Theorem 0.1

Let $a < c < b$. If f is integrable on $[a, b]$, then for any $c \in [a, b]$ f is integrable on $[a, c]$ and $[c, b]$. Conversely, if f is integrable on $[a, c]$ and $[c, b]$, then f is also integrable on $[a, b]$. Therefore, if f is integrable on $[a, b]$,

$$\int_a^c f + \int_c^b f = \int_a^b f$$

Proof. Suppose that f is integrable on $[a, b]$.

Since f is integrable, there exists a partition such that

$$U(f, P) - L(f, P) < \epsilon$$

Consider such a partition $P = \{t_0, t_1, \dots, t_n\}$ of $[a, b]$.

Suppose first that c is not of t_j. Then construct another partition Q such that $P \subset Q$ to obtain $U(f, Q) - L(f, Q) \leq U(f, P) - L(f, P) < \epsilon$. Thus, we may assume that c is equal to one of t_j.

Consider partitions $P' = [t_0, t_1, \dots, t_j]$ of $[a, c]$ and $P'' = [t_{j+1}, \dots, t_n]$ of $[c, b]$. Then by definition of $L(\cdot, \cdot)$ it follows that

$$L(f, P) = L(f, P') + L(f, P'') \tag{1}$$

$$U(f, P) = U(f, P') + U(f, P'') \tag{2}$$

Therefore, $[U(f, P'') - L(f, P'')] + [U(f, P') - L(f, P')] = U(f, P) - L(f, P) < \epsilon$.

Since each term on LHS is nonnegative, it follows that f is integrable on $[a, c]$ and $[c, b]$. Note also that

$$L(f, P') \leq \int_a^c f \leq U(f, P') \tag{3}$$

$$L(f, P'') \leq \int_c^b f \leq U(f, P''), \tag{4}$$

and thus $L(f, P) \leq \int_a^c f + \int_c^b f \leq U(f, P)$. Since P was chosen arbitrarily, $\int_a^c f + \int_c^b f = \int_a^b f$.

Conversely, if f is integrable on $[a, c]$ and $[c, b]$, it follows that

$$U(f, P') - L(f, P') < \frac{\epsilon}{2} \tag{5}$$

$$U(f, P'') - L(f, P'') < \frac{\epsilon}{2} \tag{6}$$

Construct a partition P containing both P' and P''. Then $L(f, P) = L(f, P') + L(f, P'')$ and $U(f, P) = U(f, P') + U(f, P'')$.

Therefore, from inequalities above, $U(f, P) - L(f, P) < \epsilon$, and thus f is integrable on $[a, b]$.

$\square$

Definition 0.2.

$$\int_a^a f = 0 \tag{7}$$

$$\int_b^a = -\int_a^b f \tag{8}$$

$$\tag{9}$$

Definition 0.3. Suppose $f(x)$ is integrable on $[a, b]$. Pick $x \in [a, b]$, and define the **indefinite integral** of f:

$$F(x) = \int_a^x f(t)dt$$