ASKOLD KHOVANSKII

NEWTON - OKOUNKOV BODIES

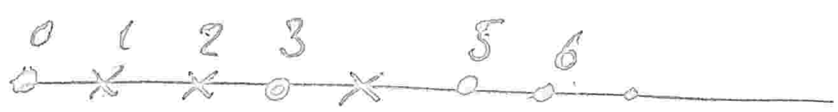
20161124

3 5

$3m+1$: $\underbrace{1, 4, 7,}$ 10

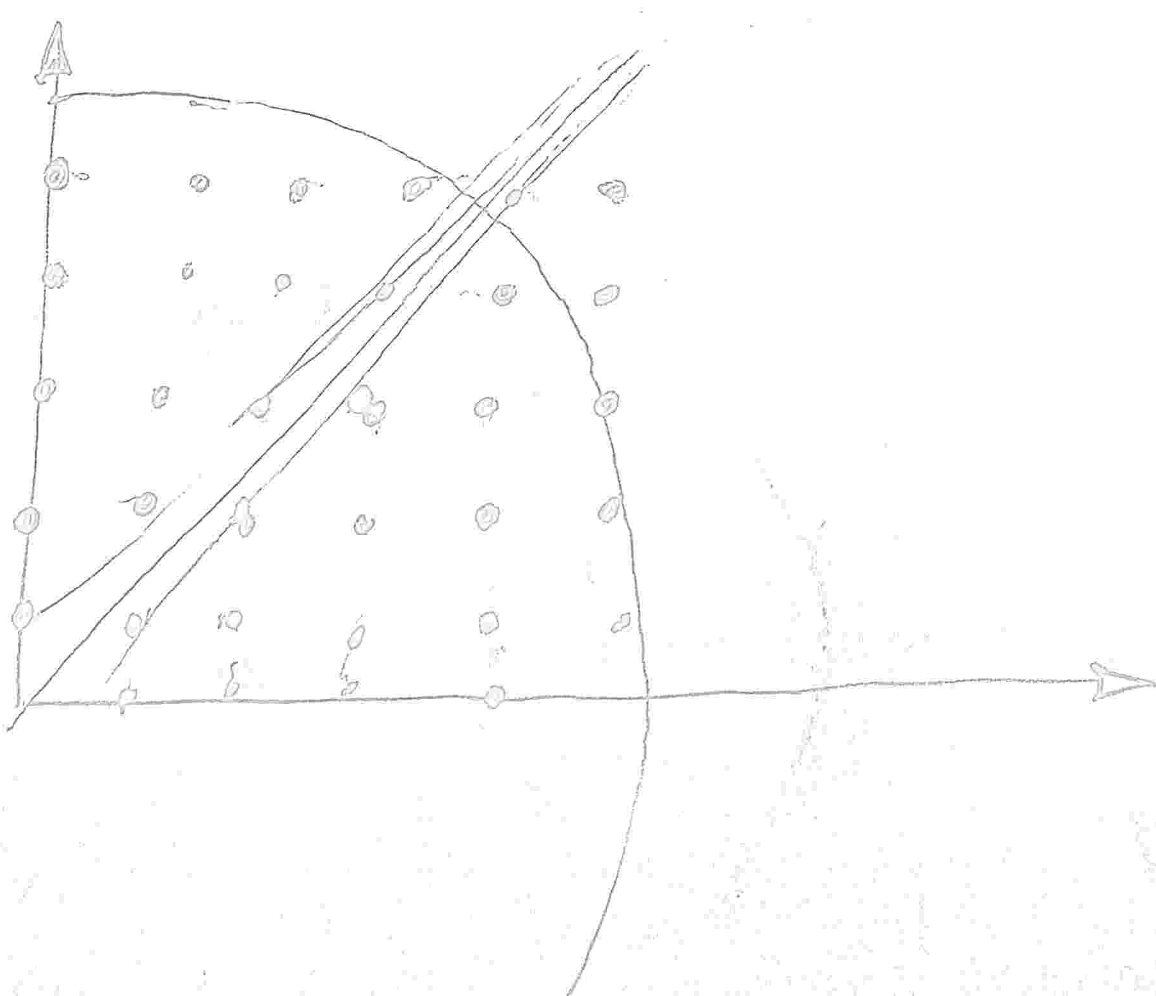
$3m+2$: 2, 5, 8

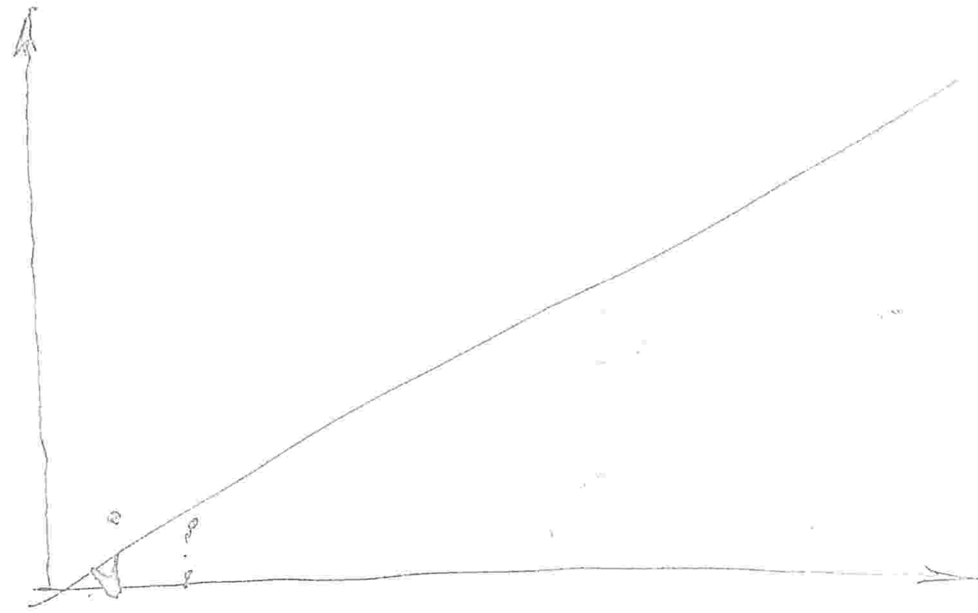
FROBENIUS



$(p-1)(q-1)$

$\alpha \approx \frac{p}{q}$ APPROXIMATION

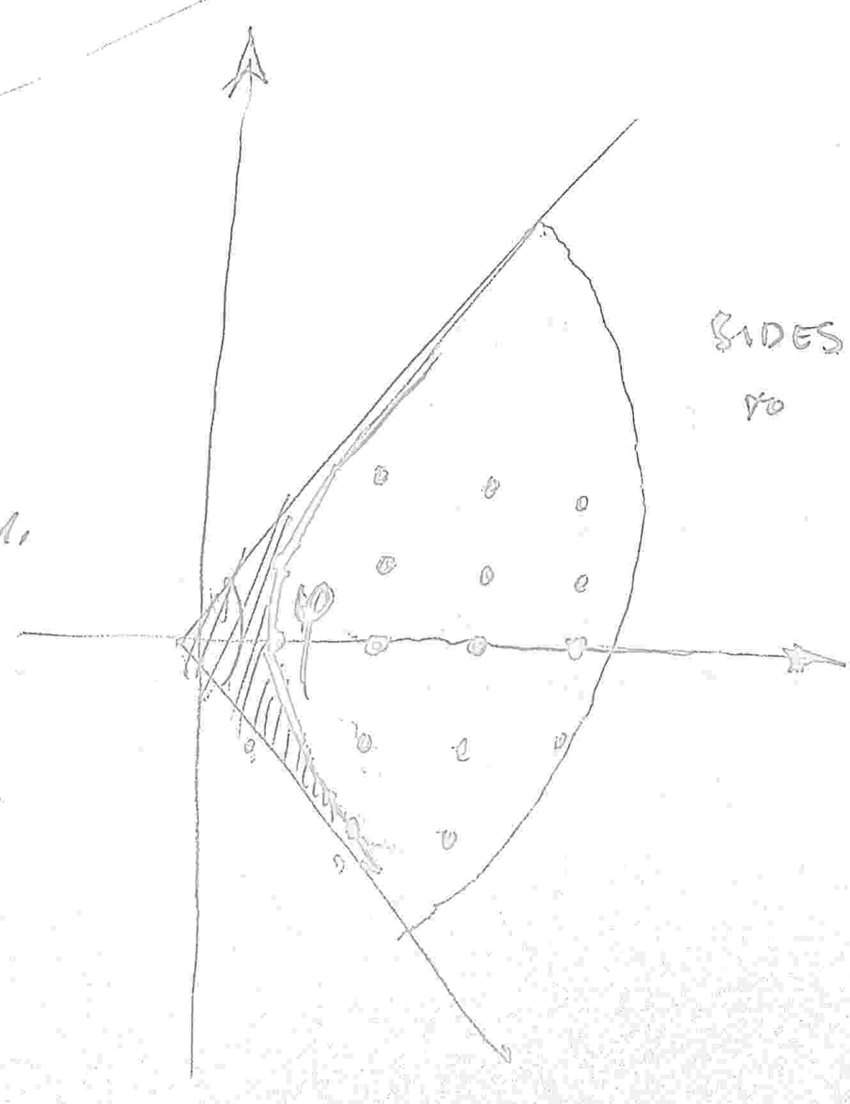




