Integral Calculus

Motivation: finding areas

Although triangulation provides a viable approach to compute the area of geometric shapes, it is not obvious that the non-unique partitions add up to the same area.

Problems also arise in the calculation of the area of a circle. What's π?

We introduce the notion of the *area under the graph between a and b*, corresponding to the signed area under the graph of $y = f(x)$ above the x-axis and between the lines $x = a$ and $x = b$.

This area can be approximated by computing f on a *partition* P of $[a, b]$, a set of points $a = t_0 < t_1 < t_2 < \cdots < t_n = b$.

For each $i = 1, \ldots, n$ define $m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$, $M_i = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$.

Note. Since f is not assumed to be continuous, note that inf and sup are used instead of min and max.

Assume f is bounded, so that m_i and M_i exist.

Define the *lower sum* $L(f, P)$ of the partition as follows:

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

Define the *upper sum* similarly:

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

e.g. for a constant function $f(x) = c > 0$, $U(f, P) = L(f, P)$ for any partition P.

Note. For a partition P,

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \leq \sum_{i=1}^{n} M_i(t_i - t_{i-1}) = U(f, P),$$

since $m_i \leq M_i$ for all i by definition.