1 Twisted Rabbits

Twisted rabbits have arised in the study of discrete dynamic systems.

1.1 Introduction

Definition 1.1. The *Douady rabbit* is a fractal defined by $P_c(z) = z^2 + c$ such that 0 is periodic with the period 3, which means that $P_c P_c P_c(0) = 0$, and $P_c \neq 0$.

Thus, $(c^2 + c)^2 + c = 0$, which means that $c^3 + 2c^2 + c + 1 = 0$. We require that $\operatorname{im} c > 0$.

The mapping P_c of a point defines its orbit, which is a set $\{z, P_c(z), P_c(P_c(z)) = P_c^{\circ 2}(z), \ldots\}$.

Only a finite area of the complex plane does not approach infinity exponentially fast. This set is called a **Julia set**.

Definition 1.2. A filled Julia set is a set $K(P_c) = \{z \mid P_c^{\circ n}(z) \not\to \infty\}$.

To proceed, we will need a concept of a Jordan curve, which is obtained when a circle is mapped continuously to a plane in such a way that the mapped points do not coincide.

The *body* of a rabbit looks like a circle. It also has a pair of *ears*. Then a scaled copy of itself looks like a pair of paws. A fractal structure is obtained. Our rabbit is an example of a filled Julia set. Note that no ears of any degree intersect each other.

The zero point is located at the centre.

Let's $v = P_c(0) = c$ and $w = P_c^{\circ 2}(0) = P_c(v)$. We can see that $0 \mapsto v \mapsto w \mapsto 0$.

Since the derivative at zero is zero, all the neighbouring points on mapping get even closer.

It is worthwhile to note that the picture shows the dynamic behaviour of the system. Thus, all the points get eventually into the periodic cycle of $0, v, w$, which act as attractors. There is also a stable point α, to which the ears are attached, and around which all the points jump around. If α is omitted, our rabbit would fall apart into three parts.

Another important point is β, which is a limit of the images of 0.

Drawing a rabbit is an easy and relatively fast programming task, which is partially due to the fast exponential runaway of the mapped points.

Now, we introduce a concept of a *Thurston lamination.*

Draw a unit circle. Now we take points and assign the angle measure, which, for our purposes, would range from $(0, 1]$. First, take 0, and assign the angle of 1. Then mark the points with the angles from $\frac{1}{7}$ to $\frac{6}{7}$.

Denote a map doubling the angle as σ_2.

Connect the points with angles $\frac{1}{7}$, $\frac{2}{7}$ and $\frac{4}{7}$ with lines, from $\frac{1}{7}$ to $\frac{2}{7}$, from $\frac{2}{7}$ to $\frac{4}{7}$, and from $\frac{4}{7}$ to $\frac{1}{7}$.

To obtain the representation of a rabbit on a circle, we need to find an equivalence relation which is invariant with respect to our mapping.

We map our triangle to a new triangle connecting the points with angles of $\frac{1}{14}$, $\frac{9}{14}$ and $\frac{11}{14}$.

Our goal is to construct the images of our triangles so that they do not intersect. We can prove that it can always be achieved.

There is also a *stripe* defined by the images, which lacks triangles covering the entire stripe.

Note. A filled Julia set is connected.

In this way, we obtain a countable number of triangles in the unit circle.

1.2 Relatives

What happens if we take other values of c?

If, for example, $\mathrm{im}(c) < 0$, we obtain an antirabbit, which can be identified by the order of the ears.

If, however, $\mathrm{im}(c_*) = 0$, then we obtain an *aeroplane*.

To study the aeroplane, we construct a Cantor set, which is a segment with a countable number of disjoint open intervals omitted in such a way that there is no interval left at the end. Cantor sets are also identified as *Cantor dust*. In particular, we build a Cantor necklace, where circles are inserted into the omitted intervals, and then another necklaces grow on the circles themselves.

Again, let's draw a unit circle and divide it into seven parts.

In the case of the aeroplane, however, we would not get triangles in the lamination. First, we connect the points with angles $\frac{1}{7}$ and $\frac{6}{7}$, then with angles $\frac{2}{7}$ and $\frac{5}{7}$, and finally with $\frac{3}{7}$ and $\frac{4}{7}$. The critical stripe is obtained in the region defined by the chords connecting $\frac{2}{7}$ and $\frac{5}{7}$, and $\frac{3}{14}$ and $\frac{11}{14}$. The difference of the lamination of the aeroplane from the lamination corresponding to the rabbit is that the number of chords is uncountable.

1.3 Generalisations

The mapping that we used for the rabbit plays an important role. However, to prove results in complex analysis, only the intrinsic properties of the mapping are used, not the explicit equation. This is due to the fact that the form of the equation does not tell much about the topological properties of the system obtained.

This is why we use such tools like Thurston laminations.

The mapping $z^2 + c$ can be viewed as the mapping of the sphere S^2 into itself.

The preimage of a small disc in the topological sense (an area bound by a Jordan curve) consists of several discs, on which the mapping is $z \mapsto z^k$, where $k \in \mathbb{Z}^+$.

Suppose that we take a map f between two discs, and we define two homeomorphisms conserving the orientation, ϕ_1 and ϕ_2, on each of the discs.

Note that $z^2 + c$ is a branched covering.

All branch coverings have critical points, of which there is a finite amount.

To proceed, we need a concept of a postcritically finite branched covering, which is a set of orbits of the critical points:

$$P(f) = \cup \text{ orbits of all critical points.}$$

The theory of postcritically finite branched coverings was actively studied by Thurston, and led to important results in Thurston theory.