1 Curry-Howard Correspondence

1.1 Introduction

Definition 1.1. Let $x_1, x_2, \ldots$ be some variables. We say that any variable x_i is a λ-term. Moreover, if M and N are λ-terms, then an application (MN) is also a λ-term. If P is a λ-term and x is a variable, then an abstraction $(\lambda x.P)$ is also a λ-term.

Note. $(\lambda x.x)$ can be interpreted as an identity function, while $(\lambda x.y)$ – as a constant function.

Note. We assume left-associativity of application, and right-associativity of abstraction. Thus, $(\lambda x.(\lambda y.(xy)))z$ can be rewritten as $(\lambda x.\lambda y.xy)z$.

We are going to differentiate between the use of variables in abstractions and applications. In a sense, abstractions *bind* variables, while variables in applications are *free*. Nevertheless, abstractions may contain free variables.

Furthermore, we can talk about equivalence of λ-terms. For example, $(\lambda x.x)$ and $(\lambda t.t)$ both represent the identity function. In this case, we denote the equivalence as $(\lambda x.x) \equiv (\lambda t.t)$.

It is worthwhile to denote the set of free variables of λ-term M as $FV(M)$.

We denote a substitution as $[N/x]P$, where P is a λ-term.

$$[N/x]x = (N) \tag{1}$$
$$[N/x]y = y \tag{2}$$
$$[N/x](MP) = ([N/x]M)([N/x]P) \tag{3}$$
$$[N/x](\lambda x.P) = (\lambda x.P) \tag{4}$$
$$[N/x](\lambda y.P) = \lambda y.[N/x]P, \text{ if } y \notin FV(N) \tag{5}$$
$$[N/x](\lambda y.P) = \lambda z.[N/x][z/y]P, \text{ if } y \in FV(N), z \notin FV(NP) \tag{6}$$

Let the application in the form $(\lambda x.M)N$ be called a *redex*. We define a computation, intuitively understood as $(\lambda x.x)t \underset{\beta}{\rightarrow} t$, as $(\lambda x.M)N \underset{\beta}{\rightarrow} [N/x]M$.

Example 1.2

$(\lambda x.x(xy))N \underset{\beta}{\rightarrow} [N/x](x(xy)) = (N)$

We define a $\lambda-$term as N, if $FV(N) = \emptyset$.

If we substitute an abstraction, λ's are omitted.

Theorem 1.3 (Church-Rosser)

If in the process of reduction we have chosen different order of reducing terms, there exists a unique λ-term which can be obtained as a final result of all the reductions.

Let $true$ denote $\lambda x.\lambda y.x$, and let $false$ denote $\lambda x.\lambda y.y$. We also introduce an operator $if\ C\ then\ E_1\ else\ E_2 = CE_1E_2$. Thus, for example,

$$\begin{aligned} trueE_1E_2 &= (\lambda x.\lambda y.x) && (7)\\ &= [E_1/x](\lambda y.x)E_2 && (8)\\ &= (\lambda y.E_1)E_2 && (9)\\ &E_1. && (10)\end{aligned}$$

Now, let $notC = if\ C\ then\ false\ else\ true = C\ false\ true$, and let introduce one of the simplest data structures, a pair:

$$(E_1, E_2) = \lambda z.zE_1E_2.$$

Now, it is worthwhile to note the concept of the Church numerals:

$$\begin{aligned} \overline{0} &= \lambda x.\lambda y.y && (11)\\ \overline{1} &= \lambda x.\lambda y.xy && (12)\\ \overline{2} &= \lambda x.\lambda y.x(xy) && (13)\end{aligned}$$