MAT240: LINEAR MAPS II 20161020

$T: V \to W$ IS LINEAR.

$N(T) = \{ v \in V \mid T(v) = 0 \}$ :: NULL SPACE

$R(T) = \{ w \in W \mid \exists v \in V : w = T(v) \}$ :: RANGE

NULLITY $(T) = \dim(N(T))$

RANK $(T) = \dim(R(T))$

EXAMPLE $T: \mathbb{R}^3 \to \mathbb{R}^3$, $(a_1, a_2, a_3) \mapsto (a_2 - a_3, a_1 - 2a_2, -a_1 + a_2 + a_3)$.

$N(T) = \{ (2t, t, t) \mid t \in \mathbb{R} \}$

$R(T) = \{ (t_1, t_2, t_3) \mid t_1 + t_2 + t_3 = 0 \}$

EXAMPLE $T: P_5(\mathbb{R}) \to P_4(\mathbb{R})$

T	NULLITY (T)	RANK (T)
$p \mapsto p'$	1	5
$p \mapsto p''$	2	4
$p \mapsto p'''$	3	3

DIMENSION (RANK-NULLITY) THEOREM:

LET V, W BE VECTOR SPACES, $\dim V < \infty$, AND $T: V \to W$ BE A LINEAR MAP.

$$\text{RANK}(T) + \text{NULLITY}(T) = \dim(V)$$

PROOF: LET $\bigodot_{i=1}^{k} v_i$ BE A BASIS OF $N(T)$.

AND $\bigodot_{i=k+1}^{n} v_i$ AN EXTENSION TO A BASIS OF V.

$$R(T) = \text{SPAN}\left\{ \bigodot_{i=1}^{n} T(v_i) \right\}$$

$$= \text{SPAN}\left\{ \bigodot_{i=k+1}^{n} T(v_i) \right\}$$

SUPPOSE $b_{k+1} T(v_{k+1}) + \dots + b_n T(v_n) = 0$.

THEN, $T\left(\sum_{i=k+1}^{n} b_i v_i\right) = 0$.

$$\Rightarrow \sum_{i=k+1}^{n} b_i v_i \in N(T), \text{ so}$$

$$\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} a_i v_i$$

$$\Rightarrow a_1 = 0, a_2 = 0, \dots, b_{k+1} = 0, \dots, b_n = 0.$$

RANK $(T) = \text{DIM } R(T) = n-k$

EXAMPLE

$$T: M_{n \times n}(F) \to M_{n \times n}(F)$$

$$A \longmapsto A - A^t$$

$$N(T) = \{\text{SYMMETRIC MATRICES}\} \Rightarrow$$

$$\text{NULLITY}(T) = \sum_{i=1}^{n} i = \frac{n}{2}(n+1)$$

IF $1+1 \neq 0$, THEN $R(T) = \{$ SKEW-SYMMETRIC MATRICES $\}$.

IF $1+1=0$, THEN THE FORMULA STILL HOLDS, BUT

$R(T) \neq \{$ SKEW-SYMM. MATRICES $\}$.

TERMINOLOGY

LET A, B BE SETS. A FUNCTION $f: A \to B$.

$(f \in \mathcal{F}(A,B))$ IS CALLED

- ONE-TO-ONE (INJECTIVE) IF $\forall a, a' \in A: f(a) = f(a') \iff a = a'$.
- ONTO (SURJECTIVE) IF $\forall b \in B \ \exists a \in A: b = f(a)$
- INVERTIBLE (BIJECTIVE) IF IT'S ONE-TO-ONE AND ONTO.

NOTE: f IS INVERTIBLE $\iff \forall b \in B \ \exists! : a \in A: f(a) = b$.

$f^{-1}: B \to A \ (f \circ f^{-1}(b) = b)$

THM: IF $T: V \to W$ IS LINEAR, THEN

a) T IS ONTO $\iff R(T) = W$

b) T IS ONE-TO-ONE $\iff N(T) = \{0\}$

PROOF: a) IMMEDIATE FROM DEFINITIONS

b) SUPPOSE T IS ONE-TO-ONE; AND $v \in N(T)$. THEN $T(v) = 0 = T(0)$, HENCE $v = 0$ (ONE-TO-ONE CORR.). THEN $N(T) = \{0\}$.

CONVERSELY, IF $N(T) = \{0\}$, SUPPOSE

$T(v_1) = T(v_2) \Rightarrow T(v_1 - v_2) = 0$

$\Rightarrow v_1 - v_2 \in N(T) \in \{0\}$, SO $v_1 = v_2$.

THM: IF $T: V \to W$ IS AN INERTIBLE LINEAR MAP, THEN $T^{-1}: W \to V$ IS AGAIN LINEAR.

PROOF:

$$
\begin{aligned}
T(T^{-1}(w_1 + w_2)) &= w_1 + w_2 \\
&= T(T^{-1}(w_1)) + T(T^{-1}(w_2)) \\
&= T(T^{-1}(w_1) + T^{-1}(w_2))
\end{aligned}
$$

SINCE T IS ONE-TO-ONE, GET $T^{-1}(w_1+w_2) = T^{-1}(w_1)+T^{-1}(w_2)$