1 Power Series

1.1 Review

Consider $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$.

We have shown that $f(x)$ converges on an open interval $(a-r, a+r)$. Moreover, $f(x)$ converges uniformly on any closed subintreval $[a-s, a+s]$ such that $0 < s < r$.

In particular, $f(x)$ is continuous on $(a-r, a+r)$.

Our guess is that $f'(x) = \sum_{n=0}^{\infty} a_n n(x-a)^n$.

1.2 Weierstrass M-Test

Theorem 1.1

Suppose that $|f_n(x)| \leq M_n$ for all $x \in I$, where I is some interval.

Suppose also that $\sum_{n=0}^{\infty} M_n$ converges.

Then $\sum_{n=0}^{\infty} f_n(x)$ converges absolutely and uniformly on I.

Proof.

Proceed with the comparison test. □

Lemma 1.2

If $f_n(x)$ is continuous for all $x \in D(f)$, then $\sum_{n=0}^{\infty} f_n(x)$ is continuous.

Theorem 1.3

Suppose $f_n(x) \to f(x)$ for all $x \in I$, and suppose that $f_n(x)$ is differentiable for all $n \in \mathbb{N}$ and $x \in I$. Suppose also that $f_n'(x)$ converges uniformly to $g(x)$ on I and $g(x)$ is continuous.

Then g is differentiable and $g(x) = f'(x)$.

Proof.

$$\int_a^x g(t)\,\mathrm{d}t = \int_a^x \lim_{n\to\infty} f_n'(t)\,\mathrm{d}t = \lim_{n\to\infty} \int_a^x f_n'(t)\,\mathrm{d}t,$$

where the second equality holds because of uniform convergence.

Therefore, $\lim_{n\to\infty}(f_n(x) - f_n(a)) = f(x) - f(a)$, so $f'(x) = g(x)$. □

For $f(x) = \sum_{n=0}^{\infty}(x-a)^n$, let $f_n(x) = \sum_{m=0}^{n} a_m(x-a)^m$.

Then by the ratio test we can guarantee that $f_n(x) \to f(x)$ uniformly on $[a-s, a+s]$.

Note that $f_n'(x) = \sum_{m=0}^{n} a_m m(x-a)^{m-1}$, and $f_m'(x)$ converges uniformly to $f'(x)$.

Remark 1.4.

On the interval $(a-r, a+r)$, where $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely, its behaviour is $\sum_{n=1}^{\infty} na_n(x-a)^{n-1}$ which also converges absolutely.

Example 1.5

Consider $s(x) = \sum_{n=1}^{\infty} n^2 x^n$.

Let $a_n = n^2 x^n$.

Using the ratio test, we can see that

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^2 x^{n+1}}{n^2 x^n} \right| = |x|$$

as $n \to \infty$.

Therefore, $s(x)$ converges for $|x| < 1$.

Note that $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$.

Differentiating both sides, we obtain that $\frac{1+x}{(1-x)^2} = \sum_{n=1}^{\infty} n^2 x^{n-1}$, and thus $fracx + x^2(1-x)^2 = \sum_{n=1}^{\infty} n^2 x^n$.

We can now construct a hierarchy of functions: $C^\infty \subset \ldots \subset$ Continuously Differentiable Functions $\subset$ Differentiable Functions $\subset$ Continuous Functions $\subset$ Functions.

There is also another class, the class of *analytic functions*, with the corresponding power series convergent at each point.

We know that there are functions in C^∞ that are not analytic, for example, e^{-1/x^2}.