1 Markov Triples

Definition 1.1. A **Markov triple** is an integer solution of the quation $x^2 + y^2 + z^2 = 3xyz$.

Note that the equation $x^2 + y^2 + z^2 = 3xyz$ is quadratic in each variable.

In this sence, it can be rearranged, for example, to $x^2 - (3bc)x + b^2 + c^2 = 0$.

Using the Vieta formula, we know that, if a and a' are two solutions, then $a + a' = 3bc$.

Note that we can construct a tree of solutions starting with a trivial solution $(1, 1, 1)$: $(1, 1, 1) \to (1, 1, 2) \to (1, 2, 5) \to \dots$.

Question. Are Markov triples determined by its maximal element?

1.1 Approximations of Irrational Numbers $\alpha \in \mathbb{R} \setminus \mathbb{Q}$

Theorem 1.2

For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $N \in \mathbb{N}$, there exists $\frac{p}{q} \in \mathbb{Q}$ such that $\gcd(p, q) = 1$ and $q < N$ and $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q \cdot N} \leq \frac{1}{q^c}$ for some $c \in \mathbb{N}$.

Corollary 1.3

For each $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exist infinitely many $\frac{p}{q}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$.

Definition 1.4. α is **approximable** of degree t if and only if there exist infinitely many $\frac{p}{q}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^t}$.

Theorem 1.5

If $\alpha \in \mathbb{R}$, $\epsilon > 0$ and there exist infinitely many $\frac{p}{q}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$, then α is transcendental.

Remark 1.6. See also the works of Liouville, Thue, Siegel and Dyson.

Let $L(\alpha)$ be a supremum of $\{c \mid$ there exist infinitely many $\frac{p}{q} \in \mathbb{Q}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{cq^2}\}$.

Definition 1.7. A **Lagrange spectrum** is the set $\mathfrak{L} = \{L(\alpha) \mid \alpha \in \mathbb{R} \setminus \mathbb{Q}\}$.

Denote the set of integers which appear in Markov triples as $\mathfrak{M}$.

Theorem 1.8

Consider $\mathfrak{L}_{<3} = \{\frac{\sqrt{9m^2-4}}{m} \mid m \in \mathfrak{M}\}$, and let $\gamma_m = \frac{a_m + \sqrt{9m^2-4}}{b_m}$ for some integer a_m and b_m.

Thus, if

$$\inf Q = \inf_{\substack{x, y \in \mathbb{Z} \\ Q(x,y) \neq 0}} \left|Q(x, y)\right| = m,$$

then

$$\mathfrak{L}_{<3} = \{\frac{\Delta(Q)}{\inf Q}\}_{<3}\Delta(Q) = 9m^2 - 4.$$

See further: multiplicative commutator, cluster algebra