Lemma 0.1

If $T = T^*$, then every eigenvalue of T is real.

Proof.

Suppose $T(x) = \lambda x$ and $x \neq 0$.

Since $T = T^*$, then T is normal, and hence $T^*(x) = \overline{\lambda} x = T(x) = \lambda x$. Since $x \neq 0$, then $\lambda = \overline{\lambda}$. ☐

Lemma 0.2

For $\mathbb{F} = \mathbb{R}$, if $T = T^*$, then the characteristic polynomials of T splits.

Proof.

Let β be an orthonormal basis of V, let $A = [T]_\beta$. Then $A = A^* \in M_{n\times n}(\mathbb{R}) \subseteq M_{n\times n}(\mathbb{C})$. Define $T_A : \mathbb{C}^n \to \mathbb{C}^n$ such that $[T_A]_\gamma = A$, where γ is a standard basis.

Since γ is an orthonormal basis, we obtain that $T_A^* = T_A$.

By Lemma 0.1, every eigenvalue of T_A is real, so every root of the characteristic polynomial of T_A is real.

Since the field is complex, the characteristic polynomial of T_A splits, so it is $(-1)^n \prod_{i=1}^n (t - \lambda_i)$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

But the characteristic polynomial of T_A is equal to the characteristic polynomial of A and the characteristic polynomial of T, so it splits over $\mathbb{R}$. ☐

Theorem 0.3

Let $\mathbb{F} = \mathbb{R}$. T is self-adjoint if and only if V has an orthonormal basis of eigenvectors for T.

Proof.

Let β be an orthonormal basis of eigenvectors.

Then

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

where $\lambda_i \in \mathbb{R}$, so

$$[T^*]_\beta = [T]_\beta^* = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^* = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = [T]_\beta,$$

and thus $T^* = T$.

Now we use induction on n.

If $n = 1$, then any unit vectors satisfies the claim.

If $n > 1$, assume the theorem holds for $n - 1$.

By Lemma 0.2, the characteristic polynomial of T splits, so T has an eigenvalue $\lambda \in \mathbb{R}$. Then $T(v) = \lambda v$ and $v \neq 0$. We may scale v so that $\|v\| = 1$.

Let $W = \operatorname{span}(v)$. It is T-invariant, and so $W^\perp$ is T- and T^*-invariant.

Observe that $\dim W^\perp = n - 1$, and $T_{W^\perp} : W^\perp \to W^\perp$.

Claim. $T_{W^\perp}$ is self-adjoint if and only if $\langle T_{W^\perp} x, y \rangle = \langle x, T_{W^\perp} y \rangle$ for all $x, y \in W^\perp$.

Proof.

Since $T = T^*$, the conclusion follows by the definition of T^*. □

By induction, $W^\perp$ has an orthonormal basis $v_2, \ldots, v_n$ of eigenvectors for $T_{W^\perp}$.

Therefore, $v, v_2, \ldots, v_n$ is an orthonormal basis of eigenvectors for T. □

Corollary 0.4

Any real symmetric matrix in $M_{n \times n}(\mathbb{R})$ is diagonalisable.

Proof.

$A = A^t = A^*$, and therefore $L_A^* = L_A : \mathbb{R}^n \to \mathbb{R}^n$, then $\mathbb{R}^n$ has an orthonormal basis of eigenvectors. Thus, A is diagonalisable. □

Theorem 0.5

T is self-adjoint if and only if T is normal and every eigenvalue is real.

Proof.

By Lemma 0.1, if T is self-adjoint, then T is normal and every eigenvalue is real.

By the Complex Spectral Theorem, pick an orthonormal basis β of eigenvectors such that $[T]_\beta = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ and λ_i are eigenvalues of T, and hence $\lambda_i \in \mathbb{R}$.

Thus,

$$[T^*]_\beta = [T]_\beta^* = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = [T]_\beta,$$

and thus $T = T^*$. □

Definition 0.6. $A \in M_{n \times n}(\mathbb{F})$ is *hermitian* ($\mathbb{F} = \mathbb{C}$) / *symmetric* ($\mathbb{F} = \mathbb{R}$) if $A = A^*$ if $A = A^*$.

1 Unitary and Orthogonal Transformations

Let V be a finite-dimensional inner product space.

Definition 1.1. $T \in \mathrm{Hom}(V, V)$ is said to be *unitary* ($\mathbb{F} = \mathbb{C}$) / *orthogonal*($\mathbb{F} = \mathbb{R}$) if $\|T(x)\| = \|x\|$ for all $x \in V$.

e.g. rotation and reflection in $\mathbb{R}^2$, $(x, y, z) \mapsto (-x, -y, -z)$ in $\mathbb{R}^3$.

