Suppose that $W_1, \dots, W_k$ are subspaces of a finite-dimensional vector space V such that $W_1 + \cdots + W_k = V$.

Claim.

$$\sum_{i=1}^{k} \dim W_i \geq \dim V$$

Proof. Consider $v \in V$.

Let $\beta = \bigcup_{i=1}^{k} \beta_i$ be the union of the bases β_i of all W_i. Let w_i for $i \in [1,k] \cap \mathbb{N}$ be vectors in W_i such that

$$v = \sum_{i=1}^{k} w_i$$

Since each w_i can be represented as a linear combination of vectors in β_i, it follows that $v \in \operatorname{span}\{\bigcup_{i=1}^{k} \beta_i\} = \operatorname{span} \beta$.

Hence, $\sum_{i=1}^{k} \dim W_i \geq \dim V$. ☐

Claim. $\sum_{i=1}^{k} \dim W_i = \dim V$ if and only if $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.

Proof. Suppose first $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$. Thus, $W_i \cap (\sum_{j \neq i} W_j) = \{0\}$.

From the previous claim, $\sum_{i=1}^{k} \dim W_i \geq \dim V$, and thus it is sufficient to show that $\sum_{i=1}^{k} \dim W_i \leq \dim V$. Note that since all W_i are subspaces of V, if γ is a basis of V, then all W_i are also subsets of the span of γ.

Consider $\beta = \bigcup_{i=1}^{k} \beta_i$, where β_i is a basis of W_i.

Since, $W_i \cap (\sum_{j \neq i} W_j) = \{0\}$, then $\operatorname{span} \beta_i \cap \bigcup_{i \neq j} (\operatorname{span} W_j) = \{0\}$, and thus, since all β_i are linearly independent, then the union of the bases is linearly independent as well. Hence, β is the basis for V, which gives that $\dim V = \sum_{i=1}^{k} \dim W_i$, as required.

☐