# 1 Review

Suppose $T \in \mathrm{End}(V)$.

The minimal polynomial of $T$ is a unique monic polynomial $p(t)$ of the least degree such that

- $g(T) = 0$, which is equivalent to saying that $p(t)|g(t)$ and thus $p(t)$ divides the characteristic polynomial of $T$
- $p(t)$ has the same roots as the characteristic polynomial.

# 2 Rational Canonical Forms

Let $T \in \mathrm{End}(V)$ be a transformation with an arbitrary characteristic polynomial. Recall that, if $x \in V \setminus \{0\}$, then let $W$ be a $T$-cyclic subspace generated by $x$.

Then $W$ has a basis $\{x, Tx, \dots, T^{k-1}x\}$ for some $k \geq 1$.

A companion matrix $C_{g(t)}$ of $g(t) = t^k + a_{k-1}t^{k-1} + \cdots + a_0$ can be defined in the following form:

$$[T]_\beta = \begin{pmatrix} 0 & & \dots & & -a_0 \\ 1 & 0 & \dots & & -a_1 \\ 0 & \ddots & & & \\ \vdots & & & & \vdots \\ 0 & \dots & & 1 & -a_{k-1} \end{pmatrix},$$

where $a_i \in \mathbb{F}$ are such that $a_0 + \sum_{i=1}^{k} a_i T^i(x) = 0$.

We want to show that there exists a basis such that the matrix is representable as a multiblock matrix, with companion matrices $C_{g_i}$as blocks such that $g_i = \phi_i^{n_i}$, where $\phi$ is an irreducible monic polynomial.

**Example 2.1**

Suppose $\mathbb{F} = \mathbb{R}$. Then

$$\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 2 \end{pmatrix}$$

is in Rational Canonical Form, since blocks are $C_{t^2+1}$ and $C_{(t-1)^2}$ are such that $t^2+1$ and $t-1$ are irreducible.

**Remark 2.2.** $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ is in RCF with the characteristic polynomial $(t-1)^2$, while JCF is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

**Example 2.3**

We can find all possible RCFs for $\mathbb{F} = \mathbb{Z}_2$ and $M_{2\times 2}(\mathbb{F})$.

First, we need to find all irreducible polynomials of degree 1 and 2. It is easy to see that $t$ and $t-1$ are the only irreducible polynomials of degree 1, while $t^2+t+1$ is the only irreducible polynomial of degree 2.

The only two possibilities for $2\times 2$ matrices are $\begin{pmatrix} C_{g_1} & \\ & C_{g_2} \end{pmatrix}$, where $g_1$ and $g_2$ have the degree of 1, and $\begin{pmatrix} C_g \end{pmatrix}$, where $g$ has the degree of 2.

Therefore, there are three matrices of the first kind:

$$\begin{pmatrix} C_t & \\ & C_t \end{pmatrix} = \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} C_t & \\ & C_{t-1} \end{pmatrix} = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}, \begin{pmatrix} C_{t-1} & \\ & C_{t-1} \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}. \tag{1}$$

Similarly, there are three matrices of the second kind:

$$\begin{pmatrix} C_{t^2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} C_{(t-1)^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} C_{t^2+t+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{2}$$

**Definition 2.4.**

Suppose $T \in \operatorname{End}(V)$.

If $\phi(t)$ is a monic irreducible polynomial, then $K_\phi = \{x \in V \mid \phi(T)^n(x) = 0 \text{ for some } n \geq 1\}$.

**e.g.** If $\phi(t) = t - \lambda$, then $K_\phi = K_\lambda$.

It can be shown that $V = \bigoplus_{i=1}^{s} K_{\phi_i}$, where the characteristic polynomial $f$ of $T$ is such that $f(t) = \pm\prod_{i=1}^{s} \phi_i^{m_i}$ for some $m_i \in \mathbb{Z}^+$ and distinct monic irreducible $\phi_i$.

**Definition 2.5.**

Let $x \in V$ be arbitrary.

$T$**-annihilator of** $x$ is a monic polynomial $p(t)$ of the least degree such that $p(T)(x) = 0$.