$$\alpha = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}$$

GAUSS:
STATISTICS
OF THE ONE-DIM.
EXT. FIELD

SEMIGROUP OF
INTEGRAL POINTS



SIDES CORRESPOND
TO $a_1, a_2, \ldots$

ASSUME THERE ARE TWO SETS IN THE VECTOR SPACE.

Minkowski SET $A+B=\{x \mid x=a+b, a\in A, b\in B\}$.

$$\underbrace{A+A+A}_{3*A}$$

$$\boxed{k*A}$$

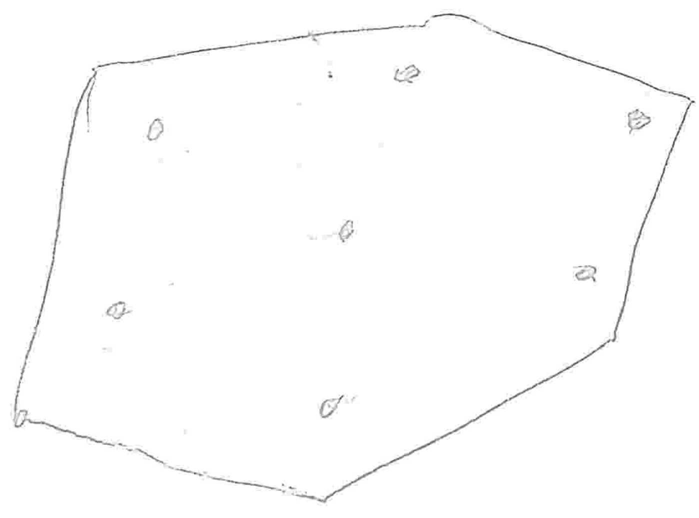
$$\lim_{k\to\infty} \frac{k*A}{k^{n}} = V(\Delta A)$$

What is the asymptotic as $k \to \infty$?

$a_1, \dots, a_k$ have no common divisors.

$\Rightarrow$ generate $\mathbb{Z}$

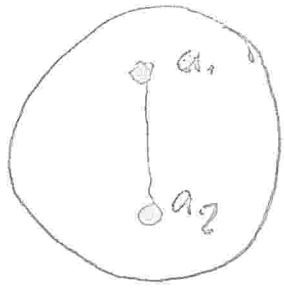
Consider $a_i - a_j$. Assume A generates a lattice



