MAT240:
SUBSPACES II

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A NONEMPTY SUBSET W OF A VECTOR SPACE V IS A SUBSPACE IF IT IS CLOSED UNDER $+$ AND $\cdot$. THAT IS,

$v, w \in W \Rightarrow v + w \in W$

$v \in W,\ a \in F \Rightarrow av \in W$

EXAMPLES

$V = M_{n \times n}(F)$.

A SQUARE MATRIX $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

IS CALLED:

- SYMMETRIC IF $a_{ij} = a_{ji}$.
- SKEW-SYMMETRIC IF $a_{ij} = -a_{ji}$.
- UPPER TRIANGULAR IF $a_{ij} = 0$ FOR $i > j$.
- LOWER TRIANGULAR IF $a_{ij} = 0$ FOR $i < j$.
- DIAGONAL IF $a_{ij} = 0$ FOR $i \neq j$.
- TRACE-FREE IF $\mathrm{TRACE}(A) = \sum_{i=1}^{n} a_{ii} = 0$

ALL THESE ARE SUBSPACES OF $M_{n\times n}(F)$.

EXAMPLE

$\because V = \mathbb{C}^n$, $W = \mathbb{R}^n$.

W IS NOT A SUBSPACE OF V, VIEWED AS A VECTOR SPACE OVER $\mathbb{C}$

W IS A SUBSPACE OF V, VIEWED AS A VECTOR SPACE OVER $\mathbb{R}$.

THEOREM

IF W_1, W_2 ARE SUBSPACES OF V, THEN $W_1 \cap W_2$ IS A SUBSPACE OF V.

PROOF

$W_1 \cap W_2 \neq \emptyset$, since $0 \in W_1$ and $0 \in W_2$.

IF $v, w \in W_1 \cap W_2$, THEN $v, w \in W_1$ so $v+w \in W_1$ AND $v, w \in W_2$, so $v+w \in W_2$.

HENCE $v+w \in W_1 \cap W_2$.

SIMILAR FOR SCALAR MULTIPLICATION.

EXAMPLES

① $V = M_{n\times n}(F)$.

$W_1 = \{$ UPPER TRIANGULAR MATRICES $\}$

$W_2 = \{$ LOWER TRIANGULAR MATRICES $\}$

$W_1 \cap W_2 = \{$ DIAGONAL MATRICES $\}$

② $V = M_{n\times n}(F)$.

$W_1 = \{$ SYMMETRIC MATRICES $\}$

$W_2 = \{$ SKEW SYMMETRIC MATRICES $\}$

$W_1 \cap W_2 = \{0\}$ PROVIDED THAT $1+1 \neq 0$ IN F.

NOTE: IF $1+1=0$, EVERY SYMMETRIC MATRIX IS SKEW-SYMMETRIC

MORE GENERALLY, FROM SIMILAR ARGUMENT:

IF $W_\alpha, \alpha \in \Lambda$ ARE SUBSPACES OF V, THEN

$$\bigcap_{\alpha \in \Lambda} W_\alpha = \{ v \in V \mid v \in W_\alpha \ \forall \alpha \in \Lambda \}$$ IS A SUBSPACE

EXAMPLES

① $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$

$$W_n = \{ f : \mathbb{R} \to \mathbb{R} \mid f(n) = 0 \}, \ n \in \mathbb{Z}$$

$$\bigcap_{n \in \mathbb{Z}} W_n = \{ f : \mathbb{R} \to \mathbb{R} \mid f(n) = 0 \text{ for all } n \in \mathbb{Z} \}$$

② V ANY VECTOR SPACE, $S \subset V$ SUBSET

LET W_α, $\alpha \in \lambda$, ALL SUBSPACES $S \subseteq W_\alpha$.

THEN $W = \bigcap_{\alpha \in \lambda} W_\alpha$ IS A SUBSPACE

THEN $S \subseteq W$, EXERCISE: W IS THE SMALLEST SUBSPACE OF CONTAINING S.

$\longrightarrow$ ANOTHER CONSTRUCTION IS $\text{SPAN}(S)$.

③ $W_1 \cup W_2$! IS IT THE CASE THAT IT IS A SUBSPACE FOR $W_1, W_2 \subseteq V$?

ONLY IF $W_1 \subseteq W_2$ OR $W_2 \subseteq W_1$

SUM OF SUBSPACES

FOR ANY SUBSPACES $W_1, W_2 \subset V$, THE SUM

$$W_1 + W_2 = \{ w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2 \}$$

PROOF:

① $W_1 + W_2 \neq \emptyset$ BECAUSE $0 + 0 = 0 \in (W_1 + W_2)$.

② $v, w \in W_1 + V_2$, WRITE $v = v_1 + v_2$ $v_1 \in W_1, v_2 \in W_2$
$w = w_1 + w_2$, $w_1 \in W_1$, $w_2 \in W_2$

$\Rightarrow$ $v + w = (v_1 + w_1) + (w_2 + v_2)$, $\in W_1 + W_2$.

③ SIMILARLY, $v \in W_1 + W_2$, $a \in F$ THEN $av \in W + W_2$.

EXAMPLE:

EXERCISE

$W_1 + W_2 = \text{span}(W_1 \cup W_2)$