Problem.

Suppose that $\lambda \in F$ and that $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Prove by induction that $J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$ for all $n \geq 1$.

Proof.

Let $P(n) =$"$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$".

In case $n = 1$, the claim holds, since $J = \begin{pmatrix} \lambda^1 & 1 \cdot \lambda^{1-1} \\ 0 & \lambda^1 \end{pmatrix}$. Thus, $P(1)$ is true.

Suppose now $P(k)$ holds for some $k \in \mathbb{Z}^+$.

Therefore, $J^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix}$.

Note that, by inductive hyptothesis,

$$
\begin{aligned}
JJ^k &= \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} && (1) \\
&= \begin{pmatrix} \lambda^k \lambda + k\lambda^{k-1} \cdot 0 & 1 \cdot \lambda^k + k\lambda^{k-1}\lambda \\ 0 \cdot \lambda + \lambda^k \cdot 0 & 0 \cdot 1 + \lambda^k \lambda \end{pmatrix} && (2) \\
&= \begin{pmatrix} \lambda^{k+1} & (k+1)\lambda^k \\ 0 & \lambda^{k+1} \end{pmatrix}, && (3)
\end{aligned}
$$

which is exactly the claim in case $n = k + 1$. Therefore, $P(k + 1)$ holds.
Since the claim is also true in case $n = 1$, then $\forall n \in \mathbb{Z}^+.P(n)$ is true by induction. $\square$

Problem.

Suppose $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{pmatrix}$ over $F = \mathbb{Q}$.

Find an invertible matrix Q such that $Q^{-1}AQ$ is in Jordan canonical form.

Solution.

Note that $A - tI = \begin{pmatrix} -t & 1 & 1 \\ 2 & 1-t & -1 \\ -6 & -5 & -3-t \end{pmatrix}$.

Expanding along the first row, we see that

$$
\begin{aligned}
\det(A - tI) &= -t \begin{pmatrix} 1-t & -1 \\ -5 & -3-t \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ -6 & -3-t \end{pmatrix} + \begin{pmatrix} 2 & 1-t \\ -6 & -5 \end{pmatrix} && (4) \\
&= -t\left((1-t)(-3-t) - 5\right) - (-6 - 2t - 6) + (-10 + 6 - 6t) && (5) \\
&= t(1-t)(3+t) + 5t + 2(t+3) + 6 - 4 - 6t && (6)
\end{aligned}
$$

$$= -\Big((t^2 - t)(t+3) - 6 - 7t - 6 + 6t + 4\Big) \tag{7}$$
$$= -\Big(t^3 + 2t^2 - 3t - 8 - t\Big) \tag{8}$$
$$= -\Big(t^3 + 2t^2 - 4t - 8\Big) \tag{9}$$
$$= -\Big(t^2(t+2) - 4(t+2)\Big) \tag{10}$$
$$= -(t-2)(t+2)^2 \tag{11}$$

Thus, $\lambda = 2$ and $\lambda = -2$ are the only eigenvalues.

Consider $A - 2I$:

$$A - 2I = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -1 & -1 \\ -6 & -5 & -5 \end{pmatrix} \tag{12}$$
$$\tag{13}$$

Now we solve $(A - 2I)|0$:

$$R_1 \to R_1 + R_2, R_3 \to -\frac{1}{8}(R_3 + 3R_2) \mid \rightsquigarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \tag{14}$$
$$R_2 \to \frac{1}{2}(R_2 + R_3) \rightsquigarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \tag{15}$$

Therefore, $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ spans E_2, and there is one column in the dot diagram corresponding to K_2.

Consider $A + 2I$:

$$A + 2I = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & -1 \\ -6 & -5 & -1 \end{pmatrix} \tag{16}$$
$$\tag{17}$$

Now we solve $(A + 2I)|0$:

$$R_2 \to \frac{1}{4}(R_1 + R_2), R_3 \to \frac{1}{2}(R_3 + 3R_1) \mid \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \tag{18}$$
$$R_1 \to (R_1 - R_2) \rightsquigarrow \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \tag{19}$$

$$R_3 \to (R_3 + R_1) \ \rightsquigarrow \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{20}$$

Therefore, $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ spans E_{-2}, and there is one column in the dot diagram corresponding to K_{-2}.

Since the algebraic multiplicity of -2 is 2 and it is equal to $\dim K_{-2}$, we know there must be a cycle of length two containing an element $v \in V$ such that $(A+2I)v = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and v is a generalised eigenvector in K_{-2}.

