1 Taylor Polynomial

For $n \in \mathbb{N}$, $a \in \mathcal{D}(f)$ and an n-differentiable function f, we define a Taylor polynomial as

$$P_{n,a,f} = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{n!}(x-a)^i.$$

We have already shown that $\lim_{x\to a} \frac{f(x)-P_{n,a,f}(x)}{(x-a)^n} = 0$.

If f is n-times differentiable at $x = a$, $P_{n,a,f}$ is well-defined.

However, even if f is not n-times differentiable, a polynomial $Q(x)$ can be found such that $\lim_{x\to a} \frac{f(x)-P(x)}{(x-a)^n} = 0$.

In this situation, f is said to agree mith P up to order n.

Theorem 1.1

If P, Q are polynomials of degree less that or equal to n, and P agrees with Q up to order n, then $P = Q$.

Proof.

Let $f = P - Q$ be a polynomial of degree less than or equal to n.

Write $f(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n$. Note that $\lim_{x\to a} \frac{f(x)}{(x-a)^n} = 0$, and thus for all $k \leq n$ we have $\lim_{x\to a} \frac{f(x)}{(x-a)^k} = 0$.

Therefore, we obtain that $\lim_{x\to a} f(x) = 0$, and thus $a_0 = 0$.

By induction, $a_k = a_1$ for $k \in [0, n] \cap \mathbb{N}$. □

Corollary 1.2

Suppose that $f(x)$ is a n-times differentiable at $x = a$ and P is a polynomial of degree less than or equal to n, which agrees with f up to order n. Then $P = P_{n,a,f}$.

Proof.

Observe that $P(x)$ and $P_{n,a,f}$ both agree with f up to order n.

Therefore,

$$\lim_{x\to a} \frac{P(x)-f(x)}{(x-a)^n} + \frac{f(x)-P_{n,a,f}(a)}{(x-a)^n} = 0.$$

Then $\lim_{x\to a} \frac{P-P_{n,a,y}}{(x-a)^n} = 0$, which proves the result. □

Consider $\arctan x = \int_0^k \frac{1}{1+x^2}$ for $x \in (-1, 1)$.

Note that $\frac{1}{1+t^2} = 1 - t^2 + t^4 s - \cdots + (-1)^2 p^{2n}$.

Therefore, $\arctan x = \int_0^x \frac{1}{1+t^2} = \int_0^a (1 - t^2 + t^4 + \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1}t^{2n+2}}{1+t^2}\, \mathrm{d}t$.

Thus,

$$\arctan x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\,\mathrm{d}t$$

.

The right hand side but for the last term *should be* $P_{2n+1,0,\arctan}$.

This would hold if it agrees with $\arctan x$ up to order $2n+1$.

Consider $\lim_{x\to 0} \frac{\frac{(-1)^{n+1}x^{2n+2}}{1+x^2}}{(2n+1)x^{2n}}$.

Thus, taking an absolute value, we obtain that

$$\lim_{x\to 0} \frac{\frac{|x|^{2n+2}}{1+x^2}}{(2n+1)|x|^{2n}} = \lim_{x\to 0} \frac{\frac{|x|^2}{1+x^2}}{2n+1} < \lim_{x\to 0} \frac{|x|^2}{2n+1} = 0.$$