15

MAT 157 PSET 8

PMM:

THE MAXIMUM AND MINIMUM OF f ON A CLOSED INTERVAL $[a,b]$ IS IN THE SET OF

(1) THE CRITICAL POINTS OF f IN $[a,b]$

(2) THE END POINTS a AND b

(3) POINTS x IN $[a,b]$ SUCH THAT f IS NOT DIFFERENTIABLE AT x.

DI

IF $f'(x) > 0$ FOR ALL x IN AN INTERVAL, THEN f IS INCREASING ON THE INTERVAL; IF $f'(x) < 0$ FOR ALL x IN THE INTERVAL, THEN f IS DECREASING ON THE INTERVAL.

LZ0. $\forall\ c \in \mathbb{R},\ \gamma \in \mathbb{Q}$,

$$\lim_{x\to\infty} \frac{c}{x^\gamma} = 0 \quad \text{AND} \quad \lim_{x\to-\infty} \frac{c}{x^\gamma} = 0.$$

PROOF. SUPPOSE $\varepsilon > 0$ IS GIVEN.

Take $N = \sqrt[\gamma]{\frac{|c|}{\varepsilon}}$. Suppose $|x| > N > 0$

Thus, $|x| > \sqrt[\gamma]{\frac{|c|}{\varepsilon}} > 0 \Rightarrow |x|^\gamma > \frac{|c|}{\varepsilon} \Rightarrow \varepsilon >$

LZ. Let $p(x) = a_n x^n + \dots + a_1 x + a_0$ be a polynomial with $a_i \in \mathbb{R}^n$ and $n \in \mathbb{N}$.
$\forall c \in \mathbb{R}$, $\gamma \in \mathbb{Q}$,

$$\lim_{x \to \infty} \frac{c}{(p(x))^\gamma} = 0 \quad \text{AND} \quad \lim_{x \to -\infty} \frac{c}{(p(x))^\gamma} = 0$$

PROOF

NOTE THAT FOR $x \neq 0$:

$$p(x) = x^n \left(a_n + \dots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)$$

BY LZ_0, $\lim_{x \to \pm\infty} a_n + \dots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} = a_n$

THEREFORE, $\lim_{x \to \pm\infty} \frac{c}{p(x)^\gamma} =$

$$= \lim_{x \to \pm\infty} \frac{c}{x^{n\gamma} \left(a_n + \dots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)} = \frac{\lim_{x \to \pm\infty} c}{\lim_{x \to \pm\infty} a_n}$$

①

(i) $f(x) = x^3 - 3x^2 + 6x - 1$ ON $[0,1]$

$$f'(x) = 3x^2 - 6x + 6$$
$$= 3(x^2 - 2x + 1 + 1) = 0$$
$$= 3((x-1)^2 + 1) > 0 \ \forall x \in \mathbb{R}$$

$\Rightarrow$ NO CRITICAL POINTS ON $[0,1]$

SINCE $f(x)$ IS A POLYNOMIAL, IT IS DIFFERENTIABLE ON $[0,1]$.

$\Rightarrow f(0) = -1$ IS A MINIMUM
$f(1) = 3$ IS A MAXIMUM ON $[0,1]$ BY PMM AND $f(c$

(ii)

$$g(x) = \frac{1}{1+x^2} \quad \text{on } (-\infty, +\infty)$$

$$g'(x) = \frac{-2x}{(1+x^2)^2} \Rightarrow g'(x) = 0 \text{ IFF } x = 0.$$

NOTE THAT $\forall x \in \mathbb{R} : x^2 \geq 0$

$$\Leftrightarrow 1 + x^2 \geq 1 \Leftrightarrow \frac{1}{1+x^2} \leq 1 \Leftrightarrow g(x) \leq g(0)$$

$\Rightarrow x = 0$ IS A MAXIMUM POINT

NOTE THAT $\lim_{x \to \pm\infty} g(x) = 0 \Rightarrow \forall \varepsilon > 0 \ \exists N > 0 : |x| >$

SINCE $g(x) \neq 0 \ \forall x \in \mathbb{R}$, THERE IS NO MINIMUM POI

(iii)

$h(x) = x - \sin x$ on $[0, \pi/2]$.

