Suppose first that $1+1=0$.

$$\begin{bmatrix}1&1&1&0\\0&1&1&1\\1&0&1&1\\1&1&0&1\end{bmatrix} = \tag{1}$$

$$L_1 \to L_1 - L_2 \mid\ = \begin{bmatrix}1&0&0&1\\0&1&1&1\\1&0&1&1\\1&1&0&1\end{bmatrix} \tag{2}$$

$$L_3 \to L_1 + L_3 \mid\ = \begin{bmatrix}1&0&0&1\\0&1&1&1\\0&0&1&0\\1&1&0&1\end{bmatrix} \tag{3}$$

$$L_4 \to L_1 + L_4 \mid\ = \begin{bmatrix}1&0&0&1\\0&1&1&1\\0&0&1&0\\0&1&0&0\end{bmatrix} \tag{4}$$

$$L_2 \to L_2 - L_3 - L_4 \mid\ = \begin{bmatrix}1&0&0&1\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix} \tag{5}$$

$$L_1 \to L_1 + L_2 \mid\ = \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix} \tag{6}$$

$$L_2 \leftrightarrow L_4 \mid\ = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \tag{7}$$

$$\tag{8}$$

Thus, if $2=0$, then $\operatorname{rank}(A)=4$. Suppose now that $1+1\neq 0$.

$$\begin{bmatrix}1&1&1&0\\0&1&1&1\\1&0&1&1\\1&1&0&1\end{bmatrix} = \tag{9}$$

$$L_1 \to L_1 - L_2 \mid \; = \begin{bmatrix}1&0&0&-1\\0&1&1&1\\1&0&1&1\\1&1&0&1\end{bmatrix} \tag{10}$$

$$L_2 \to L_2 - L_3 \mid \; = \begin{bmatrix}1&0&0&-1\\-1&1&0&0\\1&0&1&1\\1&1&0&1\end{bmatrix} \tag{11}$$

$$L_4 \to L_4 - L_2 \mid \; = \begin{bmatrix}1&0&0&-1\\-1&1&0&0\\1&0&1&1\\2&0&0&1\end{bmatrix} \tag{12}$$

$$L_2 \to L_2 + L_1 \mid \; = \begin{bmatrix}1&0&0&-1\\0&1&0&-1\\1&0&1&1\\2&0&0&1\end{bmatrix} \tag{13}$$

$$L_3 \to L_3 - L_1 \mid \; = \begin{bmatrix}1&0&0&-1\\0&1&0&-1\\0&0&1&2\\2&0&0&1\end{bmatrix} \tag{14}$$

If $3=0$, then

$$L_4 \to L_4 - 2L_1 \mid \; = \begin{bmatrix}1&0&0&-1\\0&1&0&-1\\0&0&1&2\\0&0&0&0\end{bmatrix} \tag{15}$$

$$= \begin{bmatrix}1&0&0&-1\\0&1&0&-1\\0&0&1&-1\\0&0&0&0\end{bmatrix} \tag{16}$$

1 Thus, if $3=0$, then $\operatorname{rank}(A)=3$.
Suppose now that $3\neq 0$.

$$L_4 \to 3^{-1}(L_4 - 2L_1) \mid \; = \begin{bmatrix}1&0&0&-1\\0&1&0&-1\\0&0&1&2\\0&0&0&1\end{bmatrix} \tag{17}$$

$$L_2 \to L_2 + L_4, L_1 \to L_1 + L_4, L_3 \to L_3 - 2L_4 \mid \; = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \tag{18}$$

Thus, if $3 \neq 0$, then $\operatorname{rank}(A) = 4$.

Therefore, if $3 = 0$, then $\operatorname{rank}(A) = 3$, and if $3 \neq 0$, then $\operatorname{rank}(A) = 4$.

Suppose now $\operatorname{rank}(A) = 4$ and thus $3 \neq 0$.

Consider the following augmented matrix, with the elementary row operations applied as in steps 9 to 15 above (note that they remain valid in case $0 = 2$ as well, if 3^{-1} is taken equal to $3 = 1$, and 2 is taken as 0):

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = \tag{19}$$

$$L_1 \to L_1 - L_2 \mid \; = \left[\begin{array}{ccccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{20}$$

$$L_2 \to L_2 - L_3 \mid \; = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{21}$$

$$L_4 \to L_4 - L_2 \mid \; = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \end{array}\right] \tag{22}$$

$$L_2 \to L_2 + L_1 \mid \; = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \end{array}\right] \tag{23}$$

$$L_3 \to L_3 - L_1 \mid \; = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \end{array}\right] \tag{24}$$

$$L_4 \to 3^{-1}(L_4 - 2L_1) \mid \; = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 \cdot 3^{-1} & 3^{-1} & 3^{-1} & 3^{-1} \end{array}\right] \tag{25}$$

$$\begin{array}{l} L_2 \to L_2 + L_4 \\ L_1 \to L_1 + L_4 \\ L_3 \to L_3 - 2L_4 \end{array} \mid \; = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 - 2 \cdot 3^{-1} & -1 + 3^{-1} & 3^{-1} & 3^{-1} \\ 0 & 1 & 0 & 0 & 1 - 2 \cdot 3^{-1} & 3^{-1} & -1 + 3^{-1} & 3^{-1} \\ 0 & 0 & 1 & 0 & -1 + 4 \cdot 3^{-1} & 1 - 2 \cdot 3^{-1} & 1 - 2 \cdot 3^{-1} & -2 \cdot 3^{-1} \\ 0 & 0 & 0 & 1 & -2 \cdot 3^{-1} & 3^{-1} & 3^{-1} & 3^{-1} \end{array}\right] \tag{26}$$

Since $[3] \cdot [3^{-1}] = 1$, then

$$(A|I) = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 - 2 \cdot 3^{-1} & -1 + 3^{-1} & 3^{-1} & 3^{-1} \\ 0 & 1 & 0 & 0 & 1 - 2 \cdot 3^{-1} & 3^{-1} & -1 + 3^{-1} & 3^{-1} \\ 0 & 0 & 1 & 0 & -1 + 4 \cdot 3^{-1} & 1 - 2 \cdot 3^{-1} & 1 - 2 \cdot 3^{-1} & -2 \cdot 3^{-1} \\ 0 & 0 & 0 & 1 & -2 \cdot 3^{-1} & 3^{-1} & 3^{-1} & 3^{-1} \end{array}\right] \tag{27}$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3^{-1} & -2 \cdot 3^{-1} & 3^{-1} & 3^{-1} \\ 0 & 1 & 0 & 0 & 3^{-1} & 3^{-1} & -2 \cdot 3^{-1} & 3^{-1} \\ 0 & 0 & 1 & 0 & 3^{-1} & 3^{-1} & 3^{-1} & -2 \cdot 3^{-1} \\ 0 & 0 & 0 & 1 & -2 \cdot 3^{-1} & 3^{-1} & 3^{-1} & 3^{-1} \end{array}\right] \tag{28}$$

Since the standard basis is

$$I_{\mathbb{F}^4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then

$$A^{-1} = 3^{-1} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \end{bmatrix}.$$

If $\operatorname{rank}(A) < 4$, then $3 = 0$, and hence

$$A' = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

hence the vectors in the solution space, $x = (x_1, x_2, x_3, \lambda)$, are in the following form:

$$\begin{cases} x_1 - \lambda = 0 \\ x_2 - \lambda = 0 \\ x_3 - \lambda = 0 \end{cases}$$

Therefore, a basis of the solution space is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$