MAT157: Limits III

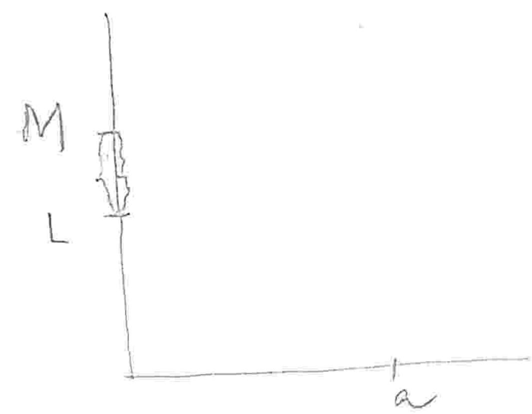
20161007

SUPPOSE $\lim_{x \to a} f(x) = L$ AND $\lim_{x \to a} f(x) = M$

$\Rightarrow L = M$

PROOF

SUPPOSE $L \neq M$.



LET $\varepsilon = \dfrac{|M-L|}{2}$.

BY THE HYPOTHESIS,

$\exists \delta : |f(x) - L| < \varepsilon$ IF $0 |x-a| < \delta$

AND

$\exists \delta' : |f(x) - M| < \varepsilon$, IF $0 < |x-a| < \delta'$

LET $\delta'' = \min(\delta, \delta')$, SO THAT BOTH INEQUALITIES HOLD.

$$2\varepsilon = |M-L| = |M - f(x) + f(x) - L|$$
$$\leq |M - f(x)| + |f(x) - L|$$
$$< \varepsilon + \varepsilon \quad \#$$

THEOREM

$$\left\{\begin{array}{l}\lim_{x \to a} f(x) = L \\ \lim_{x \to a} g(x) = M\end{array}\right.$$

$$\Rightarrow \lim_{x \to a} (f+g)(x) = L + M$$

PROOF

CONSIDER

$|(f+g)(x) - (L+M)| =$

$\leq |f(x) - L| + |g(x) - M|$

GIVEN $\varepsilon > 0$, CHOOSE $\delta > 0$

$|f(x) - L| < \frac{\varepsilon}{2}$ AND

$|g(x) - M| < \varepsilon/2$ WHENEVER

$0 < |x-a| < \delta$.

IF $0 < |x-a| < \delta$

$|(f+g)(x) - (L+M)| =$

$\leq |f(x) - L| + |g(x) - M|$

$< \varepsilon$

THEOREM

IF $p(x)$ IS A POLYNOMIAL, $a \in \mathbb{R}$,
THEN $\lim_{x \to a} p(x) = p(a)$

THEOREM

$\lim_{x \to a} f(x) = L, \ \lim_{x \to a} g(x) = M$

$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, PROVIDED $M \neq 0$

PROOF.

$\left(\frac{f(x)}{g(x)} - \frac{L}{M}\right) \leq \left|\frac{f(x) - L}{g(x)}\right| + |L| \left|\frac{1}{g(x)} - \frac{1}{M}\right|$

$= \left|\frac{f(x) - L}{g(x)}\right| + |L| \left|\frac{M - g(x)}{g(x) M}\right|$ ①

Choose δ : $|g(x) - M| \leq \frac{M}{2}$

$\Rightarrow |g(x)| > \frac{M}{2}$

$\Rightarrow \frac{1}{|g(x)|} < \frac{2}{M}$

NEED: $|f(x) - L| < \frac{|M|}{2} \frac{\varepsilon}{2}$

$|g(x) - M| < \frac{|M|^2}{2|L|} \frac{\varepsilon}{2}$

$\Rightarrow$ ① $< |f(x) - L| \frac{2}{|M|} + \frac{|L|}{|M|} |g(x) - M| \frac{2}{|M|}$

$\Rightarrow \varepsilon$

THEOREM

$$\lim_{x\to a} f(x)g(x) = L\cdot M$$

PROOF

$|(f(x)g(x) - L\cdot M)| =$

$= |f(x)g(x) - Lg(x) + Lg(x) - LM|$

$= |(f(x)-L)\,g(x) + L\,(g(x)-M)|$

$\leq \quad |(f(x)-L)\,g(x)| + |L(g(x)-M)|$

Choose $\delta > 0$:

$|g(x)-M| < \dfrac{\varepsilon}{2|L|}$ AND

$|g(x)-M| < 1.$

$|g(x)-M| < \min\left(1, \dfrac{\varepsilon}{2|L|}\right)$

$\Rightarrow \quad |g(x)-M| \geq |g(x)| - |M|$

$\Rightarrow \quad 1 > |g(x)-M| \geq |g(x)| - |M|$

$\Rightarrow \quad |g(x)| < |M| + 1$

IF δ IS CLOSE SO THAT IN ADDITION,

$|f(x)-L| < \dfrac{\varepsilon}{2(|M|+1)}$, THEN

$|f(x)g(x) - LM| \leq |f(x)-L|\,|g(x)| + |L|\,|g(x)-M|$

$< \dfrac{\varepsilon}{2|M+1|}|M+1| + |L|\dfrac{\varepsilon}{2|L|},$

IF $|L| \neq 0$. IF $L=0$,

$|f(x)g(x)| \leq |f(x)g(x)|$