$h'(x) = 1 - \cos x$ on $[0, \pi/2]$ $\Rightarrow$ $h'(x) = 0 \Leftrightarrow x = 0$, WHICH IS A CRITICAL [illegible]

NOTE THAT $x=0$ IS THE ONLY CRI[illegible] POINT ON $[0, \pi/2]$.

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$h(\pi/2) = \pi/2 - 1$. SINCE $\pi > 3$, $h(\pi/2) > 0 = h(0)$

SINCE x IS A POLYNOMIAL AND $\sin(x)$ IS CONTINUOUS, THEN $h(x)$ IS CONTINUOUS ON $[0, \pi/2]$.

BY PMM, $\pi/2$ IS A MAXIMUM AND
0 IS A MINIMUM.

(iv)

$k(x) = \dfrac{x}{8+x^2}$ ON $[0, +\infty)$.

$k'(x) = \dfrac{8+x^2 - 2x^2}{(8+x^2)^2} = \dfrac{8-x^2}{(8+x^2)^2}$ $\Rightarrow$ $k'(x) > 0$ [illegible] $k'(x) < 0$ IF [illegible]

$\Rightarrow \left(k'(0) \Leftarrow \begin{bmatrix} x = 2\sqrt{2} \\ x = -2\sqrt{2} \end{bmatrix} \right)$. SINCE $x \in [0, \ldots$ $k'(0)$ IFF [illegible]

Note that $\lim\limits_{x\to+\infty} \dfrac{x}{8+x^2} = \lim\limits_{x\to+\infty} \dfrac{1/x}{\frac{8}{x^2}+1} =$ [illegible]

Since $\lim\limits_{x\to+\infty} \dfrac{1}{x} = 0$.

NOTE ALSO THAT $k(x) \geq 0$, SINCE $x \geq 0$ AND $8+x^2$ [illegible]

SINCE $k(0) = 0$ AND $k(2\sqrt{2}) = \dfrac{\sqrt{2}}{8}$, WHILE $k'(x) <$ [illegible]
$x > 2\sqrt{2}$, THEN $k(2\sqrt{2}) \geq k(x)$ FOR x BY DI. SIMILA[illegible]
FOR $[0, 2\sqrt{2}]$, k IS INCREASING, SINCE $k'(x) > 0$. THER[illegible]

(v) $p(x) = \dfrac{1}{\sqrt{1+x^4}} = (1+x^4)^{-1/2}$

$$p'(x) = \frac{-4x^3}{2(1+x^4)^{3/2}} = \frac{-2x^3}{(1+x^4)^{3/2}} \Rightarrow p'(0) \Leftrightarrow x=0.$$

BY LZ, $\lim_{x\to\pm\infty} p(x) = 0$

$$\Rightarrow \forall \varepsilon > 0 \ \exists N > 0 : |x| > N \Rightarrow |p(x)| < \varepsilon$$

SINCE $\sqrt{1+x^4} > 0 \quad \forall x \in \mathbb{R}$, THERE IS NO MINIMUM POINT IN THE DOMAIN OF $p(x)$.

SINCE $x^4 \geq 0 \quad \forall x \in \mathbb{R}$,

$$1 + x^4 \geq 1 \Rightarrow \sqrt{1+x^4} \geq 1$$

$$\Leftrightarrow \frac{1}{\sqrt{1+x^4}} \leq 1, \quad \text{SINCE } p(0) = 1,$$

THEN $p(x) \leq p(0) \quad \forall x \in \mathbb{R}$, AND THUS $x=0$ IS THE MAXIMUM POINT.

(iv)

$q(x) = \sqrt{x^3 - 3x + 32}$ ON $[-3, 0]$.

$q'(x) = \dfrac{3x^2 - 3}{2\sqrt{x^3 - 3x + 32}} = \dfrac{3(x-1)(x+1)}{2\sqrt{x^3 - 3x + 32}}$

2

$\Rightarrow \begin{vmatrix} x = 1 \\ x = -1 \end{vmatrix}$ IFF $q'(x) = 0$

$q(1) = \sqrt{30}$, $q(-1) = \sqrt{34}$

$q(-3) = \sqrt{-27 + 9 + 32} = \sqrt{14} < q(1)$

$q(-3) < q(0) = \sqrt{32} < q(1)$

GIVEN $q(x)$ IS WELL-DEFINED ON $[-3, 0]$, THERE ARE NO DISCONTINUITIES.

