1 Sandpile Model and Divisors in Graphs

1.1 Revision

We have seen the concepts of topplings, relaxations, revertible states and forbidden configurations. We have also stidued the Riemann-Roch theorem, waves, ΔF, the sandpile group and its unit.

1.2

Exercise 1.1. Suppose that $\phi = (3+\psi)^0$. Then for all λ there exists $\psi \geq 0$ such that $(\lambda+\psi)^0 = \phi$.

Solution.

Note that $\lambda \to \lambda^0 \to 3 \to 3+\psi \to (3+\psi)^0$. Thus, $\phi = (\lambda + (3-\lambda^0) + \psi)^0$. □

Exercise 1.2. Find ΔF for $F(i,j) = i(i+1) + j(j+1)$.

Solution.

Note that $\Delta(i(i+1)) = -4(i+1)i + 2(i+1)i + (i-1)i + (i+1)(i+2)$, which means that $\Delta(i(i+1)) = -4i + 2i - i + 3i + 2 = 2$.

Therefore, $\Delta F = 4$.

□

Now we prove that the sandpile group *is* a group.

There are three ways to look at this problem.

Consider an integer lattice, and choose two linearly independent vectors.

We say that $(i,j) \sim (i',j')$ if $(i-i', j-j') = k_1 v_1 + k_2 v_2$ for $k_1, k_2 \in \mathbb{Z}$.

Let ϕ and ϕ' be two states in the form $\Gamma \setminus \delta\Gamma \to \mathbb{Z}$. We say that $\phi \sim \phi'$ if $\phi - \phi' = \Delta F$, where F is in the form $F : \Gamma \to \mathbb{Z}$. In other words, ϕ and ϕ' are equivalent if there is a sequence of topplings and antitopplings transforming ϕ into ϕ'.

Let N be the number of vertices in $\Gamma \setminus \delta\Gamma$, and consider $\mathbb{Z}^n/\mathbb{Z}\langle v_1 v_2 \dots v_N\rangle$, where $\mathbb{Z}^n$ is a set all states. Note that $\mathbb{Z}^n/\mathbb{Z}\langle v_1, v_2, \dots, v_N\rangle$ is a group.

Lemma 1.3

Each equivalence class has exactly one revertible state.

Note. In this way, the set of equivalence classes is in one-to-one correspondence with the set of revertible states of the group.

Proof.

Consider ΔF, where $F \equiv -1$. Suppose that Γ is rectangluar. Then ΔF is such that there are 2's at the vertices of a rectangular, and 1's at the other border positions. The rest is filled with 0's. This state is called the *Kreitz unity* and denoted as β. Each equivalence class has more than or equal to 1 revertible state. Let ϕ be arbitrary, then $(\phi + k\beta)^0$ is revertible, where k is some big natural number.

Exercise 1.4. Approximate k.

Now we prove that each equivalence class has less than or equal to 1 state.

Let ϕ_1 and ϕ_2 be revertible.

Let $D = \{v \in \Gamma \mid F(v) = \min F\}$.

Then D is a forbidden configuration for ϕ_2.

Suppose that $\min F = 0$. Choose a corner in D. Then $3 \leq \phi_1(v) \leq \phi_2(v) + 2$. Then $\phi_2(v) \leq 1$.

We, however, need to account for the case when the region defined by D is not compleely inside $\Gamma \setminus \delta\Gamma$. □

Note. $\Delta F \sim \langle 0 \rangle$. Thus, $\langle 0 \rangle = \phi + \Delta 1$.

Exercise 1.5. If two states are equivalent, there exists another state from which the first two states receive sand.

1.3 Group Unity Element

We canq prove that the sandpile group unity element E is $(\langle 8 \rangle - \langle 8 \rangle^0)^0$.

Theorem 1.6

Consider a rectangle n by m such that $m > 10n$. Then there is a stripe of 2s in the middle, with the distance from the left and range margin less than or equal to n.

Proof.

Consider $((n^2 + n)\beta = \phi)$.

Lemma 1.7

$\phi^0 = E$.

Proof.

$\phi^0 \sim E$, and thus we only need to show that $\phi^0 \geq \langle 2 \rangle$, which means that ϕ^0 is revertible.

Exercise 1.8. Show that $\langle 2 \rangle$ is revertible.

Exercise 1.9. If $\phi \geq \psi$ and ψ is revertible, then ψ^0 is revertible.

Exercise 1.10. ϕ is revertible if and only if $(\phi + \beta)^0 = \phi$, where β is the Kreitz unity. (Hint: if $(\phi + \beta)^0 \neq \phi$, then there exists a forbidden state.)

Let $F(i, k) = (n - k)^2 k + (n - k)$. It is easy to check that $\phi + \Delta F$ is in the form with a stripe of twos in the middle, which means that $\Delta F = 2$ almost everywhere.

Define a map G as follows for $k > [n\sqrt{2}]$:.

$$G(k, i) = \frac{([n\sqrt{2} - k])([n\sqrt{2}] - k - 9)}{2} = 0,$$

Exercise 1.11. Continue the proof by using G on a rectangle with a stripe of twos in the middle.

$\square$

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