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1 Preliminaries

1.1 Set Theory

- $\mathbb{N}$, $\mathbb{Q}$
- Operations:
  - union
  - intersection
  - symmetric difference
  - difference
- Equivalence relations and equivalence classes

1.2 Logic

- Quantifiers
  - presentation style: quantifiers first and assertions last
  - quantifier order is crucial
- Operators
  - inclusivity of $\vee$

1.3 Cuts

Theorem 1.1
$\sqrt{2}$ is irrational.

Definition 1.2. A **cut** in $\mathbb{Q}$ is a pair of subsets of A, B of $\mathbb{Q}$ such that:

(a) $A \cup B = \mathbb{Q}, A \neq \emptyset, B \neq \emptyset, A \cap B = \emptyset$.

(b) If $a \in A$ and $b \in B$, then $a < b$.

(c) A contains no largest element.

Definition 1.3. A **real number** is a cut in $\mathbb{Q}$.

Definition 1.4. The cut $x = A|B$ is **less than or equal** to the cut $y = C|D$, if $A \subset C$.

Definition 1.5. $M \in \mathbb{R}$ is an **upper bound** for a set $S \subset \mathbb{R}$ if each $s \in S$ satisfies $s \leq M$. Thus, we say the set S is **bounded above** by M.

An upper bound for S that is less than all other upper bounds for S is a **least upper bound** for S.

Theorem 1.6 (Least Upper Bound Property)

The set $\mathbb{R}$, constructed by the means of Dedekind cuts, is **complete**:

If S is a non-empty subset of $\mathbb{R}$ and is bounded above, then in $\mathbb{R}$ there exists a least upper bound for S.

Proof. Let $\mathscr{C} \in \mathbb{R}$ be any non-empty collection of cuts. Suppose $\mathscr{C}$ is bounded above by some $X|Y$.

Define two sets as follows:

$$C = \{a \in \mathbb{Q} \,|\, \exists (A|B \in \mathscr{C}) : a \in A\} \tag{1}$$
$$C' = \mathbb{Q} \setminus C \tag{2}$$

We claim that $C|C'$ is a cut, checking three conditions.

(a) $C \cup C' = \mathbb{Q}$, $C \neq \emptyset$, since $\mathscr{C}$ is not empty by definition, $C' \neq \emptyset$, since $\mathscr{C}$ is bounded above, and $C \cap C' = \emptyset$ by definition of C and C'.

(b) If $c \in C$ and $c' \in C'$, then $c < c'$, since, for all $d \in C'$, $d \notin C$.

(c) C contains no largest element, since any A in $A|B$ of $\mathscr{C}$ contains no greatest element.

Note that, for all A in $A|B$ of $\mathscr{C}$, $A \subset C$, and hence $C|C'$ is an upper bound for $\mathscr{C}$. Let $D|D'$ be any upper bound for $\mathscr{C}$. Therefore, for all $A|B \in \mathscr{C}$, $A \subset D$, and hence $C \subset D$, giving $C|C' \leq D|D'$. Thus, of all upper bounds for $\mathscr{C}$, $C|C'$ is the least. □

Theorem 1.7

The set $\mathbb{R}$ of all cuts in $\mathbb{Q}$ is a complete ordered field that contains $\mathbb{Q}$ as an ordered subfield.

Theorem 1.8 (Triangle Inequality)
$\forall(x, y \in \mathbb{R}) : |a+b| \leq |a| + |b|$

Definition 1.9. Let $a_1, a_2, a_3, \dots = (a_n)$, $n \in \mathbb{N}$, be a sequence of real numbers.

The sequence (a_n) **converges to a limit** $b \in \mathbb{R}$ as $n \to \infty$ provided that for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$|a_n - b| < \epsilon$$

Definition 1.10 (Cauchy Condition)**.** $\forall(\epsilon < 0)\exists(N \in \mathbb{N}) : n, m \geq N \Rightarrow |a_n - a_m| < \epsilon$.

Theorem 1.11
$\mathbb{R}$ is **complete** with respect to Cauchy sequences, that is, if (a_n) is a sequence of real numbers obeying a Cauchy condition, then it converges to a limit in $\mathbb{R}$.

Proof. Let A be the set of real numbers comprising the sequence (a_n),

$$A = \{x \in \mathbb{R} \,|\, \exists n \in \mathbb{N} : a_n = x\}.$$

Since A obeys the Cauchy condition, then for $\epsilon = 1$ there exists an integer N_1 such that for all $n, m \geq N_1$, $|a_n - a_m| < 1$. Then, for each $n \geq N_1$,

$$|a_n - a_{N_1}| < 1. \tag{3}$$

Therefore, for $n \geq N_1$, $a_n \in (a_{N_1} - 1, a_{N_1} + 1)$.

