Remark 0.1. On a vector space V there can be many inner products.

Example 0.2

- if $\langle \cdot, \cdot \rangle$ is an inner product, then so is $c\langle \cdot, \cdot \rangle$ for $c > 0$ in $\mathbb{R}$
- if $\phi : V \to V$ is an isomorphism, then also $\langle x, y \rangle' = \langle \phi(x), \phi y \rangle$
- if $V = C[a, b]$ is a vector space of continuous products $(a < b)$, where $\mathbb{F} = \mathbb{R}$). Then $\langle f(x), f(y) \rangle = \int_a^b f(t)g(t)\,\mathrm{d}t$ is an inner product.
- if $V = C[a, b]$ is a vector space of continuous products $(a < b)$, where $\mathbb{F} = \mathbb{C}$). Then $\langle f(x), f(y) \rangle = \int_a^b f(t)\overline{g(t)}\,\mathrm{d}t$ is an inner product.

  Here, if $f(x) \in C[a, b]$, write $f(x) = f_1(x) + i f_2(x)$, where $f_1, f_2 \in \mathbb{R}$. Define
  $$\int_a^b f(t)\,\mathrm{d}t = \int_a^b f_1\,\mathrm{d}t + i\int_a^b f_2\,\mathrm{d}t.$$
  Then
  $$\overline{\int f(t)\,\mathrm{d}t} = \int \overline{f(t)\,\mathrm{d}t}$$
  and
  $$\int (f(t) + cg(t))\,\mathrm{d}t = \int f(t)\,\mathrm{d}t + c\int g(t)\,\mathrm{d}t.$$

Definition 0.3.
H is the inner product space $C[0, 2\pi]$, $\mathbb{F} = \mathbb{C}$, with $\langle f, g, =\rangle \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,\mathrm{d}t$.

Theorem 0.4

Let V be an inner product space with the inner product $\langle \cdot, \cdot \rangle$.

a) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

b) If for all $x \in V.\langle x, y \rangle = \langle x, z \rangle$, then $y = z$

Proof.

a) If $x = 0$, then $\langle x, x \rangle = 0$. Otherwise, $\langle x, x \rangle > 0$

b) If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$, then $\langle x, y - z \rangle = 0$. Therefore, taking $x = y - z$, we obtain $y - z = 0$.

□

Definition 0.5.
The **norm** or **length** of $x \in V$ is $\left\|\sqrt{\langle x, x \rangle}\right\| \geq 0$

Example 0.6

If $V = \mathbb{F}^n$ with the standard inner product $\langle a, b \rangle = \sum a_i \overline{b_i}$, then

$$\sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}$$

Theorem 0.7

Let V be an inner product space. Then the following holds:

a) $\|cx\| = |c|\|x\|$ for all $c \in \mathbb{F}$ and $x\ inV$

b) $\|x\| = 0 \Leftrightarrow x = 0$

c) $\left|\langle x, y\rangle\right| \leq \|x\|\|y\|$ (*Cauchy-Schwarz*)

d) $\|x + y\| \leq \|x\| + \|y\|$ (*Triangle Inequality*)

Proof. a) $\|cx\| = \sqrt{\langle cx, cx\rangle} = \sqrt{c\bar{c}\langle x, x\rangle} = |c|\|x\|$

b) Exercise.

c) Note that if $y = 0$, the inequality holds. Suppose now $y \neq 0$. Note the following:

$$0 \leq \langle x + cy, x + cy\rangle = \langle x, x + cy\rangle + c\langle y, x + cy\rangle \tag{1}$$

$$= \langle x, x\rangle + \bar{c}\langle x, y\rangle + c\langle y, x\rangle + c\bar{c}\langle y, y\rangle \tag{2}$$

Plugging in $c = -\frac{\langle x,y\rangle}{\langle y,y\rangle} = -\langle x, (\|y\|)^2\rangle$, we obtain

$$0 \leq \|x\|^2 - \frac{\left|\langle x, y\rangle\right|}{\|y\|^2},$$

and thus $\left|\langle x, y\rangle \leq \|x\|\|y\|\right|$

Consider $\|x + y\|^2 \leq (\|x\| + \|y\|)^2$. Note that $\langle x+y, x+y\rangle = \langle x, x\rangle + 2\Re\langle x, y\rangle + \langle y, y\rangle$ $\square$

For the standard innerp product on $\mathbb{F}^n$, from Cauchy-Schwarz it follows that $\left|\sum_{i=1}^n a_i b_i\right| \leq \sqrt{\sum|a_i|^2}\sqrt{\sum|b_i|^2}$ and by Triangle Inequality $\left|\sum_{i=1}^n a_i b_i\right| \leq \sqrt{\sum|a_i|^2}\sqrt{\sum|b_i|^2}$