Claim. Suppose that $M, N \in M_{n\times n}(\mathbb{C})$ and N is invertible. Then there exists $a \in \mathbb{C}$ such that $M + aN$ is not invertible.

Lemma 0.1

Suppose V is a finite-dimensional non-zero vector space over $\mathbb{C}$ and $T \in \operatorname{Hom}(V, V)$. Then T has an eigenvalue.

Proof of Lemma 0.1. From the Fundamental Theorem of Algebra it follows that the characteristic polynomial $f(t) = \det(T - tI)$ splits. Therefore there exists at least one eigenvalue. $\square$

Lemma 0.2

Suppose $T \in \operatorname{Hom}(V, V)$ and $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis of a vector space V. Then $[T]_\beta$ is upper-triangular if and only if $Tv_j \in \operatorname{span}(v_1, v_2, \dots, v_j)$ for each $j = 1, \dots, n$.

Proof of Lemma 0.2. Suppose first that $M = [T]_\beta$ is upper-triangular. Evaluating Tv_j, we obtain that $Tv_j = \sum_{i=1}^{j} M_{ij} \in \operatorname{span}(v_1, v_2, \dots, v_j)$.

Conversely, if $Tv_j = \sum_{i=1}^{j} M_{ij} \in \operatorname{span}(v_1, v_2, \dots, v_j)$, then by definition M is upper-triangular. $\square$

Lemma 0.3

Suppose V is a finite-dimensional vector space over $\mathbb{C}$ and $T \in \operatorname{Hom}(V, V)$. Then there exists an ordered basis of V such that $[T]_\beta$ is upper-triangular.

Proof of Lemma 0.3. Let $n = \dim V$. We proceed by induction on n.

Note that the lemma holds trivially in case $n = 1$.

Suppose now that $k > 1$ and the lemma holds for all dimensions less than k. By Lemma 0.1, there exists an eigenvalue λ. Let

$$U = \operatorname{range}(T - \lambda I).$$

Note that $T - \lambda I$ is not injective, and thus not surjective, making $\dim U < \dim V$. Note also that U is invariant under T, which can be seen as follows. Suppose $u \in U$, and thus

$$Tu = (T - \lambda I)u + \lambda u$$

Since $(T - \lambda I)u \in U$ and also $u \in U$, it follows that $Tu \in U$. Therefore, U is invariant under T. Note that a restriction of T on U, denoted as $T|_U$ is thus an operator, i.e $T|_U \in \operatorname{Hom}(V, V)$, which, by inductive hypothesis, means that there exists a basis $\gamma = \{u_1, u_2, \dots, u_m\}$ such that $[T|_U]_\gamma$ is upper-triangular. By Lemma 0.2, for each j we have

$$Tu_j = (T|_U)v_j \in \operatorname{span}\gamma.$$

Extend γ to a basis of V, so that $\beta = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_k\}$.

For each k, $Tv_k = (T - \lambda I)v_k + \lambda v_k$. By definition, $(T - \lambda I)v_k \in U$, while $\lambda v_k \in \operatorname{span}(\beta)$, and thus $Tv_k \in \operatorname{span}(\beta)$.

Therefore, using Lemma 0.2, T has an upper-triangular matrix representation.

Thus, the Lemma holds for all $k \in \mathbb{N}$ by induction. □

Lemma 0.4

Suppose $T \in \operatorname{Hom}(V, V)$ has an upper-triangular matrix representation for some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of this matrix.

Proof of Lemma 0.4. Suppose that β is a basis of V such that $M = [T]_\beta$ is upper-triangular:

$$M = \begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Therefore,

$$M - \lambda I = \begin{pmatrix} \lambda_1 - \lambda & & & * \\ & \lambda_2 - \lambda & & \\ & & \ddots & \\ 0 & & & \lambda_n - \lambda \end{pmatrix}$$

$\det(M - \lambda I) = 0$ if and only if some diagonal entry is equal to an eigenvalue. Since there are n entries on the diagonal, all the eigenvalues must be there as well. □

Proof of the Claim. Consider $\det(M + aN)$.

From Lemmas 0.3 and 0.4, there exists a basis β for which N is upper-triangular with all the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ on the diagonal. Note also that 0 is not one of the eigenvalues, since N would not be invertible otherwise.

