Problem 1

(a) Consider the following system of equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

$$= M \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

Let's compute the eigenvalues of the matrix:

$$\det \begin{pmatrix} 1-\lambda & -5 \\ 2 & -5-\lambda \end{pmatrix} = (\lambda+5)(\lambda-1)+10 \tag{3}$$

$$= \lambda^2 + 4\lambda + 5 \tag{4}$$

Therefore, $\lambda = -2 + i$ is an eigenvalue.

Moreover, $w = \begin{pmatrix} 3+i \\ 2 \end{pmatrix}$ is a corresponding eigenvector, since

$$\begin{aligned}(M - \lambda I)w &= \begin{pmatrix} 3-i & -5 \\ 2 & -3-i \end{pmatrix} \begin{pmatrix} 3+i \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 10-10 \\ 2(3+i)-2(3+i) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}\end{aligned}$$

(b) Note that $\overline{\lambda} = -2 - i$ is also a solution of the Equation 4, implying that $\overline{\lambda}$ is also an eigenvalue. Moreover, since $\overline{(M - \lambda I)w} = (\overline{M} - \overline{(\lambda I)})\overline{w} = (M - \overline{\lambda} I)\overline{w}$, then $\overline{w} = \begin{pmatrix} 3-i \\ 2 \end{pmatrix}$ is another eigenvector of M.

(c) Now, consider the change of coordinates matrix $C = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$, so that $\begin{pmatrix} x \\ y \end{pmatrix} = C \begin{pmatrix} u \\ v \end{pmatrix}$.

Note that $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = C \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}$, and thus

$$C \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = MC \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}, \tag{5}$$

which implies that

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = C^{-1}MC \begin{pmatrix} u \\ v \end{pmatrix} \tag{6}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -5 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{7}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & -5 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{8}$$

$$= \frac{1}{2}\begin{pmatrix} -4 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{9}$$

$$= \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{10}$$

(d) Consider the following equation:

$$\dot{z} = \lambda z. \tag{11}$$

Set $z = u+iv$ for real differentiable functions u and v. Similarly, suppose that $\lambda = \alpha+i\beta$ for real scalars α and β.

Note that

$$\dot{u} + i\dot{v} = (\alpha + i\beta)(u + iv) \tag{12}$$

$$= \alpha u - \beta v + i(\beta u + \alpha v), \tag{13}$$

so the equation 11 becomes

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{15}$$

(e) Equation 15 coincides with Equation 10.

(f) Notice that all solutions of Equation 11 are in the form $z = z(0)\exp(\lambda t)$.

(g) Let's find the real and imaginary components of the solution above. Note the following:

$$z = u + iv = (u(0) + iv(0))\exp((\alpha + i\beta)t) \tag{16}$$

$$= \exp(\alpha t)(u(0) + iv(0))(\cos(\beta t) + i\sin(\beta t)) \tag{17}$$

$$= \exp(\alpha t)(u(0)\cos(\beta t) - v(0)\sin(\beta t) + i(u(0)\sin(\beta t) + v(0)\cos(\beta t))) \tag{18}$$

Hence, $u(t) = \exp(\alpha t)(u(0)\cos(\beta t) - v(0)\sin(\beta t)) = \exp(\alpha t)\begin{pmatrix}\cos(\beta t) & -\sin(\beta t)\end{pmatrix}\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}$, and $v(t) = \exp(\alpha t)\begin{pmatrix}\sin(\beta t) & \cos(\beta t)\end{pmatrix}\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}$

(h) Since we have shown already that real solutions of Equation 10 coincide with the real and imaginary components of complex solutions to Equation 15, then, given that $\alpha = -2$ and $\beta = 1$, we obtain

$$\begin{aligned}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} &= \exp(\alpha t) \begin{pmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} &(19)\\ &= \exp(-2t) \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} &(20)\end{aligned}$$

