1 Sandpile Model and Divisors in Graphs

1.1 Revision

Solution. [1]

Let $N = \sum_{v\in\Gamma} \phi(v)$. Note that at the point closest to the boundare less than or equal to $N+1$ topplings. For the point second closest to the boundary the number of topplings is bounded from above by $4(N+1)$. As we move closer and closer to the centre, we can define a bound, which means that the total number of topplings is finite.

□

Exercise 1.1. Let ϕ be a revertible state. Then there exists ψ such that $\phi = (\langle 100\rangle + \psi)^0$.

Exercise 1.2. If for ϕ there exists ψ such that $(3+\psi)^0 = \phi$, then for all β there exists ψ such that $\phi = (\beta + \psi)^0$.

Solution. [4]

We are proving that a revertible state does not have forbidden configurations.

Let v be a vertex in D of final toppling in the relaxation leading to ϕ, where ϕ is revertible.

Then $D \setminus v$ from the previous step is also in a forbidden state, which is a contradiction.

□

1.2 Decomposition of Relaxations into Waves

Think about the following statement, and try to make it work:

Theorem 1.3 (ill-defined)

Let ϕ be stable. Then $(\phi + \delta_v)^0 = (W_v)^\infty \phi + \delta_v$, where $\delta_v = \begin{cases} 1 \text{ in } v \\ 0 \text{ otherwise} \end{cases}$ w_v is a *wave operator* at v, defined by the equation $w_v\phi = \begin{cases} \phi \to \phi + \delta \\ \text{toppling in } v \text{ if possible} \to \phi \\ \phi' - \delta_v \to \text{ relaxation} \end{cases}$, and v is a vertex in $\Gamma \setminus \delta\Gamma$.

Note. If v has a neighbouring vertex with $c \geq 3$ grains, then the definition of w_v is as noted above. Otherwise, $w_v\phi = \phi$.

Exercise 1.4. When ϕ is mapped to $w_v\phi$, there is either 1 or no topplings at each vertex.

1.3 Discrete Harmonic Functions

Recall that $\Delta F(v) = \sum_{w\sim v} F(w) - \deg(v)F(v)$.

Exercise 1.5. If $F : \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$ is harmonic (i.e. $\Delta F = 0$),then F is a constant.

Exercise 1.6. If $F : \mathbb{Z}^2 \to \mathbb{R}$ is harmonic, then F is a constant.

Exercise 1.7. Compute $\Delta F(i,j)$, where F is a linear function such that $F(i,j) = A_i + B_j + C$, $F(i,j) = ij$, $F(i,j) = \frac{1}{2}(i(i+1) + j(j+1))$, $[\frac{1}{3}i^2]$, and $[\frac{1}{3}(i^2 + j^2 + 7ij + i)]$.

Exercise 1.8 (Mega)**.** Find all $F(i,j) = [\alpha i^2 + \beta j^2 + \gamma ij + \cdots + \delta i + \lambda j + \mu]$ such that $0 \leq \Delta F \leq 3$.

1.4 Riemann-Roch's Formula

Definition 1.9. Let D and D' be divisors.

We say that $D \sim D'$ if there exists a function F from the set of nodes to $\mathbb{Z}$ such that $D - D' = \Delta F$.

Definition 1.10. A divisor $D = \sum a_i v_i$ over the edges $\{v_i\}$ is called *effective*, if $a_i \geq 0$.

Definition 1.11. Let D be a divisor.

A rank $r(D)$ is defined as $\max_{\text{effective divisors } D' \text{ of degree } s} s$, if $D - D' \sim$ effective divisor, and -1 otherwise.

Theorem 1.12

Suppose that a graph G is given. Let g be the genus of G, which is the number of edges minus the number of nodes plus 1.

Then

$$r(D) - r(k - D) = d - g + 1,$$

where D is a divisor, which is a formal sum in the form $D = \sum a_i v_i$, with $a_i \in \mathbb{Z}$ and $\{v_i\}$ are the nodes of G, and $d = \sum a_i$ is the degree of a divisor, and K is also a divisor in the form $k = \sum_{r \in G} (\deg v - i)v$.

1.5 Key Words

discrete Laplacian, discrete harmonic functions