1 Geometry and Dynamics of Fractals

1.1 Douady Rabbit as a Branched Covering $S^2 \to S^2$

We can show that the number of preimages of non-critical points does not depend on the choice of a non-critical point. The number of these preimages is called a *degree* of a covering.

For our Douady rabbit, the degree is 2.

We have noted previously that a Douady rabbit is a postcritically finite branched covering, which means that all rabbit points eventually fall into a cyclic orbit. These kind of mappings are called *Thurston mappings*.

Note that we can look at rational mappings as a postcritically finite branched covering. Can we say anything useful with this method?

1.2 Dehn Twist

Definition 1.1. Dehn twist is a homeomorphism $S^2 \to S^2$ of a curve which is identity outside of its annulus A.

We have noted several important points in a Douady rabbit: 0, v, w such that $0 \mapsto v \mapsto w \mapsto 0$. The mapping τ defined earlier is a Dehn twist with respect to the annulus containing the ears with v and w.

Define $f = \tau^{\circ m} \circ P_c$. Note that f is a postcritically finite branched covering. Thurston theory allows us to affirm that the topological dynamics of such a mapping is equivalent to the polynomial topological dynamics.

1.3 Thurston and a Twisted Rabbit

Consider the critical values of a Douady rabbit. We know that the point at infinity is mapped to itself, while the other three points $(0, v, w)$ cycle over each other. The graph representation of these mappings is called a *critical portrait*.

It can be shown that a Thurston mapping with such a critical portrait is *topologically equivalent* to a rational function, which also means that it is topologically equivalent to a polynomial.

We say that two Thurston mappings f and g are topologically equivalent, if there is a diagram such that $(S^2, P(f)) \xrightarrow{f} (S^2, P(f))$, $(S^2, P(g)) \xrightarrow{g} (S^2, P(g))$, $(S^2, P(f)) \xrightarrow{\phi_1} (S^2, P(g))$, and $(S^2, P(f)) \xrightarrow{\phi_2} (S^2, P(g))$, where $P(f)$ is a postcritical set, and ϕ_1 and ϕ_2 are dually oriented homeomorphisms. Note that ϕ_1 can be continuously deformed into ϕ_2 without changing $\phi_1|_{P(f)} = \phi_2|_{P(f)}$.

We will show that twisting a Douady rabbit only once yields an aeroplane.

Douady and Hubbard wanted to know to what the critical portrait for a Douady rabbit is equivalent. Bartholdi and Nekrashevych gave an answer.

Let $m = \sum_{k=0}^{N} a_k 4^k > 0$ denote the number of Dehn twists, with $a_k \in \{0, 1, 2, 3\}$. If there are a_k equal to 1 or 2, we get an aeroplane. Otherwise, we obtain a Douady rabbit.

How can we see an aeroplane in a twisted rabbit?

To a Douady rabbit corresponds the following tree of invariance. Place α at the centre, with edges going to the nodes labelled with $0, v, w$. Note that v is a critical value. Add a point at infinity as a node, which is also critical. We draw an edge from 0 to ∞. Label the edge from 0 to α as A, from α to v as B, from α to w as C, and from 0 to ∞ as D. The mapping for a Douady rabbit is then $A \to B \to C \to A$ and $D \to BAD$.

We can also represent these points on a real line. For this, mark the points from left to right in the order v, 0, w, ∞, and denote the segments from v to 0 as $\widehat{A}$, from 0 to w as $\widehat{B}$, and from w to ∞ as $\widehat{C}$. Then $\widehat{A} \to \widehat{A}\widehat{B}, \widehat{B} \to \widehat{A}$, and from $\widehat{C} \to \widehat{B}\widehat{C}$.

If we twist a Douady rabbit once, then a different tree is obtained. Edges B', C' and D' correspond to the positions of the edges B, C and D, while the edge A_* will twist over the nodes v and w and connect to the node α', corresponding to the point α.

Now, we want to construct a new tree T^* such that $T^* \to T$ and $P(f) \subset V(T^*)$, where V is a set of nodes of T^*.

The preimage of an edge B is an edge C, and we draw it first. The edge A is mapped to B, so we can also draw it. C^* would, however, go underneath the preimages of B and A to α. To make the tree connected, we introduce an edge $-A$ from w to α. We draw the final edge D^* underneath all the edges already drawn, from w to ∞. The tree T^* is invariant up to a homotopy.

Note. Nekrashevych theory allows us to bypass the explicit drawing-out of the twists by utilising algebraic methods.