(i) To find the solutions to Equation 2, we return to the original coordinates:

$$\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= C \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} &(21)\\ &= \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} \exp(-2t) \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} &(22)\\ &= \exp(-2t) \begin{pmatrix} 3\cos(t) - \sin(t) & -3\sin(t) - \cos(t) \\ 2\cos(t) & -2\sin(t) \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} &(23)\end{aligned}$$

(j) Set $w = a + ib$, where $a = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Note the following:

$$\begin{aligned} w\exp(\lambda t) &= w\exp((-2+i)t) &(24)\\ &= \exp(-2t)(a+ib)(\cos(t) + i\sin(t)) &(25)\\ &= \exp(-2t)(a\cos(t) - b\sin(t) + i(a\sin(t) + b\cos(t))), &(26)\end{aligned}$$

so

$$\begin{aligned}\Re(w\exp(\lambda t)) &= \exp(-2t)(a\cos(t) - b\sin(t)) &(27)\\ \Im(w\exp(\lambda t)) &= \exp(-2t)(a\sin(t) + b\cos(t)) &(28)\end{aligned}$$

Then Equation 23 can be written as

$$\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \exp(-2t) \begin{pmatrix} 3\cos(t) - \sin(t) & -3\sin(t) - \cos(t) \\ 2\cos(t) & -2\sin(t) \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} &(29)\\ &= u(0)\exp(-2t)\left(\cos(t)\begin{pmatrix} 3 \\ 2 \end{pmatrix} - \sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &(30)\\ &- v(0)\exp(-2t)\left(\sin(t)\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &(31)\\ &= u(0)\Re(w\exp(\lambda t)) - v(0)\Im(w\exp(\lambda t)) &(32)\end{aligned}$$

(k) See the notebook.

Problem 2

See the notebook for phase portraits.

(a) Consider the following system of equations M:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{33}$$

Notice that the matrix is a Jordan block corresponding to the eigenvalue $\lambda = -2$. Then the solution is in the form of a matrix exponent, which we compute as follows:

$$\begin{aligned} z(t) &= \exp(Mt)z_0 && (34)\\ &= \exp\left(\left(-2I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) t\right) z_0 && (35)\\ &= \exp(-2tI)\left(I + \frac{1}{1!}\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}\right) z_0 && (36)\\ &= \exp(-2t)\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} z_0 && (37) \end{aligned}$$

Hence, the solutions are in the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \exp(-2t) \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \tag{38}$$

The stationary point is a stable dicritic node.

(b) Consider the following system of equations M:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 12 & 12 \\ -9 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{39}$$

Note that it is degenerate, so there exist no non-degenerate critical points. Let's compute the characteristic polynomial of M:

$$\begin{aligned} (12 - \lambda)(-9 - \lambda) + 9 \cdot 12 = 0 && (40)\\ \Leftrightarrow \lambda^2 - 3\lambda = 0 && (41) \end{aligned}$$

Therefore, $\lambda = 0$ and $\lambda = 3$ are eigenvalues, with $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ as corresponding eigenvectors.

Changing the coordinates, we obtain the following system of equations:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{42}$$

which has solutions in the form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \exp(3t) \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \tag{43}$$

Changing the coordinates back, we obtain:

$$\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \exp(3t) \end{pmatrix} \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} && (44) \\
&= \begin{pmatrix} 0 & 4\exp(3t) \\ 0 & -3\exp(3t) \end{pmatrix} \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} && (45) \\
&= \begin{pmatrix} 4\exp(3t) & 4\exp(3t) \\ -3\exp(3t) & -3\exp(3t) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} && (46) \\
&= \exp(3t) \begin{pmatrix} 4 & 4 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} && (47)
\end{aligned}$$

(c) Consider the following system of equations M:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2 & -13 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{48}$$

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Let's compute the characteristic polynomial:

$$\begin{aligned}
(-2-\lambda)(8-\lambda) + 26 &= 0 && (49) \\
(\lambda+2)(\lambda-8) + 26 &= 0 && (50) \\
\lambda^2 - 6\lambda + 10 &= 0 && (51)
\end{aligned}$$

Then the eigenvalues are $3+i$ and $3-i$.

