Theorem. *Let $V \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the subspace of the space of $\mathbb{R}$-valued functions on $\mathbb{R}$ spanned by the functions $\sin(x)$ and $\cos(x)$. Let $f \in V$.*

Then $\forall a \in \mathbb{R} : g(x) = f(x + a) \in V$

Proof. Since $f \in V$, then $\exists (m, n \in \mathbb{R}) : m \sin(x) + n \cos(x)$.

Therefore, $g(x) = f(x + a) = m \sin(x + a) + n \cos(x + a)$.

Suppose $n = 0$.

Therefore, $g(x) = m \sin(x + a) = (m \cos(a)) \sin(x) + (m \sin(a)) \cos(x)$. Since $m \sin(a)$ and $m \cos(a)$ are in $\mathbb{R}$, then $g(x) \in V$.

Suppose now that $n \neq 0$.

Let $\phi \in \mathbb{R}$ be such that $\cos(\phi) = \frac{m}{\sqrt{m^2+n^2}}$ and $\sin(\phi) = \frac{n}{\sqrt{m^2+n^2}}$. Since $\cos(\phi) < 1$, such ϕ exists.

Thus, $g(x) = \sqrt{m^2 + n^2}(\sin(x + a) \cos(\phi) + cos(x + a) \sin(\phi))$. Hence, $g(x) = \sqrt{m^2 + n^2}(\sin(x + (a + \phi)))$.

But then $g(x) = (\sqrt{m^2 + n^2} \cos(a + \phi)) \sin(x) + (\sqrt{m^2 + n^2} \sin(a + \phi)) \cos(x) \in V$.

Hence, $\forall a \in \mathbb{R} : g(x) \in V$, as required. $\square$