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MAT 157 Problem Set II 02016/09/23

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Definition 1.1. $\forall x \in \mathbb{Q} : \overline{x} = \{y \in \mathbb{Q} : y < x\}$

Theorem 1.1. $\forall \alpha \in \mathbb{R} : \alpha = \{x \in \mathbb{Q} : \overline{x} < \alpha\}$

Proof. Consider the set $\alpha = \{x \in \mathbb{Q} : \overline{x} < \alpha\}$.

1. Let $x \in \overline{x}$. Therefore, $\exists y \in \overline{x}$ such that $y < x$ ($\because \overline{x} \in \mathbb{R}$). Since $\overline{x} < \alpha$, $\overline{x} \subset \alpha$, from $y \in \overline{x}$ it follows that $y \in \alpha$.
2. Since $\overline{x} \neq \emptyset$ and $\overline{x} \in \alpha$, then $\alpha \neq \emptyset$.
3. Since $\overline{x} \neq \mathbb{Q}$, $\forall (\overline{x} \subset \alpha)\ \exists\ (y \notin \overline{x}) \Rightarrow \alpha \neq \mathbb{Q}$.
4. Since $\overline{x} \in \mathbb{R}$, $\forall (x \in \overline{x})\ \exists\ y : y > x$. Since $\overline{x} \subset \alpha$, $y \in \alpha$. Therefore, there is no greatest element in α.

Thus, $\alpha \in \mathbb{R}$. Moreover, since each q such that $\forall q \in \mathbb{Q} : q < x$ is in $\overline{x}$ by definition, and since $\overline{x} \subset \alpha$ then $\exists y \in \alpha : y \notin \overline{x}$, therefore for some element u in α all the elements in $\overline{x}$ are less than u. Therefore, the *usual* definition is equivalent to the aforementioned. □

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Definition 2.1. The least element in the set A is denoted as $\min(A)$.

Lemma 2.1. *Suppose* $\alpha \in \mathbb{R}, z \in \mathbb{Q}, z > 0$. *Then* $\exists x \in \alpha, y \in \mathbb{Q}\backslash\alpha : y - x = z \wedge y \notin \min(\mathbb{Q}\backslash\alpha)$.

Proof. See notes of Professor Repka's lecture on September 22, 2016. □

Lemma 2.2. *If* $\alpha \in \mathbb{R} \wedge \beta \in \mathbb{R} \Rightarrow \alpha + \beta \in \mathbb{R}$.

Lemma 2.3. $\forall \alpha \in \mathbb{R} : -\alpha \in \mathbb{R}$.

Theorem 2.4. $\alpha \neq \beta, \alpha < \beta \Rightarrow \exists x \in \mathbb{Q} : \alpha < \overline{x} < \beta$

Proof. By Lemma 2.2 and 2.3, $\delta \in \mathbb{R}$.

Since $\alpha \subset \beta$, then $\exists y \in \beta : y \notin \alpha$. Hence, suppose $y \in \beta, x \in \alpha$ are such that $y - x > 0$. Since $y, x \in \mathbb{Q}$, then $\exists n \in \mathbb{N} : y - x > \frac{1}{n}$

Suppose that there is no rational number between α and β.

Therefore, by Archimedean Property of Rational Numbers,

$$\exists\ k \in \mathbb{N}\ \forall (z_\alpha \in \alpha \wedge z_\beta \in \beta) : (\frac{k-1}{n}) < z_\alpha \wedge \frac{k}{n} > z_\beta \tag{1}$$

But $\frac{k}{n} - \frac{k-1}{n} = \frac{1}{n} < y - x$, hence there is $y > x + \frac{1}{n} > x$, which is contradictory.

□

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Lemma 3.1. *$\sqrt{2}$ is irrational.*

Theorem 3.2. *If $\alpha \neq \beta, \alpha \in \mathbb{R}, \beta \in \mathbb{R}$, then $\exists \gamma \in (\mathbb{R} \backslash \mathbb{Q}) : \alpha < \gamma < \beta$*

Proof. $\sqrt{2} \in \mathbb{R} \backslash \mathbb{Q} \Rightarrow \frac{\sqrt{2}}{2} \in \mathbb{R} \backslash \mathbb{Q}$. Moreover, by definition, $0 < \sqrt{2} < 2$, hence $0 < \frac{\sqrt{2}}{2} < 1$.

Suppose $y \in \beta, x \in \alpha$ are such that $y - x > 0$.

By Archimedean Property of Rational Numbers, $\exists n \in \mathbb{N} : n(y - x) > 1$.

Choose such n such that $(n - 1)(y - x) < 1$ and $n(y - x) \geq 1$.

Since $\overline{0} < \frac{\sqrt{2}}{2} < \overline{1} \Rightarrow \overline{0} < \frac{\sqrt{2}}{2} < \overline{n(y - x)}$.

Hence, $\overline{y} - \overline{x} > \frac{\sqrt{2}}{2n}$. Thus, $\overline{x} < \overline{x} + \frac{\sqrt{2}}{2n} < y$. Therefore, $\alpha \subset \frac{\sqrt{2}}{2n} \subset \beta$, as required. □

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Theorem 4.1. *If $\alpha \neq \beta, \alpha \in \mathbb{R}, \beta \in \mathbb{R}$, then there are infinitely many rational numbers x so that*

$$\alpha < \overline{x} < \beta.$$

Proof. Suppose that the set $T = \{x \in \mathbb{Q} : \alpha < \overline{x} < \beta\}$ is finite. By Theorem 2.4, $T \neq \emptyset$.

