1 Computational Aspects of RCF

1.1 Review

Suppose $x \in V$, $x \neq 0$, and let $T \in \operatorname{End}(V)$.

The T-annihilator of x is a monic polynomial $p(t)$ of the least degree such that $p(T)x = 0$.

Note that the characteristic polynomial of T, $f(t)$, can be represented in the form $(-1)^{\dim V} \prod_{i=1}^{s} \phi_s(t)^{m_s}$, so that $V = \bigoplus_{i=1}^{s} K_{\phi_i}$ and $\dim K_{\phi_i} = m_i \deg \phi_i$.

Each K_{ϕ_i} has a basis β_i which is a disjoint union of T-cyclic bases in the form $\{x, Tx, \ldots, T^{k-1}x\}$.

Let $\beta = \bigcup_{i=1}^{s} \beta_i$. Then $[T]_\beta$ is in RCF.

1.2 How to Find RCF

For each K_{ϕ_i} we can write a dot diagram consisting of representations of cycles inside some β_i.

The T-annihilator of x_j is $\phi_j(t)^{k_j}$ for some $k_j \in \mathbb{Z}^+$.

Order these x_j's so that $k_1 \geq k_2 \geq \cdots \geq k_s$.

Note that $\left|\beta_{x_j}\right| = k_j \deg \phi_j$.

Moreover, the number of dots in the first s rows is $\frac{\operatorname{nullity} \phi_i(T)^s}{\deg \phi_i} = \frac{\dim K_{\phi_i}}{\deg \phi_i} = m_i$.

Example 1.1

Suppose that $\mathbb{F} = \mathbb{Z}_5$ and let $T \in \operatorname{End}(V)$ be such that the characteristic polynomial of T is $f(t) = (t^2+1)(t^2+2)(t^3+3)$.

Since $\mathbb{F} = \mathbb{Z}_5$, then $f(t) = (t-2)(t+2)(t^2+2)(t^3+3)$.

Since $m_1 = m_2 = m_3 = m_4 = 1$, we know that each dot diagram consists of only one dot.

Each K_{ϕ_i} has a T-cyclic basis, and a T-annihilator is $\phi_i(t)$.

Thus, the RCF is $\begin{pmatrix} C_{t-2} & & & \\ & C_{t+2} & & \\ & & C_{t^2+2} & \\ & & & C_{t^3+3} \end{pmatrix}$.

Example 1.2

Suppose that $\mathbb{F} = \mathbb{Z}_5$.

Let $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 4 & 2 & 4 & 0 \\ & & 2 & 1 & 1 \\ & & 4 & 3 & 0 \\ & & & & 0 \end{pmatrix}$.

Then $f(t) = -(t^2+2)(t^2+2)t = -t(t^2+2)^2$.

Note that $\phi_1(t) = t$ and $m_1 = 1$, while $\phi_2(t) = t^2 + 2$ and $m_2 = 2$.

The number of dots in the first row is $\frac{\text{nullity}(A^2+2I)}{\deg \phi_2} = \frac{4}{2} = 2$.

So the dot diagram of K_{ϕ_2} consists of two dots in a row, which means that $R(A) = \begin{pmatrix} 0 & -2 & & & \\ 1 & 0 & & & \\ & & 0 & -2 & \\ & & 1 & 0 & \\ & & & & 0 \end{pmatrix}$.

1.3 How to Find a Rational Canonical Basis

In Example 1.2, $K_{\phi_2} = \ker(A^2 + 2I)$ is 4-dimensional, so each element $x \in K_{\phi_2} \setminus \{0\}$ has L_A-annihilator $t^2 + 2$, so a 2-dimensional L_A-cyclic subspace is generated.

Pick any vector $x \in K_{\phi_2} \setminus \{0\}$ to obtain the first cycle, and pick any other vector not in the span of β_x to get another cycle.