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Forecasting seasonals and trends by exponentially weighted moving averages

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Abstract

The paper provides a systematic development of the forecasting expressions for exponential weighted moving averages. Methods for series with no trend, or additive or multiplicative trend are examined. Similarly, the methods cover non-seasonal, and seasonal series with additive or multiplicative error structures. The paper is a reprinted version of the 1957 report to the Office of Naval Research (ONR 52) and is being published here to provide greater accessibility.

Keywords: Exponential smoothing; Forecasting; Local seasonals; Local trends

1. Introduction

An exponentially weighted moving average is a means of smoothing random fluctuations that has the following desirable properties: (1) declining weight is put on older data, (2) it is extremely easy to compute, and (3) minimum data is required. A new value of the average is obtained merely by computing a weighted average of two variables, the value of the average from the last period and the current value of the variable. This paper utilizes these desirable properties both to smooth current random fluctuations and to revise continuously seasonal and trend adjustments. These may then be extrapolated into the future for forecasts. The flexibility of the method combined with its economy of computation and data requirements make it especially suitable for industrial situations in which a large number of forecasts are needed for sales of individual products.

The simplest application of an exponentially moving average would be to the following stochastic process. Consider the problem of making an expected value forecast of a random variable whose mean changes between successive drawings. The following rule might be proposed: take a weighted average of *all* past observations and use this as your forecast of the present mean of the distribution, i.e.,

$$\bar{S}_t = B[S_t + AS_{t-1} + A^2 S_{t-2} + A^3 S_{t-3} + A^4 S_{t-4} + \ldots]]$$

where B is a constant between 0 and 1, A is $(1-B)$, the S's are observations of the variable and the t subscript indicates the time ordering of the observations. $\bar{S}_t$ is the estimate of the expected value of the distribution, ES_t. If the distribution mean is subject to large changes, A should be small so as to quickly attenuate the effect of old observations. However, if A is too small, $\bar{S}_t$ is subject to so much random variation

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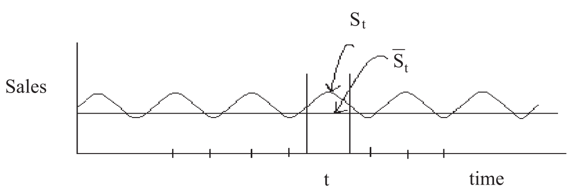
that it would be a poor estimator of the mean. The following relation is convenient in minimizing computations:

$$\bar{S}_t = BS_t + (1 - B)\bar{S}_{t-1}$$

A weighted moving average with exponential weights is clearly a sensible mode of behavior in dealing with this simple forecasting problem. An exploration of the exact conditions under which this behavior is optimal will not be considered, but rather the question of whether this approach to forecasting seems to hold promise in coping with trends and seasonals in forecasting.

2. Forecasting ratio seasonals



Let S_t = sales in period t and let $\bar{S}_t$ = smoothed and seasonally adjusted sales rate in period t. This is an estimate of $E\bar{S}_t$. P_t = periodic (seasonal) adjustment ratio for the tth period. This is an estimate of $E\bar{S}_t/ES_t$.

The sales rate is obtained by combining the current seasonally adjusted sales with the sales rate from the previous period

$$\bar{S}_t = AP_tS_t + (1 - A)\bar{S}_{t-1} \tag{1}$$

where the constant, A, determines how fast the exponential weights decline over the past consecutive periods.

$0 \leq A \leq 1.$

The current seasonal adjustment ratio is obtained by combining the current ratio of sales rate to sales with the seasonal adjustment rate from a year ago:

$$P_t = B\frac{\bar{S}_t}{S_t} + (1 - B)P_{t-N}. \tag{2}$$

where the constant, B, determines how fast the exponential weights decline over the past years—one period drawn from each year. N is the number of periods in a year (or whatever is the length of the periodic pattern).