BY PMM, -3 IS THE MINIMUM POINT AND 1 IS THE MAXIMUM.

(2)

$g(x) = \sin(2\pi x) + 2x^3 - 3x^2 + 6x - 5$ ON $[0,1]$.

(i) $g(0) = \sin(0) - 5 = -5$

2 $g(1) = \sin(2\pi) + 2 - 3 + 6 - 5 = 0$.

$\dfrac{g(1) - g(0)}{1 - 0} = 5$. BY MVT, $\exists\, x \in [0,1]: g(x) =$

1 (ii) $g'(x) = 2\pi\cos(2\pi x) + 6x^2 - 6x + 6$

SUPPOSE $g'(x) = 10 \Rightarrow$

NOTE THAT $6x^2+6x-4$ IS A POLYNOMIAL AND $2\pi\cos(2\pi x)$ IS CONTINUOUS $\Rightarrow f(x)$ IS CONTINUOUS EVERYWHERE. IN PARTICULAR, IT IS CONTINUOUS ON [

NOTE THAT $f(0) = 2\pi - 4 > 0$ AND $f(1) = 2\pi - 4 > 0$.

THEREFORE, BY IVT, SUCH x EXISTS. THUS, THERE EXISTS $x \in [0,1]$ SUCH THAT $g'(x) = 10$, A

REQUIRED.

③

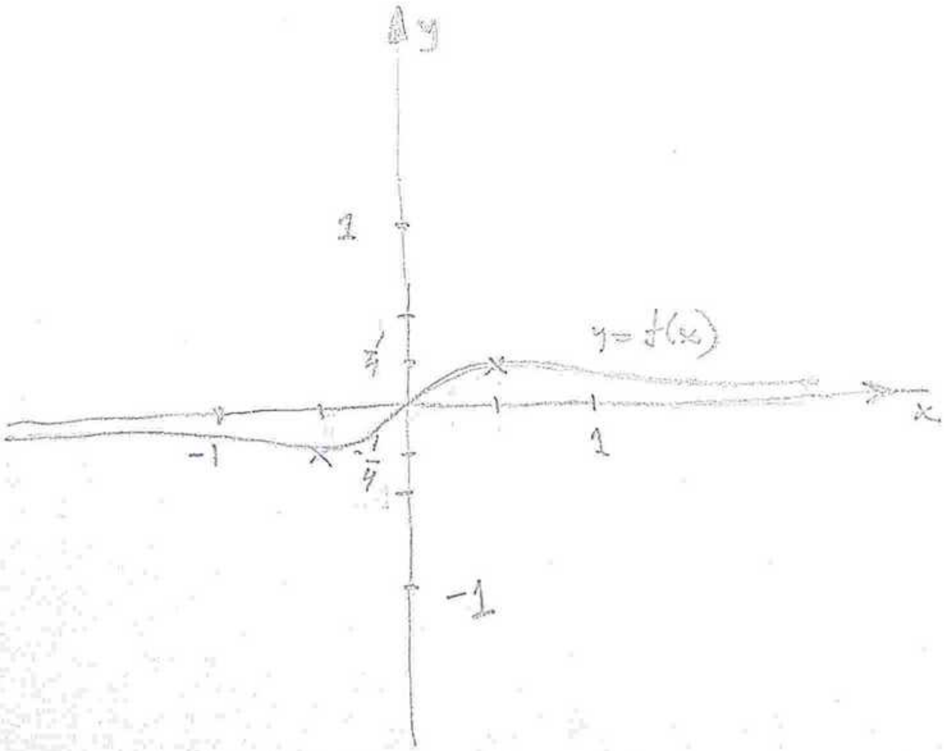
(i) $f(x) = \dfrac{x}{1+4x^2}$. Note that $f(x) = -f(-x)$.

$$f'(x) = \frac{1+4x^2 - 8x^2}{(1+4x^2)^2} = \frac{1-4x^2}{(1+4x^2)^2}.$$

$$\Rightarrow \begin{cases} x = \frac{1}{2} \\ x = -1/2 \end{cases} \iff f'(x) = 0.$$

$$f\left(\tfrac{1}{2}\right) = \frac{\frac{1}{2}}{1 + 4\cdot\frac{1}{4}} = \frac{1}{4} \Rightarrow f\left(-\tfrac{1}{2}\right) = -\frac{1}{4}$$

$f(0) = 0$. $\lim_{x \to \pm\infty} f(x) = 0$ BY LZ.



