MAT2140 20161103

CHANGE OF BASIS II

EXAMPLE $V = \mathcal{P}_n(F)$

$\beta = \{1, x, x^2, \ldots, x^n\}$

$\beta' = \{p_0, p_1, p_2, \ p_n\}$

BASIS OF LAGRANGE INTERPOLATION POLYNOMIALS CORRESPONDING TO DISTINCT $c_0, c_1, c_2, \ldots, c_n \in \mathbb{F}$.

FIND $[I_{P_n(F)}]_\beta^{\beta'} = \left[[1]_{\beta'}, [x]_{\beta'}, \ldots, [x^n]_{\beta'} \right]$

NOTE: $\forall p \in P_n(F): p(x) = \sum_{i=0}^{n} \lambda(p) \cdot p_i(x)$

$= \sum_{i=0}^{n} p(c_i) \cdot p_i(x)$

$\rightsquigarrow [p]_{\beta'} = \begin{bmatrix} p(c_0) \\ p(c_1) \\ \ldots \\ p(c_n) \end{bmatrix}$

Therefore, $\left[I_{\varphi_n(F)} \right]_{\beta}^{\beta'} =$

Vandermonde Matrix
$$\begin{bmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{bmatrix}$$

Remark
$$\left(\left[I_{\rho_n(F)} \right]_{\beta}^{\beta'} \right)^{-1} = \left[I_{\rho_n(F)} \right]_{\beta'}^{\beta}$$

$$= \left[[p_0]_\beta \; [p_1]_\beta \; [p_2]_\beta \; [p_n]_\beta \right] \quad (*)$$

HENCE, BY INVERTING $(*)$ THE COEFFICIENTS OF THE LAGRANG INTERPOLATING POLYNOMIALS.

NOTATION

- $\mathcal{L}(V) = \mathcal{L}(V,V)$
- FOR $T \in \mathcal{L}(V)$ AND β AN ORDERED BASIS OF V,

$$[T]_\beta^\beta = [T]_\beta$$

PROPOSITION LET V BE A FINITE DIMENTIONAL VECTOR SPACE, Let $T \in \mathcal{L}(V)$, AND β, β' BE ORDERED BASES OF V.

THEOREM $$[T]_{\beta'} = [I_V]_\beta^{\beta'} [T]_\beta \left([I_V]_\beta^{\beta'}\right)^{-1}$$

PROOF

$$[I_V]_\beta^{\beta'} [T]_\beta^\beta \left([I_V]_\beta^{\beta'}\right)^{-1} =$$

$$= [I_V \circ T]_\beta^{\beta'} \left([I_V]_\beta^{\beta'}\right)^{-1}$$

$$= [I_V \circ T \circ I_V]_{\beta'}$$

$$= [T]_{\beta'}$$

DEFINITION $A, A' \in M_{n \times n}(F)$ are said to be **SIMILAR**, if $A' = Q A Q^{-1}$ for some $Q \in M_{n,n}(F)$.

REMARK By the previous proposition, any two matrices have the similar representatives.

ELEMENTARY MATRIX ELIMINATION

MOTIVATION Solving systems of linear equations with Gaussian elimination

EXAMPLE

$$\left[\begin{array}{ccc|c} 0 & -2 & 1 & 5 \\ 1 & -4 & 6 & 10 \\ 4 & -11 & 11 & 12 \end{array}\right] =$$

$$= \left[\begin{array}{ccc|c} 4 & -16 & 24 & 40 \\ 0 & -2 & 1 & 5 \\ 4 & -11 & 11 & 12 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & -5 & 13 & 28 \\ 0 & -2 & 1 & 5 \\ 1 & -4 & 6 & 10 \end{array}\right]$$

$$
= \left[\begin{array}{ccc|c} 1 & -4 & 6 & 10 \\ 0 & 1 & -1/2 & -5/2 \\ 0 & -5 & 13 & 28 \end{array}\right]
$$

$$
= \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1/2 & -5/2 \\ 0 & 0 & \frac{21}{2} & \frac{31}{2} \end{array}\right]
$$