Course Webpage:
Agenda: Chapter 5-7 in Friedberg et al
Marking Scheme: HW, Term Test (Thu, Feb 16), Final (13.3/26.7/60 OR 20/40/40)
Office Hours: Florian Herzig, Wednesday 3-4 pm (BA6186)

1 Review of Determinants

Let F be a field.
Let $A \in M_{n\times n}(F) \to \det(A) \in F$.
Note that for $n = 1$, $\det(a) = a$.
If $n = 2$, $\det(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = ad - bc$
In general, compute the determinant by expanding along a row/column.
For example, expansion along row:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A_{ij}})$$

For example,

$$\det(\begin{bmatrix} 0 & 1 & 0 & 2 \\ 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ -1 & 2 & 3 & 4 \end{bmatrix}) = -1 \cdot \det(\begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 3 \\ -1 & 3 & 4 \end{bmatrix}) + 2\det(\begin{bmatrix} 0 & 0 & 2 \\ 3 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix})$$

Other properties:

- $\det A$ is zero if two rows are linearly dependent $\leftrightarrow \operatorname{rank} A < n$
- if rows are interchanged, then det changes sign
- if a row is multiplied by k, then det is scaled by k
- if a multiple of a row i is added to row j, then det is unchanged
- det is linear along each row and column
- $\det(AB) = \det(A) \cdot \det(B)$
- $\det(\begin{bmatrix} A' & * \\ 0 & A'' \end{bmatrix}) = \det(A')\det(A'')$ – a simiar result holds for any number of *blocks*
- $\det(A^t) = \det(A)$
- A is invertible $\Leftrightarrow \det A \neq 0$
- If A is invertible, then $\det(A^{-1}) = (\det A)^{-1}$
- If A, B are similar, then $\det(A) = \det(B)$.
  Note 1.1. A, B are similar iff there exists an invertible Qsuch that $A = Q^{-1}BQ$
- if A is invertible, then

$$A^{-1} = \frac{1}{\det A}(\begin{bmatrix} \det(\widetilde{A_{11}}) & -\det(\widetilde{A_{21}}) & \dots \\ -\det(\widetilde{A_{12}}) & \dots & \\ \det(\widetilde{A_{13}}) & \dots & \end{bmatrix})$$

2 Diagonalization

2.1 Eigenvalues, Eigenvectors

Motivation: simplification of the matrix form, decomposition of automorphisms (eg computation of A^{100})

Recall that A is diagonal if $A = \begin{bmatrix} A_{11} & 0 & 0 & \dots \\ 0 & A_{22} & 0 & \dots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & A_{nn} \end{bmatrix}$

Definition 2.1. For V a finite dimensional vector space, $T : V \to V$ a linear transformation, T is diagonalisable if there exists an ordered basis β such that $[T]_\beta$ is diagonal.

If $A \in M_{n\times n}(F)$, then A is diagonalisable if $L_A : F^n \to F^n$ is diagonalisable. Equivalently, A is similar to a diagonal matrix.

If $T$is diagonalisable and $[T]_\beta = \begin{bmatrix} D_{11} & 0 & \dots \\ & \ddots & \\ 0 & \dots & D_{nn} \end{bmatrix}$, where $\beta = (v_1, v_2, \dots, v_n)$, then $Tv_1 = D_{11}v_1, \dots, Tv_n = D_{nn}v_n$.

Definition 2.2. $Tv = \lambda v$ with $v \neq 0, \lambda \in F$, then v is an **eigenvector of** T with corresponding **eigenvalue** λ.

Similarly, an eigenvalue of A is an eigenvalue of L_A.

Example 2.3

If $T = \lambda I_v$ (ie $T(v) = \lambda v \forall v \in V$), then any nonzero $v \in V$ is an eigenvector of T.

Example 2.4

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then e_1, e_2, e_3 are eigenvectors with eigenvalues $1, 2, 3$.

Example 2.5

If T is arbitrary, then eigenvectors with eigenvalue 0 are the nonzero elements of $\ker(T)$.

Example 2.6

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation by angle $\alpha \in (0, \pi)$, thne there are no eigenvectors $\Rightarrow T$ is not diagonalisable.

Example 2.7

If $A = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$, then $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector with eigenvalue 1.

From above, if T is diagonalisable, then V has a basis consisting of eigenvectors of T.

Conversely, if $\beta = (v_1, v_2, v_3, \dots)$ i sa basis of eigenvectors $T(v_1) = \lambda v_1, \dots, T(v_2) = \lambda v_2$, then T is diagonalisable.

2.2 Finding Eigenvectors and Eigenvalues

If $T(v) = \lambda v, v \neq 0$, then $0 = T(v) - \lambda v = T(v) - \lambda I_v(v) = (T - \lambda I_v)(v) = 0 \Leftrightarrow v \in \ker(T - \lambda I_v)$.

Thus, T has an eigenvalue $\lambda \Leftrightarrow \ker(T - \lambda I_v) \neq 0 \Leftrightarrow T - \lambda I_v$ is not injective $\Leftrightarrow T - \lambda I_v$ is not invertible.

In that case, the eigenvectors of an eigenvalue λ are the non-zero elements of $\ker(T - \lambda I_v)$.

Similarly for $A \in M_{n \times n}(F)$.

Example 2.8

Let $A = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$, $\lambda \in F$.

Then $\det(A - \lambda I) = (4 - \lambda)(-1 - \lambda) + 6 = (\lambda - 1)(\lambda - 2)$.

Thus $\lambda_1 = 1, \lambda_2 = 2$.

Find the corresponding eigenvectors to obtain $(1, -1)$ corresponding to $\lambda = 1$ and $(3, -2)$ corresponding to $\lambda = 2$.

Since they span F^2, A is diagonalisable.

Definition 2.9. The **characteristic polynomial** of A is the polynomial $f(\lambda) = \det(A - \lambda I)$.

Example 2.10

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $f(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc)$.

Remark 2.11. If A, B are similar, they have the same characteristic ploynomial. Hence if $T \in \text{Hom}$, then the characteristic polynomial can be defined for T as the characteristic polynomial of $[T]_\beta$ for any β.