For the finite set $B = \{a_1, a_2, \dots, a_{N_1} + 1\}$ choose $M = \max\{|\min B|, |\max B|\}$.

By definition of M and from the equation 3 it follows that all the elements of A are in $[-M, M]$, and so A is bounded.
Consider now the set

$$S = \{s \in [-M, M] : \exists \text{ infinitely many } n \in \mathbb{N}, \text{ for which } a_n \geq s\ \}$$

.

Since $-M$ is in S, S is not empty. Moreover, S is bounded above by M. Therefore, by the Least Upper Bound property for $\mathbb{R}$, there exists $b \in \mathbb{R}$ such that $b = \sup S$. We claim that the sequence a_n converges to b.

Suppose some $\epsilon > 0$ is given. Since (a_n) satisfies the Cauchy condition,

$$\exists N_2 : m, n \geq N_2 \Rightarrow |a_m - a_n| < \frac{\epsilon}{2}$$

Since b is the least upper bound, $b + \frac{\epsilon}{2}$ is not in S. Thus, terms in (a_n) are greater or equal to $b + \frac{\epsilon}{2}$ only finitely many times. Therefore, there exists N_3 such that

$$n \geq N_3 \Rightarrow a_n \leq b + \frac{\epsilon}{2}.$$

Since b is the least upper bound, note also that $b - \frac{\epsilon}{2}$ is not an upper upper bound. Since real numbers are dense, there exists $s \in S$ such that $s > b - \frac{\epsilon}{2}$, which implies that there exists $N \geq N_3$ such that $a_N > b - \frac{\epsilon}{2}$.

Since $N \geq N_3$,

$$a_N \in (b - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}].$$

□

Theorem 1.12 (Cauchy Convergence Criterion for sequences)

A sequence (a_n) in $\mathbb{R}$ converges if and only if

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N}) : n, m \geq N \Rightarrow |a_n - a_m| < \epsilon$$

Definition 1.13. Let $a < b$ be given in $\mathbb{R}$. Define the **intervals** (a, b) and $[a, b]$ as follows:

$$(a, b) = \{x \in \mathbb{R} : a < x < b\} \tag{4}$$
$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \tag{5}$$

Theorem 1.14

Every interval (a, b) contains both rational and irrational numbers.

Lemma 1.15

$\mathbb{R}$ has the **Archimedean property**: for each $x \in \mathbb{R}$ there is an integer n that is greater than x.

Theorem 1.16 (ϵ-principle)

If a, b are real numbers and if, for each $\epsilon > 0$, $a \leq b + \epsilon$, then $a \leq b$.
If x, y are real numbers and, for each $\epsilon > 0$, $|x - y| \leq \epsilon$, then $x = y$.

1.4 Euclidean Space

Definition 1.17. Given sets A and B, the **Cartesian product** of A and B is the set $A \times B$ of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.
The Cartesian product of $\mathbb{R}$ with itself m times is denoted as $\mathbb{R}^m$.

Definition 1.18. The **dot product** of $x = (x_1, x_2, \dots, x_m), y = (y_1, y_2, \dots, y_m) \in \mathbb{R}^m$ is defined as

$$\langle x, y\rangle = \sum_{i=1}^{m} x_i y_i$$

Lemma 1.19

The dot product operation is bilinear, symmetric, and positive definite, i.e., for any $x, y, z \in \mathbb{R}^m$ and any $c \in \mathbb{R}$,

$$\langle x, y + cz\rangle = \langle x, y\rangle + c\langle x, z\rangle \tag{6}$$

$$\langle x, y\rangle = \langle y, x\rangle \tag{7}$$

$$\langle x, x\rangle \geq 0, \text{ and } \langle x, x\rangle = 0 \Leftrightarrow x = \mathbf{0} \tag{8}$$

Definition 1.20. The **length** or **magnitude** of a vector $x \in \mathbb{R}^m$ is defined to be

$$|x| = \sqrt{\langle x, x\rangle}$$

Theorem 1.21 (Cauchy-Schwarz Inequality)

For all $x, y \in \mathbb{R}^m$, $\langle x, y\rangle \leq |x|\,|y|$.

Corollary 1.22

For all $x, y \in \mathbb{R}^m$,

$$|x + y| \leq |x| + |y|$$

Definition 1.23. The **Euclidean distance** between $x, y \in \mathbb{R}^m$ is defined as the length of their difference.

$$|x - y| = \sqrt{\langle x - y, x - y\rangle}$$

Definition 1.24. The jth coordinate of the point $(x_1, \dots, x_m)$ is the number x_j appearing in the jth position.

Definition 1.25. The jth coordinate axis is the set of $x \in \mathbb{R}^m$ which kth coordinates are zero for all $k \neq j$.

