MAT240: INVERTIBILITY II | 20161025

REVIEW : $T \in \mathcal{L}(V,W)$
IF $T: V \to W$ INVERTIBLE,
$T^{-1}: W \to V$ IS AGAIN LINEAR.

DEFINITION: AN INVERTIBLE LINEAR MAP $T: V \to W$ IS CALLED A LINEAR <u>ISOMORPHISM</u> FROM V TO W.

NOTATION: $T \xrightarrow{\cong} W$ OR $V \cong W$.

EXAMPLES :

a) $\mathcal{P}_n(F) \to F^{n+1}$

SO THAT IF

$$p(x) = a_0 + \sum_{i=1}^{n} a_i x^i,$$

$$p \mapsto (a_0, \ldots, a_n)$$

Also, the map $p \mapsto (b_0, \ldots, b_n)$

IF $p(x) = b_0 + b_1(x+1) + \ldots + b_n(x+1)^n$

IS AN ISOMORPHISM.

b) THE MAP $\mathcal{P}_n(F) \to F^{n+1}$,

$$p \mapsto (p(c_0), \ldots, p(c_n))$$

WHERE $c_0, \ldots, c_n \in F$ DISTINCT

IS AN ISOMORPHISM

EXAMPLES (CONT.)

(c) LET $V \subset F^{\infty}$ SUBSPACE OF FINITE SEQUENCES: $(a_0, a_1, a_2, \dots)$ (i.e.: ONLY FINITELY MANY NONZERO ENTRIES)

$T: \mathcal{P}(F) \to V$, $p(x) = a_0 + a_1 x + \dots + a_n x^n$

IS AN ISOMETRY.

What is the linear map $\mathcal{P}(F) \to \mathcal{P}(F)$?

THEOREM IF $T: V \to W$ IS AN ISOMORPHISM THEN $\dim(V) = \dim(W)$

PROOF. IF $\dim(V) < \infty$:

$\dim(N(T)) + \dim(R(T)) = \dim(W)$

SIMILARLY, IF $\dim(W) < \infty$, THE SAME ARGUMENT HOLDS FOR T^{-1}. $\square$

THEOREM [$\square$] IF $\dim(V) = \dim(W) < \infty$

$\Longrightarrow \Big[T \in \mathcal{L}(V, W)$ IS AN ISOMORPHISM

$\Longleftrightarrow N(T) = \{0\} \Leftrightarrow R(T) = W \Big]$

PROOF. $\dim(N(T)) = 0$, $R(T) = W \Leftrightarrow \dim(R(T)) = \dim W$

FROM RNT,

$\dim(R(T)) + \dim(N(T)) = \dim(V)$

$\Leftrightarrow \dim(V) = \dim(W)$

REMARK IN [□] IT IS IMPORTANT THAT $\dim(V) < \infty$

EXAMPLE $T: \mathcal{P}(\mathbb{R}) \to F^\infty$ IS

1. ONE-TO-ONE BUT NOT AN ISOMORPHISM.

2. $S: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ IS AN ONTO ISOMORPHISM.

3. $T: F^\infty \to F^\infty$

$$(a_1, \dots, a_n) \mapsto (0, a_1, \dots, a_n)$$

4. $S: F^\infty \to F^\infty$

$$(a_0, a_1, \dots) \mapsto (a_1, a_2, \dots)$$

T IS INJECTIVE, NOT SURJECTIVE

S IS SURJECTIVE NOT INJECTIVE.

DEFINITION. LET THE SPACE $\mathcal{L}(V, W)$ OF LINEAR MAPS $T: V \to W$ BE ENDOWED WITH THE FOLLOWING OPERATIONS

$$(T_1 + T_2)(v) = T_1(v) + T_2(v)$$

$$(aT)v = a\,Tv$$

SPECIAL CASES

$\mathcal{L}(F,V) \cong V$ BY THE ISOMORPHISM:

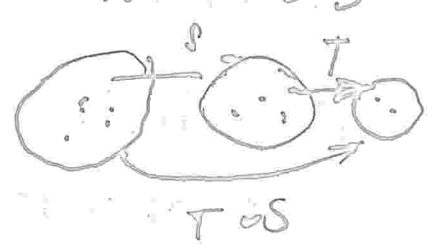
$V \longrightarrow \mathcal{L}(F,V)$, $v \to T_v : a \to T_v(a) = av$

$\mathcal{L}(V,F) = V^*$ DUAL SPACE, WITH ELEMENTS CALLED LINEAR FUNCTIONS.

LEMMA

IF $T \in \mathcal{L}(V,W)$, $S \in \mathcal{L}(U,V)$

THEN $T \circ S$ IS A LINEAR MAP



PROOF

$$T\circ S(x+y) = T(S(x+y)) = T(S(x)+S(y)) = T\circ S(x) + T\circ S(y)$$

$$T\circ S(cx) = T(cS(x)) = cT(S(x)) = cT\circ S(x)$$

CONSIDER A MAP:

$$\mathcal{L}(V,W) \times \mathcal{L}(U,V) \longrightarrow \mathcal{L}(U,W)$$

$$(T,S) \longmapsto T \circ S$$

THEOREM

LINEAR IN T:

$$(T_1 + T_2) \circ S = T_1 \circ S + T_2 \circ S$$

$$(aT) \circ S = a(T \circ S)$$

LINEAR IN S:

$$T \circ (S_1 + S_2) = T \circ S_1 + T \circ S_2$$

$$T \circ (aS) = a(T \circ S)$$

$$\begin{tikzcd} U \arrow[r] \arrow[rr, bend right, "T \circ S"'] & V \arrow[r] & W \end{tikzcd}$$

$$u \longmapsto S(u) \longmapsto T(S(u))$$

IN GENERAL, $T \circ S$ IS NOT DEFINED

IF $S: U \to V$

$T: V \to W.$

IF $U = V = W$, DENOTE $\mathcal{L}(V,W) = \mathcal{L}(V)$,

THEN FOR $S, T \in \mathcal{L}(V)$, SO

$T, T \circ S$ BOTH DEFINED.

