Problem. Consider the matrix over $\mathbb{F} = \mathbb{R}$: $A = \begin{pmatrix} -1 & 0 & -3 \\ 1 & 2 & 3 \\ 4 & 0 & 6 \end{pmatrix}$. Determine its eigenvalues.

Solution. Consider $\det(A - \lambda I)$.
Expanding over the second column, we obtain

$$\begin{aligned} \det(A - \lambda I) = (-1)^{2+2}(2-\lambda) \begin{pmatrix} -1-\lambda & -3 \\ 4 & 6-\lambda \end{pmatrix} &= (2-\lambda)((\lambda+1)(\lambda-6)+12) \\ &= (2-\lambda)(\lambda^2 - 5\lambda + 6) \\ &= -(2-\lambda)^2(\lambda - 3) \end{aligned}$$

Thus, possible eigenvalues are 2 and 3. □

Problem. Determine all eigenvectors for corresponding eigenvalues of A.

Solution. For $\lambda = 2$ and $x, y, z \in \mathbb{F}$:

$$(A - \lambda I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 0 & -3 \\ 1 & 0 & 3 \\ 4 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$$

Thus,

$$-3x + 0 - 3z = 0 \tag{1}$$
$$x + 0 + 3z = 0 \tag{2}$$
$$4x + 0 + 4z = 0, \tag{3}$$

and hence $x = -z, z = 0 = x$. Thus, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector spanning E_2.

For $\lambda = 3$,

$$(A - \lambda I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 & 0 & -3 \\ 1 & -1 & 3 \\ 4 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$$

Thus,

$$-4x + 0 - 3z = 0 \tag{4}$$
$$x - y + 3z = 0 \tag{5}$$
$$4x + 0 + 3z = 0 \tag{6}$$

and thus $z = -\frac{4}{3}x, y = x + 3z = -3x$, giving a corresponding eigenvector $\begin{pmatrix} 3 \\ -9 \\ -4 \end{pmatrix}$ spanning E_3. □