(ii) $g(x) = \dfrac{x+1}{x^2-2x} = \dfrac{x+1}{x(x-2)}$.

NOTE THAT $g(-1) = 0$.

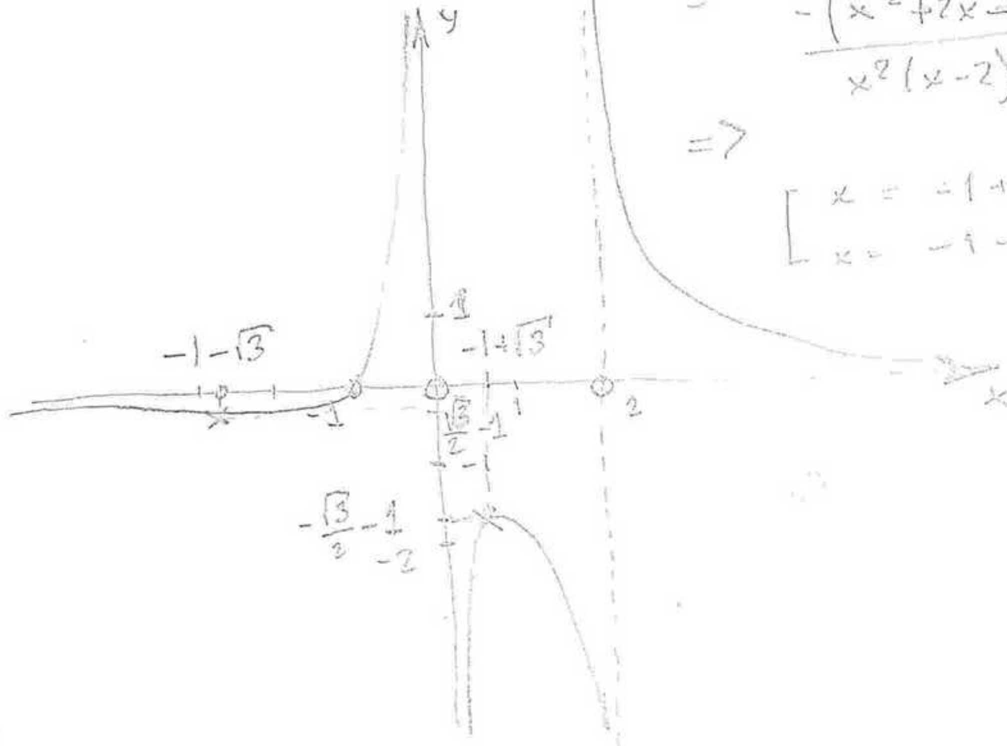
$$g'(x) = \frac{x^2-2x-(2x-2)(x+1)}{x^2(x-2)^2}$$

$$= \frac{x^2-2x-2x^2-2x+2x+2}{x^2(x-2)^2}$$

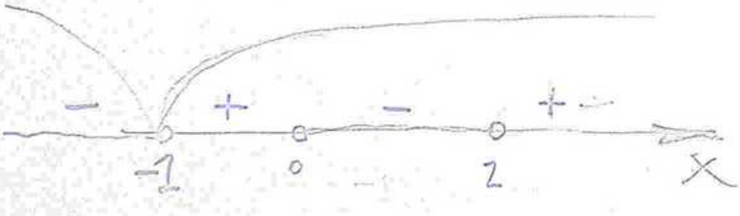
$$= \frac{-(x^2+2x-2)}{x^2(x-2)^2}$$

$\Rightarrow$

$\left[\begin{array}{l} x = -1+\sqrt{3} \\ x = -1-\sqrt{3} \end{array}\right. \Leftrightarrow$



