1 Spectral Theorem

Definition 1.1. Let V be a finite-dimensional inner product space, and let $T \in \mathrm{Hom}(V, V)$. Then T is **normal** if $TT^* = T^*T$ and **self-adjoint** if $T = T^*$.

Theorem 1.2

If $\mathbb{F} = \mathbb{C}$, V has an orthonormal basis of eigenvectors if and only if T is normal.

If $\mathbb{F} = \mathbb{R}$, V has an orthonormal basis of eigenvectors if and only if T is self-adjoint.

Example 1.3

- Let $\mathbb{F} = \mathbb{C}$. Note that if T is self-adjoint, then it is normal, and by Theorem 1.2 T is diagonalisable.

  However, if T is normal, it is not necessarily self-adjoint, since for $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $A^* = -A$.

  Moreover, if T is diagonalisable, it is not necessarily normal, since for $B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $\lambda = 1$ is an eigenvalue with the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda = 2$ is an eigenvalue with the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which are not orthogonal.

- Suppose now $\mathbb{F} = \mathbb{R}$. Then $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is normal, but not diagonalisable, since the characteristic polynomial is $t^2 + 1$. Moreover, $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ is diagonalisable, but not self-adjoint.

1.1 Infinite-Dimensional Inner Product Spaces

In general, infinite-dimensional inner product spaces do not necessarily have an orthonormal basis.

The following can be proven:

Example 1.4

Let $l^2 = \{\sigma(1), \sigma(2), \ldots, \in \mathbb{F} \,|\, \sum_{i=1}^{\infty} \left|\sigma(i)\right|^2 < \infty\}$ be an inner product space with $\langle \sigma, \tau \rangle = \sum_{i=1}^{\infty} \sigma(i)\overline{\tau(i)} \in \mathbb{F}$. Then l^2 does not have any orthonormal basis. For example, $(e_i)_{i=1}^{\infty}$ is a maximal orthonormal subset, but does not span.

Some linear transformations do not have an adjoint (see Friedberg *et al*, 6.3/ex 24).

Moreover, $V = W \oplus W^{\perp}$ and $W^{\perp\perp} = W$ can fail, and thus the spectral theorem fails.

Remark 1.5. If β is an orthonormal basis, then T is normal if and only if $[T]_\beta$ is normal, since $[T^*]_\beta = [T]_\beta^*$.

Theorem 1.6

Let $T \in \text{Hom}(V, V)$ be normal.

a) $\left\|T(x)\right\| = \left\|T^*(x)\right\|$ for all $x \in V$.
b) $T - \lambda I$ is normal, with $\lambda \in \mathbb{F}$.
c) If $T(x) = \lambda x$, then $T^*(x) = \overline{\lambda} x$
d) Eigenvectors for *distinct* eigenvalues are orthogonal to each other.

Proof.

a) $\left\|T(x)\right\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle = \langle T^*x, T^*x\rangle = \|T^*x\|^2$
b) Note that $(T - \lambda I)^* = T^* - \overline{\lambda} I^* = T^* - \overline{\lambda} I$.

   Therefore,

$$(T - \lambda I)(T^* - \overline{\lambda} I) = TT^* - \lambda T^* - \overline{\lambda} T + \lambda\overline{\lambda} \tag{1}$$
$$= T^*T - \lambda T^* - \overline{\lambda} T + \lambda\overline{\lambda} \tag{2}$$
$$= (T^* - \lambda I)(T - \overline{\lambda} I) \tag{3}$$

c) Since $T(x) = \lambda x$, then $(T - \lambda I)(x) = 0$ and thus $\left\|(T - \lambda I)(x)\right\| = 0$, which means that $\left\|(T - \lambda I)^*\right\| = 0$, and therefore, $\left\|T^* - \overline{\lambda} x\right\| = 0$, and thus $T^* = \overline{\lambda} x$.
d) If $T(x) = \lambda x$ and $T(y) = \mu y$ and $\lambda \neq \mu$:

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \tag{4}$$
$$\lambda\langle x, y\rangle = \mu\langle x, y\rangle \tag{5}$$
$$\langle x, y\rangle = 0 \tag{6}$$

□

Lemma 1.7

Suppose $T \in \text{Hom}(V, V)$. If $W \subseteq V$ is T-invariant, then $W^\perp$ is T^*-invariant.

Remark 1.8. If the lemma is applied to T^* instead, then $W^\perp$ is T-invariant, because $T^{**} = T$.

Proof. Take any $x \in W^\perp$. We need to show that $T^* \in W^\perp$, i.e. $\langle T^*x, w\rangle = 0$ for all $w \in W$. Note that $\langle T^*x, w\rangle = \langle x, Tw\rangle = 0$, since $Tw \in W$ and $x \in W^\perp$. □

Theorem 1.9

If V is finite-dimensional, $\mathbb{F} = \mathbb{C}$, $T \in \text{Hom}(V, V)$, then T is normal if and only if V has an orthonormal basis of eigenvectors for T.

Proof. The $\Leftarrow$ direction from the theorems proven before.

We use induction on $\dim V$ to prove $\Rightarrow$.

Note that if $n = 1$ then any unit vector is an orthonormal basis of eigenvectors.

Assume the theorem holds for $(n-1)$-dimensional spaces.

Since $\mathbb{F} = \mathbb{C}$, we can pick an eigenvector for T. By scaling, we may assume that it has length 1. Then we know that $Tv = \lambda v$ for some $\lambda \in \mathbb{C}$. Therefore, $T^* = \overline{\lambda} v$.

Let $W = \text{span}(v)$ of dimension 1.

Then W is T-invariant and also T^*-invariant, since v is an eigenvector.

By Lemma 1.7, $W^\perp$ is T-invariant and also T^*-invariant.

Note that $\dim W^\perp = \dim V - \dim W = n - 1$.

Claim. $(T_{W^\perp})^* = (T^*)_{W^\perp}$.

Proof. Since $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in W^\perp$, this also hold if T is restricted to $W^\perp$. □

From the claim we obtain that $T|_{W^\perp}$ is normal.

By induction hypothesis applied to $W^\perp$ we obtain an orthonormal basis $\{v_2, \ldots, v_n\}$ of eigenvectors for $T_{W^\perp}$.

Therefore, $\{v, v_2, \ldots, v_n\}$ is an orthonormal basis of eigenvectors for T. □