1 Problem II

Recall that two matrices $A, B \in M_{n\times n}(\mathbb{F})$ are unitarily/orthogonally equivalent if there exists a unitary/orthogonal matrix Q such that $A = Q^{-1}BQ$ (or equivalently, $A = Q^*BQ$). Let us write $A \sim B$ if this is the case.

Suppose now that $A, B, C \in M_{n\times n}(\mathbb{F})$.

Lemma 1.1

$A \sim A$

Proof.

Since for all $x \in V$ $\|Ix\| = \|x\|$, I is unitary/orthogonal. Moreover, $I^{-1} = I$, and hence $A = I^{-1}AI$, which means that $A \sim A$. □

Lemma 1.2

If Q is unitary/orthogonal, then Q^* is also unitary/orthogonal.

Proof.

For any $x \in V$, $\langle Q^*x, Q^*x\rangle = \langle x, QQ^*x\rangle$, and since $QQ^* = Q^*Q = I$, then $\langle x, QQ^*x\rangle = \langle x, Q^*Qx\rangle = \langle Qx, Qx\rangle$.

Because Q is unitary/orthogonal, $\langle Q^*, Q^*\rangle = \langle Qx, Qx\rangle = x$. □

Lemma 1.3

$A \sim B$ is equivalent to $B \sim A$.

Proof.

Suppose there exists a unitary/orthogonal matrix Q such that $A = Q^*BQ$.

Therefore, since Q is invertible, $AQ^* = Q^*BQQ^* = Q^*B$.

Moreover, $QAQ^* = QQ^*B = B$.

Let $P = Q^*$. Therefore, $P^* = Q^{**} = Q$, and thus $B = P^*AP$.

Note that P is unitary/orthogonal from Lemma 1.2.

Relabelling A as B and B as A, we obtain the conclusion in the other direction. □

Lemma 1.4

If $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof.

Suppose there exists a unitary/orthogonal matrix Q such that $A = Q^*BQ$, and suppose there exists a unitary/orthogonal matrix P such that $B = P^*CP$.

Therefore, $A = (Q^*P^*)C(PQ)$.

Note that $(PQ)^* = Q^*P^*$. Moreover, for any $x \in V$, $\langle PQx, PQx\rangle = \langle Qx, Qx\rangle$, because P is unitary/orthogonal, and since Q is unitary/orthogonal, then $\langle Qx, Qx\rangle = \langle x, x\rangle$. Therefore, $\langle PQx, PQx\rangle = \langle x, x\rangle$, and thus PQ is unitary/orthogonal.

From Lemma 1.2, $(PQ)^* = Q^*P^*$ is also unitary/orthogonal.

If $R = PQ$, then $A = R^*CR$ and R is unitary/orthogonal, and thus $A \sim C$. $\square$