ARCHIMEDEAN PROPERTY FOR $\mathbb{Q}$:

IF $x, y \in \mathbb{Q}$, $x \neq 0$,

$\exists\, n \in \mathbb{Z}: \; nx > y$.

PROOF

SUPPOSE $x, y > 0$.

LET $\exists \; a, b, c, d \in \mathbb{Z}$:

$x = \frac{a}{b}$, $y = \frac{c}{d}$; $a, b, c, d > 0$.

WANTED

$n \frac{a}{b} > \frac{c}{d} \iff nad > bc$

$\Leftarrow$ IF $n = 2bc$, $nad = 2bcad \geq 2bc \cdot 1 > bc$.

LEMMA

SUPPOSE $\alpha \in \mathbb{R}$ AND $z \in \mathbb{Q}$, $z > 0$.

THEN $\exists\, x \in \alpha$, $y \in \mathbb{Q} \setminus \alpha$,

$[x, y \in \mathbb{Q}]$ SUCH THAT $y - x = z$

AND $y \neq$ LEAST ELEMENT OF $y \in \mathbb{Q} \setminus \alpha$.

PROOF

CONSIDER $z, 2z, 3z$.

BY THE ARCHIMEDEAN PROPERTY FOR $\mathbb{Q}$,

FOR SOME n, $nz \notin \alpha$.

CHOOSE $m \in \mathbb{Z}$: $mz \notin \alpha \wedge (m-1)z \in \alpha$.

LET $y = mz$, $x = (m-1)z$.

$y - x = z$.

IF $y =$ LEAST ELEMENT OF $\mathbb{Q} \setminus \alpha$

REPLACE y BY $y + \frac{z}{2}$, x BY $x + \frac{z}{2}$.

$\square$

THEOREM $(\alpha+(-\alpha))=0, \forall \alpha \in \mathbb{R}$.

RECALL $-\alpha = \{x \in \mathbb{Q} \mid -x \notin \alpha, x \neq \text{LEAST ELEMENT OF } \mathbb{Q}\setminus\alpha\}$.

ANY ELEMENT OF $\alpha+(-\alpha)$ IS OF THE FORM $x+y$ WITH $x \in \alpha, y \in -\alpha$, $-y \notin \alpha$. $\Rightarrow$

$-y > x \Leftrightarrow x+y<0 \Leftrightarrow x+y \in 0$. $\Rightarrow$

$\alpha+(-\alpha) \subseteq 0$.

TO SHOW THAT $0 \subseteq \alpha+(-\alpha)$,

SUPPOSE $z \in 0 \Leftrightarrow z<0, z \in \mathbb{Q}$.

$\Rightarrow \exists x \in \alpha, -y \in -\alpha ; \ y-x=-z$.

$\Leftrightarrow x-y=z$.

So $0 \subseteq \alpha+(-\alpha)$, hence $(\alpha+(-\alpha))=0$.

DEFINITION $P=\{\alpha \in \mathbb{R} \mid \alpha>0\}$

MULTIPLICATION IF $\alpha, \beta>0$,

$\alpha \cdot \beta = \{x \mid x \leq 0 \text{ OR } x=u \cdot v \text{ with } u \in \alpha, v \in \beta, u,v>0\}$

DEFINITION

$$|\alpha| = \begin{cases} 0 & \text{IF } \alpha=0 \\ \alpha & \text{IF } \alpha>0 \\ -\alpha & \text{IF } \alpha<0 \end{cases}$$

DEFINITION

$$\alpha\beta = \begin{cases} \alpha\beta, & \alpha,\beta>0 \\ -|\alpha|\beta, & \alpha<0, \beta>0 \\ -\alpha|\beta|, & \text{IF } \alpha>0, \beta<0 \\ |\alpha||\beta|, & \text{IF } \alpha,\beta<0 \end{cases}$$