1 Math Plum

Let $\{x\}$ be the distance from x to the nearest integer.

Consider $\frac{1}{10}\{10x\}$.

Note that this function scales the graph of $\{x\}$ down by the factor of 10 in both directions.

Define the following function:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{10^n}\{10^n x\}. \tag{1}$$

Note that $f(x)$ is well-defined by the Weierstrass M-test, since $\frac{1}{10^n}\{10^n x\} \leq \frac{1}{10^n}\frac{1}{2}$. Moreover, it converges uniformly and f is continuous.

However, f is **not** differentiable **anywhere**.

Consider $\frac{f(a+h)-f(a)}{h}$.

We will find a sequence $\{h_m\}$ so that the sequence $\frac{f(a+h_m)-f(a)}{h_m}$ diverges, which would imly the difference quotient has no limit.

Fix a such that $0 < a < 1$, and let $a = 0.a_1a_2\ldots$.

Take $h_m = \begin{cases} 10^{-m}, \text{ if } a_m \neq 4, 9 \\ -10^{-m}, \text{ if } a_m = 4, 9 \end{cases}$ (think about why the nonuniqueness of decimal representation would not affect the overall argument).

It is easy to see that $h_m \to 0$.

Consider $\frac{f(a+h_m)-f(a)}{h_m}$:

$$= \pm 10(f(a+h_m) - f(a)) \tag{2}$$

$$= \pm 10^m(\sum_{n=0}^{\infty} 10^{-n}(\{10^n(a+h_m)\} - \{10^n a\})). \tag{3}$$

If $n \geq m$, then $10^n h_m = 10^{n-m} \in \mathbb{Z}$, $\{10^k(a+h_m)\} = \{10^n a\}$.

So only finitely many terms are nonzero, and thus it equals $\sum_{n=0}^{m} \pm 10^{-n}(\{10^n(a+b_m) - \{10^n a\}\})$

If $0.a_{n+1}a_{n+2}\ldots a_m < \frac{1}{2}$, then

Thus,$\frac{f(a+h_m)-f(a)}{h_m}$, which is equal to $\pm 10^m \sum_{n=0}^{m} 10^n(\pm 10^{n-m}) = \pm \sum_{n=0}^{m} \pm 1$.

It is even if m is odd, and *odd* if m is even.

Therefore, it does not converge.