Suppose that $\mathbb{F} = \mathbb{Z}_2$ and $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$.

Problem.

Find a rational canonical form R of A.

Solution.

First we find a characteristic polynomial of A.

Note that

$$A - tI = \begin{pmatrix} -t & 1 & 0 & 1 \\ 1 & -t & 1 & 0 \\ 0 & 1 & 1-t & 1 \\ 1 & 0 & 1 & 1-t \end{pmatrix}.$$

Using the Laplacian expansion along the first column, we see that

$$\begin{aligned}
\det(A - tI) &= -t \det \begin{pmatrix} -t & 1 & 0 \\ 1 & 1-t & 1 \\ 0 & 1 & 1-t \end{pmatrix} && (1)\\
&\quad - \det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1-t & 1 \\ 0 & 1 & 1-t \end{pmatrix} - \det \begin{pmatrix} 1 & 0 & 1 \\ -t & 1 & 0 \\ 1 & 1-t & 1 \end{pmatrix} && (2)\\
&= -t\left(-t[(1-t)^2 - 1] - [1 - t - 0]\right) && (3)\\
&\quad - \left((1-t)^2 - 1 - 1(0-1)\right) && (4)\\
&\quad - \left(1 + (-t(1-t) - 1)\right) && (5)\\
&= -t(-t(-t)(2-t) + t - 1) - \left((-t)(2-t) + 1\right) + t - t^2 && (6)\\
&= -t(2t^2 - t^3 + t - 1) - (-2t + t^2 + 1) + t - t^2 && (7)\\
&= t^4 - 2t^3 - t^2 + t + 2t - t^2 - 1 + t - t^2 && (8)\\
&= t^4 - 2t^3 - 3t^2 + 4t - 1 && (9)\\
&= t^4 - 3t^3 + t^2 + t^3 - 4t^2 + 4t - 1 && (10)\\
&= t^2(t^2 - 3t + 1) + t^3 - 3t^2 + t - t^2 + 3t - 1 && (11)\\
&= t^2(t^2 - 3t + 1) + t(t^2 - 3t + 1) - (t^2 - 3t + 1) && (12)\\
&= (t^2 - 3t + 1)(t^2 + t - 1) && (13)
\end{aligned}$$

Since $\mathbb{F} = \mathbb{Z}_2$, then $f(t) = \det(A - tI) = (t^2 + t + 1)^2$.

Note that

$$
\begin{aligned}
A^2 &= \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} & (14) \\
&= \begin{pmatrix} 1+1 & 0 & 1+1 & 1 \\ 0 & 1+1 & 1 & 1+1 \\ 1+1 & 1 & 1+1+1 & 1+1 \\ 1+1 & 1+1 & 1+1 & 1+1+1 \end{pmatrix} & (15) \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, & (16)
\end{aligned}
$$

and thus

$$
\begin{aligned}
A^2 + A + I &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & (17) \\
&= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. & (18)
\end{aligned}
$$

Since $A^2 + A + I \neq 0$, while the only divisor of the the minimal polynomial $p(t)$ is $t^2 + t + 1$, we deduce that $p(t) = f(t) = (t^2 + t + 1)^2 = t^4 + t^2 + 1$.

Therefore, there exists a canonical basis β such that

$$
[A]_\beta = R = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},
$$

where the signs were omitted since $-1 = 1$.

$\square$

Problem.

Find an invertible matrix Q such that $R = Q^{-1}AQ$.

Solution.

First we find the basis of K_ϕ, where $\phi = t^2 + t + 1$.

Note that, by Theorem 7.18, since $p(t) = (t^2 + t + 1)^2$ and $t^2 + t + 1$ is irreducible, then $K_\phi = \ker \phi(A)^2$. Since $\phi A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, we know that $phi(A)^2 = 0$ and thus $\ker \phi(A)^2 = V = K_\phi$.

We now look for the cycle basis of $K_\phi = V$, which has a length of $\dim V = 4$.

Take $v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Therefore, $Av = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $A(Av) = A^2 v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $A^3 v = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$.

Let $\beta = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \}$.

Note that, since $\phi(A)^2 x = 0$ for any $x \in \beta$, then $\operatorname{span} \beta \subseteq K_\phi$.

We now prove that β is linearly independent:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \rightsquigarrow \tag{19}$$

$$R_1 \to R_1 - R_3, R_4 \to R_4 - R_2 - R_3 \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]. \tag{20}$$

Therefore, β is linearly independent.

Since $|\beta| = 4$ and $\operatorname{span} \beta \subseteq K_\phi$, then β is a cyclic basis of $K_\phi = V$.

Let $Q = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

We invert Q:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \tag{21}$$

$$\begin{array}{l} R_1 \to R_1 - R_3, \\ R_4 \to R_4 - R_2 - R_3 \end{array} \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \tag{22}$$

$$R_3 \Leftrightarrow R_4 \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \tag{23}$$

Thus, $Q^{-1} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

By the change-of-matrix formula, we know that $R = Q^{-1}AQ$, since β is a cycle basis of K_ϕ and thus a rational canonical basis. $\square$

Problem.

Find an L_A-invariant subspace $W \subseteq \mathbb{F}^4$ of dimension 2.

Solution.

Let $\gamma = \{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \} \subseteq \beta$, and let $T = L_A$.

Note that γ is linearly independent, since β is linearly independent.

Let $W = \operatorname{span} \gamma$.

Since $T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \in W$ and $T \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in W$, then W is T-invariant (T is defined by its action on a basis). $\square$