MAT 157: BOUNDEDNESS 2016/10/?

THEOREM

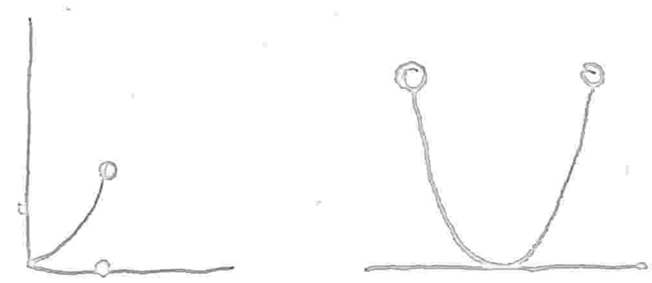
IF $f(x)$ IS CONTINUOUS ON $[a,b]$,
THEN IT ACHIEVES ITS MAXIMUM.

$$\exists c \in [a,b] : f(c) \geq f(x) \; \forall x \in [a,b]$$

NOTE

f MUST BE CONTINUOUS AND DEFINED ON THE CLOSED INTERVAL.

NON-EXAMPLES



COROLLARY

IF $f(x)$ IS CONTINUOUS THEN IT ACHIEVES ITS MIN.

PROOF

BY II HARD THEOREM,

$$\exists N : f(x) \leq N \; \forall x \in [a,b]$$

LET $A = \{ f(x) \mid x \in [a,b] \}$.

A IS A BOUNDED SET.

LET $\alpha = \sup(A)$

SUPPOSE $f(x) \neq \alpha \; \forall x \in [a,b]$

THUS $f(x) - \alpha \neq 0 \ \forall x \in [a, b]$.

f is continuous $\Rightarrow$ $f - \alpha$ continuous

$\Rightarrow \dfrac{1}{f(x) - \alpha}$ is continuous (since $f(x) - \alpha \neq 0$)

$\Rightarrow \exists M : \left| \dfrac{1}{f(x) - \alpha} \right| < M$.

SINCE $\alpha = \sup \{ f(x) \mid x \in [a, b] \}$,

$\forall \varepsilon > 0 \ \exists x \in [a, b] : |f(x) - \alpha| < \varepsilon$,

i.e. $\dfrac{1}{|f(x) - \alpha|} > \dfrac{1}{\varepsilon}$.

IF $\dfrac{1}{\varepsilon} > M$, THIS IS A CONTRADICTION.

Choose $\varepsilon = \dfrac{1}{2M}$, THEN $\dfrac{1}{|f(x) - \alpha|} > 2M > M$

But $\dfrac{1}{|f(x) - \alpha|} < M$ #

GIVEN A CONTINUOUS FUNCTION $f(x)$, GIVEN $\varepsilon > 0$, YOU CAN FIND A $\delta > 0$: $|f(x) - f(a)| < \varepsilon$ WHENEVER $|x - a| < \delta$.

HOWEVER, δ REQUIRED MAY BE DIFFERENT FOR DIFFERENT a'S.

GIVEN $\varepsilon > 0$, CAN $\delta > 0$ BE FOUND SUCH THAT $|f(x) - f(a)| < \varepsilon$ WHENEVER $|x - a| < \delta$ FOR EVERY a?

DEFINITION

A FUNCTION $f(x)$ IS UNIFORMLY CONTINUOUS IF GIVEN $\varepsilon > 0$ $\exists\, \delta > 0$: $|f(x) - f(a)| < \varepsilon$ WHENEVER $|x - a| < \delta$ FOR ANY a IN THE DOMAIN OF f.

THEOREM

IF $f(x)$ IS CONTINUOUS ON $[a, b]$, THEN IT IS UNIFORMLY CONTINUOUS ON $[a, b]$.

PROOF

GIVEN $\varepsilon > 0$,

SAY f IS ε-GOOD ON $[a, c]$

IF $\exists\, \delta > 0 : |f(x) - f(d)| < \varepsilon$

WHENEVER $|x - d| < \delta$ AND

$x, d \in [a, c]$.

LET $A = \{ x \mid f$ IS ε-GOOD ON $[a, x] \}$.

SINCE $a \in A$, $A \neq \emptyset$.

LET $\alpha = \sup(A)$.

$\exists\, \delta_1 : | f(x) - f(\alpha)| < \frac{\varepsilon}{2}$ IF $|x - \alpha| < \delta_1$.

CHOOSE A POINT $c \in (\alpha - \delta_1, \alpha)$, $c \in A$,

SO f IS ε-GOOD ON $[a, c]$

SUPPOSE $x \in [a, c]$ AND $y \in (c, \alpha + \delta_1)$:

$$|f(x) - f(y)| = | f(x) - f(c) + f(c) - f(y)|$$

$$\leq |f(x) - f(c)| + |f(c) - f(y)|$$

IF $|f(x) - f(c)| < \varepsilon$ WHENEVER $|x - c| < \delta$