Remark 0.1. On a vector space V there can be many inner products.

Example 0.2

- if $\langle\cdot,\cdot\rangle$ is an inner product, then so is $c\langle\cdot,\cdot\rangle$ for $c > 0$ in $\mathbb{R}$
- if $\phi : V \to V$ is an isomorphism, then also $\langle x, y\rangle' = \langle\phi(x), \phi y\rangle$
- if $V = C[a, b]$ is a vector space of continuous products ($a < b$), where $\mathbb{F} = \mathbb{R}$). Then $\langle f(x), f(y)\rangle = \int_a^b f(t)g(t)\,\mathrm{d}t$ is an inner product.
- if $V = C[a, b]$ is a vector space of continuous products ($a < b$), where $\mathbb{F} = \mathbb{C}$). Then $\langle f(x), f(y)\rangle = \int_a^b f(t)\overline{g(t)}\,\mathrm{d}t$ is an inner product.

  Here, if $f(x) \in C[a, b]$, write $f(x) = f_1(x) + if_2(x)$, where $f_1, f_2 \in \mathbb{R}$. Define $\int_a^b f(t)\,\mathrm{d}t = \int_a^b f_1\,\mathrm{d}ti\int_a^b f_2\,\mathrm{d}t$. Then $\overline{\int f(t)\,\mathrm{d}t} = \int \overline{f(t)\,\mathrm{d}t}$ and $\int(f(t) + cg(t))\,\mathrm{d}t = \int f(t)\,\mathrm{d}t + c\int g(t)\,\mathrm{d}t$.

Definition 0.3. **H** is the inner product space $C[0, 2\pi]$, $\mathbb{F} = \mathbb{C}$, with $\langle f, g, =\rangle \frac{1}{2\pi}\int_0^{2\pi} f(t)g(t)\,\mathrm{d}t$.

Theorem 0.4

Let V be an inner product space with the inner product $\langle\cdot,\cdot\rangle$.

a) $\langle x, x\rangle = 0 \Leftrightarrow x = 0$

b) If for all $x \in V.\langle x, y\rangle = \langle x, z\rangle$, then $y = z$

Proof. a) If $x = 0$, then $\langle x, x\rangle = 0$. Otherwise, $\langle x, x\rangle > 0$

b) If $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in V$, then $\langle x, y - z\rangle = 0$. Therefore, taking $x = y - z$, we obtain $y - z = 0$.

□

Definition 0.5. The **norm** or **length** of $x \in V$ is $\|\|$