Set $\lambda = 3+i$. Now we find the eigenvector $v = \begin{pmatrix} a \\ b \end{pmatrix}$:

$$\begin{aligned}
\begin{pmatrix} -5-i & -13 \\ 2 & 5-i \end{pmatrix} v = 0 && (52) \\
\Leftrightarrow \begin{pmatrix} -5a - 13b - ai \\ 2a + 5b - bi \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (53) \\
\Leftrightarrow \begin{pmatrix} -2(5+i)a - 26b \\ 2(5+i)a + 26b \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} && (54)
\end{aligned}$$

Therefore, $v = \begin{pmatrix} 13 \\ -5-i \end{pmatrix}$ is an eigenvector. By the derivation in Problem 1, the real solutions are in the following form (for $\alpha = 3$ and $\beta = 1$):

$$\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 13 & 0 \\ -5 & 1 \end{pmatrix} \exp(3t) \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \frac{1}{13} \begin{pmatrix} 1 & 0 \\ 5 & 13 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} && (55) \\
&= \frac{\exp(3t)}{13} \begin{pmatrix} 13\cos(t) & 13\sin(t) \\ -5\cos(t) + \sin(t) & 5\sin(t) + \cos(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 5 & 13 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} && (56) \\
&= \frac{\exp(3t)}{13} \begin{pmatrix} 13\cos(t) + 65\sin(t) & 169\sin(t) \\ 26\sin(t) & 65\sin(t) + 13\cos(t) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} && (57) \\
&= \exp(3t) \begin{pmatrix} \cos(t) + 5\sin(t) & 13\sin(t) \\ 2\sin(t) & 5\sin(t) + \cos(t) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} && (58)
\end{aligned}$$

The critical point is an unstable focus.

(d) Consider the following system of equations M:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 10 & 7 \\ -14 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{59}$$

Let's compute the characteristic polynomial:

$$\lambda^2 + \lambda + (-110 + 98) = 0 \tag{60}$$
$$\lambda^2 + \lambda + -12 = 0 \tag{61}$$

Then -4 and 3 are eigenvalues. For $\lambda = -4$, the eigenvector u is such that

$$\begin{pmatrix} 14 & 7 \\ -14 & -7 \end{pmatrix} u = 0. \tag{62}$$

Set $u = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

For $\lambda = 3$, the eigenvector v is such that

$$\begin{pmatrix} 7 & 7 \\ -14 & -14 \end{pmatrix} u = 0. \tag{63}$$

Set $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Changing the basis to that of eigenvectors, we obtain the following system of equations:

$$\begin{pmatrix} \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix}, \tag{64}$$

which has solutions in the form:

$$\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \exp(-4t) & 0 \\ 0 & \exp(3t) \end{pmatrix} \begin{pmatrix} q_0 \\ r_0 \end{pmatrix}. \tag{65}$$

Changing the coordinates back, we obtain:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} \exp(-4t) & 0 \\ 0 & \exp(3t) \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{66}$$

$$= \begin{pmatrix} \exp(-4t) & \exp(3t) \\ -2\exp(-4t) & -\exp(3t) \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{67}$$

$$= \begin{pmatrix} -\exp(-4t) + 2\exp(3t) & -\exp(-4t) + \exp(3t) \\ 2\exp(-4t) - 2\exp(3t) & 2\exp(-4t) - \exp(3t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{68}$$

The critical point is a saddle.

(e) Consider the following system of equations M:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{69}$$

Consider the characteristic polynomial of M:

$$\lambda^2 - 3\lambda + 2 = 0 \tag{70}$$

Then 1 and 2 are eigenvalues.

