Lemma

For partitions Q, P such that $P \subset Q$

$$L(f,P) \leq L(f,Q)$$
$$U(f,P) \geq U(f,Q)$$

Proof. Consider a special case when Q have just one extra point u in addition to all the points in P ($t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n$).

Note. The inf on $[t_k, u]$ and $[u, t_{k+1}]$ may be larger than the inf on $[t_k, t_{k+1}]$.
The sup on $[t_k, u]$ and $[u, t_{k+1}]$ may be less than the sup on $[t_k, t_{k+1}]$.

$$L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \tag{1}$$
$$L(f,Q) = \sum_{i=1}^{k-1} m_i(t_i - t_{i-1}) + m'(u - t_{k-1}) + m''(t_k - u) + \sum_{i=k+1}^{n} m_i(t_i - t_{i-1}), \tag{2}$$

where $m' = \inf\{f(x) \,|\, t_{k-1} \leq x \leq u\}$ and $m'' = \inf\{f(x) \,|\, u \leq x \leq t_k\}$.

By the remark given above, $m_k \leq m'$ and $m_k \leq m''$, and thus

$$m_k(t_k - t_{k-1}) = m_k(t_k - u) + m_k(u - t_{k-1}) \leq m'(t_k - u) + m''(u - t_{k-1}),$$

which shows that $L(f,P) \leq L(f,Q)$.

The argument for $U(f,P) \geq U(f,Q)$ is similar.

The general case is deduced by considering a sequence of partitions differing only by one point:

$$P = P_1, P_2, \ldots, P_\alpha = Q$$

Then by the lemma $L(f,P_1) \leq L(f,P_2) \leq \cdots \leq L(f,P_\alpha) = L(f,Q)$ and $U(f,P_1) \geq U(f,P_2) \geq \cdots \geq U(f,P_\alpha) = U(f,Q)$. $\square$

Theorem 0.1

Let P_1 and P_2 be partitions of $[a,b]$, and let the function f be bounded on $[a,b]$. Then

$$L(f,P_1) \leq U(f,P_2)$$

Proof. Consider a partition P which contains points both in P_1 and P_2. By the lemma above it follows that

$$L(f,P_1) \leq L(f,P) \leq U(f,P) \leq U(f,P_2).$$

$\square$

From Theorem 0.1 it follows that $\sup\{L(f,P)\} \leq \inf\{U(f,P)\}$.

Example 0.2

Let's find an example such that $\sup\{L(f,P)\} < \inf\{U(f,P)\}$. Consider the following function:

$$f(x) = \begin{cases} 0, & \text{x irrational} \\ 1, & \text{x rational} \end{cases}$$

For any partition $P = [t_0, t_1, \ldots, t_n]$, $m_i = 0$, since the irrational numbers are dense, and $M_i = 1$, since the rational numbers are dense.

Therefore,

$$L(f,P) = \sum_{i=1}^{n} 0(t_i - t_{i-1}) = 0 \tag{3}$$

$$U(f,P) = \sum_{i=1}^{n} 1(t_i - t_{i-1}) = b - a \tag{4}$$

We now can define the area $R(f,a,b)$ formally.

Definition 0.3. A function f which is bounded on $[a,b]$ is called **integrable** on $[a,b]$ if $\sup\{L(f,P) : \text{ P is a partition of } [a,b]\} = \inf\{U(f,P) : \text{ P is a partition of } [a,b]\}$.

In this case, the unique number $\sup\{L(f,P)\} = \inf\{U(f,P)\}$ is denoted as $\int_a^b f$.

Theorem 0.4

If f is bounded on $[a,b]$, then f is also integrable on $[a,b]$ if and only if for all $\epsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \epsilon$$

This restatement is convenient, as shown by the following example.

Example 0.5

Consider f defined on $[0,2]$ by

$$f = \begin{cases} 0, x \neq 1 \\ 1, x = 1 \end{cases}$$

Consider a partition $P = [t_0, t_1, \ldots, t_n]$ such that $t_{k-1} < 1 < t_k$.

Note that m_i and M_i are both zero for all intervals in P but for $[t_{k-1}, t_k]$, for which $m_k = 0$ and $M_k = 1$.

Therefore, $U(f,P) - L(f,P) = t_k - t_{k-1}$. Since a partition can be chosen such that $t_k - t_{k-1}$ is arbitrarily small and positive, it follows that f is integrable.

Since $L(f,P) \leq 0 \leq U(f,P)$ and the integral is unique, it is equal to zero.