1 Irrationality of π

1.1 Observations

1. Consider the function $f_n(x) = \frac{x^n(1-x)^n}{n!}$.

   Note that it satisfies the following inequality:

$$0 < f_n(x) < \frac{1}{n!} \text{ for } 0 < x < 1$$

   Observe also that f_n can be written as follows:

$$f_n(x) = \frac{1}{n!}\sum_{i=n}^{2n} c_i x^i, \tag{1}$$

   where $c_i \in \mathbb{Z}$.

   Therefore, $f_n^{(k)}(0) \in \mathbb{Z}$. Moreover, since $f_n(x) = f_n(1-x)$, then $f_n^{(k)}(0) = (-1)^k f_n^{(k)}(1-x)$. Therefore, $f_n^k(1) \in \mathbb{Z}$.

2. For any $a \in \mathbb{R}$ and $\epsilon > 0$, then for sufficiently large n we have $\frac{a^n}{n!} < \epsilon$.

   To see this, observe that if $n \geq 2a$, then

$$\frac{a^{n+1}}{(n+1)!} = \frac{a}{n+1}\frac{a^n}{n!} < \frac{1}{2}\frac{a^n}{n!}. \tag{2}$$

   Now let n_0 be any natural number with $n_0 \geq 2a$.

   Therefore, applying the inequality (2), we obtain that there exists $k \in \mathbb{N}$ such that

$$\frac{a^{n_0+k}}{(n_0+k)!} < \epsilon.$$

Now we are ready to proceed with the proof.

1.2 Proof

Theorem 1.1

π and π^2 are both irrational.

Proof. Suppose, on the other hand, that $\pi = \frac{a}{b}$ for some $a, b \in \mathbb{N}$. Consider the following function:

$$G(x) = b^n(\pi^{2n} f_n(x) - \pi^{2n-2} f_n''(x) + \pi^{2n-4} f_n^{(4)}(x) - \cdots + (-1)^n f_n^{(2n)}(x). \tag{3}$$

Since $b^n\pi^{2n-k} = a^{n-k}b^k$ is an integer, while $f_n^{(k)}(0)$ and $f_n^{(k)}(1)$ are also integers, then $G(0)$ and $G(1)$ are integers.

Notice that

$$G''(x) = b^n(\pi^{2n} f_n''(x) - \pi^{2n-2} f_n^{(4)}(x) + \cdots + (-1)^n f_n^{(2n+2)}(x). \tag{4}$$

Since $2n + 2 > 2n$, the last term is zero. Therefore, adding (3) and (4), we obtain that

$$G''(x) + \pi^2 G(x) = b^n \pi^{2n+2} f_n(x) = \pi^2 a^2 f_n(x) \tag{5}$$

Now let

$$H(x) = G'(x) \sin(\pi x) - \pi G(x) \cos(\pi x). \tag{6}$$

Therefore,

$$H'(x) = \pi^2 a^n f_n \sin(\pi x) \tag{7}$$

Thus, by the Second Theorem of Calculus,

$$\pi^2 \int_0^1 a^n f_n(x) \sin(\pi x)\, \mathrm{d}x = H(1) - H(0) \tag{8}$$

$$= \pi[G(1) + G(0)], \tag{9}$$

and therefore $\pi \int_0^1 a^n f_n(x) \sin(\pi x)\, \mathrm{d}x$ is an integer.

On the other hand, since $0 < f_n(x) < \frac{1}{n!}$ for $0 < x < 1$, it follows that for $0 < x < 1$ $0 < \pi a^n f_n(x) \sin(\pi x)\, \mathrm{d}x < \frac{\pi a^n}{n!}$, which means that :

$$0 < \pi \int_0^1 a^n f_n(x) \sin(\pi x)\, \mathrm{d}x < \frac{\pi a^n}{n!} < 1.$$

This is a contradiction, since $\pi \int_0^1 a^n f_n(x) \sin(\pi x)\, \mathrm{d}x$ was shown to be an integer. Thus, the original assumption that π^2 is rational does not hold, and hence π is irrational.

$\square$