Let $A \in M_{n\times n}(\mathbb{F})$. Recall that A and its transpose A^t have the same characteristic polynomial, hence have the same eigenvalues. For any eigenvalue λ, let E_λ denote the λ-eigenspace of A and E'_λ the -eigenspace of A^t.

Note that we can have $E_\lambda \neq E'_\lambda$

Example

Take $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$. Then $f_A(\lambda) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$ and hence $\lambda = -1$ or $\lambda = 3$.

Thus, for $\lambda = 3$,

$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x + 4y \\ x - 2y \end{pmatrix} = 0 \tag{1}$$

Thus, $x = 2y$, and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ spans E_3.

Consider now E'_3. $A^t = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x + y \\ 4x - 2y \end{pmatrix} = 0$, and thus $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ spans E'_λ. This means that $E'_\lambda \neq E_\lambda$, since $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ are linearly independent.

Suppose $T \in \mathrm{Hom}(V, V)$ is a linear transformation corresponding to the matrix A, where V is a vector space over $\mathbb{F}$.

Observe that $\mathrm{rank}(A - \lambda I) = \mathrm{rank}(A^t - \lambda I)$, since $(A - \lambda I)^t = A^t - \lambda I$. Therefore, by Rank-Nullity Theorem, $\dim(E_\lambda) = \mathrm{nullity}(A - \lambda I) = \mathrm{nullity}(A^t - \lambda I) = \dim(E'_\lambda)$.

Lemma

For a finite-dimensional vector space V and $T \in \mathrm{Hom}(V, V)$ with the distinct eigenvalues denoted as $\lambda_1, \lambda_2, \dots, \lambda_k$, T is diagonalisable if and only if $\dim V = \sum_{i=1}^m \dim E_{\lambda_m}$.

Proof. Since T is diagonalisable, it has a basis consisting of eigenvectors of T. Since all λ_i are distinct, $E_{\lambda_i} \cap E_{\lambda_j} = \{0\}$ for any $i, j \in [1, k] \cap \mathbb{N}$. Therefore, all the eigenvectors are in one and only one eigenspace, and thus $\dim V = \sum_{i=1}^m \dim E_{\lambda_m}$.

Suppose now $\dim V = \sum_{i=1}^m \dim E_{\lambda_m}$.

Choose a basis for each E_{λ_i} and take their union, obtaining a set of eigenvectors

$$\beta = \{v_1, v_2, \dots, v_n\}.$$

There are n of them by assumption. It has been proven earlier that it is linearly independent, and thus β is a basis of V. Therefore, T is diagonalisable. $\square$

Since for all λ $\dim(E_\lambda) = \dim(E'_\lambda)$, it follows that $\dim V = \sum_{i=1}^m \dim E'_{\lambda_m}$. Thus, by the Lemma, A^t is also diagonalisable.