Clearly, Eqs. (1) and (2) are interdependent, and hence must be solved simultaneously. This is a result of the fact that a new observation leads both to a revision of the sales rate. However, the determination of each of these depends on the other.Substitute Eq. (2) in Eq. (1).

$$\bar{S}_t = A[B\bar{S}_t + (1 - B)P_{t-N}S_t] + (1 - A)\bar{S}_{t-1}. \tag{3}$$

Solve for the current sales rate:

$$\bar{S}_t = \left[\frac{A(1 - B)}{1 - AB}\right]P_{t-N}S_t + \left[\frac{1 - A}{1 - AB}\right]\bar{S}_{t-1}. \tag{4}$$

Thus the current sales rate turns out to be a weighted average of the sales rate of the last period and the current sales seasonally adjusted by the index of a year ago. Numerical values of $\bar{S}_t$ from Eq. (4) can be substituted in Eq. (2) to determine P_t. However, substituting Eq. (4) in Eq. (2) gives us an explicit analytic expression for the new seasonal ratio:

$$P_t = \left[\frac{1 - B}{1 - AB}\right]P_{t-N} + \left[\frac{B(1 - A)}{1 - AB}\right]\frac{\bar{S}_{t-1}}{S_t}. \tag{5}$$

The new seasonal ratio is a weighted average of the old seasonal ratio and the ratio of the past sales rate to current sales.

The values of A and B can be chosen independently depending upon how fast the level of sales changes and how fast the seasonal patterns change. A high value of A will minimize lag in following changes in the level of sales, but there will be less smoothing of random fluctuations. A high value of A gives weight only to very recent observations thereby minimizing lag in following changes in the expected value of sales. A low value of A gives weight to many and therefore older observations; the larger sample size will reduce the random variability thereby giving greater accuracy in estimating the mean—unless the mean has changed in the meantime. Thus A is chosen for the best combination of smoothness and lag in following changes in seasonal patterns. The trial of several values of A and B with past sales data should suggest suitable values. Furthermore, if a large number of products were involved, it is likely that some relationships can be found between the stochastic parameters of the time series that would indicate desirable values for A and B.

Forecasts may be made of the expected value of sales T periods in the future by using the following extrapolation formula:

$$ES_{t+T} = \frac{\bar{S}_t}{P_{t+T-N}} \qquad T = 1, 2, \ldots N^{1}. \tag{6}$$

This amounts to assuming that the present sales rate will persist in the future and the sales forecast is made by applying the seasonal ratio that is applicable to the period.

Examination of Eq. (5) shows that the seasonal ratio is gradually modified by the factor, $\bar{S}_{t-1}/S_t$. While this is intended as a current estimate of the seasonal ratio for the tth period, actually there is a time difference of one period between the two variables. Thus, if there is any steady trend in sales, it will become incorporated in the seasonal ratio. This would not appear undesirable until it is remembered that there is a lag of N periods in the adjustment of the seasonal ratios.

This is clear in Eq. (5) where P_t is obtained by modifying P_{t-N}. With a periodic pattern repeating every N periods, there is little alternative to introducing a lag of N periods in adjusting the seasonal ratios—for example, January comes but once a year and to learn anything new about Januarys requires a year's wait in between. In contrast, some new information is available on the trend every period, and hence there is a decided disadvantage in lumping trend and seasonal factors together and introducing the lag of N periods in modifying the trend factor. This problem can be avoided by introducing a trend variable into the analysis.

[1] The forecasts can be readily extended beyond N period by reusing the N seasonal ratios, $P_{t+1-N}, \ldots, P_t$.

3. Forecasting a ratio trend

In order to explore the application of the exponentially weighted moving average to forecasting a trend, we will first consider the simplest case in which there is no seasonal fluctuation. The sales rate is obtained by combining the current sales with the sales rate from the previous period corrected for trend.

$$\bar{S}_t = AS_t + (1-A)R_t\bar{S}_{t-1}, \tag{7}$$

where R_t is the trend adjustment ratio for the tth period. This is an estimate of $E\bar{S}_t/ES_{t-1}$. Note the implicit assumption that the trend has a constant percentage change.

The current trend ratio is obtained by combining the current trend ratio with the trend ratio from the previous period.

$$R_t = C\frac{\bar{S}_t}{\bar{S}_{t-1}} + (1-C)R_{t-1}, \tag{8}$$

where the constant, C, determines how fast the exponential weights applied to trend ratios decline over the past consecutive periods. Substitute Eq. (8) in Eq. (7) to obtain:

$$\bar{S}_t = \left[\frac{S}{1-(1-A)C}\right]S_t + \left[\frac{(1-A)(1-C)}{1-(1-A)C}\right]R_{t-1}\bar{S}_{t-1}. \tag{9}$$

Substitute Eq. (9) in Eq. (8) to obtain:

$$R_t = \left[\frac{AC}{1-(1-A)C}\right]\frac{S_t}{\bar{S}_{t-1}} + \left[\frac{1-C}{1-(1-A)C}\right]R_{t-1}. \tag{10}$$

Forecasts may be made of the expected value of sales T periods in the future by using the following extrapolation formula:

$$ES_{t+T} = S_tR_t^T \qquad T = 1, 2, \ldots, N. \tag{11}$$

This assumes that the present estimate of sales rate will continue in the future modified by the percentage trend factor that is currently estimated. The two analyses can now begin.

