20161117

MAT240:
LINEAR SYSTEMS OF EQUATIONS

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

ABOUT INVERSES OF MATRICES

$$(A \mid I) \xrightarrow[\text{ROW OPERATIONS}]{} (I \mid B)$$

EXAMPLE

$V = \mathcal{P}_n(\mathbb{R})$; $c_0, \ldots, c_n$ distinct

$\gamma = \{p_0, \ldots, p_n\}$ LAGRANGE pol's

$\beta = \{1, x, \ldots, x^n\}$ STANDARD BASIS

$$Q = [I_V]_\beta^\gamma = \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ \vdots & & & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}$$

$Q^{-1} = [I_\gamma]_\gamma^\beta$ HAS COLUMNS AS THE COEFFICIENTS OF THE LAGRANGE POLYNOMIALS

EXAMPLE : $n=2$, $c_0=0$, $c_1=1$, $c_2=3$

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{pmatrix} \underset{\text{ON AUGM. MATRIX}}{\overset{\text{row op's}}{\rightsquigarrow}} Q^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -4/3 & 3/2 & -1/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix}$$

CONSIDER $Ax=b$.

IF A IS INVERTIBLE, $x = A^{-1}b$.

TERMINOLOGY FOR $Ax=b$.

IF $b=0$, THE SYSTEM IS CALLED HOMOGENEOUS.

OTHERWISE, IT IS CALLED INHOMOGENEOUS.

PROPOSITION THE SOLUTION SET OF THE HOMOGENEOUS EQUATION $Ax=0$ IS A SUBSPACE OF F^n, OF DIMENSION $n - \mathrm{rank}(A)$.

PROOF THE SOLUTION SET IS THE NULL SPACE $N(L_A)$ AND $\mathrm{nullity}(L_A) = n - \mathrm{rank}(L_A)$.

EXAMPLE: FIND THE SOLUTION SPACE OF

$x_1 + 2x_2 + x_3 = 0$

$x_1 - x_2 - x_3 = 0$

$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$. $\text{rank}(A) = 2 \Rightarrow$ THE SOLUTION SET IS 1 DIMENSIONAL.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

AN INHOMOGENEOUS EQUATION $Ax = b$ DOES NOT HAVE ANY SOLUTIONS IN GENERAL, BUT OFTEN THERE ARE.

THEOREM: IF $Ax = b$ HAS A SOLUTION s, THEN THE SOLUTION SET OF $Ax = b$ CONSISTS OF ALL $x = s + y$, WHERE y SOLVES THE HOMOGENEOUS EQUATION $Ay = 0$.

PROOF: IF x IS A SOLUTION OF $Ax = b$, THEN $y = x - s$ SATISFIES

$Ay = Ax - As = b - b = 0$

CONVERSELY, IF $Ay = 0$, THEN $x = s + y$ SATISFIES $Ax = As + Ay = b + 0 = b$.

THUS, THE SOLUTION SET IS THE SUBSPACE $N(L_A)$ SHIFTED BY s ("AFFINE SUBSPACE")

(E.G. LINES IN $\mathbb{R}^n$ ARE 1-DIMENSIONAL AFFINE SUBSPACES.)

EXAMPLE

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 3 \end{array}\right)$$

$$\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

EXAMPLE

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & -3 & 2 & 3 \\ 2 & 2 & -4 & 5 & 11 \\ 2 & 2 & -4 & 4 & 10 \end{array}\right) =$$

$$= \left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & -3 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & -2 & 0 & 5 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_4 = 1$

$x_2 = t$

$x_3 = 2 - t$

$x_1 - 3x_3 = 1$

$\therefore x_1 = 7 - 3t$

$$x = \begin{pmatrix} 7 \\ 0 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$