MAT240: $\mathbb{C}$

2016/1920

1. $P \Rightarrow Q$

$\Leftrightarrow$

2. $Q \Leftarrow P$

AXIOMS

- F1. COMMUTAT.
- F2. ASSOC
- F3 NEUTRAL ELEMENT
- F4. INVERSE ELEMENT
- F5. DISTRIB. LAW

EXAMPLE

ROOM IS MADE OF CHEESE

=> ELVIS IS ALIVE

"P IMPLIES Q",

WHERE P AND Q ARE TRUE/FALSE STATEMENTS

"IF P THEN Q"

$$B(P \Rightarrow Q) = T \text{ IF } \big(B(P) = T \wedge B(Q) = T\big) \vee B(P) = F$$

THEOREM:

Let F be a field $\forall a, b \in \mathbb{F}$.

$$a \cdot b = 0 \Rightarrow [a = 0 \text{ or } b = 0]$$

PROOF

IF $a = 0$, WE ARE DONE.

ASSUME $a \neq 0$, BY F4, a^{-1} EXISTS

THEN

$$\begin{aligned} b &= 1 \cdot b && \text{F3} \\ &= (a^{-1} \cdot a) \cdot b && \text{F4} \\ &= a^{-1}(a \cdot b) && \text{F2} \\ &= a^{-1} \cdot 0 && \text{SINCE } a \cdot b = 0 \\ &= 0 \end{aligned}$$

FURTHER PROPERTIES OF FIELDS

1. $-(-a)=a$

$(a^{-1})^{-1}$ IF $a \neq 0$

2. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$;

$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ IF $b, d \neq 0$

3. $(-a) \cdot b = -(a \cdot b) = a \cdot (-b)$

4. $(a+b)(a-b) = a^2 - b^2$

REMARK

FROM NOW ON, WE CAN OMIT $\cdot$:

$a \cdot b = ab$

$(\ )$:

$(a+b)+c = a+b+c$

EXAMPLE

$ab + c = (a \cdot b) + c$

ALREADY CONSIDERED FIELDS

$\{a + \sqrt{2}b \mid a, b \in \mathbb{Q}\}$ EXERCISE

$\mathbb{Q}, \mathbb{R}, \mathbb{Z}_p$

INTERESTING FIELDS : $f(x) = \frac{p(x)}{q(x)}$, $x \in \mathbb{R}$, p, q are polynomials

$\mathbb{C}$

MOTIVATION

CONSIDER

$Ax^2 + Bx + C = 0$;

$A, B, C \in \mathbb{R}$,

THE EQUATION HAS SOLUTIONS

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

IF $A \neq 0$.

IF $B^2 - 4AC < 0$, $x \notin \mathbb{R}$.

DEFINITION

$\mathbb{C} = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ IS A SET WITH ELEMENTS IN THE FORM

$a + bi$.

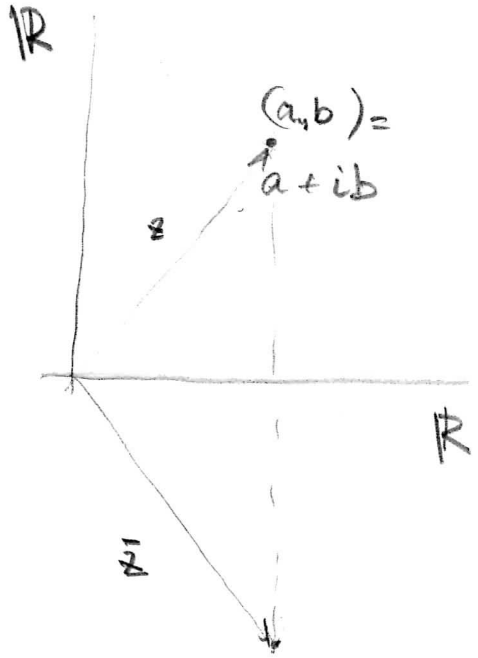
DEFINE $+$ AND $\cdot$:

$(a+ib) + (c+id) = a+c+i(b+d)$

$(a+ib)(c+id) = ac - bd + i(ad+bc)$

NOTATION FOR $\mathbb{C}$

$z = a + ib$



$a = \mathrm{Re}(z)$ "real part"
$b = \mathrm{Im}(z)$ "IMAGINARY PART"

z IS CALLED REAL IF $\mathrm{Im}(z) = 0$
IMAGINARY IF $\mathrm{Re}(z) = 0$

$\bar{z} = a - ib$
IS THE COMPLEX CONJUGATE.