Example 1.2

Let $V = H = C[0, 2\pi]$ and $\mathbb{F} = \mathbb{C}$.

Fix a continuous function $h \in H$ such that $|h(x)| = 1$ for all $x \in [0, 2\pi]$.

Let $T(f) = hf$. Then $T : H \to H$ is unitary.

Example 1.3

Let $V = M_{n \times n}(\mathbb{F})$, with $\langle A, B \rangle = \mathrm{tr}\, AB^*$.

Fix $Q \in V$ such that $QQ^* = I$. Then $T \in \mathrm{Hom}(V, V)$ such that $A \mapsto AQ$ is unitary / orthogonal.

Reason: $\|T(A)\| = \langle TA, TA \rangle = \langle AQ, AQ \rangle = \mathrm{tr}\, AQ(AQ)^* = \|A\|^2$.

Theorem 1.4

Let $T \in \mathrm{Hom}(V, V)$. The following are equivalent:

a) T is unitary / orthogonal, i.e. $\|T(x)\| = \|x\|$
b) $TT^* = T^*T = I$, and hence T is normal
c) $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all x and y in V
d) If β is any orthonormal basis of V, then $T(\beta)$ is an orthonormal basis
e) There exists an orthonormal basis β such that $T(\beta)$ is an orthonormal basis

Proof.

We prove first that (a) implies (b). Note that $\|Tx\|^2 = \|x\|^2$, and therefore $\langle Tx, Tx \rangle = \langle x, x \rangle$, which is equivalent to $\langle T^*Tx, x \rangle = \langle x, x \rangle$, and therefore $\langle (T^*T - I)x, x \rangle = 0$ for all $x \in V$.

Therefore, $S^* = (T^*T - I)^* = (T^*T)^* - I^* = T^*T^{**} - I = S$.

Claim. If $S = S^*$, and $\langle Sx, x \rangle = 0$,then $S = 0$.

Proof.

If x is an eigenvector of S, then $Sx = \lambda x$, and thus $0 = \langle Sx, x \rangle = \langle \lambda x, x \rangle = \lambda \|x\|^2$. Thus, $\lambda = 0$. $\square$

We have proven that there exists an orthonormal basis of eigenvectors for S. Then $S = 0$. $\square$

By Lemma , $T^*T = I$, and therefore T, T^* is invertible and $T^* = T^{-1}$. Therefore, $TT^* = I$.

We prove now that (b) implies (c). Note that $\langle Tx, Ty \rangle = \langle T^*Tx, y \rangle = \langle x, y \rangle = I$ by (b), and the conclusion follows.

Now, (c) implies (d). If $\beta = \{v_1, \dots, v_n\}$ is an orthonormal basis, then $\langle Tv_i, Tv_j \rangle = \delta_{i,j}$ by (c), and hence $T(\beta)$ is orthonormal.

(d) trivially implies (e).

(e) implies (a), because if $\beta = \{v_1, \dots, v_n\}$ is an orthonormal basis and $\{Tv_1, \dots, Tv_n\}$ is orthonormal, then for any $x = \sum_{i=1}^n c_i v_i$:

$$\|Tx\|^2 = \langle T(\sum_i c_i v_i), \sum_j c_j v_j \rangle \tag{1}$$

$$= \sum_i c_i \langle v_i, \sum_j c_j v_j \rangle \tag{2}$$

$$= \sum_i \sum_j c_i \overline{c_j} \langle Tv_i, Tv_j \rangle = \sum_i |c_i|^2 \tag{3}$$

Corollary 1.5

If T is unitary or orthogonal, then T is normal and any eigenvalue λ of T has $|\lambda| = 1$.

Proof.

The fact that T is normal follows from 1.4(b).

If $Tv = \lambda v$, then $\|Tv\| = \|v\| = \|\lambda v\|$. Therefore, $|\lambda| = 1$. □

Corollary 1.6

If $\mathbb{F} = \mathbb{C}$, T is unitary if and only if V has an orthonormal basis of eigenvectors with eigenvalues such that $|\lambda| = 1$

Proof.

The ($\Rightarrow$) direction follows from Corollary 1.5 (see Theorems 6.16 and 6.17 in Friedberg *et al.*

Let β be such an orthonormal basis. Then $[T]_\beta = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, where $|\lambda_i| = 1$.

Then $[T^*]_\beta = [T]_\beta^* = \begin{pmatrix} \overline{\lambda_1} & & \\ & \ddots & \\ & & \overline{\lambda_n} \end{pmatrix}$,

and hence $[T^*T]_\beta = \begin{pmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_n|^2 \end{pmatrix} = I$.

Theerefore, $T^*T = I$. □