Let V be a vector space over F of infinite dimension.
Let $T : V \to (V^*)^*$ be the linear map given by $T(v)(\phi) = \phi(v)$ for all $v \in V$ and $\phi \in V^*$.

Claim. T is injective.

Proof. Take some $\phi \in V^*$.
Note that if $\phi = \mathbf{0}$, then, for all $v \in V$, $T(v)(\mathbf{0}) = 0$.
Suppose now $\phi \neq \mathbf{0}$.
Suppose, for some $v, w \in V$, $T(v) = T(w)$, and thus $\phi(v) = \phi(w)$ for all $\phi \in V^*$.
Hence, $\phi(v - w) = 0$, since ϕ is a linear functional.
By way of contradiction, suppose that $v - w$ is non-zero. Therefore, if $\beta = \{v_1, v_2, \ldots\}$ is a basis of V, there exist non-zero $a_i \in F$, for a non-empty finite index set $\Lambda \subset \mathbb{N}$, such that

$$v - w = \sum_{i \in \Lambda} a_i v_i.$$

Therefore, since ϕ is linear,

$$\phi\left(\sum_{i \in \Lambda} a_i v_i\right) = \sum_{i \in \Lambda} a_i \phi(v_i) = 0. \tag{1}$$

Therefore, the image of all such v_i under ϕ is linearly dependent, since a_i are non-zero.
Consider a basis $\beta^* = \{\phi_1, \phi_2, \ldots\} \subset V^*$ of V^*.
If for $b_i \in F$ with i in some index set $\Lambda^* \subset \mathbb{N}$

$$\sum_{i \in \Lambda^*} b_i \phi_i = 0,$$

then, by linear independence of ϕ_i, all b_i must be equal to zero.
Note, however, that by assumption Equation 1 holds for any $\phi \in V^*$, and thus:

$$\sum_{j \in \Lambda^*} \sum_{i \in \Lambda} a_i \phi_j(v_i) = \sum_{i \in \Lambda} a_i \sum_{j \in \Lambda^*} \phi_j(v_i) = 0, \tag{2}$$

which is a contradiction, since a_i are non-zero by assumption.
Therefore, $v - w = 0$, and T is injective. □

Let $\beta \subset V$ be a basis of V. For each $v \in \beta$, let $v^* \in \beta^*$ be the linear functional which value on basis vectors $w \in \beta$ is given by $v^*(w) = 1$ if $w = v$, $v^*(w) = 0$ if $w \neq v$.

Claim. β^* is linearly independent, but is *not* a basis of V^*.

Proof. Let Γ be an infinite index set of β, so that if $v_i \in \beta$, then $i \in \Gamma$.
Thus, by definition, if $v_i^* \in \beta^*$ and $i \in \Gamma$, then $v_i^*(v_i) = 1$, and if $j \in \Gamma$ and $i \neq j$, then $v_i^*(v_j) = 0$.
Suppose for $a_i \in F$ with i in some index set $\Lambda^* \subset \Gamma$

$$\sum_{i \in \Lambda^*} a_i v_i^* = \mathbf{0}.$$

Therefore, for any $v_j \in V$ with $j \in \Lambda^*$, applying the above linear functionals,

$$\sum_{i \in \Lambda} a_i v_i^*(v_j) = \mathbf{0}(v_j) = 0,$$

and hence $a_j = 0$. Since j is arbitrary, all a_i with $i \in \Lambda^*$ must be equal to zero.

Therefore, any finite linear combination of v_i^* for $i \in \Gamma$ must be linearly independent.

Since any vector in β^* can be represented by a finite linear combination of suitable v_i^*, then β^* is linearly independent.

Consider now a map ψ such that, for any $v_i \in \beta$, $\psi(v_i)$ corresponds to the same arbitrary non-zero $z_0 \in F$. Suppose also that ψ is linear, which exists and is uniquely determined by its action on the basis. Therefore, ψ is a linear functional, $\psi = \mathrm{Hom}(V, F)$.

Note that any linear functional ψ can be represented by a *finite* linear combination of $v_i^* \in \beta^*$ so that

$$\psi = \sum_{i \in \Psi} a_i v_i^*,$$

for some finite $\Psi \subset \Gamma$ and $a_i \in F$.

Note also that, if $j \in \Gamma$ but $j \notin \Psi$, then $v_i^*(v_j) = 0$ and thus $\sum_{i \in \Psi} a_i v_i^*(v_j) = 0$ by linearity of ψ. However, by definition of ψ, $\psi(v_j) = z_0 \neq 0$. Therefore, ψ cannot be represented as a linear combination of $v_i^* \in \beta^*$, and thus β^* is not the basis of V^*. $\square$