1 Geometry of Discrete Painleve Equations

1.1 Discrete Integrable Systems

Suppose there is a parameter $t : \mathbb{Z} \to X$, where X is the configuration space. We will denote the nth step x_n as x, and let x_{n-1} be denoted as $\underline{x}$ and x_{n+1} as $\overline{x}$.

We will look at the mapping from $\mathbf{x}$ to $\overline{\mathbf{x}}$, from (x, y) to $(\overline{x}, \overline{y})$.

Moreover, we will require that $\overline{x} = \frac{p(x,y)}{q(x,y)}$. We will require that a similar condition holds for $\overline{y}$. This kind of relation is described as a *birational mapping*.

1.2 QRT

Now, let's consider a biquadratic curve Γ, where Γ is a set of zeros of some polynomial $p(x, y) \in \mathbb{C}[x, y]$. Since this polynomial is bivariate, there are two characteristics of its degree, which can be described as $(\deg_x p, \deg_x p)$.

Suppose that $p(x, y) = a_{00}x^2y^2 + a_{10}xy^2 + a_{20}y^2 + a_{01}x^2y + \cdots + a_{22}$.

Note that this polynomial can be written in the matrix form

$$p(x, y) = X^T AY = x^2; x; 1 \begin{pmatrix} a_{00} & a_{01} & \cdots \\ \vdots & & \vdots \\ \cdots & & a_{22} \end{pmatrix} \begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix}. \tag{1}$$

Consider two involutions r_x and r_y, so that $r_x^2 = r_y^2 = \mathrm{id}$.

The transformation QRT is then $r_x \circ r_y$.

It is worthwhile to note that Γ is isomorphic (?) to a torus, which in turn is isomorphic to $\mathbb{C}/\Lambda$ (cf. Hans Dustermaat 2010).

Moreover, we can explicitly state the form of r_x and r_y by applying standard methods of solving a quadratic to $p(x, y)$.

Now, let A and B be two complex 3×3 matrices.

We can study *bundles*, families of curves defined by Γ_A and Γ_B. Thus, $\lambda_0 XAY + \lambda_1 X^T BY = 0$.

Suppose we choose a point in plane. Then there exists a curve which contains this point and cuts Γ_A and Γ_B at the points of intersections, called base points, so that we obtain $[\lambda_0 : \lambda_1] = -[X_*^T BY_* : X_*^T AY_*]$. This mapping is QRT.

Now, suppose that r_y is such that it maps (x_*, y_*) to (x_*, y_*'). Then $y_*' = \frac{f_1(x_*)y_* - f_0(x_*)}{f_2(x_*)y_* - f_1(x_*)}$, where f_0, f_1 and f_2 are such that $(X^T A \times X_*^T B) = \langle f_0(x_*), f_1(x_*), f_2(x_*) \rangle$. Then $QRT = r_x \circ r_y = \phi \circ \phi$, where $\phi = \sigma \circ r_y$ and σ maps $(x, y; A, B)$ to $(y, x; A^T, B^T)$.

Considering the previous comments, we can now the form for ϕ:

$$\overline{x} = \frac{f_1(x)y - f_0(x)}{f_2(x)y - f_1(x)} \overline{y} = x \tag{2}$$

1.3 Technical Details

Consider a mapping $\mathbb{C} \times \mathbb{C} \to \mathbb{P}'_{\mathbb{C}} \times \mathbb{P}'_{\mathbb{C}}$, where $\mathbb{P}'_{\mathbb{C}}$ is the Riemann sphere. In this context we talk about the equivalence classes $\mathbb{C}^2 - \{(0,0)\}/\sim$, where $(x_0, x_1)\tilde{(}\mu x_0, \mu x_1)$, and $\mu \neq 0$.

To treat the subject matter properly, we also need a concept of a *divisor*.

Let X be an algebraic variety. Then $\mathrm{Div}(X) = \mathrm{span}_{\mathbb{Z}}\{\text{continuous algebraic varieties of codimension } 1\}$.

In our case, $X = \mathbb{P}'_{\mathbb{C}}$. Then we can evaluate $D = \sum_{i=1}^{n} a_i\{p_i\}$, where $a_i \in \mathbb{Z}$.

Let's find a divisor of a rational function $f = 3\frac{(x-1)^2(x+2)}{(x-i)(x+4)(x-5)}$. A divisor of f, (f), is then $(f) = 2\{1\} + -2 - \{i\} - \{(\} - 4) - \{5\}$.

Suppose now that $f(x) = x(x-1)$. Let $X = \frac{1}{x}$. Then $f(x) = \frac{1}{X}(\frac{1}{X} - 1)$. Notice that the degree of a divisor is 0. Indeed, it is zero always on the Riemann sphere.

Let $X = \mathbb{P}' \times \mathbb{P}'$. If we look at its classes, we can define a Picard lattice as follows:

$$Pic(X) = Cl(X) = \mathrm{span}\, \mathcal{H}_x, \mathcal{H}_y, \tag{3}$$

where $[H_x] = \mathcal{H}_x$ and $[H_y] = \mathcal{H}_y$.

For example, the class $[p(x,y) = X^T AY]$ is equivalent to $2\mathcal{H}_x + 2\mathcal{H}_y$.

We can also look at the intersection forms. For example, $\mathcal{H}_x \cdot \mathcal{H}_y = 1$ and $\mathcal{H}_x \cdot \mathcal{H}_x = 0 = \mathcal{H}_y \cdot \mathcal{H}_y$.

It is also worthwhile to note that $(2\mathcal{H}_x + 2\mathcal{H}_y)^2 = 8$.

1.4 Concrete Example

Suppose $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & \alpha & 1 \\ \alpha & 0 & -\alpha \\ 1 & -\alpha & 1 \end{pmatrix}$.

Then $\phi = \begin{cases} \overline{x} = \frac{(x-a)(x-a^{-1})}{y(x+a)(x+a^{-1})} \\ \overline{y} = x \end{cases}$.

Then for A we can write that $x_0x_1y_0y_1 = 0$, and for B we have $x^2y^2 + \alpha(x^2y + xy^2) + (x^2 + y^2) - \alpha(x+y) + 1 = 0$, which can be rewritten as $(xy + a^{-1}(x+y) - 1)(xy + a(x+y) - 1) = 0$.

We can also write $\phi^{-1} : \begin{cases} \underline{x} = y \\ \underline{y} = \frac{(y-a)(y-a^{-1})}{x(y+a)(y+a^{-1})}. \end{cases}$