Suppose a linear map $T : \mathbb{P}_3(\mathbb{R}) \to R^4$ is given such that

$$p \mapsto (p(0), p(1), p'(0), p'(1)).$$

Let β be the standard basis of $\mathbb{P}_3(\mathbb{R})$:

$$\beta = \{1, x, x^2, x^3\}$$

and let γ be the standard basis of $\mathbb{R}^4$.

Since

$$\begin{aligned}
(1)' = 0 &\Rightarrow p(0) = p(1),\ p'(0) = 0 = p'(1) && (1)\\
(x)' = 1 &\Rightarrow p(0) = 0,\ p(1) = 1,\ p'(0) = 1 = p'(1) && (2)\\
(x^2)' = 2x &\Rightarrow p(0) = 0,\ p(1) = 1,\ p'(0) = 0,\ p'(1) = 2 && (3)\\
(x^3)' = 3x^2 &\Rightarrow p(0) = 0,\ p(1) = 3,\ p'(0) = 0,\ p'(1) = 3 && (4)
\end{aligned}$$

then

$$\begin{aligned}
T(1) &= (1, 1, 0, 0) && (5)\\
T(x) &= (0, 1, 1, 1) && (6)\\
T(x^2) &= (0, 1, 0, 2) && (7)\\
T(x^3) &= (0, 1, 0, 3). && (8)
\end{aligned}$$

Therefore, since the matrix is determined uniquely by its action on the basis of the domain,

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Note that to show that T is an isomorphism, it is necessary and sufficient to show that it is linear and invertible.

Claim. T is linear.

Proof. Suppose $p, q \in \mathbb{P}_3(\mathbb{R})$ are two polynomials.

Consider $T(p + q)$:

$$T(p + q) = ((p + q)(0), (p + q)(1), (p + q)'(0), (p + q)'(1)).$$

Since p, q are functions, then $(p + q)(x) = p(x) + q(x)$.

Similarly, from the sum rule for derivatives, $(p + q)' = p' + q'$.

Thus,

$$\begin{aligned}
T(p + q) &= \big(p(0) + q(0), p(1) + q(1), p'(0) + q'(0), p'(1) + q'(1)\big) && (9)\\
&= \big(p(0), p(1), p'(0), p'(1)\big) + \big(q(0), q(1), q'(0), q'(1)\big) && (10)\\
&= T(p) + T(q) && (11)
\end{aligned}$$

Therefore, T is additive.

Suppose $a \in \mathbb{R}$. Consider now $T(ap)$.

$$T(ap) = \big((ap)(0), (ap)(1), (ap)'(0), (ap)'(1)\big) \tag{12}$$

Since p is a function, $(ap)(x) = a(p(x))$ for all $a, x \in \mathbb{R}$.

Similarly, since a is a real constant, $(ap)' = a(p')$.

Thus,

$$\begin{aligned} T(ap) &= \big(a(p(0)), a(p(1)), a(p'(0)), a(p'(1))\big) && (13)\\ &= a\big(p(0), p(1), p'(0), p'(1)\big) && (14)\\ &= aT(p) && (15) \end{aligned}$$

Therefore, T is homogeneous.

Hence, T is linear. □

Claim. T is invertible.

Proof. Note that T is invertible if and only if T is injective and surjective.

We first show that T is injective.

Suppose p and q are given, with $p, q \in \mathbb{P}_3(\mathbb{R})$.

Suppose also $T(p) = T(q)$. Therefore,

$$(p(0), p(1), p'(0), p'(1)) = (q(0), q(1), q'(0), q'(1)).$$

Let $a_0, a_1, a_2, a_3 \in \mathbb{R}$ be such that $p = a_3x^3 + a_2x^2 + a_1x + a_0$.

Therefore, $p'(x) = 3a_3x^2 + 2a_2x + a_1$.

Let $b_0, b_1, b_2, b_3 \in \mathbb{R}$ be such that $q = b_3x^3 + b_2x^2 + b_1x + b_0$.

Therefore, $q'(x) = 3b_3x^2 + 2b_2x + b_1$.

Since $p(0) = q(0)$, then $a_0 = b_0$.

Since $p(1) = q(1)$ and $p(0) = q(0)$, then $a_3 + a_2 + a_1 = b_3 + b_2 + b_1$.

Since $p'(0) = q'(0)$, then $a_1 = b_1$, and hence from above $a_3 + a_2 = b_3 + b_2$.

Since $p'(1) = q'(1)$, then $3a_3 + 2a_2 + a_1 = 3b_3 + 2b_2 + b_1$.

Since $a_1 = b_1$ from above, then $3a_3 + 2a_2 = 3b_3 + 2a_2$.

But also $a_3 + a_2 = b_3 + b_2$ from above, and hence $a_3 = b_3$.

Therefore, $p = q$, and T is injective.

Secondly, we prove that T is surjective.

Suppose that $t = (c_0, c_1, c_2, c_3)$ is given, with $t \in \mathbb{R}^4$.

Consider a polynomial with the following properties:

1. $p(0) = c_0$

2. $p(1) = c_1$
3. $p'(0) = c_2$
4. $p'(1) = c_3$

We claim that

$$p = (2c_0 - 2c_1 + c_2 + c_3)x^3 + (-3c_0 + 3c_1 - 2c_2 - c_3)x^2 + c_2 x + c_0$$

is a polynomial satisfying this properties.

Note that p satifies the property 2 and 3, since the constant term and the coefficient before x are c_0 and c_2 respectively, as required.

Note also that $p(1) = (2 - 3 + 1)c_0 + (-2 + 3)c_1 + (1 - 2 + 1)c_2 + (1 - 1)c_3 = c_1$, as required by the property 2.

Moreover, $p'(1) = 3(2c_0 - 2c_1 + c_2 + c_3) + 2(-3c_0 + 3c_1 - 2c_2 - c_3) + c_2$.

Thus, $p'(1) = (6 - 6)c_0 + (-6 + 6)c_1 + (3 - 4 + 1)c_2 + (3 - 2)c_3 = c_3$, as required by the property 3.

Since $p \in \mathbb{P}_3(\mathbb{R})$ and p satisfies the properties above, then for each t in $\mathbb{R}^4$ there exists p so that $T(p) = t$. Thus, T is surjective.

Since T is injective and surjective, it is invertible. $\square$

Since T is an invertible linear map, it is an isomorphism.