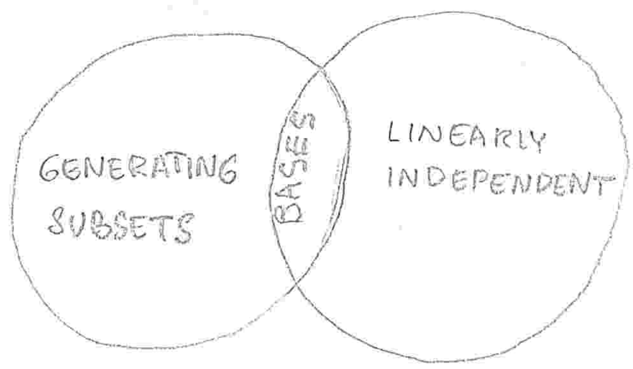
MAT240: BASES OF VECTOR SPACES

20161011

RECALL

$\beta \subseteq V$ IS A BASIS OF V

$\Longleftrightarrow$
- β LINEARLY INDEPENDENT
- β GENERATES V ($\mathrm{span}(\beta) = V$)



FACT

EVERY VECTOR SPACE ADMITS A BASES.

THEOREM (☆)

ANY TWO BASES OF A VECTOR SPACE V HAVE THE SAME NUMBER OF ELEMENTS.

DEFINITION

THE NUMBER OF ELEMENTS IN A BASIS OF V IS CALLED THE DIMENSION OF V, DENOTED $\mathrm{DIM}(V)$

EXAMPLES

$\mathrm{DIM}(F^n) = n$

$\lim_{n\to\infty} (\dim(F^n)) \longrightarrow \infty.$

$\dim(M_{m\times n}(F)) = mn$

$\dim(\mathcal{P}(F)) \longrightarrow \infty$ FOR $F = \mathbb{R}, \mathbb{C}, \mathbb{Q}$.

$\dim(P_n(F)) = n+1$

REMARK

IF $\mathcal{P}_n(F)$ AS A "FORMAL EXPRESSION",

$$a_0 + \sum_{i=0}^{n} a_i x^i$$ WITH $a_i \in F$,

THEN $\dim(\mathcal{P}_n(F)) = n+1$ IN GENERAL.

IF $\mathcal{P}_n(F)$ IS DEFINED AS A FUNCTION $F \to F$, $x \mapsto p(x)$,

$\to$ NOT NECESSARILY TRUE.

LEMMA

REPLACEMENT LEMMA

SUPPOSE V IS A VECTOR SPACE OVER F.

LET $S \subseteq V$ BE A SUBSET WITH $\mathrm{SPAN}(S) = V$, AND $v_1, \ldots, v_m$ ARE LINEARLY INDEPENDENT.

THEN THERE EXIST DISTINCT $u_1, \ldots, u_m \in S$ SUCH THAT

$$\mathrm{SPAN}\left(\left(S \setminus \{u_1, \ldots, u_m\} \right) \cup \{v_1, \ldots, v_m\} \right) = V.$$

AND $m \leq \# S$.

PROOF OF (☆)

LET β, γ BE TWO BASES OF V.

IF BOTH INFINITE, NOTHING TO PROVE,

ASSUME $\#\beta$ IS FINITE. USE REPLACEMENT LEMMA WITH $S = \beta$, $\{v_1, \ldots, v_m\} = \gamma$ (?)

THEN $\#\gamma = m \leq \# S = \#\beta$.

IN PARTICULAR, γ IS FINITE.

REVERSING ROLES, $\#\beta \leq \#\gamma$, SO $\#\beta = \#\gamma$.

PROOF OF THE REPLACEMENT LEMMA

USE INDUCTION ON m, STARTING WITH $m=0$, I.E $\{\bigodot_{i=1}^{m} v_i\} = \emptyset$.

SUPPOSE THE HYPOTHESIS HOLDS FOR m.

SUPPOSE $\bigodot_{i=1}^{m+1} v_i$ ARE LINEARLY INDEPENDENT

SINCE $\bigodot_{i=1}^{m} v_i$ ARE LINEARLY INDEPENDENT,

$$\exists \bigodot_{i=1}^{m} u_i \in S : \operatorname{SPAN}\left(\left(S \setminus \{\bigodot_{i=1}^{m} u_i\}\right) \cup \{\bigodot_{i=1}^{m} v_i\}\right) = V$$

IN PARTICULAR,

$$v_{m+1} = \bigoplus_{i=1}^{m} a_i v_i + \bigoplus_{i=1}^{r} b_r w_r,$$

WHERE $\bigodot_{i=1}^{r} w_i \in S \setminus \{u_1, \dots, u_m\}$ AND

$a_i, b_j \in F$.

NOTE THAT NOT ALL $\bigodot_{i=1}^{r} b_i$ ARE ZERO, BECAUSE v_{m+1} IS NOT A LINEAR COMBINATION OF $\bigodot_{i=1}^{m} v_i$.