**Theorem 2.6**

a) $T$-annihilator of $x$ is unique

b) The $T$-annihilator of $x$ divides any $g(t)$ such that $g(T)(x) = 0$, and thus a $T$-annihilator divides the minimal polynomial.

c) Let $W_x$ be a $T$-cyclic subspace generated by $x$. Then $T$-annihilator of $x$ is the minimal polynomial of $T|_{W_x}$.

**Ex. 1** If $x \in E_\lambda$ and $x \neq 0$, then $T$-annihilator is $t - \lambda$.

**Ex. 2** If $x \neq 0$ and $x \in E_\phi = \ker(\phi(T))$, where $\phi$ is monic irreducible, then $T$-annihilator is $\phi(t)$.

**Theorem 2.7**

Let $\phi(t)$ be monic irreducible.

a) $K_\phi$ is a $T$-invariant subspace.

b) $K_\phi \neq 0$ if and only if $\phi(t)|p(t)$, where $p(t)$ is a minimal polynomial

c) $K_\phi = \ker(\phi(T)^d)$, where $\phi(t)^d$ is the largest power of $\phi$ dividing $p(t)$.

*Proof.*

a) Exercise.

b) Pick $x \in K_\phi \setminus \{0\}$.

Then $\phi(T)^n(x) = 0$ for some $n \in \mathbb{Z}^+$ by definition of $K_\phi$.

Thus, $T$-annihilator divides $\phi(T)^n$ and thus $T$-annihilator is equal to $\phi(T)^k$ for $1 \leq k \leq n$, since $\phi$ is irreducible.

Note that $T$-annihilator divides the minimal polynomial $p(t)$, and hence $\phi(T)^k$ divides the minimal polynomial. In particular, we see that $\phi(T)$ divides the minimal polynomial.

If $\phi(t)|p(t)$, then $p(t) = \phi(t)q(t)$ for some $q(t)$ such that $q(T) \neq 0$ (since $\deg q < \deg p$).

Suppose that $q(T)(y) \neq 0$ for some $y \in V$.

Then $\phi(T)(q(T)(y)) = p(T)(y) = 0$. Therefore, $q(T)y \in K_\phi$ and $q(T)y \neq 0$.

c) From the discussion above, if $x \in K_\phi \setminus \{0\}$, then $\phi(T)^k(x) = 0$ for some $k \in \mathbb{Z}^+$ such that $\phi(T)^k|p(t)$, which means that $k \leq d$.

Therefore, $\phi(T)^d(x) = 0$, and thus $x \in \ker(\phi(T)^d)$, which show that $K_\phi \subseteq \ker \phi(T)^d$.

The inclusion in the other direction follows from the definition of $K_\phi$.

□

The following facts can be shown:

a) If $f(t) = \pm \prod_{i=1}^{s} \phi_1(t)^{m_i}$, where $\phi_i$ are distinct monic irreducble polynomials and $n_1 \geq 1$.

b) $\dim K_{\phi_i} = m_i$ has a basis $\beta_i$ that is a disjoint union of $T$-cyclic bases.

Note that if $x \in K_{\phi_i} \setminus \{0\}$, we have a $T$-cyclic basis $\beta_x = \{x, Tx, \ldots, T^{k-1}x\}$ if the $T$-cyclic basis is generated by $x$.

Then $[T_{W_x}]_{\beta_x} = C_{g(t)}$, where $g(t)$ is the minimal polynomial of $T_{W_x}$. We have also seen that $T$-annihilator of $x$ is equal to $\phi_i(t)^k$ for some $k \in \mathbb{Z}^+$.

Thus, we can conclude that

$$[T_{K_{\phi_i}}]_{\beta_i} = \begin{pmatrix} C_{\phi_1^{k_1}} & & \\ & \ddots & \\ & & C_{\phi_s}^{k_s} \end{pmatrix}$$

c) The union $\beta = \bigcup_{i=1}^{s} \beta_i$ is a rational canonical basis of $V$ such that $[T]_\beta$ is in RCF.

d) The minimal polynomial is $\prod_{i=1}^{s} \phi_i(t)^{d_i}$, where $1 \leq d_i \leq n_i$ for all $i$, and thus the characteristic polynomial has the same irreducible factors.