Problem. Prove $q(n)$ is true for all even natural numbers.

Solution. Let $p(k) = q(2k)$.

$\forall k \in \mathbb{N}.p(k)$

means the same as

$\forall k \in \mathbb{N}.q(2k)$, which is the same as

$\forall n \in \mathbb{N}.(n$ is even IMPLIES $q(n)$.

Base Case:

$p(0) = q(0)$

Induction Step:

$p(k)$ IMPLIES $p(k+1)$,

which is the same as

$q(2k)$ IMPLIES $q(2k+2)$.

It is sufficient to prove

$q(0)$ AND $\forall n \in \mathbb{N}(q(n)$ IMPLIES $q(n+2))$. □

Theorem 0.1

For all $n \in \mathbb{Z}^+$ and all $a_1, \ldots, a_n \in \mathbb{R}^+$,

$$(\prod_{i=1}^{n} a_i)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} a_i}{n}$$

Proof. We prove $\forall n \in \mathbb{Z}^+.P(n)$.

Base Case:

$n = 2$

Let $a_1, a_2 \in \mathbb{R}^+$ be arbitrary.

Then $0 \leq (a_1 - a_2)^2 = a_1^2 - 2a_1a_2 + a_2^2$.

Hence, $a_1^2 + a_2^2 \geq 2a_1a_2$.

Thus,

$$(\frac{a_1 + a_2}{2})^2 = \frac{a_1^2 + a_2^2 + 2a_1a_2}{4} \geq a_1a_2$$

Hence, $P(2)$ is true by generalisation.

Induction Step:

Let $n \in \mathbb{Z}^+$ be arbitrary and suppose $n \geq 2$.

Assume $P(n)$.

Let $a_1, \ldots, a_{n-1} \in \mathbb{R}^+$ be arbitrary.

Let $b_i = a_i$ for $i = 1, \ldots, n-1$.

Let $b_n = \frac{a_1+a_2+\cdots+a_{n-1}}{n-1}$.

By specialisation of $p(n)$,

$$\begin{aligned}
b_1 \cdots b_{n-1} b_n \leq (\frac{b_1 + \cdots + b_n}{n})^n &= (\frac{b_1 + \cdots + b_n}{n})^n && (1)\\
&= (\frac{a_1 + \cdots + a_{n-1} + b_n}{n})^n && (2)\\
&= (\frac{(n-1)b_n + b_n}{n})^n && (3)\\
&= b_n^n && (4)
\end{aligned}$$

Therefore, $b_1 b_2 \cdots b_{n-1} \leq b_n^{n-1}$.

Hence, $P(n-1)$ is true by generalisation.

Let $a_1, \ldots, a_n \in \mathbb{R}^+$ be arbitrary.

Let $b_1 = \frac{a_1 + \cdots + a_n}{n}$ and $b_2 = \frac{a_{n+1} + \cdots + a_{2n}}{n}$.

By specialisation of $P(n)$,

$$\prod_{i=1}^{n} a_i \leq (\frac{1}{n} \sum_{i=1}^{n} a_i)^n$$

and

$$\prod_{i=n+1}^{2n} a_i \leq (\frac{1}{n} \sum_{i=n}^{2n} a_i)^n$$

and by specialisation of $P(2)$,

$$b_1 b_2 \leq (\frac{b_1 + b_2}{2})^2$$

Hence

$$\prod_{i=1}^{2n} a_i \leq (\frac{\sum_{i=1}^{n} a_i}{n})(\frac{\sum_{i=n+1}^{2n} a_i}{n})^n = (b_1 b_2)^n \leq (\frac{b_1 + b_2}{n})^{2n}.$$

Note that $(\frac{b_1+b_2}{n})^{2n} = (\frac{1}{2n} \sum_{i=1}^{2n} a_i)^{2n}$.

By generalisation, $P(2n)$ is true.

$\forall n \in \mathbb{N}[(n \geq 2 \text{ AND } P(n)) \text{ IMPLIES } P(2n)]$.

Therefore, by induction,

$\forall n \in \mathbb{Z}^+ . P(n)$ □

0.1 Induction in Finite Sets

Problem. Prove $\forall i \in \{0, \ldots, n\}.P(i)$.

Solution. Base Case:

$p(0)$

Induction Step:

Let $i \in \{0, \ldots, n-1\}$ be arbitrary.

Assume $p(i)$.

$\vdots$

$p(i+1)$.

$\forall i \in \{0, \ldots, n-1\}.[p(i) \text{ IMPLIES } p(i+1)]$

$\forall i \in \{0, \ldots, n\} p(i)$ by induction

□

0.2 Strong Induction

To prove $\forall i \in \mathbb{N}.p(i)$ it suffices to prove that

$$\forall i \in \mathbb{N}.\forall j \in \mathbb{N}.[\big((j < i) \text{ IMPLIES } p(j)\big) \text{ IMPLIES } P(i)]$$

The only difference of the strong induction from the weak induction is $p(0), \ldots, p(i-1)$.

A template proof follows.

Proof. Let $i \in \mathbb{N}$ be arbitrary.

Assume $\forall j \in \mathbb{N}.(j < i \text{ IMPLIES } p(i)$.

$\ldots$ *various cases, including the base case* $\ldots$

$p(i)$

$\forall i \in \mathbb{N}[\forall j \in \mathbb{N}.(j < i) \text{ IMPLIES } p(j)] \text{ IMPLIES } p(i)$ by direct proof and generalization.

$\forall i \in \mathbb{N}.p(i)$ by strong induction □

Theorem 0.2

For all $n \geq 4$, exactly a sum of n can be exchanged in coins with nomination 2 and 5\$ bills.

Proof. Let $p(n) = \exists f \in \mathbb{N}.\exists g \in \mathbb{N}.(n = 2f + 5g)$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be arbitrary.

Suppose $n \geq 4$ and $\forall j \in \mathbb{N}.(4 \leq j < n \text{ IMPLIES } p(j))$.

If $n = 4$, then $n = 2 \cdot 2 + 0 \cdot 5$.

If $n = 5$, then $n = 0 \cdot 2 + 1 \cdot 5$.

If $n \leq 6$, then $4 \leq n - 2 < n$. Then $P(n-2)$ is true by specialisation. □