Let $B = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$

a) Suppose $A = (a_{ij})$ for $i, j \in \{1, 2\}$. Therefore,

$$T = \begin{pmatrix} a_{11} + 2a_{21} & a_{12} + 2a_{22} \\ -3a_{11} + 4a_{21} & -3a_{12} + 4a_{44} \end{pmatrix}$$

Take $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{4}{10} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \frac{3}{10} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

Then $Tv = a_{11} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + a_{21} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

If, for any $\sigma \in \mathbb{C}$, $a_{11} \neq \sigma \frac{4}{10}$ and $a_{21} \neq \sigma \frac{3}{10}$, then v and Tv are linearly independent, and thus span $\{v, Tv\}$ is 2-dimensional.

b) Consider the characteristic polynomial of B, $g(\lambda) = (1 - \lambda)(4 - \lambda) + 6$:

$$g(\lambda) = \lambda^2 - 5\lambda + 10$$

By Cayley-Hamilton Theorem, $g(B) = 0$.

c) Suppose $T \in \mathrm{Hom}(V, V)$ such that $T(A) = BA$. Suppose also h is a polynomial such that $h(T) = 0$. Observe that $\det B = 10 > 0$, and thus B is invertible.

From b),

$$B^2 - 5B + 10I = \mathbf{0}, \tag{1}$$

and then

$$B^2A - 5BA + 10A = \mathbf{0}. \tag{2}$$

Thus,

$$BT - 5T = -10A,$$

and hence

$$A = \frac{-1}{10}(B - 5I)T \tag{3}$$

Moreover,

$$B^2A^2 - 5BA^2 + 10A^2 = \mathbf{0}. \tag{4}$$

Therefore,

$$T^2 = 5TA - 10A^2 = (5T - 10A)A \tag{5}$$

which, when combined with the equation (3), implies that

$$\begin{align} T^2 &= (5T + BT - 5T)A \tag{6} \\ &= BTA \tag{7} \\ &= B^2A^2 \tag{8} \end{align}$$

Therefore, $(BA)^2 = B^2A^2$, and hence $ABA = BAA$, which gives $TA = AT$. Thus, from (5),

$$T^2 - 5AT + 10A^2 = 0,$$

which means that $h(A)(t) = t^2I - 5At + 10A^2$.

d)

Claim. Any T-cyclic subspace of V has a dimension of at most 2.

Proof. Consider $v \in V$. Let W be a T-cyclic subspace generated by v.

If v is zero, then $T^n(v) = 0$ for all $n \in \mathbb{N}$, and thus they are all linearly dependent, which means that $\dim W < 2$.

Suppose now that v is non-zero.

Let $d \geq 1$ be the largest integer such that $v, T(v), \ldots, T^{d-1}(v)$ are linearly independent. The largest d exists, since $\dim V$ is finite.

Let $U = \operatorname{span}(v, T(v), \ldots, T^{d-1}(v)) \subseteq W$.

Lemma

U is T-invariant.

Proof.

Suppose $u = c_0 v + c_1 T(v) + \cdots + c_{d-1} T^{d-1}(v)$ and $T(u) = 0$.

Note that $T(c_0 v + c_1 T(v) + \cdots + c_{d-1} T^{d-1}(v)) = c_0 T v + c_1 T^2 v + \cdots + c_{d-1} T^d v = 0$.

Since d is the largest integer such that $v, T(v), \ldots, T^{d-1}(v)$ are linearly independent, then c_{d-1} is non-zero, and thus $T^d(w) \in U$. $\square$

U is T-invariant, and thus if $v \in U$, then $W \subseteq U$, because W is the smallest T-invariant subspace containing v. By definition of U, $U \subseteq W$, and thus $U = W$.

Therefore, $\dim W = d$. Note that since a maximally linearly independent set in V has a dimension of 2, then $d \leq 2$. $\square$