$$\lim_{x\to\pm\infty} \frac{x+1}{x(x-2)} = \frac{\frac{1}{x}+\frac{1}{x^2}}{1-\frac{2}{x}} = 0 \text{ BY L2.}$$



$$g(-1+\sqrt{3}) = \frac{\sqrt{3}}{(\sqrt{3}-1)(\sqrt{3}-3)} = \frac{\sqrt{3}}{6-4\sqrt{3}} = \frac{6\sqrt{3}+12}{-12} =$$

$$g(-1-\sqrt{3}) = \frac{-\sqrt{3}}{+(1+\sqrt{3})(3+\sqrt{3})} = \frac{-\sqrt{3}}{6+4\sqrt{3}} = +\frac{\sqrt{3}(6-}{+12}$$

(iii)

$$h(x) = \frac{(x-1)(x+1)}{x-3} = x+3+\frac{8}{x-3} \Rightarrow \lim_{x\to\pm\infty} h(x) = \lim_{x\to\pm\infty}$$

3

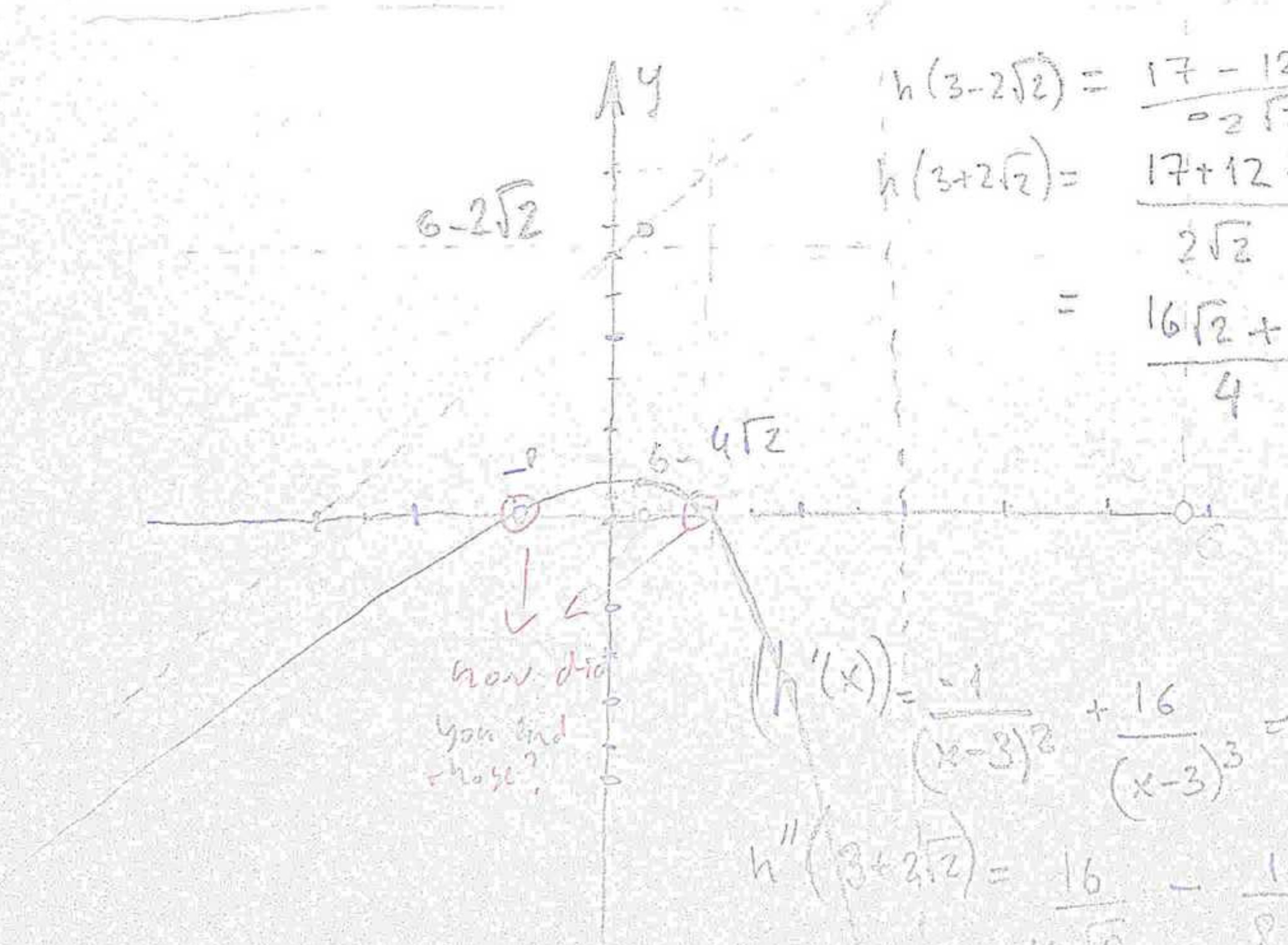
$$h(0) = \frac{1}{3}.$$

$$h'(x) = \frac{2x(x-3)-(x^2-1)}{(x-3)^2} = \frac{2x^2-6x-x^2+1}{(x-3)^2} =$$

