1 Convergence of Series

Consider $\sum_{n=1}^{\infty} a_n(x-a)^n$.

Suppose the series converges at $x = x_0$.

The terms $a_n(x_0 - a)^n$ must go to 0.

So there exists $M \in \mathbb{R}$ such that $\left|a_n(x_0 - a)^n\right| < M$ for all $n \in \mathbb{N}$.

Consider a pont r such that $|r - a| < |x_0 - a|$:

$$\sum_{i=1}^{\infty} a_n(r-a)^n = \sum_{i=1}^{\infty} a_n(x_0-a)^n \cdot (\frac{r-a}{x_0-a})^n. \tag{1}$$

Then $\left|a_n(r-a)^n\right| = \left|a_n(x_0-a)^n(\frac{r-a}{x_0-a})^n\right| \leq M\left|\frac{r-a}{x_0-a}\right|^n$.

Thus, $\sum_{i=1}^{\infty} a_n(r-a)^n$ converges absolutely.

Now consider the series as a function:

$$f_n(x) = \sum_{n=0}^{n} a_n(x-a)^n.$$

As n goes to infinity, what is its derivative?

Note that $f_n'(x) = \sum_{n=0}^{\infty} na_n(x-a)^{n-1}$.

Then the ratio test says that $\frac{f_{n+1}'(x)}{f_n'(x)} = \frac{n+1}{n}\frac{a_{n+1}}{a_n}(x-a)$.

Suppose $f_n(x) \to f(x)$ for all $x \in [a,b]$. Suppose that $f_n(x)$ is continuous for all $n \in \mathbb{N}$.

On $[0,1]$, as $n \to \infty$, x^n tends to zero for $x < 1$ and to one for $x = 1$.

Example 1.1

Let $f_n(x)$ be such that $f_n(x) = \begin{cases} n \leq x \leq n+1 \\ \text{otherwise} \end{cases}$.

Suppose now that $\lim_{n\to\infty} g_n(x) = 0$ for all $x \in \mathbb{R}$ such that

$$\frac{1}{2} = \lim_{n\to\infty} \int_0^1 g_n(x)\,\mathrm{d}x + \int_0^1 \lim_{n=1}^{\infty} g_n = 0$$

Definition 1.2. A sequence of functions f_n *converges uniformly* to $f(x)$ if, given $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $\left|f_n(x) - f(x)\right| < \epsilon$ for any $n > N$ and for any $x \in D(f)$.

Remark 1.3. We can also define postive convergence, if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in D(f)$.

Theorem 1.4

Suppose that f_n and f are integrable.

If $f_n \to f$ uniformly on $[a,b]$, then $\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x$.

Proof.

Given $\epsilon > 0$, we can find N such that $\left| f_n(x) - f(x) < \frac{\epsilon}{b-a} \right|$ for all $x \in [a, b]$ and for all $n > N$.

Then $\left| \int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x \right| \leq \int_a^b \left| f_n(x) - f(x) \right| \mathrm{d}x < \frac{\epsilon}{b-a} \int_a^b \mathrm{d}x = \epsilon.$ $\square$