MAT240: 2016

DETERMINANTS II

REVIEW

PERMUTATION IS AN INVERTIBLE MAP
$$\sigma : \{1, \dots, n\} \to \{1, \dots, n\}.$$

$$\operatorname{sign}(\sigma) = (-1)^{\#\{(i,j) \mid i<j,\ \sigma(i) > \sigma(j)\}}$$

EXERCISE: FOR ANY PERMUTATIONS σ, τ,

$$\operatorname{sign}(\sigma \circ \tau) = \operatorname{sign}(\sigma) \cdot \operatorname{sign}(\tau)$$

$$\operatorname{sign}(\sigma^{-1}) = \operatorname{sign}(\sigma)$$

IF $n=2$,

σ	$\operatorname{sign}(\sigma)$
(12)	$+1$
(21)	-1

$$\det(v_1, v_2) = A_{11}A_{22} - A_{21}A_{12}$$

$n=3$,

σ	$\operatorname{sign}\sigma$
1 2 3	$+1$
2 3 1	$+1$
3 1 2	$+1$
2 1 3	-1
1 3 2	-1
3 2 1	-1

$$\det(v_1, v_2, v_3) = A_{11}A_{22}A_{33} + A_{21}A_{32}A_{13} + A_{31}A_{12}A_{23} - A_{21}A_{12}A_{33} - A_{11}A_{32}A_{23} - A_{31}A_{22}A_{13}$$

Proof of Thm (cont'd).

Use $\det(\ldots) = \sum_{\sigma} \operatorname{sign}(\sigma) \ldots$

as a definition.

Then $\det(e_1, \ldots, e_n) = 1$

Suppose $v_r = v_s$ for $r < s$. | Therefore, $A_{ir} = A_{is}$

The permutations come in pairs σ, σ', where

$\sigma'(j) = \sigma(j)$ if $j \neq r, s$,

$\sigma'(r) = \sigma(s)$, $\sigma'(s) = \sigma(r)$.

Then $\operatorname{sign}(\sigma) = -\operatorname{sign}(\sigma')$,

and $A_{\sigma'(r) r} = A_{\sigma(s) r} = A_{\sigma(s) s}$,

$A_{\sigma'(s) s} = A_{\sigma(r) s} = A_{\sigma(r) r}$

Hence $A_{\sigma'(1) 1} \ldots A_{\sigma'(n) n} = A_{\sigma(1) 1} \ldots A_{\sigma(n) n}$,

so corresponding terms in the sum over permutations cancel. Thus, $\det(v_1, \ldots, v_n) = 0$

Definition. The determinant of $A \in M_{n \times n}(F)$ is defined by $\det(A) = \det(v_1, \ldots, v_n)$, where $v_1, \ldots, v_n$ are columns of A.

From the proof of the theorem, we get:

$$\det(A) = \sum_{\sigma} \operatorname{sign}(\sigma) A_{\sigma(1) 1} A_{\sigma(2) 2} \cdots A_{\sigma(n) n}$$

(d) IF A' IS OBTAINED FROM A BY ADDING A MULTIPLE OF ONE COLUMN TO ANOTHER COLUMN $\Rightarrow \det(A') = \det(A)$.

(e) $\det(A^t) = \det(A)$ (thus (a)-(d) ALSO HOLD FOR ROW OPERATIONS)

(f) $\det(AB) = \det(A) \cdot \det(B)$

PROOF (b), (c), (d) FOLLOW FROM DISCUSSION ABOVE:

det IS MULTILINEAR IN COLUMNS OF A, AND ZERO IF TWO COLUMNS ARE EQUAL.

(a) ~~these (b), (c), (d)~~

USING b, c, d, CHANGE A INTO REDUCED COLUMN ECHELON FORM. (THIS CHANGES $\det(A)$ BY A NON-ZERO SCALAR). THEREFORE, IT'S INVERTIBLE IFF all DIAGONAL ENTRIES ARE NON-ZERO, SO $\det(A) \neq 0$.

(e) $$\sum_{\sigma} \underbrace{\mathrm{sign}(\sigma)}_{\mathrm{sign}(\sigma^{-1})} \underbrace{A_{\sigma(1)1} \cdots A_{\sigma(n)n}}_{A_{1\sigma^{-1}(1)} \cdots A_{n\sigma^{-1}(n)}}$$

PROPOSITION: Suppose $A \in M_{n \times n}(F)$ is upper triangular. Then $\det(A)$ is the product of the diagonal entries.

PROOF Upper-triangular $\Rightarrow A_{ij} = 0$ whenever $i > j$.

Thus, a permutation σ does not contribute to $(*)$ unless

$\sigma(1) \geq 1, \sigma(2) \geq 2, \ldots, \sigma(n) \geq n.$

But this means $\sigma(n) = n$, $\sigma(n-1) = n-1$, etc.

$\Rightarrow \det(A) = A_{11} \cdot A_{22} \cdot A_{33} \cdots A_{nn}$

$\square$

In summary, $\det(A) = \sum_{\sigma} \operatorname{sign}(\sigma) A_{\sigma(1) 1} A_{\sigma(2) 2} \cdots A_{\sigma(n) n}$.

PROPERTIES OF A DETERMINANT:

(a) $\det(A) \neq 0 \Leftrightarrow$ columns of A are linearly independent

$\Leftrightarrow A$ invertible.

(b) If A' is obtained from A by interchanging two columns, then $\det(A') = -\det(A)$.

(c) If A' is obtained from A by multiplying a column by $c \in F$, then $\det(A') = c \det(A)$