1 Polynomial Approximation

The tangent line at $x = a$ can be thought as the best linear approximation to $f(x)$ when $x = a$. Can we do better approximations with polynomials?

We choose polynomials because of their properties which make them amenable to differentiation and integration.

Suppose $a = 0$.

Consider a polynomial of degree 2:

$$P_2(x) = a_0 + a_1 x + a_2 x^2 \tag{1}$$
$$P_2'(x) = a_1 + 2a_2 x \tag{2}$$
$$P_2''(x) = 2a_2 \tag{3}$$

If $a_0 = f(0)$, $a_1 = f'(0)$ and $a_2 = \frac{1}{2} f''(0)$, then P_2 would satisfy

$$P_2(0) = f(0) \tag{4}$$
$$P_2'(x) = f'(0) \tag{5}$$
$$P_2''(x) = f''(0) \tag{6}$$

We shall prove that $P_2(x)$ is the best approximation of f in terms of the polynomial of the second degree.

Now, consider $P_n(x) = \sum_{k=0}^{n} a_k x^k$. Then it is easy to show that $P_n^{(m)}(0) = m! a_m$.

Take $a_m = \frac{1}{m!} f^{(m)}(0)$.

This approximation has a name.

Definition 1.1. The *Taylor polynomial* of degree n for $f(x)$ near $x = 0$ is

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{m!} f^{(m)}(0) x^k$$

Note that a Taylor polynomial can also be defined at any point a by translation:

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{m!} f^{(m)}(a)(x-a)^k$$

Example 1.2

Let $f(x) = \sin x$. Then $f(x)$ for $x < 1$ is approximated well by $g_n(x) = \sum_{k=0}^{n} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$.

Suppose now that $f(x) = \begin{cases} e^{-\frac{1}{x^2}-}, x \neq 0 \\ 0, x = 0 \end{cases}$