$V(\Delta(A))$

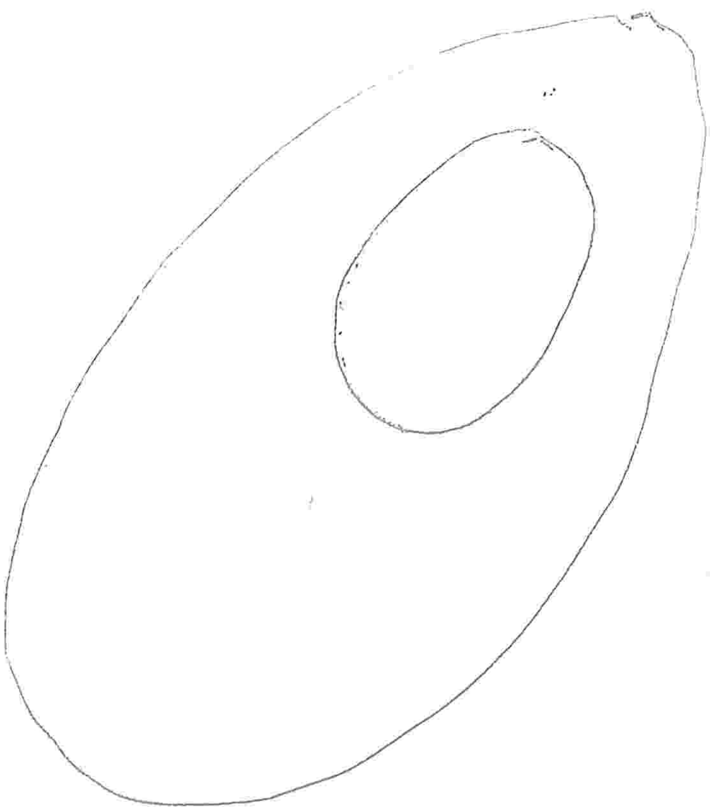
Observations

1. Minkowski sum of two convex sets is convex



PARAMETRIZATION
$a_2 + (1-b)a_1$

$k * A \subset k\,\Delta(A)$



Newton body
volume $\approx$ the number of integral points in the body

$x_i \neq 0$.

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consider the system of equations in $(\mathbb{C} \setminus 0)^n$.

If the coefficients are generic enough, then the system of eq. have the solution.

In a space of ~~polynoms~~ coefficients, there exists a hypersurface of solutions.

The number of solutions is $n! \, V(\Delta)$

Theorem (Hilbert)

Consider $\mathbb{C}^n$.

Algebraic variety:

$X \subset \mathbb{C}^n$

defined by

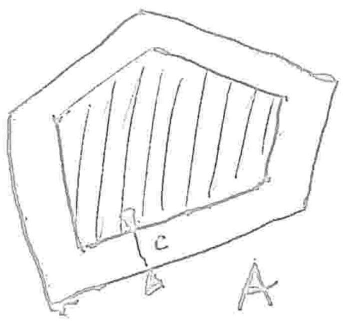
$$\begin{cases} p_1(\ ) = 0 \\ \vdots \\ p_q(\ ) = 0 \end{cases}$$

- Bernstein theorem ?
- Kovshirenko ?
- Newton_Okounkov body

Claim

$$\frac{\#(k\Delta(A) \cap \mathbb{Z}^n)}{k^n} = \text{Volume of } \Delta$$

Need another estimator!



$$(k\Delta - c \cap \mathbb{Z}^n) \subseteq$$
$$ck * A \subseteq (k\Delta \cap \mathbb{Z}^n)$$

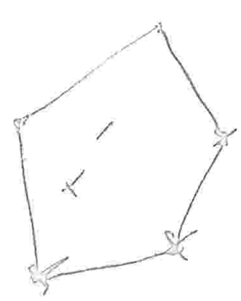
IF POINTS ARE DEEP ENOUGH, THEY CAN BE REPRESENTED IN THIS FORM, c IS INDEPENDENT OF k!

What is a Laurent polynomial?

MONOMIAL: $x^m = x_1^{m_1} \cdot x_2^{m_2} \cdots x_n^{m_n}$

LAURENT POLYNOMIAL : $P = \sum_{m \in A} c_m x^m$

if m is non-negative, then



NEWTON POLYTOPE

a simplex corresponding to a polynomial

dees the volume $\frac{d^n}{n!}$ | $p_1 = 0$ ⋮ $p_n = 0$

Consider a polynomial of degree $\leq d$.

$Q_1 P_1 + \cdots Q_m P_X$

What is the dimension of $\cdot$? X

$H_X(d)$ — HILBERT FUNCTION.

If d is big enough, $H_X(d)$ ~~[illegible]~~ is a HILBERT POLYNOMIAL.