Definition 1.26. The **integer lattice** is the set $\mathbb{Z}^m \subset \mathbb{R}^m$ of ordered m-tuples of integers.

Definition 1.27. The **first orthant** of $\mathbb{R}^m$ is the set of points $x \in \mathbb{R}^m$ all of which coordinates are nonnegative.

Definition 1.28. A **box** is a Cartesian product of intervals in $\mathbb{R}^m$

$$[a_1, b_1] \times \cdots \times [a_m, b_m]$$

Definition 1.29. The **unit cube** in $\mathbb{R}^m$ is the box $[0,1]^m = [0,1] \times \cdots \times [0,1]$.

Definition 1.30. The **unit ball** in $\mathbb{R}^m$ is the set

$$B^m = \{x \in \mathbb{R}^m \,|\, |x| \leq 1\}.$$

Definition 1.31. The **unit sphere** in $\mathbb{R}^m$ is the set

$$S^{m-1} = \{x \in \mathbb{R}^m \,|\, |x| = 1\}.$$

2 Continuity

Definition 2.1. The function $f : [a,b] \to \mathbb{R}$ is **continuous** if for each $\epsilon > 0$ and each $x \in [a,b]$ there is a $\delta > 0$ such that

$$t \in [a,b] \text{ and } |t - x| < \delta \Rightarrow |f(t) - f(x)| < \epsilon$$

2.1 Three Hard Theorems

Theorem 2.2

If f is a continuous function on $[a,b]$ and $f(a) < 0 < f(b)$, then its values form a bounded subset of $\mathbb{R}$. Thus, there exist $m, M \in \mathbb{R}$ such that for all $x \in [a,b]$, $m \leq f(x) \leq M$.

Proof. For $x \in [a,b]$, let

$$V_x = \{y \in \mathbb{R} \,|\, \exists(t \in [a,x]) : y = f(t)\}.$$

Set

$$X = \{x \in [a,b] \,|\, V_x \text{ is a bounded set of } \mathbb{R}\}.$$

We prove now that b is in X.

Since $a \in X$, X is not empty. Note also that b is an upper bound for X.
Thus, there exists in $\mathbb{R}$ a least upper bound $c \leq b$ for X.

Since f is continuous, consder the neighbourhood of c for $\epsilon = 1$. By definition of continuity, there exists a $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < 1$ Since c is the least upper bound for X, there is some $x \in X$ in the interval $[c - \delta, c]$ (otherwise $c - \delta$ is a smaller upper bound for X).

With t varying from a to c, t is first mapped to $f(t) \in V_x$, and then $f(t)$ varies in the bounded set $J = (f(c) - 1, f(c) + 1)$.

The union of two bounded set is a bounded set, and thus V_c is bounded. Therefore, $c \in X$.

If $c < b$, then by continuity, for some $t > c$, $f(t)$ still varies in the bounded set J, which contradicts the fact that that c is an upper bound for X. Thus, $c = b$, $b \in X$, and the values of f form a bounded subset of $\mathbb{R}$. $\square$

Theorem 2.3

If f is a continuous function on $[a, b]$, then there exist some numbers x_0, x_1 in $[a, b]$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all x in $[a, b]$.

Proof. Let $M = \sup\{f(t) \,|\, t \in [a, b]\}$. By theorem 2.2, M exists.

Let $X = \{x \in [a, b] \,|\, \sup\{V_x\} < M\}$, where

$$V_x = \{y \in \mathbb{R} \,|\, \exists(t \in [a, x]) : y = f(t)\}.$$

We first prove that f achieves a maximum on $[a, b]]$.

Case $(\mathbf{1})$**.** $f(a) = M$

Thus, f takes on a maximum at a.

Case (2)**.** $f(a) < M$

Thus, X is not empty and $\sup\{X\}$ exists. Suppose $\sup\{X\} = c$.

If $f(c) < M$, choose $\epsilon > 0$ such that $\epsilon < M - f(c)$. By continuity, there exists a $\delta > 0$ such that $|t - c| < \delta$ implies $|f(t) - f(c)| < \epsilon$. Thus, $\sup\{V_c\} < M$.

If $c < b$, then there exists a point $t > c)$ at which $\sup\{(V_c\} < M$, which contradicts the fact that c is an upper bound of such points.

Thus, $c = b$, and hence $M < M$, which is a contradiction. Therefore, $f(c) = M$, so f achieves a maximum at c.

□

Theorem 2.4

A continuous function defined on an interval $[a, b]$ achieves all intermediate values: if $f(a) = \alpha, f(b) = \beta$, and γ is given such that $\alpha \leq \gamma \leq \beta$ or $\beta \leq \gamma \leq \alpha$, then there exists some $c \in [a, b]$ such that $f(c) = \gamma$.