1 Dynamical Systems and Bifurcations

1.1 Introduction

Suppose that (X, f) is given, where X is a manifold and f is a mapping from X to X. Usually we also say that f is continuous, and X is compact. (X, f) defines a *dynamical system*.

If f is C^k-smooth, then a dynamical system is *C^k-smooth*.

If f is invertible and f^{-1} is continuous, then (X, f) is an *invertible* dynamical system.

1.2 Examples

Example 1.1

When a link between differential equations and dynamical systems was realised, a problem of Laplacian determinism came to the attention, which led to the study of order and chaos.

We can imagine a circle as a glued interval from 0 to 1, or, equivalently, as $\mathbb{R}/\mathbb{Z}$.

Take $f(x) = 2x \mod 1$. If we know the initial point with an error of 10^{-30}, after every step the error is doubled. Therefore, after 100 iterations, we cannot say anything certain about the position of our dot. This is an example of *dynamical chaos*, or *trajectory divergence*. The same behaviour is seen, for example, in weather forecasts, since we cannot measure temperature and pressure everywhere all the time.

There are, however, examples with more regular behaviour. Take, for instance, $R_\alpha : X \mapsto X + \alpha \mod 1$, with $\alpha \notin \mathbb{Q}$.

Definition 1.2. A sequence $x_0, x_1 = f(x_0), \dots, x_n = f^n(x_0), \dots$ is called a *trajectory* of a point x_0.

Definition 1.3. A point x is called *periodic*, if there exists n such that $f^n(x) = x$.

Theorem 1.4

If $\alpha \notin \mathbb{Q}$, then for all $x_0 \in S^1$ the trajectory of x_0 under the action of R_α is dense everywhere.

Definition 1.5. A periodic point x of period 1, $f(x) = x$, is called a fixed point.

Definition 1.6. For dim = 1, a multiplicative fixed point is a fixed point such that $f'(x_0) = \lambda$.

This means that $f(x) - x_0 \approx (x - x_0)$, so that $|\lambda| < 1$ is an *attracting point*, and if $|\lambda| > 1$, then x_0 is *repelling*.

1.3 Revertible Chaos

Take $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Note that $\sqrt{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ maps $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} x + y \\ x \end{pmatrix}$.

Since a Fibonacci sequence is asymptotically geometric with the golden ratio as the multiplicative factor, then in the long term a standard plane with the Descartes coordinate system, then the slope of the horizontal axes coincides with ϕ. Notice that the inverse maps $\binom{u}{v}$ to $\binom{v}{u-v}$.

Take a two-dimensional torus on $\Pi^2 = \mathbb{R}^2/\mathbb{Z}^2$, and map the trajectories defined by the aforementioned f. Then due to the topology of a torus chaos would emerge, and its development is easy to track.

1.4 Complex Dynamicals

Suppose that we are given $f : \mathbb{C}P^1 \to \mathbb{C}P^1$.

Assume further that f is a polynomial.

Let $K_f = \{z \,|\, f^n(z) \not\to \infty\}$.

Notice that in a projective space the point at infinity is attracting with respect to f. We can simplify the study of dynamicals in a projective space by making a substitution $z = \frac{1}{w}$:

$$\frac{1}{f(\frac{1}{w})} = \frac{1}{(\frac{1}{w})^n + a_{n-1}(\frac{1}{w})^{n-1} + \dots} \tag{1}$$

$$= \frac{w^n}{1 + a_{n-1}w + a_{n-2}w^2 + \dots}. \tag{2}$$

For example, $f(z) = z^2$ is mapped to a disc.

Definition 1.7. A Julia set of a polynomial f is $J_f = \partial K_f$.

Note. Julia sets are often fractals.

1.5

Definition 1.8. Let f be a rational function, $f : \mathbb{C}P^1 \to \mathbb{C}P^1$.

Note. Rational functions are *good* mappings of a Riemann sphere to itself.

Consider a torus E in $\mathbb{C}/\Lambda$,where $\Lambda = \mathbb{Z} \oplus i\mathbb{Z}$, and take a mapping $f : E \to E$ such that $z \mapsto 2z \mod \Lambda$.

Let's make a sphere out of our torus: $E/_{(z\sim -z)} \simeq \mathbb{C}P^1$. Define a map $p : E \to \mathbb{C}P^1$, which is also known as Weirstraß p-function, $p(z) = \frac{1}{z^2} + \sum_{\gamma\in\Lambda \ \{0\}} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2}\right)$.

This map defines an exciting example of a dynamical system, which plays a key role in the Lattès construction.

1.6 Bifurcations

Definition 1.9. A *bifurcation* is a rapid change in the qualitative behaviour of a dynamical system.

Define $f_\epsilon = x + x^2 + \epsilon$. Fixed points of f_ϵ are thus at $x = \pm\sqrt{-\epsilon}$. Note that, for example, if ϵ is less than 0, then $-\sqrt{\epsilon}$ is attracting, and $\sqrt{\epsilon}$ is repelling. This is an example of a saddle-knot bifurcation.

If the mapping is perturbed, we can study what happens with fixed points. For example, a fixed point can degenerate into an orbit with a period 2. On the other hand, the stability interval can eventually narrow down to a point at which a small perturbation pushes the trajectory to drastic instability. This is, for instance, what happens in the *Andronov-Hopf* regime.