CHOOSE i SUCH THAT $b_i \neq 0$, AND LET $u_{m+1} = w_i$.

$$u_{m+1} = \frac{1}{b_i}\left(v_{m+1} - \left(\bigoplus_{k=1}^{m} a_k v_k + \bigoplus_{i=1}^{r} b_j w_j - b_i w_i\right)\right)$$

$$\in \operatorname{SPAN}\left(\left(S \setminus \{\bigodot_{i=1}^{m+1} u_i\}\right) \cup \{\bigodot_{i=1}^{m+1} v_i\}\right).$$

$$\Rightarrow \operatorname{SPAN}\left(\left(S \setminus \{u_1, \dots, u_{m+1}\}\right) \cup \{v_1, \dots, v_{m+1}\}\right) =$$

$$= \operatorname{SPAN}\left(\left(S \setminus \{u_1, \dots, u_m\}\right) \cup \{v_1, \dots, v_{m+1}\}\right)$$

$$\supseteq \operatorname{SPAN}\left(\left(S \setminus \{\bigodot_{i=1}^{m} u_i\}\right) \cup \{\bigodot_{i=1}^{m} v_i\}\right) = V$$

THEOREM

LET V BE A VECTOR SPACE,
$\dim(V)$ IS FINITE, AND $S \subseteq V$.

a) IF $\operatorname{SPAN}(S) = V$, THEN
$\#S > \dim V \Rightarrow S$ CONTAINS A BASIS
$\#S = \dim V$ HOLDS IF S IS A BASIS!

b) IF S IS LINEARLY INDEPENDENT, THEN
$\#S \leq \dim V$, AND S CAN BE EXTENDED TO A BASIS.

GAP! WHAT IF S IS INFINITE?

$\#S = \dim \iff S$ IS A BASIS.

THEOREM

$\forall (S \subseteq V)$, $\dim(V)$ IS FINITE,
ANY TWO OF THE CONDITIONS

- $\operatorname{SPAN}(S) = V$
- S LINEARLY INDEPENDENT
- $\#S = \dim(V)$

$\iff S$ IS A BASIS

EXAMPLE

LAGRANGE INTERPOLATION

F IS A FIELD, $\bigodot_{i=0}^{n} c_i \in F$ ARE DISTINCT.

SUPPOSE $\bigodot_{i=0}^{n} a_i \in F$ ARE "VALUES".

PROBLEM

FIND A POLYNOMIAL $p \in \mathcal{P}_n(F)$

WITH $\bigodot_{i=0}^{n} p(c_i) = a_i$

SUBEXAMPLE

FIND A POLYNOMIAL $p \in \mathcal{P}_3(\mathbb{R})$

WITH $p(0)=0,\ p(1)=2,\ p(2)=0,\ p(3)=2$

SOLUTION

CONSIDER POLYNOMIALS $p_0, \dots, p_n \in \mathcal{P}_n(F)$

WITH $p_i(c_j) = \begin{cases} 0 & \text{IF } i \neq j \\ 1 & \text{IF } i = j \end{cases}$

THEN $$p_i(x) = \frac{\prod_{\substack{j \neq i \\ j=0}}^{n} (x - c_j)}{\prod_{\substack{j \neq i \\ j=0}}^{n} (c_i - c_j)}$$

IN GENERAL,

$$p_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{m} \frac{(x - c_j)}{(c_i - c_j)}$$

THEN $p(c_j) = a_0 p_0(c_j) + \dots + a_j \underbrace{p_j(c_j)}_{1} + 0 = a_j!$

IN THE SUBEXAMPLE,

$$p_0(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = \frac{x^3 - 6x^2 + 11x - 6}{-6}$$

PROOF

a) NOTE

ALREADY PROVEN:

IF $V = \text{SPAN}(S)$, THEN S CONTAINS A BASIS γ. THEN $\# S \geq \# \gamma = \text{DIM } V$.

EQUALITY MEANS THAT $\# S = \# \gamma$.

$\iff S = \gamma \implies S$ IS A BASIS

b)

PICK ANY BASIS β OF V.

APPLY THE REPLACEMENT LEMMA WITH $V_1, \dots, V_m$ AS THE ELEMENTS OF S.

$$\exists \bigodot_{i=1}^{m} u_i \in \beta :$$

$$\text{SPAN}\left\{ \left(\beta \setminus \left\{ \bigodot_{i=1}^{m} u_i \right\}\right) \cup \left\{ \bigodot_{i=1}^{m} v_i \right\}\right\} = V$$

$$\text{THEN } \gamma := \left(\beta \setminus \left\{ \bigodot_{i=1}^{m} u_i \right\}\right) \cup \left\{ \bigodot_{i=1}^{m} V_n \right)$$

SPANS V AND $\# \gamma = \# \beta = \text{DIM } V$,

THEN BY a), γ IS A BASIS AND γ CONTAINS S. $\# S = m = \# \gamma$

$\iff \gamma = S$, i.e. IFF S IS A BASIS

EXAMPLE

$M_{2\times 2}(\mathbb{R})$ :: DIM $= 2 \times 2 = 4$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

FORM A BASIS.

THEOREM THE LAGRANGE INTERPOLATION POLYNOMIALS ARE A BASIS OF $\mathcal{P}_n(\mathbb{F})$ (IF $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$)

PROOF SUPPOSE $\sum_{i=0}^{n} a_i p_i = 0$.

EVALUATE LHS AT $\sum_{i=0}^{n} c_i$ TO GET $\sum_{i=0}^{n} a_i = 0$.

SINCE $\# \{p_0, \dots p_n\} = n+1 = \dim \mathcal{P}_n(F)$, WE'RE DONE.

$\square$

EXERCISE FOR A FINITE FIELD F, $\# F = q$; SHOW THAT $\mathcal{F}(\mathbb{F}, \mathbb{F}) = \mathcal{P}_{q-1}(F)$

HINT USE LAGRANGE POLYNOMIALS

MORE ON DIMENSIONS!

THEOREM: IF V IS A VECTOR SPACE, W A SUBSPACE, THEN $\dim W \leq \dim V$. IF $\dim W = \dim V$, THEN $W = V$.

PROOF CHOOSE A BASIS β OF W. SINCE β IS LINEARLY INDEPENDENT, IT CAN BE EXTENDED TO THE BASIS γ OF V.

SO $\dim W = \# \beta \leq \# \gamma = \dim V$.
EQUALITY IF $\beta = \gamma$, THUS $W = V$

THE [illegible] DIMENSIONAL [illegible] ARE
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