Let $A \in M_{n\times n}(F)$ be the matrix:

$$\begin{bmatrix} t & 1 & 0 & \dots & 0 & 0 \\ 0 & t & 1 & \dots & 0 & 0 \\ 0 & 0 & t & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & t & 1 \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \end{bmatrix}$$

Therefore, expanding the determinant,

$$\det(A) = \det(A^t) \tag{1}$$

$$= \begin{vmatrix} t & 1 & 0 & \dots & 0 & 0 \\ 0 & t & 1 & \dots & 0 & 0 \\ 0 & 0 & t & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & t & 1 \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \end{vmatrix} \tag{2}$$

$$= \sum_{j=1}^{n} (-1)^{n+j} A_{nj} \det(\widetilde{A}_{nj}) \tag{3}$$

Notice that $A_{nj} = a_j$ and $\widetilde{A}$ is an upper triangular matrix with j number of t's in the diagonal, which means that $\det(\widetilde{A}_{nj}) = t^j$.

Therefore,

$$det(A) = \sum_{j=1}^{n} (-1)^{n+j} A_{nj} \det(\widetilde{A}_{nj}) \tag{4}$$

$$= \sum_{j=1}^{n} (-1)^{n+j} a_j t^j \tag{5}$$