By definition of linearity, a map S from $\mathscr{L}(V,W)$ to W, with V,W defined on $\mathbb{F}$, is linear, if the following conditions are satisfied:

1. **additivity**:
$$\forall(T,T' \in \mathscr{L}(V,W)) : S(T+T') = S(T) + S(T')$$
2. **homogeneity**:
$$\forall(T \in \mathscr{L}(V,W))\forall(a \in \mathbb{F}) : S(aT) = aS(T)$$

Suppose S is such that $T \mapsto T(v)$.

Thus,

$$\begin{aligned} \forall(T,T' \in \mathscr{L}(V,W))\forall(v \in V) : S(T+T') &= (T+T')(v) & (1)\\ \text{by definition of addition for linear maps } | &= T(v) + T'(v) & (2)\\ \text{by definition of } S \ | &= S(T) + S(T') & (3) \end{aligned}$$

Hence, S is additive.

Moreover,

$$\begin{aligned} \forall(T \in \mathscr{L}(V,W))\forall(a \in \mathbb{F}) : S(aT) &= (aT)(v) & (4)\\ \text{by definition of scalar multiplication for linear maps } | &= a(T(v)) & (5)\\ \text{by definition of } S \ | &= aS(T) & (6) \end{aligned}$$

Hence, S is homogeneous.

Thus, S is linear.

Note that by the Rank-Nullity Theorem, $\dim(\mathscr{L}(V,W)) = \operatorname{rank}(S) + \operatorname{nullity}(S)$.

Claim. $\operatorname{rank}(S) = \dim(W)$, i.e. S is surjective.

Proof. Note that S is a linear map of $T \in \mathscr{L}(V,W)$ to W.

Suppose $w \in W$, and suppose a map U is given with the following properties:

- $\exists(v \in V) : U(v) = w$
- $\forall(a \in \mathbb{F})\forall(u \in V) : U(au) = aU(u)$
- $\forall(u,u' \in V) : U(u+u') = U(u) + U(u')$.

Thus, by definition of linearity, U is linear and $U \in \mathscr{L}(V,W)$.

Hence, $\forall(w \in W)\exists(U \in \mathscr{L}(V,W))\exists(v \in V) : U(v) = w$, and thus S is surjective and therefore $W = \operatorname{Im}(S)$. □

Since $\dim(\mathscr{L}(V,W)) = \dim(V)\dim(W)$, then by the claim and Rank-Nullity Theorem,

$$\operatorname{nullity}(S) = \dim(W)(\dim(V) - 1).$$