MAT240: 20161101

MATRIX MULTIPLICATION II

MOTIVATION:

Suppose U, V, W are finite-dimensional vector spaces.

Let $T \in \mathcal{L}(V,W)$, $S \in \mathcal{L}(V,W)$. Thus, $T \circ S \in \mathcal{L}(V,W)$.

Let α, β, γ be ordered bases of U, V, W.

$$\begin{array}{ccccc} U & \xrightarrow{S} & V & \xrightarrow{T} & W \\ \alpha & & \beta & & \gamma \end{array} \quad \Big| \quad \rightsquigarrow \quad [T \circ S]_\alpha^\gamma, \; [T]_\beta^\gamma, \; [S]_\alpha^\beta$$

GOAL: RELATE THESE 3 MATRICES

IDEA: TRY TO DEFINE A NOTION OF "MATRIX MULTIPLICATION" SO THAT $[T \circ S]_\alpha^\gamma = [T]_\beta^\gamma \cdot [S]_\alpha^\beta$.

PROPOSITION: THE FOLLOWING DEFINITION OF THE MATRIX MULTIPLICATION WILL ALWAYS MAKE $[T \circ S]_\alpha^\gamma = [T]_\beta^\gamma \; [S]_\alpha^\beta$ TRUE.

DEFINITION

$$M_{m \times n}(F) \times M_{n \times l}(F) \longrightarrow M_{m \times l}(F)$$

$$(A, B) \longmapsto AB$$

$$(AB)_{ij} = \sum_{k=1}^{4} A_{ik} B_{kj}$$

1. $$\overset{2\times 4}{\begin{pmatrix} 1 & -3 & 4 & 3 \\ 0 & 4 & 2 & -1 \end{pmatrix}} \overset{4\times 3}{\begin{pmatrix} 1 & 2 & -1 \\ 2 & -2 & 0 \\ 0 & 0 & -1 \\ 1 & 4 & 3 \end{pmatrix}}$$

$$= \begin{pmatrix} -2 & 20 & 4 \\ 7 & -12 & -10 \end{pmatrix}$$

2. $\in \mathbb{R}$

$$\begin{pmatrix} -2 & 20 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \end{pmatrix}$$

BUT

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} \begin{pmatrix} -2 & 20 \end{pmatrix} = \begin{pmatrix} -10 & 100 \\ -2 & 20 \end{pmatrix}$$

REMARKS

(0) $\dim(U) = l$

$$\begin{array}{ccccc} U & \xrightarrow{S} & V & \longrightarrow & W \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ F^l & \to & F^n & \to & F^m \end{array}$$

DIAGRAM COMMUTES

(1) (MATRIX)(VECTOR IN F^n) := (MATRIX)($n \times 1$ MATRIX)

(2) α, β STANDARD BASES OF F^n, F^m,

$\varphi : \mathcal{L}(F^n, F^m) \xrightarrow{\cong} M_{m \times n}(F)$

$T \longmapsto [T]_\alpha^\beta$ IS AN ISOMORPHISM, WHERE

$A = [T(e_1), T(e_2), \ldots, T(e_n)]$

$\varphi^{-1} : M_{m \times n}(F) \longrightarrow \mathcal{L}(F^n, F^m)$.

$A \longmapsto$ THE LINEAR MAP $T: F^n \to F^m$

WITH $A = [T(e_1), T(e_2), \ldots, T(e_n)]$,

WHICH IS THE LINEAR MAP

$T: F^n \to F^m$ WITH

$[Ae_1, A(e_2), \ldots, Ae_n] = [T(e_1), T(e_2) \ldots T(e_n)]$

DEFINITION: $L_A: F^n \to F^m$,

$L_A(v) = Av$.

THUS, $\varphi^{-1}(A) = L_A$.

(3) PROPERTIES OF MATRIX MULTIPLICATION

1) $(A_1 + A_2)B = A_1B + A_2B$

2) $A(B_1 + B_2) = AB_1 + AB_2$

3) $(\alpha A)B = \alpha(AB) = A(\alpha B)$

4) MATRIX ALGEBRA IN $M_{n \times n}(\mathbb{R})$:

- TWO OPERATIONS: ADDITION, MATRIX MULTIPLICATION
- ADDITIVE UNIT: zero matrix
- MULTIPLICATIVE UNIT: IDENTITY MATRIX

Q: IS $M_{n \times n}(F)$ A FIELD?

A: CASE ① : $n=1 \Rightarrow$ YES, SINCE
$$M_{1\times 1}(F) \cong F$$

CASE ② : $n \geq 2$. NO

#1
$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & & 0 \\ & & & \vdots \\ & & & \vdots \end{pmatrix} \begin{pmatrix} 0 & \cdots & & 1 \\ \vdots & \ddots & & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \end{pmatrix} = \begin{pmatrix} 0 & \cdots & \\ \vdots & \ddots & \\ \vdots & & \end{pmatrix}$$

$\rightsquigarrow$ $AB = 0$. YET $A \neq 0$ AND $B \neq 0$,
WHILE $AB' = 0$ #

#2 MATRIX MULTIPLICATION IS NOT COMMUTATIVE IN GENERAL

#3 NOT EVERY A HAS A MULTIPLICATIVE INVERSE

CHANGE OF BASIS

SUPPOSE V IS A F. VEC. SP., $\dim(V) = n$.

