1 Convergence of Series

Consider $\sum_{n=1}^{\infty} a_n(x-a)^n$.

Suppose the series converges at $x = x_0$.

The terms $a_n(x_0-a)^n$ must go to 0.

So there exists $M \in \mathbb{R}$ such that $\left|a_n(x_0-a)^n\right| < M$ for all $n \in \mathbb{N}$.

Consider a pont r such that $|r-a| < |x_0-a|$:

$$\sum_{i=1}^{\infty} a_n(r-a)^n = \sum_{i=1}^{\infty} a_n(x_0-a)^n \cdot (\frac{r-a}{x_0-a})^n. \tag{1}$$

Then $\left|a_n(r-a)^n\right| = \left|a_n(x_0-a)^n(\frac{r-a}{x_0-a})^n\right| \leq M\left|\frac{r-a}{x_0-a}\right|^n$.

Thus, $\sum_{i=1}^{\infty} a_n(r-a)^n$ converges absolutely.

Now consider the series as a function:

$$f_n(x) = \sum_{n=0}^{n} a_n(x-a)^n.$$

As n goes to infinity, what is its derivative?

Note that $f_n'(x) = \sum_{n=0}^{\infty} na_n(x-a)^{n-1}$.

Then the ratio test says that $\frac{f_{n+1}'(x)}{f_n'(x)} = \frac{n+1}{n}\frac{a_{n+1}}{a_n}(x-a)$.

Suppose $f_n(x) \to f(x)$ for all $x \in [a,b]$. Suppose that $f_n(x)$ is continuous for all $n \in \mathbb{N}$.