MAT240 2016 11 30

DETERMINANTS

RECALL: IF $A \in M_{n\times n}(\mathbb{F})$ IS INVERTIBLE,
THEN $Ax=b$ HAS A UNIQUE SOLUTION.

QUESTION: HOW TO DETERMINE INVERTIBILITY OF A MATRIX?

ANSWER: $\det(A) \neq 0 \iff A^{-1}$ EXISTS

e.g. $\underline{n=2}$ $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc$

THEOREM: THE MATRIX $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ IS INVERTIBLE $\iff \det(A) \neq 0$. IN THIS CASE,

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

PROOF:

LET $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \det(A) \cdot I$$

$\Rightarrow$ $\det(A)^{-1}B$ IS ~~A~~ THE INVERSE OF A.

IF $\det(A) = 0$, THEN $AB = 0$.

THEREFORE, IF A IS INVERTIBLE, THEN $A^{-1}AB = A^{-1}0$

$\Rightarrow B = 0$. BUT THEN $A=0$, AND HENCE

A IS NOT INVERTIBLE, WHICH IS A CONTRADICTION

EXAMPLE

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\cancel{X} = \frac{1}{-4}\begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -5 \\ -2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

MOTIVATION FOR $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc$:

CLAIM IF $F = \mathbb{R}$, THEN $\det(A)$ IS the SIGNED AREA OF A PARALLELOGRAM SPANNED BY COLUMNS $v_1 = \begin{pmatrix} a \\ c \end{pmatrix}$, $v_2 = \begin{pmatrix} b \\ d \end{pmatrix}$.

LET $\mathrm{vol}(v_1, v_2)$ BE THE SIGNED AREA OF THE PARALLELOGRAM SPANNED BY v_1, v_2

P1) $\mathrm{vol}(av_1, v_2) = a\,\mathrm{vol}(v_1, v_2) = \mathrm{vol}(av_1, av_2)$

P2)

P3 $\mathrm{Vol}(v_1 + av_2) = \mathrm{Vol}(v_1, v_2)$

P4 $\mathrm{Vol}(v_1, v_2 - av_1) = \mathrm{vol}(v_1, v_2)$

P3 $\mathrm{vol}(e_1, e_2)$

CONSEQUENCES:

$\mathrm{Vol}(0, v) = 0$

$\mathrm{Vol}(v, v) = 0$ by P 2 WITH $v_1 = 0, a = 1$

LEMMA vol IS BI-linear (LINEAR IN BOTH v_1 AND v_2.

PROOF NEED TO SHOW $\mathrm{vol}(v_1 + v', v_2) = \mathrm{vol}(v_1, v_2) + \mathrm{vol}(v_1', v_2)$.

IF $v_2 = 0$, THIS IS CLEAR. IF ONE OF v_1, v_1' IS A SCALAR MULTIPLE OF v_2, IT FOLLOWS FROM P2.

THEREFORE, WE MAY ASSUME THAT IS ... THE ...

SO v_1, v_2 ARE A BASIS, WRITE

$v_1' = \lambda v_1 + \mu v_2$, WITH $\lambda, \mu \in \mathbb{R}$.

$$\mathrm{vol}(v_1 + v_1', v_2) = \mathrm{vol}(v_1 + \lambda v_1 + \mu v_2, v_2)$$

$$= \mathrm{vol}((1+\lambda) v_1 + \mu v_2, v_2)$$

$$= \mathrm{vol}((1+\lambda) v_1, v_2) \text{ BY } \ldots$$

$$= (1+\lambda)\mathrm{vol}(v_1, v_2) \text{ BY D1}$$

$$= \mathrm{vol}(v_1, v_2) + \mathrm{vol}(\lambda v_1, v_2)$$

$$= \mathrm{vol}(v_1, v_2) + \mathrm{vol}'(v_1', v_2)$$

ANOTHER CONSEQUENCE $\quad \circ\ \mathrm{vol}(v_1, v_2) = -\mathrm{vol}(v_2, v_1)$

