1 Problem III

Let V be a finite-dimensional inner product space over $\mathbb{F}$. Suppose that $T \in \operatorname{End}(V)$.

Lemma 1.1

If T is an orthogonal projection, then $\|Tx\| \leq \|x\|$.

Proof.

Fix $x \in V$.

Since T is an orthogonal projection, then $V = \operatorname{im} T \oplus \ker T$ and $\operatorname{im}(T)^{\perp} = \ker T$. Let $v \in \operatorname{im} T$ and $w \in \ker T$ be such that $x = v + w$. Since T is a projection, $Tv = v$.

Since $\operatorname{im} T$ and $\ker T$ are orthogonal subspaces of V, then $\langle v, w \rangle = 0 = \langle w, v \rangle$, and therefore

$$\begin{aligned} \langle x, x \rangle &= \langle v + w, v + w \rangle && (1) \\ &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle && (2) \\ &= \langle v, v \rangle + \langle w, w \rangle. && (3) \end{aligned}$$

Since $\langle Tx, Tx \rangle = \langle v, v \rangle$, then $\langle Tx, Tx \rangle \leq \langle x, x \rangle$, with equality only when $x \in \operatorname{im} T$. Therefore, $\|Tx\|^2 \leq \|x\|^2$, and since $\|y\| \geq 0$ for all $y \in V$, $\|Tx\| \leq \|x\|$.

□

Lemma 1.2

If T is a projection such that $\|Tx\| \leq \|x\|$ for all $x \in V$, then T is an orthogonal projection.

Proof.

Let $W_1 \subseteq V$ and $W_2 \subseteq V$ be such that $V = W_1 \oplus W_2$, and suppose T is a projection on W_1 along W_2.

Let $D = \{z \in V \mid Tz = z\}$.

If $y \in W_1$, then, by definition of T, $Ty = y$, and hence $y \in D$ and $W_1 \subseteq D$.

If $y \in D$, let $y_1 \in W_1$ and $y_2 \in W_2$ be such that $y = y_1 + y_2$. Since $y \in D$, $T(y_1 + y_2) = y_1 + y_2$, and by definition of T $T(y_1 + y_2) = y_1$. Therefore, $y_2 = 0$ and hence $y_1 = y \in W_1$. Therefore, $D \subseteq W_1$ and thus $D = W_1$.

Suppose $x \in V$.

Let $w_1 \in W_1$ and $w_2 \in W_2$ be such that $x = w_1 + w_2$.

For any $v_1 \in W_1$, $Tv_1 = v_1$ and thus $v_1 \in \operatorname{im} T$ and $W_1 \subseteq \operatorname{im} T$. Similarly, if $v \in \operatorname{im} T$, let $v_1 \in W_1$ and $v_2 \in W_2$ be such that $v = v_1 + v_2$. Therefore, $v_2 = v - v_1 \in \operatorname{im} T$, and since $W_1 \cap W_2 = \{0\}$, then $v_2 = 0$, which means that $v_1 = v \in W_1$ and thus $\operatorname{im} T \subseteq W_1$. Hence, $\operatorname{im} T = W_1$.

Note that $x = Tx + (x - Tx)$.

Therefore, $Tx = T^2x + T(x - Tx) = Tw_1 + T(x - Tx) = w_1 + T(x - Tx) = T(w_1 + w_2) = w_1$, and thus $T(x - Tx) = 0$ and hence $x - Tx \in \ker T$.

Noting that $W_1 = \operatorname{im} T = D$, suppose $z \in W_1 \cap \ker T$. Therefore, $Tz = z = 0$, and thus $V = \operatorname{im} T \oplus \ker T$.

Let $v \in \operatorname{im} T$ and $w \in \ker T$ be such that $x = v + w$.

Note that $V = \ker T \oplus \ker(T)^\perp$.

Let $v' \in \ker(T)^\perp$ and $w' \in \ker T$ be such that $x = v' + w'$.

Note that, by definition of T, $Tx = T(v' + w') = Tv' + Tw' = Tv' = v = Tv$. Thus, $T(v' - v) = 0$, and hence $v' - v \in \ker T$. But $\langle v', v' - v\rangle = 0$, because $v' \in \ker(T)^\perp$, and thence $\langle v', v'\rangle = \langle v', v\rangle$.

Moreover, $\|Tv'\| \le \|v'\|$, which means that $\langle Tv', Tv'\rangle = \langle v, v\rangle \le \langle v', v'\rangle = \langle v', v\rangle$.

Hence, $\langle v - v', v\rangle \le 0$.

However, since $\langle v', v' - v\rangle = 0$, then $-\langle v', v - v'\rangle = -\overline{\langle v - v', v'\rangle} = 0$.

Thus, $\langle v - v', v'\rangle = 0$.

Therefore, $\langle v - v', v - v'\rangle \le 0$, which is possible only when $v = v'$ (by positive definiteness of $\langle \cdot, \cdot\rangle$). Therefore, $v \in \ker(T)^\perp$ and $v' \in \operatorname{im} T$, and thus $\operatorname{im} T = \ker(T)^\perp$.

Hence, T is an orthogonal projection. $\square$

Lemma 1.3

If T is a projection and T is normal, then T is an orthogonal projection.

Proof.

Since T is a projection, we have shown in the proof of Lemma 1.2 that $V = \operatorname{im} T \oplus \ker T$.

Note that $\ker T^* = \operatorname{im}(T)^\perp$ and $\operatorname{im} T^* = \ker(T)^\perp$ by the properties of T^*.

Suppose $x \in \ker T^*$. Therefore, since T is normal, then $T^*Tx = TT^*x = 0$, and thus $Tx \in \ker T^* = \operatorname{im} T^\perp$. Since $Tx \in \operatorname{im} T$, then $\langle Tx, Tx\rangle = 0$, which means that $Tx = 0$ and thence $x \in \ker T$. Therefore, $\ker T \subseteq \ker T^*$.

Suppose now that $x \in \ker T$. Again, since T is normal, then $T^*Tx = 0 = TT^*x$, and thus $T^*x \in \ker T$. Since $T^*x \in \operatorname{im} T^* = \ker(T)^\perp$, then $\langle T^*x, T^*x\rangle = 0$, which means that $T^*x = 0$ and thence $x \in \ker T^*$. Therefore, $\ker T^* \subseteq \ker T$, and hence $\ker T^* = \ker T$. But $\ker T^* = \operatorname{im} T^\perp$, and thus $\ker T = \operatorname{im} T^\perp$ and T is an orthogonal projection. $\square$