1 More on Sequences

Definition 1.1. A sequence $\{a_n\}$ is a **Cauchy sequence** if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ whenever $m, n > N$.

Theorem 1.2

A sequence is Cauchy if and only if it converges.

Proof.

The converse follows easily from the definition of convergence and triangle inequality.

Suppose now that $\{a_n\}$ is Cauchy.

Consider $\lim_{n\to\infty} \sup a_n$, $\lim_{n\to\infty} \inf a_n$. We want to show that they exist and are equal to each other.

Given $\epsilon > 0$, choose N as above. Then $|a_n - a_{n+1}| < \epsilon$ for any $n > N$. Thus $|a_n - a_m| < 2\epsilon$ for any $m, n > N$.

We know that $|\lim_{n\to\infty} \sup a_n - a_{N+1}| < \epsilon$ and $\liminf_{n\to\infty} a_n - a_{N+1} < \epsilon$.

Therefore, $|\limsup a_n - \liminf a_n| < 2\epsilon$, and thus $\limsup a_n = \liminf a_n$ and hence there exists $\lim a_n$ exists. $\square$

We now consider nonnegative series.

We have defined the nth partial sum of a sequence as $s_n = \sum_{k=1}^{n} a_k$. Note that they are non-decreasing, since $a_k \geq 0$.

Theorem 1.3

If a sequence $\{a_i\}$ is nonnegative, then $\sum_{n=1}^{\infty} a_n$ converges if and only if the partial sums are bounded.

Remark 1.4. There is no equivalent result for sequences that are not nonnegative. Take for example, the alternating sequence of 1 and -1. Then s_n are bounded, but there is no limit.

Theorem 1.5

Suppose that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.

If $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{i=1}^{\infty} a_i$ converges.

Proof.

Note that $S_k(\{a_i\}) \leq S_k(\{b_i\})$.

Suppose that $\sum_{i=1}^{\infty} b_i$ converges and thus $S_k(\{b_i\})$ is bounded.

Therefore, $S(\{a_i\})$ is bounded and hence $\sum_{i=1}^{\infty} a_i$ converges. $\square$

Example 1.6

$\sum_{n=1}^{\infty} \frac{1}{6 \cdot 2^n + \cos(n)} \leq \sum_{n=1}^{\infty} \frac{1}{5 \cdot 2^n}$ and thus $\{a_i = \frac{1}{6 \cdot 2^n + \cos(n)}\}$ converges.

Consider the graph of $f(x) = \frac{1}{x}$ and the corresponding harmonic series.

Note that
$$\sum_{i=1}^{\infty} \frac{1}{n} > \int_{i=1}^{\infty} \frac{\mathrm{d}t}{t} = \lim_{R \to \infty} \int_{i=1}^{R} \frac{\mathrm{d}t}{t} = \lim_{R \to \infty} \log(R).$$

We can then define an **integral test**:

Definition 1.7.

Suppose f is a non-increasing punction and $a_n = f(n)$. Then $\sum_{i=1}^{\infty} a_n$ converges if and only if $\int_{i=1}^{\infty} f(x)\,\mathrm{d}x$ converges.

We can still use this test as long as the hypothesis is satisfied after some ponts.