MAT240: DUAL SPACES 20161124

LET V BE A VECTOR SPACE OVER $\mathbb{F}$.
A DUAL SPACE V^* IS A SPACE OF LINEAR MAPS $\varphi: V \to \mathbb{F}$ ("LINEAR FUNCTIONALS").

$$V^* = \mathcal{L}(V, F)$$

PROPOSITION: $\dim(V^*) = \dim(V)$

PROOF:
$$\dim(V^*) = \dim(\mathcal{L}(V,F)) = \dim(V)\cdot\dim(F) = \dim(V).$$

EXAMPLES:

① $tr: M_{n\times n}(F) \to F$,
$A \mapsto A_{11} + A_{22} + \dots + A_{nn}$

② X a SET, $V = \mathcal{F}(X, F)$, $c \in X$
$ev_c : \mathcal{F}(X, F) \to F$,
$f \mapsto f(c)$

③ COORDINATE FUNCTIONS
$F^n \to F$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto x_i$
for any $i = 1, \dots, n$

MORE GENERALLY, GIVEN $b_1, \dots, b_n$, THE MAP $x \mapsto b_1x_1 + \dots + b_nx_n$

$F^n \to (F^n)^*$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \mapsto \left(x \mapsto \sum_{i=1}^{n} b_i x_i = b^t x \right.$
LINEAR

④ $F^\infty \to \left(F^\infty_{fin}\right)^*$

F^∞_{fin} FINITE SEQUENCES

$F^\infty_{fin} \neq F^\infty$

$(b_1, b_2, \dots) \longmapsto \varphi$

$\varphi(x_1, x_2, \dots) \longmapsto b_1 x_1 + b_2 x_2 + \dots$

this is an isomorphism

⑤ $F^\infty_{fin} \longrightarrow \left(F^\infty\right)^*$

NOT AN ISOMORPHISM.

$\boxed{\text{EXERCISE:}}$ WRITE AN EMAIL

DEFINITION FOR ANY $T \in \mathcal{L}(V, W)$, ONE DEFINES A DUAL MAP

$T^* \in \mathcal{L}(W^*, V^*)$ AS

$T^*(\varphi)(v) = \varphi(T(v))$

for all $\varphi \in W^*$, $v \in V$

$$V \xrightarrow[T]{} W \xrightarrow{\varphi} F$$

$$T^*(\varphi) = \varphi \circ T$$

EXERCISE:

IF $T \in \mathcal{L}(V, W)$, $S \in \mathcal{L}(U, V)$,

then $(T \circ S)^* = S^* \circ T^*$

THEOREM Let V, W be finite dimensional vector spaces with bases $\beta = \{v_1, \dots, v_n\}$, $\gamma = \{w_1, \dots, w_m\}$, and $T \in \mathcal{L}(V, W)$.

Then the matrices $[T]^{\gamma}_{\beta}$ and $[T^*]^{\beta^*}_{\gamma^*}$ are transposes to each other, i.e. if $A = [T]^{\gamma}_{\beta}$, $B = [T^*]^{\beta^*}_{\gamma^*}$, then $B = A^t$.

PROOF

$$T(v_j) = \sum_{i=1}^{m} A_{ij} w_i$$

$$T^*(w_i^*) = \sum_{j=1}^{n} B_{ji} v_j^*$$

$$w_i^*(T(v_j)) = A_{ij} = T^*(w_i^*)(v_j) = B_{ji}$$

For this reason, T^* is also called a "transpose map", denoted as T^t.

Suppose $S \subseteq V$ is a subspace.

Let $S^\circ = \{\varphi \in V^* \mid \varphi(v) \text{ for all } v \in S\}$.

Such S° is called an "annihilator".

LEMMA. For $T \in \mathcal{L}(V, W)$, with the dual $T^* \in \mathcal{L}(W^*, V^*)$,

$$N(T^*) = R(T)^\circ$$

BASES

Suppose $\dim(V) < \infty$.

Let $\beta = \{v_1, \dots, v_n\}$ be a basis

LEMMA

The dual space V^* has a unique basis

$$\beta^* = \{v_1^*, \dots, v_n^*\} \text{ (dual basis)}$$

with the property

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (*)$$

PROOF

Any linear map is uniquely determined by its values on the basis vectors.

Hence $(*)$ define linear functionals $v_1^*, \dots, v_n^*$. To check they are linearly independent, suppose $a_1 v_1^* + \dots + a_n v_n^* = 0$ for some $a_1, \dots, a_n \in F$.

Then $a_i = 0$, by evaluation of both sides on v_i

$\square$

PROOF

$$\psi \in N(T^*) \iff T^*(\psi) = 0 \iff T^*(\psi)(v) = 0$$

FOR ALL $v \in V$.

$$\iff \psi(T(v)) = 0 \ \forall v \in V.$$

$$\iff \psi(w) = 0 \text{ FOR ALL } w \in R(T)$$

$$\iff \psi \in R(T)^0.$$

EXERCISE: $\dim S^0 = \dim V - \dim S$

IN FINITE DIMENSIONS, THIS SHOWS THAT

$$\dim(N(T^*)) = \dim(R(T)^0) = \\ = \dim W - \dim R(T)$$

$$\dim R(T^*) = \dim(W) - \dim N(T^*) = \\ = \dim R(T)$$

Matrices: $\operatorname{rank}(A^t) = \operatorname{rank}(A)$