MAT240: 20161122

APPLICATIONS OF ROW ECHELON REDUCTION

DEFINITION

A MATRIX $A \in M_{m \times n}(F)$ IS IN REDUCED ROW ECHELON FORM IF

(i) THE ROWS WITH NON-ZERO ENTRIES COME BEFORE THOSE WITH ALL ZERO ENTRIES.

(ii) THE FIRST NON-ZERO ENTRY OF ANY ROW IS A "1".

(iii) IN THE COLUMN WITH THE FIRST NON-ZERO ENTRY IN THE ROW HAS NO OTHER NON-ZERO ENTRIES.

THE FIRST NON-ZERO ENTRY OF ANY ROW IS CALLED A PIVOTAL ENTRY, AND THE CORRESPONDING COLUM IS CALLED A "PIVOTAL COLUMN.

THEOREM

EVERY $A \in M_{m \times n}(F)$ CAN BE TRANSFORMED INTO $A' \in M_{m \times n}(F)$ IN REDUCED ROW ECHELON FORM BY ROW OPERATIONS.

PROOF

CONSIDER IN THE FIRST NON-ZERO COLUMN THE FIRST NON-ZERO ENTRY. USE R2 TO MAKE THAT ENTRY EQUAL TO 1. INTERCHANGE THIS ROW WITH THE FIRST ROW. USE R3 TO MAKE ALL OTHER ENTRIES IN THIS COLUMN EQUAL TO ZERO. NOW, CONSIDER THE FIRST COLUMN WITH NON-ZERO ENTRY BELOW THE FIRST ROW. REPEAT THE PROCEDURE.

MOST QUESTIONS ABOUT MATRICES CAN BE ANSWERED USING THIS METHOD:

RANK: IF $A \in M_{m \times n}(F)$ IS IN REDUCED ROW ECHELON FORM, THEN

$$\text{rank}(A) = \#\text{ pivotal elements} = \#\text{ non-zero rows} = \#\text{ pivotal columns}$$

Thus, $\text{nullity}(A) = \#$ non-pivotal COLUMNS

SOLUTIONS OF HOMOGENEOUS EQUATIONS $Ax = 0$.

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 & 3 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccc} 1 & 2 & & 4 & \end{array}$$

CORRESPONDS TO

$$x_1 + 3x_3 + 3x_5 = 0$$
$$x_2 + 2x_3 + 2x_5 = 0$$
$$x_4 - x_5 = 0$$

SUPPOSE $x_3 = \alpha,\ x_5 = \beta \Rightarrow x = \begin{pmatrix} -3(\alpha+\beta) \\ -2(\alpha+\beta) \\ \alpha \\ \beta \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

IN FACT, THE $N(L_A)$ BASIS IS ALSO OBTAINED.

IN GENERAL IF A IS IN REDUCED ROW-ECHELON FORM, COMPUTE THE SOLUTION FOR $Ax=0$ BY ASSIGNING ARBITRARY VALUES FOR x_i'S WHERE i CORRESPONDS TO THE NON-PIVOTAL COLUMN, AND EXPRESS THE SOLUTION IN TERMS OF THESE VALUES AND PIVOTAL COLUMNS.

REMARK

EX 1 IF $A \in M_{m \times n}(F)$ IS GIVEN, AND $A' \in M_{m \times n}(F)$ IS IN REDUCED ROW ECHELON FORM OBTAINED BY ROW OPERATIONS, THEN A' IS UNIQUELY DETERMINED BY A.

EX 2 IF $A, B \in M_{m \times n}(F)$ ARE IN REDUCED ROW ECHELON FORM, AND $N(L_A) = N(L_B)$, THEN $A = B$.

SOLUTIONS OF INHOMOGENEOUS EQUATIONS $Ax = b$.

FORM AN AUGMENTED MATRIX $(A|b)$, USE ROW OPERATIONS TO REDUCE IT TO ROW ECHELON FORM $(A'|b')$.

SUPPOSE $(A|b)$ IS IN REDUCED ROW ECHELON FORM.

THEOREM: $Ax = b$ HAS A SOLUTION

$\iff$ THE NUMBER OF ZERO ROWS IN A EQUALS THE NUMBER OF ZERO ROWS IN $(A|b)$.

$\iff$ THE LAST COLUMN b OF $(A|b)$ IS NOT PIVOTAL.

PROOF. EXERCISE

SUPPOSE THERE IS A SOLUTION. THE GENERAL SOLUTION IS OBTAINED BY TAKING ARBITRARY x_i FOR NON-PIVOTAL COLUMNS i, AND HENCE THE VALUES FOR PIVOTAL x_i ARE DETERMINED.

RECALL : GENERAL SOLUTION OF $Ax=b$ IS $x = s+y$, WHERE s IS ANY FIXED SOLUTION, AND y IS A GENERAL SOLUTION OF $Ay=0$.

FOR $Ax=b$, WITH $(A|b)$ IN REDUCED ECHELON FORM, CAN TAKE

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$ WHERE ALL x_i FOR NON-PIVOTAL i ARE EQUAL TO 0.

DETERMINING $R(L_A)$.

WARNING: ROW OPERATIONS ON A CHANGE $R(L_A)$.

RECALL: $R(L_A) \leq F^m$ IS A SUBSPACE SPANNED BY ITS COLUMNS.

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 & 2 \\ 1 & 1 & 3 & 3 & -1 \\ 3 & 0 & 6 & 1 & -1 \\ -2 & -1 & -5 & -3 & -1 \\ 3 & 6 & 6 & 1 & -1 \end{pmatrix}.$$

WANT: BASIS OF $R(L_A)$ CONSISTING OF COLUMN VECTORS.

SUPPOSE A HAS COLUMNS $v_1, \dots, v_n \in F^m$.

$$A = (v_1 \ v_2 \ \dots \ v_n)$$

SO THAT $A' = (v_1' \ v_2' \ v_3' \ \dots \ v_n')$.

CLAIM: $$\sum_{i=1}^{n} a_i v_i = 0 \iff \sum_{i=1}^{n} a_i v_i' = 0$$

REASON: $\sum_{i=1}^{n} a_i v_i = A \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

$\Longleftrightarrow$ $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ IS A SOLUTION OF $Ax = 0$, AND THE SOLUTION SPACE DOES NOT CHANGE UNDER ROW OPERATIONS

IF $A \underset{\text{ROW OPERATIONS}}{\leadsto} A'$ WITH A' IN REDUCED ROW ECHELON FORM, THEN THE PIVOTAL COLUMNS OF A' ARE A MAXIMAL LINEARLY INDEPENDENT SET OF COLUMN VECTORS, HENCE SO ARE THE CORRESPONDING COLUMNS OF A.

LINEAR INDEPENDENCE

PROBLEM: GIVEN $v_1, \dots, v_n \in V$, FIND A MAXIMAL LINEARLY INDEPENDENT SET.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ -2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 6 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 3 \\ 6 \\ -5 \\ 6 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 3 \\ 1 \\ -3 \\ 1 \end{pmatrix},$$

$$v_5 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$