$$= \frac{x^2-6x+1}{(x-3)^2} = \frac{1}{x-3} - \frac{8}{(x-3)^2} \Rightarrow \lim_{x\to\pm\infty} h'(x) = 0$$

$$\Delta(x^2-6x+1) = 36-4 = 16\cdot 2$$

$$\Rightarrow \begin{cases} x_1 = 3+2\sqrt{2} \\ x_2 = 3-2\sqrt{2} \end{cases} \Longleftrightarrow h'(x) = 0$$



how did you find those?

$$h(3-2\sqrt{2}) = \frac{17-12}{-2\sqrt{}}$$

$$h(3+2\sqrt{2}) = \frac{17+12}{2\sqrt{2}}$$

$$= \frac{16\sqrt{2}+}{4}$$

$$h''(x) = \frac{-1}{(x-3)^2} + \frac{16}{(x-3)^3} =$$

$$h''(3+2\sqrt{2}) = \frac{16}{} - $$

(w)

$k(x) = 2x + \sin(2\pi x)$. $\quad k(0) = 0$.

$k'(x) = 2 + 2\pi\cos(2\pi x)$. $\quad k(\frac{1}{2}) = 1 + \sin(\pi) =$

$2 + 2\pi\cos(2\pi x) = 0$.

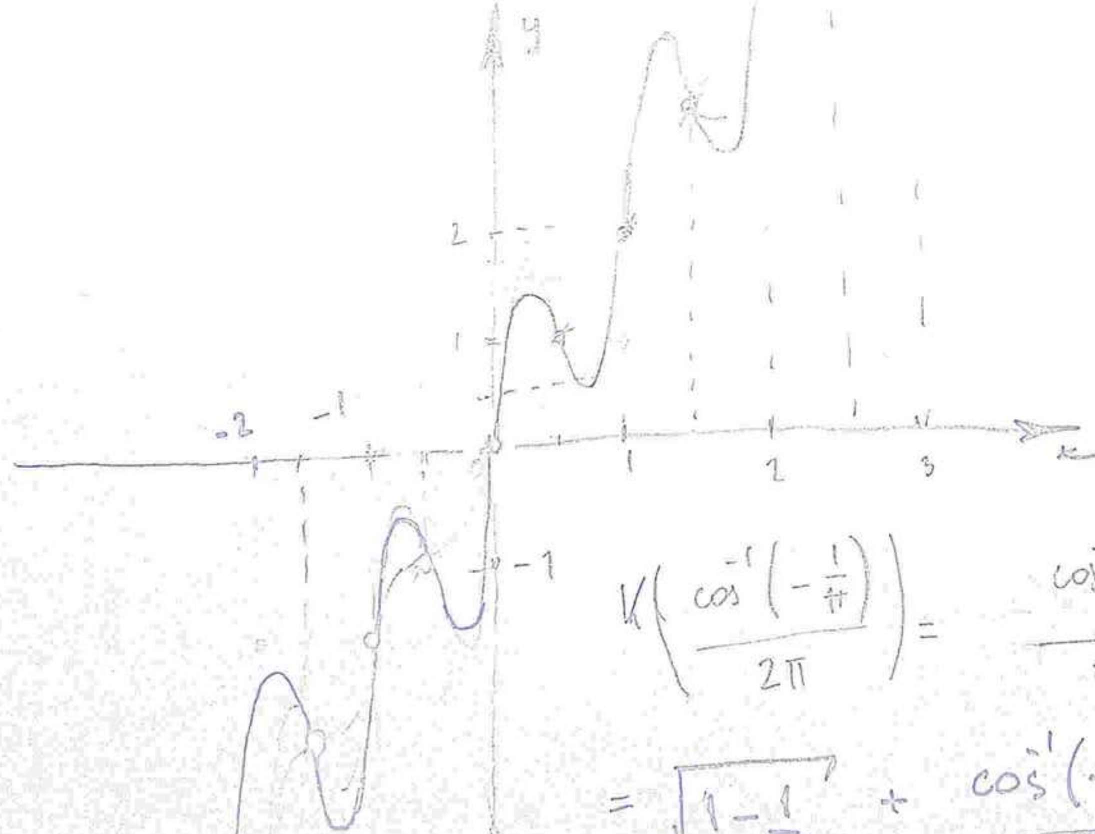
$-1 < \cos(2\pi x) = -\frac{1}{\pi} < 1$.

