1 Problem

Problem. Suppose $n \geq 1$ and $V = F^n$. Find a linear transformation $T : V \to V$ such that V has a basis consisting of just one cycle of generalised eigenvectors. Write down such a basis (for your T).

Solution.

Suppose $V = F$.

Let $T = I$. Then (1) is an eigenvector, and thus $(T - I)(1) = 0$. Moreover, $\{1\}$ is a basis, and it consists of one cycle. $\square$

Problem. Suppose that $S : V \to V$ satisfies $S^r = 0$ for some integer $r \geq 1$. Show that the characteristic polynomial $f(t)$ of S is $(-1)^n t^n$, where $n = \dim V$.

Lemma 1.1

Suppose that $S : V \to V$ satisfies $S^r = 0$ for some integer $r \geq 1$. Then $S^{\dim V} = 0$.

Proof.

Since S is nilpotent, we know that $K_0(S) = V$, by definition of a generalised eigenspace. Therefore, by Theorem proved in the last assignment (because there exists r such that $\operatorname{rank} S^{r+1} = \operatorname{rank} S$), we know that $S^{\dim V} = 0$. $\square$

Solution.

Choose a basis of $\ker S$. Extend it to a basis of $\ker S^2$. Repeating the procedure, finally obtain a basis β of V. Note that such a basis exists, because $S^{\dim V} = 0$ and thus the process terminates.

Consider $[S]_\beta$. Since the first elements in β by construction are in $\ker S$, we know that at least the first column of $[S]_\beta$ consists of all zeroes.

The next subset of elements belongs to $\ker S^2$. Suppose $v \in \ker S^2$. Then $Sv \in \ker S$, because $S(Sv) = 0$, and thus $\ker S^2 \subseteq \ker S$ and Sv can be represented as a linear combination of the basis elements of $\ker S$, and thus all the corresponding nonzero entries lie above the diagonal. Repeating the procedure, we see that all the elements in $[S]_\beta$ lie above the diagonal. Since $S^{\dim V} = 0$, we know that the process terminates.

Hence, there is a matrix representation such that $[S]_\beta$ is upper-triangular with the diagonal entries all equal to zero, which by Laplacian expansion means that the characteristic polynomial is equal to $(-1)^{\dim V} t^n$, as required. $\square$

Problem. Suppose that $S : V \to V$ satisfies $S^r = 0$ for some integer $r \geq 1$ and suppose that $\text{nullity}(S) = 1$. Find a Jordan canonical form of S.

Solution.

By the previous problem, we know that there is a basis β of V such that $[S]_\beta$ is upper triangular with all the values on the diagonal equal to 0. Therefore, all the eigenvalues of S are equal to 0.

Since nullity $S = 1$, we know that there is only one element in $\ker S$. Since if $v \in \ker S^2$, thus $Sv\ in \ker S$, and we know that there is only one possible nonzero entry, right above the diagonal. Similarly, for $v \in \ker S^3$, $Sv \in \ker S^2$, and the same reasoning applies.

Therefore, JCF of S is a single Jordan block with the zeroes on the diagonal. $\square$