We now solve $A + 2I \mid \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$:

$$A + 2I \mid \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 2 & 3 & -1 & -1 \\ -6 & -5 & -1 & -1 \end{array}\right] \tag{21}$$

$$R_2 \to \frac{1}{4}(R_1 + R_2), R_3 \to \frac{1}{2}(R_3 + 3R_1) \ \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right] \tag{22}$$

$$R_1 \to \frac{1}{2}(R_1 - R_3) \ \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right] \tag{23}$$

$$R_1 \to \frac{1}{2}(R_1 - R_2) \ \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right] \tag{24}$$

Therefore, if $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $x + y = 0$ and $-y + z = 1$.

Therefore, if $x = \tau$, then $v = \begin{pmatrix} \tau \\ -\tau \\ 1 - \tau \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \tau \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

Now consider $(A + 2I)^2$:

$$(A + 2I)^2 = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & -1 \\ -6 & -5 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & -1 \\ -6 & -5 & -1 \end{pmatrix} \tag{25}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 16 & 14 & 0 \\ -20 & -16 & 0 \end{pmatrix} \tag{26}$$

Note that $(A+2I)^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$, which means that $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a generalised eigenvector.

Since $\dim K_{-2} = 2$, $(T+2I)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and $\beta = \{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \}$ is linearly independent (if there exist $a_1 \in \mathbb{F}$, $a_2 \in \mathbb{F}$ such that

$$a_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 0,$$

then from the first row $a_2 = 0$ and from the third $a_1 = a_2 = 0$), then β spans K_{-2}.
Since the characteristic polynomial of A splits, then $V = K_2 \oplus K_{-2}$.
Let $\gamma = \{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$. Since $\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \}$ is a basis of K_2, while β is a basis of K_{-2}, since $V = K_2 \oplus K_{-2}$, we have that γ is a cycle basis of V.
Therefore, $[A]_\gamma$ is in Jordan Canonical Form, and therefore, by the change-of-basis formula, $Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$.

Expanding along the first row, we obtain that $\det[A]_\gamma = -\det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = -1$.

Therefore, Q is invertible.
We find Q^{-1} by row-reduction:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \tag{27}$$

$$R_1 \to R_1 + R_2 \mid \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \tag{28}$$

$$\begin{array}{r} R_2 \to -(-R_1 + R_2) \\ R_3 \to R_1 + R_3 \end{array} \mid \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 1 \end{array}\right] \tag{29}$$

$$R_2 \to R_2 + R_3 \mid \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array}\right] \tag{30}$$

$$\tag{31}$$

Therefore, $Q^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}$. $\square$

Problem.

Using the previous parts, compute A^n for any $n \geq 1$.

Solution.

From the previous discussion of the dot diagram, the Jordan Canonical Form for a basis γ is as follows:

$$[A]_\gamma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} \tag{32}$$

Let $P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$.

Note that $P^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, because the only nonzero entry is obtained by multiplying the first row with the first column.

Moreover, $PR = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} = 0$

and $RP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$. Therefore, P and R commute.

Since $[A]_\gamma = P + R$, then $A^n = P^n + \sum_{i=1}^{n-1}(P^{n-i}R^i) + R^n$ by Binomial Theorem, which is applicable since P and R commute.

The middle sum is equal to 0, because $PR = RP = 0$.

We now calculate R^n.

We claim that $R^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-2)^n & n(-2)^{n-1} \\ 0 & 0 & (-2)^n \end{pmatrix}$.

The claim holds in case $n = 1$.

Suppose that the claim holds in case $n = k$.

Then $R^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-2)^k & k(-2)^{k-1} \\ 0 & 0 & (-2)^k \end{pmatrix}$ and thus

$$R^{k+1} = R^k R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-2)^k & k(-2)^{k-1} \\ 0 & 0 & (-2)^k \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} \tag{33}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-2)^{k+1} & (-2)^k + k(-2)^k \\ 0 & 0 & (-2)^{k+1} \end{pmatrix} \tag{34}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-2)^{k+1} & k+1(-2)^{k+1-1} \\ 0 & 0 & (-2)^k + 1 \end{pmatrix}, \tag{35}$$

which is exactly the claim in case $n = k+1$, similarly to the result in the problem 1.

Let $m_{21} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $m_{12} = (0,0)$. Then $R = \begin{pmatrix} 0 & m_{12} \\ m_{21} & J \end{pmatrix}$ in the notation of the first problem.

From the discussion above,

$R^n = \begin{pmatrix} 0 & m_{12} \\ m_{21} & J^n \end{pmatrix}$ by induction.

Therefore, $[A]_\gamma^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-2)^n & n(-2)^{n-1} \\ 0 & 0 & (-2)^n \end{pmatrix}$ and thus, since by the change-of-basis formula we have $[A]_\gamma = Q^{-1}AQ$, and thus

$$A^n = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-2)^n & n(-2)^{n-1} \\ 0 & 0 & (-2)^n \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}.$$

$\square$