Administrativia: no discussions, no extra material consulted

1 Problem I

Let $\mathcal{N}$ denote the nonempty finite subsets of $\mathbb{Z}^+$ that do not contain any consecutive numbers.

Let $F : \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined recursively as follows:

- $F(1) = 1$
- $F(2) = 2$
- $\forall n \geq 3.F(n) = F(n-1) + F(n-2)$

Problem. Give a recursive definition of $\mathcal{N}$.

Solution.

Let $\mathcal{N}$ be the set of nonempty finite subsets of $\mathbb{Z}^+$ that do not contain any consecutive numbers.

Base Case

Singlets, sets containing only one positive integer in $\mathbb{Z}^+$, are in $\mathcal{N}$:

$$\forall n \in \mathbb{Z}^+.\{n\} \in \mathcal{N}$$

Constructor Case

$$\forall M \in \mathcal{N}.\forall i \in M.(i+1 \notin M)$$

AND

$$\left[\forall P \in \mathcal{N}.\forall Q \in \mathcal{N}.\left(\forall k_P \in P.\forall k_Q \in Q.\left(\left|k_P - k_Q\right| \neq 1\right)\right) \text{ IMPLIES } P \cup Q \in \mathcal{N}\right]$$

□

2 Problem II

Lemma 2.1

$\forall n \in \mathbb{Z}^+.F(n) > 0.$

Proof.

Let $P(n)$="$F(n) > 0$" for any $n \in \mathbb{Z}^+$.

Base Case

Note that $F(1) = 1 > 0$ and $F(2) = 2 > 0$. Therefore, $P(1)$ and $P(2)$ hold.

Inductive Step

Suppose, for some $k \in \mathbb{Z}^+$ such that $k \geq 3$, $\forall i \in [1, k] \cap \mathbb{Z}^+.P(i)$.

In particular, $P(k)$ and $P(k-1$ by specialisation, and thus

$F(k) > 0$ and $F(k-1) > 0$.

Note that, by definition of F, since $k \geq 3$, $F(k+1) = F(k) + F(k-1) > 0$. Therefore, $P(k+1)$.

Conclusion

Therefore, $\forall n \in \mathbb{Z}^+.F(n) > 0$ by strong induction.

□

> **Corollary 2.2**
>
> $\forall n \in \mathbb{Z}^+.F(n+1) > F(n) > 0$.

Proof.

Let $P(n)$="$F(n+1) > F(n) > 0$" for any $n \in \mathbb{Z}^+$.

Base Case

Note that $F(1) = 1$ and $F(2) = 2 > 1 = F(1)$. Therefore, $P(1)$ and $P(2)$.

Inductive Step

Suppose, for some $k \in \mathbb{Z}^+$ such that $k \geq 3$, $\forall i \in [1, k] \cap \mathbb{Z}^+.P(i)$.

In particular, $P(k+1)$ and $P(k)$ by specialisation, and thus $F(k+1) > F(k)$ and $F(k) > F(k-1)$.

Therefore, $F(k+2) = F(k+1) + F(k) > F(k) + F(k-1) = F(k+1)$. Thus, $P(k+1)$.

Conclusion

Hence, $\forall n \in \mathbb{Z}^+.F(n+1) > F(n) > 0$ by strong induction and Lemma 2.1.

□

> **Corollary 2.3**
>
> Let $n \in \mathbb{Z}^+$, $k \in \mathbb{Z}^+$ be such that $n > k$ and $n \geq 2$. Then $F(n-1) \geq F(k)$.

Proof.

Since $n > k$, then $n - 1 \geq k$.

Suppose $n - 1 = k$. Then $F(n-1) = F(k)$ by substitution.

Suppose now $n - 1 > k$. Then, $F(n-1) > F(n-2) > \cdots > F(k)$ by repeated application of Corollary 2.2.

Therefore, $F(n-1) > F(k)$.

