1 Problem II

Lemma 1.1

If T is diagonalisable and $W \subseteq V$ is a T-invariant subspace, then the restriction $T|_W$ is also diagonalisable.

Proof.

Since T is diagonalisable, the characteristic polynomial of T splits.

Let $f(t)$ be a characteristic polynomial of T. By Cayley-Hamilton Theorem, $f(T) = 0$. Therefore, we obtain that $f(T)|_W = 0$, which by homogeneity of T and from the fact that W is T-invariant means that $f(T|_W) = 0 = g(T)$. Since $f(T)$ splits, the characteristic polynomial of $T|_W$ also splits.

We now show that for every eigenvalue μ of $T|_W$, the condition $E_\mu|_W = K_\mu|_W$ must hold, where $E_\mu|_W$ is an eigenspace and $K_\mu|_W$ is a generalised eigenspace corresponding to the eigenvalue μ of $T|_W$.

Consider an eigenvalue μ of $T|_W$. Since $g(T)$ splits, at least one μ and a corresponding eigenvector $v_\mu \in E_\mu$ exist.

Since $K_\mu|_W = \{v \,|\, v \in \ker(T|_W - \mu I)^m \text{ for some } m \in \mathbb{Z}^+\}$, then $E_\mu|_W \subseteq K_\mu|_W$ by definition.

Suppose now $w \in K_\mu|_W$. Note that $K_\mu|_W = K_\mu \cap W$, and thus $w \in K_\mu$ and $w \in W$. Since $f(T)$ splits, then $K_\mu = E_\mu$. Therefore, $w \in E_\mu$. Since W is T-invariant, then $w \in E_\mu \cap W = E_\mu|_W$. Thus, $K_\mu|_W \subseteq E_\mu|_W$, and hence $K_\mu|_W = E_\mu|_W$. Because $g(T)$ splits, we conclude that $T|_W$ is diagonalisable.

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