For $\lambda = 1$, the corresponding eigenvector u satisfies the following:

$$\begin{pmatrix} 4 & -4 \\ 3 & -3 \end{pmatrix} u = 0. \tag{71}$$

Set $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = 2$, the corresponding eigenvector v satisfies the following:

$$\begin{pmatrix} 3 & -4 \\ 3 & -4 \end{pmatrix} v = 0. \tag{72}$$

Set $v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

Changing the basis to that of eigenvectors, we obtain the following system of equations:

$$\begin{pmatrix} \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix}, \tag{73}$$

which has solutions in the form:

$$\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \exp(t) & 0 \\ 0 & \exp(2t) \end{pmatrix} \begin{pmatrix} q_0 \\ r_0 \end{pmatrix}. \tag{74}$$

Changing the coordinates back, we obtain:

$$\begin{align} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \exp(t) & 0 \\ 0 & \exp(2t) \end{pmatrix} (-1) \begin{pmatrix} 3 & -4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{75} \\ &= - \begin{pmatrix} \exp(t) & 4\exp(2t) \\ \exp(t) & 3\exp(2t) \end{pmatrix} \begin{pmatrix} 3 & -4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{76} \\ &= - \begin{pmatrix} 3\exp(t) - 4\exp(2t) & -4\exp(t) + 4\exp(2t) \\ 3\exp(t) - 3\exp(2t) & -4\exp(t) + 3\exp(2t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{77} \end{align}$$

The critical point is an unstable node.

Problem 3

(a) Consider the following system of equations parametrised by $s \in \mathbb{R}$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -6 & -7 \\ s & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{78}$$

Notice that the corresponding characteristic polynomial is

$$\lambda^2 + 2\lambda + (-24 + 7s) = 0. \tag{79}$$

Now, the discriminant of the quadratic is $\Delta = 4 - 4(7s - 24) = 100 - 28s$.

If $s = \frac{25}{7}$, then there is only one eigenvalue ($\lambda = -1$), and since the corresponding matrix is not diagonal, then the critical point is a stable degenerate node.

If $s > \frac{25}{7}$, then $\Delta < 0$, and thus the critical value is a stable focus (since $\operatorname{tr} M < 0, \det M > \frac{(\operatorname{tr} M)^2}{2}$), and the matrix has two complex eigenvalues having a negative real component.

If $s < \frac{25}{7}$, but $\sqrt{\Delta} < 2$, which is equivalent to $s > \frac{24}{7}$, then $\det M > 0$, and the critical point is a stable node, with the matrix having two real negative eigenvalues.

If, on the other hand, $\sqrt{\Delta} > 2$, so that $s < \frac{24}{7}$, then the critical point is a saddle, with the matrix having two real eigenvalues of opposite sign.

Otherwise, if $s = \frac{24}{7}$, then the matrix is degenerate. There are no non-degenerate critical points, and the matrix has two real eigenvalues, one zero and one negative.

(b) If $s < \frac{24}{7}$, then the critical point $(0, 0)$ is neither asymptotically nor Lyapunov stable.

If $s = \frac{24}{7} \approx 3.4286$, then the critical point is unstable as well, since in this case the phase space consists of straight lines, which do not 'concentrate' around $(0, 0)$.

If $s > \frac{24}{7}$, then the critical point is both Lyapunov and asymptotically stable, since the real components of eigenvalues are negative.

(c) See the notebook for phase portraits.

Problem 4

Consider the following system of equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \sin(x) + \exp(y) - 1 \\ \sin(x - y) \end{pmatrix} \tag{80}$$

(a) See the notebook.

(b) Let's compute the Jacobian matrix at (0,0):

$$J(0, 0) = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} |_{(0,0)} \tag{81}$$

$$= \begin{pmatrix} \cos(x) & \exp y \\ \cos(x - y) & -\cos(x - y) \end{pmatrix} |_{(0,0)} \tag{82}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{83}$$

The corresponding linear system of equations is thus:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{84}$$

(c) See the notebook.

(d) See the notebook.