4. Forecasting a ratio trend and seasonals

The sales rate is obtained by combining the current sales adjusted for seasonal with the sales rate of the previous period adjusted for trend:

$$\bar{S}_t = AS_tP_t + (1-A)R_t\bar{S}_{t-1}. \tag{12}$$

Each seasonal ratio is revised every N periods as before in Eq. (2). The trend ratio is revised each period as before in Eq. (8). Now substituting Eqs. (2) and (8) in Eq. (12) we obtain:

$$\bar{S}_t = \left[\frac{A(1-B)}{1-AB-(1-A)C}\right]S_tP_{t-N} + \left[\frac{(1-A)(1-C)}{1-AB-(1-A)C}\right]\bar{S}_{t-1}R_{t-1}. \tag{13}$$

Substituting Eq. (13) in Eq. (2) yields:

$$P_t = \left[\frac{[1-B][1-(1-A)C]}{1-AB-(1-A)C}\right]P_{t-N} + \left[\frac{B(1-A)(1-C)}{1-AB-(1-A)C}\right]\frac{\bar{S}_{t-1}}{S_t}R_{t-1}. \tag{14}$$

Substituting Eq. (13) in Eq. (8) yields:

$$R_t = \left[\frac{AC(1-B)}{1-AB-(1-A)C}\right]\frac{S_tP_{t-N}}{\bar{S}_{t-1}} + \left[\frac{[1-C][1-AB]}{1-AB-(1-A)C}\right]R_{t-1}. \tag{15}$$

Forecasts may be made of the expected value of sales T periods in the future by using the following extrapolation formula:

$$ES_{t+T} = \frac{\bar{S}_tR_t^T}{P_{t+T-N}} \qquad T = 1, 2, \ldots N. \tag{16}$$

This assures that the present sales rate may be adjusted for a percentage trend and a seasonal both of which continue into the future.

The exponential weighting coefficients, A, B, and C, may be adjusted independently depending on random variability on one hand and on the other, the speed with which the sales rate changes aside from trend, the speed with which the seasonal pattern changes, and the speed with which the trend changes respectively. The use of exponentially weighted moving averages for forecasting is not limited to ratio trend and seasonal adjustments. Trends may change by constant increments rather than constant percentages and seasonal fluctuations may be additive rather than multiplicative.

5. Forecasting a linear trend and additive seasonals

We consider the case of both trend and seasonal, but as before either of these can be worked out separately. The sales rate is obtained from the following relation:

$$\bar{S}_t = a(S_t + P_t) + (1-a)(\bar{S}_{t-1} + r_t). \tag{17}$$

where P_t = periodic (seasonal) adjustment increment for the tth period. This is an estimate of $E\bar{S}_t - ES_t$. r_t = trend adjustment increment for the tth period. This is an estimate of $E\bar{S}_t - E\bar{S}_{t-1}$.

Note that the lower case letters are used to denote the additive case while upper case letters denote the multiplicative case above.

The periodic adjustment increment is estimated as follows:

$$P_t = b(\bar{S}_t - S_t) + (1-b)P_{t-N} \tag{18}$$

The trend adjustment increment is estimated as follows:

$$r_t = c(\bar{S}_t - \bar{S}_{t-1}) + (1-c)r_{t-1}. \tag{19}$$

Substituting Eqs. (18) and (19) in Eq. (17), we obtain:

$$\bar{S}_t = \left[\frac{a(1-b)}{1-ab-(1-a)c}\right]S_t + \left[\frac{(1-c)(1-a)}{1-ab-(1-a)c}\right][\bar{S}_{t-1} + r_{t-1}] + \left[\frac{(1-b)a}{1-ab-(1-a)c}\right]P_{t-N}. \tag{20}$$