Thus, for any $n \in \mathbb{Z}^+$, $k \in \mathbb{Z}^+$ we obtain that $F(n-1) \geq F(k)$. □

Corollary 2.4

$\forall k \in \mathbb{Z}^+ . F(k) \geq k$

Proof.

Base Case

Note that $F(1) = 1$ and thus $F(1) \geq 1$.

Note also that $F(2) = 2$ and thus $F(2) \geq 2$.

Inductive Step

Suppose for some $k \in \mathbb{Z}^+$ $\forall k \in [1, k] \cap \mathbb{Z}^+ . F(k) \geq k$.

In particular, $F(k) \geq k$.

Since the claim has been shown to hold in case $k = 1$ and $k = 2$, suppose $k \geq 3$. Therefore, $F(k+1) = F(k) + F(k-1)$ by definition of F.

By Corollary 2.2, $F(k-1) \geq F(1)$. Therefore, $F(k-1) \geq 1$.

Therefore, $F(k+1) = F(k) + F(k-1) \geq k+1$ by inductive hypothesis, which is exactly the claim in case $n = k+1$.

Conclusion

Hence, $\forall n \in \mathbb{Z}^+ . F(n) > n$ by induction.

□

Problem. Prove that

$$\forall S \in \mathcal{N} . \left(\sum_{i \in S} F(i) < F(1 + \max(S)) \right). \tag{1}$$

Solution.

Let $P(S) = \sum_{i \in S} F(i) < F(1 + \max(S))$ for any $S \in \mathcal{N}$.

For all $i \in \mathbb{Z}^+$, denote the set of all sets in $\mathcal{N}$ of cardinality i as $T(i)$.

Base Case

Let $n \in \mathbb{Z}^+$ be arbitrary. By definition of $\mathcal{N}$, $\{n\} \in \mathcal{N}$.

If $n = 1$, then $F(1) = 1$ and $F(2) = 2$ by definition of F. Hence

$$\sum_{i \in \{1\}} F(i) = F(1) = 1 < F(2) = 2.$$

Then $P(\{1\})$ holds.

Note that $F(3) = F(1) + F(2) = 3$ by definition of F.

If $n = 2$, then $F(2) = 2$ and $F(3) = 3$. Hence

$$\sum_{i \in \{2\}} F(i) = F(2) = 2 < F(3) = 3.$$

Then $P(\{2\})$ holds.

Suppose now $n \geq 3$ and $P(\{n\})$ holds.

By definition of F, $F(1+n) = F(n) + F(n-1)$. Hence by Lemma 2.1:

$$\sum_{i \in \{n\}} F(i) = F(n) < F(n) + F(n-1) = F(1+n).$$

Thus, $P(\{n+1\})$ holds.

Therefore, $\forall S \in T(1).P(S)$ by induction, and thus

$$\forall n \in \mathbb{Z}^+.\left(\sum_{i \in \{n\}} F(i) < F\Big(1 + \max\{n\}\Big)\right).$$

Let now $m \in \mathbb{Z}^+$, $n \in \mathbb{Z}^+$ be such that $|m-n| \neq 1$ and $m \neq n$.

By definition of $\mathcal{N}$, $\{m, n\} \in \mathcal{N}$. Without loss of generality, assume $n > m$.

Since $|m-n| \neq 1$ and m, n are distinct, then $|n-m| > 1$, and thus $n > m+1$.

Observe that, since $m \geq 1$ by definition and $n > m+1$, then $n \geq 3$.

