MAT240: 20161027

MATRIX MULTIPLICATION

RECALL: V IS A VECTOR SPACE, $\beta = \{v_1, \dots, v_n\}$ ORDERED BASIS, $\dim V = n \in \mathbb{N}$.

$$V \longrightarrow F^n, \; v \mapsto [v]_\beta$$

WITH INVERSE $F^n \longrightarrow V$,

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \longmapsto \sum_{j=1}^{m} a_j v_j .$$

SUPPOSE V, W ARE GIVEN WITH BASES $\{v_1, \dots, v_n\}$, $\{w_1, \dots, w_m\}$.

DEFINE AN ISOMORPHISM $\mathcal{L}(V, W) \rightarrow M_{m \times n}(F)$
WITH INVERSE $M_{m \times n}(F) \longrightarrow \mathcal{L}(V, W)$

$$A = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn} \end{pmatrix} \longmapsto T$$

WHERE T IS GIVEN AS

$$T\left(\sum_{j=1}^{n} a_j v_j\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij}\, a_j\right) w_i$$

THUS, THE j^{TH} COLUMNS OF $[T]_\beta^\gamma = A$ ARE THE COEFFICIENTS OF $T(v_j)$ IN BASIS $\{w, \ldots, w_m\}$.

<u>EXAMPLE</u>

FIND THE COORDINATE REPRESENTATION FOR

$$T: \mathcal{P}_3(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R}),$$

$$p \longmapsto p''' + p'$$

FOR THE STANDARD BASES

$\beta = \{1, x, x^2, x^3\}$, $\gamma = \{1, x, x^2\}$:

$T(1) = 0$

$T(x) = 1$

$T(x^2) = 2x$

$T(x^3) = 6 + 3x^2$

$$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

REMARK $[T]_{\beta}^{\gamma}$ DEPENDS ON THE CHOICE OF ORDERED BASES.

INTERCHANGING w_{j_1}, w_{j_2} OF W INTERCHANGES THE j_1, j_2-TH <u>ROW</u> OF $[T]_{\beta}^{\gamma}$.

INTERCHANGING v_{j_1}, v_{j_2} OF V INTERCHANGE THE j_1, j_2-TH <u>COLUMN</u> OF $[T]_{\beta}^{\gamma}$.

NOTATION IF $W = V$, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

IF $\gamma = \beta$, $[T]_{\beta} = [T]_{\beta}^{\beta}$

EXAMPLE THE MATRIX OF $I_V \in \mathcal{L}(V)$

$v \longmapsto v$ HAS A MATRIX

$$[T]_{\beta} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \; 1 \end{pmatrix} = I$$

MORE GENERALLY, GIVEN AN ISOMORPHISM

$T : V \longrightarrow W$, AND $\beta = \{v_1, \dots, v_n\}$,

$\gamma = \{w_1, \dots, w_n\}$.

ARE RELATED BY $w_i = T(v_i)$, THEN

$$[T]_\beta^\gamma = I_{n \times n}$$

QUESTION: HOW TO DESCRIBE $w = T(v)$ TOS IN TERMS OF THEIR BASES?

IF $[v]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $[w]_\gamma = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

$$\boxed{b = \sum_{j=1}^{n} A_{ij} \; a_j}$$

IN SHORT,

$$[T]_\beta^\gamma [v]_\beta = [T(v)]_\gamma$$

THE i-TH ENTRY OF $[T(v)]_\gamma$

IS THE "PRODUCT" OF THE i-TH ROW OF $[T]_\beta^\gamma$ WITH $[v]_\beta$.

QUESTION: HOW TO DESCRIBE $w = T(v)$, $T \circ S$?

$$
\begin{tikzcd}
U \arrow[r, "S"] \arrow[rr, bend right, "T \circ S"'] & V \arrow[r, "T"] & W
\end{tikzcd}
$$

$\alpha = \{u_1, \ldots, u_n\}$: A BASIS FOR U

$\beta = \{v_1, \ldots, v_n\}$: A BASIS FOR V

$\gamma = \{w_1, \ldots, w_m\}$: A BASIS FOR W

$[T \circ S]_\alpha^\gamma$: $A = [T]_\beta^\gamma$, $B = [S]_\alpha^\beta$

$$
\begin{aligned}
T \circ S(u_k) &= T\left(\sum_{j=1}^{n} B_{jk} v_j\right) \\
&= \sum_{j=1}^{n} B_{jk} \sum_{i=1}^{m} A_{ij} w_i \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} B_{jk}\right) w_i
\end{aligned}
$$

$$[T \circ S]_{\alpha}^{\gamma} = C \,, \qquad \boxed{C_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}}$$

↳ DEFINITION OF A MATRIX PRODUCT