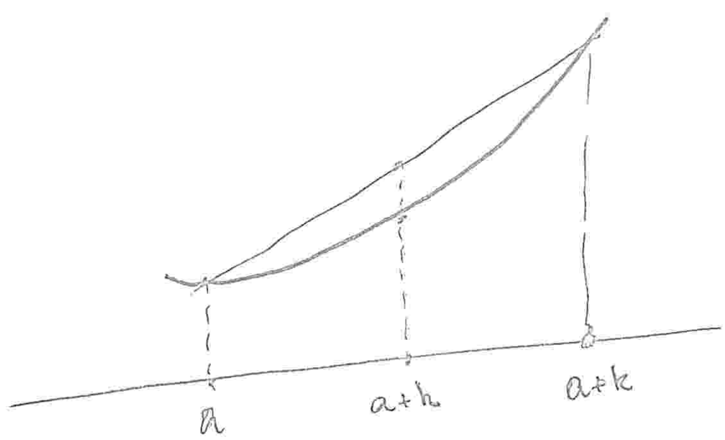
20161130

MAT 157:

CONVEXITY

THM If f is convex and it is differentiable at $x=a$, then the graph of $y=f(x)$ lies above the tangent line at $(a, f(a))$, except at the point of contact.

PROOF



Suppose $0<h<k$

Convexity: $f(a+h) < f(a) + \frac{f(a+k)-f(a)}{k} h$

$$\frac{f(a+h)-f(a)}{h} < \frac{f(a+k)-f(a)}{k}$$

Slope of the secant through $(a, f(a))$ and $(a+h, f(a+h))$ decreases as $h \to 0^+$.

Thus, the limit is

$$f'(a) < \frac{f(a+h)-f(a)}{h}$$

The slope of the tangent is less than the slope of the secant, so tangent is below the secant, and hence below $(a+h, f(a+h))$.

The argument is similar for $h<0$.

IF $f(x)$ IS CONVEX AND DIFFERENTIABLE ON AN INTERVAL, THEN FOR ANY $a<b$ IN THE INTERVAL $f'(a) < f'(b)$.

PROOF.

$f'(a) <$ SLOPE OF THE SECANT FROM $(a, f(a))$ to $(b, f(b))$ $< f'(b)$

LEMMA.

IF f IS DIFFERENTIABLE AND $f(a) = f(b)$, FOR SOME $a<b$, AND $f'(x)$ IS INCREASING, THEN $f(x) < 0$ $\forall x \in (a,b)$.

PROOF

SUPPOSE NOT. SUPPOSE THERE ARE POINTS AT WHICH f IS POSITIVE.

f HAS A MAX ON $[a,b]$, WHICH CANNOT BE AT a OR b, WHERE $f=0$, SINCE OTHERWISE IT IS A CONSTANT function.

So $\exists\, c \in (a,b)$, WHERE f HAS A MAX,

So $f'(c) = 0$.

APPLY MVT TO $[a,c]$ SO THAT THERE EXISTS $d \in (a,c)$ SUCH THAT $f'(d) > 0$.

$$\Rightarrow \begin{cases} f'(d) > f'(c) = 0. \\ d < c. \end{cases}$$

SINCE $f'(x)$ IS INCREASING.

THEOREM. Suppose f' is differentiable, f' increasing on some interval, then f is convex.

PROOF. Define $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$

$g(a) = f(a)$, $g(b) = f(b)$

$g'(x) = f'(x) - \text{const}$

$\Rightarrow$ IF f' IS INCREASING, THEN g' IS INCREASING.

<u>LEMMA</u>. $g(x) < f(a)$ $\forall$ $a < x < b$.

$\Rightarrow$ CAN CHOOSE ANY $a < b$ IN THE DOMAIN OF f

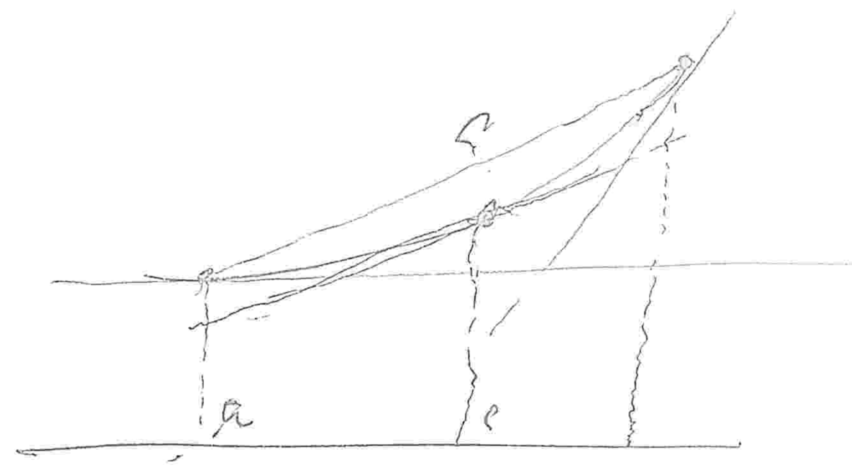
$$f(x) \geq \frac{f(b) - f(a)}{b-a}(x-a) + f(x)$$

$\Rightarrow$ $y = f(x)$ LIES ABOVE THE SECANT

$\Rightarrow$ CONVEX

THEOREM

SUPPOSE $f(x)$ IS DIFFERENTIABLE ON AN INTERVAL AND $y = f(x)$ IS ABOVE EACH OF ITS TANGENT LINES, EXCEPT AT THE POINTS OF CONTACT.



$$f(c) > f(a) + f'(a)(c-a)$$

$$+ \quad f(a) > f(c) + f'(c)(a-c)$$

$$\left(f'(a) - f'(c)\right)(c-a) < 0 \Rightarrow$$

$$f'(a) - f'(c) < 0 \Rightarrow$$

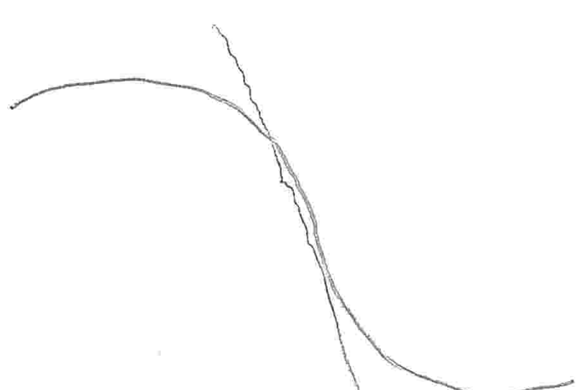
$\Rightarrow f'$ IS INCREASING

PREVIOUS THM $\Rightarrow$ f IS CONVEX

IF f IS TWICE DIFFERENTIABLE ON AN INTERVAL AND $f''(x) > 0$ ON THE INTERVAL THEN $f'(x)$ IS INCREASING, SO f IS CONVEX, IF $f''(x) < 0$ ON THE INTERVAL.

INFLECTION POINTS

IF $f(x)$ IS DIFFERENTIABLE AT $x=a$ AND THE TANGENT LINE AT $x=a$ CROSSES THE GRAPH AT $x=a$, THEN f HAS AN INFLECTION POINT AT $x=a$.



IF f IS TWICE DIFFERENTIABLE, f'' CHANGES SIGN AT a.