By Corollary 2.3, we obtain that $F(n-2) \geq F(m)$ (because $n-1 > m$), which means that

$$F(n) + F(n-2) \geq F(n) + F(m) = \sum_{i \in \{m,n\}} F(i)$$

Note that by Corollary 2.2 we have $F(n-1) > F(n-2)$, and thus

$$F(n) + F(n-1) > \sum_{i \in \{m,n\}} F(i).$$

Since $n \geq 3$, which means that $F(1+n) = F(n) + F(n-1)$, we obtain that

$$F(1+n) > \sum_{i \in \{m,n\}} F(i).$$

Because n was assumed to be the greatest of the two, $\max\{m, n\} = n$, and thus

$$F(1 + \max\{m, n\}) > \sum_{i \in \{m,n\}} F(i).$$

Therefore, since m, n were chosen arbitrarily, by generalisation the claim holds in case of any sets in $\mathcal{N}$ consisting of two elements, i.e.

$$\forall S \in T(2).P(S).$$

Constructor Case

Suppose now for some $k \in \mathbb{Z}^+$ such that $k \geq 3$ $\forall i \in [1, k] \cap \mathbb{Z}^+ . \forall S \in T(i).P(i)$.

Let $Q \in T(k)$ be arbitrary.

Let $a_1, a_2, \ldots, a_k \in \mathbb{Z}^+$ be such that $a_i \in Q$ for all $i \in [1, k] \cap \mathbb{Z}^+$ and $a_i \neq a_j$ for all $i \neq j$ such that $i \in [1, k] \cap \mathbb{Z}^+$ and $j \in [1, k] \cap \mathbb{Z}^+$.

Let $A = \mathbb{Z}^+ \setminus \{a_1, \ldots, a_k, a_1 - 1, a_1 + 1, a_2 - 1, a_2 + 1, \ldots, a_k - 1, a_k + 1\}$.

Let $q \in A$ be arbitrary.

By definition of A, since $\forall a \in Q . |a - q| \neq 1$. Thus, using the definition of $\mathcal{N}$ and modus ponens we obtain that $Q \cup \{q\} \in \mathcal{N}$.

Note that $Q \cup \{q\} \in T(k+1)$, since q is distinct from any element in Q by definition of A. Moreover, by definition of $\mathcal{N}$, for any $i \in \mathbb{Z}^+$ each element in $T(i+1)$ can be constructed by adding a suitable element to $T(i)$ from the set $\overline{T(i)}$, where $\overline{T(i)}$ is the subset of $\mathbb{Z}^+$ complementary to $T(i)$, which means that our construction is generalisable for each element in $T(k+1)$.

Let $Q' = Q \cup \{q\}$

Suppose first that $q = \max(Q')$.

Therefore, since $k \geq 3$, then $q \geq 3$ (note that $1, 3, 5$ is the *minimal* set in T_3, i.e the set the sum of elements of which is minimal).

Therefore,

$$\sum_{i \in Q'} F(i) = \sum_{i \in Q} F(i) + F(q) < F\big(1 + \max(Q)\big) + F(q), \tag{2}$$

by specialisation of inductive hypothesis for $Q \in T(k)$.

Observe that $\max(Q) + 1 < q$ by construction of q (q is distinct from all elements in Q and $\big|\max(Q) - q\big| \neq 1$) and assumption ($\max(Q) < q$).

Therefore, $\max(Q) + 1 \leq q - 1$.

Thus, from Corollary 2.3, $F(1 + \max(Q)) \leq F(q - 1)$, and hence from (2) we have

$$\sum_{i \in Q'} F(i) = \sum_{i \in Q} F(i) + F(q) < F(q - 1) + F(q), \tag{3}$$

and therefore from Corollary 2.4 we obtain

$$\sum_{i \in Q'} F(i) < F(q - 1) + F(q), \tag{4}$$

and since $q \geq 3$,

$$\sum_{i \in Q'} F(i) < F(1 + q), \tag{5}$$

which by assumption that $q = \max(Q')$ is equivalent to

$$\sum_{i \in Q'} F(i) < F(1 + \max(Q')). \tag{6}$$

Suppose now $q \neq \max(Q')$.

Thus, one of $\{a_1, a_2, \ldots, a_k\}$ is equal to $\max(Q')$.

Let $m = a_j$ be such that $a_j = \max(Q')$.

Let $U = \{a_1, a_2, \ldots, q, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k\}$

Note that $Q' = U \cup \{m\}$ and $U \in T(k)$.

