Problem. Suppose that $V = M_{2\times 2}(\mathbb{F})$ with $\mathbb{F} = \mathbb{Z}_5$.

Let $T \in \mathrm{End}(V)$ be such that $TA = A\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Find a basis of V that consists of a disjoint union of cycles of generalized eigenvalues. Find a Jordan canonical form.

Solution.

Let $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $a, b, c, d \in \mathbb{Z}_5$.

Note that

$$\det(B - \lambda I) = (1 - \lambda)^2. \tag{1}$$

Let $f(t) = (1 - t)^2$.

By Cayley-Hamilton Theorem, we know that $f(B) = 0$.

Consider $f(T)$:

$$\begin{aligned} f(T)(A) &= (I - 2T + T^2)A & (2) \\ &= A - 2T(A) + T^2(A) & (3) \\ &= A - 2AB + T(AB) & (4) \\ &= A - 2AB + AB^2 & (5) \\ &= A(1 - 2B + B^2) & (6) \\ &= Af(B) & (7) \\ &= 0. & (8) \end{aligned}$$

Since (2)-(7) holds for any A, then $f(T)(A)$ is the zero homomorphism.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $a, b, c, d \in \mathbb{F}$.

Therefore, if $T(A) = AB = \lambda A$, then

$$\begin{cases} a + 2b &= \lambda a \\ c + 2d &= \lambda c \\ b &= \lambda b \\ d &= \lambda d. \end{cases}$$

Therefore, from the equation 4, since $\lambda \neq 0$ (because then $a = b = c = d = 0$), we have $\lambda = 1$.

Hence, $b = d = 0$, and thus $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ span $\ker(T - I)$.

We have shown that $T^2 - 2T + I = 0$, and therefore $T^2 = 2T - I$ and $(T - I)^2 = 0$ for any A, which means that $\ker(T - I)^2 = V$.

Therefore, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are generalised eigenvectors.

Since nullity$(T - I) = 2$, we know that there are exactly two Jordan blocks in the corresponding Jordan canonical form.

Now we find a cycle basis.

Note that $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2b & b \\ c+2d & d \end{pmatrix}$.

Take $v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Then $(T - I)v = Tv - Iv = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, which is in E_1, and thus $p = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ form a cycle.

Take now $w = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $(T - I)w = Tw - Iw = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$, which is in E_1, and thus $q = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and $w = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ form a cycle.

It is easy to see that $\beta = \{p, v, q, w\}$ is linearly independent and has the length of 4, which is equal to $\dim W$. Therefore, β is a cycle basis and hence

$$[A]_\beta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\square$