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    - presentation style: quantifiers first and assertions last
    - quantifier order is crucial

- Operators
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1.3 Cuts

> **Theorem 1.1**
> $\sqrt{2}$ is irrational.

Definition 1.2. A **cut** in $\mathbb{Q}$ is a pair of subsets of A, B of $\mathbb{Q}$ such that:

(a) $A \cup B = \mathbb{Q}, A \neq \emptyset, B \neq \emptyset, A \cap B = \emptyset$.

(b) If $a \in A$ and $b \in B$, then $a < b$.

(c) *A* contains no largest element.

Definition 1.3. A **real number** is a cut in $\mathbb{Q}$.

Definition 1.4. The cut $x = A|B$ is **less than or equal** to the cut $y = C|D$, if $A \subset C$.

Definition 1.5. $M \in \mathbb{R}$ is an **upper bound** for a set $S \subset \mathbb{R}$ if each $s \in S$ satisfies $s \leq M$. Thus, we say the set S is **bounded above** by M.

An upper bound for S that is less than all other upper bounds for S is a **least upper bound** for S.

Theorem 1.6

The set $\mathbb{R}$, constructed by the means of Dedekind cuts, is **complete**:

If S is a non-empty subset of $\mathbb{R}$ and is bounded above, then in $\mathbb{R}$ there exists a least upper bound for S.

Theorem 1.7

The set $\mathbb{R}$ of all cuts in $\mathbb{Q}$ is a complete ordered field that contains $\mathbb{Q}$ as an ordered subfield.

Theorem 1.8 (Triangle Inequality)

$\forall (x, y \in \mathbb{R}) : |a + b| \leq |a| + |b|$

Definition 1.9. Let $a_1, a_2, a_3, \cdots = (a_n)$, $n \in \mathbb{N}$, be a sequence of real numbers.

The sequence (a_n) **converges to a limit** $b \in \mathbb{R}$ as $n \to \infty$ provided that for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$|a_n - b| < \epsilon$$

Definition 1.10 (Cauchy Condition). $\forall (\epsilon < 0) \exists (N \in \mathbb{N}) : n, m \geq N \Rightarrow |a_n - a_m| < \epsilon.$

Theorem 1.11

$\mathbb{R}$ is **complete** with respect to Cauchy sequences in the sense that if (a_n) is a sequence of real numbers obeying a Cauchy condition, then it converges to a limit in $\mathbb{R}$.

Theorem 1.12 (Cauchy Convergence Criterion for sequences)

A sequence (a_n) in $\mathbb{R}$ converges if and only if

$$\forall (\epsilon > 0) \exists (N \in \mathbb{N}) : n, m \geq N \Rightarrow |a_n - a_m| < \epsilon$$

Definition 1.13. Let $a < b$ be given in $\mathbb{R}$. Define the **intervals** (a, b) and $[a, b]$ as follows:

$$(a, b) = \{x \in \mathbb{R} : a < x < b\} \tag{1}$$
$$[a, b] = \{x \in \mathbb{R} : a \le x \le b\} \tag{2}$$

Theorem 1.14

Every interval (a, b) contains both rational and irrational numbers.

Lemma 1.15

$\mathbb{R}$ has the **Archimedean property**: for each $x \in \mathbb{R}$ there is an integer n that is greater than x.

Theorem 1.16 (ϵ-principle)

If a, b are real numbers and if, for each $\epsilon > 0$, $a \le b + \epsilon$, then $a \le b$.
If x, y are real numbers and, for each $\epsilon > 0$, $|x - y| \le \epsilon$, then $x = y$.

1.4 Euclidean Space

Definition 1.17. Given sets A and B, the **Cartesian product** of A and B is the set $A \times B$ of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.
The Cartesian product of $\mathbb{R}$ with itself m times is denoted as $\mathbb{R}^m$.

Definition 1.18. The **dot product** of $x = (x_1, x_2, \ldots, x_m), y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$ is defined as

$$\langle x, y \rangle = \sum_{i=1}^{m} x_i y_i$$

Lemma 1.19

The dot product operation is bilinear, symmetric, and positive definite, i.e., for any $x, y, z \in \mathbb{R}^m$ and any $c \in \mathbb{R}$,

$$\langle x, y + cz \rangle = \langle x, y \rangle + c \langle x, z \rangle \tag{3}$$
$$\langle x, y \rangle = \langle y, x \rangle \tag{4}$$
$$\langle x, x \rangle \ge 0, \text{ and } \langle x, x \rangle = 0 \Leftrightarrow x = \mathbf{0} \tag{5}$$

Definition 1.20. The **length** or **magnitude** of a vector $x \in \mathbb{R}^m$ is defined to be

$$|x| = \sqrt{\langle x, x \rangle}$$

> **Theorem 1.21** (Cauchy-Schwarz Inequality)
>
> For all $x, y \in \mathbb{R}^m, \langle x, y \rangle \leq |x| \, |y|$.

> **Corollary 1.22**
>
> For all $x, y \in \mathbb{R}^m$,
> $$|x + y| \leq |x| + |y|$$

Definition 1.23. The **Euclidean distance** between $x, y \in \mathbb{R}^m$ is defined as the length of their difference.
$$|x - y| = \sqrt{\langle x - y, x - y \rangle}$$

Definition 1.24. The jth coordinate of the point $(x_1, \ldots, x_m)$ is the number x_j appearing in the jth position.

