MAT240 : DETERMINANTS II 20161206

REVIEW:

THERE EXISTS A UNIQUE MULTILINEAR FUNCTIONAL

$$\det : \underbrace{F^n \times \dots \times F^n} \longrightarrow F$$

SUCH THAT

- $\det(v_1, \dots, v_n) = 0$ WHENEVER $v_r = v_s$ FOR SOME $r < s$.
- $\det(e_1, \dots, e_n) = 1$ FOR STANDARD BASIS

DEFINITION.

$A \in M_{n \times n}(F)$, $\det(A) = \det(v_1, \dots, v_n)$,

WHERE $v_j = A e_j$ ARE COLUMNS OF A

FORMULA

$$\det(A) = \sum_{\sigma} \operatorname{sign}(\sigma) A_{\sigma(1)1} \cdots A_{\sigma(n)n}$$

THEOREM:

$A, B \in M_{n \times n}(F)$,

$\det(AB) = \det(A) \cdot \det(B)$

PROOF: WE MAY ASSUME A IS INVERTIBLE, FOR IF A IS NOT INVERTIBLE, THEN AB IS NOT INVERTIBLE AND THEN $\det(A) = 0$, $\det(AB) = 0$.

ASSUME NOW $\det(A) \neq 0$.

CONSIDER THE MULTILINEAR FUNCTIONAL

$$\varphi(w_1, \dots, w_n) = \frac{1}{\det(A)} \det(Aw_1, \dots, Aw_n)$$

IF $w_r = w_s$ FOR SOME $r < s$ THEN

$\varphi(w_1,\dots,w_n) = \dfrac{\varphi(Aw_1,\dots,Aw_n)}{\det A} = 0$, AND $\varphi(e_1,\dots,e_n) = 1$

BECAUSE $v_j = Ae_j$ IS j-th COLUMN OF A.

BY THE THEOREM PROVEN BEFORE,

$$\varphi(w_1,\dots,w_n) = \det(w_1,\dots,w_n).$$

PUT $w_j = Be_j$ (COLUMNS OF B).

$$\varphi(w_1,\dots,w_n) = \det(w_1,\dots,w_n) = \det(B).$$

$$\det(Aw_1,\dots,Aw_n) = \det(ABe_1,\dots,ABe_n) =$$
$$= \det(AB).$$

THUS, WE HAVE SHOWN

THAT $\det(B) = \dfrac{1}{\det A} \det(AB)$.

<u>RECALL</u>

IF A IS UPPER TRIANGULAR OR LOWER TRIANGULAR,
THEN $\det(A) = A_{11} A_{22} \cdots A_{nn}$.

EXAMPLE: $F = \mathbb{Z}_3$. FIND

$$\det(A) = \det \begin{pmatrix} 2 & 0 & 1 & 2 & 0 \\ 2 & 1 & 2 & 0 & 2 \\ 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 2 & 2 & 2 & 2 & 1 \end{pmatrix}$$

$$\det(A) = \det \begin{pmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{pmatrix}$$

$$= \det \begin{pmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \cancel{\det}\ 1$$

LEMMA: SUPPOSE A ~~IS~~ IN THE BLOCK FORM

$$A = \left(\begin{array}{c|c} A' & * \\ \hline 0 & A'' \end{array}\right) \quad \text{WHERE } A' \in M_{k\times k}(F), \; A'' \in M(l\times l)(F)$$

$$\det(A) = \det(A')\det A''.$$

PROOF: CONSIDER FIRST THE CASE WHEN $A'' = I_{l\times l}$.

THEN BY ROW OPERATIONS

$$\det\left(\begin{array}{c|c} A' & * \\ \hline 0 & I \end{array}\right) = \det\left(\begin{array}{c|c} A' & 0 \\ \hline 0 & I \end{array}\right)$$

$= \det A'$. (USING AN EXPLICIT FORMULA FOR det, OR SIMPLIFYING FURTHER TO MAKE A' UPPER TRIANGULAR)

