MAT240 Problem Set IV: Problem 2 Thursday 13^{th} October, 2016

Suppose V is a subspace of $\mathbb{R}^4$, and β its basis. The required β must satisfy the following two properties:

$$V = \operatorname{span}(\beta) \tag{1}$$
$$\beta \text{ is linearly independent.} \tag{2}$$

Represent the given system of equations in the table and simplify:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{array}\right] \tag{3}$$
$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \qquad L_2 - L_1 \tag{4}$$
$$= \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \qquad L_1 - L_2 \tag{5}$$

Thus, from L_1, $x_1 = x_3 + 2x_4$ and $x_2 = -2x_3 - 3x_4$. Hence any vector $\boldsymbol{x} \in V$ must satisfy the following equation:

$$\boldsymbol{x} = (x_3 + 2x_4,\ -2x_3 - 3x_4,\ x_3,\ x_4)$$

which is equivalent to

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} x_4 \tag{6}$$

Let $\alpha = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Suppose that $\exists (a, b \in \mathbb{R})$, not both zero, such that

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} a + \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} b = 0$$

Therefore, from the 3rd row it follows that $a = 0$ and from the 4th row it follows that $b = 0$, which is a contradiction.

Thus, the set α is linearly independent.

From Equation 6 it follows that $\forall (x_3, x_4 \in \mathbb{R}) : \boldsymbol{x}$ satisfies the conditions given by the system of equations. Therefore all elements of $\operatorname{span}(\alpha)$ satisfy these conditions.

Moreover, from Equation 6 it follows that all such $\boldsymbol{x}$ are in $\operatorname{span}(\alpha)$.

Thus, $V = \operatorname{span}(\alpha)$. But α is also linearly independent, hence $\alpha = \beta$.