1 Problem

Suppose that $T \in \operatorname{End}(V)$.

Lemma 1.1

The T-annihilator of x is unique.

Proof.

Suppose that $q(t)$ and $r(t)$ are T-annihilators of x.

Since they are both monic polynomials of the least degree, then $\deg q(t) = \deg r(t)$.

By the division algorithm, there exist unique $u(t)$ and $v(t)$ such that $\deg v(t) < \deg r(t)$ and $q(t) = u(t)r(t) + v(t)$.

Therefore, $q(T) = u(T)r(T) + v(T)$, and since $q(T) = r(T) = 0$, then $v(T) = 0$.

Since $v(t)$ is such that $\deg v(t) < \deg r(t)$ and $r(t)$ is a T-annihilator, while $v(T) = 0$, then $v(t)$ is a zero polynomial.

Hence, $q(t) = u(t)r(t)$.

Since $\deg q(t) = \deg u(t) + \deg r(t)$, then $\deg u(t) = 0$. Therefore, $u(t) = c \in \mathbb{F}$. Since $q(t)$ and $r(t)$ are both monic, $u(t) = c = 1$, and thus $q(t) = r(t)$. $\square$

Theorem 1.2

The T-annihilator of x divides any polynomial $g(t)$ such that $g(T)(x) = 0$.

Proof.

Suppose $x \in V$.

Let $p(t)$ be a minimal polynomial of T, and let $q(t)$ be the T-annihilator of x.

By Theorem 7.12, $p(t)$ divides any polynomial $g(t)$ such that $g(T)(x) = 0$. Thus, we only need to show that $q(t)$ divides $p(t)$, and the claim follows.

By the division algorithm, there exist $u(t)$ and $v(t)$ such that $\deg v(t) < \deg q(t)$ and $p(t) = u(t)q(t) + v(t)$.

Since $p(T)x = 0$ by definition and we are given that $q(T)x = 0$, then by additivity and homogeneity of T we know that $p(T)x = 0 = u(T)q(T)x + v(T)x = v(T)x$. Therefore, $v(T)x = 0$, and since $\deg v(t) < \deg q(t)$, where $q(t)$ is the monic polynomial of the *minimal* degree such that $q(T)x = 0$, then $v(t)$ is the zero polynomial, which means that $p(t) = u(t)q(t)$. Therefore, $q(t)$ divides $p(t)$, as required.

$\square$

Theorem 1.3

If W is the T-cyclic subspace generated by x, then the T-annihilator of x equals the minimal polynomial of $T|_W$ and can be represented in the form $(-1)^{\dim}$ times the characteristic polynomial of $T|_W$.

Proof.

Let T-annihilator of x be $q(t)$ and let $p(t)$ be the minimal polynomial of $T|_W$.

Let $n = \dim W$.

By definition of a minimal polynomial, $p(T|_W) = 0$.

By definition of a T-annihilator, $q(T)x = 0$.

Note that x generates a cyclic basis $\beta_x = \{x, Tx, \ldots, T^{n-1}x\}$.

Since the product of $q(T)$ and any power of T is commutative, we know that, for any $j \in [1, n-1] \cap \mathbb{N}$, we have $T^j(q(T)x) = T(0) = 0 = q(T)T^jx$, and hence $q(T)$ restricted to W is a zero transformation, which means that $q(T|_W) = 0$.

From Theorem 1.2, we know that $q(t)|p(t)$ and thus there exists a polynomial $u(t)$ such that $p(t) = u(t)q(t)$.

On the other hand, by Theorem 7.12, since $q(T|_W) = 0$, we obtain that $p(t)|q(t)$. Therefore, there exists a polynomial $u'(t)$ such that $q(t) = u'(t)p(t)$.

Since $p(t) = u(t)u'(t)p(t)$, we see that $\deg u(t) + \deg u'(t) = 0$, and thus $\deg u(t) = 0 = \deg u'(t)$. Since $q(t)$ and $p(t)$ are also monic, then $p(t) = q(t)$.

By Theorem 7.15, since W is an n-dimensional cyclic vector space, then the characteristic polynomial $f(t)$ of $T|_W$ is $(-1)^n p(t)$, which, from the previous discussion, means that $f(t) = (-1)^n q(t)$ and thus $q(t) = (-1)^{-n} f(t) = (-1)^n f(t)$, as required.

$\square$