Suppose V is the infinite-dimensional vector space of sequences $\sigma : \mathbb{N} \to F$ that have only a finite number of non-zero terms. In other words, $\sigma(k) \neq 0$ for only finitely many positive integers k.

Define $\langle \sigma, \tau \rangle = \sum_{k=1}^{\infty} \sigma(k)\overline{\tau(k)}$.

Problem

Show that this is an inner product on V.

Solution

Positivity

Since the sum of products of nonnegative integers is nonnegative, then for any $\sigma, \tau \in V$ we have $\langle \sigma, \tau \rangle \geq 0$.

Definiteness

Let $\sigma \in V$ be a sequence a_i.

Suppose $\langle v, v \rangle = 0$. Therefore, $\sum_{i=1}^{n} |a_i|^2 = 0$. Hence, $\langle v, v \rangle$ is 0 if and only if $a_i = 0$ for all i, and thus $v = 0$.

Additivity in the First Slot

Note that for all $\sigma, \tau, \upsilon \in V$

$$\begin{aligned} \langle \sigma + \tau, \upsilon \rangle &= \sum_{i=1}^{\infty} (\sigma(i) + \tau(i))\overline{\upsilon(i)} && (1) \\ &= \sum_{i=1}^{\infty} \sigma(i)\overline{\upsilon(i)} + \sum_{i=1}^{\infty} \sigma(i)\overline{\upsilon(i)} && (2) \\ &= \langle \sigma, \upsilon \rangle + \langle \tau, \upsilon \rangle && (3) \end{aligned}$$

Homogeneity in the First Slot

For all $\sigma, \tau \in V$ and $\lambda \in \mathbb{F}$,

$$\langle \lambda\sigma, \tau \rangle = \sum_{i=1}^{\infty} \lambda\sigma(i)\overline{\tau(i)} = \lambda \sum_{i=1}^{\infty} \sigma(i)\overline{\tau(i)} = \lambda\langle \sigma, \tau \rangle$$

Conjugate Symmetry

For all $\sigma, \tau \in V$

$$\langle \sigma, \tau \rangle = \sum_{i=1}^{\infty} \sigma(i)\overline{\tau(i)} = \sum_{i=1}^{\infty} \overline{\overline{\sigma(i)}\tau(i)} = \sum_{i=1}^{\infty} \overline{\tau(i)\overline{\sigma(i)}} = \overline{\langle \tau, \sigma \rangle}$$

Therefore, $\langle \cdot, \cdot \rangle$ is an inner product.

Problem

For $n \geq 1$ define $e_n \in V$ by $e_n(k) = 1$ if $k = n$ and $e_n(k) = 0$ if $k \neq n$. Show that the set $\{e_n\}$ is an orthonormal basis of V

Solution

Since for each $k \in \mathbb{N}$, $\sigma \in V$ and any $i \in \mathbb{N}$ there exists $\lambda \in \mathbb{N}$ such that $\sigma(i) = \lambda k$ by the fundamental theorem of arithmetic, while each σ contains only finitely many nonzero elements, then V is spanned by e_i. By definition, the set of all e_i is linearly independent.

We prove now that it is also orthonormal.

Take e_i, e_j for any $i, j \in \mathbb{N}$ such that $i \neq j$

Note that $\langle e_i, e_j \rangle = \sum_{k=1}^{\infty} e_i(k)\overline{e_j(k)} = 0$, since e_i is nonzero only at the i^{th} position and e_j is nonzero at the j^{th} position.

If $i = j$, however, we obtain $\langle e_i, e_i \rangle = 1$, since the value of e_i at the ith position is 1, and thus $e_i\overline{e_i} = 1$.

Therefore, $\{e_n\}$ is an orthonormal basis.

Problem

Let W be the subspace spanned by the elements $e_1 + e_n$ for all $n \geq 2$.

Show that $e_1 \notin W$ and that $W^{\perp} = \{0\}$.

Deduce that $(W^{\perp})^{\perp} \neq W$.

Solution

Note that for any sequence $w \in W$, $w = a_1 e_1 + \sum_{i=2}^{\infty} a_i e_i$ by definition of W. Moreover, if $a_1 = 0$, then $w = 0$, since W is spanned by $e_1 + e_i$ for $i \geq 2$.

If, however, $a_1 \neq 0$, then there exists at least one $k \geq 2$ such that

$$w = a_1 e_1 + a_k e_k + \sum_{i=2, i \neq k}^{\infty} a_i e_i$$

and $a_k \neq 0$.

Since e_1 and e_k are linearly independent, the first element and the k^{th} in the sequence of w are nonzero.

Therefore, there does not exist an element $v \in V$ such that the first element in v is nonzero and the rest are zero. Therefore, $e_1 \notin W$.

Consider now $W^{\perp}$.

Let $w \in W$ be a sequence in $W^{\perp}$.

Since $\{e_i\}$ is a basis of V,

$$w = \sum_{i=1}^{\infty} a_i e_i,$$

where there are only finitely many nonzero a_i.

Suppose that there exists $k \in \mathbb{N}$ such that some a_k is nonzero.

By definition, $\forall y \in W. \langle w, y \rangle = \sum_{i=1}^{\infty} a_i \overline{b_i} = 0$, where b_i are such that $y = \sum_{i=1}^{\infty} b_i e_i$.

Take some $y \in W$ such that $y \neq 0$ and the k^{th} element in y is nonzero. By the argument above and definition of W as a span of $e_1 + e_i$ for $i \geq 2$, such an element exists.

Therefore, $\langle w, y \rangle = \sum_{i=1}^{\infty} a_i \overline{b_i}$, and thus $\langle w, y \rangle = a_k b_k$, since $b_k \in \mathbb{N}$ and thus $b_k = \overline{b_k}$.

Since $a_k \neq 0$ and $b_k \neq 0$, while $a_k, b_k \in \mathbb{N}$, we get that $\langle w, y \rangle > 0$. But $w \in W^{\perp}$ by assumption, so our assumption must be false and hence there exists no such $k \in \mathbb{N}$ such that some a_k is nonzero. Therefore, $W^{\perp} = \{0\}$.

Since $\{0\}^{\perp} = V$, because each vector in V is orthogonal to 0, while $e_1 \notin W$ and thus $V \neq W$, it follows that $W^{\perp\perp} \neq W$.