Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x+y \end{pmatrix}$$

Note that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Consider $T^n\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Claim.

$$T^n\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$$

Proof. If $n = 1$, then

$$\begin{aligned}
T &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} && (1) \\
&= \begin{pmatrix} 1 \\ 1 \end{pmatrix} && (2) \\
&= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} && (3)
\end{aligned}$$

Suppose now the claim is true for $n = k \in \mathbb{N}$.

Consider $T^{k+1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\begin{aligned}
T^{k+1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= T\left(T^k\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) && (4) \\
&= T\left(\begin{pmatrix} a_k \\ a_{k+1} \end{pmatrix}\right) && (5) \\
&= \begin{pmatrix} a_{k+1} \\ a_k + a_{k+1} \end{pmatrix} && (6) \\
\text{definition of the Fibonacci sequence } | &= \begin{pmatrix} a_{k+1} \\ a_{k+2} \end{pmatrix}, && (7)
\end{aligned}$$

which is exactly the claim in case $n = k + 1$.

Therefore, if the claim is true in case $n = k$, it is true for $n = k + 1$.

But the claim is true in case $n = 1$, therefore

$$\forall (n \in \mathbb{N}) : T^n\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}.$$

$\square$

Let $\omega = \frac{\sqrt{5}+1}{2}$.

Let $\gamma = \{f_1, f_2\}$, where $f_1 = \begin{pmatrix} 1 \\ \omega \end{pmatrix}$, $f_2 = \begin{pmatrix} -\omega \\ 1 \end{pmatrix}$.

Consider $[T]_\gamma$.

Thus,

$$T(f_1) = \begin{pmatrix} \omega \\ 1+\omega \end{pmatrix} = \omega f_1 \tag{8}$$

$$T(f_2) = \begin{pmatrix} 1 \\ 1-\omega \end{pmatrix} = -\frac{1}{\omega} f_2 \tag{9}$$

Hence,

$$[T]_\gamma = \begin{bmatrix} \omega & 0 \\ 0 & -\frac{1}{\omega} \end{bmatrix}$$

Therefore,

$$[T^2]_\gamma = [T]_\gamma [T]_\gamma = \begin{bmatrix} \omega^2 & 0 \\ 0 & \frac{1}{\omega^2} \end{bmatrix} \tag{10}$$

and, in general,

$$[T^n]_\gamma = \begin{bmatrix} \omega^n & 0 \\ 0 & (-1)^n \frac{1}{\omega^n} \end{bmatrix} \tag{11}$$

Suppose now that there exist $c, d \in \mathbb{R}$ such that $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = cf_1 + df_2$. Thus,

$$\begin{cases} c = d\omega \\ d(1+w^2) = 1 \end{cases}$$

Note that $1+\omega^2 = \sqrt{5}\omega$, and $\omega - 1 = \frac{1}{\omega}$. Therefore,

$$\begin{cases} d = \frac{1}{1+\omega^2} = \frac{1}{\sqrt{5}\omega} = \frac{\omega-1}{\sqrt{5}} \\ c = \frac{1}{\sqrt{5}} \end{cases}$$

Hence,

$$[T^n]_\gamma \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}\omega} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \omega^n & 0 \\ 0 & (-1)^n \frac{1}{\omega^n} \end{bmatrix} \begin{pmatrix} 1 \\ \frac{1}{\omega} \end{pmatrix} \tag{12}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} \omega^n \\ (-1)^n \frac{1}{\omega^{n+1}} \end{pmatrix} \tag{13}$$

Therefore, from above,

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \frac{1}{\sqrt{5}} \left(\omega^n f_1 + (-1)^n \frac{1}{\omega^{n+1}} f_2 \right) \tag{14}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} \omega^n + (-1)^{n+1} \frac{1}{\omega^n} \\ \omega^{n+1} + (-1)^n \frac{1}{\omega^{n+1}} \end{pmatrix} \tag{15}$$

Thus,

$$a_n = \frac{1}{\sqrt{5}} \left(\omega^n - (-1)^n \frac{1}{\omega^n} \right).$$