b) $V = \mathcal{P}(\mathbb{R})$

$T \in \mathcal{L}(V)$, $p(x) \longmapsto x\,p(x)$

$S \in \mathcal{L}(V)$, $p(x) \longrightarrow p'(x)$

$S \circ T - T \circ S = I_V$

$$p(x) \longmapsto \frac{d}{dx}\left(x\,p(x)\right) - x\frac{d}{dx}p(x) = p(x)$$

a) $T : F^\infty \longrightarrow F^\infty$

$(a_0, a_1, \ldots) \longmapsto (0, a_0, a_1, \ldots)$

$S : F^\infty \longrightarrow F^\infty$

$(a_0, a_1, \ldots) \longmapsto (a_1, a_2, \ldots)$

$S \circ T = I_V$ IDENTITY

$T \circ S = (a_0, a_1, \ldots) \longrightarrow (0, a_1, a_2, \ldots)$

THEOREM: A linear map $T \in \mathcal{L}(V, W)$ is an isomorphism

$\iff \exists\, S \in \mathcal{L}(W, V)$ with $T \circ S = I_W$ and $S \circ T = I_V$.

PROOF:

$(\Longrightarrow)$: If T is an isomorphism, then $S := T^{-1}$ satisfies $T \circ S = I_W$, $S \circ T = I_V$.

$(\Longleftarrow)$: Suppose $T \circ S = I_W$, $S \circ T = I_V$.

$N(T) = 0$ because $T(v) = 0$ implies $S \circ T(v) = S(0) = 0 = v$.

$R(T) = 0$ because $\forall\, w \in W$ is $w = T(v)$ with $v = S(w)$, since $T \circ S(w) = I_W(w) = w$

REMARK

WOULD BE ENOUGH TO HAVE $S_1, S_2 \in \mathcal{L}(W,V)$:

$T \circ S_1 = I_W$, $S_2 \circ T = I_V$

AND EED, $S_1 = S_2$ BECAUSE

$$S_1 = I_V \circ S_1 = (S_2 \circ T) \circ S_1 =$$
$$= S_2 \circ (T \circ S_1)$$
$$= S_2 .$$

REMARK

IF $\dim V = \dim W < \infty$

THEN $T \circ S = I_W \iff S \circ T = I_V$

BECAUSE $S \circ T = I_V \Rightarrow$

$N(T) = \{0\}$ AND

HENCE $R(T) = W$ BY EARLIER THEOREM.

IF $\dim V = \dim W = \infty$

THEN $T \circ S = I_W \not\Leftrightarrow S \circ T = I_V$

COUNTEREXAMPLE: SHIFT OPERATIONS

c) IF $\dim W \neq \dim V < \infty$,

THEN $T \circ S = I_W$

$\not\Leftrightarrow$ $S \circ T = I_V$.

COUNTEREXAMPLES

① $T: \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t, 0)$

$S: \mathbb{R}^2 \to \mathbb{R}$, $(t_1, t_2) \mapsto t_1$

② $T: V \to \{0\}$, $v \mapsto 0$

$S: \{0\} \to V$, $0 \mapsto 0$

MATRIX REPRESENTATION OF LINEAR MAPS

LET V BE FINITE DIMENSIONAL, $\dim(V) = n < \infty$,

$\beta = \{v_1, v_2, v_3, \dots, v_n\}$ AN ORDERED BASIS

FOR $v = \sum_{i=1}^{n} a_i v_i$ WITH $a \in \mathbb{F}$.

DENOTE
$$[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$
"THE COORDINATE VECTOR IN THE BASIS β"

THE MAP $\varphi_\beta : V \to F^n$, $v \mapsto [v]_\beta$.

IS A LINEAR ISOMORPHISM WHICH "IDENTIFIES" V WITH F^n.

EXAMPLE:

$$V = \mathcal{P}_3(\mathbb{R}), \quad \beta = \{1, x, \ldots, x^3\}$$

$$\varphi(x) = 1 + 2x - 3x^3$$

SUPPOSE V HAS AN ORDERED BASIS

$$\beta = \{v_1, v_2, \ldots, v_n\}.$$

W —— " ——

$$\gamma = \{w_1, w_2, \ldots, w_m\}$$

$T \in \mathcal{L}(V, W)$ IS DETERMINED

BY ITS ACTION ON ON BASIS VECTORS.

FOR $j \in [1, n]$:

$$T(v_j) = \sum_{i=1}^{m} A_{ij} w_i$$

$$[T]_\beta^\gamma = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix}$$

THE MAP $\mathcal{L}(V,W) \to M_{m \times n}(\mathbb{F})$,

$$T \longmapsto [T]_\beta^\gamma$$

IS A LINEAR ISOMORPHISM