$|z|$ IS THE ABSOLUTE VALUE,
$|z| = \sqrt{a^2 + b^2}$

PROPERTIES

- $\bar{\bar{z}} = z$
- $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z} \cdot \bar{w}$
- $z \in \mathbb{R} \Leftrightarrow \bar{z} = z$
- $z \in \mathbb{C} \setminus \mathbb{R} \Leftrightarrow \bar{z} = -\bar{z}$
- $\mathrm{Re}(z) = \dfrac{z + \bar{z}}{2}$

  $\mathrm{Im}(z) = \dfrac{z - \bar{z}}{2}$
- i HAS THE MULTIPLICATIVE INVERSE $\frac{1}{i} = -i$
- $z \cdot \bar{z} = |z|^2$
- $z^{-1} = \dfrac{\bar{z}}{z\bar{z}}$

GEOMETRIC INTERPRETATION OF MULTIPLICATION

$(a+ib)(c+id) =$

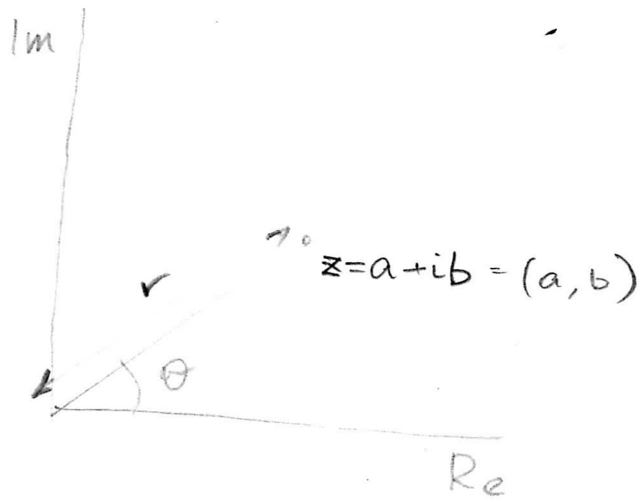
$= ac - bd$

VIE POLAR COORDINATES:

$a = r\cos\theta$

$b = r\sin\theta,$

$r = |z|$



LET $w = s(\cos\varphi + i\sin\varphi),$
$s = |w|$

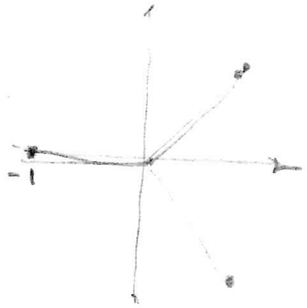
$\Rightarrow$

$zw = r(\cos\theta + i\sin\theta) \times s(\cos\varphi + i\sin\varphi)$

$= rs(\cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\sin\theta\cos\varphi + \cos\theta\sin\varphi))$

$= rs(\cos(\varphi+\theta) + i\sin(\theta+\varphi))$

EXAMPLE



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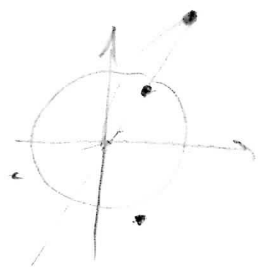

FIND $z \in \mathbb{C}$ WITH $z^3 = -1$

$z_1 = -1,\ z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i;\ z_3 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

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TRIANGLE INEQUALITY

$z, w \in \mathbb{C}$,

$|z+w| \leq |z| + |w|$

PROOF

$$|z+w|^2 = (z+w)(\bar{z}+\bar{w})$$

$$= z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w$$

SINCE $(w+\bar{w}) = 2\,\mathrm{Re}(w)$

$$= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w})$$

$$\leq |z|^2 + |w|^2 + 2|z\bar{w}|$$

$$= |z|^2 + |w|^2 + 2|z||w|$$

$$= (|z| + |w|)^2$$

THEOREM:

$\mathbb{C}$ IS A FIELD. WITH $0 = 0 + i0$.

$1 = 1 + i \cdot 0$

PROOF

EXERCISE.

THM

$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$