MAT240:

$\mathbb{C}$: PART II

2016 09 22

RECAP

IN $\mathbb{R}$ $\neg\exists$ $\sqrt{-1}$.

$\Rightarrow$ INTRODUCE A FORMAL ELEMENT WITH $x^2 = -1$.

$z = a + ib$, $a, b \in \mathbb{R}$,

$$z = |z|(\cos\theta + i\sin\theta)$$
$$w = |z|^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$
$$\Rightarrow w^2 = z.$$

TRICKING

$$1 = \sqrt{1}$$
$$= \sqrt{(-1)(-1)}$$
$$= \sqrt{-1}\sqrt{-1}$$
$$= -1$$

$\Rightarrow$ BE CAREFUL!

PROBLEM

θ IS DEFINED UP TO MULTIPLES OF 2π.

CONVENTION

RESTRICT TO ONE INTERVAL OF 2π.

$-\pi < \theta \leq \pi$

$0 \leq \theta < 2\pi$

HIGHER ORDER POLYNOMIALS

$$p(z) = \sum_{i=0}^{n} a_i x^i,$$

$$a_i \in \mathbb{C} \wedge a_n \neq 0,$$

DEFINITION

A NUMBER $c \in \mathbb{C}$ CALLED A ROOT OF p IF $p(c) = 0$

FUNDAMENTAL THEOREM OF ALGEBRA

EVERY POLYNOMIAL OF DEGREE $n > 0$ HAS AT LEAST ONE ROOT

PROOF

SEE APPENDIX D

THEOREM

IF $c_1 \in \mathbb{C}$ IS A ROOT OF p, THEN p IS DIVISIBLE BY $z - c_1$, ~~i.e.~~

$$q(z) = \frac{p z}{z - c_1}$$ IS A POLYNOMIAL OF DEGREE $n-1$.

THEOREM

IF $n-1>0$, q HAS AT LEAST ONE ROOT WHICH IS ALSO A ROOT OF p

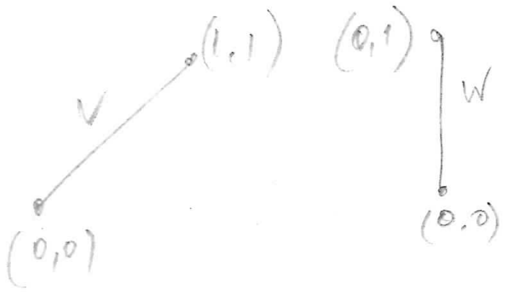
THEREFORE,

$$p(z)=a_n\prod_{i=1}^{n}(z-c_i),$$

WHERE c_i ARE THE ROOTS.

VECTOR SPACES

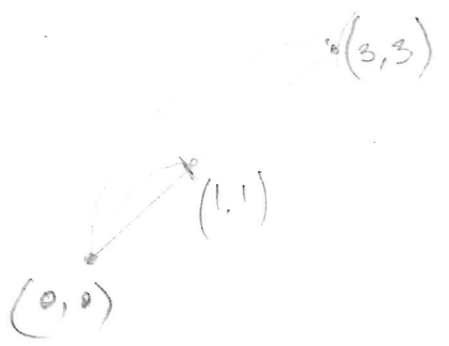
MOTIVATION: COORDINATE VECTORS IN $\mathbb{R}^2$



AS A SET, THE COMBINATION OF THESE VECTORS FORM $\mathbb{R}^2$.

ADDITION

SCALAR MULTIPLICATION



IN COODINATES:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1,\ a_2 + b_2)$$

IN $\mathbb{F}^n$:

$$(a_1, a_2, a_3, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$$
$$t(a_1, a_2, \dots, a_n) = (ta_1, ta_2, \dots, ta_n)$$

BASIC PROPERTIES

$$v + w = w + v$$
$$(u + v) + w = u + (v + w)$$
$$v + 0 = v$$
$$v + (-v) = 0$$
$$s(tv) = (st)v$$
$$(s + t)v = sv + tv$$

EXAMPLES

LET S BE A SET.

$\mathcal{F}(S, \mathbb{R})$ SET OF FUNCTIONS $f: S \to \mathbb{R}$.