Suppose first that N is diagonalizable. Therefore,

$$[M + aN]_\beta = [M]_\beta + a[N]_\beta = \begin{pmatrix} M_{11} & \dots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \dots & M_{nn} \end{pmatrix} + a \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix}$$

Denote the ith column of $[M]_\beta$ as m_i and the jth column of $[N]_\beta$ as l_j.

Therefore,

$$\begin{aligned} \det[M + aN]_\beta &= \det(m_1 + al_1, m_2 + al_2, \dots, m_n + al_n) \\ &= \det(m_1, m_2 + al_2, \dots, m_n + al_n) + a\det(l_1, m_2 + al_2, \dots, m_n + al_n) \end{aligned}$$

By expanding $a\det(l_1, m_2 + al_2, \ldots, m_n + al_n)$ along the first column we obtain

$$\begin{aligned}\det[M + aN]_\beta &= \det(m_1 + al_1, m_2 + al_2, \ldots, m_n + al_n)\\ &= \det(m_1, m_2 + al_2, \ldots, m_n + al_n) + a\lambda_1 \det \widetilde{A_{11}}\end{aligned}$$

where $A_{11} = (l_1, m_2 + al_2, \ldots, m_n + al_n)$. Thus, $\widetilde{A_{11}} = (\widetilde{M + aN})_{11}$.

Repeating the procedure, first we use multilinearity for the kth of det and then apply the Laplacian expansion to the kth column of the second term for all k in $[2, n] \cap \mathbb{N}$:

$$\det[M + aN]_\beta = \det(m_1, m_2, \ldots, m_n) + a\sum_{i=1}^{n} \lambda_i \det \widetilde{A_{ii}},$$

where $A_{ii} = (m_1, \ldots, m_{i-1}, l_i, m_{i+1} + al_{i+1} \ldots, m_n + al_n)$.

Note that for $i = n$, $A_{nn} = (m_1, m_2, \ldots, m_{n-1}, l_n)$, and thus $\widetilde{A}_{nn} = \widetilde{M}_{nn}$.

Note also that, by multilinearity again,

$$\det \widetilde{A_{11}} = \det(m_2, m_3, \ldots, m_n) + a\sum_{i=2}^{n} \lambda_i \det \widetilde{B_{ii}},$$

where B_{ii} is a matrix such that

$$B_{ii} = (m_2, \ldots, m_{i-1}, l_i, m_{i+1} + l_{i+1}, \ldots, m_n + al_n).$$

Similarly,

$$\det \widetilde{A_{ii}} = \det(m_1, m_2, \ldots, m_{i-1}, m_{i+1}, m_n) + a\sum_{j=i+1}^{n} \lambda_j \det \widetilde{B_{jj}},$$

where B_{jj} is a matrix such that

$$B_{jj} = (m_{i+1}, \ldots, m_{j-1}, l_j, m_{j+1} + l_{j+1}, \ldots, m_n + al_n).$$

In turn, $\det(\widetilde{B_{jj}})$ can be defined similarly. Let's call A_{ii}, B_{jj} and similarly defined matrix coefficients as *plaques*. Call a determinant of a permutation of m_i corresponding to each plaque as a *wall*. Let's also call cofactor of each plaque as a *fat*. Observe that in each iteration the dimension of a newly added plaque decreases, since a fat of the previous iteration is a plaque of the current iteration. Each fat has a product of a and some eigenvalue of $[N]_\beta$ as the coefficient before it. Note that the terminal plaque is thus equal to (m_n) with the corresponding coefficient of $a\lambda_n$.

By definition of a plaque, there are n plaques in total. Before each plaque there is a factor of a. Since each wall is a well-defined complex number, $\det[M + aN]_\beta$ is a polynomial in a of degree n. Therefore, by the Fundamental Theorem of Algebra, there exists at least one $a \in \mathbb{C}$ such that $\det[M + aN]_\beta$ is equal to zero.

In case of the non-diagonalizable matrix N, the argument is similar, since by a similar procedure of *expanding* the determinant of a sum of the matrices still gives a polynomial in a of degree not greater than n, for which a root is guaranteed by the Fundamental Theorem of Algebra. $\square$

Problem. Find non-zero 2×2 matrices M, N over $\mathbb{C}$ such that $M + aN$ is invertible for all $a \in \mathbb{C}$.

Solution. By the Claim above, if $M + aN$ is invertible for all $a \in \mathbb{C}$, then N is not invertible.

Take $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\det(M + aN) = \det \begin{pmatrix} 1 & 1+a \\ 0 & 1 \end{pmatrix} = 1 \neq 0$. $\square$