Substituting Eq. (20) in Eq. (18), we obtain:

$$P_t = \left[\frac{ab(1-b)}{1-ab-(1-a)c} - b\right] S_t + \left[\frac{(1-c)(1-a)b}{1-ab-(1-a)c}\right][\bar{S}_{t-1} + r_{t-1}] + \left[\frac{ab(1-b)}{1-ab-(1-a)c} + (1-b)\right] P_{t-N}. \quad (21)$$

Substituting Eq. (20) in Eq. (19), we obtain:

$$r_t = \left[\frac{ac(1-b)}{1-ab-(1-a)c}\right] S_t + \left[\frac{(1-c)(1-a)c}{1-ab-(1-a)c} - c\right] \bar{S}_{t-1} + \left[\frac{(1-c)(1-a)c}{1-ab-(1-a)c} + (1-c)\right] r_{t-1} + \left[\frac{(1-b)ac}{1-ab-(1-a)c}\right] P_{t-N}. \quad (22)$$

Forecasts of sales T periods in the future may be made with the following relation:

$$ES_{t+T} = \bar{S}_t + r_t T - P_{t+T-N} \quad T = 1, 2, \ldots, N. \quad (23)$$

The trend adjustment solution without the seasonal may be obtained from Eqs. (20)–(22) by letting b and P_{t-N} equal zero. The seasonal adjustment solution without the trend may be obtained by letting c and r_{t-1} equal zero. The combination of a multiplicative seasonal and a linear trend is very frequently found to be suitable for forecasting. The following derivation extends the exponentially weighted moving average to this case.

6. Forecasting a linear trend and ratio seasonals

The sales rate is estimated by the following relation:

$$\bar{S}_t = AS_t P_t + (1-A)(\bar{S}_{t-1} + r_t) \quad (24)$$

The seasonal rate is estimated by Eq. (2), and the trend increment is estimated by Eq. (19). Substituting these in Eq. (24) yields the formula for calculating $\bar{S}_t$:

$$\bar{S}_t = \left[\frac{A(1-B)}{1-AB-(1-A)c}\right] S_t P_{t-N} + \left[\frac{(1-A)(1-c)}{1-AB-(1-A)c}\right][\bar{S}_{t-1} + r_{t-1}]. \quad (25)$$

Values of P_t and r_t may then be calculated numerically by substitutions in Eqs. (2) and (19), respectively. Forecasts for sales T periods in the future would be obtained by the formula:

$$ES_{t+T} = \frac{\bar{S}_t + r_t T}{P_{t+T-N}} \qquad T = 1, 2, \ldots, N. \quad (26)$$

7. Comments on the theory

The foregoing derivations illustrate the flexibility of exponentially weighted moving averages in forecasting. They make possible a simple integrated approach to the estimation of both trends and seasonals.

The underlying stochastic theory has not been explored here. A great deal of work has been done on optimal filter design for stationary time series using the criteria of minimizing the sum of the square of the errors.[2] The solution of this problem leads to the optimality of linear filters of which exponential weights are but a special case. However, the implicit models assumed by the analysis of this paper are a good deal more complicated than those assumed in Wiener's book.

Many linear forecasting rules can be constructed using the general approach that is presented here. The rules that have been developed are all based on first order difference equations. Second- and higher-order equations would allow even greater flexibility. Many other extensions are possible. For example, a macroeconomic variable like GNP could be forecasted

[2] For example, see Norbert Wiener, *The Extrapolation, Interpolation, and Smoothing of Stationary Time Series*, 1949. Note particularly Levinson's appendix on the solution for difference equations.

independently, and used to modify the forecast of an individual product through a self-adjusting parameter.

8. Conclusion

This exploratory analysis indicates the great flexibility of exponentially weighted moving averages in dealing with forecasts of seasonals and trends. Further study seems fully justified both on empirical and theoretical levels.

Biography: Charles C. HOLT is Professor of Management Emeritus at the Graduate School of Business, University of Texas at Austin. His current research is on quantitative decision methods, decision support systems, and financial forecasting. Previously he has done research and teaching at M.I.T., Carnegie Mellon University, the London School of Economics, the University of Wisconsin, and the Urban Institute. He has been active in computer applications since 1947, and has done research on automatic control, the simulation of economic systems, scheduling production, employment and inventories, and the dynamics of inflation and unemployment.