LET β, β' BE TWO ORDERED BASES OF V.

NOTE THAT $v \in V \rightsquigarrow [v]_\beta \in \mathbb{F}^n$
$\hookrightarrow [v]_{\beta'} \in \mathbb{F}^n$.

Q: HOW ARE THEY RELATED?

$v = I_V v$

$$[v]_{\beta'} = [I_V(v)]_{\beta'}$$

$$= [I_V]_{\beta}^{\beta'} [v]_{\beta}.$$

DEFINITION $[I_V]_{\beta}^{\beta'}$ IS CALLED THE CHANGE OF COORDINATE MATRIX.

LEMMA WITH ALL THE NOTATION AS ABOVE, WE HAVE THAT $[I_V]_{\beta}^{\beta'}$ IS INVERTIBLE AND $\left([I_V]_{\beta}^{\beta'}\right)^{-1} = [I_V]_{\beta'}^{\beta}$.

PROOF

$$[I_V]_{\beta}^{\beta'} [I_V]_{\beta'}^{\beta} =$$

$$= [I_V \circ I_V]_{\beta'}^{\beta'}$$

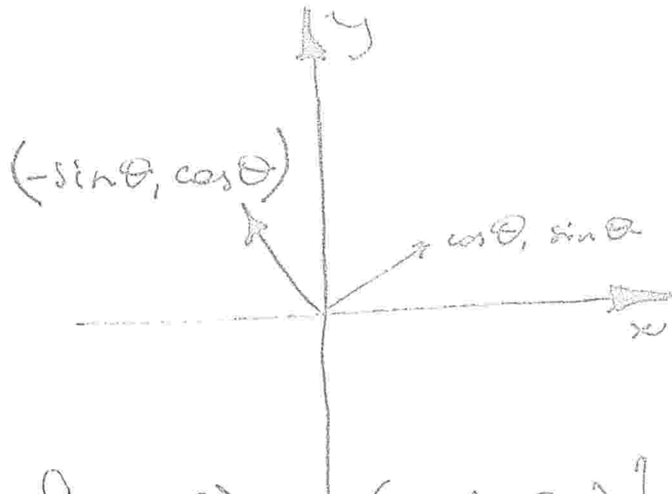
$$= [I_V]_{\beta'}^{\beta'} = I_n$$

Similarly, $[I_V]_{\beta}^{\beta'} [I_V]_{\beta'}^{\beta} = [I_V]_{\beta}^{\beta}$

$$= I_n$$

$$\Rightarrow [I_V]_{\beta}^{\beta'} [I_V]_{\beta'}^{\beta} = [I_V]_{\beta'}^{\beta} [I_V]_{\beta}^{\beta'} \quad \square$$

EXAMPLE



CONSIDER: $\beta = \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\}$

$\beta' = \left\{ \begin{pmatrix} \cos\theta' \\ \sin\theta' \end{pmatrix}, \begin{pmatrix} -\sin\theta' \\ \cos\theta' \end{pmatrix} \right\}$

FIND: $\left[I_{\mathbb{R}^2} \right]_{\beta'}^{\beta}$.

SOLUTION: $\left[I_{\mathbb{R}^2} \right]_{\beta'}^{\beta} = \Big[\underbrace{[(\cos\theta', \sin\theta')]_{\beta}, [-\sin\theta', \cos\theta']_{\beta}}_{} \Big]$

NEED TO WRITE THEM IN TERMS OF β.

RECALL: • $(\cos(\theta+\alpha), \sin(\theta+\alpha)) = (\cos\theta\cos\alpha - \sin\theta\cdot\sin\alpha, \sin\theta\cos\alpha + \cos\theta\sin\alpha)$

$= \cos\alpha(\cos\theta, \sin\theta) + \sin\alpha(-\sin\theta, \sin\alpha)$

• $(-\sin(\theta+\alpha), \cos(\theta+\alpha)) = -\sin\alpha(\cos\theta, \sin\theta) + \cos\alpha(-\sin\theta, \cos\theta)$

set $\alpha = \theta' - \theta$.

$$\left[(\cos\theta', \sin\theta')\right]_\beta = \begin{bmatrix} \cos(\theta' - \theta) \\ \sin(\theta' - \theta) \end{bmatrix}$$

$$\left[(-\sin\theta', \cos\theta')\right]_\beta = \begin{bmatrix} -\sin(\theta' - \theta) \\ \cos(\theta' - \theta) \end{bmatrix}$$

$$\Rightarrow \left[I_{\mathbb{R}^2} \right]_{\beta'}^{\beta} = \begin{bmatrix} \cos(\theta' - \theta) & -\sin(\theta' - \theta) \\ \sin(\theta' - \theta) & \cos(\theta' - \theta) \end{bmatrix}$$