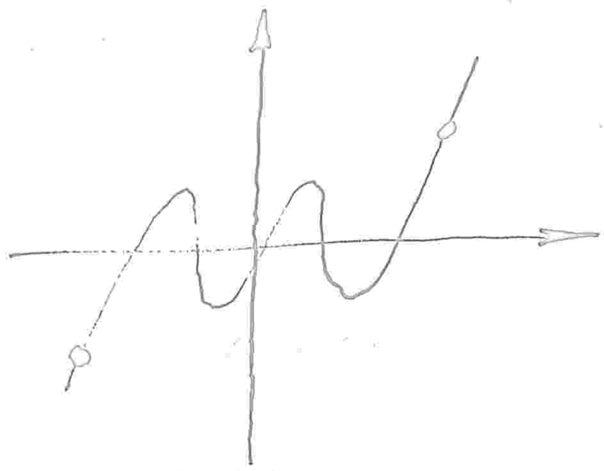
MAT157:

Applications of IVT.

20161017

THEOREM

If $p(x)$ is a polynomial of odd degree, then it must have a root.



If $|x|>1$, then $|x^m| \geq |x|$, $\forall\, m \in [1:n]$,

so $\dfrac{1}{|x^m|} \leq \dfrac{1}{|x|}$

Choose x such that $|x|>1,\ 2n\max(|a_i|)$.

Then

$$\left|\left(\sum_{i=0}^{n-1}\frac{a_i}{x^{n-i}}\right)\right| \leq \sum_{i=0}^{n-1}\left|\frac{a_i}{x^{n-i}}\right|$$

$$\leq \sum_{i=0}^{n-1}\left|\frac{a_i}{x}\right|$$

$$< \sum_{i=0}^{n-1}\left|\frac{a_i}{2n a_i}\right| = \frac{1}{2}$$

$$\boxed{\begin{aligned} p(x) &= \overset{n-1}{\underset{i=1}{+}} a_i x^i + x^n + a_0 \\ &= x^n\left(1 + \overset{n}{\underset{i=1}{+}} \frac{a_{n-i}}{x^i}\right)\end{aligned}}$$

$x>0$, $\left(1+\dots+\dfrac{a_n}{x^n}\right) > 0:$

$\Rightarrow \quad p(x) = x^n\left(1+\dots+\dfrac{a_0}{x^n}\right) > 0.$

If $|x|>1$, $x<0$, then $p(x) = \underbrace{x^n}_{<0}\underbrace{\left(1+\dots+\dfrac{a_0}{x^n}\right)}_{>0} < 0.$

$\Rightarrow$ x_1, x_2 can be found so that $f(x_1)<0$, $f(x_2)>0$,

then by IVT $\exists\, x \mid f(x) = 0$

THEOREM:

IF $f(x)$ IS CONTINUOUS ON $[a,b]$,
THEN $\exists M$ SUCH THAT $f(x) \leq M \;\forall x \in [a,b]$.

"M IS AN UPPER BOUND FOR f,
or f is BOUNDED ABOVE BY M.

LEMMA: Suppose $f(x)$ is cts. at $x=r$.

$\Rightarrow \exists \delta > 0 : f(x)$ is BOUNDED on $(r-\delta, r+\delta)$, i.e., $\exists N : f(x) < N \;\forall x \in (r-\delta, r+\delta)$.

PROOF: LET $\varepsilon = 1$, THEN $\exists \delta > 0 : |f(x) - f(r)| < 1$ WHENEVER $|x-r| < \delta$.

So $|f(x)| < |f(r)| + 1 \;\forall x$ S.T. $|x-r| < \delta$.

Let $N = |f(r)| + 1$.

D.

PROOF: LET $A = \{ x \mid f$ IS BOUNDED ABOVE ON $[a, x] \}$

Let $\alpha = \sup(A)$.

By LEMMA, $\exists \delta$ s.t. $f(x)$ IS BOUNDED ABOVE ON $(\alpha-\delta, \alpha+\delta)$, i.e.

$\exists N: f(x) < N$ on $(\alpha-\delta, \alpha+\delta)$.

Pick u, v, $\alpha - \delta < u < \alpha < v < \alpha+\delta$

COROLLARY

If $f(x)$ is cts on $[a,b]$ then $f(x)$ is bounded below on $[a,b]$, i.e.

$$\exists N: f(x) > N, \forall x \in [a,b].$$

COROLLARY

If $f(x)$ is continuous on $[a,b]$, then $f(x)$ is bounded, i.e.,

$$\exists M : |f(x)| < M, \forall x \in [a,b].$$

THEOREM

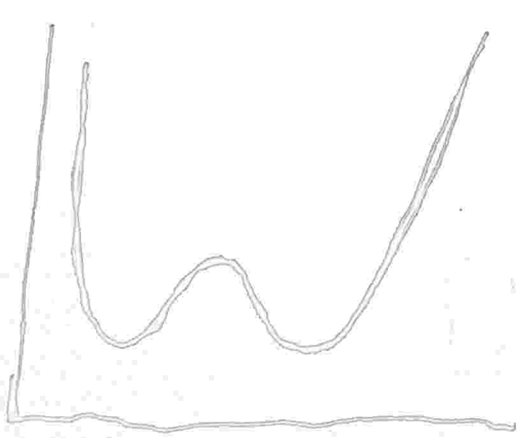
If $f(x)$ is cts. on $[a,b]$, then $\exists\ c \in [a,b]$ s.t

$$f(c) \geq f(x) \quad \forall x \in [a,b],$$

$$f(c) = \max \cdot \{ f(x) \mid x \in [a,b] \},$$

i.e. $f(x)$ achieves its max on $[a,b]$.

EVEN DEGREE POLYNOMIALS



$$p(x) = \sum_{i=0}^{n-1} x^i a_i + x^n$$ with

n even. 4ℓ degree, write

$$p(x) = x^n \left(1 + \ldots + \frac{a_0}{x^n} \right)$$

CAN ASSUME $\alpha \in A$, SINCE $\alpha = \sup A$.

$f(x) < Q$ ON $[a, \alpha]$ AND $f(x) < N$

ON $(\alpha - \delta, \alpha + \delta)$, SO

$f(x) < \max(Q, N)$ ON $[a, r]$ $\therefore r \in A$,

BUT $r > \alpha = \sup(A)$

$\#$

WHAT IF $A = \{a\}$, SO $\alpha = a$

THE ONE-SIDED VERSION OF THE LEMMA SAYS f IS BOUNDED ON $[a, a+\delta]$ FOR SOME $\delta > 0$. $\Rightarrow$ $\forall\, r \in [a, a+\delta) \in A$

So $\sup(A) = b$.

WTS: f IS BOUNDED ON $[a, b]$.

BUT THE ONE-SIDED VERSION OF THE LEMMA SAYS $\exists\ \delta > 0$ S.T. $f(x) < N$ ON $(b - \delta, b]$ AND

$\exists\ Q$ S.T. $f(x) < Q$ ON $[a, b - \frac{\delta}{2}]$

SO $f(x) < M = \max(N, Q)$ ON $[a, b]$

$\square$

CAN CHOOSE M SUCH THAT IF $|x| > M$,

THEN $\left(1 + \frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n}\right) \geq \frac{1}{2}$

OUTSIDE OF $[-M, M]$, $f(x) > 0$.