1 Uniqueness of JCF

1.1 Review

We have shown that, if λ_i are r distinct eigenvalues and $\mathbb{F}$ is algebraically closed, then $V = \bigoplus_{i=1}^{r} K_{\lambda_i}$.

For each K_λ, there is a basis of a disjoint union of cycles. The union $\beta = \bigcup_{i=1}^{r} \beta_{\lambda_i}$ is a basis of V such that $[T]_\beta$ is in JCF.

Example 1.1

Consider the matrix $A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}$ for $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{F}^3$.

The characteristic polynomial $-(t-1)^3$ splits, and thus the only eigenvalue is 1. Hence, $V = K_1$.

We know that there is a basis which is a disjoint union of cycles. There are three possibilities:

1. one cycle of length 3
2. two cycles of length 2 and 1
3. three cycles of length 1

We compute the eigenspace:

$$A - I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}, \tag{1}$$

and thus $\operatorname{rank}(A - I) = 1$, which means that $\operatorname{nullity}(A - I) = 2$.

Therefore, the third case is not possible, since in this case there must be 3 linearly independent eigenvectors.

The first case is also not possible, since it implies that there must be at least two linearly independent vectors in the range, while $\operatorname{nullity}(A - I) = 2$.

Therefore, the second case applies.

Note that therefore the basis must be of the form $\{(A - I)y, y, z\}$.

Try $y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Therefore, $(A - I)y = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

Now take any vector $z \in \ker(A - I)$ that is orthogonal to the initial vector $(A - I)y$, for example, $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

Therefore, $[A]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Remark 1.2. Note that $(A - I)^2 = 0$, since $(A - I)^2$ sends every basis vector $(A - I)y, y, z$ to 0.

We want to show that JCF is unique up to the reordering of blocks, given that the characteristic polynomial of $T \in \operatorname{End}(V)$ splits.

For any eigenvalue λ we can draw a dot diagram of $T_{K_\lambda} : K_\lambda \to K_\lambda$.

Our plan is as follows:

- We can find a basis of K_λ which is a disjoint union of cycles. Call this basis a **cycle basis** $\beta = \bigcup_{i=1}^{r} \beta_i$.

- We will order cycles by length $l_1 \geq l_2 \cdots \geq l_r$, where l_i is the length of β_i.
- Let the end vector of the cycle β_i be v_I.

The dot diagram is then can be defined as follows

$$\begin{pmatrix} \bullet(T-\lambda I)^{l_1-1}v_1 & \bullet(T-\lambda I)^{l_2-1}v_2 & \dots & \bullet(T-\lambda I)^{l_r-1}v_r \\ \bullet(T-\lambda I)^{l_1-2}v_1 & * & \dots & \vdots \\ \vdots & \vdots & \dots & \bullet v_r \\ \bullet(T-\lambda I)^{2}v_1 & \bullet(T-\lambda I)^{2}v_2 & \dots & \\ \bullet(T-\lambda I)v_1 & \bullet v_2 & \dots & \\ \bullet v_1 & & \dots & \end{pmatrix}$$

The dot diagram consists only of the dots.

We will show that the dot diagram of $T_{K_\lambda}$is unique, and does not depend on our choice of a cycle basis.

Theorem 1.3

For any $T \in \text{End}(V)$ and $s \geq 1$ such that the characteristic polynomial of T splits, vectors corresonding to the dots in the first s rows of $[T]_\beta$ form a basis of $\ker(T-\lambda I)^s$.

Proof.

Note that $\ker(T-\lambda I)^s \subseteq K_\lambda$.

Let $U = (T-\lambda I)^s \in \text{End}(K_\lambda)$, so that $\ker U = \ker(T-\lambda I)^s$.

In the dot diagram, $T-\lambda I$ moves up by one dot and sends the first row to 0. Therefore, $U = (T-\lambda I)^s$ moves up by s dots and sends first s rows to 0.

Let S_1, S_2 be such that $S_1 = \{x \in \beta \,|\, Ux = 0\}$ and $S_2 = \{x \in \beta \,|\, Ux \neq 0\}$.

Then $U \in \text{Hom}(S_2, \beta)$ is injective, because U shifs up s dots.

Therefore, the set $\{Ux \,|\, x \in S_2\}$ has a size of $|S_2|$ and is linearly independent in $\operatorname{im} U$, which means that $\dim \operatorname{im} U \geq |S_2|$.

On the other hand, S_1 is linearly independent and is inside $\ker U$. Thus, $\text{nullity}\, U \geq |S_1|$, which means that

$$\dim K_\lambda = \text{rank}\, U + \text{nullity}\, U \geq |S_1| + |S_2| = \beta = \dim K_\lambda.$$

By Rank-Nullity Theorem, $\text{nullity}\, U = |S_1|$, and thus S_1 is a basis of $\text{nullity}\, U$. □

Corollary 1.4

$\dim E_\lambda = \#$ columns in the dot diagram

Proof.

Note that $E_\lambda = \ker(T-\lambda I)$.

Apllying the theorem for the case $s = 1$, we obtain that $\ker(T-\lambda I) = \#$ dots in the first row. □

Remark 1.5. Note that $\dim E_\lambda$ is also equal to the number of cycles in the cycle basis β for K_λ, which is also equal to the number of Jordan blocks in $[T|_{K_\lambda}]_\beta$.

Theorem 1.6

Let r_j be the number of dots in the jth row of the dot diagram of T_{K_λ}.

Then

$$r_j = \operatorname{rank}(T-\lambda I)^{j-1} - \operatorname{rank}(T-\lambda I)^j = \ker(T-\lambda I)^j - \ker(T-\lambda I)^{j-1}.$$

Proof.

By Theorem 1.3, $\ker(T-\lambda I)^j$ is the number of dots in the first j rows, which is equal to $\sum_{i=1}^{j} r_i$.

Applying the theorem again, we get that $\ker(T-\lambda I)^j - \ker(T-\lambda I)^{j-1}$.

By the Rank-Nullity Theorem, the rest follows. □

Corollary 1.7

For any eigenvalue λ, the dot diagram for T_{K_λ} is unique and thus does not depend on the choice of the cycle basis β.

Corollary 1.8

Suppose that $T \in \operatorname{End}(V)$ and the characteristic polynomial of T splits.

Then the Jordan Canonical Form of T is unique up to the reordering of blocks.

If β, γ are bases of V such that $[T]_\beta$ consists of the Jordan blocks $\{J_i\}_1^t$ and $[T]_\gamma$ consists of the Jordan blocks $\{L_j\}_1^u$, then $t = u$ and we can reorder $\{J_i\}_1^t$ to obtain $\{L_j\}_1^u$.

Proof.

Suppose that $[T]_\beta$ is in JCF, and $\beta = \bigcup_{i=1}^r \beta_i$ is a cycle basis. Then the number of times the $l \times l$ block with the eigenvalue λ occurs inside $[T]_\beta$ is equal to the number of cycles β_i, which in turn equals to the number of columns of length l in the dot diagram for T_{K_λ}, then this quantity is the same for $[T]_\gamma$. □