1 Sandpile Model and Divisors in Graphs IV

1.1 Revision

Last time we contsructed the Kreitz element β.

We also wanted to show that ϕ is revertible if and only if $(\phi+\beta)^0 = \phi$. For the direction to the right, note that $(\phi+\beta)^0$ is also revertible. Since we know that each equivalence class has only one revertible state, then we know that $(\phi+\beta)^0 = \phi+0 = \phi$. In the other direction, we just need to notice that $(\phi+k\beta)^0 = \phi$ for sufficiently big k.

The other exercise from the last time was to compute the unity for a n by m rectangle such that $m >> n$. Let $\phi = (n^2+n)\beta$. Applying ΔF, where $F(i,k) = (n-k)^2 + (n-k)$ to ϕ until we get ψ such that the middle stripe is filled with 2's. Now, let $G(i,k) = \frac{([\sqrt{2}n]-i)([\sqrt{2}n]-i-9)}{2}$, for $i < [\sqrt{2}n]$, and $G = 0$ otherwise. Applying ΔG after ΔF, the middle stripe shrinks.

1.2 Concentrating Sand in a Point

Suppose that we have n grains in one point, so that $\phi = n\delta_{0,0}$.

We can show that the convex hull of all the points with the non-zero number of grains lies inside a circle of radius $\sqrt{n}$.

Note that $\phi^0 = \phi + \Delta F$, where F is the minimal function, and $\Delta F(0,0) \le 3-n$, and $0 \le \Delta F(i,j) \le 3$.

Lemma 1.1

F decreases in the directions $(2,0),(0,2)$ and $(1,-1)$.

Exercise 1.2. Suppose ϕ has been obtained as a result of relaxation such that in each vertex of D there was a toppling. Then $\sum_{v\in D}\phi(v)$ is less than or equal to the number of inner edges in D.

1.3 Rescalings

Assume that $\phi_n^0 = (nS_{(0,0)})^0$ is contained inside $\Gamma_n = \{\frac{i}{\sqrt{n}}, \frac{j}{\sqrt{n}}, i,j\in\mathbb{Z}\}$. Therefore, all ϕ_n^0 are contained in the square with the vertices $(1,-1),(1,1),(-1,1),(-1,-1)$. Thus, all $\frac{F_n}{n}$ are inside the same square.

1.4 Drawing 1D pictures

Suppose that there exist two limits. Therefore, there exist $F_2'' > k > F_1''$, which means that there exists $G(x,y) = [ax^2+bxy+cy^2+dx+ey+w]$, with $G'' = k$, $0 \le D \le 3$.

The interesting fact is that we can use Appolonius carpet of kissing circles to find G.