1 Induction

Example 1.1

Let $p : \mathbb{N} \to \{T, F\}$ be a predicate. $\forall n \in \mathbb{N}.p(n)$.

a) a. $p(0)$ is base case or basis step.

b) Let $n \in \mathbb{N}$ be arbitrary.

c) Assume $p(n)$.

d) p(n+1)

e) $p(n)$ IMPLIES $p(n+1)$ direct proof c-d

f) $\forall n \in \mathbb{N}.(p(n)$ IMPLIES $p(n+1)$ generalization b-e

g) $\forall n \in \mathbb{N}.(p(n)$ induction a-f

Theorem 1.2

Consider any square chessboard which sides have length which is a power of 2. If any one square is removed, then the resulting shape can be tiled using only 3-square L-shaped tiles.

Proof. For all $n \in \mathbb{N}$, let

$$\text{P(n)} = \begin{array}{l}\text{“any } 2^n \times 2^n \text{ with 1 square removed} \\ \text{can be tiled using 3-square L-shaped tiles. ”}\end{array}$$

Let C_n the set of all $2^n \times 2^n$ chessboards with 1 square removed.

Let “L-tile” denote a 3-square L-shaped tile.

$P(n) =$ “$\forall c \in C_n.(c$ can be tiledusing only L-tiles.)”.

$\forall n \in \mathbb{N}.P(n)$.

Basis: $P(0)$ is true.

A $2^0 \times 2^0$ chessboard with 1 square removed has no squares and, therefore, can be tiled with 0 tiles.

Let $n \in \mathbb{N}$ be arbitrary.

Suppose $P(n)$ is true.

Divide c into 4 equal $2^n \times 2^n$ chessboards. One of these has a square removed, so it is in C_n, and hence, by the induction hypothesis, it can be tiled with L-tiles.

Consider the other 3 chessboards.

Each has 1 square that is one of the 4 squares in the middle of c.

With those squares removed, the remaining three squares are also in C_n, is the inductive hypothesis implies that they can be tiled by L-tiles.

The 3 squares in the middle can be tiled with 1 L-tile.

c can be tiled using L-tiles.

$\forall c \in C_{n+1}$. c can be tiled using L-tiles.

$P(n+1)$ — generalisation

$\square$

Another theorem can be proved using the result above:

Theorem 1.3

All square chessboards with sides of length a power of 2 and with 1 square removed from the middle can be tiled using L-tiles.

Theorem 1.4

$\forall n \in (n \geq 3 \text{ IMPLIES } 2n+1 \leq 2^n)$.

Proof. For $n \in \mathbb{N}$, let $q(n) =$ "$2n+1 \leq 2^n$".

Let $n \in \mathbb{N}$ be arbitrary. Assume $q(n)$.

Base case: Let $p(n) = q(n+3)$ for all $n \in \mathbb{N}$.

Basis:

$P(0)$ is true.

Induction step:

Let $n \in \mathbb{N}$ be arbitrary.

Assume $p(n)$.

...

$p(n+1)$

$p(n)$ IMPLIES $p(n+1)$, direct proof

$\forall n \in \mathbb{N}.(p(n)$ IMPLIES $p(n+1)$, generalization

$\forall n \in \mathbb{N}.p(n)$, induction

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