Problem I

Consider an operation defined for $x = (x_1, x_2)$ and $y = (y_1, y_2)$, both in $\mathbb{C}^2$, as

$$\langle x, y\rangle = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 1+i \\ 1-i & 1 \end{pmatrix} \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \end{pmatrix} \tag{1}$$

Note that $\langle x, y\rangle$ is not an inner product, since if $x = \begin{pmatrix} -1 & i \end{pmatrix}$, then

$$\begin{aligned}
\langle x, x\rangle &= \begin{pmatrix} -1 & i \end{pmatrix} \begin{pmatrix} 1 & 1+i \\ 1-i & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -i \end{pmatrix} &(2)\\
&= \begin{pmatrix} i & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -i \end{pmatrix} &(3)\\
&= -i + i = 0, &(4)
\end{aligned}$$

and thus $\langle x, x\rangle = 0$ even though $x = \begin{pmatrix} -1 & i \end{pmatrix} \neq \mathbf{0}$.

Consider now an operation defined for $x = (x_1, x_2)$ and $y = (y_1, y_2)$, both in $\mathbb{C}^2$, as

$$[x, y] = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 4 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \end{pmatrix}. \tag{5}$$

By definition of the matrix summation and the distributive law for matrices, for any $x, y, z \in \mathbb{C}^2$, $[x + y, z] = [x, z] + [y, z]$, and thus $[\cdot, \cdot]$ is additive.

Similarly, by definition of the matrix multiplication by a scalar, $[\lambda x, y] = \lambda[x, y]$, and thus $[\cdot, \cdot]$ is homogeneous.

Observe the following:

$$\begin{aligned}
[x, y] &= \begin{pmatrix} 4x_1 - ix_2 & x_2 + ix_1 \end{pmatrix} \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \end{pmatrix} &(6)\\
&= (4x_1 - ix_2)\overline{y_1} + (x_2 + ix_1)\overline{y_2} &(7)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{[y, x]} &= \overline{(4y_1 - iy_2)\overline{x_1} + (y_2 + iy_1)\overline{x_2}} &(8)\\
&= (4\overline{y_1} + i\overline{y_2})x_1 + (\overline{y_2} - i\overline{y_1})x_2 &(9)\\
&= (4x_1 - ix_2)\overline{y_1} + (x_2 + ix_1)\overline{y_2} &(10)\\
&= [x, y] &(11)
\end{aligned}$$

Consider $[x, x]$.

$$\begin{aligned}
[x, x] &= (4x_1 - ix_2)\overline{x_1} + (x_2 + ix_1)\overline{x_2} &(12)\\
&= 4|x_1|^2 + |x_2|^2 + i(x_1\overline{x_2} - \overline{x_1}x_2) &(13)\\
&= 4|x_1|^2 + |x_2|^2 + i(x_1\overline{x_2} - \overline{x_1\overline{x_2}}) &(14)\\
&= 4|x_1|^2 + |x_2|^2 - 2\Im(x_1\overline{x_2}). &(15)
\end{aligned}$$

Since $\Im(x_1\overline{x_2}) = \Re(x_1)\Im(\overline{x_2}) + \Im(x_1)\Re(\overline{x_2})$, while

$$|x_1|^2 = \Re(x_1)^2 + \Im(x_1)^2$$

and

$$|x_2|^2 = \Re(x_2)^2 + \Im(x_2)^2,$$

it follows from (15) that

$$\begin{aligned}
[x, x] &= 4\Re(x_1)^2 + 4\Im(x_1)^2 + \Re(x_2)^2 + \Im(x_2)^2 - 2(\Re(x_1)\Im(\overline{x_2}) + \Im(x_1)\Re(\overline{x_2})) && (16)\\
&= 3(\Re(x_1)^2 + \Im(x_1)^2) + (\Re(x_1) + \Im(x_2))^2 + (\Im(x_1) - \Re(x_2))^2 \geq 0 && (17)
\end{aligned}$$

Note that, from (17), $[x, x] = 0$ if and only if $x_1 = \mathbf{0}$ and $x_2 = \mathbf{0}$, because $\Re(x_1) = 0$ and $\Im(x_1) = 0$ from the first expression in parentheses, and the rest follows from the second and third expressions in parentheses.

Therefore, $[\cdot, \cdot]$ is indeed an inner product.