NOW LET

$$v_1 = \begin{pmatrix} a \\ c \end{pmatrix} = a e_1 + c e_2$$

$$v_2 = \begin{pmatrix} c \\ d \end{pmatrix} = \mathrm{vol}\ b e_1 + d e_2$$

THEN $\quad \mathrm{vol}(v_1, v_2) = \mathrm{vol}(a e_1 + c e_2, b e_1 + d e_2) =$

$= ab\ \mathrm{vol}(e_1, e_1) + ad\ \mathrm{vol}(e_1, e_2) + bc(e_2, e_1) +$

$+ cd(e_2, e_2)$

$\Rightarrow \det A = ab - cd$

IN A SENSE,

$\det(A) = \text{vol}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right)$ measures the extent to which the two column vectors are linearly independent.

MORE GENERALLY, FOR A ANY FIELD F, THERE IS A UNIQUE BILINEAR MAP

$$\text{vol}: F^2 \times F^2 \longrightarrow F, \; v_1, v_2 \mapsto \text{vol}(v_1, v_2)$$

GIVEN THAT

$$\text{vol}(v, v) = 0 \quad \text{FOR all } v \in F^2$$
$$\text{vol}(e_1, e_2) = 1.$$

NAMELY, $\text{vol}(v_1, v_2) = ad - bc$ IF $v_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ & $v_2 = \begin{pmatrix} b \\ d \end{pmatrix}$

REMARK: $\text{vol}(v, v) = 0 \Rightarrow \text{vol}(v_1, v_2) = -\text{vol}(v_2, v_1)$

BUT $\text{vol}(v_1, v_2) = -\text{vol}(v_2, v_1) \Rightarrow 2\,\text{vol}(v, v) = 0$

IF $2 \neq 0$ IN F, THE TWO PROPERTIES ARE EQUIVALENT.

HIGHER DIMENSIONS

IN $\mathbb{R}^n$, CONSIDER THE VOLUME OF A PARALLELEPIPED SPANNED BY $v_1, \dots, v_n \in \mathbb{R}^n$.

$$\mathrm{vol}(v_1, \dots, v_n)$$

WHICH HAS THE FOLLOWING PROPERTIES

- MULTI-LINEAR (LINEAR IN EACH ARGUMENT, KEEPING OTHERS FIXED)
- $\mathrm{vol}(v_1, \dots, v_n) = 0$ WHENEVER $v_r = v_s$ FOR ANY $r \neq s$.
- $\mathrm{vol}(e_1, \dots, e_n) = 1$.

FOR $F = \mathbb{R}$, det CAN BE DEFINED AS FOLLOWS:

$$\det(A) = \mathrm{vol}(v_1, \dots, v_n)$$

WHERE $v_1, \dots, v_n$ ARE COLUMNS OF A.

<u>THEOREM</u>. THERE IS A UNIQUE MULTI-LINEAR FUNCTIONAL

$$\det : \underbrace{F^n \times \dots \times F^n}_{n} \longrightarrow F$$

WITH 2 ADDITIONAL PROPERTIES

- $\det(v_1, \dots, v_n) = 0$ WHENEVER $v_r = v_s$, SOME $r \neq s$
- $\det(e_1, \dots, e_n) = 1$

FROM THE FIRST PROPERTY,

$\mathrm{vol}(v_1, \ldots, v_n)$

CHANGES SIGN WHENEVER ANY TWO v_i'S ARE INTERCHANGED.

THUS, $\det(e_2, e_1, e_3, \ldots, e_n) = -1$.

A PERMUTATION OF $1, \ldots, n$ IS AN INVERTIBLE MAP.

$\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$.

$n = 4$: $\sigma(1) = 4$, $\sigma(2) = 2$, $\sigma(3) = 1$, $\sigma(4) = 3$

DEFINITION: A PERMUTATION IS EVEN (ODD) IF THE NUMBER OF PAIRS i, j WITH $i < j$ BUT $\sigma(i) > \sigma(j)$ IS EVEN (ODD).

