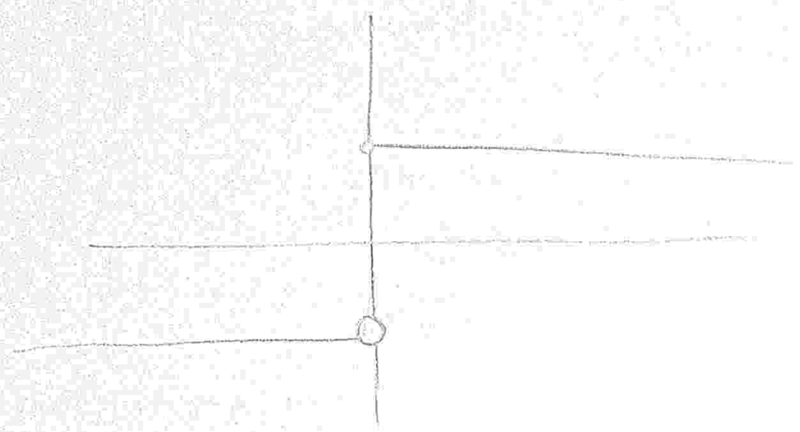
MAT157:

ONE-SIDED LIMITS

2016 10 12



$$f(x) = \frac{|x|}{x}$$

DEFINITION

ONE-SIDED LIMIT

THEOREM

$\lim_{x \to a} f(x)$ EXISTS

IFF $\lim_{x \to a^+} f(x)$ AND $\lim_{x \to a^-} f(x)$

EXIST AND ARE EQUAL.

THUS IN THIS CASE, ALL THREE ARE EQUAL.

EXAMPLE

$$f(x) = \frac{2x^3 - 5}{3x^3 + 2x + 7}$$

$$\lim_{x \to \infty} f(x) = 2/3$$

GIVEN $\varepsilon > 0$, $\exists N$ : $x > N \Rightarrow \left| f(x) - \frac{2}{3} \right| < \varepsilon$

EXERCISE

WORK OUT WHAT N HAS TO BE FOR A GIVEN $\varepsilon$

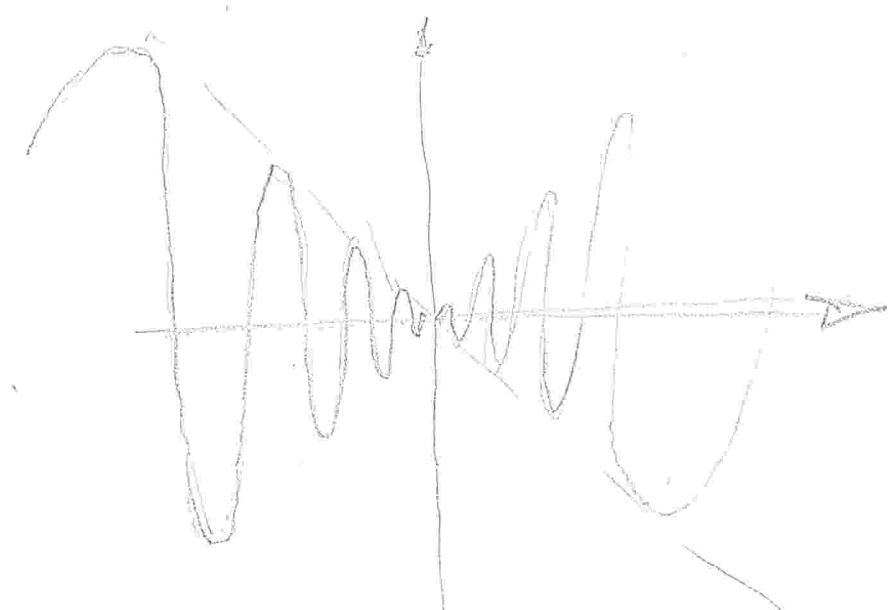
$$\left| \frac{2 - \frac{5}{x^3}}{3 + \frac{2}{x^2} + \frac{7}{x^3}} - 2/3 \right| < \varepsilon$$

=

6

EXAMPLE

$h(x) = x \cdot \sin\left(\frac{1}{x}\right)$



DEFINITION

$f(x)$ IS CONTINUOUS AT $x = a$

IF $\lim_{x \to a} f(x) = f(a)$

$\Rightarrow$ 1. LIMIT EXISTS
2. FUNCTION IS DEFINED

PROVE THE FOLLOWING:

- A POLYNOMIAL IS CONTINUOUS AT EVERY POINT
- A RATIONAL FUNCTION IS CONTINUOUS AT EVERY POINT AT WHICH IT IS DEFINED.

DEFINITION: $f(x)$ IS CONTINUOUS ON AN INTERVAL $(a,b)$ IF IT IS CONTINUOUS $\forall c \in (a,b)$

EXAMPLE

$k(x) = \frac{x^2-9}{x-3}$ HAS A LIMIT, BUT IT IS NOT DEFINED AT $x=3$.

IT COULD BE MADE CONTINUOUS AT $x=3$ BY DEFINING $k(3)=6$, I.E BY CHANGING IT AT ONE POINT $\Rightarrow$ IT HAS A REMOVABLE DISCONTINUITY.

EXAMPLE

$h(x) = x \sin\left(\frac{1}{x}\right)$ ALSO HAS A REMOVABLE DISCONTINUITY AT $x=0$, since $f(x)$ is NOT DEFINED AT $x=0$ BUT IT HAS A LIMIT $\Rightarrow$ THE DISCONTINUITY IS REMOVABLE.

$f(x) = \frac{|x|}{x}$ DOES NOT HAVE A LIMIT AT $x=0$,

SO THE DISCONTINUITY IS NOT REMOVABLE. IT HAS A JUMP DISCONTINUITY, i.e., IT HAS ONE-SIDED LIMITS THAT ARE NOT EQUAL.

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RECALL

$$f(x) = \begin{cases} 0, & \text{IF } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{n}, & \text{IF } x = \frac{m}{n} \;\middle|\; \gcd(m,n)=1,\ n>0 \end{cases}$$

$\lim_{x \to a} f(x) = 0, \ \forall a$

$\Rightarrow f(x)$ IS CONTINUOUS AT IRRATIONAL NUMBERS.

IT HAS A REMOVABLE DISCONTINUITY AT $x \in \mathbb{Q}$.

PROPOSITION

SUPPOSE $f(x)$ IS CONTINUOUS ON $(b,c)$, AND $a \in (b,c)$, AND $f(a) > 0$

$\Rightarrow \exists \ b < r < a < s < c$.

THUS $f(x) > 0 \ \forall x \in (r,s)$

PROOF

LET $\varepsilon = f(a) > 0$.

$\Rightarrow \exists \delta > 0 : |f(a) - f(x)| < \varepsilon \ \forall x$

WITH $0 < |x - a| < \delta$.

$|f(x) - f(a)| < |f(a)|$

$f(x)$ IS CLOSER TO $f(a)$

THAN $0$ IS, SO $f(x) > 0$

WHENEVER $0 < |x - a| < \delta$

LET $r = a - \delta$

$s = a + \delta$.

SO $x \in (r, s) \Leftrightarrow |x - a| < \delta$

EXAMPLE

$|x|$ :: CONTINUOUS EVERYWHERE