Special Functions: From Gamma to Zeta II

Consider an equation $x^2 + y^2 + z^2 = 3xyz$, where $x, y, z \in \mathbb{N}$.

Solutions to this equation are called *Markov triples.*

$(1, 1, 1), (1, 1, 2)$ and $(1, 2, 5)$ are one of the solutions.

Define an involution $\tau : (x, y, z) \to (x, y, 3y - z)$ mapping a Markov triple to a Markov triple. Why?

Note the following:

$$z^2 - 3xyz + x^2 + y^2 = 0 \tag{1}$$
$$z_1 + z_2 = 3xy \tag{2}$$
$$z_2 = 3xy - z_1 \tag{3}$$

Theorem 0.1 (Markov)
All Markov triples are obtained from $(1, 1, 1)$ by application of τ and permutations.

For example, $(1, 1, 1) \xrightarrow{\tau} (1, 1, 2) \to (1, 2, 1) \xrightarrow{\tau} (1, 2, 5)$.

Note that this procedure can be represented as a tree. This tree is spacial, with the numbers written in the complement of the tree, rather than the nodes themselves.

The building unit of a tree consists of z between two branches, connected to another two branches. Over the edge connecting two branches there is x, and y is under the branch, while the other branch encloses $3xy - z$.

It is easy to spot Fibonacci numbers in the resulting tree. In fact, every second number is Fibonacci, which is left as an exercise.

0.1 Tropicalisation

Tropicalisation is a transformation replacing each operator in the expression according to a predefined rules: $+$ is replaced with max, $\times$ with $+$ and $/$ with $-$.

Tropicalised equation for Markov triples is thus $\max(2a, 2b, 2c) = a + b + c$. Without loss of generality, suppose that c is maximal, and thus we obtain that $a + b = c$.

What kind of a tree do we obtain from a tropicalised Markov equation?

The surprising fact is that in this case the step-by-step results of the Euclidean algorithm are represented by the tree, because the building unit consists of three branches connected to one note, with a, b and $a + b$ in its complement.

0.2 Quantisation

The modern approach applies the same principles to matrices.

For example, take the rule $AB = C$ for 2-by-2 matrices A and B, with the inital matrices $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, which generate the commutator of $\mathrm{SL}_2(\mathbb{Z})$.

In this case, the Cohn tree (1953) is obtained.

The Markov numbers less than 1000, which constitute Markov triples, are 1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985. Zognev has shown that $m_n \sim \frac{1}{3}e^{c\sqrt{n}}$.

Unicity Conjecture (Frobenius, 1913) Each Markov triple is uniquely defined by its maximal element.

0.3 Wonders of Markov Triples

Markov has shown that the set of Markov triples is equivalent to the set of the most irrational numbers up to an equivalence class. In other words,

$$(x, y, z) \Leftrightarrow d = \frac{b}{x} + \frac{y}{xz} - \frac{3}{2} + \frac{\sqrt{9z^2 - 4}}{2z},$$

where $by - ax = z$.

For example, if we consider $(1, 1, 1)$, we obtain $b - a = 1$, $b = 1$, $a = 0$, and thus $d = \frac{1+\sqrt{5}}{2}$.

For $(1, 1, 2)$, we get $d = 1 + \sqrt{2}$, which is called the silver ratio.

For $(1, 2, 5)$, we obtain $\frac{9+\sqrt{221}}{10}$

How can we measure how irrational a number is?

One of the ways requires the notion of the Markov constant.

Define $\mu(\alpha) = \inf\{c : \left|\alpha - \frac{p}{q}\right| < \frac{c}{q^2}$ for infinitely many $\frac{p}{q}\}$.

0.3.1 Markov Spectrum

Note that $\mu(\alpha) > \frac{1}{3}$ is equivalent to $\mu(\alpha) = \frac{m}{\sqrt{9m^2-4}}$, where $m \in M$.

Markov managed to study the spectrum successfully for some $\mu > \frac{1}{3}$. The spectrum was also classified for μ from 0 to Freiman's constant. What happens between Freiman's constant and $\frac{1}{3}$ is not yet known.

0.3.2 Second Wonder

Gorshkov (1953) and Cohn (1955) studied a punctured $\mathbb{Z}_3$-symmetric torus T_*^2 with a hyperbolic metric, with a complete puncture. The result was that the lengths of simple closed geodesics is in the form $\frac{2}{3} \cosh m$, where $m \in M$.

It was also established that real Markov's surface $x^2 + y^2 + z^2 = xyz$, where $x, y, z > 0$, is a Teichmüller space of punctured tori.

0.3.3 Third Wonder

Rudakov (1988) has established that the set of Markov triples can be put into the correspondence with the unique vector bundles on $\mathbb{P}^2$.

0.3.4 Fourth Wonder

Kontsevich, Manin (1994) and Dubrovin(1996) have established that the generating function for N_d, which is the number of solutions of rational curves of degree d, containing $3d - 1$ points on $\mathbb{P}^2$, satisfies the Painleve (?) equation PVI with the Markov initial conditions.

0.3.5 Fifth Wonder

Prokhorov, Hacking (2010) have show that the set of Markov triples is equivalent to the toric degeneration of $\mathbb{P}^2$, weighted projective plane $\mathbb{P}^2(x, y, z)$, where (x, y, z) is a Markov triple.