1 More on the Exponential Functios

Note that for $a > 0$,

$$a^r = e^{r \log a}.$$

Therefore,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} x^r &= \frac{\mathrm{d}}{\mathrm{d}x} e^{r \log x} && (1) \\ &= \frac{r}{x} e^{r \log x} && (2) \\ &= r x^{r-1} && (3) \end{aligned}$$

Define f as follows:

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}}, x \neq 0 \\ 0, x = 0 \end{cases}$$

It can be proven that $f(x)$ is continuous.

In fact, it is smooth and $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$.

Consider $g(x) = \frac{(x+2)^4}{(x^2-x+1)^5} \arctan(x)$.

Even though finding g' is straightforward, it is messy. Taking logs, however, is easier:

$$\log g(x) = 4 \log(x+2) - 5 \log(x^2 - x + 1) + \log(\arctan(x)).$$

This makes differentiation cleaner.

By chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x} \log(g(x)) = \frac{g'(x)}{g(x)}.$$

So

$$\begin{aligned} g'(x) &= g(x) \frac{\mathrm{d}}{\mathrm{d}x} \log(g(x)) && (4) \\ &= \frac{(x+2)^4}{(x^2-x+1)^5} \arctan(x) [\frac{4}{x+2} - \frac{5}{x^2-x+1} + \frac{1}{(1+x^2)\arctan x}] && (5) \end{aligned}$$

2 Hyperbolic Functions

Definition 2.1.

$$\begin{aligned} \cosh(x) &= \frac{e^x + e^{-x}}{2} && (6) \\ \sinh(x) &= \frac{e^x - e^{-x}}{2} && (7) \\ \tanh(x) &= \frac{\sinh(x)}{\cosh(x)} && (8) \end{aligned}$$

Note the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x \tag{9}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x \tag{10}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \operatorname{sech}^2 x \tag{11}$$
$$\cosh^2 x - \sinh^2 x = 1 \tag{12}$$

Note that the area of the hyperbolic sector for some x is $\frac{x}{2}$.

The shape of the $\cosh x$ graph is *catenary*, which is the shape of a hanging rope with the uniform mass.

3 Techniques of Integration

3.1 Integration by Parts

Recall product rule:

$$(uv)' = u'v + uv'$$

Then

$$\int uv' = uv - \int u'v$$