Problem. Find all possible rational canonical forms with characteristic polynomial $t^6 - 1$ over $\mathbb{F} = \mathbb{Z}^2$, up to similarity.

Solution.

Note that $t^6 - 1 = (t^3 - 1)(t^3 + 1) = (t - 1)^2(t^2 + t + 1)^2 = (t + 1)^2(t^2 + t + 1)^2$.

Let $\phi_1 = t + 1$ and $\phi_2 = t^2 + t + 1$. Note that ϕ_1 and ϕ_2 are monic irreducible.

Note that $\phi_1^2 = t^2 + 1$ and $\phi_2^2 = t^4 + t^2 + 1$.

Since the characteristic polynomial and minimal polynomial have the same zeroes, there are four possibilities for a minimal polynomial $p(t)$:

1. $\phi_1\phi_2$
2. $\phi_1^2\phi_2$
3. $\phi_1\phi_2^2$
4. $\phi_1^2\phi_2^2$

Hence, there are four possible rational canonical forms built from the following companion matrices:

$$A(\phi_1) = \begin{pmatrix} 1 \end{pmatrix},\ A(\phi_1^2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

$$A(\phi_2) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\ A(\phi_2^2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2}$$

which are thus

$$1. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} 2. \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{3}$$

$$3. \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} 4. \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{4}$$

□

Problem.

Show that any two matrices in $M_{6\times 6}(\mathbb{R})$ with the characteristic polynomial $f(t) = t(t^2 + 1)(t^2 + 2t + 5)(t + 1)$ are similar to each other.

Solution.

Let $\phi_1(t) = t$, $\phi_2(t) = t^2 + 1$, $\phi_3(t) = t^2 + 2t + 5$, $\phi_4 = t + 1$.

Note that the discriminant of ϕ_3 is -21, and thus ϕ_3 is monic irreducible. Similarly, since the same argument applies to ϕ_2, while t is 0 if and only if $t = 0$ and $t + 1 = 0$ if and only if $t = -1$, which are not the roots of any other ϕ_i, all ϕ_i for $i \in [1, 4] \cap \mathbb{N}$ are monic irreducible.

Note that, since there exists a rational canonical basis which is a union of disjoint unions of K_{ϕ_i} for $i \in [1, 4] \cap \mathbb{N}$, we know that $V = \bigoplus_{i=1}^{4} K_{\phi_i}$.

Let A, B be arbitrary matrices in $M_{6\times 6}(\mathbb{R})$ such that their characteristic polynomial is $f(t)$.

Note that for A and B we have $\dim K_{\phi_1} = 1 = \dim K_{\phi_4}$ and $\dim K_{\phi_2} = 2 = \dim K_{\phi_3}$ by Theorem 7.23.

Since minimal polynomials $p(t)$ of A and $q(t)$ of B have ϕ_i for $i \in [1, 4] \cap \mathbb{N}$ as monic irreducible factors, while their multiplicity is 1 in the characteristic polynomial, then $p(t) = f(t) = q(t)$.

Since the dimension of each K_{ϕ_i} for $i \in [1, 4] \cap \mathbb{N}$ is 1, there is only one dot in the corresponding dot diagram, and thus the dot diagrams for each K_{ϕ_i} of A and B are the same. Denote this rational canonical form as R:

$$R = \begin{pmatrix} 0 & & & & & \\ & 0 & -1 & & & \\ & 1 & 0 & & & \\ & & & 0 & -5 & \\ & & & 1 & -2 & \\ & & & & & -1 \end{pmatrix} \tag{5}$$

From the discussion above, $A \sim R$ and $B \sim R$. Since similarity of matrices induces an equivalence relation, we have $A \sim B$, as required. $\square$