Let $\overline{m}$ be the element in T such that $\forall x \notin \overline{m} : \overline{m} \leq \overline{x}$. By definition, $\overline{m} \in \mathbb{R}$. But also $\alpha \in \mathbb{R}$, hence by Theorem 2.4 there exists $\overline{m'}$ such as $\alpha < \overline{m'} < \overline{m} < \beta$.

Since $\forall x \in \overline{m'} : x \in \mathbb{Q}$ by definition, $\overline{m'}$ must be in T, which is a contradicton to the assumption that $\overline{m}$ is the least element. Then, there T is not bounded below.

Similar argument is applied to the case when the assumed greatest element $\overline{n}$ in T is considered for $\overline{n} < \overline{n'} < \beta$. Since $\forall x \in \overline{n'} : x \in \mathbb{Q}$ by definition, $\overline{n'}$ must be in T, which is a contradicton to the assumption that $\overline{n}$ is the greatest element. Therefore, T is not bounded above.

□

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Theorem 5.1. $\forall x \in \mathbb{Q} : \overline{x} = \{ \bigcup_{\alpha \in \mathbb{R}} \alpha : (\forall a \in \alpha : a < x) \land LUB(\alpha) \in \mathbb{Q}\}$

Proof.

1. Suppose there are $x \in \overline{x}$ and $y \in \mathbb{Q}$ such that $y < x$. Since $\alpha \in \mathbb{R} \neq \emptyset$, such x, y exist. By definition of $\overline{x}$, $\exists y \in \alpha : y < x$. Since $y \in \alpha$ and $\alpha \subset \overline{x}$ by definition, $y < x$ and $y \in \overline{x}$.
2. Since some $\alpha \subset \overline{x}$ and $\alpha \in \mathbb{R}$, then $\alpha \neq \emptyset$. Therefore, $\overline{x} \neq \emptyset$.
3. Since $\forall \alpha \subset \overline{x} : \alpha \in \mathbb{R}$ and $\forall a \in \alpha, x \in \overline{x} : a < x$, then any $\alpha \neq \mathbb{Q}$ and by Archimedean Property of Rational Numbers $\exists y > x : (y \notin \overline{x})$ so that $\overline{x} \neq \mathbb{Q}$.
4. Since $\forall \alpha \subset \overline{x} \; \exists \; y \in \alpha : y > x$, then $y \in \overline{x}$ and hence there is no greatest element in $\overline{x}$. Therefore, $\overline{x}$ is real.

Suppose now that the opposite is true and the rational numbers are precisely those real numbers α such that their LUB is irrational.

Since $\forall\ p, q \in \mathbb{Q}\ \exists\ \gamma \in (\mathbb{R} - \mathbb{Q}) : \overline{p} < \gamma < \overline{q}$, if $LUB(\alpha) = \gamma$, then there is always some rational p such that $p < \gamma$. Therefore, there is no one-to-one correspondence between the set of rational numbers x and the set of $\overline{x}$, and several rational numbers correspond to one definition of a rational number. Since rational numbers are unique, this is a contradiction. Hence, $LUB(\alpha)$ is rational. $\square$

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1. Consider $r = \{\frac{1}{x} : x \in \mathbb{Q}, 0 < x < 1\}$.

   Then the claim is that there is no upper bound for r.

   Suppose first there is an upper bound r' such that $\forall f \in r : r' \geq f$.

   Consider $r'' = \frac{1}{\frac{1}{r'} - \epsilon} = \frac{r'}{1 - \epsilon \cdot r'}$ such that $0 < \epsilon < \frac{1}{r'}$. Then $r'' > r'$, since $r'(\frac{1}{1-\epsilon \cdot r'} - 1) = r'(\frac{\epsilon \cdot r'}{1 - \epsilon \cdot r'})$ and from $0 < \epsilon < \frac{1}{r'}$, $0 < \epsilon \cdot r' < 1$. Thus, $\frac{\epsilon \cdot r'}{1-\epsilon \cdot r'} > 0$ and hence $r'' > r'$, which is a contradiction.

2. Consider $s = \{1, 2, 3, 4, 6\}$. Then $\forall x \in s : x \leq 6$. Therefore, $v = 6$ is an upper bound of s by definition. Suppose now that there exists an upper bound u of s such that $u < v$. But then $u < 6$, which is a contradiction, hence 6 is $LUB(s)$.

3. Consider $z = \{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$. Then the claim is that $1 = LUB(z)$. First of all, $s = 1$ is an upper bound for z, since $\forall n \in \mathbb{Z} : 0 < 1 \leq n \Leftrightarrow 0 < \frac{1}{n} \leq 1$. Suppose there is another upper bound $s' < s$ of z. But then $s' < 1 \in z$, which is a contradiction. $\Rightarrow s = LUB(z)$

4. Consider $d = \{1 - \frac{1}{n+1} : n \in \mathbb{N}\}$. Then the claim is that there is no upper bound of d and hence no LUB. First, suppose there is some upper bound $t \in d$ such that $\forall x \in d : t \geq x$. Therefore, t can be written in the form $\exists k \in \mathbb{N} : t = 1 - \frac{1}{k}$. Consider $t' = 1 - \frac{1}{k+1}$. Since $k + 1 > k$, $\frac{1}{k} > \frac{1}{k+1}$, and $-\frac{1}{k} < -\frac{1}{k+1}$. Therefore, $t' \in d$,but $t' > t$, which is a contradiction. Hence, there is no such $t \in d$.