Therefore,

$$\sum_{i \in Q'} F(i) = \sum_{i \in U} F(i) + F(m) < F\big(1 + \max(U)\big) + F(m), \tag{7}$$

by specialisation of inductive hypothesis for $U \in T(k)$.

Observe that, since $m = \max(Q')$ and $\forall i \in Q' \setminus \{m\}.(|i - m| \neq 1)$, then

$$m - 1 > \max(U),$$

which means that $m > 1 + \max(U)$, and thus by Corollary 2.3,

$$F(m - 1) \geq F(1 + \max(U)).$$

Thus, by (7),

$$\sum_{i \in Q'} F(i) < F(m - 1) + F(m), \tag{8}$$

Again, since $Q' \in T(k+1)$, $m \geq 3$, because $k \geq 3$ by assumption and hence $(1, 3, 5, 7)$ is the minimal set in the sense explained earlier. Therefore, by definition of F,

$$\sum_{i \in Q'} F(i) < F(m - 1) + F(m) = F(1 + m), \tag{9}$$

and since $m = \max Q'$, we obtain that

$$\sum_{i \in Q'} F(i) < F(1 + \max Q'). \tag{10}$$

Thus, the claim holds for $S \in T(k + 1)$.

Conclusion

Since the claim holds for any set in $T(1)$ or $T(2)$, while, if $k \in \mathbb{Z}^+$,

$$\Big(\forall R \in T_k.P(R)\Big) \text{ IMPLIES } \Big(\forall S \in T_{k+1}.P(S)\Big),$$

then $\forall S \in \mathcal{N}.P(S)$ by strong induction. Therefore,

$$\forall S \in \mathcal{N}.\left(\sum_{i \in S} F(i) < F(1 + \max(S))\right). \tag{11}$$

$\square$

3 Problem III

Problem. Prove that every positive integer is equal to $\sum_{i \in S} F(i)$ for some $S \in \mathcal{N}$.

Solution.

Let $P(n) =$ "n is equal to $\sum_{i \in S} F(i)$ for some $S \in \mathcal{N}$" be defined for $n \in \mathbb{Z}^+$.

Base Case

Consider the positive integers $F(k)$ for all $k \in \mathbb{Z}^+$.

Note that, for any $k \in \mathbb{Z}^+$,

$$F(k) = \sum_{i \in \{k\}} F(i),$$

and since $\{k\} \in \mathcal{N}$ by definition of $\mathcal{N}$, then

$$\forall f \in \mathbb{Z}^+ . P(F(f)).$$

In particular, since $F(1) = 1$ and $F(2) = 2$, if $i \in [1, F(2)] \cap \mathbb{Z}^+$, then i can be written as $\sum_{i \in S} F(i)$ for some $S \in \mathcal{N}$.

Inductive Step

Suppose now there exists $k \in \mathbb{Z}^+$ such that for all $i \in [1, F(k)] \cap \mathbb{Z}^+$ the claim holds, i.e. each $i \in [1, F(k)] \cap \mathbb{Z}^+$ can be written as $\sum_{i \in S} F(i)$ for some $S \in \mathcal{N}$.

The claim has been shown to hold in case $k = 1$ or $k = 2$, so assume that $k \geq 3$.

Therefore, $F(k+1) = F(k) + F(k-1)$ by definition of F.

We show now that any number in $[F(k)+1, F(k+1)]$ can be written in the required form.

Note that $F(k+1) - F(k) = F(k-1)$. Since $F(k-1) < F(k)$ by Corollary 2.2, by inductive hypothesis each number j in $[1, F(k-1)] \cap \mathbb{Z}^+$ can be written as $\sum_{i \in S_j} F(i)$ for some $S_j \in \mathcal{N}$. Since by inductive hypothesis $F(k)$ can also be written in such a form, then each number in $[F(k)+1, F(k+1)] = [F(k)+1, F(k)+F(k-1)]$ satisfies the claim

Therefore, $P(F(k+1))$ holds.