Definition 1.25. The jth coordinate axis is the set of $x \in \mathbb{R}^m$ which kth coordinates are zero for all $k \neq j$.

Definition 1.26. The **integer lattice** is the set $\mathbb{Z}^m \subset \mathbb{R}^m$ of ordered m-tuples of integers.

Definition 1.27. The **first orthant** of $\mathbb{R}^m$ is the set of points $x \in \mathbb{R}^m$ all of which coordinates are nonnegative.

Definition 1.28. A **box** is a Cartesian product of intervals in $\mathbb{R}^m$
$$[a_1, b_1] \times \cdots \times [a_m, b_m]$$

Definition 1.29. The **unit cube** in $\mathbb{R}^m$ is the box $[0, 1]^m = [0, 1] \times \cdots \times [0, 1]$.

Definition 1.30. The **unit ball** in $\mathbb{R}^m$ is the set
$$B^m = \{x \in \mathbb{R}^m \,|\, |x| \leq 1\}.$$

Definition 1.31. The **unit sphere** in $\mathbb{R}^m$ is the set
$$S^{m-1} = \{x \in \mathbb{R}^m \,|\, |x| = 1\}.$$

2 Continuity

Definition 2.1. The function $f : [a, b] \to \mathbb{R}$ is **continuous** if for each $\epsilon > 0$ and each $x \in [a, b]$ there is a $\delta > 0$ such that
$$t \in [a, b] \text{ and } |t - x| < \delta \Rightarrow |f(t) - f(x)| < \epsilon$$

2.1 Three Hard Theorems

> **Theorem 2.2**
>
> If f is a continuous function on $[a,b]$ and $f(a) < 0 < f(b)$, then its values form a bounded subset of $\mathbb{R}$. Thus, there exist $m, M \in \mathbb{R}$ such that for all $x \in [a,b]$, $m \le f(x) \le M$.

Proof. For $x \in [a,b]$, let

$$V_x = \{y \in \mathbb{R} \mid \exists (t \in [a,x]) : y = f(t)\}.$$

Set

$$X = \{x \in [a,b] \mid V_x \text{ is a bounded set of } \mathbb{R}\}.$$

We prove now that b is in X.

Since $a \in X$, X is not empty. Note also that b is an upper bound for X. Thus, there exists in $\mathbb{R}$ a least upper bound $c \le b$ for X.

Since f is continuous, consder the neighbourhood of c for $\epsilon = 1$. By definition of continuity, there exists a $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < 1$ Since c is the least upper bound for X, there is some $x \in X$ in the interval $[c - \delta, c]$ (otherwise $c - \delta$ is a smaller upper bound for X).

With t varying from a to c, t is first mapped to $f(t) \in V_x$, and then $f(t)$ varies in the bounded set $J = (f(c) - 1, f(c) + 1)$.

The union of two bounded set is a bounded set, and thus V_c is bounded. Therefore, $c \in X$.

If $c < b$, then by continuity, for some $t > c$, $f(t)$ still varies in the bounded set J, which contradicts the fact that that c is an upper bound for X. Thus, $c = b$, $b \in X$, and the values of f form a bounded subset of $\mathbb{R}$. $\square$

> **Theorem 2.3**
>
> If f is a continuous function on $[a,b]$, then there exist some numbers x_0, x_1 in $[a,b]$ such that $f(x_0) \le f(x) \le f(x_1)$ for all x in $[a,b]$.

Proof. Let $M = \sup\{f(t) \mid t \in [a,b]\}$. By theorem 2.2, M exists.

Let $X = \{x \in [a,b] \mid \sup\{V_x\} < M\}$, where

$$V_x = \{y \in \mathbb{R} \mid \exists (t \in [a,x]) : y = f(t)\}.$$

We first prove that f achieves a maximum on $[a,b]]$.

Case (1). $f(a) = M$

Thus, f takes on a maximum at a.

Case (2). $f(a) < M$

Thus, X is not empty and $\sup\{X\}$ exists. Suppose $\sup\{X\} = c$.

If $f(c) < M$, choose $\epsilon > 0$ such that $\epsilon < M - f(c)$. By continuity, there exists a $\delta > 0$ such that $|t - c| < \delta$ implies $|f(t) - f(c)| < \epsilon$. Thus, $\sup\{V_c\} < M$.

If $c < b$, then there exists a point $t > c$) at which $\sup\{(V_c\} < M$, which contradicts the fact that c is an upper bound of such points.

Thus, $c = b$, and hence $M < M$, which is a contradiction. Therefore, $f(c) = M$, so f achieves a maximum at c.

$\square$

> **Theorem 2.4**
>
> A continuous function defined on an interval $[a, b]$ achieves all intermediate values: if $f(a) = \alpha, f(b) = \beta$, and γ is given such that $\alpha \leq \gamma \leq \beta$ or $\beta \leq \gamma \leq \alpha$, then there exists some $c \in [a, b]$ such that $f(c) = \gamma$.