ADDITION AND MULTIPLICATON DEFINED 'POINTWISE'

$$(f+g)(x) = f(x) + g(x), \quad (tf)(x) = t(f(x))$$

PROPERTIES:

ADDITION

MULTIPLICATION

EXIST. OF 'SEQUENCES' $(a_1, a_2, a_3, \dots)$.

$$(a_1, a_2, \dots) + (b_1, b_2, \dots) = (a_1 + b_1, a_2 + b_2, \dots)$$
$$t(a_1, a_2, \dots) = (ta_1, ta_2, \dots)$$

GEOMETRIC INTERPRETATION OF MULTIPLICATION

$(a+ib)(c+id) =$

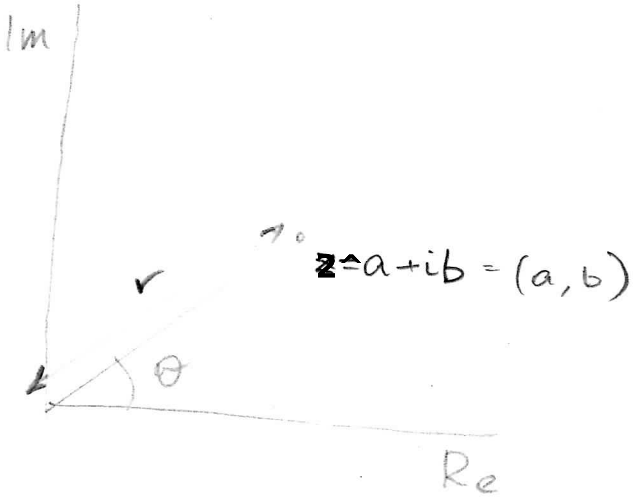
$= ac - bd$

VIE POLAR COORDINATES:

$a = r\cos\theta$

$b = r\sin\theta,$

$r = |z|$



$\Rightarrow$

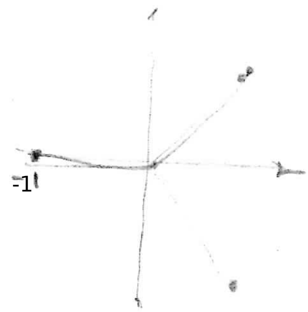
LET $w = s(\cos\varphi + i\sin\varphi),$
$s = |w|$

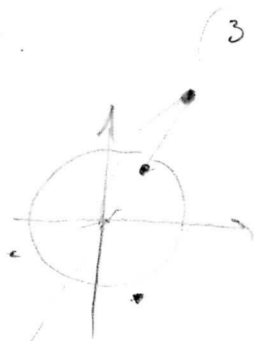
$zw = r(\cos\theta + i\sin\theta) \times s(\cos\varphi + i\sin\varphi)$

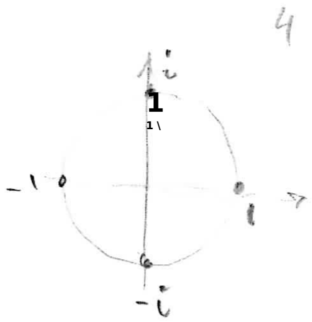
$= rs(\cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\sin\theta\cos\varphi + \cos\theta\sin\varphi))$

$= rs(\cos(\varphi+\theta) + i\sin(\theta+\varphi))$

EXAMPLE







FIND $z \in \mathbb{C}$ WITH $z^3 = -1$

$z_1 = -1, \; z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \; z_3 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

TRIANGLE INEQUALITY

$z, w \in \mathbb{C}$,

$$|z+w| \leq |z| + |w|$$

PROOF

$$|z+w|^2 = (z+w)(\bar{z}+\bar{w})$$

$$= z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w$$

$$= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w})$$

SINCE $(\bar{\bar{w}} + \bar{w}) = 2\,\mathrm{Re}(w)$

$$\leq |z|^2 + |w|^2 + 2|z\bar{w}|$$

$$= |z|^2 + |w|^2 + 2|z||w|$$

$$= (|z| + |w|)^2$$

THEOREM

$\mathbb{C}$ IS A FIELD WITH $0 = 0 + i0$
$1 = 1 + i \cdot 0$

PROOF

EXERCISE.

THM

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$