IN THIS CASE DEF. $\mathrm{sign}(\sigma) = +1 \ (-1)$.

EG. $(4\ 3\ 1\ 2)$

$(4\,3), (4\,1), (4\,2), (3\,1), (3\,2)$

IF σ' IS OBTAINED BY INTERCHANGING TWO ADJACENT ELEMENTS, THEN σ, σ' HAVE OPPOSITE PARITY.

EXAMPLE : $(4\ 1\ 3\ 2)$

$\Rightarrow (4\,1), (4\,3), (4\,2), (3\,2)$

THE COLUMN VECTORS CAN BE PUT INTO THE CORRECT ORDER BY A FINITE NUMBER OF INTERCHANGES OF DISTINCT ELEMENTS, EACH OF WHICH GIVES A MINUS SIGN.

$$\det(e_{\sigma(1)}, \ldots, e_{\sigma(n)}) = \operatorname{sign}(\sigma) \det(e_1, \ldots, e_n)$$
$$= \operatorname{sign}(\sigma)$$

FOR GENERAL $v_1, \ldots, v_n \in \mathbb{F}^n$, WRITE

$$v_j = \sum_{i=1}^{n} A_{ij} e_i$$

$$\det(v_1, \ldots, v_n) = \det\left(\sum_{i_1=1}^{n} A_{i_1,1} e_{i_1}, \ldots, \sum_{i_n=1}^{n} A_{i_n n} e_{i_n}\right)$$

$$= \sum_{i_1=1}^{n} \sum_{i_n=1}^{n} A_{i_1 1} \cdots A_{i_n n} \det(e_{i_1}, \ldots, e_{i_n})$$

$$= \sum_{\sigma} A_{\sigma(1) 1} \cdots A_{\sigma(n) n} \det(e_{\sigma(1)}, \ldots, e_{\sigma(n)}),$$

$$= \sum_{\sigma} \operatorname{sign}(\sigma) A_{\sigma(1) 1} \cdots A_{\sigma(n) n}$$

THEREFORE, FOR ANY PERMUTATION σ, $\operatorname{SIGN}(\sigma) = (-1)^N$,

WHERE $N =$ # OF INTERCHANGES OF ADJACENT ELEMENTS, PUTTING $\{1, n\}$ IN ORDER

EG: $(4\,3\,1\,2) \to (4\,1\,3\,2) \to (1\,4\,3\,2) \to (1\,3\,4\,2)$

$\downarrow$

$(1\,2\,3\,4) \leftarrow (1\,3\,2\,4)$

<u>EXERCISE</u> IF σ' IS OBTAINED FROM σ BY INTERCHANGING ANY TWO ELEMENTS, THEN σ, σ' HAVE OPPOSITE PARITY.

<u>PROOF OF THE THEOREM</u>

ASSUMING EXISTENCE, WE PROVE UNIQUENESS FIRST.

ANY MULT-LINEAR FUNCTIONAL IS DETERMINED BY ITS VALUES ON BASIS ELEMENTS (SINCE ONE CAN USE LINEARITY TO 'EXPAND' ITS IMAGE).

So $\det(\dots)$ IS DETERMINED BY

$(e_{i_1}, e_{i_2}, \dots, e_{i_n})$ FOR ALL $i_1, \dots, i_n \in \{1, \dots, n\}$.

IF ANY TWO INDICES COINCIDE THEN

$\det(e_{i_1}, \dots, e_{i_n}) = 0$ BY ASSUMPTION.

$\Rightarrow$ NEED ONLY TO CONSIDER THE CASE THEY'RE ALL DIFFERENT, i.e. A PERMUTATION OF $1, 2, \dots, n$!

$i_1 = \sigma(1), \dots, i_n = \sigma(n)$