Consider $V = \mathbb{C}^3$ with the standard inner product. Let W be the subspace of V spanned by $(2, i, 0)$ and $(0, 1, i)$.

Problem. Find an orthogonal basis β of W.

Solution.

i	v_i	$\sum_{j=1}^{i-1} \frac{\langle v_i, u_j \rangle}{\langle u_j, u_j \rangle} u_j$	u_i	$\|u_i\|^2$
1	$(2, i, 0)$	–	$(2, i, 0)$	5
2	$(0, 1, i)$	$\frac{2 \cdot 0 + (-i) \cdot 1 + 0 \cdot i}{5} u_1 = (-2/5\ i, 1/5, 0)$	$(0, 1, i) - (-2/5\ i, 1/5, 0) = (2/5\ i, 4/5, i)$	$9/5$

Therefore,

$$\beta = \{(2, i, 0), (2/5\ i, 4/5, i)\}$$

$\square$

Problem. Find an orthogonal basis of V that contains the basis β.

Solution. Suppose $(1, 0, 0) \in \operatorname{span} \beta$. Therefore, there exist $a, b \in \mathbb{C}$ such that

$$a(2, i, 0) + b(2/5\ i, 4/5, i) = (1, 0, 0) \tag{1}$$

Thus, $a \cdot 0 + bi = 0$ and hence $b = 0$. But then $ai = 0$, and thus $a = 0$. Therefore, $(1, 0, 0)$ is not in the span of β.

Let $v_3 = (1, 0, 0)$. Observe that $\beta \cup \{v_3\}$ is linearly independent.

Using the values in the table and applying the Gram-Shmidt procedure, we obtain

$$\begin{aligned}
u_3 &= v_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} u_2 \\
&= (1, 0, 0) - \frac{2}{5}(2, i, 0) - \frac{-2/5 i}{9/5}(2/5\ i, 4/5, i) \\
&= (1/5, -2/5 i, 0) + \frac{2}{9} i (2/5\ i, 4/5, i) \\
&= (1/9, -2/9 i, -2/9)
\end{aligned}$$

Therefore, $\gamma = \beta \cup \{(1/9, -2/9 i, -2/9)\}$ is an orthogonal linearly independent set of length 3 which contains β.

Thus, since γ has the right length, it is a basis of $\mathbb{C}^3$. $\square$