Conclusion p

By strong induction, $\forall m \in \mathbb{Z}^+ . \forall r \in [1, F(m)] \cap \mathbb{Z}^+ . P(r)$ holds. By definition of F, F is not bounded, and by Archimedean property of $\mathbb{Z}^+$ for all $q \in \mathbb{Z}^+$ there exists $g \in \mathbb{Z}^+$ such that $q < F(g)$. Therefore, $\forall n \in \mathbb{Z}^+ . P(n)$ must hold.

□

4 Problem IV

Problem.

Prove that every positive integer is equal to $\sum_{i\in S} F(i)$ for at most one set $S \in \mathcal{N}$.

Solution.

From Problem III, every positive integer is equal to $\sum_{i\in S} F(i)$ for at least one set $S \in \mathcal{N}$. We prove now that this set is unique.

Let $U \subseteq \mathbb{Z}^+$ be a set of numbers in $\mathbb{Z}^+$ such that they cannot be written uniquely in the required form. By way of contradiction, suppose U is not empty. Therefore, by well-ordering principle, there exists the smallest integer in U.

Let $k \in \mathbb{Z}^+$ be the smallest integer in U, and let $D \in \mathcal{N}$ be a set such that $\sum_{i\in D} F(i) = k$.

Suppose that another set $D' \in \mathcal{N}$ is such that $k = \sum_{i\in D'} F(i)$ and $D \neq D'$.

Let $C = (D \cup D') \setminus (D \cap D')$. Since D and D' are distinct, then C is not empty (because if C is empty, then $D \cup D' = D \cap D'$ and hence $D = D'$). By definition of C, each element in C belongs either to D or D', but not both.

Let $A = \{a_1, a_2, \ldots, a_p\} \subseteq C$ be such that $\{a_1, a_2, \ldots, a_p\} \subseteq D$,

and let $B = \{b_1, b_2, \ldots, b_q\} \subseteq C$ be such that $\{b_1, b_2, \ldots, b_q\} \subseteq D$, where $p \in \mathbb{Z}^+$, $q \in \mathbb{Z}^+$ and $p + q = |C|$ (so that $A \cup B = C$).

Since $k = \sum_{i\in D'} F(i)$ and $k = \sum_{i\in D} F(i)$, subtracting one from another and rearranging we obtain that

$$\sum_{i\in A} F(i) = \sum_{j\in B} F(j). \tag{12}$$

If A is a proper subset of D, then $\sum_{i\in A} F(i) < k$, and since k is the minimal positive integer which does not have a unique representation, then the number $\sum_{i\in A} F(i)$ has a unique representation. Since all elements in A are distinct from elements in B by construction, from (12) we get a contradiction, because $\sum_{i\in A} F(i)$ must have a unique representation in the required form. Therefore, $A = D$, and thus $B = D'$ by construction of A and B. Since $C = A \cup B$, we obtain that $C = D \cup D'$ and thus by the construction of C we get that $D \cap D' = \emptyset$.

Let $m = \max(C) = \max(D \cup D')$. Therefore, $\forall s \in (D \cup D') \setminus \{m\}.(s < m)$ by construction of m.

Since each element in C is either in D or D' but not both, without loss of generality assume $m \in D$. Note that, since all elements in A and B are also in $\mathcal{N}$ by construction,

$$\forall i \in [1, p] \cap \mathbb{Z}^+ . \forall j \in [1, q] \cap \mathbb{Z}^+ . (\left|a_i - b_j\right| \neq 1).$$

Therefore, by Problem II, $F(m) > F(1 + \max D') > \sum_{i\in D'} F(i)$, which is a contradiction to Equation (12).

Suppose now $m \in D'$. Similarly, by Problem II, $F(m) > F(1 + \max D') > \sum_{i\in D'} F(i)$, which is a contradiction to Equation (12).

Therefore, our assumption that there exist such D, D' must be false, and hence k can be written uniquely in the required form. Thus, every positive integer can be written as $\sum_{i \in S} F(i)$ uniquely by generalisation.

$\square$