1 Geometry of Discrete Painleve Equations

1.1 Introduction

1.1.1 What is a Painleve equation?

The classic eqample is $\frac{d^2 y}{dt^2} = 6y^2 + 1$.

The key property of Painleve equations is nonlinearity. Moreover, their general solution is free of movable critical points, which means that it is independent of any constants of integration.

1.1.2 Why should we study them?

The original motivation of Painleve et al was to find new nonlinear special functions. Examples of special functions, like the exponential or trigonometric functions arising from solutions of the first- and second-order differential equations, are well-known.

There are six Painleve equations, which have parameters, and equations from different classes can be transformed into each other by degeneration of parameters.

In turn, *discrete* Painleve equations are second-order non-autonomous nonlinear recurrence relations.

It is difficult to determine whether a given recurrence relation is a discrete Painleve equation. One of the best ways to approach the problem uses tools of algebraic geometry developed by Sakai.

Note. There is a notion of entropy in discrete Painleve equations.

Discrete Painleve equations have a much richer classification scheme. For example, each equation has a corresponding pair of orthogonal sublattices.

Note. Discrete Painleve equations have applications in the study of the longest increasing subsequences in permutations.

Solutions of Painleve equations are called *Painleve transcendents*, which are purely nonlinear special functions.

We can also approach the study of Painleve equations through geometry, which was actively developed by Okamoto in 1970-1980s.