1 More on Isometries

Corollary 1.1

T is unitary/orthonormal if and only if T is normal and every eigenvalue λ is such that $|\lambda| = 1$.

Theorem 1.2

Let $T \in \text{Hom}(V, V)$ be an operator on V with $F = \mathbb{R}$. Then T is orthogonal and self-adjoint if and only if V has an orthonormal basis of eigenvectors for T with eigenvalues ± 1.

Proof. From Corollary 1.1, all eigenvalues are ± 1. Then by Theorem 6.17 (Friedberg *et al*), there exists an orthonormal basis of eigenvectors.

Now, pick an orthonormal basis β of eigenvectors with eigenvalues ± 1.

Then $[T]_\beta = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, and $[T^*]_\beta \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, and hence $[TT^*]_\beta = I$. □

Definition 1.3. $A \in M_{n\times n}(\mathbb{F})$ is orthogonal ($\mathbb{F} = \mathbb{R}$)/unitary ($\mathbb{F} = \mathbb{C}$) if $AA^* = I = A^*A$.

Remark 1.4. If β is an orthonormal basis, then T is orthogonal/unitary if and only if $[T]_\beta u$ is orthogonal/unitary.

Remark 1.5. A is orthogonal/unitary if and only if rows or columns form an orthonormal basis of $\mathbb{F}^n$ with a standard inner product.

Proof.

$$\begin{aligned}
(AA^*)_{ij} &= \sum_k A_{ik} A^* && (1)\\
&= \sum_k A_{ik} \overline{A_{jk}} && (2)\\
&= \langle i^{\text{th}} \text{ row}, j^{\text{th}} \text{ row} \rangle && (3)
\end{aligned}$$

$$\begin{aligned}
(A^*A)_{ij} &= \sum_k A_{ik} (A^*) && (4)\\
&= \sum_k A_{ik} \overline{A_{jk}} && (5)\\
&= \langle i^{\text{th}} \text{ row}, j^{\text{th}} \text{ row} \rangle && (6)
\end{aligned}$$

□

Definition 1.6. Two matrices $A, B \in M_{n\times n}(\mathbb{F})$ are unitarily/orthogonally equivalent there exists a unitary/orthogonal Q usch that $Q^{-1}AQ = B$, which is equivalent to $Q^*AQ = B$.

Theorem 1.7

$A \in M_{n\times n}(\mathbb{F})$ is normal if and only if A is unitarily equivalent to a diagonal matrix.

Proof.

If A is normal, then $L_A : \mathbb{C}^n \to \mathbb{C}^n$ is normal, since (since $[L_A]_{\text{std}} = A$.

Thus, by Theorem 6.16 there exists an orthonormal basis β of eigenvectors for L_A. Thus, $Q^{-1}[L_A]_{\text{std}}Q$, where $Q = [I]_\beta^{\text{std}}$.

On the other hand, if $A = Q^{-1}DQ = Q^*DQ$, where Q is unitary and D is diagonal. Then $A^* = Q^*D^*Q^{**} = Q^*DQ$, and hence

$$AA^* = Q^*DQQ^*D^*Q = Q^*DD^*Q.$$

Thus, $A^*A = Q^*D^*DQ$. $\square$

Theorem 1.8

$A \in M_{n\times n}(\mathbb{F})$ is self-adjoint if and only if A is orthogonally equivalent to a diagonal matrix.

Theorem 1.9 (Schur)

If the characteristic polynomial of $T \in \operatorname{Hom}(V, V)$ splits, then there exists β such that $[T]_\beta$ is upper-triangular.

Remark 1.10. See also Exercise 5.4/32 in Friedberg *et al.*

Lemma 1.11

If $T \in \operatorname{Hom}(V, V)$ has an eigenvalue λ, then T^* has an eigenvalue $\overline{\lambda}$.

Proof.

Note that $\operatorname{rank}(T - \lambda I)^* = \operatorname{rank}(T - \lambda I)$. herefore, $\ker(T - \lambda I)^* = \ker(T - \lambda I) > 0$.

Thus T^* has an eigenvalue $\overline{\lambda}$. $\square$