NOTE THAT $k(x)=$

$\Rightarrow$ IF $x \in (0, \frac{1}{2})$, $\sin(2\pi x)$

$$\Rightarrow \quad x = \frac{\pm\cos^{-1}(-\frac{1}{\pi}) + 2\pi k}{2\pi}, \; k \in \mathbb{Z}$$

$$= \pm\frac{\cos^{-1}(-\frac{1}{\pi})}{2\pi} + k, \; k \in \mathbb{Z} \iff k'(x$$



$$k\left(\frac{\cos^{-1}(-\frac{1}{\pi})}{2\pi}\right) = \frac{\cos^{-1}(-}{\pi}$$

$$= \sqrt{1-\frac{1}{\pi^2}} + \frac{\cos^{-1}(-\frac{1}{\pi})}{\pi}$$

$$> \sqrt{1-\frac{1}{\pi^2}} + \frac{1}{2}$$

since $\cos(\frac{\pi}{2}) = 0$

⑥ SUPPOSE $f(x)$ IS DEFINED AND DIFFERENTIABLE ON (a,b).

SUPPOSE ALSO THERE EXISTS $c \in (a,b)$ SO THAT $f'(c)=0$ AN $f'(x)>0$ FOR ALL $x \in (a,b)$ BUT FOR $x=c$.

CLAIM: $f(x)$ IS INCREASING ON (a,b)

PROOF:

SUPPOSE ON THE CONTRARY THAT $f(x)$ IS NOT INCREASING.

0

THEREFORE, THERE EXISTS $(a', b') \subset (a,b)$ SUCH THAT $a \leq a' < b' \leq b$ ON WHICH $f(x)$ IS CONSTANT OR DECREASING. SINCE $f'(x)>0$ FOR ALL $x \in (a,b)$, FOR $x \in (a',b')$ $f(x)$ IS NOT CONSTANT ($f'(x) \neq 0$).

no $f'(c)=0$

THEREFORE, f MUST BE DECREASING.

PICK ANY m, n SUCH THAT $a' < m < n < b'$, AND CONSIDER f ON $[m,n]$, ON WHICH f IS DECREASING. SINCE $[m,n] \subset (a,b)$ AND f IS DIFFERENTIABLE ON (a,b), IT IS CONTINUOUS ON $[m,n]$ AND THUS $f(m)$, $f(n)$ ARE DEFINED. BY MVT, THERE IS A NUMBER $x_0 \in (m,n)$ SUCH THAT

$$f'(x) = \frac{f(n) - f(m)}{n-m}.$$

SINCE f MUST BE DECREASING ON $[m,n]$, $f(n) < f(m)$. SINCE $n > m$, $f'(x) < 0$, WHICH IS A CONTRADICTION. THEREFORE, f IS INCREASING ON (a,b), AS REQUIRED $\square$