f IS CTS. AT $b \Rightarrow \exists \delta_2>0$:

$|x-b|<\delta_2 \Rightarrow |f(x)-f(b)|<\frac{\varepsilon}{2}$.

LET $\delta_1 = \min\{\delta, \delta_2\}$.

PROOF [L1] (CONT.)

SUPPOSE $|x-y| < \delta_1$

IF $x, y \in [a,b]$ OR

IF $x, y \in [b,c]$,

THEN $|f(x) - f(y)| < \varepsilon$,

BY HYPOTHESIS.

IF $x \in [a,b]$, $y \in [b,c]$,

AND $|x-y| < \delta_1 \Rightarrow |x-y| < \delta_2$.

~~Also~~ $|(y-x)| \leq |y-b| + |b-x|$. BY DEF OF x, y, $y-x = y-b+b-x$.

So $|f(y) - f(x)| = |f(y) - f(b) + f(b) - f(x)|$

$\leq |f(y) - f(b)| + |f(b) - f(x)|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

PROOF [T1]

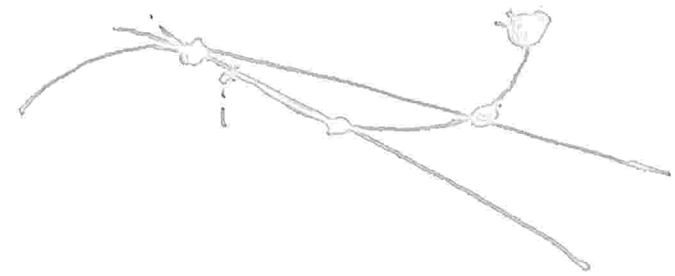
LET $A = \{x \in [a,c] : f(x)$ IS "ε-GOOD" ON $[a,x]\}$.

"ε-GOOD" ON $[a,x]$ MEANS THAT

$\exists\, \delta > 0 : |f(y) - f(z)| < \varepsilon$

WHENEVER $|y-z| < \delta$ AND $y, z \in [a,x]$

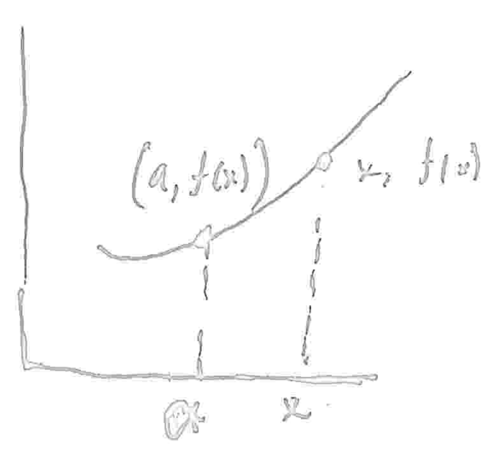
DIFFERENTIATION
&
INTEGRATION



MOTIVATION: GIVEN TWO POINTS ON A CURVE, THE SECANT IS THE LINE THROUGH THEM. IF ONE POINT APPROACHES THE OTHER, SEEMS REASONABLE TO EXPECT THAT THE SECANT WILL APPROACH THE TANGENT.

BETTER WAY: THE SLOPE OF THE SECANT SHOULD APPROACH THE SLOPE OF THE TANGENT.

CONSIDER A FUNCTION $y = f(x)$, AND SUPPOSE a IS IN ITS DOMAIN.



PROOF [T1]
(CONT.)

$a \in A \Rightarrow A \neq \emptyset$.

THUS, A HAS A L.U.B.

LET $\alpha = \sup(A)$.

f IS CONTINUOUS AT $\alpha \Rightarrow$

$\exists\, \delta_1 > 0$:

$$|f(y) - f(\alpha)| < \frac{\varepsilon}{2} \text{ IF}$$

$$|y - \alpha| < \delta.$$

GIVEN $y, z \in (\alpha, \alpha + \delta_1)$,

$$|f(y) - f(z)| \leq |f(y) - f(\alpha)| + |f(\alpha) - f(z)|$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

SO f IS ε-GOOD IN $\left[\alpha, \alpha + \frac{\delta_1}{2}\right]$. FROM LEMMA I, f IS ε-GOOD ON $\left[a, \alpha + \frac{\delta_1}{2}\right]$, WHICH CONTRADICTS THE DEFINITION OF α.

SO f IS ε-GOOD ON $[a, b]$ FOR ANY $\varepsilon > 0$,

i.e. f IS UNIFORMLY CONTINUOUS.

DEFINITION IF $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ EXISTS,

IT IS THE SLOPE OF THE TANGENT LINE TO $y = f(x)$ AT $(a, f(a))$.

DENOTE THE LIMIT AS

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

AND CALL IT THE DERIVATIVE OF $f(x)$ AT $x = a$