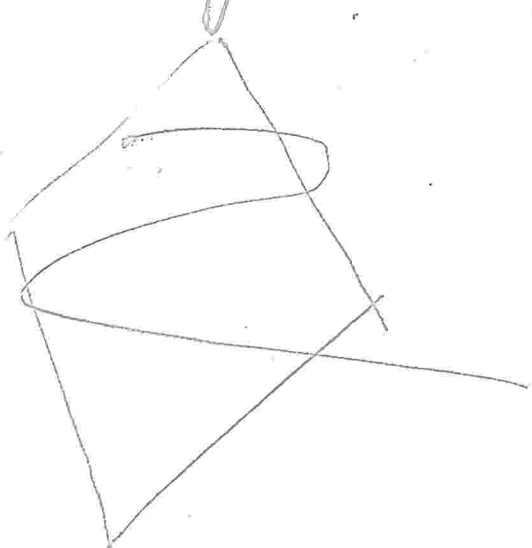
$$H_X(d) = a_k d^k + \cdots$$

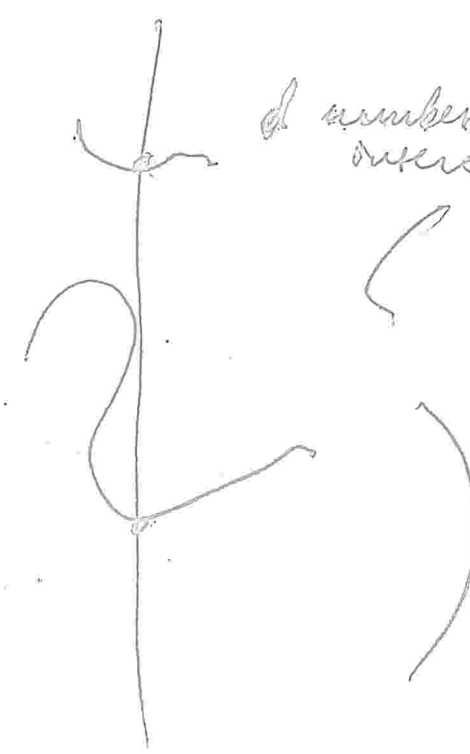
$$k = \dim X$$

$$\deg X = k!\, a_k$$

$$P(x, y) = 0$$

$$\deg P = d$$





d number of intersections

~~FIXED INTERS~~

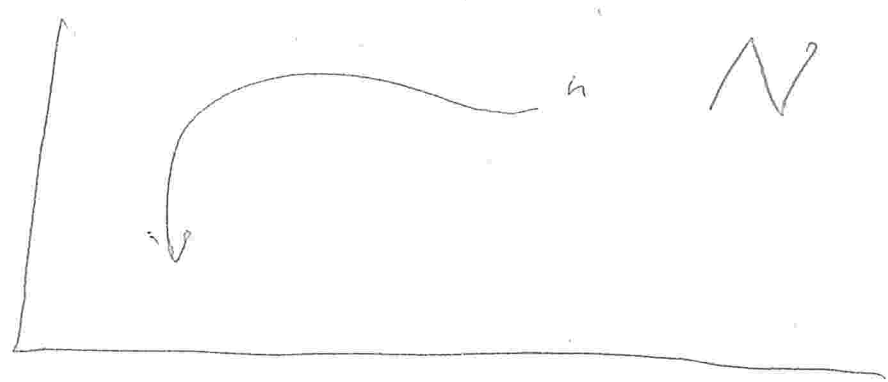
IF A HYPERPLANE INTERSECTS A GENERIC CURVE, THE NUMBER OF INTERSECTION WILL BE $\deg$ | ~~[illegible]~~ in the general case.

$$(\mathbb{C}^*)^n \longrightarrow \mathbb{C}^{*A}$$

$$\left\{ \begin{matrix} x^{m_1} \\ \vdots \\ x^{m_N} \end{matrix} \right\} N \text{ monomials}$$

Let $x \longrightarrow (x^{m_1}, \ldots, x^{m_N})$

Take any linear function $a_1 x^{m_1} + \ldots x^{m_N}$



number of linear function is equal to the number of monomials.

$$x \longrightarrow (x^{m_1}, \ldots, x^{m_N})$$

$$x^{m_1} * x^{m_1} = x^{m_1 n}$$