IN GENERAL,

$$\begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix} = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & A'' \end{array}\right)\left(\begin{array}{c|c} A' & * \\ \hline 0 & I \end{array}\right)$$

$$\therefore \text{ So } \det\begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix} = \det\begin{pmatrix} I & 0 \\ 0 & A'' \end{pmatrix}\det\begin{pmatrix} A' & * \\ 0 & I \end{pmatrix}$$

$$= \det(A')\det A''.$$

EXAMPLE

$$\det\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix} =$$

$$= \det\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & -3 & -5 & 3 \\ 0 & -4 & 4 & -8 \end{pmatrix} =$$

$$= 2\det\begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 3 \\ -4 & 4 & -8 \end{pmatrix} = 2\det\begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & -6 \\ 0 & 16 & -20 \end{pmatrix} =$$

$$= 2\det\begin{pmatrix} 4 & -6 \\ 16 & -20 \end{pmatrix} = 2\,(-80+96) = 32$$

COFACTOR EXPANSIONS

USE LINEARITY IN THE FIRST COLUMN

$$\det \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} =$$

$$= A_{11} \det \begin{pmatrix} 1 & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & & & \end{pmatrix} + A_{21} \det \begin{pmatrix} 0 & A_{12} & \cdots & A_{1n} \\ 1 & A_{22} & \cdots & \vdots \\ 0 & & & \\ \vdots & & & A_{nn} \end{pmatrix}$$

$$+ \cdots$$

$$= A_{11} \det \begin{pmatrix} A_{22} & \cdots & A_{2n} \\ \vdots & & \vdots \\ A_{n2} & \cdots & A_{nn} \end{pmatrix} - A_{21} \det \begin{pmatrix} A_{12} & \cdots & A_{1n} \\ A_{32} & \cdots & A_{3n} \\ A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

$$+ \cdots$$

$$= A_{11} \det\left(\tilde{A}_{[11]}\right) - A_{21} \det\left(\tilde{A}_{[21]}\right) + A_{31} \det\left(\tilde{A}_{[31]}\right) + \cdots$$

WHERE $\tilde{A}_{[ij]} \in M_{(n-1)(n-1)}(F)$ OBTAINED FROM A BY DELETING i-TH ROW AND j-TH COLUMN.

NOTE: SUMMANDS ARE OF THE FORM

$$(-1)^{i+j} A_{ij} \det\left(\tilde{A}_{[ij]}\right)$$

EXAMPLE

$$\det\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix} =$$

$$= 2 \det\begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} - \det\begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 4 & -4 & 4 \end{pmatrix}$$

$$= 2 \det\begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 16 & -18 \end{pmatrix} - \det\begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 0 & -10 & -6 \end{pmatrix}$$

$$= 2 \det\begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 0 & 10 \end{pmatrix} - \det\begin{pmatrix} -2 & -3 & -5 \\ 0 & -10 & -6 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= 2 \cdot 40 - \det\begin{pmatrix} -2 & -3 & -5 \\ 0 & 0 & 24 \\ 0 & 1 & 3 \end{pmatrix} =$$

$$= 80 + \det\begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{pmatrix} = 80 - 48 = 32$$

Cramer's rule.

Let $A \in M_{n\times n}(F)$ be an invertible matrix, and $b \in F^n$.

Then the unique solution $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ of $Ax = b$ is given by

$$x_i = \frac{1}{\det(A)} \det\left(v_1, \dots, v_{i-1}, b, v_{i+1}, \dots, v_n\right),$$

where $v_1, \dots, v_n$ are columns of A.

$\det\left(v_1, \dots, v_{i-1}, b, v_{i+1}, \dots, v_n\right)$ is a determinant of matrix obtained from A by replacing i-th column with b.

Proof:

$$b = Ax = x_1 v_1 + x_2 v_2 + \dots + x_n v_n, \text{ or}$$

x solves $Ax = b$.

$$\det\left(v_1, \dots, v_{i-1}, b, v_{i+1}, \dots, v_n\right) = \sum_{r=1}^{n} x_r \det\overbrace{\left(v_1, \dots, v_{i-1}, v_r, v_{i+1}, \dots, v_n\right)}^{0 \text{ unless } r=i}$$

by linearity in the i^{th} column.

$$= x_i \det\left(v_1, \dots, v_n\right) = x_i \det(A)$$

Divide by $\det(A) \neq 0$.

EXAMPLE. SOLVE $Ax = b$ FOR

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 4 \\ -3 & -2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

REMARK

CRAMER'S RULE CAN BE USED TO GIVE A FORMULA FOR THE INVERSE OF THE MATRIX, A^{-1}.

INDEED, j-TH COLUMN OF A^{-1}, $w_j = A^{-1} e_j$ IS THE SOLUTION OF $A w_j = e_j$.

THUS, $B = A^{-1}$ IS GIVEN BY

$$B_{ij} = \frac{1}{\det(A)} (-1)^{i+j} \det\left(\tilde{A}_{[ji]}\right)$$