Theorem

Suppose that the characteristic polynomial of $T \in \text{End}(V)$ splits.

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvectors of T and let p_i be the size of the largest Jordan block corresponding to λ_i in a Jordan canonical form of T.

The minimal polynomial of T is

$$p(t) = \prod_{i=1}^{k} (t - \lambda_i)^{p_i}.$$

Proof.

Since the characteristic polynomial of $T \in \text{End}(V)$ splits, $V = \bigoplus_{i=1}^{k} K_{\lambda_i}$.

For $i \in [1, k] \cap \mathbb{N}$, consider $T|_{K_{\lambda_i}}$.

Let v be an initial vector such that a cycle $\gamma = \{v, (T - \lambda_i I)v, \ldots, (T - \lambda_i I)^{p_i - 1} v\}$ corresponds to the longest column in the dot diagram for K_{λ_i}.

Let $G_i = \operatorname{span} \gamma$.

Since $(T - \lambda_i I)^{p_i} v = 0 \in G_i$, we deduce that G_i is $(T - \lambda_i I)|_{G_i}$-invariant, since any linear transformation is defined uniquely by its action on a basis.

Note that $[(T - \lambda_i I)|_{G_i}]_\beta$ is $\begin{pmatrix} 0 & \ldots & & 0 \\ 1 & & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \ldots & 1 & 0 \end{pmatrix}$.

Repeatedly using Laplacian expansion along the last row, we see that the characteristic polynomial of $(T - \lambda_i I)|_{G_i}$, $g(t)$, is such that $g(t) = (-1)^{p_i} t^{p_i}$. By Cayley-Hamilton Theorem, $(T - \lambda_i I)^{p_i}|_{G_i} = 0$.

We now show that, in fact, $(T - \lambda_i I)^{p_i} v = 0$ for any $v \in K_{\lambda_i}$.

Indeed, since there exists a cycle basis β_i of K_{λ_i} and γ is a cycle with the greatest length, all the other cycles of β_i have the length of less than or equal to p_i. Thus, by definition of a cycle, for any vector $v \in K_{\lambda_i}$, since $(T - \lambda_i I)$ is linear and homogeneous, then there exists $m \in \mathbb{Z}^+$ such that $m \leq p_i$ and $(T - \lambda_i I)^m v = 0$. In particular, since $(T - \lambda_i I)(0) = 0$, we know that $(T - \lambda_i I)^{p_i} v = 0$ for all $v \in K_{\lambda_i}$.

Let $g_i(x) = (x - \lambda_i)^{p_i}$. Let $g(x) = \prod_{i=1}^{k} g_i$.

Note that the absolute value of the leading coefficient of $g(x)$ is 1.

Note that $g(T) = 0$, from Theorem E.4, the fact that $V = \bigoplus_{i=1}^{k} K_{\lambda_i}$ and $g_i(x) = 0$ for all $x \in K_{\lambda_i}$.

By Theorem 7.12, $p(t)$ divides $g(t)$.

Therefore, there exists $c \in \mathbb{F}$ such that $g(t) = cp(t)$. Since $g(t)$ and $p(t)$ are monic up to the sign, we see that $c = 1$.

Hence, $p(t) = \prod_{i=1}^{k} (t - \lambda_i)^{p_i}$. $\square$