Let $f(x_1, x_2) = |x_1 + x_2 - 3|^2 + |2x_1 + x_2 + 1|^2 + |3x_1 + x_2 - 2|^2$.

Problem.

Find $(x_1, x_2) \in \mathbb{C}^2$ that minimize f.

Solution.

Since f is a sum of nonnegative terms, $f \geq 0$.

Therefore, f might be equal to 0 for some $(x_1, x_2) \in \mathbb{C}^2$. In this case, the following system of equations holds:

$$\begin{cases} x_1 + x_2 - 3 &= 0 \\ 2x_1 + x_2 + 1 &= 0 \\ 3x_1 + x_2 - 2 &= 0 \end{cases}, \tag{1}$$

and thus

$$\begin{cases} x_1 + x_2 &= 3 \\ 2x_1 + x_2 &= -1 \\ 3x_1 + x_2 &= 2 \end{cases}, \tag{2}$$

Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$ and let $b = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$. To minimise f, we must minimise $\left\| b - A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|^2$.

This can be achieved by the least squares approximation.

Note that $A^* = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$, and hence

$$A^*A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix} \tag{4}$$

Note that $\operatorname{rank} A = 2$, since

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \underset{R_2 \to R_2 - R_1}{\rightsquigarrow} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 3 & 1 \end{pmatrix} \tag{5}$$

$$\underset{R_3 \to R_3 - 3R_2}{\rightsquigarrow} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

$$\underset{R_1 \to R_1 - R_2 - R_3}{\rightsquigarrow} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

Therefore, $\operatorname{rank} A^*A = \operatorname{rank} A$, and thus

$$x_0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (A^*A)^{-1}A^*b \tag{8}$$

$$= \frac{1}{\det A^*A}\begin{pmatrix} 3 & -6 \\ -6 & 14 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \tag{9}$$

$$= \frac{1}{6}\begin{pmatrix} 3 & -6 \\ -6 & 14 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix} \tag{10}$$

$$= \frac{1}{6}\begin{pmatrix} -3 \\ 14 \end{pmatrix} \tag{11}$$

$$= \begin{pmatrix} -0.5 \\ 7/3 \end{pmatrix} \tag{12}$$

Since $A^*Ax_0 - b = 0$, then $\langle x, A^*Ax_0 - A^*b\rangle = 0$ for all $x \in \mathbb{C}^2$.

Therefore, $\langle Ax, Ax_0 - b\rangle = 0$ for all $x \in \mathbb{C}^2$. Hence, $Ax_0 - b \in \operatorname{Im}(L_A)^\perp$. Since $Ax_0 = b + (Ax_0 - b)$, where $b \in \operatorname{Im}(L_A)$, we find that Ax_0 is the unique vector closest to b. Therefore, x_0 is the unique solution, since A is invertible and hence injective.

□